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JEL Classification: N/A

Keywords: Bargaining, gradual sale, Coase conjecture, divisible objects, interdependent valuations, market for lemons

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Bargaining over a Divisible Good in the Market for Lemons*

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June 22, 2020

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1. Introduction

In many economic environments, agents bargain over goods that are divisible. Negotiations in financial markets typically involve both the amount of an asset and its price. Banks and institutional investors (e.g. pension funds) routinely bargain over how much of a securitized asset (pool of mortgages, credit-card debts, automotive loans) to trade and at what price. Similarly, after restructuring a company, an equity firm negotiates what fraction to sell and at what price. These negotiations are generally dynamic and decentralized. One party, typically the seller, is better informed about the quality of the asset.[1]

We study bargaining over a divisible good with asymmetric information, interdependent values and positive gains from trade. Gains from trade of a divisible good may depend not only on the quality of the asset, but also on how much of it has already been traded. We focus on the case of decreasing gains from trade, which leads to new insights into bargaining. Consider a bank negotiating a sale of a pool of mortgages to a pension fund. The quality of the asset can be either low or high, depending on its future cash flows from homeowners. As the pension fund is more interested in owning these promises of future cash flows, there are gains from trade. These gains are decreasing in the amount of the asset already traded between the parties, as they reflect the pension fund's desire to diversify its portfolio. The bank is directly involved in the process of securitization and hence has better information about the quality of these assets.

The main message of this paper is that divisibility introduces a new channel of competition between the buyer's present and future selves, and that this new channel has stark implications for the pattern of trade and for parties' payoffs. When assets are arbitrarily divisible and bargaining frictions vanish, high-quality assets are traded gradually. Divisibility is detrimental to the buyer; the competition between his present and future selves drives his payoffs to zero. This is in contrast to the outcome when the asset is indivisible. In that case, only the low-quality asset is traded in the beginning of the relationship.

[1] Consider the classic example of the synthetic CDO Hudson Mezzanine. As explained in McLean and Nocera [2011], Goldman Sachs selected all the securities in that CDO, strived to sell it as fast as possible and simultaneously bet against that security by taking a short position. See also Ashcraft and Schuermann [2008], Downing, Jaffee, and Wallace [2008] and Gorton and Metrick [2013]

A market freeze then follows, and only afterwards the high-quality asset is traded. The buyer of an indivisible asset obtains a positive payoff.

We extend the canonical bargaining model with incomplete information (Fudenberg, Levine, and Tirole [1985], and Gul, Sonnenschein, and Wilson [1986]) to account for interdependencies in values (Deneckere and Liang [2006] — DL henceforth) and divisibility. A buyer purchases a durable good from a seller who is privately informed about its quality. A high-type seller provides a high-quality good, while a low-type seller provides a low-quality good. The good is divided into finitely many units. There are positive gains from trade, which are decreasing in the number of units already traded by the parties. In every period, the buyer makes a take-it-or-leave-it offer that specifies a price and a number of units to be traded. The bargaining process continues until the parties have traded all available units. The buyer learns about the good's quality only through the seller's behavior; owning a fraction of the good does not provide the buyer with additional information about its quality.[2]

We show that stationary equilibria exist and that the equilibrium outcome is generically unique. In equilibrium, the buyer employs only two types of offers: *screening* and *universal*. Screening offers are for all remaining units at a price lower than the high-type seller's cost. Universal offers are for some (or all) of the remaining units, at a price equal to the high-type seller's cost. The buyer alternates between screening the seller and purchasing some units through universal offers. The low-type seller randomizes between accepting and rejecting screening offers, while the high-type seller always rejects them. The rejection of screening offers makes the buyer more optimistic that the good is of high quality. Eventually, he is optimistic enough to purchase some (or all) of the remaining units through a universal offer. Both seller types accept this offer. After the purchase, the units that remain (if any) are less valuable, so the buyer returns to screening the seller.

Our main result characterizes the limit equilibrium outcome when bargaining frictions vanish and the good becomes arbitrarily divisible. We first let the length of each period converge to zero and we then let the number of units grow to infinity. In the limit,

[2] In our example, the pension fund does not learn about the quality of the securitized asset by owning an additional unit of it. This information only becomes available in the future, when the asset cash flows materialize.

the buyer continuously makes both screening offers and universal offers for infinitesimal fractions of the good. At each point in time, he breaks even with either type of offer, so he obtains a payoff equal to zero. The high-type seller only accepts universal offers and thus sells the good smoothly over time. The low-type seller is indifferent between the two offers (screening and universal). He sells the good smoothly (pooling with the high-type seller) until a certain random time, and then concedes by selling the remaining fraction of the good at once.

In order to understand the driving forces behind our main result, we first describe the pattern of trade when parties bargain over an indivisible good, as in DL. When bargaining frictions vanish, if the buyer can obtain a positive payoff, the usual Coasean forces imply that trade occurs without delay. In one of their main contributions, DL show that if the buyer must screen the seller, he does it through an impasse. During an impasse the market freezes: trade occurs with probability zero. After the impasse, the buyer is optimistic enough to pay the cost of the high-quality good. The impasse introduces delay, which is necessary to lower the price of screening offers before the impasse. In their path-breaking *double delay* result, DL show that the delay is twice the time necessary to make the low-type seller indifferent between the price after the impasse (which is the low-type seller's continuation payoff then) and the buyer's valuation of the low-quality good. This result has two important implications. First, before the impasse, the price of screening offers is strictly lower than the buyer's valuation of the low-quality good, so the buyer obtains a strictly positive payoff. Second, the larger the price after the impasse, the lower the price of screening offers before the impasse.

The driving force behind the gradual sale of the high-quality good when the good is divisible is that the buyer benefits from splitting his purchases. To see this, consider a simple example with ten remaining units. Suppose that the buyer is optimistic enough so that by making a universal offer, he obtains a positive payoff from the first five units (which are more valuable), a negative payoff from the last five units (which are less valuable), and overall, obtains a positive payoff from purchasing all ten units. If the buyer could only make offers for ten units, then he would purchase all of them through a universal offer. When the good is divisible, the buyer can instead purchase the more valuable units through a universal offer and by doing so essentially commit to pay a low price for the less valuable ones. Intuitively, when only the less valuable units remain, the buyer obtains a negative payoff from a universal offer, and so he must screen the seller. As in DL, screening occurs through impasses when the good is divisible. We extend their double delay result and show that the buyer obtains a strictly positive payoff before impasses. The buyer thus prefers to split the purchases of the high-quality good, instead of purchasing all remaining units through one transaction.

The temptation to split the purchases of the high-quality good generates a new channel of competition between the buyer's present and future selves. This new channel of competition is the driving force behind the buyer's zero payoff from trading an arbitrarily divisible good. To see this, consider again the simple example from the previous paragraph. Suppose now that the buyer is so pessimistic that he suffers a loss from a universal offer even for the most valuable of the ten remaining units. He must then screen the seller through an impasse. After this impasse, the buyer splits the purchases of the high-quality good, and so the low-type seller's payoff is lower than the one he would obtain if the buyer could only make offers for ten units. As the low-type seller's payoff after the impasse is lower, then the delay is shorter, which means that the price of the screening offers for ten units before the impasse must be larger. To sum up, since the buyer splits the purchases of the high-quality good *after* the impasse is resolved, then he must pay a higher price for screening offers *before* the impasse.

We show that the competition between the buyer's present and future selves is fierce when the good becomes arbitrarily divisible. Formally, as the good becomes arbitrarily divisible, the number of impasses goes to infinity but each of them becomes short: the price of screening offers before and after each impasse are close to each other, and thus screening does not take long. Between two consecutive impasses, the buyer purchases a vanishing fraction of the good through a universal offer. The driving forces described in the previous paragraphs lead to stark results: the high-quality good is traded smoothly over time and the buyer's payoff is zero.

Our analysis highlights the importance of the shape of gains from trade. If gains from trade are constant in the number of units already traded, the buyer cannot benefit from splitting the purchases of the high-quality good. Intuitively, the buyer cannot commit to pay a lower price for the last units by purchasing the first ones through a universal offer. All units are equally valuable, so if the buyer is willing to pay the cost of the high-quality good for the first ones, he must also be willing to pay that price for the last ones. Thus, with constant gains from trade all units are traded at the same time. The same result holds for increasing gains from trade.

1.1 Related literature

There is a large literature that studies bilateral bargaining with interdependent values (Samuelson [1984], Evans [1989], Vincent [1989], DL, Fuchs and Skrzypacz [2013] and Gerardi, Hörner, and Maestri [2014]). Our paper is closely related to DL and Fuchs and Skrzypacz [2013]. DL solves the one-unit version of the model in our paper. We take DL's construction as a stepping stone and extend the analysis to multiple units when there are two types of sellers.[3] In DL, the gains from trade are bounded away from zero. Fuchs and Skrzypacz bridge the gap between the value of the good to the buyer and the cost to the seller. We find that trade happens gradually over time when the good is arbitrarily divisible. This finding is reminiscent of Fuchs and Skrzypacz [2013]. In a model with indivisibility, Fuchs and Skrzypacz show that, as the gains from trade from the good of highest quality vanish, the bursts of trade found in DL disappear. Like in Fuchs and Skrzypacz [2013], in our model the buyer slowly learns the seller's type. Unlike in Fuchs and Skrzypacz [2013], however, in our model the buyer makes two kinds of offers as he learns the seller's type. On the one hand, he gradually makes universally accepted offers for small pieces of the good at large per-unit prices. On the other, he makes offers for all remaining units at large discounts. Finally, also unlike in Fuchs and Skrzypacz [2013], the gains from trade are bounded away from zero in our model.

Our paper is also related to the burgeoning body of literature that studies the effects of adverse selection in dynamic markets. An important stream of this literature focuses on markets in which one of the players is short-run. Inderst [2005] and Moreno and Wood-

[3]We present a model with interdependent values because if instead values are private, divisibility plays no role; the Coase conjecture holds.

ers [2010] pioneered the study of adverse-selection in decentralized dynamic markets. Camargo and Lester [2014] and Moreno and Wooders [2016] focus on the effect of policy interventions on liquidity in such markets. A question that has drawn much attention is how different transparency regimes affect the bargaining outcome (see Hörner and Vieille [2009] and Fuchs, Öry, and Skrzypacz [2016] for a comparison of public and private offers, and Kim [2017] for the role of time-on-the-market information). Finally, Fuchs and Skrzypacz [2019] characterize optimal market design policies. Beyond the issue of divisibility, our paper differs from the above studies by analyzing the strategic effects that arise when two long-run players bargain under adverse selection.

A third related strand of the literature analyzes the effect of exogenous learning in the market for lemons. In the pioneering work of Daley and Green [2012], noisy information about the value of a good is revealed to the market. In Kaya and Kim [2018], the buyer observes a noisy and private signal about the quality of the good held by the seller. Daley and Green [Forthcoming] analyze the advent of exogenous news when two long-run players bargain over an indivisible good. Our model differs from these contributions as we assume that the good is divisible and abstract from exogenous learning.

The rest of the paper is organized as follows. We describe the model in Section 2. In Section 3 we present equilibrium existence and uniqueness (for generic parameters). We also describe the pattern of trade when the good is divided into a finite number of units and there are bargaining frictions. In Section 4 we present our main result. We characterize the pattern of trade when bargaining frictions vanish and the good becomes arbitrarily divisible. Section 5 presents a detailed argument behind our main result and intermediate results leading to it. In Section 6 we present comparative statics and extensions. Section 7 concludes. Most proofs are relegated to the appendix.

2. The model

A buyer and a seller bargain over a good of size one. The seller is of one of two types $i \in \{L, H\}$. A seller of high type ($i = H$) provides a high-quality good, while a low-type seller ($i = L$) provides a low-quality good. The seller knows his own type, but the buyer does not. The seller is of high type with prior probability $\hat{\beta}$ that satisfies $0 < \hat{\beta} < 1$.

The buyer and the seller can trade fractions of the good. Let $z \in [0,1]$ denote an infinitesimal unit of the good. We index units in *reverse order*. The buyer's first purchase consists of units $z \in [\overline{z}, 1]$, for some $0 \leq \overline{z} \leq 1$. A buyer who has already acquired units $z \in [\overline{z}, 1]$ can then buy subsequent units $z \in [\underline{z}, \overline{z}]$ from the seller, with $0 \leq \underline{z} \leq \overline{z}$.

2.1 Parties' valuations

The buyer's valuation for the units $z \in [\underline{z}, \overline{z}]$ when the seller is of type i is equal to $\int_{\underline{z}}^{\overline{z}} \lambda(z) v_i \, dz$, where $\lambda(z)$ is a smooth function and $\lambda(z) > 0$ for all $z \in [0,1]$. This valuation is higher if the seller is of high type: $0 < v_L < v_H$. The cost of the units $z \in [\underline{z}, \overline{z}]$ to the seller of type i is equal to $(\overline{z} - \underline{z}) c_i$. The constant marginal cost of providing the good is higher for the high-type seller: $0 = c_L < c_H = c$.

We focus on the case with decreasing gains from trade. Since we index units in reverse order, this corresponds to a strictly increasing function $\lambda(z)$.[4] Without loss of generality we assume that $\min_{z \in [0,1]} \lambda(z) = \lambda(0) = 1$. We also assume that $0 < v_L < c < v_H$, so there are always gains from trade. Furthermore, we assume that

$$[\hat{\beta} v_H + (1 - \hat{\beta}) v_L] \lambda(1) < c. \tag{1}$$

The buyer's expected valuation from the first infinitesimal unit is lower than the high-type seller's cost. This assumption allows us to focus on the most interesting case: the buyer must screen the seller even to purchase the most valuable unit.[5]

We study the equilibrium behavior of the buyer and the seller as the good becomes arbitrarily divisible. We divide the good into m equally sized units and study the equilibrium behavior as m grows large. As with $z \in [0,1]$, we also index units in reverse order, by $s \in \{1, \ldots, m\}$: $s = 1$ indicates the last unit, while $s = m$ indicates the first unit. The cost of each unit to the seller of type i is simply c_i / m. The buyer's valuation for the s'th

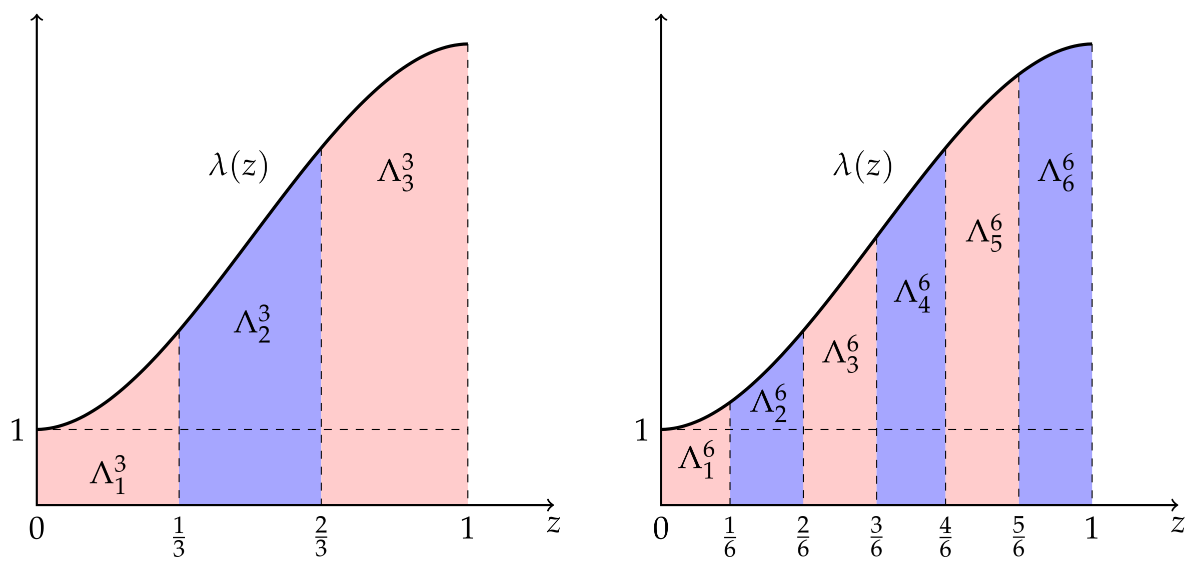
[4] There are two natural alternative environments: $\lambda(z)$ constant and $\lambda(z)$ strictly decreasing. In Section 6.2 we describe how divisibility plays no role in those cases.

[5] In Section 6.2 we extend our analysis to cases where equation (1) does not hold.

unit when the seller is of type i is $\Lambda_s^m v_i$ with

$$\Lambda_s^m \equiv \int_{(s-1)/m}^{s/m} \lambda(z)dz.$$

Figure 1 illustrates the buyer's valuation coefficients Λ_s^m of successive units of the good. In Figure 1(a) the good is divided into 3 units. Assume that the seller is of type i. The buyer's valuation for the first unit is $\Lambda_3^3 v_i$. The second unit gives the buyer intermediate valuation $\Lambda_2^3 v_i$. The last unit is the one with the lowest valuation to the buyer: $\Lambda_1^3 v_i$. Figure 1(b) illustrates the valuation coefficients of successive units of the good when it is divided into 6 units.



(a) Good divided into $m = 3$ units

(b) Good divided into $m = 6$ units

Figure 1: Valuation coefficients of successive units of a divided good

2.2 Timing, payoffs and strategies

The buyer and the seller trade sequentially over time. Time is discrete and periods are indexed by $t = 0, 1, \ldots$. In each period the buyer makes an offer $\varphi_t = (k, p)$, where $k \in \mathbb{Z}_+$ is the number of units requested and $p \in \mathbb{R}_+$ is the total payment offered. Without loss of generality, we assume that the number of units requested cannot exceed the number of remaining units. The seller can either accept ($a_t = A$) or reject ($a_t = R$) the offer. If the seller accepts, k units are traded and the buyer pays p to the seller. The buyer does not learn the quality of the good upon purchasing fractions of it. Therefore, all learning is strategic; the buyer only updates his belief based on the seller's behavior. The game ends when all m units are traded.

The buyer and the seller share a discount factor $\delta = e^{-r\Delta}$ where $\Delta > 0$ represents the length of each period and $r > 0$ represents the discount rate. Suppose that the buyer and the seller of type i agree on trading a total of D times, indexed by $d \in \{1, \ldots, D\}$. In the first trade ($d = 1$), which takes place at time t_1, the buyer pays the seller p_1, in exchange for k_1 units, so the set of traded units is $S_1 = \{m, \ldots, m - k_1 + 1\}$. A generic trade $d > 1$ takes place at time t_d and involves a total payment p_d in exchange for k_d units. The set of traded units is $S_d = \{m - k_1 - \ldots - k_{d-1}, \ldots, m - k_1 - \ldots - k_d + 1\}$. Then, the total payoff to the buyer is:

$$\sum_{d=1}^{D} \delta^{t_d} \left[\sum_{s \in S_d} \Lambda_s^m v_i - p_d \right]$$

The seller, in turn, obtains

$$\sum_{d=1}^{D} \delta^{t_d} \left[p_d - \frac{c_i}{m} k_d \right].$$

The public history h^t, with $t \geq 1$, lists all offers made, together with all responses by the seller, from period 0 through period $t - 1$: $h^t = ((\varphi_0, a_0), \ldots, (\varphi_{t-1}, a_{t-1}))$. We let $h^0 = \varnothing$ denote the initial public history and we let H^t denote the set of all possible histories h^t at the beginning of period t. Intermediate histories (h^t, φ_t) include the offer made after history h^t, but not the subsequent action chosen by the seller.

A buyer's (behavior) strategy $\sigma_B = (\sigma_B^t)_{t=0}^{\infty}$ assigns a random offer to every public history h^t, with $\sigma_B^t(h^t) \in \Delta\Phi(h^t)$, where $\Phi(h^t)$ is the set of available offers at h^t. A seller's (behavior) strategy $(\sigma_L, \sigma_H) = (\sigma_L^t, \sigma_H^t)_{t=0}^{\infty}$ assigns a random decision (A or R) to each intermediate history (h^t, φ_t), so $\sigma_i^t(h^t, \varphi_t) \in \Delta\{A, R\}$ for every $i \in \{L, H\}$. The system of beliefs $\beta(\cdot)$ is as follows. We let $\beta(h^t)$ and $\beta(h^t, \varphi_t)$ denote the buyer's belief that the seller is of high type after an arbitrary public history h^t, and an arbitrary intermediate history (h^t, φ_t), respectively.

2.3 Discussion of the equilibrium concept and preliminary results

We work with *Stationary Perfect Bayesian Equilibria*.[6] In this model, at any public history h^t there are two state variables: the number of remaining units $K(h^t)$ and the buyer's belief $\beta(h^t)$. A strict notion of stationarity would require strategies and value functions to depend only on the two state variables $K(h^t)$ and $\beta(h^t)$. As is standard in bargaining, there is no equilibrium that satisfies this strict notion. We then use a notion that places restrictions only on the seller's strategy. We require the seller's strategy to be a supply function and to depend only on state variables. In what follows we describe our definition in detail.

We present some preliminary results before presenting formally our notion of stationarity. In any Perfect Bayesian Equilibrium (PBE), the buyer's system of beliefs $\beta(\cdot)$ must satisfy the following properties. Beliefs $\beta\left(h^t, \varphi_t, a_t\right)$ are derived from $\beta(h^t)$ according to Bayes' rule whenever action a_t occurs with positive probability after intermediate history (h^t, φ_t). Moreover, beliefs after intermediate histories are not affected by the buyer's offer: $\beta(h^t, \varphi_t) = \beta(h^t)$.

Lemma 1 provides a partial characterization of equilibria whenever the seller's strategy depends only on state variables. Let $V_H(h^t)$, $V_L(h^t)$ and $V_B(h^t)$ denote the continuation payoffs for, respectively, a seller of high type, a seller of low type and the buyer.

Lemma 1. Partial Characterization. *Let $\left(\sigma_B, (\sigma_L, \sigma_H), \beta\right)$ be an arbitrary PBE. Assume that whenever histories h^t and $\tilde{h}^{t'}$ have the same state variables: $\beta(h^t) = \beta(\tilde{h}^{t'})$ and $K(h^t) = K(\tilde{h}^{t'})$, then $\sigma_i\left(h^t, \varphi\right) = \sigma_i\left(\tilde{h}^{t'}, \varphi\right)$ for all $\varphi \in \Phi(h^t) = \Phi\left(\tilde{h}^{t'}\right)$ and for both $i \in \{L, H\}$. Then,*

(a) Whenever $\beta(h^t) = 0$, the low-type seller gets zero payoffs: $V_L(h^t) = 0$.

(b) The buyer's continuation payoff $V_B(h^t)$ depends only on $\beta(h^t)$ and on $K(h^t)$.

(c) The high-type seller gets zero payoffs: $V_H(h^t) = 0$ for all h^t.

(d) The low-type seller's payoffs are bounded: $V_L(h^t) \leq \frac{c}{m} K(h^t)$ for all h^t.

See Appendix A.1 for the proof.

[6]Our definition extends the notion of stationary equilibrium (see Gul and Sonnenschein [1988], Ausubel and Deneckere [1992], DL and Fuchs and Skrzypacz [2010]) to our setting.

Lemma 1(a) states that the low-type seller cannot obtain positive payoffs after his type has been revealed. This result holds true in any PBE, so does not rely on stationarity.[7] Lemma 1(b) and (c) are direct results of the seller's strategy depending only on state variables. Lemma 1(b) states that the buyer's continuation payoff must depend only on beliefs and on the number of remaining units. Lemma 1(c) states that the high type seller always obtains zero profits. Hence, any offer of a payment larger than $\frac{c}{m}K(h^t)$ would be accepted with probability one by the high-type seller, and so also by the low type. This implies Lemma 1(d): the low-type seller continuation payoff is bounded above by $\frac{c}{m}K(h^t)$.

Our definition of stationary PBE incorporates the results from Lemma 1. The behavior of both types of sellers must be consistent with the payoffs that they obtain in a stationary environment. Following Lemma 1(c), a high-type seller accepts any offer that leads to non-negative payoffs. Similarly, following Lemma 1(d), a low-type seller accepts any offer that the high-type seller also accepts.[8] But, does the low-type seller ever accept offers that the high-type seller rejects? If he does so, he immediately reveals his own type to the buyer. Moreover, if the low-type seller mixes, then a rejection increases the belief that the seller is of high type. Then, the behavior of the low-type seller is more subtle than that of the high-type seller. We impose that the acceptance decision of the low-type seller be governed by a function $\mathcal{V}_L(K,\beta)$ that depends on the number of remaining units K and on the beliefs β induced by a rejection.

DEFINITION. STATIONARY PERFECT BAYESIAN EQUILIBRIUM. *A PBE is stationary if there exists a (left-continuous) function* $\mathcal{V}_L(K,\beta) : \{1,\ldots,m\} \times [\hat{\beta},1] \to \mathbb{R}$ *such that*

1. *The high-type seller accepts with probability one any payment greater or equal than $\frac{c}{m}k$ in exchange for any number of remaining units $k \leq K(h^t)$. The high-type seller rejects any other offer with probability one.*

2. *The behavior of the low-type seller is as follows. Take any history h^t where the remaining number of units is $K(h^t)$ and the belief is $\beta(h^t) \geq \hat{\beta}$. Assume that the buyer offers a total payment p in exchange for $k \leq K(h^t)$ remaining units. Then,*

[7] This standard result is analogous to that in a model with common knowledge of types and a buyer who always makes the offer.

[8] This in turn implies that beliefs never decrease over time, and so they are bounded below by $\hat{\beta}$.

a. *If $p \geq \frac{c}{m}k$, then the low-type seller accepts the offer with probability one.*

b. *If $p < \frac{c}{m}k$ and $p < \delta \mathcal{V}_L \left(K(h^t), \beta\right)$ for all $\beta \geq \beta(h^t)$, then the low-type seller rejects the offer with probability one.*

c. *If $p < \frac{c}{m}k$ and there exists $\beta \geq \beta(h^t)$ with $p \geq \delta \mathcal{V}_L \left(K(h^t), \beta\right)$, then the low-type seller randomizes so that $\beta' = \max\left\{\beta : \delta\mathcal{V}_L(K(h^t), \beta) \leq p\right\}$ is the next-period posterior after rejection.*

The function $\delta\mathcal{V}_L(K, \cdot)$ acts as a stationary supply when there are K units left. First, it acts as a supply function because when the buyer offers a higher price p, he induces a (weakly) higher posterior β' after rejection. Therefore, the probability of acceptance of the low-type seller is (weakly) increasing in the price offered by the buyer. Second, the function $\delta\mathcal{V}_L(K, \cdot)$ acts as a *stationary* supply because the price that the buyer needs to pay to induce a posterior belief $\beta' \geq \beta(h^t)$ is independent of the current belief $\beta(h^t)$.

The concept of stationary PBE (equilibrium henceforth), together with Lemma 1, allow for a characterization of the offers that can occur with positive probability in equilibrium. In particular, consider the family of *partial offers*. The buyer makes a partial offer when he requests less than the total number of remaining units and offers a payment that does not cover the costs of the high-type seller. These offers cannot be made and accepted with positive probability in equilibrium. The intuitive reason behind this is simple. A high-type seller never accepts a partial offer, since, by definition, a partial offer does not cover his costs. Then, only the low-type seller may accept partial offers with positive probability. The acceptance of a partial offers reveals that the seller is of low type, so remaining units are traded immediately, and the low-type seller gets no payoff from that trade. Instead of making a partial offer, the buyer could offer to buy *all remaining units* at the same (total) price. The low-type seller would get the same payoff from this alternative offer, so he would accept it, and with the same probability.[9] Trade would then speed up, with the buyer obtaining the additional surplus. Thus, the buyer could obtain a strictly

[9]Our definition of stationary PBE implies that the randomization probability of the low-type seller depends on the number of units remaining, but not on the number of units requested by the buyer. This assumption is without loss of generality. In Section T.1.1 of the Technical Addendum we allow $\mathcal{V}_L$ to depend also on the number of units requested by the buyer. For generic values of the parameters, we obtain the same equilibrium outcome as with our definition of stationary PBE.

higher payoff by making this alternative offer, i.e. asking for all remaining units, and offering the same payment. Lemma 2 formalizes this.

Lemma 2. No Partial Offers. *Fix an equilibrium. Take any history h^t with $K(h^t) > 1$. Trades (k, p) with $k < K(h^t)$ and $p < \frac{c}{m}k$ occur with zero probability.*

See Appendix A.2 for the proof.

Consider the two remaining families of offers:

Definition. Universal and Screening Offers. *The buyer makes a* universal offer *for $k \leq K(h^t)$ units when he offers a payment $p = \frac{c}{m}k$. Universal offers are then of the form $\left(k, \frac{c}{m}k\right)$ and both types accept them. The buyer makes a* screening offer *for all remaining units $K(h^t)$ when he offers a payment $p < \frac{c}{m}K(h^t)$. Screening offers are then of the form $\left(K(h^t), p\right)$ and the high-type seller never accepts them.*

It is without loss of generality to restrict attention only to universal and screening offers. To see this, suppose that in equilibrium, at history h^t, the buyer makes a partial offer (k, p), which the seller rejects (by Lemma 2). Replace this offer with the screening offer $(K(h^t), p)$. Stationarity implies that this offer is also rejected by the seller. By replacing all partial offers this way, we obtain an outcome equivalent equilibrium in which no partial offer is ever made. In this sense, there is no equilibrium with partial offers.[10]

3. Equilibrium existence and pattern of trade

We perform a convenient change of variables. We work with the transformed beliefs $q(\beta) : [\hat{\beta}, 1] \to [0, 1 - \hat{\beta}]$ given by the continuous and strictly increasing mapping

$$q(\beta) = 1 - \frac{\hat{\beta}}{\beta}.$$

For convenience we write $\hat{q} = 1 - \hat{\beta}$ and, with a slight abuse of notation, we let $q\left(h^t\right) = q\left(\beta\left(h^t\right)\right)$. This transformation allows for a simple expression for the probability that the low-type seller accepts screening offers. Assume that after the rejection of a screening

[10] In fact, in the proof of Proposition 1 we show a stronger result: for generic values of the parameters, partial offers are never made in equilibrium.

offer, the buyer updates his transformed belief from q to q'. This means that the low-type seller accepts such offer with probability $(q'-q)/(\hat{q}-q)$. Moreover, as we show in Appendix A.3, the buyer's value function is linear in transformed beliefs $q\left(h^t\right)$.[11]

We show that an equilibrium exists and is generically unique.

PROPOSITION 1. EQUILIBRIUM EXISTENCE. *There exists an equilibrium. Moreover, for generic parameters, all stationary equilibria are outcome equivalent.*

See Appendix A.3 for the proof.

We show equilibrium existence by construction. Moreover, we show that, for generic values of the parameters, any equilibrium induces the same outcome as our construction. Within our construction, we introduce the function $P(K,q) : \{1,\ldots,m\} \times [0,\hat{q}] \to \mathbb{R}$, which plays a key role in the description and the analysis of the equilibrium. We derive this function from $\mathcal{V}_L(\cdot,\cdot)$, and show that $P(K,\cdot)$ is an increasing and left-continuous step function for every $K \in \{1,\ldots,m\}$. The function $P(\cdot,\cdot)$ describes the relevant screening offers available to the buyer in equilibrium. Its interpretation is as follows. Suppose that there are K units left and that the current belief is $q \in [0,\hat{q}]$. Consider any discontinuity point q' of the function $P(K,\cdot)$ with $q' \geq q$. Then, if the buyer makes a screening offer $(K, P(K,q'))$ and it is rejected, his posterior belief is q'.

We solve the buyer's dynamic optimization problem. For any state (K,q), we let $W(K,q) : \{1,\ldots,m\} \times [0,\hat{q}] \to \mathbb{R}$ denote the (normalized) buyer's continuation payoff.[12] When it is optimal for the buyer to make a screening offer $(K, P(K,q'))$ for some discontinuity point q', the low-type seller accepts it with probability $(q'-q)/(1-q)$. The buyer's continuation payoff satisfies

$$W(K,q) = \left(q'-q\right)\left(\sum_{s=1}^{K} \Lambda_s^m v_L - P(K,q')\right) + \delta W\left(K,q'\right).$$

If instead it is optimal for the buyer to make a universal offer $\left(k, \frac{c}{m}k\right)$, the buyer's contin-

[11] This change of variables is also explored in several papers in bargaining with incomplete information. Some readers may find useful the following interpretation for the variable q. Assume that the sellers's type q is uniformly distributed in the unit interval. Whenever $q \in [0,\hat{q})$ then the seller is of low type. If instead $q \in [\hat{q},1]$, the seller is of high type. Under this interpretation for q, the function $P(K,\cdot)$ that we introduce in Appendix A.3 and describe below represents the reservation price $P(K,q)$ for type $q \in [0,\hat{q})$.

[12] $W(K,q)$ is normalized in the sense that we multiply the buyer's continuation payoff by $1-q$.

uation payoff satisfies

$$W(K,q) = \left(\sum_{s=K-k+1}^{K} \Lambda_s^m\right)[(\hat{q}-q)v_L + (1-\hat{q})v_H] - (1-q)\frac{c}{m}k + \delta W(K-k,q).$$

We show that the low-type seller is indifferent between accepting and rejecting all screening offers that he receives in equilibrium. Assume that in equilibrium the buyer makes a screening offer $(K, P(K,q))$. If the low-type seller accepts it, he obtains a continuation payoff of $P(K,q)$. If he instead rejects it, the number of units left stays at K and the buyer's posterior is q. The buyer's subsequent offer can be either screening or universal. If the buyer makes a screening offer $(K, P(K,q'))$, then the low-type seller's indifference requires that the prices of these consecutive screening offers be linked: $P(K,q) = \delta P(K,q')$. Assume instead that the buyer makes a universal offer $(k, \frac{c}{m}k)$ after the rejection of the screening offer $(K, P(K,q))$. This universal offer must be followed by a screening offer $(K-k, P(K-k,q''))$.[13] The low-type seller's indifference then requires that $P(K,q) = \delta\frac{c}{m}k + \delta^2 P(K-k,q'')$.

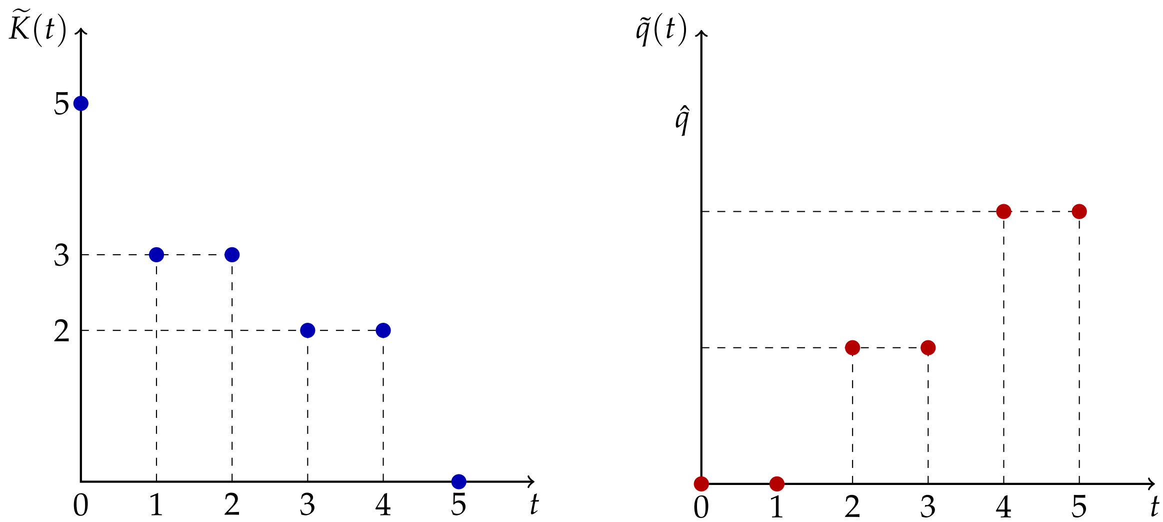
3.1 Pattern of trade

In this subsection we introduce the functions $\widetilde{K}(\cdot)$ and $\tilde{q}(\cdot)$, which describe the evolution of the number of remaining units and of beliefs over time in equilibrium. Together with $P(\cdot,\cdot)$, these functions completely characterize the pattern of trade. The proof of Proposition 1 shows that the game ends after finitely many periods. Let $h^* = ((\varphi_0, a_0), \ldots, (\varphi_{T^*-1}, a_{T^*-1}))$ denote the longest on-path history. Along the history h^*, that lasts for T^* periods, the seller rejects all screening offers. The last offer φ_{T^*-1} is universal, so the seller accepts it. For any $t \leq T^*$, let $\tilde{q}(t)$ denote the transformed beliefs at the beginning of period t along the history h^*. Similarly, $\widetilde{K}(t)$ denotes the number of units left at the beginning of period t along the history h^*. Transformed beliefs $\tilde{q}(t)$ are non-decreasing: the acceptance of a universal offer does not change beliefs, while the re-

[13]We show this result in the proof of Proposition 1. The intuition behind it is simple. In equilibrium, the buyer's continuation payoff is positive at every state. Thus, he has an incentive to combine any two consecutive universal offers.

jection of a screening offer strictly increases them. Similarly, the number of units left $\widetilde{K}(t)$ is non-increasing in t.

Figure 2 depicts a possible pattern of trade over time when the good is divided into five units. Figure 2(a) presents the number of remaining units over time, while Figure 2(b) shows the evolution of transformed beliefs. In this example, the buyer makes a universal offer for two units in period zero. The seller accepts it, and beliefs do not change. In period one, the buyer makes a screening offer which is rejected, prompting an increase in transformed beliefs. A universal offer follows suit in period two, and a screening offer in period three. The rejection of the screening offer in period three makes the buyer optimistic enough to make a universal offer for all remaining units in period four. This offer is accepted, and thus the game ends and beliefs do not change.



(a) Units left $\widetilde{K}(t)$ (b) Transformed beliefs $\tilde{q}(t)$

Figure 2: Pattern of trade $\left(\widetilde{K}(t), \tilde{q}(t)\right)$ for fixed Δ and m

The functions $\widetilde{K}(\cdot)$, $\tilde{q}(\cdot)$ and $P(\cdot,\cdot)$ characterize the equilibrium outcome. Consider two consecutive periods t and $t+1$. Whenever $\widetilde{K}(t+1) < \widetilde{K}(t)$, it means that the buyer makes a universal offer $\left(\widetilde{K}(t) - \widetilde{K}(t+1), \frac{c}{m}\left(\widetilde{K}(t) - \widetilde{K}(t+1)\right)\right)$ in period t. Both the low-type and the high-type seller accept this offer, so the belief does not change: $\tilde{q}(t+1) = \tilde{q}(t)$. If instead $\widetilde{K}(t+1) = \widetilde{K}(t)$, it means that the buyer makes a screening offer $\left(\widetilde{K}(t), P\left(\widetilde{K}(t), \tilde{q}(t+1)\right)\right)$ in period t. The high-type seller always rejects screening offers.

The transformed beliefs $\tilde{q}(t)$ and $\tilde{q}(t+1)$ pin down the probability of acceptance for the low-type seller. He accepts the screening offer with probability $[\tilde{q}(t+1)-\tilde{q}(t)]/[\hat{q}-\tilde{q}(t)]$.

4. Limit equilibrium outcome

We study the limit equilibrium outcome when bargaining frictions vanish and the good becomes arbitrarily divisible.[14] We first let the time between offers Δ converge to zero, so that the discount factor $\delta = e^{-r\Delta}$ converges to one, and we then let the number of units m grow to infinity. With this order of limits, we first solve a game where the good is divided into finitely many units. This allows us to use an inductive argument on the number of remaining units to characterize the limit equilibrium outcome as bargaining frictions vanish (Proposition 2).[15]

In order to keep track of the level of bargaining frictions and of the number of units, we index the functions defined in Section 3 by m and Δ, and write $P_m^{\Delta}(K,q)$, $W_m^{\Delta}(K,q)$, $\widetilde{K}_m^{\Delta}(t)$ and $\tilde{q}_m^{\Delta}(t)$. We also index the length of the longest history $T_m^{*\Delta}$ by m and Δ.

We characterize the equilibrium outcome as a function of time elapsed $\tau \in \mathbb{R}_+$. In a game with period-length Δ, the time elapsed τ after t periods is $\tau = t\Delta$. In order to make meaningful comparisons between games with different period-lengths Δ, we express the number of remaining units and the transformed beliefs as functions of time elapsed $\tau \geq 0$:

$$K_m^{\Delta}(\tau) = \widetilde{K}_m^{\Delta}\left(\min\left\{\lfloor \tau/\Delta \rfloor, T_m^{*\Delta}\right\}\right)$$
$$q_m^{\Delta}(\tau) = \tilde{q}_m^{\Delta}\left(\min\left\{\lfloor \tau/\Delta \rfloor, T_m^{*\Delta}\right\}\right)$$

To examine the limit equilibrium outcome as bargaining frictions vanish, we take a sequence $\{\Delta_n\}_{n=1}^{\infty} \to 0$ and study the limit of its associated sequence $\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^{\infty}$. In Lemma 3 (Appendix A.4) we show that for any $\{\Delta_n\}_{n=1}^{\infty} \to 0$, the associated sequence $\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^{\infty}$ converges pointwise to the same limit functions $(K_m(\cdot), q_m(\cdot))$.

[14] Our results hold for generic values of the parameters. We describe the genericity conditions in the proofs of Propositions 1 and 2. To ease the exposition, hereafter we omit the genericity conditions in the statement of the results.

[15] We provide a brief discussion of the implications of a different order of limits in the conclusion.

Similarly, for any $\{\Delta_n\}_{n=1}^{\infty} \to 0$, the associated sequence $\left\{\left(P_m^{\Delta_n}(K,\cdot), W_m^{\Delta_n}(K,\cdot)\right)\right\}_{n=1}^{\infty}$, with $K \in \{1,\dots,m\}$, converges pointwise to the same limit functions $(P_m(K,\cdot), W_m(K,\cdot))$. The functions $(K_m(\cdot), q_m(\cdot))$ describe the limit equilibrium outcome as bargaining frictions vanish.

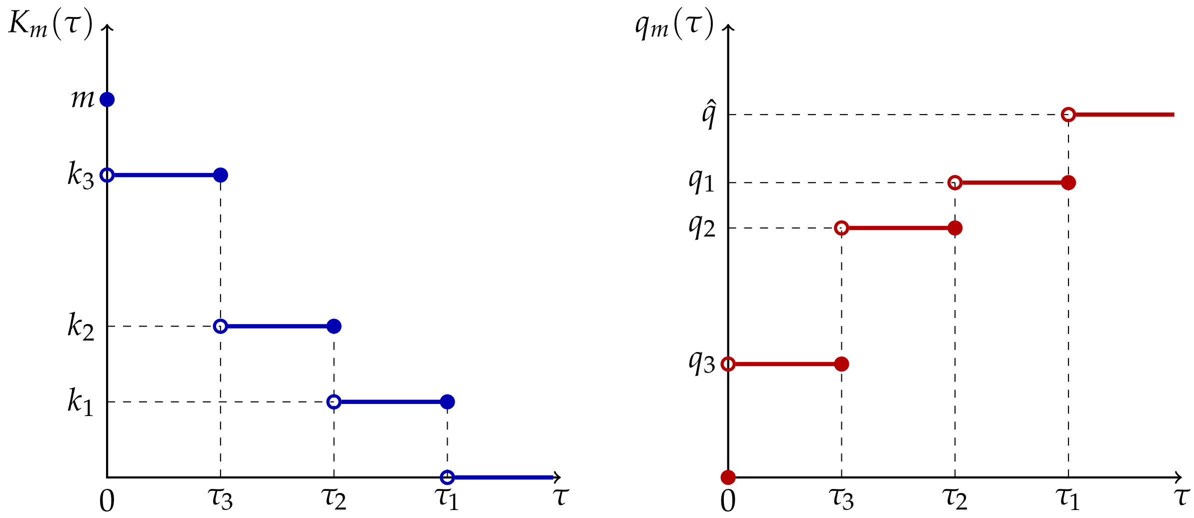
The pattern of trade that emerges as bargaining frictions vanish is simple: there is a sequence of phases of fast trade, mediated by impasses. We show this in Proposition 2 (Section 5.1) but first we provide a formal definition of this pattern of trade.

DEFINITION. PHASES OF FAST TRADE AND IMPASSES. *We say that the limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade, mediated by impasses whenever $K_m(\cdot)$ and $q_m(\cdot)$ are (left-continuous) step functions that are discontinuous at the same points in time. Moreover, we say that the collection of quantities and beliefs $\{(k_j, q_j)\}_{j=1}^{J}$ characterizes this limit equilibrium outcome as bargaining frictions vanish whenever there exist times $\tau_1 > \ldots > \tau_{J+1} = 0$ such that*

$$(K_m(\tau), q_m(\tau)) = \begin{cases} (m, 0) & \text{if } \tau = 0 \\ (k_j, q_j) & \text{if } \tau \in (\tau_{j+1}, \tau_j] \quad \text{for } j \in \{1,\dots,J\} \\ (0, \hat{q}) & \text{if } \tau > \tau_1. \end{cases}$$

The phases of fast trade correspond to jumps in $K_m(\cdot)$ and $q_m(\cdot)$, while $K_m(\cdot)$ and $q_m(\cdot)$ are constant during each impasse. Each pair (k_j, q_j) describes quantities and beliefs during an impasse. The total number of impasses is $J \leq m$. We index impasses in reverse order, so $j = 1$ corresponds to the last impasse (k_1, q_1), while $j = J$ corresponds to the impasse (k_J, q_J) that occurs first. Therefore, $k_{j+1} > k_j$ for all j and $q_{j+1} < q_j$.

Figure 3 depicts an example of the limit equilibrium outcome as bargaining frictions vanish. At the beginning of the game, there is a phase of fast trade. The transformed belief $q_m(\cdot)$ jumps to q_3 at time elapsed $\tau = 0$, which reflects that the buyer makes (a sequence of) screening offers. The low-type seller accepts with total probability $q_3/\hat{q}$. The number of units left $K_m(\cdot)$ jumps to k_3 at time elapsed $\tau = 0$, which reflects that the buyer makes a universal offer for $m - k_3$ units. Although for any given $\Delta > 0$ these offers occur in different periods, as $\Delta \to 0$ the total time it takes to jump to k_3 and q_3 converges to zero.



(a) Units left $K_m(\tau)$

(b) Transformed beliefs $q_m(\tau)$

Figure 3: Pattern of trade $(K_m(\tau), q_m(\tau))$ as bargaining frictions vanish

After the first phase of fast trade, an impasse follows. Intuitively, an impasse is an interval of time elapsed in which no trade occurs. The first impasse depicted in Figure 3 takes place in the interval $(0, \tau_3]$. Within this interval, $K_m(\cdot)$ remains constant at k_3 and $q_m(\cdot)$ remains constant at q_3. First, the fact that the number of units left is constant reflects that, in the limit, the buyer makes a sequence of screening offers after the first universal offer. As $\Delta \to 0$ the total number of such screening offers goes to infinity. Crucially, it does so sufficiently fast so that the total time elapsed while making these offers converges to $\tau_3 > 0$. Second, the fact that the belief $q_m(\cdot)$ is constant reflects that, in the limit, the low-type seller accepts these screening offers with total probability zero. This is possible because as $\Delta \to 0$, the probability of acceptance of each screening offer goes to zero fast enough to overcome that the total number of screening offers goes to infinity. Finally, Figure 3 illustrates that after the first impasse, there are three phases of fast trade, mediated by impasses.

Proposition 2 characterizes the sequence of phases of fast trade and impasses that emerges as bargaining frictions vanish. An impasse introduces delay, which is the only way for a buyer to screen the seller. After the delay, the buyer is optimistic enough to offer a high price to the seller. The delay is necessary to lower the price that the buyer must offer before the impasse. In fact, as in DL, there is double delay. The length of the impasse is twice the time necessary to make the low-type seller indifferent between his continuation payoff after the impasse and receiving the buyer's valuation for (the remaining units of) the low-quality good immediately. Because of double delay, the price before the impasse is lower than the buyer's valuation for the low-quality good. Thus, the buyer obtains a strictly positive continuation payoff before the impasse. Whenever the buyer can obtain a positive payoff, he has an incentive to speed up trade, so the usual Coasean forces kick in as bargaining frictions vanish. Trade must occur without delay, and a phase of fast trade occurs. Before developing this explanation in detail (in Section 5.1), we present the limit equilibrium outcome as the good becomes arbitrarily divisible.

4.1 Main result: characterization of the limit equilibrium outcome

How does the pattern of trade of an arbitrarily divisible good look like? Is there a finite or infinite number of phases of fast trade, mediated by impasses? Is the high-quality good traded in large portions or is it instead traded in dribs and drabs? How are the parties' payoffs affected by the good becoming arbitrarily divisible? Our main result (Theorem 1) characterizes the limit equilibrium outcome as the good becomes arbitrarily divisible. We show that the high-quality good is traded smoothly over time, the buyer's equilibrium payoff converges to zero, and the low-type seller's equilibrium payoff converges to $\left(\int_0^1 \lambda(z)dz\right)v_L$.

Two simple functions characterize the equilibrium outcome as the good becomes arbitrarily divisible. The function $z^* : \mathbb{R}_+ \to [0,1]$ describes the fraction of the good left for trade and the function $q^* : \mathbb{R}_+ \to [0,\hat{q}]$ describes the evolution of beliefs. In order to describe these functions, we first let $\bar{q}(z)$ denote the belief that makes the buyer break even when he makes a universal offer for the infinitesimal unit z:

$$[\hat{q} - \bar{q}(z)]\,[\lambda(z)v_L - c] + [1-\hat{q}]\,[\lambda(z)v_H - c] = 0$$

The function $\bar{q} : [0,1] \to [0,\hat{q})$ is strictly decreasing. We let $\psi(\cdot)$ denote its inverse.

The construction of the functions $q^*(\cdot)$ and $z^*(\cdot)$ is simple, and can be better understood through the following artificial pattern of trade. At time $\tau = 0$, the buyer makes a screening offer and breaks even. The low-type seller accepts this offer with probability $\bar{q}(1)/\hat{q}$, so the belief at time $\tau = 0$ satisfies $q^*(0) = \bar{q}(1)$. From that point on, the buyer continuously makes both screening offers and universal offers for infinitesimal units. At any point in time $\tau \in \mathbb{R}_+$, the buyer breaks even with either type of offer. Finally, the low-type seller is indifferent between accepting and rejecting any screening offer. The functions $q^*(\cdot)$ and $z^*(\cdot)$ are the results of this artificial pattern of trade.

In the artificial pattern of trade, the buyer breaks even every time he makes a universal offer for the infinitesimal unit z. Thus, at any point in time $\tau \in \mathbb{R}_+$, the belief $q^*(\tau)$ and the fraction of remaining units $z^*(\tau)$ must satisfy $q^*(\tau) = \bar{q}\left(z^*(\tau)\right)$. Furthermore, since the buyer also breaks even whenever he makes a screening offer, at any point in time $\tau \in \mathbb{R}_+$ he offers to purchase the fraction $z^*(\tau)$ at the price $v_L \int_0^{z^*(\tau)} \lambda(z)dz$. Finally, the low-type seller is indifferent between accepting a screening offer at time τ or mimicking the high-type seller's behavior from τ to $\tau + \Delta\tau$ and then accepting a screening offer at time $\tau + \Delta\tau$:

$$v_L \int_0^{z^*(\tau)} \lambda(z)dz = \int_\tau^{\tau+\Delta\tau} e^{-r(s-\tau)} c(-z^{*\prime}(s))ds + e^{-r\Delta\tau} v_L \int_0^{z^*(\tau+\Delta\tau)} \lambda(z)dz \qquad (2)$$

We next take advantage of the fact that $q^*(\tau) = \bar{q}\left(z^*(\tau)\right)$, we let $\Delta\tau \to 0$, and through a first order approximation of the right hand side of equation (2) we show that:

$$q^{*\prime}(\tau) = \frac{r v_L \int_0^{\psi(q^*(\tau))} \lambda(z)dz}{\psi'\left(q^*(\tau)\right)\left[v_L \lambda\left(\psi\left(q^*(\tau)\right)\right) - c\right]} \quad \text{and} \quad z^{*\prime}(\tau) = \psi'\left(q^*(\tau)\right) q^{*\prime}(\tau). \qquad (3)$$

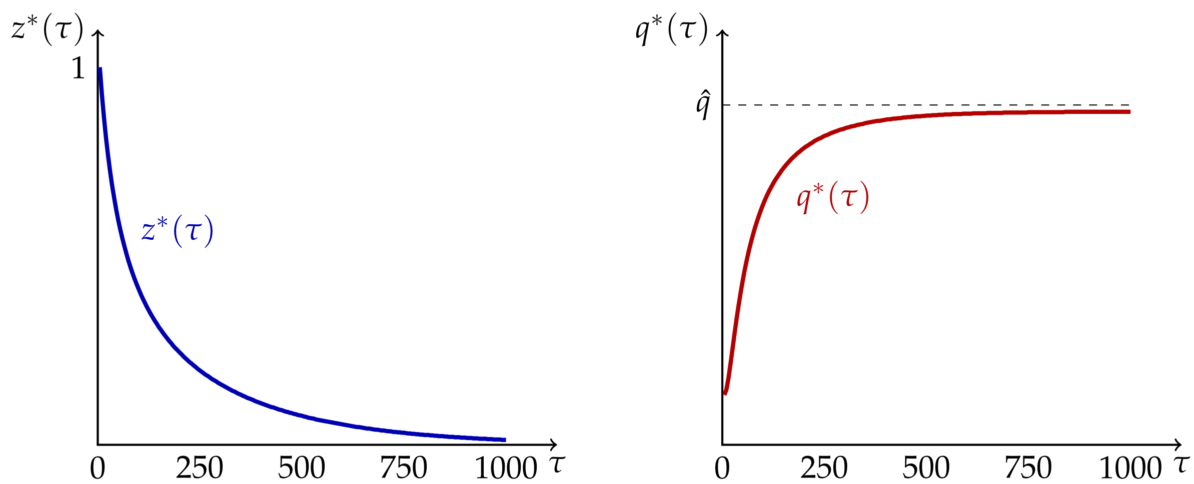
This, together with the initial conditions $q^*(0) = \bar{q}(1)$ and $z^*(0) = 1$ pins down the functions $q^*(\cdot)$ and $z^*(\cdot)$.

For any m, we define the function $z_m(\tau) : \mathbb{R}_+ \to [0,1]$ by setting $z_m(\tau) = K_m(\tau)/m$.

THEOREM 1. LIMIT EQUILIBRIUM OUTCOME. *The sequence* $\{(z_m(\cdot), q_m(\cdot))\}_{m=1}^{\infty}$ *converges pointwise to* $(z^*(\cdot), q^*(\cdot))$. *Thus, in the limit equilibrium outcome, the high-quality good is traded smoothly over time, the low-type seller's payoff is* $\left(\int_0^1 \lambda(z)dz\right)v_L$ *and the buyer's payoff is zero.*

Theorem 1 follows immediately from Proposition 3 in Section 5 (and the low-type seller's indifference between accepting and rejecting screening offers).

Figure 4 illustrates the limit equilibrium outcome $(z^*(\cdot), q^*(\cdot))$. The number of impasses grows without bound and the length of each shrinks to zero. The buyer's belief evolves smoothly and the high-quality good is sold gradually over time. The functions $z^*(\cdot)$ and $q^*(\cdot)$ are such that, at any point in time, the buyer's continuation payoff is zero and the low-type seller is indifferent between selling the remaining fraction of the good at the buyer's valuation or mimicking the high-type seller's behavior.



(a) Fraction of the good left $z^*(\cdot)$

(b) Transformed beliefs $q^*(\cdot)$

Note: These figures depict the limit equilibrium outcome for the following primitives: $v_H = 35$, $v_L = 1$, $c = 30$, $r = 0.1$ and $\hat{q} = 0.9$. Finally, $\lambda(z) = 1 + 0.1z + 15z^2 - 10z^3$ (this is the function shown in Figure 1).

Figure 4: Limit equilibrium outcome $(z^*(\cdot), q^*(\cdot))$: Pattern of trade as bargaining frictions vanish and the good becomes arbitrarily divisible

Divisibility enriches the set of offers available to the buyer. When the good is divisible the buyer does not need to wait until he is optimistic enough to purchase the whole good. Instead, he can act fast and make offers for the most valuable fractions of it. The buyer's ability to purchase fractions of the good introduces a novel channel of competition between his present and future selves. The intuition behind this competition is as follows. The double delay result guarantees that the buyer obtains a positive payoff from screening offers before each impasse. Then, he is tempted to purchase small fractions of the good whenever, by doing so, he can reach an impasse faster and then enjoy earlier the positive payoff before the impasse. However, this temptation reduces the positive payoffs from screening offers before *previous* impasses. In the next paragraphs we describe this intuition in detail and explain how this brings the buyer's payoff to zero. We present the formal analysis behind this result in Section 5.

First, for any fixed m, in the limit equilibrium outcome as bargaining frictions vanish the last impasse must occur when only one unit remains. To see why, let q_1 denote the belief that makes the buyer break even when he makes a universal offer for the last unit. Whenever one unit remains and the buyer's belief is lower than q_1, he must screen the seller through delay, and so an impasse occurs at $(1, q_1)$. At beliefs lower than q_1, the buyer makes screening offers at a price lower than his valuation of the low-quality good, and so obtains a positive payoff. Moreover, this impasse must be on-path. If it were not, there would be a last impasse, with a quantity larger than one, and after this last impasse the buyer would make a universal offer and break even.[16] Because of decreasing gains for trade, this last impasse would occur at a belief lower than q_1. But whenever the belief is lower than q_1, the buyer is always better off by making a screening offer for the last unit, instead of making a universal offer.

Next, as m grows large, the penultimate impasse must be close to the last. To see why, assume that a small fraction of the good remains and the belief is such that the buyer makes a loss if he purchases all but the last unit with a universal offer. As m grows large, the profit from screening offers before the last impasse shrinks to zero. Then, the profit from the last unit is not enough to recoup the loss from the universal offer. The buyer must then screen the seller through delay, and so there is an impasse with a small fraction left for trade and a belief lower but close to q_1. This impasse lowers the price of the screening offers for the small fraction of the good, and so the buyer takes advantage of it.

Similarly to the last impasse, the buyer's payoff before the penultimate impasse must shrink as m grows large. The fraction traded between the penultimate and the last impasse is small. Moreover, the buyer pays a price close to his valuation for the low-quality

[16]There must be at least one impasse. Otherwise, as bargaining frictions vanish, the buyer would pay c to purchase the whole good and thus obtain a negative payoff.

good in the screening offers before the last impasse. As a result, after the penultimate impasse is resolved, the low-type seller's continuation payoff is close to the buyer's valuation for the low-quality good. Then, because of double delay, it does not take long for the buyer to screen the low-type seller during the penultimate impasse. Therefore, the price of the screening offer before the impasse is also close to the buyer's valuation for the low-quality good.

The logic from the previous paragraphs extends recursively. Whenever the buyer's profit from screening offers before an impasse is low, another impasse must occur close to it, and this previous impasse must also be associated to low profits. In the limit, this generates an infinite sequence of impasses, each with length that shrinks to zero.

The buyer's inability to commit to make universal offers for large fractions is detrimental to him. He always trades small quantities of the high-quality good. As a result, the price of the screening offers after the impasse is close to the buyer's valuation of the low-quality good. The buyer can screen the seller fast, and so the price of the screening offers before the impasse is also close to the buyer's valuation, and yields a low profit to the buyer. As the good becomes arbitrarily divisible, this drives the buyer's continuation payoff to zero.

5. Mechanism behind the limit equilibrium outcome

The proof of Theorem 1 consists of two parts. In the first one we fix the number of units m and construct an algorithm that characterizes the equilibrium outcome as bargaining frictions vanish ($\Delta \to 0$). Proposition 2 highlights the key properties of this equilibrium outcome. In the second one we present Proposition 3, which characterizes the outcome uncovered by the algorithm in the previous part as the number of units m grows to infinity. This proposition immediately implies that the sequence of functions $\{(q_m(\cdot), z_m(\cdot))\}_{m\in\mathbb{N}}$ converges to $(q^*(\cdot), z^*(\cdot))$ as m grows to infinity.

5.1 Vanishing bargaining frictions: the algorithm

We construct an algorithm that pins down the phases of fast trade and the impasses that take place as bargaining frictions vanish. This algorithm also identifies some key properties of the limit functions $P_m(\cdot,\cdot)$ and $W_m(\cdot,\cdot)$. In order to state these properties, we define the belief $\bar{q}_m(K)$ for any $K \in \{1,\ldots,m\}$ as follows. Assume that the buyer makes an offer $\varphi = (1, \frac{c}{m})$ when there are K units left; that is, he offers to pay the high-type's cost in exchange of one unit. Then, $\bar{q}_m(K) \in (0,\hat{q})$ is the transformed belief that makes the buyer break even:

$$[\hat{q} - \bar{q}_m(K)] \left(\Lambda_K^m v_L - \frac{c}{m}\right) + (1-\hat{q}) \left(\Lambda_K^m v_H - \frac{c}{m}\right) = 0$$

Note that $\bar{q}_m(m) < \ldots < \bar{q}_m(1)$, as gains from trade are decreasing.[17] Finally, for any (K,q) let $P_m^-(K,q) = \lim_{q'\uparrow q} P_m(K,q)$ and $P_m^+(K,q) = \lim_{q'\downarrow q} P_m(K,q)$.

PROPOSITION 2. EQUILIBRIUM OUTCOME AS BARGAINING FRICTIONS VANISH. *Fix m. The limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade and impasses characterized by a collection of quantities and beliefs $\left\{(k_j,q_j)\right\}_{j=1}^{J}$ with $1 \le J \le m$, $(k_1,q_1) = (1,\bar{q}_m(1))$ and $\bar{q}_m(k_j+1) < q_j < \bar{q}_m(k_j)$ for all $j > 1$.*

Moreover, $W_m(k_j,q_j) = 0$ for every $j \in \{1,\ldots J\}$. Finally,

$$P_m^+(k_1,q_1) = P_m^+(1,\bar{q}_m(1)) = \frac{c}{m}, \tag{4a}$$

$$P_m^-(k_j,q_j) = \left(\frac{v_L \sum_{s=1}^{k_j} \Lambda_s^m}{P_m^+(k_j,q_j)}\right)^2 P_m^+(k_j,q_j) \qquad \forall\, j \in \{1,\ldots,J\} \quad \textit{and} \tag{4b}$$

$$P_m^+(k_{j+1},q_{j+1}) = (k_{j+1}-k_j)\frac{c}{m} + P_m^-(k_j,q_j) \qquad \forall\, j \in \{1,\ldots,J-1\}. \tag{4c}$$

See Appendix A.5 for the proof.

Proposition 2 shows that there is at least one impasse. The last impasse always occurs at $(1,\bar{q}_m(1))$, that is, when one unit remains left and the belief is $\bar{q}_m(1)$. The buyer's continuation payoff is zero at all impasses. Finally, Proposition 2 describes the limit functions

[17] For convenience, we set $\bar{q}_m(m+1) = 0$.

$P_m(\cdot,\cdot)$ around each impasse (k_j, q_j).

Equation (4c) links the limit functions $P_m(\cdot,\cdot)$ between two consecutive impasses. After the impasse (k_{j+1}, q_{j+1}) is resolved, the state shifts without delay to (k_j, q_j). To fix ideas, suppose that the shift consists of one universal offer for $k_{j+1} - k_j$ units followed by a screening offer $\left(k_j, P_m^-(k_j, q_j)\right)$.[18] The low-type seller obtains a continuation payoff $\left(k_{j+1} - k_j\right) \frac{c}{m} + P_m^-(k_j, q_j)$, which must be equal to the price $P_m^+(k_{j+1}, q_{j+1})$ that the buyer has to pay in the limit to induce a belief $q > q_{j+1}$ close to q_{j+1}. Equation (4a) follows the same logic as equation (4c): after the last impasse is resolved the buyer purchases without delay the last unit at the price $\frac{c}{m}$.

Equation (4b) shows that the limit function $P_m(k_j, \cdot)$ is discontinuous at q_j. The jump between $P_m^-(k_j, q_j)$ and $P_m^+(k_j, q_j)$ pins down the length of the impasse (k_j, q_j). Let $\tilde{\tau}$ be the necessary time elapsed for the buyer's valuation for the low-quality good $v_L \sum_{s=1}^{k_j} \Lambda_s^m$ to be equal to the discounted value of $P_m^+(k_j, q_j)$: $v_L \sum_{s=1}^{k_j} \Lambda_s^m = e^{-r\tilde{\tau}} P_m^+(k_j, q_j)$. Equation (4b) shows that the delay is of length $2\tilde{\tau}$:

$$P_m^-(k_j, q_j) = e^{-2r\tilde{\tau}} P_m^+(k_j, q_j) = \left(\frac{v_L \sum_{s=1}^{k_j} \Lambda_s^m}{P_m^+(k_j, q_j)} \right)^2 P_m^+(k_j, q_j)$$

This finding is in line with DL's double delay result, which characterizes the length of each impasse. We extend this result to the case of a divisible good.

In the next two subsections we describe the construction of the algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish. We explain the main steps of the algorithm and the intuition behind it (see Appendix A.5 for the remaining details). The algorithm follows an inductive approach. In the base step, we identify the last impasse. We show that it occurs when only one unit remains, and pin down both the length of the impasse, and the belief at which it occurs. In the inductive step we take an impasse and construct the previous one.

[18]This discussion holds regardless of the particular sequence of offers that characterizes the shift from (k_{j+1}, q_{j+1}) to (k_j, q_j).

5.1.1 The base step: the last impasse

We first focus on the case when only one unit remains and revisit the results from DL. Whenever the belief is higher than $\bar{q}_m(1)$, the buyer can guarantee a positive continuation payoff by making a universal offer for the last unit. Since his continuation payoff is strictly positive, the usual Coasean forces imply that the buyer has an incentive to speed up trade. Thus, the buyer purchases the remaining unit without delay. The next paragraphs formalize this intuition.

Suppose that starting at state $(1, q)$, it is optimal for the buyer to make two consecutive screening offers $(1, P(1, q'))$ and $(1, P(1, q''))$. The buyer prefers this to making instead the offer $(1, P(1, q''))$ right away:

$$\begin{aligned} W_m^\Delta(1, q) &= (q' - q)\left[\Lambda_1^m v_L - P(1, q')\right] + \delta W_m^\Delta(1, q') \\ &\geq (q'' - q)\left[\Lambda_1^m v_L - P(1, q'')\right] + \delta W_m^\Delta(1, q'') \\ &= (q' - q)\left[\Lambda_1^m v_L - P(1, q'')\right] + \underbrace{(q'' - q')\left[\Lambda_1^m v_L - P(1, q'')\right] + \delta W_m^\Delta(1, q'')}_{W_m^\Delta(1, q')} \end{aligned}$$

The low-type seller is indifferent between accepting any of these two offers, so $P(1, q') = \delta P(1, q'')$. Combining this with the inequality above (second and fourth term) implies

$$q' - q \geq \frac{W_m^\Delta(1, q')}{P(1, q'')} \geq \frac{W_m^\Delta(1, q')}{c/m},$$

where the last inequality results from the price $P(1, \cdot)$ being bounded above by c/m. Finally, suppose that there exists $\tilde{q} < \hat{q}$ and $\eta > 0$ such that $W_m^\Delta(1, q) \geq \eta$ for all $q \geq \hat{q}$. Then, starting at any state $(1, q)$, with $q \geq \tilde{q}$ the buyer makes at most $\lceil (c/m)(\hat{q} - \tilde{q})/\eta \rceil$ screening offers. Thus, a strictly positive continuation value for the buyer implies an upper bound on the number of offers.

The buyer in fact obtains a continuation payoff bounded away from zero when the belief exceeds $\bar{q}_m(1)$. To see this, fix $\tilde{q} > \bar{q}_m(1)$. For any $q \geq \tilde{q}$, the buyer's continuation payoff is bounded below by that from making a universal offer when the belief is $\tilde{q}$:

$$W_m^{\Delta}(1,q) \geq (\hat{q} - \tilde{q})\,[\Lambda_1^m v_L - c/m] + (1-\hat{q})\,[\Lambda_1^m v_H - c/m] > 0$$

This lower bound is independent of Δ, and therefore in the limit, when the state is $(1,\tilde{q})$, with $\tilde{q} > \bar{q}_m(1)$, trade occurs without delay.

As trade occurs immediately at states $(1,q)$ with $q > \bar{q}_m(1)$, then $P_m(1,q) = c/m$. Moreover, $P_m(1,\cdot)$ must be discontinuous at $\bar{q}_m(1)$, with $P_m(1,q) \leq \Lambda_1^m v_L$ for beliefs $q < \bar{q}_m(1)$. If this were not the case, the buyer's continuation payoff at $q < \bar{q}_m(1)$ would be negative, as $W_m(1,\bar{q}_m(1)) = 0$. The discrete jump in $P_m(1,\cdot)$ at $\bar{q}_m(1)$ implies that there must be delay, so that the low-type seller is indifferent between accepting offers lower than $v_L\Lambda_1^m$ or waiting for the offer c/m.

DL show that the length of the impasse is $2\tilde{\tau}$ (where, as before, $\tilde{\tau}$ is given by $v_L\Lambda_1^m = e^{-r\tilde{\tau}}c/m$). The intuition behind the double delay result from DL is as follows. Fix $\Delta > 0$ and assume that $P_m^{\Delta}(1,\cdot) = \Lambda_1^m v_L$ for an interval of beliefs. DL show that the function $P_m^{\Delta}(1,\cdot)$ must be symmetric around this interval in the following sense. The length of the interval with $P_m^{\Delta}(1,\cdot) = \Lambda_1^m v_L$ must be equal to the length of the interval with $P_m^{\Delta}(1,q) = \delta\Lambda_1^m v_L$. Similarly, the intervals with $P_m^{\Delta}(1,\cdot) = \delta^{-1}\Lambda_1^m v_L$ and with $P_m^{\Delta}(1,q) = \delta^2\Lambda_1^m v_L$ are also of the same length. This symmetry extends as we move away from the segment with $P_m^{\Delta}(1,\cdot) = \Lambda_1^m v_L$. As $\Delta \to 0$, the function $P_m^{\Delta}(1,\cdot)$ takes values which are arbitrarily close to $\Lambda_1^m v_L$. Therefore, in the limit, for any small $\varepsilon > 0$ it takes as much time to move from $\bar{q}_m(1) - \varepsilon$ to $\bar{q}_m(1)$ as it takes to move from $\bar{q}_m(1)$ to $\bar{q}_m(1) + \varepsilon$.

Pinning down the length of the impasse allows for the following simple expression for $P_m^-(1,\bar{q}_m(1))$:

$$P_m^-(1,\bar{q}_m(1)) = \left(\frac{v_L\Lambda_1^m}{c/m}\right)^2 c/m = \left(\frac{v_L\Lambda_1^m}{c/m}\right) v_L\Lambda_1^m < v_L\Lambda_1^m.$$

The last inequality implies that in the limit the buyer's continuation payoff at any state $(1,q)$ with $q < \bar{q}_m(1)$ must be strictly positive. Intuitively, for Δ small, the buyer can make a screening offer with a price close to $P_m^-(1,\bar{q}_m(1))$ that the low-type seller accepts with strictly positive probability. Since the buyer has a positive continuation payoff, the usual Coasean forces kick in, and the state $(1, \bar{q}_m(1))$ is reached without delay. Therefore $P_m(1, q) = P_m^-(1, \bar{q}_m(1))$ for any $q < \bar{q}_m(1)$. This concludes the explanation of the one-unit case from DL.

We next explain why there must be at least one impasse in our multi-unit model. If there were none, then in the limit the buyer would buy all units without delay, pay c/m for each of them, and obtain a negative payoff. Then, by continuity, for Δ sufficiently close to zero, the buyer's payoff would be negative, which cannot happen in equilibrium. Following a similar continuity argument, throughout this section we focus directly on the "limit game" in the sense that the low-type seller's behavior is given by the limit function $P_m(\cdot, \cdot)$.

We finally describe why the last impasse occurs at state $(1, \bar{q}_m(1))$. Assume instead that it occurs at state (K, q), with $K > 1$. First, notice that q must be strictly smaller than $\bar{q}_m(1)$. This is because for any $q \geq \bar{q}_m(1)$ the buyer can obtain a strictly positive continuation payoff by making a universal offer for all remaining units, and so there cannot be delay. Second, after the impasse (K, q) is resolved, the buyer purchases all remaining units without delay and therefore pays c/m for each of them. However, because of divisibility, there exists an alternative course of action that gives the buyer a higher continuation payoff. The buyer can instead first make a universal offer for $K - 1$ units. Then, he can make a screening offer $(1, P_m^-(1, \bar{q}_m(1)))$, which is accepted by the low-type seller with probability $(\bar{q}_m(1) - q)/(\hat{q} - q) > 0$. If instead the offer is rejected, the buyer pays c/m for the remaining unit. Divisibility allows the buyer to take advantage of the positive profits from the screening offer before the impasse $(1, \bar{q}_m(1))$. He then has a profitable deviation for Δ sufficiently close to zero.

5.1.2 The inductive step: from one impasse to the previous one

In this subsection we show in detail whether the penultimate impasse occurs when two units remain. We also explain how to identify previous impasses.

As before, we consider a simple course of action that allows the buyer to take advantage of the positive profits from a screening offer before the last impasse. This course of action brings the buyer from any state (K, q) with $K > 1$ and $q < \bar{q}_m(1)$ to the last impasse $(1, \bar{q}_m(1))$, where the buyer's continuation payoff is zero. The buyer first makes the universal offer $\left(K-1, \frac{c}{m}(K-1)\right)$ and then the screening offer $(1, P_m^-(1, \bar{q}_m(1)))$. Let $\check{q}(K)$ be the belief that makes the buyer break even when he follows this course of action:[19]

$$\left[\left[[\hat{q} - \check{q}(K)]\, v_L + (1-\hat{q})\, v_H\right] \sum_{s=2}^{K} \Lambda_s^m - [1-\check{q}(K)]\,(K-1)\frac{c}{m}\right] + \\ [\bar{q}_m(1) - \check{q}(K)]\left[\Lambda_1^m v_L - P_m^-(1, \bar{q}_m(1))\right] = 0 \tag{5}$$

As the buyer breaks even at state $(2, \check{q}(2))$, this is a natural candidate for an impasse. In fact, we show next that whenever $\check{q}(2) > \bar{q}_m(3)$, the penultimate impasse must occur at $(2, \check{q}(2))$. To do this, we characterize the function $P_m(2, \cdot)$. Note first that $\check{q}(2) < \bar{q}_m(2)$.[20] Consider any state $(2, q)$ with $\check{q}(2) < q < \bar{q}_m(2)$. As $q < \bar{q}_m(2)$, the game cannot end without delay (if the buyer were to acquire both remaining units without delay, he would obtain a negative payoff). Moreover, as $q > \check{q}(2)$, the buyer's continuation payoff is positive, so an impasse cannot occur at state $(2, q)$. Thus, the state transitions without delay to $(1, \bar{q}_m(1))$ and $P_m(2, q) = \frac{c}{m} + P_m^-(1, \bar{q}_m(1))$ for all $q \in (\check{q}(2), \bar{q}_m(2))$.

The function $P_m(2, \cdot)$ must be discontinuous at $\check{q}(2)$. To see this, note that the definition of $\check{q}(2)$ in equation (5), together with the fact that $\check{q}(2) > \bar{q}_m(3) > 0$ imply that

$$P_m^+(2, \check{q}(2)) = \frac{c}{m} + P_m^-(1, \bar{q}_m(1)) > (\Lambda_2^m + \Lambda_1^m)\, v_L.$$

This in turn implies that if $P_m^-(2, \check{q}(2)) = P_m^+(2, \check{q}(2))$, the buyer would obtain strictly negative payoffs at states $(2, q)$ with q close and lower than $\check{q}(2)$. The buyer would pay more than his valuation to reach $(2, \check{q}(2))$ through screening offers, and then would get a zero continuation payoff there. Universal offers would also lead to negative continuation payoffs, since $q < \check{q}(2) < \bar{q}_m(2)$.

The limit function $P_m(2, \cdot)$ can be discontinuous only if there is an impasse associated

[19] If this course of action leads to a positive payoff for every belief $q \in [0, \bar{q}_m(1))$ then we set $\check{q}(K) = 0$.

[20] When the buyer follows the simple course of action, he obtains a positive payoff from the last unit. Then, since he breaks even at $\check{q}(2)$, he must obtain a negative payoff from the second to last unit.

to the state $(2, \check{q}(2))$.[21] As is the case for $(1, \bar{q}_m(1))$, this impasse also requires double delay:

$$P_m^-(2, \check{q}(2)) = \left(\frac{v_L(\Lambda_2^m + \Lambda_1^m)}{P_m^+(2, \check{q}(2))} \right)^2 P_m^+(2, \check{q}(2)) < v_L(\Lambda_2^m + \Lambda_1^m)$$

Since $P_m^-(2, \check{q}(2)) < v_L(\Lambda_2^m + \Lambda_1^m)$, the buyer can guarantee a positive continuation payoff at any state $(2, q)$ with $q < \check{q}(2)$ by making the screening offer $(2, P_m^-(2, \check{q}(2)))$. Therefore, the state $(2, \check{q}(2))$ is reached without delay, so $P_m(2, q) = P_m^-(2, \check{q}(2))$ for any $q < \check{q}(2)$.

We next show that the condition $\check{q}(2) > \bar{q}_m(3)$ guarantees that the buyer is again tempted to take advantage of the positive profits from the screening offer before the impasse $(2, \check{q}(2))$ and so the penultimate impasse must occur at $(2, \check{q}(2))$. Assume towards a contradiction that it occurs at (K, q) with $K > 2$.[22] Since $\check{q}(2) > \bar{q}_m(3) > \bar{q}_m(4) > \ldots$, then it must be the case that $\check{q}(K) < \check{q}(2)$. Whenever $q > \check{q}(K)$, the buyer can obtain a positive continuation payoff by following the simple course of action described before. Then, no impasse can occur at (K, q) with $q > \check{q}(K)$. Thus, the penultimate impasse must occur at (K, q) with $q \le \check{q}(K) < \check{q}(2)$. After this impasse is resolved, the state moves without delay to $(1, \bar{q}_m(1))$. Consider next the following course of action, which is in the spirit of the one defined before, but instead takes advantage of the profits before the impasse $(2, \check{q}(2))$. The buyer first purchases $K - 2$ units, then makes the screening offer $(2, P_m^-(2, \check{q}(2)))$ and finally follows the optimal course of action at state $(2, \check{q}(2))$. This alternative course of action is more profitable than moving without delay to $(1, \bar{q}_m(1))$. The difference in payoffs is $(\check{q}(2) - q)\left[\frac{c}{m} + P_m^-(1, \bar{q}_m(1)) - P_m^-(2, \check{q}(2))\right] > 0$. Thus, we have reached a contradiction.

In Appendix A.5 we show that if instead $\check{q}(2) < \bar{q}_m(3)$, then the penultimate impasse cannot occur when two units remain. We characterize $P_m(3, \cdot)$ and show that there is a (potentially off-path) impasse when three units remain. The buyer prefers to take advantage of it and skip the impasse with two remaining units. We then move to any number of remaining units $K > 2$ and compare $\check{q}(K)$ with $\bar{q}_m(K+1)$. The penultimate impasse occurs at $(k_2, \check{q}(k_2))$, where $k_2 = \min\{K \in \{2, \ldots, m\} : \bar{q}_m(K+1) < \check{q}(K) < \bar{q}_m(K)\}$. If in-

[21] This impasse may not be part of the limit equilibrium outcome as bargaining frictions vanish, since the state $(2, \check{q}(2))$ may never be reached.

[22] The case without a penultimate impasse follows a similar logic.

stead the minimum does not exist, $(1, \bar{q}_m(1))$ is the unique impasse: this state is reached without delay.

Our algorithm proceeds by induction by taking the impasse (k_j, q_j) and identifying the previous impasse (k_{j+1}, q_{j+1}). To do this, we construct a simple course of action, analogous to the ones before, where the buyer takes advantage of the positive profits from screening offers before the impasse (k_j, q_j). This course of action brings the buyer from any state (K, q) with $K > k_j$ and $q < q_j$ to the impasse (k_j, q_j), where the buyer's continuation payoff is zero. We use properties of the continuation payoff from this alternative course of action to identify the previous impasse(k_{j+1}, q_{j+1}). The algorithm then ends in finitely many steps and there are at most m impasses.

5.2 Arbitrarily divisible good

In this subsection we describe the limit of the equilibrium outcome identified in Proposition 2 as the number of units m grows to infinity. In order to keep track of the number of units, we index impasses by m and write $\left\{\left(k_j^m, q_j^m\right)\right\}_{j=1}^{J_m}$, where J_m denotes the number of impasses when the good is divided into m units. We also let $z_j^m = k_j^m/m$ represent the fraction of the good left for trade at impasse j. Thus, we denote impasses by $\left\{\left(z_j^m, q_j^m\right)\right\}_{j=1}^{J_m}$ in this subsection.

PROPOSITION 3. IMPASSES FOR AN ARBITRARILY DIVISIBLE GOOD. *The limit equilibrium outcome satisfies*

$$\lim_{m\to\infty}\left(\max\left\{z_j^m - z_{j-1}^m\right\}_{j=2}^{J_m}\right) = 0 \tag{6a}$$

$$\lim_{m\to\infty}\left(\max\left\{q_{j-1}^m - q_j^m\right\}_{j=2}^{J_m}\right) = 0 \tag{6b}$$

$$\lim_{m\to\infty} z_{J_m}^m = 1 \tag{6c}$$

$$\lim_{m\to\infty}\left(\max\left\{\left|q_j^m - \bar{q}\left(z_j^m\right)\right|\right\}_{j=1}^{J_m}\right) = 0 \tag{6d}$$

$$\lim_{m\to\infty}\left(\max\left\{\left|P_m^-\left(mz_j^m, q_j^m\right) - v_L\int_0^{z_j^m}\lambda(z)dz\right|\right\}_{j=1}^{J_m}\right) = 0 \tag{6e}$$

$$\lim_{m\to\infty}\left(\max\left\{\left|P_m^+\left(mz_j^m,q_j^m\right)-v_L\int_0^{z_j^m}\lambda(z)dz\right|\right\}_{j=1}^{J_m}\right)=0 \tag{6f}$$

See Appendix A.6 for the proof.

Proposition 3 directly leads to Theorem 1. As the good becomes arbitrarily divisible, the number of impasses goes to infinity. The fraction that the buyer purchases through each universal offer shrinks to zero (equation (6a)). The change in the buyer's belief between two consecutive impasses also shrinks to zero (equation (6b)). Thus, in the limit, the high-quality good is traded smoothly over time, and beliefs also evolve continuously. The first impasse takes place with the whole good left for trade (equation (6c)). Let $z(\tau)=\lim_{m\to\infty}z_m(\tau)$ and $q(\tau)=\lim_{m\to\infty}q_m(\tau)$ denote respectively the limit fraction of the good left and the limit belief at time elapsed τ. Equation (6d) shows that these functions are linked: $q(\tau)=\bar{q}\left(z(\tau)\right)$. Furthermore, at any screening offer for a fraction $z(\tau)$ of the good, the buyer offers a price $v_L\int_0^{z(\tau)}\lambda(z)dz$, and so he breaks even (equations (6e) and (6f)).

The limit equilibrium outcome then coincides with the artificial pattern of trade described in Section 4.1. At time zero the buyer makes a screening offer for the whole good at price $v_L\int_0^1\lambda(z)dz$. The low-type seller accepts this offer with probability $\bar{q}_m(1)/\hat{q}$. Then, the buyer continuously makes both universal and screening offers. The low-type seller's indifference between accepting different screening offers implies that the fraction $z(\tau)$ must satisfy equation (2). This pins down the pattern of trade in the limit: $z(\tau)=z^*(\tau)$ and $q(\tau)=q^*(\tau)$, as stated in Theorem 1.

We next describe the intuition behind the proof of Proposition 3, which relies on the key properties identified in Proposition 2. First, equation (6d) directly results from the condition $\bar{q}_m(k_j^m+1)<q_j^m<\bar{q}_m(k_j^m)$ in Proposition 2. As m goes to infinity, $\frac{k_j^m+1}{m}\to\frac{k_j^m}{m}=z_j^m$. Next, we explain, through a unified argument, why equations (6a), (6b), (6e) and (6f) hold true.

We say that an impasse (z_j^m,q_j^m) is short whenever $P_m^-\left(mz_j^m,q_j^m\right)$ and $P_m^+\left(mz_j^m,q_j^m\right)$ are close (and so both are close to the valuation $v_L\int_0^{z_j^m}\lambda(z)dz$).[23] The buyer makes a

[23]Proposition 2 guarantees that $P_m^-\left(mz_j^m,q_j^m\right)<v_L\int_0^{z_j^m}\lambda(z)dz<P_m^+\left(mz_j^m,q_j^m\right)$ for every impasse

profit with a screening offer before each impasse. Whenever an impasse is short, this profit is low. The driving force behind Proposition 3 is that whenever m is large, if an impasse (z_j^m, q_j^m) is short, then the previous impasse (z_{j+1}^m, q_{j+1}^m) must also be short. Moreover, the fraction $z_{j+1}^m - z_j^m$ that the buyer purchases between these two impasses must be small. To show this, we link two consecutive impasses (z_{j+1}^m, q_{j+1}^m) and (z_j^m, q_j^m). The buyer obtains a zero continuation payoff at every impasse. Thus, the difference $W_m(mz_{j+1}^m, q_{j+1}^m) - W_m(mz_j^m, q_j^m)$, which we express in equation (7), is also zero:

$$\overbrace{\left(\hat{q} - q_{j+1}^m\right)\left[\int_{z_j^m}^{z_{j+1}^m} [\lambda(z)v_L - c]\,dz\right] + (1-\hat{q})\left[\int_{z_j^m}^{z_{j+1}^m} [\lambda(z)v_H - c]\,dz\right]}^{(*)}$$
$$+ \underbrace{\left(q_j^m - q_{j+1}^m\right)\left[\int_0^{z_j^m} \lambda(z)v_L dz - P_m^-\left(mz_j^m, q_j^m\right)\right]}_{(**)} = 0 \qquad (7)$$

From one impasse to the next one, the buyer makes a loss with a universal offer $(*)$, and a profit with a screening offer $(**)$. This profit is close to zero since the price $P_m^-\left(mz_j^m, q_j^m\right)$ of the screening offer is close to the buyer's valuation. Therefore, the loss associated to the universal offer must also be close to zero, which can only happen if z_j^m is close to z_{j+1}^m.[24]

We next show that the previous impasse (z_{j+1}^m, q_{j+1}^m) must also be short. Equations (4b) and (4c) in Proposition 2 imply that:

$$P_m^-\left(mz_{j+1}^m, q_{j+1}^m\right) = \left(\frac{v_L \int_0^{z_{j+1}^m} \lambda(z)dz}{\left(z_{j+1}^m - z_j^m\right)c + P_m^-\left(mz_j^m, q_j^m\right)}\right)^2 P_m^+\left(mz_{j+1}^m, q_{j+1}^m\right)$$

Since z_j^m and z_{j+1}^m are close and the price $P_m^-\left(mz_j^m, q_j^m\right)$ is close to the buyer's valuation, then the first term on the right hand side is close to one.

(z_j^m, q_j^m). Whenever $P_m^-\left(mz_j^m, q_j^m\right)$ and $P_m^+\left(mz_j^m, q_j^m\right)$ are close and different from zero, their ratio is close to one. This implies that it takes a short time for the price to go from $P_m^-\left(mz_j^m, q_j^m\right)$ to $P_m^+\left(mz_j^m, q_j^m\right)$ and in this sense the impasse is short.

[24] Equation (6d) implies that for m large, the buyer is close to breaking even if he makes a universal offer for an arbitrarily small unit at state $\left(mz_{j+1}^m, q_{j+1}^m\right)$. Since gains from trade are decreasing, any non-negligible universal offer would lead to a loss bounded away from zero.

We argue next that the last impasse must be short as m grows large. The last impasse occurs when only one unit remains. Equations (4a) and (4b) in Proposition 2 imply that $P_m^-(1, \bar{q}_m(1))$ and $P_m^+(1, \bar{q}_m(1))$ both converge to zero as m grows large.

To complete the argument it remains to be shown that there are no cumulative effects in the sense that if one impasse is short, all previous ones must also be short. We relegate the proof of this technical result to Appendix A.6.

Finally, the intuition behind equation (6c) is simple. If it does not hold, then for large m the buyer reaches the first impasse after purchasing a strictly positive fraction of the good though a universal offer. This offer yields a loss to the buyer. At each impasse the price of the screening offer is close to the buyer's valuation, so the buyer's profit from this offer is negligible. Therefore, if equation (6c) is violated, the buyer obtains a negative continuation payoff at the beginning, which can never happen.

6. Comparative statics and extensions

6.1 Comparative statics

In this subsection we show how the primitives of the model affect the speed of trade for both the high-quality and the low-quality good. We start with a configuration of the primitives $(\hat{\beta}, \lambda(\cdot), c, v_L, v_H, r)$, modify one of them (resulting in a new configuration that also satisfies the assumptions of our model) and compare the resulting limit equilibrium outcomes.

PROPOSITION 4. SPEED OF TRADE OF THE HIGH-QUALITY GOOD. *Let $(z^*(\cdot), q^*(\cdot))$ denote the limit equilibrium outcome associated to the primitives $(\hat{\beta}, \lambda(\cdot), c, v_L, v_H, r)$. Consider next an alternative configuration of primitives with associated limit equilibrium outcome $(\tilde{z}^*(\cdot), \tilde{q}^*(\cdot))$. For any of the following alternative configurations of primitives, the high-quality good is traded faster, i.e. $\tilde{z}^*(\tau) < z^*(\tau)$ for every $\tau > 0$:*

(a) $(\hat{\beta}, \tilde{\lambda}(\cdot), c, v_L, v_H, r)$ *with* $\tilde{\lambda}(z) > \lambda(z)$ *for all* $z \in (0, 1]$.

(b) $(\hat{\beta}, \lambda(\cdot), \tilde{c}, v_L, v_H, r)$ *with* $\tilde{c} < c$.

(c) $\left(\hat{\beta}, \lambda(\cdot), c, \tilde{v}_L, v_H, r\right)$ *with* $\tilde{v}_L > v_L$.

(d) $\left(\hat{\beta}, \lambda(\cdot), c, v_L, v_H, \tilde{r}\right)$ *with* $\tilde{r} > r$.

Finally, the parameters v_H *and* $\hat{\beta}$ *do not affect the speed of trade of the high-quality good.*

Proof. We present here the proof for (a). The cases (b), (c) and (d) follow the same argument. Assume towards a contradiction that the result does not hold for (a). Equation (3) implies that

$$\tilde{z}^{*\prime}(0) = \frac{rv_L \int_0^1 \tilde{\lambda}(z)dz}{v_L\tilde{\lambda}\left(1\right) - c} < \frac{rv_L \int_0^1 \lambda(z)dz}{v_L\lambda\left(1\right) - c} = z^{*\prime}(0).$$

Let $\underline{\tau} = \min\left\{\tau > 0 : \tilde{z}^*(\tau) = z^*(\tau)\right\}$. It follows again from equation (3) that

$$\tilde{z}^{*\prime}\left(\underline{\tau}\right) = \frac{rv_L \int_0^{\tilde{z}^*(\underline{\tau})} \tilde{\lambda}(z)dz}{v_L\tilde{\lambda}\left(\tilde{z}^*\left(\underline{\tau}\right)\right) - c} < \frac{rv_L \int_0^{z^*(\underline{\tau})} \lambda(z)dz}{v_L\lambda\left(z^*\left(\underline{\tau}\right)\right) - c} = z^{*\prime}\left(\underline{\tau}\right).$$

But then there exists $\tau' \in (0, \underline{\tau})$ with $\tilde{z}^{*\prime}\left(\tau'\right) = z^{*\prime}\left(\tau'\right)$, reaching a contradiction. Finally, notice that $z^*(0) = 1$ and that $z^{*\prime}(\cdot)$ does not depend on v_H or $\hat{\beta}$. ■

The intuition behind Proposition 4 is simple. The speed of trade of the high-quality good is such that the low-type seller is always indifferent between accepting the current screening offer or rejecting all screening offers and obtaining the discounted value of future universal offers. An increase in either v_L or in the function $\lambda(\cdot)$ makes each screening offer more attractive. Similarly, a decrease in c or an increase in r lower the value of future universal offers. In all these four cases the high-quality good must be traded faster to keep the low-type seller indifferent.

Unlike the high-quality good, the low-quality good is not always traded smoothly. Trade occurs smoothly while the low-type seller mimics the high-type seller's behavior. However, the buyer purchases the whole remaining fraction of the good as soon as the low-type seller accepts a screening offer. Therefore, the fraction of the low-quality good remaining at time elapsed τ is a random variable that takes a value of zero with probability $\frac{q^*(\tau)}{\hat{q}}$ and a value of $z^*(\tau)$ with the remaining probability. Then $g^*(\tau) = \frac{\hat{q} - q^*(\tau)}{\hat{q}} z^*(\tau)$

is the expected remaining fraction of the low-quality good at time elapsed τ and reflects the speed of trade of the low-quality good.

We next study how changes in the parameters r, v_H and $\hat{\beta}$ affect the speed of trade of the low-quality good. Consider first an increase in r, so both parties become less patient. Proposition 4 guarantees that $z^*(\cdot)$ decreases (for all $\tau > 0$). This, and the fact that $q^*(\tau) = \bar{q}(z^*(\tau))$, imply that $q^*(\cdot)$ increases. Thus, the low-quality good is traded faster, i.e. $g^*(\cdot)$ decreases. Second, consider a decrease in either v_H or in $\hat{\beta}$. Proposition 4 shows that $z^*(\cdot)$ does not change in either case. It follows from the fact that $q^*(\tau) = \bar{q}(z^*(\tau))$ and the definition of $\bar{q}(\cdot)$ that $q^*(\cdot)$ increases (if v_H decreases) and that $\hat{q} - q^*(\cdot)$ decreases (if $\hat{\beta}$ decreases). Thus, in either case, $g^*(\cdot)$ decreases. The following corollary summarizes these results.

COROLLARY 1. SPEED OF TRADE OF THE LOW-QUALITY GOOD. *Whenever either r increases, or v_H decreases, or $\hat{\beta}$ decreases, then the low-quality good is traded faster, i.e. $g^*(\tau)$ decreases for every $\tau > 0$.*

The intuition behind Corollary 1 is as follows. In the limit equilibrium outcome, at time elapsed τ the buyer's belief $q^*(\tau)$ is such that he breaks even when he makes a universal offer for the next infinitesimal unit: $q^*(\tau) = \bar{q}(z^*(\tau))$. When the parties become less patient, the remaining fraction of the good $z^*(\tau)$ decreases. Therefore, decreasing gains from trade imply that the buyer's belief $q^*(\tau)$ must increase. This is possible only if the low-type seller becomes more likely to accept screening-offers, so trade of the low-quality good occurs faster. The intuition for the other two cases is similar. A decrease in v_H or in $\hat{\beta}$ requires that the low-type seller accepts screening offers with a higher probability to guarantee that, at any time τ, the buyer breaks even with the universal offer.

The remaining primitives ($\lambda(\cdot)$, v_L and c) have ambiguous effects on the speed of trade of the low-quality good. It is easy to construct examples where changes in these primitives can either increase or decrease $g^*(\tau)$ for some τ.

We finally study the limit as gains from trade from different units become arbitrarily close to each other. Formally, we fix the parameters $\left(\hat{\beta}, c, v_L, v_H, r\right)$ and consider a sequence of strictly increasing functions $\{\lambda_n(\cdot)\}_{n=1}^{\infty}$ converging uniformly to the constant function $\underline{\lambda}(\cdot) = 1$. We assume that for every n, the configuration of primitives satisfies the assumptions of our model. Let $(z_n^*(\cdot), q_n^*(\cdot))$ denote the limit equilibrium outcome associated to configuration n. We derive the limit of $(z_n^*(\cdot), q_n^*(\cdot))$ using the characterization of the limit equilibrium outcome from Section 4.1 – see equation (2). As n goes to infinity, $q_n^*(\cdot)$ converges uniformly to a constant function with value $\bar{q}(0)$.[25] Furthermore $z_n^*(\cdot)$ converges uniformly to the function $e^{-\frac{rv_L}{c-v_L}\tau}$. In the limit as $\lambda_n(\cdot)$ becomes flat, the belief jumps to $\bar{q}(0)$ at time zero and then remains constant. This means that the low-type seller accepts the screening offer at time zero with positive probability. From that point on, although he is always indifferent, the low-type seller accepts all screening offers with zero probability. Finally, the high-quality good is traded gradually at a constant rate: $\underline{z}'(\tau)/\underline{z}(\tau) = -(rv_L)/(c - v_L)$.[26]

6.2 Extensions

In our first extension, we study the limit equilibrium outcome when equation (1) does not hold. Equation (1) reflects an extreme form of adverse selection: under the prior belief, the buyer's expected valuation from any fraction of the good exceeds the high-type seller's cost. Therefore, the buyer needs to screen the seller even to purchase the most valuable fraction of the good.

We first assume that $\left[\hat{\beta}v_H + (1-\hat{\beta})v_L\right]\lambda(\bar{z}) = c$ for some $\bar{z} \in (0,1]$, so the buyer obtains a positive payoff if he buys any infinitesimal unit $z \in [\bar{z}, 1]$ through a universal offer. Our analysis directly extends to this case.[27] In the limit equilibrium outcome, the buyer purchases the first fraction $1 - \bar{z}$ from both types without delay, paying $c(1 - \bar{z})$. The environment after the units $z \in [\bar{z}, 1]$ are traded resembles that from our baseline model. Theorem 1 pins down the pattern of trade for the remaining fraction $\bar{z}$. Similarly to the case when equation (1) holds, divisibility is detrimental to the buyer. Although he obtains a profit from the units $z \in [\bar{z}, 1]$, he must pay the high-type seller's cost for

[25]Recall that the belief $\bar{q}(0)$ makes the buyer break even when he makes a universal offer for the least valuable unit. As $\lambda_n(0) = 1$ for all n, this belief does not depend on n.

[26]This pattern of trade differs from the pattern of trade with constant gains from trade that we describe in the next subsection. As discussed in the conclusion, this difference also arises if we invert the order of limits, first letting the number of units grow to infinity and then lettting bargaining frictions vanish.

[27]The proof of the characterization of the limit equilibrium outcome in this case is analogous to the proof of Theorem 1 so we omit it.

them. He then obtains a zero profit from the remaining units. Furthermore, like in the benchmark model, the high-quality good is traded smoothly, but only for the units $z \in [0, \bar{z}]$.

We next assume that $\left[\hat{\beta} v_H + (1-\hat{\beta}) v_L\right] \lambda(0) \geq 0$. In this case, the buyer obtains a positive payoff if he buys any fraction through a universal offer, so the standard Coasean forces apply. For any m, as bargaining frictions vanish, the buyer purchases the whole good from both types without delay and pays c.

In our second extension we assume that $\lambda(\cdot)$ is either constant or strictly decreasing, which correspond, respectively, to constant gains from trade or increasing gains from trade. In either of these cases divisibility plays no role: for any m the buyer only makes offers (m, p) with $p \leq c$ in equilibrium. The equilibrium outcome is identical to the one in DL. Intuitively, whenever the buyer is happy to pay the high-type seller's cost for some units, then he is also happy to pay that cost for subsequent units. As gains from trade are constant or increasing, those subsequent units are at least as valuable as the previous ones.

Consider the case with constant gains from trade: $\lambda(z) = 1$. Fix the number of units m and the period length Δ. Let $W(1, \cdot)$ and $P(1, \cdot)$ be respectively the buyer's normalized payoff and the price function when one unit remains. These functions are as in DL, so $W(1, q) > 0$ for every $q \in [0, \hat{q}]$. Suppose that for every $K \in \{1, \ldots, m\}$ and for every $q \in [0, \hat{q}]$, $W(K, q) = KW(1, q)$ and $P(K, q) = KP(1, q)$. Finally consider a belief $q' \in [0, \hat{q}]$ such that the buyer makes a screening offer at state $(1, q')$. Next, we show that it is not optimal for the buyer to make a universal offer at any state (K, q') with $K \in \{2, \ldots, m\}$. Assume towards a contradiction that it is optimal to make a universal offer for $K - k$ units. Then,

$$
\begin{aligned}
W(K, q') = KW(1, q') &\leq \frac{K-k}{m} \left[\left(\hat{q} - q'\right) v_L + \left(1 - \hat{q}\right) v_H - (1 - q')c \right] + \delta k W(1, q') \\
&< \frac{K-k}{m} \left[\left(\hat{q} - q'\right) v_L + \left(1 - \hat{q}\right) v_H - (1 - q')c \right] + k W(1, q')
\end{aligned}
$$

This in turn, implies that

$$W(1, q') < \frac{1}{m}\left[\left(\hat{q} - q'\right) v_L + (1 - \hat{q})\, v_H - (1 - q')c\right]$$

which violates the assumption that a screening offer is optimal at state $(1, q')$. This argument directly implies Proposition 5.

PROPOSITION 5. CONSTANT GAINS FROM TRADE. *When gains from trade are constant, the buyer only makes screening offers in equilibrium.*

An argument analogous to the one in the previous paragraphs extends the result in Proposition 5 to the case of increasing gains from trade. We omit the proof here.[28]

7. Conclusion

In this paper we study bargaining over a divisible good. We characterize the limit equilibrium outcome as bargaining frictions vanish and the good becomes arbitrarily divisible. Our model generates novel and testable predictions for dynamic markets with adverse selection. When gains from trade are constant or increasing, the pattern of trade is identical to that of parties negotiating over an indivisible good. Time on the market is the main signaling device and the buyer keeps some of his bargaining power. On the other hand, when there are decreasing gains from trade, the high-quality good is traded smoothly over time and the buyer loses all the bargaining power in the limit.

In this paper we first let the time between offers shrink to zero and we then let the number of units grow to infinity. With this order of limits we can use an inductive argument on the number of remaining units and develop an algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish. The tools developed in this paper do not allow for a complete characterization of the pattern of trade if we instead invert the order of limits. However, one of our main findings extends to that environment. If we invert the order of limits, the number of transactions of the high-quality good must

[28] Further details on the cases of constant and increasing returns from trade can be found in an earlier version of our paper available at https://www.carloalberto.org/wp-content/uploads/2018/11/no.312.pdf.

also grow without bound. The intuition behind this is simple. Assume instead that in the limit there is a finite number of transactions, and take the last transaction for a positive fraction of the good. Consider the last impasse before this transaction.[29] At this impasse, the buyer's payoff is zero and his belief is such that he breaks even if he makes a universal offer for the remaining fraction of the good. But then the buyer has a profitable deviation; because of decreasing gains from trade, he obtains a positive payoff by making a universal offer for less than the remaining fraction of the good.

Our model relies on some simplifying assumptions that make the analysis tractable. First, we assume that the quality of the good can take only two values. Our results extend to a model with finitely many types provided that the buyer's valuation for a good of any intermediate quality is sufficiently close to his valuation for the good of the highest quality. Future research can shed further light on bargaining with divisibility and many types.

Second, we assume that the buyer learns about the quality of the good only through the seller's behavior. This assumption is reasonable in a number of important applications. Moreover, our model represents a useful theoretical benchmark to study bargaining over divisible objects. However, it would be interesting to extend the model to allow for additional forms of learning: endogenous, for example, in the form of learning via the consumption of parts of the good, or exogenous (as in Daley and Green [2012] and Daley and Green [Forthcoming]). We leave the study of bargaining with learning for future work, but conjecture that the driving forces behind our results would emerge even with learning, leading to the gradual sale of the high-quality good.

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[29] At least one impasse must exist before this transaction. Otherwise, the buyer would pay the high-type seller's cost for the whole good and obtain a negative payoff.

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A. Proofs

A.1 Proof of Lemma 1

Proof. We show that (a) holds for the weaker solution concept of PBE. For any $K \in \{1, \ldots, m\}$ and for any PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, let $H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$ denote the set of histories h^t with $K(h^t) = K$ and $\beta(h^t) = 0$.

We show first that (a) holds when only one unit remains. Let $\bar{u}_L$ denote the supremum, over all PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, of the low-type seller's continuation payoff at histories $h^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$. Assume towards a contradiction that $\bar{u}_L > 0$ and take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$. There must exist a PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and a history $\bar{h}^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$ at which the buyer offers $\varphi_t = (1, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$. The low-type seller must accept this offer with probability one. To see why, notice that if the low-type seller rejects this offer with positive probability, then $(\bar{h}^t, (\varphi_t, R)) \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and therefore the low-type seller's continuation payoff is at most $\bar{u}_L$. But then, since $\bar{u}_L - \varepsilon > \delta\bar{u}_L$, it is not optimal for the low-type seller to reject φ_t. For the same reason, the low-type seller must accept the offer $\varphi'_t = (1, \bar{u}_L - \frac{3}{2}\varepsilon)$ with probability one. Thus, the buyer has a profitable deviation at $\bar{h}^t$ since he strictly prefers the offer φ'_t to φ_t.

We show next that (a) holds for any number of remaining units K. We proceed by induction. Fix $K \in \{2, \ldots, m\}$ and assume that for any PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and for any $h^t \in H_1(\sigma_B, (\sigma_L, \sigma_H), \beta) \cup \ldots \cup H_{K-1}(\sigma_B, (\sigma_L, \sigma_H), \beta)$, the low-type seller"s continuation payoff is zero. Again, let $\bar{u}_L$ denote the supremum, over all PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$, of the low-type seller's continuation payoff at histories $h^t \in H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$. Towards a contradiction, assume that $\bar{u}_L > 0$ and take $\varepsilon = \left(\frac{1-\delta}{2}\right)\bar{u}_L$. There must exist a PBE $(\sigma_B, (\sigma_L, \sigma_H), \beta)$ and a history $\bar{h}^t \in H_K(\sigma_B, (\sigma_L, \sigma_H), \beta)$ at which the buyer offers $\varphi_t = (k, p)$ for some $p \in [\bar{u}_L - \varepsilon, \bar{u}_L]$ and some $k \leq K$. Using the induction hypothesis and an argument similar to the one presented in the previous paragraph, we conclude that the low-type seller must accept this offer with probability one. However, the same is true for the offer $\varphi'_t = \left(k, \bar{u}_L - \frac{3}{2}\varepsilon\right)$ which is, therefore, strictly preferred to φ_t. Again, this shows that the buyer has a profitable deviation at $\bar{h}^t$ and concludes the proof of part (*a*) of Lemma 1.

We show (*b*) by contradiction. Assume that there exist two histories h^t and $\tilde{h}^{t'}$ with the same state variables but with $V_B(h^t) < V_B\left(\tilde{h}^{t'}\right)$. The buyer then has a profitable deviation after history h^t. He can choose the same actions as he chooses after history $\tilde{h}^{t'}$. Since the seller's strategy depends only on state variables, then he reacts as he does after history $\tilde{h}^{t'}$, and so the buyer's continuation payoff increases.

We show (*c*) by contradiction. Assume instead that there is a history h^t where the high-type seller obtains a positive continuation payoff: $V_H(h^t) > 0$. Over all histories with positive continuation payoffs, pick those with the smallest number of remaining units $\underline{K} = \min\left\{K(h^t) : V_H(h^t) > 0\right\}$. Let $\alpha = \sup\left\{V_H(h^t) : K(h^t) = \underline{K}\right\}$ denote an upper bound for the high-type seller's continuation payoff when only $\underline{K}$ units remain. Finally, let $\varepsilon \equiv (1-\delta)\alpha/3$.

There must exist a history h^t with $K(h^t) = \underline{K}$ at which the buyer makes an offer (k, p) that the high-type seller accepts, and the offer satisfies $1 \leq k \leq \underline{K}$ and $p > \frac{c}{m}k + \alpha - \varepsilon$. This in turn implies that the low-type seller also accepts this offer (otherwise, by Lemma 1(*a*), he gets a total payoff of zero). Consider instead the following deviation by the buyer; he offers $\left(k, \frac{c}{m}k + \alpha - \varepsilon\right)$. If the high-type seller rejects this offer, he obtains a continuation payoff of at most $\delta\alpha < \alpha - \varepsilon$, so he accepts it. For the same reason as above, the low-type seller also accepts this offer. Both the original offer and the deviation lead to the same state variables, and therefore to the same continuation payoff to the buyer, as shown in Lemma 1(*b*). This implies that the deviation is profitable. This shows part (*c*) of Lemma 1.

Consider next part (*d*) of Lemma 1. Whenever $\beta(h^t) = 0$, the result follows immediately from Lemma 1(*a*). Otherwise, the zero bound on the continuation payoff for the high type seller directly implies a $\frac{c}{m}K(h^t)$ upper bound for the continuation payoff for the low-type seller. ■

A.2 Proof of Lemma 2

Proof. In the case $\beta(h^t) = 0$ all units are traded in the first period (this follows immediately from Lemma 1(*a*)). Assume instead that $\beta(h^t) > 0$ and consider an offer $\varphi_t = (k, p)$ with $k < K(h^t)$ and $p < \frac{c}{m}k$. We show that such an offer is not accepted with positive probability. By contradiction, assume that this offer is accepted with positive probability.

A high-type seller would never accept such an offer, so it must be the low-type seller who accepts this offer with probability $\sigma_L^t(h^t, \varphi_t) > 0$.

A rejection then leads to a posterior $\beta' \in \big(\beta(h^t), 1\big)$. Whenever the low-type seller accepts, the buyer immediately learns that the seller is of low type. Then, in the following period all remaining units are traded, at zero cost. The buyer obtains the following payoff from this offer:

$$[1-\beta(h^t)]\sigma_L^t(h^t,\varphi_t)\left[\sum_{s=K(h^t)-k+1}^{K(h^t)} \Lambda_s^m v_L - p + \delta \sum_{s=1}^{K(h^t)-k} \Lambda_s^m v_L\right] + \left[1-\beta(h^t)\left(1-\sigma_L^t(h^t,\varphi_t)\right)\right] V_L(\beta', K)$$

Consider instead an offer to pay p in exchange for *all* remaining units. If the low-type seller accepts, he obtains the same payoff as from accepting the previous offer. Moreover, because of stationarity, a rejection leads to the same belief β' as before. Then, the low-type seller accepts this offer with the same probability as the previous offer. The buyer, however, obtains the following *higher* payoff from this offer:

$$[1-\beta(h^t)]\sigma_L^t(h^t,\varphi_t)\left[\sum_{s=K(h^t)-k+1}^{K(h^t)} \Lambda_s^m v_L - p + \sum_{s=1}^{K(h^t)-k} \Lambda_s^m v_L\right] + \left[1-\beta(h^t)\left(1-\sigma_L^t(h^t,\varphi_t)\right)\right] V_L(\beta', K)$$

Then, if an offer for $k < K(h^t)$ remaining units was accepted with positive probability, the buyer would rather make an offer for all remaining units, so there would be a profitable deviation. ■

A.3 Proof of Proposition 1

The proof is divided in three parts. In Part A we define the notion of a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ of intertwined functions. We show that whenever a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ exists, then a stationary PBE must exist. Our proof is constructive: we derive equilibrium strategies and beliefs from the consistent quadruplet. In Part B we construct a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. In Part C we show that for generic parameters all equilibria are outcome equivalent to the one constructed in Parts A and B. We present Part C in Section T.1 of the Technical Addendum.

Part A. The consistent quadruplet $(\mathcal{V}_L, P, W, y)$

We first describe the components of the quadruplet $(\mathcal{V}_L, P, W, y)$. The function $\mathcal{V}_L(K, q) : \{1, \dots, m\} \times [0, \hat{q}] \to \mathbb{R}$ determines the strategy of the low-type seller, as described in the definition of stationary PBE. The function $P(K, q) : \{1, \dots, m\} \times [0, \hat{q}] \to \mathbb{R}$ pins down the screening offer $(K, P(K, q))$ that induces (transformed) posterior belief q if rejected. The function $W(K,q) : \{1,\ldots,m\} \times [0,\hat{q}] \to \mathbb{R}$ represents the buyer's (normalized) continuation payoff. Finally, the function $y(K,q) : \{1,\ldots,m\} \times [0,\hat{q}] \to \{1,\ldots,m\} \cup [0,\hat{q}]$ specifies the offers that the buyer makes on the equilibrium path.

Part A contains four steps. The first three define the notion of a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. In step 1 we derive the function P from the function $\mathcal{V}_L$. In step 2 we turn to the buyer's optimization problem. We take as given the behavior of the low-type seller, which is summarized by P. We define the buyer's value function W and his best response correspondence. From this best response correspondence, in step 3 we select the offer $y(K,q)$ that the buyer makes in state (K,q). We construct a candidate value function $\mathcal{V}_L'$ for the low-type seller from the functions y and P. Finally, we say that the quadruplet $(\mathcal{V}_L, P, W, y)$ is *consistent* if $\mathcal{V}_L' = \mathcal{V}_L$.

In step 4 we construct strategies from the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ and show that these strategies (together with appropriate beliefs) form a stationary PBE.

Step 1. From $\mathcal{V}_L$ to P. Consider a (left-continuous) candidate function $\mathcal{V}_L$ with $0 \leq \mathcal{V}_L(K,q) \leq \frac{c}{m}K$ for all (K,q). This function determines the low-type seller's behavior, following the definition of stationary PBE.[30] This same definition also pins down the high-type seller behavior: he accepts any offer for k units if and only if he receives in exchange a payment greater or equal than $\frac{c}{m}k$.

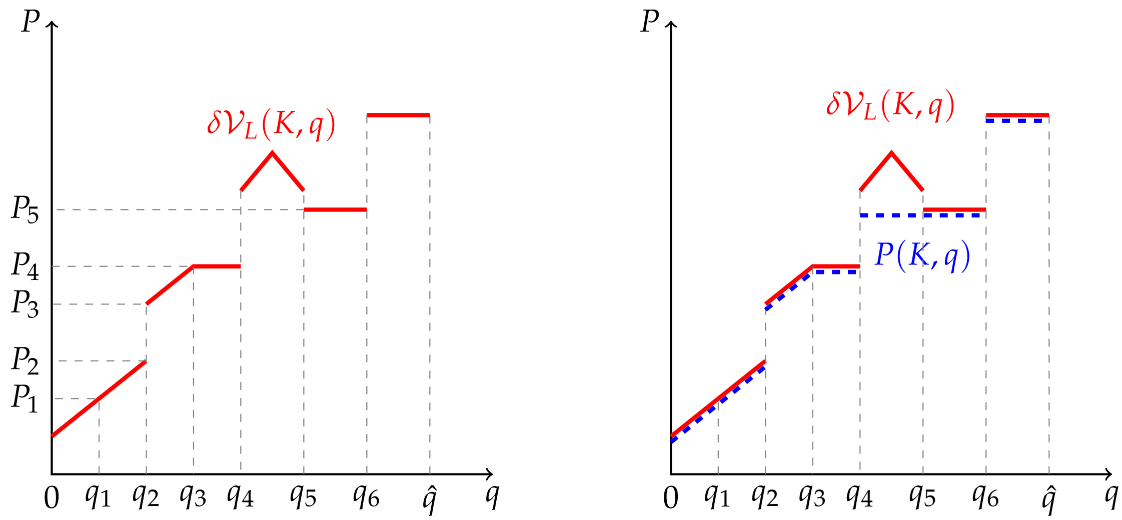
We study the buyer's best response to the seller's behavior implied by $\mathcal{V}_L(K,q)$. We can restrict attention to two types of offers: universal and screening. Universal offers are simple: the buyer offers a payment $\frac{c}{m}k$ for some (or all) remaining units $k \leq K$, both sellers accept and beliefs do not change.

Screening offers involve both a price and a transformed posterior belief. A price induces a probability of acceptance, which in turn leads to a transformed posterior belief after the offer is rejected. As we show below, different prices may induce the same posterior. Moreover, there may be some posteriors that no price can induce. We define a modified problem where the buyer who starts a period with a (transformed) belief $q \in [0,\hat{q}]$ can induce any (transformed) posterior belief $q' \in [q,\hat{q}]$ by choosing a unique price $P(K,q')$. We show in step 4 that solutions to the modified problem coincide with those of the original one.

We first illustrate how we derive $P(K,q)$ from $\mathcal{V}_L(K,q)$ and then provide the formal definition of $P(K,q)$. Consider the function $\delta\mathcal{V}_L(K,q)$ shown in Figure 5(a). It is simple to see that the price $P_1 = \delta\mathcal{V}_L(K,q_1)$ induces posterior belief q_1. This is because the function $\delta\mathcal{V}_L(K,q)$ lies above P_1 for posteriors greater than q_1. In fact, obtaining $P(K,q)$ would be straightforward if $\mathcal{V}_L(K,q)$ was continuous and strictly increasing. However, consider for example posterior belief q_2, which is induced by all prices in the range $[P_2, P_3]$. The buyer's preferred price in that range is the lowest: P_2; and thus we set $P(K,q_2) = P_2$.

The set of induced beliefs may be non-convex. The price P_4 induces posterior belief q_4, but no price induces posterior beliefs on the range $[q_3, q_4)$. To restore convexity, in the modified problem we allow the buyer to induce any belief $q \in [q_3, q_4)$ by paying the price $P(K,q) = P_4$. Similarly, the buyer cannot induce posterior beliefs in the range (q_4, q_6). We allow the buyer to induce any belief $q \in (q_4, q_6)$ by paying the price $P(K,q) = P_5$.

[30]The function $\mathcal{V}_L(K,q)$ maps one-to-one to a function $\mathcal{V}_L(K,\beta) : m \times [\hat{\beta}, 1] \to \mathbb{R}$. The definition of stationary PBE pins down the behavior of the low-type seller through the function $\mathcal{V}_L(K,\beta)$.



(a) Multiplicity and non-convexity

(b) The function $P(K,q)$

Figure 5: Derivation of $P(K,q)$ from $\mathcal{V}(K,q)$

Differently than before, $P(K,q) < \delta\mathcal{V}_L(K,q)$ for the interval $q \in (q_4, q_5]$.

Formally, we let $P(K,q)$ be the largest weakly increasing function below $\delta\mathcal{V}_L(K,q)$. As an example, the dashed line in Figure 5(b) depicts the function $P(K,q)$ derived from $\delta\mathcal{V}_L(K,q)$ in Figure 5(a). Whenever the buyer can induce a posterior q but cannot induce posteriors in some range $(q-\eta, q)$, our definition implies that $P(k,q') = \delta\mathcal{V}_L(K,q)$ for all $q' \in (q-\eta, q)$. By doing so, the function $P(K,q)$ becomes flat in some region. Claim 1 in step 4 shows that the buyer never chooses interior points in flat regions, which guarantees that the solutions to the modified problem coincide with those of the original one.

Step 2. From P to W. The buyer's modified problem. We now formalize the buyer's (modified) dynamic optimization problem. With a slight abuse of notation, let $V_B(K,q)$ denote the buyer's continuation payoff when the state is (K,q). For convenience, we work directly with the buyer's *normalized* continuation payoff

$$W(K,q) \equiv (1-q)V_B(K,q).$$

We set $W(0,q) = 0$ and

$$W(K,\hat{q}) = (1-\hat{q})\left[\left(\sum_{s=1}^{K} \Lambda_s^m\right) v_H - \frac{c}{m}K\right].$$

For all other cases, we define $W(K,q)$ recursively by:

$$W(K,q) = \max\Bigg\{ \max_{q' \in [q,\hat{q}]} \overbrace{(q'-q)\left[\left(\sum_{s=1}^{K}\Lambda_s^m\right)v_L - P(K,q')\right] + \delta W(K,q')}^{(*)\text{ Screening. Offer } P(K,q') \text{ for } K \text{ units. If rejected, induced belief is } q'},$$
$$\max_{0\le k\le K-1} \underbrace{\left\{\left(\sum_{s=k+1}^{K}\Lambda_s^m\right)\left[(\hat{q}-q)\,v_L + (1-\hat{q})\,v_H\right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q)\right\}}_{(**)\text{ Universal offer. Request next } K-k \text{ units in exchange for payment } \frac{c}{m}(K-k)} \Bigg\} \quad (8)$$

The first component $(*)$ of equation (8) provides the continuation payoff when the buyer induces belief q' through a screening offer. The second component $(**)$ of equation (8) provides the continuation payoff when the buyer makes a universal offer for $K-k$ units. The buyer compares the value of the best screening offer (optimal q') with the value of the best universal offer (optimal k) to choose which kind of offer to make.[31]

Equation (8) defines the buyer's *modified* problem. When the state is (K,q) with $q \in [0,\hat{q})$ we allow the buyer to induce any state (K,q') with $q' \ge q$ by making the screening offer $(K, P(K,q'))$. This includes states that cannot be reached in the original game, like (K,q_5) in Figure 5.

Let $Y(K,q)$ denote the set of solutions to the problem in equation (8). A screening offer that induces posterior q' is of the form $(K, P(K,q'))$. When such offer is optimal, we let $q' \in Y(K,q)$. A universal offer for $K-k$ units is of the form $\left(K-k, \frac{c}{m}(K-k)\right)$. When such offer is optimal, we let $k \in Y(K,q)$.

Step 3. From P and W to y and $\mathcal{V}_L'$. The notion of consistent quadruplet. We combine the low-type seller's behavior, implicit in P, with the buyer's optimal behavior to construct a *candidate* value function $\mathcal{V}_L'(K,q)$ for the low-type seller. Let $\mathcal{V}_L'(K,q)$ be defined recursively by:

$$\mathcal{V}_L'(K,q) = \min\left\{\min_{q'\in Y(K,q)} P\left(K,q'\right), \min_{k\in Y(K,q)} \frac{c}{m}(K-k) + \delta\mathcal{V}_L'(k,q)\right\} \quad (9)$$

As equation (9) shows, we construct $\mathcal{V}_L'$ by always selecting the offer that minimizes the low-type seller's continuation payoff from all of the buyer's optimal choices $Y(K,q)$. Let $y(K,q) \in Y(K,q)$ denote the buyer's choice that solves (9). There may be many solutions to (9), but if so, one of them is universal.[32] In such case, we let $y(K,q)$ be the universal offer associated to the lowest k.

Finally, we say that a quadruplet $(\mathcal{V}_L, P, W, y)$ is *consistent* if its components are linked as described in steps 1 to 3 and if the derived $\mathcal{V}_L'$ satisfies $\mathcal{V}_L' = \mathcal{V}_L$.

Step 4. From the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ to a stationary PBE.

[31] The buyer's continuation payoff is always positive, so his individual rationality constraint is satisfied. To see this, note that the buyer can always choose $q' = q$ in equation (8).

[32] To see why, assume that $P(K,q') = P(K,\tilde{q}')$ for $q' \in Y(K,q)$ and $\tilde{q}' \in Y(K,q)$. Since $P(K,q)$ is weakly increasing, then $P(K,q)$ is constant between q' and $\tilde{q}'$. But this cannot happen; Claim 1 shows that the buyer never chooses interior points in flat regions of $P(k,q)$.

a. Definition of strategies and beliefs. Fix a consistent quadruplet $(\mathcal{V}_L, P, W, y)$. Our definition of stationary PBE, together with $\mathcal{V}_L$, fully pin down the seller's strategy. Both types accept with probability one any offer (k, p) with $p \geq \frac{c}{m}k$. The high-type seller rejects offers (k, p) with $p < \frac{c}{m}k$ with probability one, while the low-type seller accepts them with probability pinned down by $\mathcal{V}_L$.

We next specify the buyer's strategy and beliefs. We first define for each t a set of histories $\widehat{H}^t$ that is not reached on the equilibrium path. We say that $h^t \in \widehat{H}^t$ whenever h^t contains either 1) a rejected offer (k, p) with $p \geq \frac{c}{m}k$, or 2) an accepted partial offer. Whenever $h^t \in \widehat{H}^t$, we let the buyer assign probability zero to the seller being of high type. Also, we let the buyer offer a payment of zero for all remaining units after any history $h^t \in \widehat{H}^t$.[33]

If instead $h^t \notin \widehat{H}^t$, the buyer's offer depends on the state $\left(K(h^t), q(h^t)\right)$ and on the actions (φ_{t-1}, a_{t-1}) in $t-1$. The buyer's strategy and beliefs are as follows:

1. If $(\varphi_{t-1}, a_{t-1}) = \left((k, p), A\right)$ with $p \geq \frac{c}{m}k$, then the belief is unchanged: $q(h^t) = q(h^{t-1})$. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.
2. If $(\varphi_{t-1}, a_{t-1}) = \left((k, p), R\right)$ with $p < \frac{c}{m}k$, then
    a. If $p \leq P\left(K(h^{t-1}), q(h^{t-1})\right)$, then the belief is unchanged: $q(h^t) = q(h^{t-1})$. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.
    b. If $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$ and $p = P\left(K(h^{t-1}), q\right)$ for some $q > q(h^{t-1})$, then the belief $q(h^t)$ is given by the probability of acceptance implied in the definition of stationary PBE. The buyer makes the offer implied by $y(K(h^t), q(h^t))$.
    c. If $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$ and $p \neq P\left(K(h^{t-1}), q\right)$ for all $q > q(h^{t-1})$, then the belief $q(h^t)$ is given by the probability of acceptance implied in the definition of stationary PBE. The buyer randomizes among the elements of $Y(K(h^t), q(h^t))$ to rationalize the probability of acceptance of the low-type seller in $t-1$.[34]

b. Verification that strategies and beliefs form a stationary PBE. The strategy of the high-type seller is optimal. On-path, the buyer never pays more than $\frac{c}{m}k$ for any k. Then, it is optimal to accept any offer greater or equal than $\frac{c}{m}k$ for any k and to reject otherwise.

The optimality of the low-type seller's strategy follows from $\mathcal{V}_L = \mathcal{V}_L'$. Assume that the buyer and the seller follow the equilibrium strategies specified above. Then, in any

[33]The set $\widehat{H}^t$ contains some but not all off-path histories. Below we specify the buyer's strategy and beliefs for all histories on path, and also for the remaining off-path histories.

[34]Supoose that $p > P\left(K(h^{t-1}), q(h^{t-1})\right)$, $p \neq P\left(K(h^{t-1}), q\right)$ for all $q > q(h^{t-1})$ and that the new belief is $q(h^t)$. Then, $\delta\mathcal{V}_L\left(K(h^t), q(h^t)\right) < p < \delta \lim_{q \downarrow q(h^t)} \mathcal{V}_L\left(K(h^t), q\right)$. One element of $Y\left(K(h^t), q(h^t)\right)$ yields a continuation payoff of $\mathcal{V}_L\left(K(h^t), q(h^t)\right)$ to the low-type seller, while another one yields a continuation payoff of $\lim_{q \downarrow q(h^t)} \mathcal{V}_L\left(K(h^t), q\right)$ to the low-type seller. In period t the buyer randomizes between these two elements of $Y\left(K(h^t), q(h^t)\right)$ so that the low-type seller's continuation payoff in period $t-1$ (if he rejects the screening offer) is exactly p. Note that this implies that off-the-equilibrium path the low-type seller's continuation payoff may depend not only on the state but also on the offer in the previous period.

on-path history h^t with state $(K,q) = \big(K(h^t), q(h^t)\big)$ the function $\mathcal{V}_L(K,q)$ satisfies:

$$\mathcal{V}_L(K,q) = \begin{cases} \frac{c}{m}(K-k) + \delta\mathcal{V}_L(k,q) & \text{if } \ y(K,q) = k \\ P\left(K,q'\right) = \delta\mathcal{V}_L\left(K,q'\right) & \text{if } \ y(K,q) = q' \end{cases} \tag{10}$$

Equation (10) follows from the definition of $\mathcal{V}'_L$ in equation (9), the equality $\mathcal{V}'_L = \mathcal{V}_L$, the definition of $P(K,q)$ and the fact that the buyer never chooses an induced posterior in a flat region of $P(K,q)$. Therefore, $\mathcal{V}_L(K,q)$ is the on-path continuation payoff of the low-type seller.

The low-type seller obtains a continuation payoff of zero if he rejects a universal offer. The first line of equation (10) shows that he obtains a strictly positive payoff if he instead accepts it. Then, it is optimal for the low-type seller to accept a universal offer.[35] The second line of equation (10) shows that the low-type seller is indifferent between accepting and rejecting the screening offers that the buyer makes on path. Consider instead a buyer who deviates and makes a partial offer $(k, P(K,q'))$ with $k < K$. If the low-type seller accepts, he obtains $P(K,q')$ in the current period and zero from then on. If he instead rejects, his continuation payoff is $\delta\mathcal{V}_L\left(K,q'\right)$. Thus, the low-type seller is also willing to randomize in this case.[36]

We construct the strategy of the buyer by choosing for every history h^t elements from the set $Y\left(K(h^t), q(h^t)\right)$ of best responses in the modified problem. The difference between the original and modified problem lies in the set of posteriors that screening offers can induce. While in the modified problem the buyer can induce the whole set of posteriors $[q, \hat{q}]$ at any state (K,q), the set of posteriors that he can induce in the original game may be limited. Claim 1 shows that the best response correspondence $Y\left(K,q\right)$ in the modified problem only induces posteriors that are feasible in the original game.

Claim 1. The buyer never chooses a posterior in a flat region of $P(K,\cdot)$. *If $q' \in Y(K,q)$, then $P(K,q'') > P(K,q')$ for every $q'' > q'$.*

See Section T.2 of the Technical Addendum for the proof.

This proves that the strategy of the buyer is optimal.

Part B. Construction of the consistent quadruplet $(\mathcal{V}_L, P, W, y)$

We construct a consistent quadruplet $(\mathcal{V}_L, P, W, y)$ through two processes of induction (and a fixed point argument). In the base step of the first process of induction we construct the quadruplet $(\mathcal{V}_L(1,\cdot), P(1,\cdot), W(1,\cdot), y(1,\cdot))$, which deals with the case when only one unit remains. In the inductive step there are K units left, with $1 < K \leq m$. We assume that the quadruplet $(\mathcal{V}_L(k,\cdot), P(k,\cdot), W(k,\cdot), y(k,\cdot))$ has already been constructed for all $k \in \{1,\ldots,K-1\}$ and construct the quadruplet $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$.

The second process of induction is nested within the first one. We explain this process in detail in steps 1 to 3 below. Let K be the number of remaining units and assume that the quadruplet $(\mathcal{V}_L(k,\cdot), P(k,\cdot), W(k,\cdot), y(k,\cdot))$ has already been constructed for all $k \in$

[35] For this same reason it is optimal for the low-type seller to accept any offer (k,p) with $p > \frac{c}{m}k$.

[36] The buyer could also deviate by making an offer (k,p) with $k \leq K$ and $p \neq P(K,q')$. The equilibrium strategies that we define also guarantee that the low-type seller behaves optimally. We omit the details.

$\{1, \ldots, K-1\}$. In the base step, we construct $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ for $q \in [\overline{q}, \hat{q}]$ for some $\bar{q} < \hat{q}$ (see step 1 below). In the inductive step (indexed by n), we assume that the quadruplet $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ has already been constructed for $q \in [q_n, \hat{q}]$. We extend $(\mathcal{V}_L(K,\cdot), P(K,\cdot), W(K,\cdot), y(K,\cdot))$ to $q \in [q_{n+1}, \hat{q}]$ with $q_{n+1} < q_n$ (we explain this in step 2a below). This inductive step involves a fixed point argument that we describe in detail in step 2b. Finally, we show that in a finite number ($\tilde{n}$) of steps $q_{\tilde{n}} = 0$ (step 3 below).

Step 1. Quadruplet in interval $q \in [\overline{q}, \hat{q}]$. Claim 2 describes the simple form that the quadruplet $(\mathcal{V}_L, P, W, y)$ takes when transformed beliefs are sufficiently close to $\hat{q}$. The intuition behind Claim 2 is simple. If the buyer is sufficiently convinced that the seller is of high type, he is better off trading right away. He offers to pay the high type's cost in exchange for all remaining units. Both types accept and the game ends. This leads directly to the quadruplet's form in Claim 2.

CLAIM 2. *There exists $\overline{q} < \hat{q}$, such that any consistent quadruplet $(\mathcal{V}_L, P, W, y)$ must satisfy*

$$\begin{aligned}
\mathcal{V}_L(K,q) &= \frac{c}{m}K, \\
P(K,q) &= \delta\frac{c}{m}K, \\
W(K,q) &= \sum_{s=1}^{K} \Lambda_s^m \left[(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H\right] - (1-q)\frac{c}{m}K > 0 \quad \textit{and} \\
y(K,q) &= K
\end{aligned}$$

for every $q \in [\overline{q}, \hat{q}]$ and for every $K \in \{1, \ldots, m\}$.

Proof. Assume that there are K remaining units. A buyer who makes a screening offer obtains a (normalized) continuation payoff bounded above by

$$(\hat{q}-q)\sum_{s=1}^{K}\Lambda_s^m v_L + (1-\hat{q})\,\delta\left(\sum_{s=1}^{K}\Lambda_s^m v_H - \frac{c}{m}K\right).$$

Moreover, for a sufficiently high $q < \hat{q}$, the expression above is strictly smaller than

$$\sum_{s=1}^{K}\Lambda_s^m\left[(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H\right] - (1-q)\frac{c}{m}K$$

which represents the continuation payoff for the buyer when he makes a universal offer for all remaining units. This continuation payoff is strictly positive for sufficiently high $q < \hat{q}$. This, in turn, implies that there exists $\bar{q} < \hat{q}$ such that for any $q \in [\bar{q}, \hat{q}]$ and for any $K \in \{1, \ldots, m\}$, screening offers are strictly dominated by a universal offer for all remaining units, and this universal offer leads to strictly positive payoffs. Therefore, the best universal offer is to buy all units immediately, which leads to the expressions for W and y outlined above. These expressions, in turn, imply that $\mathcal{V}_L$ and P are as above. ■

Step 2. Extension of quadruplet from interval $[q_n, \hat{q}]$ **to interval** $q \in [q_{n+1}, \hat{q}]$. The extension of the quadruplet consists of two sub-steps. In the first one (a), we only allow the buyer to make screening offers. We find an interval $[q_{n+1}, q_n]$ where the optimal screening offer induces posterior belief above q_n. If universal offers were not allowed (i.e., if there were only one unit left, as in DL), this would conclude the extension to $[q_{n+1}, q_n]$. In the second sub-step (b), we give the buyer the possibility of making universal offers. This modifies the low-type seller's continuation payoff – and therefore the function $P(K, \cdot)$ – in the interval $[q_{n+1}, q_n]$. We allow the buyer to re-optimize, given the modified function $P(K, \cdot)$, which in turn changes the low-type seller's continuation payoff. We continue this process until we reach a fixed point.

a. Only screening offers. Fix the number of remaining units K. Assume that the quadruplet $(\mathcal{V}_L(k, \cdot), P(k, \cdot), W(k, \cdot), y(k, \cdot))$ is already defined for all $1 \leq k \leq K-1$ and that the quadruplet $(\mathcal{V}_L(K, \cdot), P(K, \cdot), W(K, \cdot), y(K, \cdot))$ is defined for $q \in [q_n, \hat{q}]$.

We define two auxiliary value functions for the buyer that represent continuation payoffs from making screening offers. First, for $q \in [0, q_n]$ we let $W^I(K, q)$ represent the buyer's payoff from making a screening offer that leads to posterior $q' \geq q_n$:

$$W^I(K,q) = \max_{q' \geq q_n} \left(q' - q\right) \left(\sum_{s=1}^{K} \Lambda_s^m v_L - P\left(K, q'\right) \right) + \delta W\left(K, q'\right) \tag{11}$$

Let $X(K,q) \in [q_n, \hat{q}]$ denote the set of solutions to the above maximization problem, and let $\underline{x}(K,q)$ and $\overline{x}(K,q)$ denote respectively the smallest and largest elements of $X(K,q)$.

Second, let $P^I(K,q) = \delta P\left(K, \underline{x}(K,q)\right)$ denote an auxiliary pricing function for $q \in [0, q_n]$. The function $W^{II}(K,q)$ represents the buyer's payoff from making a screening offer $\left(K, P^I(K,q)\right)$ that leads to posterior $q' \in [q, q_n]$ (and to a continuation payoff W^I afterwards):

$$W^{II}\left(K,q\right) = \max_{q' \in [q, q_n]} \left(q' - q\right) \left(\sum_{s=1}^{K} \Lambda_s^m v_L - P^I\left(K, q'\right) \right) + \delta W^I\left(K, q'\right) \quad \text{for } q \in [0, q_n]$$

Let the endpoint q_{n+1} be defined by $q_{n+1} = \max\left\{q \in [0, q_n] : W^I(K,q) \leq W^{II}(K,q)\right\}$ if the set is non-empty and $q_{n+1} = 0$ otherwise.

CLAIM 3. *Endpoints are strictly decreasing: $q_{n+1} < q_n$. Moreover, the continuation payoff $W^I(K,q)$ is continuous and satisfies $W^I(K,q) > 0$ for all $q \in [q_{n+1}, q_n]$.*

Proof. The continuation payoff $W^I(K, q_n)$ is strictly positive because it is bounded below by $\delta W(K, q_n) > 0$. By definition, $W^{II}(K, q_n) = \delta W^I(K, q_n)$, and so $W^{II}(K, q_n) < W^I(K, q_n)$. Finally, the theorem of the maximum guarantees that the functions $W^I(K, \cdot)$ and $W^{II}(K, \cdot)$ are continuous. Therefore, $q_{n+1} < q_n$. Next, by definition, for any $q \in (q_{n+1}, q_n]$, we have $W^I(K,q) > W^{II}(K,q) \geq \delta W^I(K,q)$. Thus, for any $q \in (q_{n+1}, q_n]$, we have $W^I(K,q) > 0$. It only remains to be shown that $W^I(K, q_{n+1}) > 0$, which we do in Section T.3 of the Technical Addendum. ■

b. Fixed Point. We define a sequence of quadruplets

$$\left\{ \left(\mathcal{V}_L^\ell(K, \cdot), P^\ell(K, \cdot), W^\ell(K, \cdot), y^\ell(K, \cdot) \right) \right\}_{\ell=1,2,\ldots} \quad \text{for the interval } [q_{n+1}, \hat{q}].$$

The first element of the sequence is as follows. For $q \in (q_n, \hat{q}]$, we set

$$\left(\mathcal{V}_L^1(K,q), P^1(K,q), W^1(K,q), y^1(K,q)\right) = \left(\mathcal{V}_L(K,q), P(K,q), W(K,q), y(K,q)\right).$$

For $q \in [q_{n+1}, q_n]$ we instead set

$$W^1(K,q) = \max\Bigg\{ W^I(K,q), \max_{0\le k\le K-1} \left\{ \sum_{s=k+1}^{K} \Lambda_s^m \left[(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H\right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \right\}\Bigg\}$$

and we let $y^1(K,q)$ be the solution that gives the lowest continuation payoff to the low-type seller.[37] The screening offer in W^I leads to a state (K,q') with $q' \geq q_n$. The continuation payoff $\mathcal{V}_L(K,q')$ is already defined for this state. Similarly, a universal offer leads to a state (k,q) with $k < K$, for which the continuation payoff $\mathcal{V}_L(k,q)$ is already defined. Thus, we extend $\mathcal{V}_L^1(K,\cdot)$ to the interval $[q_{n+1}, q_n]$ as follows:

$$\mathcal{V}_L^1(K,q) = \begin{cases} \delta\mathcal{V}_L(K,q') & \text{if } y^1(K,q) = q' \\ \frac{c}{m}(K-k) + \delta\mathcal{V}_L(k,q) & \text{if } y^1(K,q) = k \end{cases}$$

Finally, in the interval $[q_{n+1}, q_n]$, we define $P^1(K,\cdot)$ to be the largest weakly increasing function below $\delta\mathcal{V}_L^1(K,\cdot)$.

We define the remaining elements of the sequence of quadruplets recursively. For any $\ell \geq 1$, we define the $\ell+1$'th element of the sequence as follows. First, we set

$$W^{\ell+1}(K,q) = \max\Bigg\{ \max_{q'\in[q,\hat{q}]} (q'-q)\left[\left(\sum_{s=1}^{K}\Lambda_s^m\right) v_L - P^\ell(K,q')\right] + \delta W^\ell(K,q'), \max_{0\le k\le K-1}\left\{\left(\sum_{s=k+1}^{K}\Lambda_s^m\right)\left[(\hat{q}-q)\, v_L + (1-\hat{q})\, v_H\right] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q)\right\}\Bigg\}.$$

Next, we let $y^{\ell+1}(K,q)$ be the solution to the above problem that gives the lowest continuation payoff to the low-type seller. Denote that continuation payoff by $\mathcal{V}_L^{\ell+1}(K,q)$. Finally, let $P^{\ell+1}(k,\cdot)$ be the largest weakly increasing function below $\delta V_L^{\ell+1}(K,\cdot)$.

Claim 4. *There exists ℓ^* such that*

$$\left(\mathcal{V}_L^{\ell^*}(K,\cdot), P^{\ell^*}(K,\cdot), W^{\ell^*}(K,\cdot), y^{\ell^*}(K,\cdot)\right) = \left(\mathcal{V}_L^{\ell^*+1}(K,\cdot), P^{\ell^*+1}(K,\cdot), W^{\ell^*+1}(K,\cdot), y^{\ell^*+1}(K,\cdot)\right).$$

[37] As in Step 3 of Part A, whenever there are many solutions with the same continuation payoff, then there must exist at least one that implies a universal offer $\left(K-k, \frac{c}{m}(K-k)\right)$. Of all such universal offers, we pick the one with the lowest k.

Proof. For every $q \geq q_{n+1}$ and for every $\ell > 1$, $W^{\ell}(k,q) \geq W^{1}(k,q) > 0$. Then, there exists $\eta > 0$ such that for $q \in [q_{n+1}, q_n]$ and for every $\ell > 1$, $W^{\ell}(k,q) > \eta$.

If the claim fails, for any positive integer T there exist $\ell, q \in [q_{n+1}, q_n)$, and a sequence $\{q^{\tau}\}_{\tau=0}^{T}$ with $q^0 = q$, $q^T < q + \frac{1}{T}$ and $y^{\ell}(K, q^{\tau-1}) = q^{\tau}$ for all $\tau \in \{1, \ldots, T\}$. The buyer's continuation payoff $W^{\ell}(K,q)$ is bounded above:

$$W^{\ell}(K,q) < \left(\frac{1}{T} + \delta^{T}\right) \sum_{s=1}^{K} \Lambda_s^m v_H$$

Finally, pick T so that

$$\left(\frac{1}{T} + \delta^{T}\right) \sum_{s=1}^{K} \Lambda_s^m v_H < \eta.$$

But $W^{\ell}(K,q) > \eta$, so we have reached a contradiction. ∎

At the end of the n'th inductive step, the quadruplet is already defined for $q \geq q_n$. We extend the quadruplet to $q \in [q_{n+1}, q_n)$ by setting it equal to the fixed point defined above:

$$(\mathcal{V}_L(K,q), P(K,q), W(K,q), y(K,q)) = \left(\mathcal{V}_L^{\ell^*}(K,q), P^{\ell^*}(K,q), W^{\ell^*}(K,q), y^{\ell^*}(K,q)\right).$$

Step 3. Extension to interval $[0, \hat{q}]$ **takes finitely many steps**. In the last step of the construction, we show that it takes finitely many steps to extend the quadruplet to the whole interval $[0, \hat{q}]$.

CLAIM 5. *There exists $\tilde{n}$ so that $q_{\tilde{n}} = 0$.*

See Section T.4 of the Technical Addendum for the proof.

Finally, note that $W(K,q) > 0$ for every (K,q). Thus it is never optimal for the buyer to make two consecutive universal offers. Formally, if $k \in Y(K,q)$ for some (K,q), then $k' \notin Y(k,q)$. Assume towards a contradiction that $k \in Y(K,q)$ and $k' \in Y(k,q)$. Then,

$$\begin{aligned}
W(K,q) &= \left(\sum_{s=k+1}^{K} \Lambda_s^m\right) [(\hat{q} - q)\, v_L + (1-\hat{q})\, v_H] - (1-q)\frac{c}{m}(K-k) + \delta W(k,q) \\
&< \left(\sum_{s=k+1}^{K} \Lambda_s^m\right) [(\hat{q} - q)\, v_L + (1-\hat{q})\, v_H] - (1-q)\frac{c}{m}(K-k) + W(k,q) \\
&= \left(\sum_{s=k'+1}^{K} \Lambda_s^m\right) [(\hat{q} - q)\, v_L + (1-\hat{q})\, v_H] - (1-q)\frac{c}{m}(K-k') + \delta W(k',q)
\end{aligned}$$

This shows that, at state (K,q), the buyer strictly prefers to make a universal offer for $K - k'$ units, instead of making one for $K - k$ units. Thus, $k \notin Y(K,q)$. ∎

A.4 Convergence as bargaining frictions vanish

LEMMA 3. CONVERGENCE AS BARGAINING FRICTIONS VANISH. *Fix m.*

(a) Consider an arbitrary sequence of vanishing frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$. *The associated sequences* $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *and* $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *have subsequences that converge pointwise.*

(b) There exist functions $K_m(\cdot)$, $q_m(\cdot)$, $\{P_m(K,\cdot)\}_{K=1}^{m}$ *and* $\{W_m(K,\cdot)\}_{K=1}^{m}$ *such that for any sequence of vanishing frictions* $\{\Delta_n\}_{n=1}^{\infty} \to 0$*, the associated sequences* $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *and* $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ *converge pointwise to* $K_m(\cdot)$, $q_m(\cdot)$, $\{P_m(K,\cdot)\}_{K=1}^{m}$ *and* $\{W_m(K,\cdot)\}_{K=1}^{m}$*, respectively, except for finitely many points.*[38]

Proof of part (a). For any $\Delta > 0$, the functions $K_m^{\Delta}(\cdot)$ and $q_m^{\Delta}(\cdot)$ are monotonic in time elapsed τ and the function $P_m^{\Delta}(K,\cdot)$ is monotonic in q for all $K \in \{1,\dots,m\}$. Therefore, they all have bounded variation. Moreover, all these functions are bounded above and below by bounds that do not depend on Δ. By Helly's First Theorem (Theorem 6.1.18 in Kannan and Krueger [1996]), $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$ and $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ all have subsequences that converge pointwise.

Fix $K \in \{1,\dots,m\}$. The functions $\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{n=1}^{\infty}$ are uniformly equicontinuous since they all have the same Lipschitz constant $v_H \sum_{s=1}^{K} \Lambda_s^m$. They are also uniformly bounded. Then, the Arzelà-Ascoli Theorem guarantees that $\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{n=1}^{\infty}$ has a subsequence that converges uniformly. ■

Proof of part (b). In Proposition 2 we show that all convergent sequences $\left\{K_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{q_m^{\Delta_n}(\cdot)\right\}_{n=1}^{\infty}$, $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ have the same limit. ■

A.5 Proof of Proposition 2

In this proof we introduce an algorithm that characterizes the limit equilibrium outcome as bargaining frictions vanish.[39] Proposition 2 follows immediately from this characterization.

We consider a sequence of vanishing bargaining frictions $\{\Delta_n\}_{n=1}^{\infty} \to 0$ with associated sequences $\left\{\left\{P_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$, $\left\{\left\{W_m^{\Delta_n}(K,\cdot)\right\}_{K=1}^{m}\right\}_{n=1}^{\infty}$ and $\left\{\left(K_m^{\Delta_n}(\cdot), q_m^{\Delta_n}(\cdot)\right)\right\}_{n=1}^{\infty}$

[38] The finitely many points where pointwise convergence may not occur correspond to impasses. At any impasse at state (K,q), $P_m^-(K,q)$ and $P_m^+(K,q)$ are well defined. We set $P_m(K,q) = P_m^+(K,q)$. This is without loss of generality, as the limit equilibrium outcome as bargaining frictions vanish does not depend on this choice.

[39] We do this for generic values of the parameters (see Remark 1 for details).

that converge pointwise, by Lemma 3(a). We characterize the limits of these associated sequences, which we denote by $\{P_m(K,\cdot)\}_{K=1}^m$, $\{W_m(K,\cdot)\}_{K=1}^m$ and $(K_m(\cdot), q_m(\cdot))$.

We describe both on-path and off-path behavior: we specify how quantities and beliefs evolve starting from any state (K,q). We let $K_m(\tau;(K,q))$ and $q_m(\tau;(K,q))$ denote respectively the number of remaining units and the belief at time elapsed τ if the starting state at time elapsed zero is (K,q).[40] The on-path limit equilibrium outcome as bargaining frictions vanish $(K_m(\tau), q_m(\tau))$ then corresponds to $(K_m(\tau;(m,0)), q_m(\tau;(m,0)))$.

Our algorithm proceeds by induction. In each step we characterize the limit functions $\{P_m(K,\cdot)\}_{K=1}^m$, $\{W_m(K,\cdot)\}_{K=1}^m$ and $(K_m(\cdot), q_m(\cdot))$ for different subsets of the state space $\{1,\dots,m\} \times [0,\hat{q}]$. In the base step ($j=0$), we identify a candidate impasse $(k_1,q_1) = (1,\bar{q}_m(1))$. We characterize the limit functions for all states $(1,q)$ with $q < q_1$ (Claim 6) and for all states (K,q) with $q \geq q_1$ (Claim 7). At each (non-final) step $j \geq 1$ of the inductive process we identify a candidate impasse (k_{j+1}, q_{j+1}) with $k_{j+1} > k_j$ and $q_{j+1} < q_j$. Claims 8, 9 and 10 characterize the limit functions for all states (K,q) with either 1) $K \in \{k_j+1,\dots,k_{j+1}\}$ and $q \in [0,q_j)$, or 2) $K \in \{k_{j+1}+1,\dots,m\}$ and $q \in [q_{j+1}, q_j)$. In particular, these claims show that the candidate impasse (k_j,q_j) is reached from the candidate impasse (k_{j+1},q_{j+1}).

The algorithm ends after finitely many steps with a characterization of the limit functions for the whole state space $\{1,\dots,m\} \times [0,\hat{q}]$ and with a collection $\left\{(k_j,q_j)\right\}_{j=1}^J$ of J candidate impasses. All candidate impasses are on-path: the limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade and impasses summarized by $\left\{(k_j,q_j)\right\}_{j=1}^J$.

The base step ($j = 0$)

In the base step we obtain the first candidate impasse $(k_1,q_1) = (1,\bar{q}_m(1))$. Claim 6 shows that the candidate impasse $(1,\bar{q}_m(1))$ is reached without delay starting from any state $(1,q)$ with $q < \bar{q}_m(1)$.

CLAIM 6. *For all $q < \bar{q}_m(1)$, we have*

$$P_m(1,q) = \frac{\left(\Lambda_1^m v_L\right)^2}{c/m},$$

$$W_m(1,q) = \left(\bar{q}_m(1) - q\right)\left(\Lambda_1^m v_L\right)\left(1 - \frac{\Lambda_1^m v_L}{c/m}\right) \quad \textit{and}$$

$$\left(K_m(\tau;(1,q)), q_m(\tau;(1,q))\right) = \begin{cases} (1,\bar{q}_m(1)) & \text{if } \tau \leq \tau_1 \\ (0,\hat{q}) & \text{if } \tau > \tau_1 \end{cases} \quad \textit{with } \tau_1 = \frac{2}{r}\ln\left(\frac{c/m}{\Lambda_1^m v_L}\right).$$

The proof of Claim 6 is in DL, so we omit it.

Claim 7 shows that starting at any state (K,q) with $K \in \{1,\dots,m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$, the game ends without delay.

[40] As in the main body of the paper, these functions are left-continuous in τ. These functions are uniquely identified at all states, except at finitely many states, which correspond to (on- and off-path) impasses. For these states, the functions $K_m(\tau;(K,q))$ and $q_m(\tau;(K,q))$ reflect the evolution after the impasse is resolved.

Claim 7. *For all (K,q) with $K \in \{1,\ldots,m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$ we have*

$$P_m(K,q) = K\frac{c}{m},$$
$$W_m(K,q) = (\hat{q}-q)\left(\sum_{s=1}^{K}\Lambda_s^m v_L - K\frac{c}{m}\right) + (1-\hat{q})\left(\sum_{s=1}^{K}\Lambda_s^m v_H - K\frac{c}{m}\right) \quad \textit{and}$$
$$\left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) = (0,\hat{q}) \quad \forall \tau \geq 0.$$

Proof. At all states (K,q) with $K \in \{1,\ldots,m\}$ and $q \in [\bar{q}_m(1), \hat{q}]$, except for $(1, \bar{q}_m(1))$, the buyer can guarantee a strictly positive continuation payoff by making a universal offer for K units. Thus, as described in section 5.1.1, the game ends without delay. The low-type seller can always mimic the high-type seller's behavior. Therefore, as bargaining frictions vanish, the price that the low-type seller is willing to accept for K units must converge to $K\frac{c}{m}$. Then, the function $P_m(1,\cdot)$ is discontinuous at $(1,\bar{q}_m(1))$. We assign $P_m(1,\bar{q}_m(1)) = P_m^+(1,\bar{q}_m(1))$. We do the same with the outcome $\left(K_m\left(\tau;(1,\bar{q}_m(1))\right), q_m\left(\tau;(1,\bar{q}_m(1))\right)\right)$, i.e. we take the limit from the right. In this way, these functions evaluated at $(1,\bar{q}_m(1))$ reflect what happens right after the impasse $(1,\bar{q}_m(1))$ is resolved. We follow this convention also for the next impasses. ■

The algorithm then continues to the first inductive step ($j = 1$).

The inductive step (j $\geq$ 1)

The previous step $j-1$ provides a (candidate) impasse (k_j, q_j) of length τ_j. The impasse (k_j, q_j) satisfies $\bar{q}_m(k_j+1) < q_j$ and $k_j < m$. All previous steps together provide a characterization of the limit functions for all states (K,q) with either $K \leq k_j$, or $q \geq q_j$, or both.

As we do in the main body of the paper, throughout this proof we focus on the "limit game" in the sense that the low-type seller's behavior is summarized by the limit function $P_m(\cdot,\cdot)$. We consider a simple course of action that brings the buyer from any state (K,q) with $K \in \{k_j+1,\ldots,m\}$ and $q \in [0,q_j]$ to the impasse (k_j,q_j). The buyer first makes the universal offer $\left(K-k_j, \frac{c}{m}(K-k_j)\right)$ and then the screening offer $\left(K, P_m^-(k_j,q_j)\right)$. The function $\mathcal{W}(K,q) : \{k_j+1,\ldots,m\} \times [0,q_j] \to \mathbb{R}$, defined in equation (12), denotes the buyer's (normalized) payoff from following this simple course of action.

$$\mathcal{W}(K,q) \equiv (\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right]$$
$$+ (q_j - q)\left[\sum_{s=1}^{k_j}\Lambda_s^m v_L - P_m^-\left(k_j,q_j\right)\right] \tag{12}$$

REMARK 1. *The following two conditions hold for generic values of the parameters:*

$$\mathcal{W}(K, 0) \neq 0 \quad \textit{for all } K \in \{k_j + 1, \ldots, m\} \tag{13a}$$

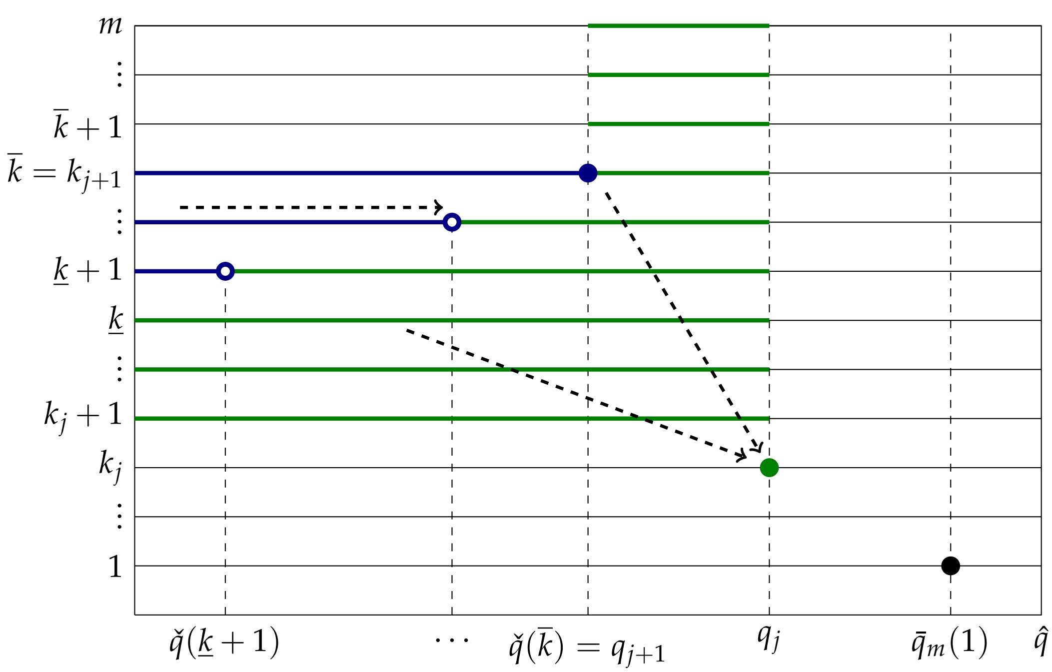
$$\mathcal{W}(K, \bar{q}_m(K)) \neq 0 \quad \textit{for all } K \in \{k_j + 1, \ldots, m\} \tag{13b}$$

Throughout this proof we restrict attention to parameters that satisfy these two conditions.

The function $\mathcal{W}(\cdot, \cdot)$ satisfies $\mathcal{W}(K, q_j) > 0$ because $\bar{q}_m(K) \leq \bar{q}_m(k_j + 1) < q_j$. Moreover, $\mathcal{W}(\cdot, 0)$ is strictly decreasing in K. Given the genericity condition (13a), we next let

$$\underline{k} = \begin{cases} \max\left\{K \in \{k_j + 1, \ldots, m\} : \mathcal{W}(K, 0) > 0\right\} & \text{if } \mathcal{W}(k_j + 1, 0) > 0 \\ k_j & \text{if } \mathcal{W}(k_j + 1, 0) < 0 \end{cases}$$

We split the remainder of the inductive step into two parts, a and b. If $\underline{k} = m$, the algorithm proceeds with part a and then ends. If $k_j < \underline{k} < m$, the algorithm proceeds first with part a and then with part b. If $\underline{k} = k_j$, the algorithm skips part a and moves directly to part b. Throughout the description of these two parts, we refer to Figure 6 to facilitate their exposition.



Notes: The green circle at state (k_j, q_j) denotes the candidate impasse from the previous step $j - 1$. Thick green lines represent states (K, q) with $\mathcal{W}(K, q) > 0$, while thick blue lines represent states (K, q) with $\mathcal{W}(K, q) < 0$. Dashed black arrows illustrate transitions without delay. Filled circles represent on-path impasses, while empty circles represent off-path impasses.

Figure 6: The inductive step ($j \geq 1$) of the algorithm

Part a. In this part we characterize the equilibrium outcome for all states (K, q) with $K \in \{k_j+1, \dots, \underline{k}\}$ and $q \in [0, q_j)$. At any such state, the buyer can guarantee a positive continuation payoff by following the simple course of action described above. We represent this area of the state space with thick green lines in Figure 6. We show in Claim 8 how starting an any state (K, q) with $K \in \{k_j+1, \dots, \underline{k}\}$ and $q \in [0, q_j)$, the state (k_j, q_j) is reached without delay. The state remains there for time elapsed τ_j, i.e. there is an impasse of length τ_j at state (k_j, q_j). After the impasse is resolved, the evolution of the number of remaining units and of beliefs is as specified in the previous step of the induction process.

CLAIM 8. *For all $K \in \{k_j+1, \dots, \underline{k}\}$ and for all $q \in [0, q_j)$ we have:*

$$
\begin{aligned}
P_m(K,q) &= (K-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right) \\
W_m(K,q) &= \mathcal{W}(K,q) \\
\left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) &= \begin{cases} \left(k_j, q_j\right) & \text{if } \tau \le \tau_j \\ \left(K_m\left(\tau-\tau_j;(k_j,q_j)\right), q_m\left(\tau-\tau_j;(k_j,q_j)\right)\right) & \text{if } \tau > \tau_j \end{cases}
\end{aligned}
$$

See Section T.5 of the Technical Addendum for the proof.

If $\underline{k} = m$, then (k_j, q_j) is the first impasse and the algorithm ends. Otherwise, the algorithm proceeds to part *b*.

Part b. We first let[41]

$$
\bar{k} = \max\left\{K \in \{\underline{k}+1, \dots, m\} : \mathcal{W}\left(K, \bar{q}_m(K)\right) > 0\right\}.
$$

Furthermore, for all $K \ge \underline{k}+1$ we let $\check{q}(K) \in (0, q_j)$ be defined by $\mathcal{W}(K, \check{q}(K)) = 0$. In this part we derive the functions of interest for all states (K, q) with either 1) $K \in \{\underline{k}+1, \dots, \bar{k}\}$ and $q < q_j$ or 2) $K > \bar{k}$ and $q \in [\check{q}(\bar{k}), q_j)$. To do so, we first prove the following fact.

FACT 1. *The following inequalities hold:*

$$
\frac{\partial \mathcal{W}(K,q)}{\partial q} = (K-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right) - \sum_{s=1}^{K} \Lambda_s^m v_L > 0 \quad \forall\, K > \underline{k} \tag{14a}
$$

$$
\bar{q}_m(\bar{k}+1) < \check{q}(\bar{k}) < \bar{q}_m(\bar{k}) \tag{14b}
$$

$$
\check{q}(\underline{k}+1) < \check{q}(\underline{k}+2) < \cdots < \check{q}(\bar{k}-1) < \check{q}(\bar{k}) \tag{14c}
$$

where if $\bar{k} = m$, replace (14b) by $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$.

Proof. First, for (14a), note that $\mathcal{W}(K, 0) < 0$ and $\mathcal{W}(K, q_j) > 0$ for all $K > \underline{k}$. Moreover, $\mathcal{W}(K, q)$ is linear in q. Thus, $\mathcal{W}(K, q)$ is strictly increasing in q for all $K > \underline{k}$.[42] Second, for (14b), note that by the definition of $\bar{k}$, $\mathcal{W}\left(\bar{k}, \bar{q}_m(\bar{k})\right) > 0$. Since $\mathcal{W}(K, q)$ is strictly increas-

[41] This definition of $\bar{k}$ is equivalent to the one suggested in Section 5.1.2: $\bar{k}$ is the lowest K such that $\bar{q}_m(K+1) < \check{q}(K) < \bar{q}_m(K)$.

[42] The strict monotonicity of $\mathcal{W}(K, q)$ together with the equality $\mathcal{W}\left(K, \bar{q}_m(K)\right) = \mathcal{W}\left(K-1, \bar{q}_m(K)\right)$ imply that $\mathcal{W}\left(K, \bar{q}_m(K)\right) > 0$ for all $K \in \{\underline{k}+1, \dots, \bar{k}\}$. Furthermore, $\check{q}(K) < \bar{q}_m(K+1)$ for all $K \in \{\underline{k}+1, \dots, \bar{k}-1\}$.

ing, then $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$. If $\bar{k} = m$, this finishes the proof of (14b). Otherwise, note that the definition of $\bar{k}$ (and the genericity condition (13b)) imply that $\mathcal{W}\left(\bar{k}+1, \bar{q}_m(\bar{k}+1)\right) < 0$. Since $\mathcal{W}\left(\bar{k}+1, \bar{q}_m(\bar{k}+1)\right) = \mathcal{W}\left(\bar{k}, \bar{q}_m(\bar{k}+1)\right)$, then $\bar{q}_m(\bar{k}+1) < \check{q}(\bar{k})$. Finally, regarding equation (14c), note that:

$$\mathcal{W}(K,q) = \mathcal{W}(K-1,q) + (\hat{q}-q)\left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1-\hat{q})\left[\Lambda_K^m v_H - \frac{c}{m}\right]$$

Then, $\mathcal{W}(K,q) \geq \mathcal{W}(K-1,q) \Leftrightarrow q \geq \bar{q}_m(K)$. Suppose that $\check{q}(K) < \bar{q}_m(K)$. Then, $0 = \mathcal{W}(K,\check{q}(K)) < \mathcal{W}(K-1,\check{q}(K))$ and so $\check{q}(K-1) < \check{q}(K)$. Since, $\check{q}(\bar{k}) < \bar{q}_m(\bar{k})$, an inductive argument shows equation (14c). ■

The buyer can guarantee a positive continuation payoff at any state (K,q) with $K \in \{\underline{k}+1, \ldots, \bar{k}\}$ and $q \in (\check{q}(K), q_j)$. This follows directly from the definition of $\check{q}(\cdot)$. The buyer can also guarantee a positive continuation payoff at any state (K,q) with $K \in \{\bar{k}+1, \ldots, m\}$ and $q \in [\check{q}(\bar{k}), q_j)$. This follows from the first inequality in equation (14b) and the fact that $\bar{q}_m(\cdot)$ is strictly decreasing in K. We represent these areas of the state space with thick green lines in Figure 6. As in Claim 8, starting from any state (K,q) with $\mathcal{W}(K,q) > 0$, the state (k_j, q_j) is reached without delay and an impasse of length τ_j occurs. Claim 9 summarizes these findings.[43] We omit the proof of Claim 9 since it is analogous to that of Claim 8.

Claim 9. *For all (K,q) with either 1) $K \in \left\{\underline{k}+1, \ldots, \bar{k}\right\}$ and $q \in \left[\check{q}(K), q_j\right)$ or 2) $K \in \left\{\bar{k}+1, \ldots, m\right\}$ and $q \in \left[\check{q}(\bar{k}), q_j\right)$ we have*

$$\begin{aligned} P_m(K,q) &= (K-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right), \\ W_m(K,q) &= \mathcal{W}(K,q) \quad \text{and} \\ \left(K_m\left(\tau;(K,q)\right), q_m\left(\tau;(K,q)\right)\right) &= \begin{cases} \left(k_j, q_j\right) & \text{if } \tau \leq \tau_j \\ \left(K_m\left(\tau-\tau_j;(k_j,q_j)\right), q_m\left(\tau-\tau_j;(k_j,q_j)\right)\right) & \text{if } \tau > \tau_j. \end{cases} \end{aligned}$$

Claim 10 completes the description of the limit functions in the inductive step. States (K,q) with $K \in \{\underline{k}+1, \ldots, \bar{k}\}$ and $q < \check{q}(K)$ have $\mathcal{W}(K,q) < 0$. We represent these states with thick blue lines in Figure 6. Claim 10 shows that starting from any such (K,q), the state shifts without delay to $(K, \check{q}(K))$, where an impasse of length $\rho(K)$ occurs. The reason behind this impasse is that the function $P_m(K,\cdot)$ must be discontinuous at $\check{q}(K)$ for any $K \in \{\underline{k}+1, \ldots, \bar{k}\}$. If it were continuous, the buyer's continuation payoff would be negative at states (K,q) with q close (and to the left) of $\check{q}(K)$. This impasse makes the price $P_m^-\left(K, \check{q}(K)\right)$ low enough so that the buyer finds it optimal to move to state $(K, \check{q}(K))$ without delay.

[43] In Claim 10 we show that there is a (potentially off-path) impasse at every state $(K, \check{q}(K))$ with $K \in \left\{\underline{k}+1, \ldots, \bar{k}\right\}$. Following the convention established in Claim 7, the limit functions evaluated at $(K, \check{q}(K))$ reflect the outcome after the impasse is resolved.

Claim 10. *For all (K, q) with $K \in \{\underline{k}+1, \ldots, \overline{k}\}$ and $q \in [0, \check{q}(K))$ we have:*

$$P_m(K,q) = \frac{\left(\sum_{s=1}^{K} \Lambda_s^m v_L\right)^2}{(K-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right)},$$
$$W_m(K,q) = (\check{q}(K) - q)\left(\sum_{s=1}^{K} \Lambda_s^m v_L\right)\left(1 - \frac{\sum_{s=1}^{K} \Lambda_s^m v_L}{(K-k_j)\frac{c}{m} + P_m^-\left(k_j, q_j\right)}\right)$$

and

$$(K_m(\tau;(K,q)), q_m(\tau;(K,q))) = \begin{cases} (K, \check{q}(K)) & \text{if } \tau \leq \rho(K) \\ \Big(K_m(\tau - \rho(K); (K, \check{q}(K))), & \\ \qquad q_m(\tau - \rho(K); (K, \check{q}(K)))\Big) & \text{if } \tau > \rho(K) \end{cases}$$
$$\text{with } \rho(K) = \frac{2}{r}\log\left(\frac{(K-k_j)\frac{c}{m} + P_m^-(k_j, q_j)}{\left(\sum_{s=1}^{K} \Lambda_s^m\right) v_L}\right).$$

We finally describe how the inductive step concludes. We let $(k_{j+1}, q_{j+1}) = \left(\overline{k}, \check{q}\left(\overline{k}\right)\right)$ and $\tau_{j+1} = \rho(\overline{k})$. If $\overline{k} < m$, then the algorithm proceeds to the next inductive step. If $\overline{k} = m$, then the algorithm ends. Since m is finite, the algorithm ends in finitely many steps.

When the algorithm ends, it provides a collection $\left\{(k_j, q_j)\right\}_{j=1}^{J}$ of candidate impasses and a complete characterization of the limit functions. The last inductive step shows that starting at the initial state $(m, 0)$, the state (k_J, q_J) is reached without delay and an impasse of length τ_J ensues. Each inductive step shows how after the impasse in state (k_j, q_j) is resolved, the state shifts without delay to (k_{j-1}, q_{j-1}), where an additional impasse of length τ_{j-1} occurs. The base step shows that the game ends after the last impasse $(1, \bar{q}_m(1))$ is reached.

To sum up, all impasses in $\left\{(k_j, q_j)\right\}_{j=1}^{J}$ occur on-path.[44] Thus, the limit equilibrium outcome as bargaining frictions vanish consists of a sequence of phases of fast trade an impasses characterized by $\left\{(k_j, q_j)\right\}_{j=1}^{J}$. ■

A.6 Proof of Proposition 3

We first show equation (6d). We then proceed with the proof of equation (6b), which is the most involved part of the proof of Proposition 3 and includes several steps. We finally show how the remaining equations in Proposition 3 follow from equations (6b) and (6d).

Proof of equation (6d). Any impasse (k_j^m, q_j^m) must satisfy $\bar{q}_m(k_j^m + 1) < q_j^m < \bar{q}_m(k_j^m)$ (see Proposition 2). Together with the definitions of $\bar{q}(\cdot)$ and $\bar{q}_m(\cdot)$, and replacing $z_j^m =$

[44] All other impasses identified in Claim 10 in each inductive step are off-path.

k_j^m/m when needed, this implies

$$\bar{q}\left(z_j^m+\frac{1}{m}\right)=\bar{q}\left(\frac{k_j^m+1}{m}\right)<\bar{q}_m(k_j^m+1)<q_j^m<\bar{q}_m(k_j^m)<\bar{q}\left(\frac{k_j^m-1}{m}\right)=\bar{q}\left(z_j^m-\frac{1}{m}\right)$$

Notice that $\left|\frac{d\bar{q}(z)}{dz}\right|$ is bounded by some constant $\check{\rho}<\infty$ (because $\frac{d\lambda(\cdot)}{dz}$ is continuous). Thus,

$$\left|q_j^m-\bar{q}(z_j^m)\right|<\max\left\{\left|\bar{q}\left(z_j^m-1/m\right)-\bar{q}(z_j^m)\right|;\left|\bar{q}\left(z_j^m+1/m\right)-\bar{q}(z_j^m)\right|\right\}<\check{\rho}/m.$$

The bound $\check{\rho}$ is independent of j, so $\max\left\{\left|q_j^m-\bar{q}(z_j^m)\right|\right\}_{j=1}^{J_m}<\check{\rho}/m$, which leads to equation (6d):

$$\lim_{m\to\infty}\max\left\{\left|q_j^m-\bar{q}(z_j^m)\right|\right\}_{j=1}^{J_m}=0$$

Proof of equation (6b). We split this proof in two parts. In the first one we construct a sequence of limits of consecutive impasses and show how to link these limits. In the second one we use this construction to show that limits of consecutive impasses must be arbitrarily close.

Construction of the sequence of limits of consecutive impasses. Assume towards a contradiction that

$$\limsup_{m\to\infty}\left(\max\left\{q_{j-1}^m-q_j^m\right\}_{j=2}^{J_m}\right)>0.$$

Then, by taking a subsequence if necessary, we may assume that a sequence of consecutive impasses $\left\{\left(z_{j_m}^m,q_{j_m}^m\right),\left(z_{j_m-1}^m,q_{j_m-1}^m\right)\right\}_{m=1}^{\infty}$ that converges to $((z_0,q_0),(z_{-1},q_{-1}))$ with $q_0>q_{-1}$ exists. Equation (6d) guarantees that $q_0=\bar{q}(z_0)$ and $q_{-1}=\bar{q}(z_{-1})$

The buyer obtains a zero continuation payoff at every impasse. Thus, the difference $W_m(mz_{j_m}^m,q_{j_m}^m)-W_m(mz_{j_m-1}^m,q_{j_m-1}^m)$, which we express in equation (15), is also zero:[45]

$$\begin{aligned}&\left(q_{j_m-1}^m-q_{j_m}^m\right)\left[\int_0^{z_{j_m}^m}\lambda(z)v_L dz-P_m^+\left(mz_{j_m}^m,q_{j_m}^m\right)\right]\\&+\left(\hat{q}-q_{j_m-1}^m\right)\int_{z_{j_m-1}^m}^{z_{j_m}^m}\left[\lambda(z)v_L-c\right]dz+(1-\hat{q})\int_{z_{j_m-1}^m}^{z_{j_m}^m}\left[\lambda(z)v_H-c\right]dz=0\end{aligned}\tag{15}$$

The left hand side of equation (15) is continuous in $\left(z_{j_m}^m,q_{j_m}^m\right)$, $\left(z_{j_m-1}^m,q_{j_m-1}^m\right)$ and $P_m^+\left(mz_{j_m}^m,q_{j_m}^m\right)$. Moreover it strictly decreases in $P_m^+\left(mz_{j_m}^m,q_{j_m}^m\right)$, with derivative bounded away from zero. Hence, since $\left\{\left(z_{j_m}^m,q_{j_m}^m\right)\right\}_{m=1}^{\infty}$ and $\left\{\left(z_{j_m-1}^m,q_{j_m-1}^m\right)\right\}_{m=1}^{\infty}$ converge, then $\left\{P_m^+\left(mz_{j_m}^m,q_{j_m}^m\right)\right\}_{m=1}^{\infty}$ must also converge. We let P_0^+ denote its limit. Equation (16) ex-

[45] We use equation (4c) to obtain equation (15).

presses equation (15) in the limit:

$$(q_{-1} - q_0)\left[\int_0^{\psi(q_0)} \lambda(z) v_L dz - P_0^+\right] \tag{16}$$
$$+ (\hat{q} - q_{-1}) \int_{\psi(q_{-1})}^{\psi(q_0)} [\lambda(z)v_L - c]\, dz + (1-\hat{q}) \int_{\psi(q_{-1})}^{\psi(q_0)} [\lambda(z)v_H - c]\, dz = 0$$

with a change of variables taking advantage of $z_\ell = \psi(q_\ell)$ for $\ell \in \{0, -1\}$, where $\psi(\cdot)$ is the inverse of $\bar{q}(\cdot)$. Equation (16) links the limits (z_0, q_0) and (z_{-1}, q_{-1}).

We show next that $q_{-1} < \bar{q}(0)$ (and so $z_{-1} > 0$). Assume towards a contradiction that $q_{-1} = \bar{q}(0)$ and $z_{-1} = 0$. This implies that $P_0^+ = z_0 c$.[46] Using this, we rewrite the left hand side of equation (16) as

$$(\hat{q} - q_0)\left[\int_0^{\psi(q_0)} [\lambda(z)v_L - c]\, dz\right] + (1-\hat{q}) \int_0^{\psi(q_0)} [\lambda(z)v_H - c]\, dz < 0$$

where the inequality follows from the definition of $\psi(\cdot)$. This leads to a contradiction.

For every (large enough) m there exists an impasse $\left(z^m_{j_m-2}, q^m_{j_m-2}\right)$ that occurs after $\left(z^m_{j_m-1}, q^m_{j_m-1}\right)$ is resolved. This is because the last impasse occurs at $z = \frac{1}{m}$ and $z_{-1} > 0$. Assume, by taking a subsequence if necessary, that the sequence $\left\{\left(z^m_{j_m-2}, q^m_{j_m-2}\right)\right\}_{m=1}^{\infty}$ converges to (z_{-2}, q_{-2}). By an argument like the one for q_{-1}, then also $q_{-2} < \bar{q}(0)$.

We show next that $q_{-1} < q_{-2}$. Assume instead that $q_{-1} = q_{-2}$ (so $z_{-1} = z_{-2}$). Equation (4c) then implies $\lim_{m\to\infty} P_m^+\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) - P_m^-\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) = 0$. Proposition 2 guarantees that in general

$$P_m^-\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) < v_L \int_0^{z^m_{j_m-2}} \lambda(z) dz < v_L \int_0^{z^m_{j_m-1}} \lambda(z) dz < P_m^+\left(mz^m_{j_m-1}, q^m_{j_m-1}\right).$$

Thus, $q_{-1} = q_{-2}$ implies $\lim_{m\to\infty} P_m^+\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) = \lim_{m\to\infty} P_m^-\left(mz^m_{j_m-2}, q^m_{j_m-2}\right) = v_L \int_0^{z_{-1}} \lambda(z)dz$. Finally, we link $P_m^+\left(mz^m_{j_m}, q^m_{j_m}\right)$ and $P_m^+\left(mz^m_{j_m-1}, q^m_{j_m-1}\right)$ using equations (4b) and (4c) and take limits to obtain

$$P_0^+ = (z_0 - z_{-1})\, c + v_L \int_0^{z_{-1}} \lambda(z) dz.$$

We plug this expression for P_0^+ in the left hand side of equation (16) and obtain the fol-

[46] Equation (4b) implies that $P_m^-\left(mz^m_{j_m-1}, q^m_{j_m-1}\right) < v_L \int_0^{z^m_{j_m-1}} \lambda(z)dz$, which converges to zero as $m \to \infty$. This and equation (4c) imply that $P_m^+\left(mz^m_{j_m}, q^m_{j_m}\right)$ becomes arbitrarily close to $(z^m_{j_m} - z^m_{j_m-1})c$ as $m \to \infty$.

lowing contradiction:

$$(\hat{q}-q_0)\int_{\psi(q_{-1})}^{\psi(q_0)}[\lambda(z)v_L-c]\,dz+(1-\hat{q})\int_{\psi(q_{-1})}^{\psi(q_0)}[\lambda(z)v_H-c]\,dz<0$$

The same argument that shows that the sequence $\left\{P_m^+\left(mz_{j_m}^m,q_{j_m}^m\right)\right\}_{m=1}^{\infty}$ must converge to P_0^+ also guarantees that the sequence $\left\{P_m^+\left(mz_{j_m-1}^m,q_{j_m-1}^m\right)\right\}_{m=1}^{\infty}$ must converge, and its limit, which we denote by P_{-1}^+ must satisfy an equation like (16):

$$\begin{aligned}(q_{-2}-q_{-1})\left[\int_0^{\psi(q_{-1})}\lambda(z)v_L dz-P_{-1}^+\right]\\+(\hat{q}-q_{-2})\int_{\psi(q_{-2})}^{\psi(q_{-1})}[\lambda(z)v_L-c]\,dz+(1-\hat{q})\int_{\psi(q_{-2})}^{\psi(q_{-1})}[\lambda(z)v_H-c]\,dz=0\end{aligned}$$

The previous equation links the limits (z_{-1},q_{-1}) and (z_{-2},q_{-2}) of the sequences of consecutive impasses $\left\{\left(z_{j_m-1}^m,q_{j_m-1}^m\right)\right\}_{m=1}^{\infty}$ and $\left\{\left(z_{j_m-2}^m,q_{j_m-2}^m\right)\right\}_{m=1}^{\infty}$.

We next link the limit prices P_0^+ and P_{-1}^+ using equations (4b) and (4c). Equation (4c) links $P_m^+\left(mz_{j_m}^m,q_{j_m}^m\right)$ and $P_m^-\left(mz_{j_m-1}^m,q_{j_m-1}^m\right)$. Equation (4b) links $P_m^-\left(mz_{j_m-1}^m,q_{j_m-1}^m\right)$ and $P_m^+\left(mz_{j_m-1}^m,q_{j_m-1}^m\right)$. Using these equations together, and taking limits, we obtain

$$P_0^+=[\psi(q_0)-\psi(q_{-1})]\,c+\frac{\left(v_L\int_0^{\psi(q_{-1})}\lambda z dz\right)^2}{P_{-1}^+}. \tag{17}$$

We proceed recursively and construct, taking subsequences if necessary, a collection of sequences of impasses $\left\{\left\{\left(z_{j_m-\ell}^m,q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}\right\}_{\ell=0}^{\infty}$, where, for every ℓ, the sequence $\left\{\left(z_{j_m-\ell}^m,q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}$ converges to $(z_{-\ell},q_{-\ell})$ as m grows to infinity. Furthermore, for every ℓ, the sequence $\left\{P_m^+\left(mz_{j_m-\ell}^m,q_{j_m-\ell}^m\right)\right\}_{m=1}^{\infty}$ converges to $P_{-\ell}^+$.

For every $\ell=0,1,\ldots$ the limits of consecutive impasses must satisfy equations (18) and (19).

$$\begin{aligned}&\left(q_{-(\ell+1)}-q_{-\ell}\right)\left[\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz-P_{-\ell}^+\right] && (18)\\&\quad+\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]\,dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]\,dz=0\end{aligned}$$

$$P_{-\ell}^+=\left[\psi(q_{-\ell})-\psi(q_{-(\ell+1)})\right]c+\frac{\left(v_L\int_0^{\psi(q_{-(\ell+1)})}\lambda(z)dz\right)^2}{P_{-(\ell+1)}^+} \tag{19}$$

These conditions mirror equations (16) and (17). Finally, limit beliefs satisfy

$$q_0 < q_{-1} < \ldots < q_{-\ell} < \ldots < \bar{q}(0). \tag{20}$$

Bounding the distance between limits of consecutive impasses. In the remainder of the proof we focus on the collection $\left\{\left(q_{-\ell}, P^+_{-\ell}\right)\right\}_{\ell=0}^{\infty}$ which satisfies equations (18), (19), and (20). We show that the limit beliefs $\{q_{-\ell}\}_{\ell=0}^{\infty}$ are arbitrarily close to each other. To do this, we obtain explicit bounds that link successive limit impasses by using equations (18) and (19). These bounds link differences between consecutive beliefs and also differences between prices and valuations. Facts 2 and 3 state the first bounds (see Section T.6 of the Technical Addendum for their proof).

FACT 2. *There exists $\eta^* > 0$ such that for every $\ell \geq 1$, if $q_{-(\ell+1)} - q_{-\ell} < \eta^*$, then $q_{-\ell} - q_{-(\ell-1)} < \frac{4}{3}\left(q_{-(\ell+1)} - q_{-\ell}\right)$.*

FACT 3. *There exists constants $b_1 > 0$ and $b_2 > 0$ such that for every $\ell = 0, 1, \ldots,$ we have:*

$$\frac{\left[P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] - \left[P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right]}{q_{-(\ell+1)} - q_{-\ell}} \leq b_1 \left(q_{-(\ell+1)} - q_{-\ell}\right) \tag{21}$$

$$q_{-(\ell+1)} - q_{-\ell} \leq b_2 \left[P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] \tag{22}$$

Using Facts 2 and 3 we prove Claims 11 and 12, which provide further bounds. Claim 11 links successive differences between prices and valuations and Claim 12 links differences between successive beliefs.

CLAIM 11. *Consider ℓ' and ℓ'' with $0 \leq \ell' < \ell''$. Let $\varepsilon > 0$ and $\eta > 0$ be such that $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ for all $\ell \in \{\ell', \ldots, \ell'' - 1\}$ and $q_{-\ell''} - q_{-\ell'} < \eta$. Then, for every $\ell \in \{\ell', \ldots, \ell'' - 1\}$, we have:*

$$P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz < P^+_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon \eta b_1$$

Proof. For every $\ell \in \{\ell', \ldots, \ell'' - 1\}$ we have

$$\begin{aligned}
P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz &= P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz \\
&\quad + \left(q_{-(\ell+1)} - q_{-\ell}\right) \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz - \left(P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right)}{q_{-(\ell+1)} - q_{-\ell}} \\
&< P^+_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz + \varepsilon b_1 \left(q_{-(\ell+1)} - q_{-\ell}\right)
\end{aligned}$$

where the inequality follows from $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ and equation (21) in Fact 3. Applying the same argument recursively leads to

$$P^{+}_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz < P^{+}_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon b_1 \sum_{\tilde{\ell}=\ell}^{\ell''-1} \left(q_{-(\tilde{\ell}+1)} - q_{-\tilde{\ell}} \right)$$
$$< P^{+}_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon \eta b_1 \qquad \blacksquare$$

Claim 12. *Consider ℓ' and ℓ'' with $1 \leq \ell' < \ell''$. Let $0 < \varepsilon < \eta^*$ and $0 < \eta < (3b_1 b_2)^{-1}$ be such that $q_{-(\ell+1)} - q_{-\ell} < \varepsilon$ for all $\ell \in \{\ell', \ldots, \ell''-1\}$, $q_{-\ell''} - q_{-\ell'} < \eta$ and $P^{+}_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz < (3b_2)^{-1}\varepsilon$. Then, $q_{-\ell'} - q_{-(\ell'-1)} < \varepsilon$.*

Proof. We have

$$q_{-(\ell'+1)} - q_{-\ell'} \leq b_2 \left[P^{+}_{-\ell'} - \int_0^{\psi(q_{-\ell'})} \lambda(z) v_L dz \right] < b_2 \left(P^{+}_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz + \varepsilon \eta b_1 \right)$$
$$< b_2 \left((3b_2)^{-1}\varepsilon + \varepsilon (3b_1 b_2)^{-1} b_1 \right) < \frac{2}{3}\varepsilon$$

where the first inequality follows from equation (22) in Fact 3 and the second one from Claim 11. This, together with Fact 2, implies that

$$q_{-\ell'} - q_{-(\ell'-1)} < \frac{4}{3} \left(q_{-(\ell'+1)} - q_{-\ell'} \right) < \left(\frac{4}{3} \right) \left(\frac{2}{3} \right) \varepsilon < \varepsilon \qquad \blacksquare$$

Claim 12 provides the last intermediate result to complete the proof of equation (6b). The sequence $\{q_{-\ell}\}_{\ell=0}^{\infty}$ is strictly increasing and bounded above by $\bar{q}(0)$. Then, it has a limit, which we denote by $q_{-\infty}$. With this, applying L'Hôpital's rule to equation (18) we obtain

$$\lim_{\ell \to \infty} P^{+}_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz = 0.$$

We focus on elements of the sequence $\{q_{-\ell}\}_{\ell=0}^{\infty}$ which are sufficiently close to $q_{-\infty}$. Let $\ell' = \min\left\{ \ell : q_{-\ell} \geq q_{-\infty} - (6b_1 b_2)^{-1} \right\}$. Fix $\varepsilon = \frac{1}{2} \min\{q_{-\ell'} - q_{-\ell'+1}; \eta^*\} > 0$ and pick ℓ'' such that:

$$\max \left\{ q_{-(\ell''+1)} - q_{-\ell''}; P^{+}_{-\ell''} - \int_0^{\psi(q_{-\ell''})} \lambda(z) v_L dz \right\} < \min \left\{ \varepsilon, (3b_2)^{-1}\varepsilon \right\}$$

Then, applying Claim 12 recursively, we obtain $q_{-\ell'} - q_{-\ell'+1} < \varepsilon$, which is a contradiction and completes the proof of equation (6b).

Proof of equations (6a)**,** (6c)**,** (6e) **and** (6f). Equations (6b) and (6d) together imply equation (6a). Equation (15) links *any* sequence of consecutive impasses. We take the limit of equation (15) as m grows large, use equations (6b) and (6d) and apply L'Hôpital's rule to obtain equation (6f). Equation (6e) follows from equation (6f) and equation (4b) in Proposition 2. Finally, we show equation (6c) by contradiction. Assume instead that, taking subsequences if necessary, $\lim_{m \to \infty} z^m_{J_m} = \bar{z} < 1$. This, together with equation (6e), implies that, in the limit, the buyer's continuation payoff at the beginning of the game is negative:

$$\lim_{m\to\infty} W_m(m,0) = \hat{q}\left[\int_{\bar{z}}^{1} [\lambda(z)v_L - c]\,dz\right] + (1-\hat{q})\left[\int_{\bar{z}}^{1} [\lambda(z)v_H - c]\,dz\right] < 0$$

This can never happen, so we have reached a contradiction. ■

Technical Addendum to "Bargaining over a Divisible Good in the Market for Lemons"

T.1 Part C of the proof of Proposition 1. Generic uniqueness

Throughout this proof, we fix arbitrary values for the parameters $\{\Lambda_s^m\}_{s=1}^m$ and v_H and show that for generic values of the parameters $(\delta, v_L, c, \hat{\beta})$ the equilibrium outcome is unique. We present this proof through two claims. In Claim T1 we show that for a generic set $\mathfrak{G}$ of parameters, our consistent quadruplet satisfies some uniqueness properties. In Claim T2 we use these properties to show that the equilibrium outcome is unique.

Claim T1. *There exists a generic set $\mathfrak{G}$ of the parameters $(\delta, v_L, c, \hat{\beta})$ such that the correspondence $Y(\cdot,\cdot)$ associated to the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ constructed in Part B (of the proof of Proposition 1) is a singleton everywhere except for finitely many (K, q). Furthermore, $Y(K, 0)$ is a singleton for every $K \in \{1, \ldots, m\}$. Finally, even at states (K, q) where $Y(K, q)$ is not a singleton, $y(K, q)$ is the unique element of $Y(K, q)$ that minimizes the low-type seller's payoff.*

Proof. Consider the consistent quadruplet $(\mathcal{V}_L, P, W, y)$ constructed in Part B and fix the number of remaining units $K \in \{1, \ldots, m\}$. There are finitely many possible universal offers. Also, Claim 1 guarantees that the buyer never chooses in a flat region of $P(K, \cdot)$. Then, the buyer needs to compare only finitely many screening offers. Thus, the set $\cup_{q \in [0,\hat{q}]} Y(K, q)$ is finite.

We show next that for generic values of the parameters (δ, v_L, c), two arbitrary elements of $\cup_{q \in [0,\hat{q}]} Y(K, q)$ can both be optimal at most at one state (K, q). In particular, assume that $q'' \in Y(K, q')$ and $k \in Y(K, q')$ for some (K, q'). If the buyer makes the screening offer $(K, P(K, q''))$ when the state is (K, q) with $q < q''$, he obtains the continuation payoff

$$(q'' - q)\left[\left(\sum_{s=1}^{K} \Lambda_s^m\right) v_L - P(K, q'')\right] + \delta W(K, q''). \tag{T1}$$

This continuation payoff is linear in the starting belief q, with derivative

$$P(K, q'') - \left(\sum_{s=1}^{K} \Lambda_s^m\right) v_L = \sum_{s=1}^{K} \delta^{T^1(s)} \frac{c}{m} - \left(\sum_{s=1}^{K} \Lambda_s^m\right) v_L, \tag{T2}$$

where $(T^1(1), \ldots, T^1(K))$ are K integers that satisfy $T^1(1) \geq \ldots \geq T^1(K) \geq 1$. We use two facts to establish equation (T2). First, $P(K, q'') = \delta V_L(K, q'')$. Second, since the low-type seller is always indifferent between accepting or rejecting any screening offer, his continuation payoff can be computed by assuming that he rejects all screening offers (and accepts all universal offers). We let $T^1(s)$ represent the time it takes the buyer to make a universal offer for unit s.

Consider next the universal offer $\left(K - k, \frac{c}{m}(K - k)\right)$. It is never optimal for the buyer to make two consecutive universal offers, so this offer is followed by a screening offer $(k, P(k, q'''))$ for some $q''' > q'$. Thus, if the buyer makes the offer $\left(K - k, \frac{c}{m}(K - k)\right)$

when the state is (K,q) with $q<q'''$, he obtains the continuation payoff

$$\left(\sum_{s=k+1}^{K}\Lambda_s^m\right)\left[\left(\hat{q}-q\right)v_L+\left(1-\hat{q}\right)v_H\right]-\left(1-q\right)\frac{c}{m}(K-k)$$
$$+\delta\left[\left(q'''-q\right)\left[\left(\sum_{s=1}^{k}\Lambda_s^m\right)v_L-P\left(k,q'''\right)\right]+\delta W\left(k,q'''\right)\right]. \tag{T3}$$

This continuation payoff is linear in the starting belief q, with derivative

$$\begin{aligned}&\frac{c}{m}(K-k)+\delta P\left(k,q'''\right)-\left[\delta\sum_{s=1}^{k}\Lambda_s^m+\sum_{s=k+1}^{K}\Lambda_s^m\right]v_L\\&=\frac{c}{m}(K-k)+\sum_{s=1}^{k}\delta^{T^2(s)+1}\frac{c}{m}-\left[\delta\sum_{s=1}^{k}\Lambda_s^m+\sum_{s=k+1}^{K}\Lambda_s^m\right]v_L,\end{aligned} \tag{T4}$$

where $\left(T^2(1),\ldots,T^2(k)\right)$ are k integers that satisfy $T^2(1)\geq\ldots\geq T^2(k)\geq 1$. As before, $T^2(s)$ represents the time it takes the buyer to make a universal offer for unit s.

For generic values of the parameters (δ,v_L,c), the right hand side of equation (T2) is different from the right hand side of equation (T4). Thus, the continuation payoff in equation (T1) is equal to the continuation payoff in equation (T3) if and only if the starting belief is q'. The cases where either two universal offers or two screening offers are optimal for some (K,q') are analogous. Therefore, since $\cup_{q\in[0,\hat{q}]}Y(K,q)$ is a finite set, for a generic set of the parameters (δ,v_L,c) there are finitely many states (K,q) where $Y(K,q)$ is not a singleton.

Next, fix a triple (δ,v_L,c) from the generic set defined in the previous paragraph. Consider a prior $\breve{\beta}$ and compute the quadruplet $(\mathcal{V}_L,P,W,y)$ with associated $Y(K,q)$. The set $O_K=\{(K,q):Y(K,q)\text{ is not a singleton}\}$ is finite. Fix a small $\varepsilon>0$. For any prior $\hat{\beta}\in\left[\breve{\beta},\breve{\beta}+\varepsilon\right]\setminus\left(\cup_{(K,q)\in O_K}\left\{\frac{\breve{\beta}}{1-q}\right\}\right)$, $Y(K,0)$ is a singleton. Thus, for generic values of the prior $\hat{\beta}$, $Y(K,0)$ is a singleton.

Finally, consider states (K,q') such that $Y\left(K,q'\right)$ is not a singleton. First, if $Y\left(K,q'\right)$ contains multiple screening offers, then their prices must be different. So at most one screening offer can minimize the low-type seller's continuation payoff. Second, assume that $q''\in Y\left(K,q'\right)$ and $k\in Y\left(K,q'\right)$. If the buyer makes the screening offer $(K,P(K,q''))$, then the low-type seller obtains the continuation payoff $P\left(K,q''\right)=\sum_{s=1}^{K}\delta^{T^3(s)}\frac{c}{m}$.[47] If instead the buyer makes the universal offer $\left(K-k,\frac{c}{m}(K-k)\right)$, then the low-type seller obtains a continuation payoff $\frac{c}{m}(K-k)+\sum_{s=1}^{k}\delta^{T^4(s)+1}\frac{c}{m}$. Both offers yield the same continuation payoff to the low-type seller only if

$$\sum_{s=1}^{K}\delta^{T^3(s)}=(K-k)+\sum_{s=1}^{k}\delta^{T^4(s)+1},$$

[47] As before, $\left(T^3(1),\ldots,T^3(k)\right)$ are k integers that satisfy $T^3(1)\geq\ldots\geq T^3(k)\geq 1$ and represent the time it takes the buyer to make a universal offer for unit s. The mapping T^4 below plays an analogous role.

which holds only for finitely many values of δ, by the fundamental theorem of algebra. In a similar way, if $Y(K, q')$ contains multiple universal offers, they yield the same continuation payoff to the low-type seller only for finitely many values of δ.

To sum up, for a generic set $\mathfrak{G}_K$ of the parameters $(\delta, v_L, c, \hat{\beta})$ there are finitely many states (K, q) such that $Y(K, q)$ is not a singleton. Next, $Y(K, 0)$ is not a singleton. Moreover, even in states in which $Y(K, q)$ is not a singleton, only one element of $Y(K, q)$ minimizes the continuation payoff of the low-type seller. Finally by letting $\mathfrak{G} = \cap_{K=1}^{m} \mathfrak{G}_K$ we complete the proof of Claim T1. ■

CLAIM T2. *For every element of $\mathfrak{G}$ the equilibrium outcome is unique.*

Proof. Fix an element of $\mathfrak{G}$ and compute the quadruplet $(\mathcal{V}_L, P, W, y)$ with associated $Y(K, q)$ from Part B. Consider next an arbitrary (stationary Perfect Bayesian) equilibrium $\left(\widetilde{\sigma}_B, (\widetilde{\sigma}_L, \widetilde{\sigma}_H), \widetilde{\beta}\right)$ and let $\widetilde{\mathcal{V}}_L$ be the (left-continuous) function that governs the acceptance decision of the low-type seller in that equilibrium. Following step 1 from Part A (of the proof of Proposition 1), we obtain $\widetilde{P}$ from $\widetilde{\mathcal{V}}_L$ and following step 2 from Part A, we compute $\widetilde{W}$ and $\widetilde{Y}$ from $\widetilde{P}$.

The remainder of the proof is divided into two parts. In the first one, we show that $\left(\widetilde{\mathcal{V}}_L, \widetilde{P}, \widetilde{W}, \widetilde{Y}\right) = (\mathcal{V}_L, P, W, Y)$. In the second one, we show that the on-path equilibrium behavior under $\left(\widetilde{\sigma}_B, (\widetilde{\sigma}_L, \widetilde{\sigma}_H), \widetilde{\beta}\right)$ coincides with the one from the consistent quadruplet.

In any equilibrium, when the belief is sufficiently high, the buyer makes a universal offer for all remaining units. Formally, it follows from Claim 2 that there exists $\overline{q} < \hat{q}$, such that for every $q \in [\overline{q}, \hat{q}]$ and for every $K \in \{1, \ldots, m\}$, $\widetilde{\mathcal{V}}_L(K, q) = \mathcal{V}_L(K, q)$, $\widetilde{P}(K, q) = P(K, q)$, $\widetilde{W}(K, q) = W(K, q)$, and $\widetilde{Y}(K, q) = Y(K, q) = \{K\}$.

Assume towards a contradiction that $\left(\widetilde{\mathcal{V}}_L, \widetilde{P}, \widetilde{W}, \widetilde{Y}\right) \neq (\mathcal{V}_L, P, W, Y)$ and let

$$\begin{aligned}
\underline{K} = \max\Big\{ & K \in \{1, \ldots, m\} : \left(\widetilde{\mathcal{V}}_L(k, \cdot), \widetilde{P}(k, \cdot), \widetilde{W}(k, \cdot), \widetilde{Y}(k, \cdot)\right) = \\
& \left(\mathcal{V}_L(k, \cdot), P(k, \cdot), W(k, \cdot), Y(k, \cdot)\right) \text{ for all } k < K \Big\} \quad \text{and} \\
\underline{q} = \inf\Big\{ & q' \in [0, \hat{q}] : \left(\widetilde{\mathcal{V}}_L(\underline{K}, q), \widetilde{P}(\underline{K}, q), \widetilde{W}(\underline{K}, q), \widetilde{Y}(\underline{K}, q)\right) \\
& = \left(\mathcal{V}_L(\underline{K}, q), P(\underline{K}, q), W(\underline{K}, q), Y(\underline{K}, q)\right) \text{ for all } q > q' \Big\}.
\end{aligned}$$

Note that by definition, $\underline{q} \in (0, \bar{q}]$.

The function $\widetilde{W}(\underline{K}, \cdot)$ is Lipschitz with coefficient less than $\mathfrak{v} \equiv \sum_{s=1}^{\underline{K}} \Lambda_s^m v_H + \underline{K}\frac{c}{m}$. Moreover, $\widetilde{W}\left(\underline{K}, \underline{q}\right) = W\left(\underline{K}, \underline{q}\right) = \xi$ for some $\xi > 0$ (since $W(\underline{K}, \cdot)$ is bounded away from zero). Then, there exists $\varepsilon_1 > 0$ such that $\widetilde{W}(\underline{K}, q) > \xi/2$ for all $q \in \left[\underline{q} - \varepsilon_1, \underline{q}\right]$. Fix $0 < \varepsilon_2 \leq \min\{\varepsilon_1, \xi(1-\delta)/8\mathfrak{v}\}$ such that for any $q \in \left(\underline{q} - \varepsilon_2, \underline{q}\right)$, $Y(\underline{K}, q)$ is a singleton.

For any $q \in \left(\underline{q} - \varepsilon_2, \underline{q}\right]$ with $q' \in \widetilde{Y}\left(\underline{K}, q\right)$ we have

$$\frac{\xi}{2} < \widetilde{W}\left(\underline{K}, q\right) \leq \left(q' - q\right)\mathfrak{v} + \delta \widetilde{W}\left(\underline{K}, q'\right) \leq \left(q' - q\right)\mathfrak{v} + \left(q' - q\right)\mathfrak{v} + \delta \widetilde{W}\left(\underline{K}, q\right).$$

Then, $\frac{\xi}{2} < \widetilde{W}\left(\underline{K}, q\right) \leq \frac{2(q'-q)\mathfrak{v}}{1-\delta}$, which implies $q' > \underline{q} + \varepsilon_2$. This, together, with the left-continuity of $\mathcal{V}_L$ and $\widetilde{\mathcal{V}}_L$ implies that for all $q \in \left(\underline{q} - \varepsilon_2, \underline{q}\right]$,

$$\left(\widetilde{\mathcal{V}}_L(\underline{K}, q), \widetilde{P}(\underline{K}, q), \widetilde{W}(\underline{K}, q), \widetilde{Y}(\underline{K}, q)\right) = \left(\mathcal{V}_L(\underline{K}, q), P(\underline{K}, q), W(\underline{K}, q), Y(\underline{K}, q)\right)$$

which contradicts the definition of $\underline{q}$ and proves that $\left(\widetilde{\mathcal{V}}_L, \widetilde{P}, \widetilde{W}, \widetilde{Y}\right) = (\mathcal{V}_L, P, W, Y)$.

We proceed next with the second part of the proof. Let $\left(K\left(h^t\right), q\left(h^t\right)\right)$ denote the state at the history h^t under the equilibrium $\left(\widetilde{\sigma}_B, (\widetilde{\sigma}_L, \widetilde{\sigma}_H), \widetilde{\beta}\right)$. Consider the consistent quadruplet $(\mathcal{V}_L, P, W, y)$. We say that $\widetilde{\sigma}_B$ agrees with y at the history h^t if the following two conditions hold. First, whenever $y\left(K\left(h^t\right), q\left(h^t\right)\right) = q'$ then $\widetilde{\sigma}_B(h^t) = \left(K\left(h^t\right), P(K\left(h^t\right), q')\right)$. Second, whenever $y\left(K\left(h^t\right), q\left(h^t\right)\right) = k$ then $\widetilde{\sigma}_B(h^t) = \left(K\left(h^t\right) - k, \frac{c}{m}\left(K\left(h^t\right) - k\right)\right)$.

To conclude the proof, we show that if $\widetilde{\sigma}_B$ does not agree with y at the history h^t, then the history h^t is off-the-equilibrium path under the equilibrium $\left(\widetilde{\sigma}_B, (\widetilde{\sigma}_L, \widetilde{\sigma}_H), \widetilde{\beta}\right)$. Assume towards a contradiction that there exist on-path histories at which $\widetilde{\sigma}_B$ does not agree with y and let h^t be the shortest of those histories. Thus, $\widetilde{\sigma}_B$ agrees with y at all sub-histories h^τ of h^t.

As $\widetilde{\sigma}_B$ does not agree with y at the history h^t, our genericity condition guarantees that

$$V_L(h^t) > \mathcal{V}_L(K\left(h^t\right), q\left(h^t\right)), \tag{T5}$$

where $V_L(h^t)$ denotes the low-type seller's continuation payoff at history h^t under the equilibrium $\left(\widetilde{\sigma}_B, (\widetilde{\sigma}_L, \widetilde{\sigma}_H), \widetilde{\beta}\right)$.

Since $\widetilde{\sigma}_B$ does not agree with y at the history h^t, then $Y\left(K\left(h^t\right), q\left(h^t\right)\right)$ cannot be a singleton, and so $0 < q(h^t) < \bar{q}$. The first inequality holds since the parameters belong to the set $\mathfrak{G}$ identified in Claim T1. Since $q(h^t) > 0$, then the buyer makes a screening offer at some point along the history h^t. Under the consistent quadruplet, the buyer never makes two universal offers in a row. Moreover, $\widetilde{\sigma}_B$ and y agree up to h^{t-1}. Then, the buyer makes at least one screening offer in the periods $\{t-1, t-2\}$. Let $t' \in \{t-1, t-2\}$ denote the period of the last screening offer which is equal to $\left(K(h^{t'}), P\left(K(h^{t'}), q(h^t)\right)\right)$. Note that $q\left(h^{t'}\right) < q\left(h^t\right)$ and therefore in equilibrium the low-type seller accepts this offer with positive probability at $h^{t'}$.

Assume first that $t' = t - 1$. Then,

$$P\left(K\left(h^{t-1}\right), q\left(h^t\right)\right) = \delta \mathcal{V}_L\left(K\left(h^t\right), q\left(h^t\right)\right) < \delta V_L(h^t),$$

where the equality follows from the definition of consistent quadruplet, and the inequality follows from equation (T5). Thus, it is not optimal for the low-type seller to accept the screening offer at h^{t-1}.

Similarly, if $t' = t-2$ we have

$$P\left(K\left(h^{t-2}\right), q\left(h^t\right)\right) = \delta\frac{c}{m}\left(K\left(h^{t-1}\right) - K\left(h^t\right)\right) + \delta^2\mathcal{V}_L\left(K\left(h^t\right), q\left(h^t\right)\right)$$
$$< \delta\frac{c}{m}\left(K\left(h^{t-1}\right) - K\left(h^t\right)\right) + \delta^2 V_L\left(h^t\right)$$

which, again implies that it is not optimal for the low-type seller to accept the screening offer at h^{t-2}. ∎

T.1.1 Alternative definition of stationarity

In this subsection, we let the $\mathcal{V}_L$ depend not only on the number of remaining units but also on the number of units requested by the buyer. As a result, the randomization probability of the low-type seller may depend on the number of units requested by the buyer.

DEFINITION. STATIONARY* PERFECT BAYESIAN EQUILIBRIUM. *A PBE is stationary* if for each $K \in \{1, \dots, m\}$ and for each $k \in \{1, \dots, K\}$, there exists a (left-continuous) function $\mathcal{V}_L(K, k, \cdot) : [\hat{\beta}, 1] \to \mathbb{R}$ such that*

1. *The high-type seller accepts with probability one any payment greater or equal than $\frac{c}{m}k$ in exchange for any number of remaining units $k \leq K(h^t)$. The high-type seller rejects any other offer with probability one.*

2. *The behavior of the low-type seller is as follows. Take any history h^t where the remaining number of units is $K(h^t)$ and the belief is $\beta(h^t) \geq \beta$. Assume that the buyer offers a total payment p in exchange for $k \leq K(h^t)$ remaining units. Then,*

   a. *If $p \geq \frac{c}{m}k$, then the low-type accepts with probability one.*
   b. *If $p < \frac{c}{m}k$ and $p < \delta\mathcal{V}_L\left(K(h^t), k, \beta\right)$ for all $\beta \geq \beta(h^t)$, then the low-type rejects with probability one.*
   c. *If $p < \frac{c}{m}k$ and there exists $\beta \geq \beta(h^t)$ with $p \geq \delta\mathcal{V}_L\left(K(h^t), k, \beta\right)$, then the low-type seller randomizes so that $\beta' = \max\left\{\beta : \delta\mathcal{V}_L\left(K(h^t), k, \beta\right) \leq p\right\}$ is the next-period posterior after rejection*

CLAIM T3. *For every element of $\mathfrak{G}$, any stationary* PBE is outcome equivalent to a stationary PBE.*

Proof. This proof follows the same logic as the one of Claim T2. Fix an element of $\mathfrak{G}$ and compute the quadruplet $(\mathcal{V}_L, P, W, y)$ with associated $Y(K, q)$ from Part B. Consider next an arbitrary stationary* equilibrium and let $\widetilde{\mathcal{V}}_L(\cdot, \cdot, \cdot)$ be the function that governs the acceptance decision of the low-type seller in that equilibrium. For each (K, k) with $K \in \{1, \dots, m\}$ and $k \in \{1, \dots, K\}$, we derive $\widetilde{P}(K, k, \cdot)$ from $\widetilde{\mathcal{V}}_L(K, k, \cdot)$ following the same logic as in step 1 from Part A. Consider next the buyer's optimization problem, given $\widetilde{P}(\cdot,\cdot,\cdot)$. The buyer now has the option to make partial offers. If the low-type seller accepts a partial offer, we let the buyer obtain all remaining units in the following period in exchange for a payment of zero. We let $\widetilde{W}$ denote the buyer's normalized continuation payoff. We also let $\widetilde{Y}$ denote the set of solutions to the buyer's optimization problem.

The same argument as in the first part of the proof of Claim T2 guarantees that for every $K \in \{1,\ldots,m\}$ and for every $k \in \{1,\ldots,K\}$, $\widetilde{\mathcal{V}}_L(K,k,\cdot) = \mathcal{V}_L(K,\cdot)$. This implies that the buyer never makes partial offers in equilibrium. The outcome of the stationary* PBE is equivalent to the outcome of the stationary PBE constructed in Part B. ■

T.2 Proof of Claim 1

Proof. By contradiction. Assume that $q' \in Y(K,q)$ and that $P(K,q'') = P(K,q')$ for some $q'' > q'$ and consider the course of action started by choosing q'. We show that there exists an alternative course of action that leads to a strictly higher payoff than that from the course of action started by choosing q'. To simplify the algebra, in what follows we focus on a specific (optimal) course of action started by choosing q'. Assume that in the two periods following the screening offer $(P(K,q'),q')$, the buyer makes offers implied by $y\,(K,q') = k$ and $y\,(k,q') = q'''$ with $q''' > q''$. The alternative course of action involves inducing the belief q'' in the first period. In the following two periods, the buyer mimics the behavior from the first course of action. He makes a universal offer for $K-k$ units in the second period and induces belief q''' in the third period.

The difference in payoffs between the alternative course of action and the original one is given by:

$$(q''-q')\left[\overbrace{v_L\left[(1-\delta)\sum_{j=k}^{K}\Lambda_j+(1-\delta^2)\sum_{j=1}^{k}\Lambda_j\right]}^{>0}+\overbrace{\left[\delta\frac{c}{m}(K-k)+\delta^2 P\left(k,q'''\right)-P(K,q')\right]}^{\geq 0}\right]>0$$

The weak inequality in the second term is a direct consequence of the definitions of P and $\mathcal{V}'_L$, together with the equality $\mathcal{V}_L = \mathcal{V}'_L$.[48] ■

T.3 Details of the proof of Claim 3

Proof. In what follows we show, by contradiction, that $W^I(K,q_{n+1}) > 0$. Take $\varepsilon > 0$ with $q_{n+1}+\varepsilon < q_n$. Then, the buyer's continuation payoff $W^I(K,q_{n+1}+\varepsilon)$ is bounded below

[48] In general, consider an arbitrary optimal course of action started by choosing q' in period t. Let $t+T$ denote the first period in which the buyer makes a screening offer that leads to a posterior $q''' > q''$. The behavior in periods $t' \in \{t+1,\ldots,t+T-1\}$ encompasses screening and universal offers. Let T^1 be the subset of $\{t+1,\ldots,t+T-1\}$ at which the buyer makes screening offers and T^2 be those periods at which the buyer makes universal offers. We define an alternative course of action as follows. First, the buyer induces posterior q'' in period t. Second, the buyer makes no offers in periods $t' \in T^1$. Third, the buyer makes the same universal offers as in the optimal course of action in periods $t' \in T^2$. Finally, the buyer induces belief q''' in period $t+T$. The definitions of P and $\mathcal{V}'_L$, together with the equality $\mathcal{V}_L = \mathcal{V}'_L$ imply that the alternative course of action leads to a strictly higher payoff.

by the value of choosing the posterior $\overline{x}(K, q_{n+1})$:

$$W^I(K, q_{n+1} + \varepsilon) \geq [\overline{x}(K, q_{n+1}) - q_{n+1} - \varepsilon] \left(\sum_{j=1}^{K} \Lambda_j^m v_L - P\left(K, \overline{x}(K, q_{n+1})\right) \right)$$

$$+ \delta W\left(K, \overline{x}(K, q_{n+1})\right)$$

$$= W^I(K, q_{n+1}) - \varepsilon \left(\sum_{j=1}^{K} \Lambda_j^m v_L - P\left(K, \overline{x}(K, q_{n+1})\right) \right) \tag{T6}$$

Similarly, the buyer's continuation payoff $W^{II}(K, q_{n+1})$ is bounded below by the value of choosing the posterior $q_{n+1} + \varepsilon$:

$$W^{II}(K, q_{n+1}) \geq \varepsilon \left(\sum_{j=1}^{K} \Lambda_j^m v_L - P^I\left(K, q_{n+1} + \varepsilon\right) \right) + \delta W^I\left(K, q_{n+1} + \varepsilon\right) \tag{T7}$$

Assume towards a contradiction that $W^I(K, q_{n+1}) = W^{II}(K, q_{n+1}) = 0$. This, together with equations (T6) and (T7) leads to:

$$0 \geq \varepsilon \left[(1-\delta) \sum_{j=1}^{K} \Lambda_j^m v_L - \delta \left[P\left(K, \underline{x}(K, q_{n+1} + \varepsilon)\right) - P\left(K, \overline{x}(K, q_{n+1})\right) \right] \right]$$

We show next that $\lim_{\varepsilon \downarrow 0} P\left(K, \underline{x}\left(K, q_{n+1} + \varepsilon\right)\right) = P\left(K, \overline{x}(K, q_{n+1})\right)$. This implies that the right hand side is strictly positive for $\varepsilon > 0$ low enough, which implies a contradiction. To show this, note that the objective function in (11) has strictly increasing differences in q at all maximizers. Thus, $X(K, \cdot)$ is a nondecreasing correspondence: if $q' > q$, then $\underline{x}\left(K, q'\right) \geq \overline{x}\left(K, q\right)$. Moreover, the theorem of the maximum guarantees that $X(K, \cdot)$ is upper hemicontinuous.

First, since $X(K, \cdot)$ is a non-decreasing upper hemicontinuous correspondence, then $\lim_{\varepsilon \downarrow 0} \underline{x}\left(K, q_{n+1} + \varepsilon\right) = \overline{x}(K, q_{n+1})$. If $P\left(K, \cdot\right)$ is continuous at $\overline{x}(K, q_{n+1})$, then this implies that $\lim_{\varepsilon \downarrow 0} P\left(K, \underline{x}\left(K, q_{n+1} + \varepsilon\right)\right) = P\left(K, \overline{x}(K, q_{n+1})\right)$. Second, if instead $P\left(K, \cdot\right)$ is discontinuous at $\overline{x}(K, q_{n+1})$, then $\underline{x}(K, q_{n+1} + \varepsilon) = \overline{x}(K, q_{n+1})$ for ε sufficiently small. This guarantees that $\lim_{\varepsilon \to 0} P\left(K, \underline{x}\left(K, q_{n+1} + \varepsilon\right)\right) - P\left(K, \overline{x}(K, q_{n+1})\right) = 0$, so $W^I(K, q_{n+1}) > 0$. ■

T.4 Proof of Claim 5

Proof. We show by contradiction that $q_{\tilde{n}} = 0$ for some $\tilde{n}$. Assume instead that $\lim_{n \to \infty} q_n = q^* > 0$. We split this proof in two exhaustive cases.

Case 1. Assume that there exists a sequence of transformed beliefs $\{q_j\}_{j=1}^{\infty}$ with $q_j > q^*$ for all j, with $\lim_{j \to \infty} q_j = q^*$, and such that at all those beliefs, the buyer makes screening offers: $y(K, q_j) = q_j'$. This implies that for any $\eta > 0$, there exists j with $q_j' - q_j < \eta$. Take a subsequence $\{q_{j_r}\}_{r=1}^{\infty}$ with $q_{j_r} < q_{j_r}' < q_{j_{r-1}}$. The function $P(K, \cdot)$ is non-decreasing and satisfies $P(K, q) \leq \delta \mathcal{V}_L(K, q)$ for all q. Moreover, whenever the buyer makes a screening offer $y(K,q) = q'$, it must be true that $\mathcal{V}_L(K,q) = P(K,q')$. Then $P(K,q_{j_r}) \leq \delta P(K,q'_{j_r}) \leq \delta P(K,q_{j_{r-1}})$. This implies that $\lim_{q\to q^*} P(K,q) = 0$, and so $\inf_{q\in(q^*,\hat{q}]} W(K,q) > 0$.

Fix $\varepsilon > 0$ so that

$$\left[\sum_{j=1}^{K} \Lambda_j^m v_H + \delta\right] \varepsilon < (1-\delta) \inf_{q\in(q^*,\hat{q}]} W(K,q). \tag{T8}$$

Uniform continuity of $W(K,\cdot)$ guarantees that there exists $\tilde{\eta} \in (0,\varepsilon)$ such that for every $(q,\tilde{q}) \in (q^*,\hat{q}] \times (q^*,\hat{q}]$, whenever $|q-\tilde{q}| < \tilde{\eta}$, then $|W(K,q) - W(K,\tilde{q})| < \varepsilon$. Pick $q_{\hat{j}} \in \{q_j\}_{j=1}^{\infty}$ such that $q'_{\hat{j}} - q_{\hat{j}} < \tilde{\eta}$. Then,

$$\min\left\{W(K,q_{\hat{j}}), W(K,q'_{\hat{j}})\right\} \leq W(K,q_{\hat{j}}) \leq \sum_{j=1}^{K} \Lambda_j^m v_H \varepsilon + \delta W(K,q'_{\hat{j}}) \leq$$
$$\sum_{j=1}^{K} \Lambda_j^m v_H \varepsilon + \delta\left(\min\left\{W(K,q_{\hat{j}}), W(K,q'_{\hat{j}})\right\} + \varepsilon\right) < \min\left\{W(K,q_{\hat{j}}), W(K,q'_{\hat{j}})\right\}$$

where the last inequality follows from equation (T8). We have reached a contradiction. If there is only one unit left ($K=1$), case 1 covers all possibilities (as in DL). If there is more than one unit left ($K \geq 2$), the buyer may make no screening offers close to q^*. The following case covers this remaining possibility.

Case 2. Assume there exists an interval $(q^*, q^*+\eta')$ where the buyer only makes universal offers for some number $K-k$ of remaining units: $y(K,q) = k$ for all $q \in (q^*, q^*+\eta')$.

$W(k,\cdot)$ is bounded away from zero for all $k < K$. Thus, any universal offer for $K-k$ units must be followed by a screening offer. Furthermore, the low-type seller accepts the screening offer that the buyer makes with probability bounded away from zero. These two facts together imply that there exist n' and $\tilde{q} > q_{n'}$ such that for all $n \geq n'$ we have that $y(K,q_n) = k$ and $y(k,q_n) = \tilde{q}$. In what follows we show that $\lim_{n\to\infty} W(K,q_n) = 0$.

Consider a small $\varepsilon > 0$. Uniform continuity of $W^I(K,\cdot)$ guarantees that there exists $\eta \in (0,\varepsilon)$ such that for every $(q,\tilde{q}) \in (q^*,\hat{q}] \times (q^*,\hat{q}]$, whenever $|q-\tilde{q}| < \eta$, then $\left|W^I(K,q) - W^I(K,\tilde{q})\right| < \varepsilon$. Furthermore, there exists n'' such that $q_n - q_{n+1} < \eta$ for every $n \geq n''$. Therefore, for every $n \geq \overline{n} \equiv \max\{n', n''\}$ we have

$$\begin{aligned}
W^I(K,q_{n+1}) &= W^{II}(K,q_{n+1}) \\
&= \max_{q'\in[q_{n+1},q_n]} (q' - q_{n+1}) \left(\sum_{j=1}^{K} \Lambda_j v_L - P^I\left(K,q'\right)\right) + \delta W^I\left(K,q'\right) \\
&\leq \varepsilon \sum_{j=1}^{K} \Lambda_j v_L + \delta \max_{q'\in[q_{n+1},q_n]} W^I\left(K,q'\right) \\
&\leq \varepsilon \sum_{j=1}^{K} \Lambda_j v_L + \delta\left(W^I\left(K,q_{n+1}\right) + \varepsilon\right)
\end{aligned}$$

Then,

$$W^I(K, q_{n+1}) \leq \frac{\varepsilon}{1-\delta}\left(\sum_{j=1}^{K} \Lambda_j v_L + \delta\right)$$

This implies that $\lim_{n\to\infty} W^I(K, q_n) = 0$. Moreover, for all $n \geq \overline{n}$ we have

$$W_B^I(K, q_{n+1}) \geq -\varepsilon \frac{c}{m} K + \delta W(K, q_{n+1})$$

which in turn implies that $\lim_{n\to\infty} W(K, q_n) = 0$.

We have

$$P(K, q_{\overline{n}}) \leq \delta \mathcal{V}_L(K, q_{\overline{n}}) = \delta\left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q})\right]. \tag{T9}$$

Suppose that the state is (K, q_n) and consider a screening offer $(K, P(K, q_{\overline{n}}))$. Then,

$$\begin{aligned} W(K, q_n) &\geq (q_{\overline{n}} - q_n)\left[\sum_{j=1}^{K} \Lambda_j^m v_L - P(K, q_{\overline{n}})\right] \\ &\geq (q_{\overline{n}} - q_n)\left[\sum_{j=1}^{K} \Lambda_j^m v_L - \delta\left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q})\right]\right] \end{aligned}$$

Since $\lim_{n\to\infty} W(K, q_n) = 0$ and $q_{\overline{n}} - q_n$ is positive and bounded away from zero, then it must be true that

$$\delta\left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q})\right] \geq \sum_{j=1}^{K} \Lambda_j^m v_L.$$

In what follows we describe a course of action for the buyer that, when started in state (K, q) with $q < q_{\bar{n}}$ provides the buyer with continuation payoff of at least $R(q)$. We show next that $\lim_{n\to\infty} R(q_n)$ is positive and bounded away from zero. This contradicts our previous result that $\lim_{n\to\infty} W(K, q_n) = 0$. The course of action is as follows. In the first period, the buyer makes the screening offer $(K, P(K, q_{\bar{n}}))$. In the second period, the buyer makes the universal offer $\left(K-k, \frac{c}{m}(K-k)\right)$. In the third period the buyer is in state $(k, q_{\bar{n}})$. From that period on, he follows the optimal strategy. The continuation payoff from this alternative course of action at state (K, q) with $q < q_{\bar{n}}$ is bounded below by $R(q)$, given by:[49]

$$\begin{aligned} R(q) = (q_{\bar{n}} - q)&\left(\sum_{j=1}^{K} \Lambda_j^m v_L - \delta\left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q})\right]\right) \\ &+ \delta\left[\left[(\hat{q} - q_{\bar{n}})\, v_L + (1-\hat{q})\, v_H\right] \sum_{j=k+1}^{K} \Lambda_j^m - (1 - q_{\bar{n}})\, \frac{c}{m}(K-k)\right] \\ &+ \delta^2\, (\tilde{q} - q_{\bar{n}})\left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k, \tilde{q})\right) + \delta^3 W(k, \tilde{q}) \end{aligned}$$

[49]The bound is a direct consequence of equation (T9).

For all states (K, q) with $q \in (q^*, q_{\bar{n}}]$ the buyers's continuation payoff is given by:

$$
\begin{aligned}
W(K,q) = {} & (q_{\bar{n}} - q)\, v_L \sum_{j=k}^{K} \Lambda_j^m - (q_{\bar{n}} - q)\, \frac{c}{m}(K-k) \\
& + [(\hat{q} - q_{\bar{n}})\, v_L + (1-\hat{q})\, v_H] \sum_{j=k+1}^{K} \Lambda_j^m - (1 - q_{\bar{n}})\, \frac{c}{m}(K-k) \\
& + \delta \left[(q_{\bar{n}} - q) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k, \tilde{q}) \right) + (\tilde{q} - q_{\bar{n}}) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k, \tilde{q}) \right) \right] \\
& + \delta^2 W(k, \tilde{q})
\end{aligned}
$$

Let $\bar{q} \leq q_{\bar{n}}$ be such that $R(\bar{q}) = W(K, \bar{q})$. Such $\bar{q}$ is well defined since it solves:

$$
\begin{aligned}
& (q_{\bar{n}} - q) \left(\sum_{j=1}^{k} \Lambda_j^m v_L + \left[\frac{c}{m}(K-k) + \delta P(k, \tilde{q}) \right] \right) \\
& \quad = [(\hat{q} - q_{\bar{n}})\, v_L + (1-\hat{q})\, v_H] \sum_{j=k+1}^{K} \Lambda_j^m - (1 - q_{\bar{n}})\, \frac{c}{m}(K-k) \\
& \qquad + \delta \left[(\tilde{q} - q_{\bar{n}}) \left(\sum_{j=1}^{k} \Lambda_j^m v_L - P\,(k, \tilde{q}) \right) \right] + \delta^2 W(k, \tilde{q})
\end{aligned}
\tag{T10}
$$

The right hand side of equation (T10) exceeds the left hand side for all $q \in (q^*, q_{\hat{n}}]$ because $W(K, q)$ is the value from following the optimal course of action. As $q \to -\infty$ the left hand side increases continuously without bound, while the right hand side is constant. Thus, there exists $\bar{q} \leq q^*$ with $R(\bar{q}) = W(K, \bar{q})$. Moreover, from the definition of $R(q)$ and equation (T10), we obtain

$$
R(\bar{q}) = V(\bar{q}) \geq (q_{\bar{n}} - \bar{q}) \sum_{j=1}^{k} \Lambda_j^m v_L > 0.
$$

Finally, note that $R(\cdot)$ is weakly increasing in q, so for all $n \geq \bar{n}$:

$$
W(K, q_n) \geq R(q_n) \geq R(\bar{q}) > 0
$$

which contradicts the fact that $\lim_{n \to \infty} W(K, q_n) = 0$. ■

T.5 Proof of Claims 8 and 10

We briefly discuss the links between the proofs of Claims 8 and 10 before presenting them. For the inductive step $j = 1$, the proof of Claim 8 only requires Claim 6 to hold. For inductive steps $j > 1$, the proof of Claim 8 uses the results (including Claims 9 and 10) from previous inductive steps. Similarly, for any inductive step j, the proofs of Claims 9 and 10 use results from previous inductive steps and Claim 8 from the current step j.

Throughout the following proofs we proceed as follows. We first provide an explicit characterization of the limit functions $(K_m(\tau;(K,q)), q_m(\tau;(K,q)))$. In equilibrium, the low-type seller is always indifferent between accepting or rejecting a screening offer. This, together with the limit functions $(K_m(\tau;(K,q)), q_m(\tau;(K,q)))$ pins down the function $P_m(K,q)$. We explicitly express $P_m^-(K,q)$ whenever there is an impasse at (K,q). For all other states, the expression of $P_m(K,q)$ is immediate. The buyer's continuation payoff $W_m(K,q)$ can be easily computed from the limit functions $(K_m(\tau;(K,q)), q_m(\tau;(K,q)))$ and $P_m(K,q)$ so we omit it.

Proof of Claim 8. For Δ sufficiently small, the buyer has a course of action with continuation payoff arbitrarily close to $\mathcal{W}(K,q)$. For all (K,q) with $K \in \{k_j+1,\ldots,\underline{k}\}$ and $q \in [0,q_j]$, $\mathcal{W}(K,q)$ is bounded away from zero. Then, for Δ sufficiently small the buyer can guarantee a strictly positive continuation payoff. This implies, as shown in section 5.1.1, that there is no delay: $K_m(0;(K,q)) \leq k_j$. In what follows, we show that $(K_m(0;(K,q)), q_m(0;(K,q))) = (k_j, q_j)$ and that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} = \tau_j$.

First, assume by contradiction that $(K_m(0;(K,q)), q_m(0;(K,q))) = (k, q')$ with $k < k_j$. This leads to a continuation payoff (weakly) bounded above by

$$\begin{aligned}
&(\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] \\
&\quad + (\hat{q}-q)\left[\sum_{s=k+1}^{k_j}\Lambda_s^m v_L - (k_j-k)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k+1}^{k_j}\Lambda_s^m v_H - (k_j-k)\frac{c}{m}\right] \\
&\quad + W_m(k,q) \\
&< (\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] \\
&\quad + W_m(k_j,q)
\end{aligned}$$

In the previous induction step we show that at state (k_j,q) there exists a unique course of action that yields $W_m(k_j,q)$. This leads to the strict inequality in the expression above. Thus, $K_m(0;(K,q)) = k_j$.

Second, assume towards a contradiction that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} = 0$. If so, the buyer's continuation payoff results from 1) making a universal offer for $K-k_j$ units and then 2) reaching the state (k_j,q'), with $q' > q_j$ without delay. Therefore, the buyer's continuation payoff is strictly bounded above by

$$(\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] + W_m(k_j,q).$$

Thus, $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} > 0$. We know from the previous inductive step that there is no delay at any state (k_j,q) with $q < q_j$. Thus $q_m(0;(K,q)) = q_j$.

We finally show that $\inf\{\tau : q_m(\tau;(K,q)) > q_j\} = \tau_j$. The characterization of the limit functions from the previous inductive step implies that $\inf\left\{\tau : q_m(\tau;(K,q)) > q_j\right\} \leq \tau_j$. Assume by contradiction that $\inf\left\{\tau : q_m(\tau;(K,q)) > q_j\right\} \in (0,\tau_j)$. Then, in state (K,q) the low-type seller obtains a limit continuation payoff $\widetilde{V}_L(K,q)$ that satisfies:

$$\widetilde{V}_L(K,q) > (K-k_j)\frac{c}{m} + P_m^-(k_j,q_j) \tag{T11}$$

The buyer obtains a continuation payoff

$$\begin{aligned}
&(\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] \\
&+ (q_j - q)\left[\sum_{s=1}^{k_j}\Lambda_s^m v_L - \left[\widetilde{V}_L(K,q) - (K-k_j)\frac{c}{m}\right]\right] + W_m(k_j,q_j) \\
&< (\hat{q}-q)\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_L - (K-k_j)\frac{c}{m}\right] + (1-\hat{q})\left[\sum_{s=k_j+1}^{K}\Lambda_s^m v_H - (K-k_j)\frac{c}{m}\right] \\
&+ (q_j - q)\left[\sum_{s=1}^{k_j}\Lambda_s^m v_L - P_m^-(k_j,q_j)\right] + W_m(k_j,q_j) \\
&= \mathcal{W}(K,q)
\end{aligned}$$

where the strict inequality follows from equation (T11). Thus, we have reached a contradiction. ■

Proof of Claim 10. We first characterize the limit functions for all states (K,q) with $K \in \{\underline{k}+1,\ldots,\overline{k}\}$, $q < \check{q}(K)$ and $q \geq \check{q}(K-1)$ if $K \neq \underline{k}+1$. In particular, we show that starting from any such state (K,q), the state $(K,\check{q}(K))$ is reached without delay. At state $(K,\check{q}(K))$ a (potentially off-path) impasse of length $\rho(K)$ occurs.

First, assume towards a contradiction that $K_m(0;(K,q)) < K$. Then, the buyer's continuation payoff is bounded above by:

$$(\hat{q}-q)\left[\Lambda_K^m v_L - \frac{c}{m}\right] + (1-\hat{q})\left[\Lambda_K^m v_H - \frac{c}{m}\right] + W_m(K-1,q) = \mathcal{W}(K,q) < 0. \tag{T12}$$

Since the continuation payoff cannot be strictly negative, we have reached a contradiction.

We next show that $P_m(K,\cdot)$ is discontinuous at $\check{q}(K)$. If it were not, then $P_m(K,q) > \sum_{s=1}^{K}\Lambda_s^m v_L$ for all $q \in [\check{q}(K)-\eta,\check{q}(K))$ for some $\eta > 0$.[50] This together with equation (T12), implies that the buyer's continuation payoff would be strictly negative at any state (K,q) with $q \in [\check{q}(K)-\eta,\check{q}(K))$, leading to a contradiction.

The discontinuity of $P_m(K,\cdot)$ at $\check{q}(K)$ implies that an impasse occurs at $(K,\check{q}(K))$. Because of an argument analogous to that in DL, the length of the impasse must be $\rho(K)$, as defined in Claim 10. The expression for $P_m^-(K,\check{q}(K))$ is a direct consequence

[50] This follows from $P_m^+(K,\check{q}(K)) = (K-k_j)\frac{c}{m} + P_m^-(k_j,q_j) > \sum_{s=1}^{K}\Lambda_s^m v_L$, see equation (14a).

of $P^{+}_{m}\left(K,\check{q}(K)\right)$ and the length of the impasse:

$$P^{-}_{m}\left(K,\check{q}(K)\right)=\frac{\left(\sum_{s=1}^{K}\Lambda^{m}_{s}v_{L}\right)^{2}}{(K-k_{j})\frac{c}{m}+P^{-}_{m}\left(k_{j},q_{j}\right)}<\sum_{s=1}^{K}\Lambda^{m}_{s}v_{L}$$

where the inequality follows from equation (14a). Since shifting to state $\left(K,\check{q}(K)\right)$ gives the buyer a positive continuation payoff, there cannot be delay at state (K,q) with $K\in\{\underline{k}+1,\ldots,\overline{k}\}$, $q<\check{q}(K)$ and $q\geq\check{q}(K-1)$ if $k\neq\underline{k}+1$. Together with equation (T12) this implies that in fact the impasse $\left(K,\check{q}(K)\right)$ is reached without delay. This concludes the characterization of the limit functions for all (K,q) with $K\in\{\underline{k}+1,\ldots,\overline{k}\}$, $q<\check{q}(K)$ and $q\geq\check{q}(K-1)$ if $K\neq\underline{k}+1$.

The remainder of the proof is by induction. The base step is that Claim 10 holds for $\underline{k}+1$, which follows from the first part of the proof of this claim. The inductive step is as follows. Assume that Claim 10 holds for all $k\in\{\underline{k}+1,\ldots,K-1\}$ with $\underline{k}+1\leq K-1<\overline{k}$. We show next that then it must also hold for K and $q<\check{q}(K-1)$.

Consider any state (K,q) with $q<\check{q}(K-1)$. The continuation payoff of the buyer is bounded away from zero:

$$W_{m}(K,q)\geq[\check{q}(K)-q]\left[\sum_{s=1}^{K}\Lambda^{m}_{s}v_{L}-P^{-}_{m}\left(K,\check{q}(K)\right)\right]>0$$

This implies that there cannot be delay at state (K,q). To conclude this proof, we show that $K_{m}(0;(K,q))=K$. Assume towards a contradiction that $K_{m}(0;(K,q))<K$. Then the buyer's continuation payoff is bounded above by:[51]

$$\begin{aligned}
&(\hat{q}-q)\left[\Lambda^{m}_{K}v_{L}-\frac{c}{m}\right]+(1-\hat{q})\left[\Lambda^{m}_{K}v_{H}-\frac{c}{m}\right]+W_{m}(K-1,q)\\
&=(\hat{q}-q)\left[\Lambda^{m}_{K}v_{L}-\frac{c}{m}\right]+(1-\hat{q})\left[\Lambda^{m}_{K}v_{H}-\frac{c}{m}\right]\\
&\qquad+(\check{q}(K-1)-q)\left[\sum_{s=1}^{K-1}\Lambda^{m}_{s}v_{L}-P^{-}_{m}(K-1,\check{q}(K-1))\right]\\
&\qquad+(\hat{q}-\check{q}(K-1))\left[\sum_{s=k_{j}+1}^{K-1}\Lambda^{m}_{s}v_{L}-(K-1-k_{j})\frac{c}{m}\right]\\
&\qquad+(1-\hat{q})\left[\sum_{s=k_{j}+1}^{K-1}\Lambda^{m}_{s}v_{H}-(K-1-k_{j})\frac{c}{m}\right]\\
&\qquad+\left(q_{j}-\check{q}(K-1)\right)\left[\sum_{s=1}^{k_{j}}\Lambda^{m}_{s}v_{L}-P^{-}_{m}\left(k_{j},q_{j}\right)\right]\\
&<(\hat{q}-q)\left[\Lambda^{m}_{K}v_{L}-\frac{c}{m}\right]+(1-\hat{q})\left[\Lambda^{m}_{K}v_{H}-\frac{c}{m}\right]
\end{aligned}$$

[51] In the expression to the right of the equality sign, the third, fourth and fifth lines add up to zero. Nevertheless, we include them to make the comparison between payoffs easier. We proceed in a similar fashion in the expression for Ω_{2} below.

$$
\begin{aligned}
&+ (\check{q}(K) - q) \left[\sum_{s=1}^{K-1} \Lambda_s^m v_L - P_m^-(K-1, \check{q}(K-1)) \right] \\
&+ (\hat{q} - \check{q}(K)) \left[\sum_{s=k_j+1}^{K-1} \Lambda_s^m v_L - (K-1-k_j)\frac{c}{m} \right] \\
&+ (1-\hat{q}) \left[\sum_{s=k_j+1}^{K-1} \Lambda_s^m v_H - (K-1-k_j)\frac{c}{m} \right] \\
&+ \left(q_j - \check{q}(K)\right) \left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^- \left(k_j, q_j\right) \right] \\
\equiv\ & \Omega_1
\end{aligned}
$$

where the strict inequality follows from

$$
P_m^-(K-1, \check{q}(K-1)) < P_m^+(K-1, \check{q}(K-1)) = (K-1-k_j)\frac{c}{m} + P_m^-(k_j, q_j).
$$

Starting in state (K, q), the buyer could instead follow an alternative course of action and reach the state $(K, \check{q}(K))$ without delay. This would lead to a continuation payoff equal to

$$
\begin{aligned}
&(\check{q}(K) - q) \left[\sum_{s=1}^{K} \Lambda_s^m v_L - P_m^-(K, \check{q}(K)) \right] \\
&= (\check{q}(K) - q) \left[\sum_{s=1}^{K} \Lambda_s^m v_L - P_m^-(K, \check{q}(K)) \right] \\
&\quad + (\hat{q} - \check{q}(K)) \left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_L - (K-k_j)\frac{c}{m} \right] + (1-\hat{q}) \left[\sum_{s=k_j+1}^{K} \Lambda_s^m v_H - (K-k_j)\frac{c}{m} \right] \\
&\quad + \left(q_j - \check{q}(K)\right) \left[\sum_{s=1}^{k_j} \Lambda_s^m v_L - P_m^- \left(k_j, q_j\right) \right] \\
&\equiv \Omega_2
\end{aligned}
$$

The difference $\Omega_2 - \Omega_1$ takes the following form

$$
\begin{aligned}
\Omega_2 - \Omega_1 &= (\check{q}(K) - q) \left[\frac{c}{m} + P_m^- \left(K-1, \check{q}(K-1)\right) - P_m^- \left(K, \check{q}(K)\right) \right] \\
&= (\check{q}(K) - q) \left[\frac{c}{m} + \frac{\left(\sum_{s=1}^{K-1} \Lambda_s^m v_L \right)^2}{(K-1-k_j)\frac{c}{m} + P_m^- \left(k_j, q_j\right)} - \frac{\left(\sum_{s=1}^{K-1} \Lambda_s^m v_L + \Lambda_K^m v_L \right)^2}{\frac{c}{m} + (K-1-k_j)\frac{c}{m} + P_m^- \left(k_j, q_j\right)} \right] > 0,
\end{aligned}
$$

where the inequality holds because $\frac{c}{m} > \Lambda_K^m v_L$. Thus, we have reached a contradiction. ■

T.6 Proof of Facts 2 and 3

Proof of Fact 2. We first plug the expression for $P^+_{-\ell}$ from equation (18) for ℓ into equation (19). We obtain an expression for $P^+_{-(\ell+1)}$ that we plug into equation (18) for $\ell-1$. The resulting expression links the (limit) beliefs of three consecutive impasses $q_{-(\ell-1)}$, $q_{-\ell}$ and $q_{-(\ell+1)}$:

$$\begin{aligned}&\left(q_{-\ell}-q_{-(\ell-1)}\right)\left[\int_0^{\psi(q_{-(\ell-1)})}\lambda(z)v_L dz-\left[\psi(q_{-(\ell-1)})-\psi(q_{-\ell})\right]c\right]\\&-\left(q_{-\ell}-q_{-(\ell-1)}\right)\left[\frac{\left(v_L\int_0^{\psi(q_{-\ell})}\lambda(z)dz\right)^2}{\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz+\frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]dz}{q_{-(\ell+1)}-q_{-\ell}}}\right]\\&+\left(\hat{q}-q_{-\ell}\right)\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[\lambda(z)v_L-c\right]dz+(1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[\lambda(z)v_H-c\right]dz=0\end{aligned}\tag{T13}$$

Rearranging terms, we obtain the following expression for the ratio of the difference of consecutive beliefs:

$$\begin{aligned}\frac{q_{-\ell}-q_{-(\ell-1)}}{q_{-(\ell+1)}-q_{-\ell}}=&\frac{\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz}{\int_0^{\psi(q_{-\ell})}\lambda(z)v_L dz+\frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]dz}{q_{-(\ell+1)}-q_{-\ell}}}\\&\times\frac{\frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]dz}{\left(q_{-(\ell+1)}-q_{-\ell}\right)^2}}{\frac{\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[c-\lambda(z)v_L]dz}{q_{-\ell}-q_{-(\ell-1)}}-\frac{(\hat{q}-q_{-\ell})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_H-c]dz}{\left(q_{-\ell}-q_{-(\ell-1)}\right)^2}}\end{aligned}$$

It follows from the definition of $\psi(\cdot)$ that the first term in the right hand side of previous equation is less than one. Therefore, the ratio $\frac{q_{-\ell}-q_{-(\ell-1)}}{q_{-(\ell+1)}-q_{-\ell}}$ is bounded above as follows:

$$\frac{q_{-\ell}-q_{-(\ell-1)}}{q_{-(\ell+1)}-q_{-\ell}}\leq\frac{\frac{\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}[\lambda(z)v_H-c]dz}{\left(q_{-(\ell+1)}-q_{-\ell}\right)^2}}{\frac{\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[c-\lambda(z)v_L]dz}{q_{-\ell}-q_{-(\ell-1)}}-\frac{(\hat{q}-q_{-\ell})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_L-c]dz+(1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}[\lambda(z)v_H-c]dz}{\left(q_{-\ell}-q_{-(\ell-1)}\right)^2}}\tag{T14}$$

We next provide convenient expressions for the terms in the right hand side of the inequality above. To do so, we define the function $\hat{\lambda}(\cdot)$ by $\hat{\lambda}(q)=\lambda(\psi(q))$ and the function $v(\cdot)$ by

$$v(q)=\begin{cases}v_L & \text{if } q\in[0,\hat{q}]\\ v_H. & \text{if } q\in(\hat{q},1]\end{cases}$$

Using these definitions, we express

$$
\begin{aligned}
&\left(\hat{q}-q_{-(\ell+1)}\right)\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}\left[\lambda(z)v_L-c\right]dz+(1-\hat{q})\int_{\psi(q_{-(\ell+1)})}^{\psi(q_{-\ell})}\left[\lambda(z)v_H-c\right]dz\\
&=\int_{q_{-\ell}}^{q_{-(\ell+1)}}-\psi'(q)\left[\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q)v(s)-c\right]ds\right]dq\\
&=\left(\int_{q_{-(\ell+1)}}^{1}v(s)ds\right)\int_{q_{-\ell}}^{q_{-(\ell+1)}}\psi'(q)\left(\int_{q}^{q_{-(\ell+1)}}\hat{\lambda}'(u)du\right)dq\\
&=\left(\int_{q_{-(\ell+1)}}^{1}v(s)ds\right)\psi'\left(q'_{\ell,\ell+1}\right)\hat{\lambda}'(q''_{\ell,\ell+1})\frac{\left(q_{-(\ell+1)}-q_{-\ell}\right)^2}{2},
\end{aligned}
\tag{T15}
$$

for some $\left(q'_{\ell,\ell+1},q''_{\ell,\ell+1}\right)\in\left[q_{-\ell},q_{-(\ell+1)}\right]^2$. The first equality follows from a change of variables. For the second we use the fact that for all $q<q_{-(\ell+1)}$, then $\hat{\lambda}(q)=\hat{\lambda}(q_{-(\ell+1)})-\int_q^{q_{-(\ell+1)}}\hat{\lambda}'(s)ds$ and also that $\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q_{-(\ell+1)})v(s)-c\right]ds=0$. The third equality follows from the mean value theorem.

In a similar way we obtain

$$
\begin{aligned}
&\left(\hat{q}-q_{-\ell}\right)\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[\lambda(z)v_L-c\right]dz+(1-\hat{q})\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[\lambda(z)v_H-c\right]dz\\
&=\left(\int_{q_{-\ell}}^{1}v(s)ds\right)\psi'\left(q'_{\ell-1,\ell}\right)\hat{\lambda}'(q''_{\ell-1,\ell})\frac{\left(q_{-\ell}-q_{-(\ell-1)}\right)^2}{2}
\end{aligned}
\tag{T16}
$$

for some $\left(q'_{\ell-1,\ell},q''_{\ell-1,\ell}\right)\in\left[q_{-(\ell-1)},q_{-\ell}\right]^2$. Finally, again with a change of variables and using the mean value theorem, we obtain

$$
\int_{\psi(q_{-\ell})}^{\psi(q_{-(\ell-1)})}\left[c-\lambda(z)v_L\right]dz=-\psi'(q'''_{\ell-1,\ell})\left[c-\hat{\lambda}(q'''_{\ell-1,\ell})v_L\right]\left(q_{-\ell}-q_{-(\ell-1)}\right)
\tag{T17}
$$

for some $q'''_{\ell-1,\ell}\in\left[q_{-(\ell-1)},q_{-\ell}\right]$.

We plug equations (T15), (T16) and (T17) into equation (T14) and obtain

$$
\begin{aligned}
\frac{q_{-\ell}-q_{-(\ell-1)}}{q_{-(\ell+1)}-q_{-\ell}}&\leq\frac{\frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1}v(s)ds\right)\psi'\left(q'_{\ell,\ell+1}\right)\hat{\lambda}'(q''_{\ell,\ell+1})}{-\psi'(q'''_{\ell-1,\ell})\left[c-\hat{\lambda}(q'''_{\ell-1,\ell})v_L\right]-\frac{1}{2}\left(\int_{q_{-\ell}}^{1}v(s)ds\right)\psi'\left(q'_{\ell-1,\ell}\right)\hat{\lambda}'(q''_{\ell-1,\ell})}\\
&\equiv\Xi\left(q_{-(\ell-1)},q_{-\ell},q_{-(\ell+1)}\right)
\end{aligned}
\tag{T18}
$$

where we do not express explicitly that $q'_{\ell,\ell+1},q''_{\ell,\ell+1},q'_{\ell-1,\ell},q''_{\ell-1,\ell},q'''_{\ell-1,\ell}$ also depend on $q_{-(\ell-1)},q_{-\ell}$ and $q_{-(\ell+1)}$.

Fact 2 links $q_{-\ell} - q_{-(\ell-1)}$ and $q_{-(\ell+1)} - q_{-\ell}$ when $q_{-(\ell+1)} - q_{-\ell}$ is small. We study the function $\Xi(\cdot,\cdot,\cdot)$ when this difference is small. We fix $q_{-(\ell+1)}$, let $q_{-\ell} = q_{-(\ell+1)} - h$ and define $q_{-(\ell-1)}(q_{-(\ell+1)}, h)$ implicitly by equation (T13). Therefore, we directly study the function $\widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) \equiv \Xi\left(q_{-(\ell-1)}(q_{-(\ell+1)}, h), q_{-(\ell+1)} - h, q_{-(\ell+1)}\right)$ in a neighborhood of $h = 0$.

First, we show that $\lim_{h\to 0} \widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) = 1$ for every $q_{-\ell} < \bar{q}(0)$. It follows from equation (T13) that $\lim_{h\to 0} q_{-(\ell-1)}(q_{-(\ell+1)}, h) = q_{-(\ell+1)}$. Thus,

$$
\begin{aligned}
&\lim_{h\to 0} \widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) \\
&\quad = \frac{\frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right)\psi'\left(q_{-(\ell+1)}\right)\hat{\lambda}'(q_{-(\ell+1)})}{-\psi'(q_{-(\ell+1)})\left[c - \hat{\lambda}(q_{-(\ell+1)})v_L\right] - \frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right)\psi'\left(q_{-(\ell+1)}\right)\hat{\lambda}'(q_{-(\ell+1)})} \\
&\quad = \frac{\frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right)\frac{\hat{\lambda}(q_{-(\ell+1)})v_L - c}{\int_{q_{-(\ell+1)}}^{1} v(s)ds}}{-\left[c - \hat{\lambda}(q_{-(\ell+1)})v_L\right] - \frac{1}{2}\left(\int_{q_{-(\ell+1)}}^{1} v(s)ds\right)\frac{\hat{\lambda}(q_{-(\ell+1)})v_L - c}{\int_{q_{-(\ell+1)}}^{1} v(s)ds}} = 1
\end{aligned}
$$

where the second equality follows from $\hat{\lambda}'(q) = \frac{\hat{\lambda}(q)v_L - c}{\int_q^1 v(s)ds}$.[52]

Second, it follows from the fact that $\lambda(\cdot)$ is smooth that there exists $\xi > 0$ and $\tilde{h} > 0$ such that $h' < \tilde{h}$ implies that $\left|\left.\frac{\partial \widetilde{\Xi}\left(h, q_{-(\ell+1)}\right)}{\partial h}\right|_{h=h'}\right| < \xi$ for every $q_{-(\ell+1)} < \bar{q}(0)$.

Putting together the last two results, it follows that for any $\varepsilon > 0$ there exists $\tilde{h}$ such that if $h < \tilde{h}$ then $\widetilde{\Xi}\left(h, q_{-(\ell+1)}\right) < 1 + \varepsilon$ for every $q_{-(\ell+1)} < \bar{q}(0)$. This directly leads to Fact 2. ■

Proof of Fact 3. It is straightforward to establish the first result in Fact 3 if $q_{-(\ell+1)} - q_{-\ell}$ is bounded away from zero. Therefore, we restrict attention to the case in which $q_{-(\ell+1)} - q_{-\ell}$ is small.

Consider the following three consecutive limit beliefs: $\left(q_{-\ell}, q_{-(\ell+1)}, q_{-(\ell+2)}\right)$. Equation (T18) guarantees

$$
\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} \geq \frac{1}{\Xi\left(q_{-\ell}, q_{-(\ell+1)}, q_{-(\ell+2)}\right)}.
$$

We fix $q_{-(\ell+1)}$ and let $h \equiv q_{-(\ell+1)} - q_{-\ell}$. We define $q_{-(\ell+2)}\left(h, q_{-(\ell+1)}\right)$ implicitly by equation (T13), but linking the consecutive limit beliefs: $\left(q_{-\ell}, q_{-(\ell+1)}, q_{-(\ell+2)}\right)$. We also

[52]This, in turn, follows from $\int_q^1 \left[\hat{\lambda}(q)v(s) - c\right] ds = 0$ for every q in the domain of $\hat{\lambda}(\cdot)$.

define

$$\widehat{\Xi}\left(h, q_{-(\ell+1)}\right) \equiv \frac{1}{\Xi\left(q_{-(\ell)} - h, q_{-(\ell+1)}, q_{-(\ell+2)}\left(h, q_{-(\ell+1)}\right)\right)}.$$

The function $\widehat{\Xi}(\cdot,\cdot)$ satisfies $\lim_{h\to 0}\widehat{\Xi}\left(h, q_{-(\ell+1)}\right) = 1$. Moreover, for every $\tilde{h} > 0$ there exists ξ such that if $0 \le h' < \tilde{h}$ then $\left|\left.\frac{\partial \widehat{\Xi}\left(h, q_{-(\ell+1)}\right)}{\partial h}\right|_{h=h'}\right| < \xi$ for every $q_{-(\ell+1)} < \bar{q}(0)$. Thus, through a Taylor approximation, there must exist $\tilde{h} > 0$ such that for all $h < \tilde{h}$:

$$\widehat{\Xi}\left(h, q_{-(\ell+1)}\right) > 1 - \xi h \quad \text{for every } q_{-(\ell+1)} < \bar{q}(0)$$

We restrict attention to $q_{-(\ell+1)} - q_{-\ell} < \tilde{h}$, which implies

$$\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} \ge 1 - \xi\left(q_{-(\ell+1)} - q_{-\ell}\right). \tag{T19}$$

We put together equation (18) and the first equality in (T15) to express the left hand side in (21). First, note that

$$P^{+}_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz = \frac{\int_{q_{-\ell}}^{q_{-(\ell+1)}} -\psi'(q)\left[\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q)v(s) - c\right] ds\right] dq}{\left(q_{-(\ell+1)} - q_{-\ell}\right)} \tag{T20}$$

and so

$$\begin{aligned}
&\frac{\left[P^{+}_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz\right] - \left[P^{+}_{-(\ell+1)} - \int_0^{\psi(q_{-(\ell+1)})} \lambda(z) v_L dz\right]}{q_{-(\ell+1)} - q_{-\ell}} \\
&= \frac{\int_{q_{-\ell}}^{q_{-(\ell+1)}} -\psi'(q)\left[\int_{q_{-(\ell+1)}}^{1}\left[\hat{\lambda}(q)v(s) - c\right] ds\right] dq}{\left(q_{-(\ell+1)} - q_{-\ell}\right)^2} \\
&\quad - \frac{\int_{q_{-(\ell+1)}}^{q_{-(\ell+2)}} -\psi'(q)\left[\int_{q_{-(\ell+2)}}^{1}\left[\hat{\lambda}(q)v(s) - c\right] ds\right] dq}{\left(q_{-(\ell+2)} - q_{-(\ell+1)}\right)^2}\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right) \\
&= R\left(q_{-\ell}, q_{-(\ell+1)}\right) - R\left(q_{-(\ell+1)}, q_{-(\ell+2)}\right)\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right)
\end{aligned} \tag{T21}$$

where for any $(q, q') \in [\bar{q}(1), \bar{q}(0)]^2$ with $q \le q'$, we let:

$$R\left(q, q'\right) \equiv \begin{cases} \frac{\int_q^{q'} -\psi'(u)\left[\int_{q'}^{1}\left[\hat{\lambda}(u)v(s) - c\right] ds\right] du}{(q'-q)^2} & \text{if } q < q' \\ \frac{1}{2}\psi'(q)\hat{\lambda}'(q)\int_q^1 v(s) ds & \text{if } q = q' \end{cases}$$

The function $R(\cdot,\cdot)$ is continuous. We let $\underline{R} \equiv \min_{\bar{q}(1) \leq q \leq q' \leq \bar{q}(0)} R\left(q, q'\right) > 0$ and $\overline{R} \equiv \max_{\bar{q}(1) \leq q \leq q' \leq \bar{q}(0)} R\left(q, q'\right)$. If $\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} > \overline{R}/\underline{R}$ then the right hand side of equation (T21) is negative and the first inequality in Fact 3 holds trivially. Therefore, we restrict attention to:

$$\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}} \leq \overline{R}/\underline{R} \tag{T22}$$

The function $R(\cdot,\cdot)$ has bounded partial derivatives. Then, there exist constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that:

$$\begin{aligned}
& R\left(q_{-\ell}, q_{-(\ell+1)}\right) - R\left(q_{-(\ell+1)}, q_{-(\ell+2)}\right)\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right) \\
& \leq R\left(q_{-(\ell+1)}, q_{-(\ell+1)}\right) + \kappa_1\left(q_{-(\ell+1)} - q_{-\ell}\right) \\
& \quad - \left[R\left(q_{-(\ell+1)}, q_{-(\ell+1)}\right) - \kappa_2\left(q_{-(\ell+2)} - q_{-(\ell+1)}\right)\right]\left(\frac{q_{-(\ell+2)} - q_{-(\ell+1)}}{q_{-(\ell+1)} - q_{-\ell}}\right) \\
& \leq \left(\overline{R}\xi + \kappa_1 + \kappa_2\overline{R}/\underline{R}\right)\left(q_{-(\ell+1)} - q_{-\ell}\right) - \kappa_2\xi\overline{R}/\underline{R}\left(q_{-(\ell+1)} - q_{-\ell}\right)^2
\end{aligned}$$

where the second inequality follows from the inequalities in (T19) and (T22), plus the definition of $\overline{R}$. This directly leads to (21) in Fact 3.

Next, we obtain the following simple bound for $q_{-(\ell+1)} - q_{-\ell}$ from equation (T20) and the definition of $R(\cdot,\cdot)$:

$$q_{-(\ell+1)} - q_{-\ell} = \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz}{R\left(q_{-\ell}, q_{-(\ell+1)}\right)} \leq \frac{P^+_{-\ell} - \int_0^{\psi(q_{-\ell})} \lambda(z) v_L dz}{\underline{R}}.$$

This directly leads to (22) in Fact 3. ■