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JEL Classification: C72, D74

Keywords: Landscape theory, landscape equilibrium, blocs, gradual deviation, potential functions, hedonic games

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A Game-Theoretical Model of the Landscape Theory*

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In this paper we examine a game-theoretical generalization of the landscape theory introduced by Axelrod and Bennett (1993). In their two-bloc setting each player ranks the blocs on the basis of the sum of her individual evaluations of members of the group. We extend the Axelrod-Bennett setting by allowing an arbitrary number of blocs and expanding the set of possible deviations to include multi-country gradual deviations. We show that a Pareto optimal landscape equilibrium which is immune to profitable gradual deviations always exists. We also indicate that while a landscape equilibrium is a stronger concept than Nash equilibrium in pure strategies, it is weaker than strong Nash equilibrium.

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1 Introduction

This paper examines the class of strategic environments covered by the "landscape theory" introduced by Axelrod and Bennett (1993) (AB — henceforth). In the landscape setting the actors (countries) are partitioned into two mutually exclusive blocs on the basis of their propensity to work together with other players on bilateral basis. All players rank groups according to the sum of her individual evaluations of all members of the group. In this sense the AB approach to international interactions is related to Bueno de Mesquita (1975, 1981) who constructed a proximity matrix for every pair of nations based on history of their defense cooperation.

Each actor i is characterized by the value of her size/strength/influence parameter s_i. For each pair of countries i and j there is a parameter p_{ij} (positive or negative), the value of which represents the propensity for collaboration between i and j. Thus, the data of the model consists of a n-dimensional vector of countries' strength parameters and an $n \times n$ matrix P of pairwise proximity coefficients.

For an arbitrary partition of all countries into two blocs, AB define the *frustration* of a country i as the sum of the proximity coefficients p_{ij} for all members outside her bloc weighted by their strength parameter s_j. Obviously, the country frustration will be reduced if it avoids countries with whom it has a strong negative propensity to align. The *energy* of any two-bloc partition is then determined as the sum of individual frustrations of all countries weighted by their size. The objective of the theory is to identify the configurations that yield, as they call it, a local and *global* minimum of energy. To attain these outcomes, AB used the *incremental* or *gradual* approach by allowing single countries to switch their membership, one at a time, to generate a new configuration with the reduced energy level. Assuming the symmetry of the proximity matrix P, i.e. $p_{ij} = p_{ji}$ for all pairs of players i, j, AB showed that for any initial bloc structure, the sequential gradual reduction of energy does not contain cycles and is terminated when a stable configuration is attained. Note that the symmetry of the proximity matrix P is essential to obtain stable configurations. Indeed, consider a game with two players, where player 1 prefers to join player 2, i.e., $p_{12} > 0$, whereas player 2 would like to avoid being together with 1, i.e., $p_{21} < 0$. The game obviously does not admit a stable partition, as the partition in two groups would be challenged by player 1, while the creation of a two-country bloc would be rejected by player 2.

AB provide a spectacular application of the landscape theory to European alliances prior to World War II. By using the Correlates of War data and estimating the propensity for cooperation based on ethnic and border conflicts, history, etc., AB calibrate a matrix P to conclude that that there were two stable configurations. One is the expected partition to the Axis and Allies of World War II, while the other separates USSR, Yugoslavia and Greece from the rest of Europe! Axelrod et al. (1995) also illustrate and test the landscape theory by estimating the choices of nine computer companies to join one of two alliances sponsoring competing UNIX operating system standards in 1988.

Even though AB have not done that explicitly, it is straightforward to present their setting in game-theoretical terms. Each actor i is a player with two available pure strategies corresponding to two blocs, X and Y, and her payoff function is represented by her frustration level derived from the two-bloc partition. Thus, after minor adjustments, proper reformulation, and clarifications, AB in fact show the existence of a pure strategies Nash equilibrium in landscape games. As Bennett (2000, p. 51) points out: "A local optimum is defined as a configuration for which every adjacent configuration has higher (worse) energy. When the system reaches one of those points, no further improvement in energy is possible given a single step (change of coalition by one actor). This optimum is akin to a Nash equilibrium in game theory, wherein no single actor can improve its own payoff by choosing a different move." Interestingly enough, the AB energy function E could be viewed as a potential, so that symmetric landscape games belong to the class of potential games examined by Monderer and Shapley (1996).

Notice that landscape games belong to the class of hedonic games pioneered by Banerjee, Konishi and Somnez (2001) and Bogomolnaia and Jackson (2002). Hedonic games are coalition formation games, where the payoff of any player depends solely upon the composition of the coalition to which she belongs, and a strategic choice made by the coalition does not impact its members' payoffs. This is the case for landscape games, where each player possesses a precise evaluation of every potential partner and then ranks groups according to the sum of her individual evaluations of all members of the group she may join. In the case of equal values of the strength parameter s_i for all countries, this model belongs to the class of *additively separable* games in Banerjee, Konishi and Somnez (2001). By constructing a potential function, as in AB, Bogomolnaia and Jackson (2002) show that the symmetric additively separable hedonic games, including the landscape games, admit a Nash stable configuration.

In this paper, we consider the class of landscape games that expands the AB framework in two aspects. First, we allow for an arbitrary number of blocs to form, without limiting ourselves to two-bloc configurations.[1] The configurations with more than two blocs have a place in various environments. In fact, during the Cold War between East and West that followed the end of the World War II, an important role has been played by the third bloc of non-aligned countries. And nowdays, when the world is often described as a multi-polar environment, the study of multi-bloc settings becomes even more relevant. Another distinction with regard to the AB model is that we expand the notion of incremental or gradual deviations in AB. AB allowed for individual countries to switch their bloc membership, one at a time. In the same time the right of any country to deviate could not be permanently denied. In our framework we take the gradual approach further by allowing several countries to switch their blocs at the same time. However, the cost of absorption of new members from different blocs could be quite high. Thus, a switch will be allowed only for a subgroup from one bloc to another. We call such a deviation gradual and define a *landscape equilibrium* as a configuration immune to gradual deviations. Note that individual deviations are obviously allowed under the umbrella of gradualism.

Our main result shows that, under the symmetry assumption, there is a landscape equilibrium. Moreover, there exists a Pareto optimal landscape equilibium under which there is no other strategy profile to yield at least the same utility for all players, with a strictly higher utility level for, at least, one player. It is important to underscore that the existence of a Pareto optimal landscape equilibrium rules out an emergence of prisoner's dilemma where countries acting in their own self-interests do not produce the optimal outcome. Interestingly, some aspects of Pareto optimality have been discussed by AB, who searched for the global optimum as the lowest energy level of any configuration. Since the concept of landscape equilibrium is stronger than Nash equilibrium, our result reconfirms the existence of a Nash equilibrium in the landscape games. On the other hand, we also consider a more demanding notion of strong Nash equilibrium introduced by Aumann (1959), which requites immunity against any deviation by any group of players. However, as is implied by the result in Banerjee, Konishi and Sonmez

[1]See Florian and Galam (2000) for a discussion on a three-bloc extension of the landscape theory.

(2001), a strong Nash equilibrium in landscape games may fail to exist. Thus, the unrestricted extension of the set of feasible deviations not only violates the concept of gradualism, but also diminishes the likelihood of obtaining a meaningful existence result.

The paper is organized as follows. In the next section we offer a brief review of the literature. In Section 3 we present a model and state our result on existence of a Pareto optimal landscape equilibrium. In Section 4 we discuss the links of our equilibrium concept with other modifications of Nash equilibrium. The proof of the main result is relegated to the Appendix.

2 Related Literature

Our paper has roots in two branches of the existing literature. On one hand, as pointed out above, it belongs to the research area on hedonic games. The specificity of landscape games is that players compare coalitions of partners on the basis of pairwise evaluations[2]. The more distant is a player from her potential partner in a coalition, the less likely are the prospects of their cooperation. In the international relations setting Bueno de Mesquita (1975, 1981) constructed a matrix that captures the proximity between pairs of nations according to their alliances on defense issues and defined “indicators of tightness” which are used as a key determinant to evaluate the war proneness of the international system. Le Breton and Weber (1994) consider such a setting in the case where only a two-player coalitions can be formed. Desmet et al. (2011) consider a nation formation game where pairwise hedonic heterogeneity is described by the matrix of genetic distances between nations as calculated by scholars in population genetics.

On the other hand, our paper is related to the literature on congestion games. Indeed, when all non-diagonal entries p_{ij} are negative, we obtain a congestion game, where all individuals are negatively impacted by the presence of others in their coalition[3]. Any such congestion game belongs to the class of games considered by Quint and Shubik (1994), Milchtaich (1996)

[2]This is different from other assumptions like, for instance, the one considered by Milchtaich and Winter (2002) and Kukushkin (2019). They consider instead hedonic games where each player, characterized by a one-dimensional parameter (status), evaluates every potential group on the basis of the status values of its members.

[3]Obviously, if all off-diagonal entries of the matrix D are positive, the unique Nash equilibrium (and therefore strong Nash equilibrium) is the grand coalition.

and Konishi, Le Breton and Weber (1997a,b) who prove the existence of a Nash equilibrium in pure strategies for anonymous congestion games. For the latter class of games Konishi, Le Breton and Weber (1997a), in fact, show the existence of a strong Nash equilibrium. Rosentahl explores a different class of anonymous congestion games. On one hand, it is more general since the players do not have the set of strategies (a strategy is a path in a network). The payoff of a player is defined as the sum of her payoffs on each segment of the path that she ultimately selects. Rosenthal assumes that on any given segment, the payoffs of the players who have access to that segment in their strategy set are identical, and proves that any game in his class is a potential game, and, thus, admits a Nash equilibrium in pure strategies. Rosentahl was the first to introduce this notion which was later systematically explored by Monderer and Shapley (1996). Note that the Nash equilibrium does not need to be strong. To address this point, Holzman and Law-Yone (1997) introduce the notion of strong potential and obtain conditions on the network that guarantee the existence of a strong Nash equilibria. This topic is further explored in Voorneveld et al. (1999) and Harks, Klimm and Möhring (2013). Finally, we would like to point out that strong Nash equilibria has received recently a lot of attention in algorithmic game theory, see, e.g., Andelman, Feldman and Mansour (2009), Chien and Sinclair (2009), Epstein, Feldman and Mansour (2009) who compute strong versions of the price of anarchy for various classes of games.

3 The Model and The Result

The landscape game Γ^0 is defined as follows. Let $N = \{1, 2, \dots, n\}$ be a finite set of players. Each player i is associated with the positive value s_i which represents her influence, or in the case of countries, the population size or military power. For every pair of players i and j in N there is a value p_{ij} (positive or negative) that represents the strength of ties between i and j and their benefit of being members of the same coalition. It is assumed that this value is symmetric for every pair i and j, i.e., $p_{ij} = p_{ji}$ with $p_{ii} = 0$ for every player i. The data on pairwise propensities is therefore represented by the symmetric $n \times n$ matrix P.

The set of alternatives $X = \{x^1, \dots, x^m\}$ is common for all players. Each player $i \in N$ chooses an alternative $x_i \in X$.[4] Two players i and j belong

[4]The lower and upper indices indicate players and alternatives, respectively. The ex-

to the same bloc if $x_i = x_j$. A vector of players' choices $\mathbf{x} = (x_1, \ldots, x_n)$ generates the partition $G(\mathbf{x})$ of the set N, which consists of no more than m non-empty blocs. We denote $G^i(\mathbf{x}) \in G(\mathbf{x})$ the bloc that contains player i.

The payoff $U_i^0(\mathbf{x})$, $i \in N$, of each player solely depends upon the bloc to which she belongs[5]. More specifically, in the landscape theory setting, we assume that

$$U_i^0(\mathbf{x}) = \sum_{j \in G^i(\mathbf{x})} p_{ij} s_j. \quad (1)$$

We shall analyze the stability of emerging bloc formations. To do so, we need to examine a threat of feasible deviations. For a given strategy profile $\mathbf{x}$, a group of players S would deviate if each member i of S would switch to another bloc, while raising her utility. Formally.

Definition 1 - Deviation: Let a strategy profile $\mathbf{x} = \{x_1, \ldots, x_n\}$ be given. A *(feasible)* deviation from $\mathbf{x}$ by a group of players S is a profile $\mathbf{x}' = \{\{x_i'\}_{i \in S}, \{x_i\}_{i \notin S}\}$ consisting of "new" choices for players in S and unaltered choices for the rest of the players. It is *profitable* if:

$$U_i^0(\mathbf{x}') > U_i^0(\mathbf{x}) \quad \text{for all } i \in S,$$

However, in our setting, as in many others, one needs to impose some restrictions on feasible deviations. The coordination challenges, switching costs and other factors may limit the size and the composition of deviating groups. In fact, AB argued for need for *incrementalism* and allowed only for one single actor to switch bloc. Thus, we first consider the case where the only feasible deviating coalitions are singletones. It immediately leads to the notion of Nash equilibrium.

Definition 2: Nash equilibrium: A strategic profile $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a Nash equilibrium if for all $i \in N$, there is no profitable deviation from $\mathbf{x}$ by an individual in N.

Following AB, we impose severe limitations on the range of possible deviations. However, we allow for a wider set of deviations than simply singletons.

pression $x_i \in X$ means that there is $x^k \in X$ such that $x_i = x^k$.

[5]Note that in our paper, we use a more common notion of *utility* rather than frustation. The utility maximization and the frustration minimization are, obviously, equivalent objectives.

In our view, the essence of deviation costs in the landscape theory boils down to the absorption cost of a bloc incurred by the acceptance of new members. Thus, we allow only for subgroups of a bloc to switch to another bloc. We rule out the situations where members from two blocs, say, A and B, join another bloc C. It goes along the lines of *gradual approach* supported by AB. It is important to point out that no country should be prevented from switching to a bloc of its choice. What we examine here is the first stage of possible realignment process.

Definition 3: Gradualism and Landscape Equilibrium: Let a strategy profile $\mathbf{x} = \{x_1, \ldots, x_n\}$ be given. Assume that a strategy $\mathbf{x}'$ represents a deviation from $\mathbf{x}$ by a group of players S. The deviation is called *gradual* if the following condition is satisfied for every of two players i and j in S:

$$(GR)\ \text{ If } i, j \in S \text{ and } x'_i = x'_j \quad \text{then} \quad x_i = x_j.$$

A profile $\mathbf{x}$ is called a *landscape equilibrium* if it does not allow a profitable gradual deviation.

In addition to landscape equilibrium, we introduce the notions of a strong Nash equilibrium (Aumann (1959)), which is immune against an unrestricted set of coalitional deviations.

Definition 4: Strong Nash Equilibrium: A strategy profile $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a *strong Nash equilibrium* if for all $S \subseteq N$, there there is no profitable deviation from $\mathbf{x}$ by group S.

We have our main result:

Theorem 1: The game Γ^0 admits a landscape equilibrium, which is strongly Pareto optimal.[6]

The proof of the Theorem is relegated to the Appendix.

[6]A profile $\mathbf{x}$ is strongly Pareto optimal if there is no other strategy profile $\mathbf{y}$ such that $U_i^0(\mathbf{y}) \geq U_i^0(\mathbf{x})$ for all $i \in N$ with a strict inequality for at least one i.

4 Comments

Let usfirst offer some comments on the connection between landscape equilibria and the two other equilibrium notions introduced above. First, the notion of gradual deviation in Definition 3 is stronger than individual deviations, and, therefore, the set of landscape equilibria is smaller than the set of Nash equilibria. Thus, Theorem 1 yields the existence of a Nash equilibrium as well. The following example shows that, in general, the sets of landscape and Nash equilibria do not coincide, and a Nash equilibrium does not necessarily constitute a landscape equilibrium.

Example: Consider the game Γ^0 with four players and two alternatives ($n = 4, m = 2$). The influence parameter s_i is assumed to be equal to 1 for all players, and the distance matrix P is given by:

$$P = \begin{pmatrix} 0 & 1000 & -100 & -50 \\ 1000 & 0 & -100 & -50 \\ -100 & -100 & 0 & -400 \\ -50 & -50 & -400 & 0 \end{pmatrix}$$

It is easy to see that the strategy profile $\{a, a, a, b\}$ is a Nash equilibrium. However, players 1 and 2 would benefit by switching to b and joining player 4. By Definition 3, it is a profitable gradual deviation, and this profile is not a landscape equilibrium. Notice that in this example the profile $\{a, a, b, a\}$ is a landscape equilibrium.

Similarly, the notion of gradual deviation is weaker than the unrestricted notion of deviation utilized at strong Nash equilibrium. Thus, the set of landscape equilibria is larger than the set of Nash equilibria. However, in the case of two blocs ($m = 2$), a threat of gradual deviation is vacuous, and the sets of landscape equilibria and strong Nash equilibria coincide. Thus, Theorem 1 yields the existence of a strong Nash equilibria in the case of two blocs (see Dower at al. (2020)). We would like to complete this section by pointing out that in the case, where the number of alternatives m exceeds the number of players n, the set of strong Nash equilibria is nonempty if and only if the set of core stable configurations in the Banerjee, Konishi and Sonmez (2001) model.[7] Thus, their example of nonexistence of a core stable configuration yields the nonexistence of a strong Nash equilibria in

[7]Even though they do not explicitly include a set of alternatives.

our model. The latter conclusion reinforces the importance of examination of landscape equilibria.

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Appendix

This section contains the proof of Theorem 1. We first modify the game Γ^0 and construct a potential function P over the set of all strategies profiles. We then show that its maximum cannot be improved upon via gradual deviations. Due do the finiteness of our game, a maximum exists, and, thus, represents a landscape equilibrium.

Proof of Theorem 1: For each player $i \in N$ multiply the utility function U_i^0 by i's own influence parameter s_i:

$$s_i U_i^0(\mathbf{x}) = \sum_{j \in G^i(\mathbf{x})} p_{ij} s_i s_j.$$

While this modification does not alter the equilibrium structure of the game, it makes the summation term $p_{ij}s_is_j$ symmetric in i and j: $p_{ij}s_is_j = p_{ji}s_js_i$. We therefore define the game Γ that differs from Γ^0 only with respect to individual utility functions, that, with some abuse of notation, are represented as

$$U_i(\mathbf{x}) = \sum_{j \in G^i(\mathbf{x})} p_{ij}. \tag{2}$$

It suffices to prove that the game Γ admits a landscape equilibrium. As we indicated above, the proof relies on the introduction of the utilitarian social welfare function $P(x) \equiv \sum_{i \in N} U_i(x)$, which is *a landscape strong potential* in the following sense: If there is a gradual deviation of group S from $\mathbf{x}$ to $\mathbf{x}'$ in game Γ , then $P(\mathbf{x}') > P(\mathbf{x})$.

Let us introduce an additional assumption on feasible coalitional deviations. The deviation of group S from $\mathbf{x}$ via $\mathbf{x}'$ is feasible if

(G) for any $i, j \in S$ the equality $x_i = x_j$ implies $x_i' = x_j'$.

Condition (G) means that in order for deviation to be feasible, the deviating members of the same bloc should stay together in the new bloc as well.

The proof of Theorem 1 proceeds in three steps. First, by constructing a potential function we show the existence of a strategy profile which is immune against deviations that satisfy not only (GR) but also (G). Second, we demonstrate that condition (G) can be suspended with, which guarantees the existence of a landscape equilibrium. Finally, the choice of the landscape equilibrium as a maximizer of the total payoff function completes the proof of Theorem 1.

Lemma 1 *Let $\tilde{\mathbf{x}}$ be a maximum of the function $P(\mathbf{x})$ over the set of all strategy profiles. Then there is no group S and profile $\mathbf{x}'$ such that S profitably deviates from $\tilde{\mathbf{x}}$ via $\mathbf{x}'$ while satisfying (GR) and (G).*

Proof: Let $\tilde{\mathbf{x}}$ be the maximum of the function $P(\mathbf{x})$ over the set of all strategy profiles. A maximum exists given that the set of alternatives is finite. Players' choices at the maximum induce the partition of the set N onto m (not necessarily nonempty) coalitions. Each such coalition, denoted by $G^j(\tilde{\mathbf{x}})$, $j = 1, \ldots, m$, consists of players $i \in N$ who choose the strategy $\tilde{x}_i = x^j$ in X.

Assume, to the contrary, that there is a coalition $S \subset N$ and strategy choices $x'_i, i \in S$ such that S profitably deviates from $\tilde{\mathbf{x}}$ via $\mathbf{x}'$ in line with (GR) and (G), where $\mathbf{x}' = \{\{x'_i\}_{i \in S}, \{x_i\}_{i \notin S}\}$.

For every pair of alternatives $x^k \neq x^l$ in X denote by $T^{kl} \subset S$ the set of players who changed their choice from x^k to x^l. Then there are (at most) $m(m-1)$ such groups. For every $x^k \in X$ denote three sets of players :

Q^k — those who left the bloc that chose x^k at $\tilde{\mathbf{x}}$: $Q^k = \cup_{l \neq k} T^{kl}$,

R^k — those who choose x^k at $\mathbf{x}'$ but not at $\tilde{\mathbf{x}}$: $R^k = \cup_{l \neq k} T^{lk}$,

Ψ^k — those who choose x^k at both $\mathbf{x}'$ and $\tilde{\mathbf{x}}$.

Note that $\Psi^k = G^k(\tilde{\mathbf{x}}) \setminus Q^k$. Since each player from $T^{kl} \subset S$ increases her payoff by switching from x^k to x^l, it follows that

$$U_i(\mathbf{x}') - U_i(\tilde{\mathbf{x}}) > 0 \tag{3}$$

for all $i \in T^{kl}$.

To simplify the notation, we introduce the mapping $\sigma(\cdot,\cdot) : N \times N \longrightarrow \Re$ for every two subsets N_1 and N_2 of N as follows:

$$\sigma(N_1, N_2) = \sum_{i \in N_1} \sum_{j \in N_2} p_{ij}.$$

In particular, for every $i \in N$ and a strategy profile $\mathbf{x}$ we have

$$\sigma(\{i\}, G^i(\mathbf{x})) = \sum_{j \in G^i(\mathbf{x})} p_{ij} = U_i(\mathbf{x}) \tag{4}$$

(recall that G^i is the set of players who share the choice of player i). Moreover, the symmetry of D induces the symmetry of σ:

$$\sigma(N_1, N_2) = \sigma(N_2, N_1). \tag{5}$$

In addition, for every triple $N_1, N_2, N_3 \subset N$, we have

$$\sigma(N_1, N_2 \cup N_3) = \sigma(N_1, N_2) + \sigma(N_1, N_3) \tag{6}$$

We extend the notation $\sigma(N_1, N_2)$ to the case of $N_1 = \emptyset$ or / and $N_2 = \emptyset$ by assigning $\sigma(\emptyset, N_2) = \sigma(N_1, \emptyset) = \sigma(\emptyset, \emptyset) = 0$.

Let player i switch from x^k to x^l. Then combining observation (6) with the decompositions

$$G^k(\tilde{\mathbf{x}}) = \Psi^k \cup_{r=1, r\neq k}^m T^{kr} \quad \text{and} \quad G^l(\mathbf{x}') = \Psi^l \cup_{q=1, q\neq l}^m T^{ql},$$

we rewrite inequality (**??**) in the following way:

$$\sigma(\{i\}, \Psi^l \cup_{q=1,q\neq l}^m T^{ql}) - \sigma(\{i\}, \Psi^k \cup_{r=1,r\neq k}^m T^{kr}) > 0 \quad \text{for all } i \in T^{kl}. \tag{7}$$

Summing up inequalities (7) over all $i \in T^{kl}$, we have

$$\sigma(T^{kl}, \Psi^l \cup_{q=1,q\neq l}^m T^{ql}) > \sigma(T^{kl}, \Psi^k \cup_{r=1,r\neq k}^m T^{kr}). \tag{8}$$

By using the properties (5) and (6) of σ and cancelling out the identical terms, we expand the last inequality to obtain:

$$\sigma(T^{kl}, \Psi^l) + \sum_{q=1,q\neq l}^{m} \sigma(T^{kl}, T^{ql}) > \sigma(T^{kl}, \Psi^k) + \sum_{r=1,r\neq k}^{m} \sigma(T^{kl}, T^{kr}). \tag{9}$$

Condition (GR) does not allow players from two different blocs at $\tilde{\mathbf{x}}$ to join the same bloc at $\mathbf{x}'$, i. e., either T^{kl} or T^{ql} is empty set for each pair (T^{kl}, T^{ql}), $q = 1, \ldots, m$, $q \neq l$, $q \neq k$. The corresponding terms $\sigma(T^{kl}, T^{ql})$ are equal to zero. In the same way, according to (G), $\sigma(T^{kl}, T^{kr}) = 0$, $r = 1, \ldots, m$, $r \neq l$, $r \neq k$. Therefore, (9) is simplified to

$$\sigma(T^{kl}, \Psi^l) > \sigma(T^{kl}, \Psi^k). \tag{10}$$

There are $m(m-1) - 1$ (not necessarily nonempty) other groups of players from S that alter their strategies and raise their payoff by shifting from $\tilde{\mathbf{x}}$ to $\mathbf{x}'$. Summing up all inequalities (10) obtained for different pairs (k, l) we end up with

$$\sum_{i\in S} (U_i(\mathbf{x}') - U_i(\tilde{\mathbf{x}})) = \sum_{q=1}^{m} (\sigma(R^q, \Psi^q) - \sigma(Q^q, \Psi^q)) > 0. \tag{11}$$

Now we evaluate the difference of the potential at the two points that represent the players' choices before and after the deviation:

$$P(\tilde{\mathbf{x}}) - P(\mathbf{x}') = \sum_{q=1}^{m} \sigma(\Psi^q \cup Q^q, \Psi^q \cup Q^q) - \sum_{q=1}^{m} \sigma(\Psi^q \cup R^q, \Psi^q \cup R^q).$$

Once more, the symmetry property (5) is applied and the terms $\sigma(\Psi^q, \Psi^q)$ are cancelled out. Then

$$P(\tilde{\mathbf{x}}) - P(\mathbf{x}') = \sum_{q=1}^{m} (2\sigma(Q^q, \Psi^q) + \sigma(Q^q, Q^q)) - \sum_{q=1}^{m} (2\sigma(R^q, \Psi^q) + \sigma(R^q, R^q)). \tag{12}$$

The conditions (GR) and (G) allow us to conclude that

$$\sum_{q=1}^{m} \sigma(Q^q, Q^q) - \sum_{q=1}^{m} \sigma(R^q, R^q) = 0. \tag{13}$$

Indeed, each (non-empty) R^q consists of a single sub-group T^{kq}: for any $q = 1, \ldots, m$ there is $k = k(q)$: $R^q = T^{kq}$ (the existence of the second sub-group would violate condition (GR)). According to (G), each Q^q also consists of a single sub-group T^{ql}: for any $q = 1, \ldots, m$ there is $l = l(q)$: $Q^q = T^{ql}$. Thus, the both sums in (13) consist of the same terms. From (12) and (13) it follows that

$$P(\tilde{\mathbf{x}}) - P(\mathbf{x}') = 2 \sum_{q=1}^{m} (\sigma(Q^q, \Psi^q) - \sigma(R^q, \Psi^q)). \tag{14}$$

Combining (11) and (14), we conclude that

$$P(\tilde{\mathbf{x}}) - P(\mathbf{x}') = 2 \sum_{i \in S} (U_i(\tilde{\mathbf{x}}) - U_i(\mathbf{x}')) < 0.$$

This contradicts the fact that the point $\tilde{x}$ maximizes the function P and completes the proof of Lemma 1. **Q.E.D.**

To complete the proof of Theorem 1, we first show that the profile $\tilde{\mathbf{x}}$, which is a maximum of the function P, represents a landscape equilibrium. Suppose, it is not a landscape equilibrium. Then there is a group $\hat{S}$ of players with strategies $\hat{x}_i, i \in \hat{S}$, such that S deviates from $\tilde{\mathbf{x}}$ via $\hat{\mathbf{x}}$, where $\hat{\mathbf{x}} = \{\{\hat{x}_i\}_{i \in \hat{S}}, \{\tilde{x}_j\}_{j \notin \hat{S}}\}$. Moreover, $\hat{S}$ satisfies (GR) but not (G).

Take a player $i \in \hat{S}$. Let $T = \{j \in \hat{S} : \hat{x}_j = \hat{x}_i,\}$ be the set of players in $\hat{S}$ who choose the same bloc as i does at $\hat{\mathbf{x}}$. We define a strategy profile $\mathbf{x}''$ generated by the deviation of the coalition T from the strategy profile $\tilde{\mathbf{x}}$, namely, $\mathbf{x}'' = \{\{\hat{x}_i\}_{i \in T}, \{\tilde{x}_j\}_{j \notin T}\}$. Then

$$U_i(\mathbf{x}'') = U_i(\hat{\mathbf{x}})$$

for all $i \in T$. Indeed, the utility of player i after deviation depends on the players who make the same choice $\hat{x}_i$ at $\mathbf{x}''$. These players are divided into two groups: (i) those who chose $\hat{x}_i$ at both profiles, $\tilde{\mathbf{x}}$ and $\mathbf{x}''$, and (ii) those who alter original choice for $\hat{x}_i$. By (GR), the second group consists solely of the players who have chosen the strategy $\tilde{x}_i$ of the player i at $\tilde{\mathbf{x}}$. Note that the condition (G) holds for T. But since

$$U_i(\mathbf{x}'') = U_i(\hat{\mathbf{x}}) > U_i(\tilde{\mathbf{x}}), \quad i \in T.$$

it contradicts the statement of Lemma 1 on nonexistence of a profitable deviation that satisfies condition (G).

Finally, the choice of $\tilde{\mathbf{x}}$ as a maximizer of the function $P(\mathbf{x})$ over the set of all strategy profiles guarantees that $\tilde{\mathbf{x}}$ is strongly Pareto optimal. This completes the proof of Theorem 1. **Q.E.D.**