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JEL Classification: F12, R12

Keywords: Economic Geography, Cities, Racetrack economy, Multiplicity of stable equilibria, Commuting costs, Transportation Costs

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Stability and sustainability of urban systems under commuting and transportation costs*

Yuki Takayama, † Kiyohiro Ikeda, ‡ and Jacques-François Thisse §

May 4, 2020

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1 Introduction

This paper explores the conditions for the emergence of a system of cities in a general equilibrium setting that accounts for the transportation cost of goods between non-equidistant cities, the mobility of consumers across space, and the commuting cost borne by consumers within cities. To achieve our goal, we use a bare-bones framework with one monopolistically competitive sector and a finite number of locations equidistantly distributed along a circle where monocentric cities can be developed.[1] Consumers are mobile and choose a city where to live and work, as well as a location within this city where they consume land and a tradable good. We investigate the impact of spatial linkages between cities (trade) and labor markets (commuting and migration) on the structure of stable equilibria and the transition from one equilibrium to another. We pay a special attention to commuting costs as these costs, unlike transportation costs, remain without question high. For example, 139 million American workers have spent a collective 3.4 million years in commuting during the year 2014 (*The Washington Post*, February 25, 2016). In addition, in 2006, the total value of urban land in the U.S. exceeded twice the American GDP (Albouy et al., 2018). These are sizable numbers that justify why we focus on land use in cities.

The new fundamental ingredient that a multi-location setting brings is the (implicit) existence of a transport network so that accessibility varies across spatially dispersed locations. In particular, the two-city setting makes the (stability) analysis simple as moving away from one region automatically implies that consumers and firms necessarily go to the other. By contrast, when the spatial economy involves several locations, what happens in one location has different impacts on the others because the accessibility to markets varies across cities. In other words, the relative position of cities within the transportation system matters: any change in parameters that directly involves only two cities generates spatial spillover effects that are unlikely to leave the remaining cities unaffected. This in turn further affects the other cities and so on. As a result, firms and consumers may be more or less agglomerated – or dispersed – across a variable number of cities.

Our main findings may be summarized as follows. First, several spatial equilibria may coexist. In this case, it is common place to consider stability as a selection device. However, for various domains of transportation and commuting costs, there exist *multiple stable equilibria*. This makes it hard a priori to predict the evolution

[1] It is difficult to trace back the origin of modeling space by means of a circle. Though not the first one, Salop's (1979) circular city model is probably the most well-known reference.

of the urban system when the intensity of spatial frictions changes. Furthermore, one may wonder which equilibrium outcome is associated with the data at hand and what are the results to be tested. As a remedy for these problems, we combine two devices. The first one relies on the historic evidence that shows the resilience of cities and the resulting persistence of spatial equilibria to various kinds of shocks (Davis and Weinstein, 2002; Brakman et al., 2004; Bosker and Buringh, 2017). Therefore, among the plethora of stable equilibria, we will pay a special attention to *invariant* equilibria in which the urban system remains the same for non-negligible sets of transportation and/or commuting cost values.

Somewhat unexpectedly, two types of invariant equilibria exist, i.e., the *symmetric* and *pairwise-symmetric* patterns. The former suggests itself and involves equidistant cities that have the same size. The latter is symmetric about an oblique axis and involves one or several pairs of cities having the same size. The main distinctive feature of a pairwise-symmetric patterns is that *the distance between two adjacent cities may vary with the city pair.* What is more, when two cities are close to each other, it is reasonable to consider them as forming a *megalopolis.* Our analysis uncovers a still different type of equilibrium, that is, *non-invariant* equilibria in which cities are distributed according to a hierarchy. Stable pairwise-symmetric or hierarchical equilibria can appear only in multi-location settings.

The coexistence of multiple stable equilibria that differ in nature points to the need of a second selection device to assess the impact of shocks on the urban system. In this paper, we use the concept of *stability area* of a spatial equilibrium, which is defined as the domain of parameters over which this equilibrium is stable. When a shock renders the prevailing equilibrium unstable, the stability area of this equilibrium shares a boundary with the stability area of another spatial equilibrium. This one is the natural candidate in the transition to an alternative pattern. By applying this argument to the subsequent stability areas, it is possible to select a path generated by decreasing transportation or commuting costs. We will see that such a path often involves invariant and non-invariant equilibria. In what follows, we call this selection rule the *principle of path dependency.*

In sum, we square the circle of multiple stable equilibria by combining historic evidence and simple stability analysis. Putting results together shows that the urban system may vastly differ as they are described either by *a finite number of identical, but not necessarily equidistant, cities* or by *a hierarchical system of cities.* Since we consider a seamless space, these patterns are the cheer outcome of interactions among agents. Equally important, since the selected path depends on the initial conditions, all paths do not necessarily contain all possible stable states.

For example, the invariant patterns involving m_1 and $m_2 > m_1$ cities may be stable equilibria while the invariant pattern with m cities such that $m_1 < m < m_2$ need not emerge as an equilibrium outcome. We also find that raising the number of locations entails a rapid widening of the range of spatial equilibria. This concurs with the idea that, by restricting the number of potential settlements, different physical environments and the nonreplicability of scarce resources needed for establishing cities may lead to different types of urban systems.

We then study the effects of decreasing commuting and transportation costs and show that these costs have opposite impacts on the location of activities. This extends Murata and Thisse (2005) and Tabuchi and Thisse (2006) who consider two-location settings. The multi-location setting generates a richer set of results that are more likely to emerge than the perennial cases of full agglomeration or full dispersion. When commuting costs are very high, the economy involves a dispersed pattern of small cities because urban costs become too high when the number of cities is smaller. When commuting costs decrease, *cities are fewer and larger* because the home market effect remains a significant agglomeration force when transportation costs are not too low. Note also that cities need not have the same size nor be equidistant when commuting costs steadily decline. Consequently, *although it seems natural to expect symmetric patterns to emerge in a setting like ours, they do not come to light during the agglomeration process.*

Finally, we turn our attention to the standard thought experiment of geographical economics in which transportation costs decrease. According to Krugman (1991) and Fujita et al. (1999), falling transportation costs would foster the geographical concentration of activities. Ikeda et al. (2012) have extended this result to the case of a racetrack economy by showing that, as transportation costs steadily decrease, the number of market centers is reduced by half, doubling the spacing between them. As suggested by Helpman (1998), when urban costs are the dispersion force, this prediction ceases to hold. To be precise, when commuting costs are not too high, *decreasing transportation costs leads to more and smaller cities.* Indeed, since the level of urban costs is unaffected when the population distribution remains the same, it is no surprise that, eventually, dispersion overcomes agglomeration. This is what Brülhart et al. (2019) observe in developed countries — but not in developing countries — where the market potential effect is significantly weaker than what it used to be thanks to the provision of very efficient transportation infrastructures. Furthermore, the paths generated by decreasing transportation costs display a richer set of possible outcomes than what Ikeda et al. (2012) obtain in the core-periphery model. We may thus safely conclude that different agglomeration and dispersion forces do not necessarily generate the same global effects.

Before proceeding, the following comment is in order. The multiplicity of stable equilibria could be driven by the fact that a racetrack economy retains a great deal of symmetry. In contrast, many cities have been developed at locations endowed with specific natural advantages or are the outcome of historical accidents, such as the existence of a colonial transport networks that beget a lock-in effect on the location of economic activities. All of this points to the need to work with more general spaces. The work of Allen et al. (2020) shows how difficult it is to work with a general matrix of spatial frictions. So, one should not expect a silver bullet to solve the dimensionality problem in geographical economics. This is why we want to argue in this paper that working with simple geographies remains a useful departure from the canonical two-location setting.

Related literature. Following Krugman (1991) and Fujita et al. (1999), a great many number of theoretical works in geographical economics focus on transportation costs between two locations. Among the main exceptions are Tabuchi et al. (2005) and Gaspar et al. (2018), who consider a finite number of equidistant locations, Mossay and Picard (2011), which we discuss in the next section, and Akamatsu et al. (2012) and Ikeda et al. (2012), who work with locations that are equidistantly distributed along a circle. The last two papers extend Krugman's core-periphery model and rely on numerical analysis to study the stability and sustainability of particular patterns. By contrast, we carry out a more developed analytical analysis to investigate the stability and sustainability of a richer set of patterns, which are based on Ikeda et al. (2019). Furthermore, we do not postulate the existence of a rural sector whose output is shipped at zero cost.

Starting with Helpman (1998) and Tabuchi (1998), geographical economics now pays more attention to land than the canonical models (additional references are given in the text). However, this is often accomplished in two-location settings. Spatial quantitative models also account for both commuting and transportation costs in multi-location settings (Redding and Rossi-Hansberg, 2017; Monte et al., 2018; Allen et al., 2020). However, apart from existence and uniqueness of a spatial equilibrium, these models do not tell us much about the properties of the urban system as their main purpose is to quantify the impact of various shocks on the spatial equilibrium.

Our paper differs from the existing literature in three major respects. First, it proposes a multi-location model in which locations are not equidistant. Second, it focuses on urban rather than rural land use. Third, it unveils new stable patterns that can arise only in multi-location settings that take into account that transportation costs between cities, consumers' mobility, and commuting costs within cities. The remaining of the paper is organized as follows. Section 2 presents the model. The set of stable equilibria is characterized in Section 3 when there are 4 locations. In Section 4, we study the properties of invariant patterns in the case of an arbitrary number of locations. The stable equilibria in the case of 8 locations are identified in Section 5. In Section 6, we discuss several possible extensions.

2 The model

2.1 The economy

The economy features a unit mass of consumers/workers, $n \geq 2$ locations equidistantly distributed along the circle $\mathbf{C}$ of length $C > 1$, and three goods, land, labor and a horizontally differentiated product.[2] The amount of land available at each location of $\mathbf{C}$ is equal to one and the opportunity cost of land is zero. A consumer supplies one unit of labor and consumes one unit of land, so that $C - 1 > 0$ units of land are unused. Let $\mathcal{I} \equiv \{x_0, x_1, \cdots, x_{n-1}\}$ be the set of locations $x_i = iC/n$. With a slight abuse of notation, we also use $\mathcal{I}$ to describe the set of indices of these locations. The distance between locations x_i and x_j is measured by the minimum path length, i.e., $\ell_{ij} = \min\{|x_i - x_j|, C - |x_i - x_j|\}$.

Using a circular space has two major advantages. First, it allows us to consider distances between two locations that vary with the location pair. Second, it rules out boundary effects that act as an agglomeration force. For example, Beckmann (1976) shows that land use and social interaction generate a bell-shaped distribution of agents over a compact interval. This result is driven by this assumption as the peak tends to vanish when the interval becomes arbitrarily wide, thus showing how borders matter. Furthermore, Mossay and Picard (2011) have revisited Beckmann's model when individuals are distributed over a circle. They showed that there are multiple equilibria, which involve any odd number of identical and evenly spaced cities. We show that their equilibrium patterns also arise in our setting.

The mass of consumers residing at location i is denoted by $h_i \geq 0$ with $\sum_{i \in \mathcal{I}} h_i = 1$. A location $x_i \in \mathcal{I}$ that hosts a positive population is called a *city*. City i has a spaceless central business district (CBD) at x_i and a spatial extension described by an arc $\mathcal{C}_i$ of $\mathbf{C}$. A consumer is free to choose the city i where she wants to live and

[2] Using a linear-quadratic utility, Picard and Tabuchi (2010) show that spatial equilibria always involve the distribution of firms and workers in a finite number of cities in a setting where farmers are uniformaly distributed along a circle.

her residential location $x \in \mathcal{C}_i$ in that city. The fixed lot size assumption implies that the consumers who reside in city i choose to be uniformly distributed over the arc $\mathcal{C}_i \equiv [x_i - h_i/2, x_i + h_i/2]$ at the residential equilibrium.

The differentiated good is produced by using labor under monopolistic competition and increasing returns. Each variety is provided by a single firm and each firm supplies a single variety. To operate, a firm in city i needs a fixed requirement of $\alpha > 0$ units and a marginal requirement of $\beta > 0$ units of labor. Any variety is shipped according to an iceberg transportation technology, i.e., for each unit of the variety shipped from city i to city j, only a fraction $1/\tau_{ij} \leq 1$ arrives at destination. The transportation cost τ_{ij} between locations i and j is given by $\tau_{ij} = \exp(\tau \ell_{ij}) \geq 1$, where $\tau \geq 0$ is the uniform transportation rate.

Commuting costs have the nature of an iceberg. More specifically, the effective labor supply $l(x)$ by a consumer living at a distance $|x - x_i| \leq h_i/2 \leq 1/2$ from the CBD is given by

$$l(x) = 1 - 4\theta|x - x_i| \qquad \forall x \in \mathcal{C}_i,$$

where $\theta \geq 0$ denotes the commuting rate. Since $|x - x_i|$ is at most equal to $1/2$, we assume that $\theta \leq 1/2$ for $l(x) \geq 0$ to hold regardless of the size of city i. Using an iceberg commuting cost captures the fact that individuals who have a longer commute are more prone to being absent from work, to arrive late at the workplace and/or to make less work effort (van Ommeren and Gutiérrez-i-Puigarnau, 2011). An iceberg cost is also consistent with the empirical literature that shows that commuting costs increase with income.

The effective labor supply in city i is:

$$L_i(h_i) = \int_{-h_i/2}^{h_i/2} l(x)\mathrm{d}x = h_i\left(1 - \theta h_i\right).$$

Since L_i is maximized at $h_i = 1/(2\theta) \geq 1$, $L_i(h_i)$ increases at a decreasing rate. On the other hand, $L_i(h_i)$ decreases when the commuting rate increases.

Let $R_i(x)$ be the land rent at location $x \in \mathcal{C}_i$. Since workers are free to choose their residential location in city i, the wage net of both commuting costs and land rent must be equal across $\mathcal{C}_i$ at the residential equilibrium:

$$l(x)w_i - R_i(x) = l(\tilde{x})w_i - R_i(\tilde{x}) \qquad \forall x, \tilde{x} \in \mathcal{C}_i,$$

where w_i denotes the wage rate paid at the CBD of city i. Since $R_i(x_i + h_i/2) = 0$, the equilibrium land rent in city i is given by

$$R_i(x) = 2\theta\left(h_i - 2|x - x_i|\right) w_i \qquad \forall x \in \mathcal{C}_i,$$

so that the aggregate land rent in city i is as follows:

$$ALR_i \equiv \int_{-h_i}^{h_i} R_i(x)\mathrm{d}x = \theta w_i h_i^2.$$

Land is owned by consumers residing in each city. Therefore, the income net of commuting costs and land rents of a consumer residing at x in city i is equal to

$$y_i = l(x)w_i - R_i(x) + \frac{ALR_i}{h_i} = \left(1 - \theta h_i\right) w_i.$$

Each consumer residing in city i is endowed with CES preferences:

$$U_i = \left[\sum_{j\in\mathcal{I}} \int_0^{M_j} q_{ji}(k)^{(\sigma-1)/\sigma}\mathrm{d}k\right]^{\sigma/(\sigma-1)},$$

where M_j is the mass of varieties produced in city j, $q_{ji}(k)$ the consumption of variety $k \in [0, M_j]$, and σ the constant elasticity of substitution between any two varieties.

The budget constraint is given by

$$y_i = \sum_{j\in\mathcal{I}} \int_0^{M_j} p_{ji}(k)q_{ji}(k)\mathrm{d}k,$$

where $p_{ji}(k)$ denotes the price in city i of variety k produced in city j. This price is independent of the consumer's location in city i.

Utility maximization yields the following demand functions:

$$q_{ji}(k) = p_{ji}(k)^{-\sigma}P_i^{\sigma-1}y_i, \tag{1}$$

where

$$P_i \equiv \left[\sum_{j\in\mathcal{I}} \int_0^{M_j} p_{ji}(k)^{1-\sigma}\mathrm{d}k\right]^{1/(1-\sigma)}$$

denotes the price index in city i.

Since the total income in city i is $h_i y_i = L_i w_i$, city i's total demand $Q_{ji}(k)$ for variety k produced in city j is equal to

$$Q_{ji}(k) = p_{ji}(k)^{-\sigma} P_i^{\sigma-1} L_i w_i.$$

Market clearing implies that the total output $X_i(k)$ of variety k produced in city i is such that

$$X_i(k) = \sum_{j \in \mathcal{I}} \tau_{ij} Q_{ij}(k). \tag{2}$$

Producing $X_i(k)$ units of variety k requires $\alpha + \beta X_i(k)$ units of labor. The total production cost of a firm in city i is thus given by $[\alpha + \beta X_i(k)]\, w_i$. Each firm located in city i maximizes its profits given by

$$\Pi_i(k) = \sum_{j \in \mathcal{I}} p_{ij}(k) Q_{ij}(k) - w_i \left[\alpha + \beta X_i(k)\right]. \tag{3}$$

Under monopolistic competition, the first order condition for profit maximization yields the equilibrium price

$$p_{ij}^*(k) = \frac{\sigma}{\sigma - 1} \beta w_i \tau_{ij}, \tag{4}$$

which is the same across varieties $k \in [0, M_i]$. Since all firms set up in city i charge the same price in equilibrium, we drop the variety index k.

2.2 The market equilibrium

We first assume that consumers are immobile across cities, so that the spatial distribution of consumers $\mathbf{h} \equiv (h_i)_{i \in \mathcal{I}}$ is given. The market equilibrium conditions involve the differentiated product and labor market clearing conditions and the zero-profit condition associated with free entry. The first condition is given by (2), while the second is such that

$$(\alpha + \beta X_i) M_i = L_i.$$

The third condition implies that a firm's operating profits are entirely absorbed by the sum of wages paid to its workers:

$$(\alpha + \beta X_i) w_i = \sum_{i \in \mathcal{I}} p_{ij} Q_{ij}.$$

These three conditions and (4) yield the equilibrium output, $X_i^* = (\sigma - 1)\alpha/\beta$, and the equilibrium mass of firms in city i, $M_i^*(\mathbf{h}) = L_i/(\alpha\sigma)$. Hence, the mass of firms established in city i increases when the commuting rate θ decreases because more labor becomes available for production.

Substituting X_i^*, $M_i^*(\mathbf{h})$ and (4) into (1) yields the price index in city i:

$$P_i^*(\mathbf{h}) = \frac{\beta\sigma}{\sigma - 1}\left(\frac{1}{\alpha\sigma}\sum_{j\in\mathcal{I}} L_j w_j^{1-\sigma}\phi_{ji}\right)^{\frac{1}{1-\sigma}},$$

where $\phi_{ji} \equiv \tau_{ji}^{1-\sigma} \in [0,1]$ measures the freeness of trade between cities i and j. Since $\tau_{ij} = \exp(\tau\ell_{ij})$, we have $\phi_{ij} \equiv \phi^{t_{ij}}$ where $t_{ij} \equiv \min\{|i-j|, n-|i-j|\}$ and $\phi \equiv \exp(-(\sigma-1)\tau C/n) \in [0,1]$ decreases when the transportation rate τ increases, while ϕ_{ij} decreases when the distance $\ell_{ij} = (C/n)t_{ij}$ rises. In what follows, we will use $\phi \in [0,1]$ as the index of transportability of the differentiated good. Set $S \equiv \{(\phi,\theta); (\phi,\theta) \in [0,1]\times[0,1/2]\}$.

Evaluating the output (2) at the equilibrium prices (4) and setting (3) equal to zero yields the wage equation in city i:

$$L_i w_i = \sum_{j\in\mathcal{I}} \frac{L_i w_i^{1-\sigma}\phi_{ij}}{\sum_{k\in\mathcal{I}} L_k w_k^{1-\sigma}\phi_{kj}} L_j w_j, \qquad \forall i \in \mathcal{I}. \tag{5}$$

Applying Appendix B.3 of Monte *et al.* (2018) implies that the system of equations (5) has a unique (up-to-scale) positive solution $\mathbf{w}^*(\mathbf{h}) \equiv (w_i^*(\mathbf{h}))_{i\in\mathcal{I}}$. Therefore, the indirect utility of a consumer residing in city i is uniquely determined by

$$v_i(\mathbf{h}) = \zeta\Delta_i(\mathbf{h})^{\frac{1}{\sigma-1}} y_i \geq 0 \quad \forall i \in \mathcal{I} \tag{6}$$

where $y_i(\mathbf{h}) = (1-\theta h_i)w_i^*(\mathbf{h})$, $\Delta_i(\mathbf{h}) \equiv \sum_{j\in\mathcal{I}} L_j w_j^{1-\sigma}\phi_{ji}$, and $\zeta \equiv \frac{\sigma-1}{\beta\sigma}\left(\frac{1}{\alpha\sigma}\right)^{1/(\sigma-1)}$.

To illustrate, consider the case in which there is an even number m of identical and equidistant cities of size $1/m$, i.e., $h_i = 1/m$ for $i \in \{0, \frac{C}{m}, 2\frac{C}{m}, ..., (m-1)\frac{C}{m}\}$ and $h_i = 0$ otherwise. In this case, (6) implies that the *equilibrium indirect utility* is given by

$$v^* = \zeta\left[\left(\frac{m-\theta}{m}\right)^{\sigma} \cdot \frac{1}{m} \cdot \Phi_0\right]^{\frac{1}{\sigma-1}}, \tag{7}$$

where $\mathcal{I}_m \equiv \{i; h_i = 1/m > 0\}$ is the set of cities while

$$\Phi_0 \equiv \sum_{i\in\mathcal{I}_m} \phi^{\ell_{0i}} = \begin{cases} \frac{\left(1-\phi^{n/2}\right)\left(1+\phi^{n/m}\right)}{1-\phi^{n/m}} \geq 1 & \text{for } 0 \leq \phi < 1, \\ m & \text{for } \phi = 1. \end{cases}$$

Since the total mass of varieties supplied in the economy is equal to $(m-\theta)/m$, *a higher number of cities leads to a wider range of varieties* because lower commuting costs make available for production a bigger amount of labor. However, the mass of varieties produced in each city, i.e., $(m-\theta)/m^2$, decreases with the number of cities. The term $(\frac{m-\theta}{m})^{\sigma}\frac{1}{m}$ on the right-hand side of (7) takes into account these two effects; it may increase or decrease with m. Regarding Φ_0, it stands for the global accessibility of city i to the other cities; it always increases with m.

Differentiating (7) with respect to m yields

$$\frac{\mathrm{d}v^*}{\mathrm{d}m} = \begin{cases} f(m) \equiv \frac{v^*}{\sigma-1}\left[\frac{(\sigma+1)\theta-m}{m(m-\theta)} - \frac{2n}{m^2}\frac{2\phi^{n/m}}{1-\phi^{2n/m}}\ln\phi\right] & \text{for } 0 \le \phi < 1, \\ \frac{\sigma\zeta}{\sigma-1}\left(\frac{m-\theta}{m}\right)^{\frac{1}{\sigma-1}}\frac{\theta}{m^2} > 0 & \text{for } \phi = 1. \end{cases}$$

Since

$$\lim_{\phi\to 1} f(m) = \frac{(\sigma-1)\theta+m}{(\sigma-1)m(m-\theta)}v^* > 0,$$

while the first term in the square brackets in $f(m)$ is independent of ϕ and the second one decreases in ϕ, the function $f(m)$ is positive over $[0,1)$. Therefore, $\mathrm{d}v^*/\mathrm{d}m > 0$, and thus the equilibrium utility level increases with m. Put differently, the dispersion of production and consumption in a growing number of cities makes consumers better-off. This has the following somewhat unexpected implication: in the absence of urban spillovers, the concentration of activities in a smaller number of large cities is detrimental to consumers for all values of ϕ. This runs against the prediction that agglomeration is welfare-enhancing in the core-periphery model when transportation costs are low because the winners are able to compensate the losers.

To reduce the proliferation of parameters, we choose the unit of the differentiated good for $\alpha = 1$ to hold.

2.3 Stable spatial equilibria

Assume now that consumers are mobile across cities. Since wages are adjusted in each city for each firm to break even, the spatial equilibrium is such that firms and workers are concentrated in an endogenous number of cities which is such that the utility level is the same across cities. For any given $(\phi,\theta) \in S$, $\mathbf{h}^*(\phi,\theta)$ is a *spatial* *equilibrium* if v^* exists such that the following two conditions hold:

$$\begin{cases} v^* - v_i(\mathbf{h}^*(\phi, \theta)) = 0 & \text{if} \quad h_i^*(\phi, \theta) > 0 \\ v^* - v_i(\mathbf{h}^*(\phi, \theta)) \geq 0 & \text{if} \quad h_i^*(\phi, \theta) = 0 \end{cases} \qquad \forall i \in \mathcal{I}, \tag{8}$$

and

$$\sum_{i \in \mathcal{I}} h_i^*(\phi, \theta) = 1, \tag{9}$$

where $v^* > 0$ denotes the equilibrium utility level. The condition (8) means that no consumer may get a higher utility level by moving to another location, while (9) is the population constraint.

The uniform distribution $h_i = 1/n$ is always a spatial equilibrium because $v_i(\mathbf{h}_n) = \bar{v}(\mathbf{h}_n)$ for all $i \in \mathcal{I}$. Therefore, we have:

Proposition 1. *For any given $(\phi, \theta) \in S$, there exists at least one spatial equilibrium.*

Note that our model admits multiple equilibria over some subset of S as in Murata and Thisse (2005) and Tabuchi and Thisse (2006). As usual, we use the concept of stability to rule out some equilibria.

It is reasonable to assume that workers are attracted (repulsed) by locations that provide high (low) utility levels. Formally, we follow the literature and model the migration process by the replicator dynamics:

$$\frac{\mathrm{d}h_i}{\mathrm{d}t} = F_i(\mathbf{h}) \equiv [v_i(\mathbf{h}) - \bar{v}(\mathbf{h})]\, h_i \qquad \forall i \in \mathcal{I}, \tag{10}$$

where

$$\bar{v}(\mathbf{h}) = \sum_{i \in \mathcal{I}} h_j v_j(\mathbf{h})$$

denotes the average utility level.[3] The pattern $\bar{\mathbf{h}}$ is a *steady-state* of (10) if $\left[v_i(\overline{\mathbf{h}}) - \bar{v}(\overline{\mathbf{h}})\right] \overline{h}_i = 0$ for all $i \in \mathcal{I}$. Therefore, a spatial equilibrium is always a steady-state of (10), but a steady-state need not be a spatial equilibrium.

Given $\mathbf{F}(\mathbf{h}) \equiv (F_i(\mathbf{h}))_{i \in \mathcal{I}}$, the stability of a spatial equilibrium $\mathbf{h}^*(\phi, \theta)$ is studied

[3]The replicator dynamics (10) assumes that workers care only about their current utility level. Admittedly, this is a fairly restrictive assumption to the extent that migration decisions are often made on the grounds of current and future utility flows and various costs as a result of search, mismatch, and homesickness. Nevertheless, most analyses of the migration of forward-looking agents are conducted under the assumption of perfect foresights, an assumption hardly less restrictive than that of myopic agents.

by linearizing the system (10) in a neighborhood of $\mathbf{h}^*(\phi, \theta)$:

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \nabla\mathbf{F}(\mathbf{h}^*(\phi, \theta)) \cdot [\mathbf{h} - \mathbf{h}^*(\phi, \theta)],$$

where $\nabla\mathbf{F}(\mathbf{h})$ is the Jacobian matrix of $\mathbf{F}(\mathbf{h})$.

A spatial equilibrium $\mathbf{h}^*(\phi, \theta)$ is said to be (locally) *stable* if any small perturbations away from the equilibrium dies out over time; otherwise it is said to be unstable. Let $\delta\mathbf{h} \equiv \mathbf{h} - \mathbf{h}^*(\phi, \theta)$ be a small perturbation of the equilibrium $\mathbf{h}^*(\phi, \theta)$. Since $\mathrm{d}\mathbf{h}^*(\phi, \theta)/\mathrm{d}t = 0$, we obtain the following differential equations for $\delta\mathbf{h}$:

$$\frac{\mathrm{d}\delta\mathbf{h}}{\mathrm{d}t} = \nabla\mathbf{F}(\mathbf{h}^*(\phi, \theta))\delta\mathbf{h}$$

whose solution is given by

$$\delta\mathbf{h} = \sum_{i \in \mathcal{I}} c_i \exp(\lambda_i t)\boldsymbol{\eta}_i,$$

where c_i is a constant, λ_i is the i-th eigenvalue of $\nabla\mathbf{F}(\mathbf{h}^*(\phi, \theta))$, and $\boldsymbol{\eta}_i$ is the associated eigenvector. It follows immediately from this expression that $\mathbf{h}^*(\phi, \theta)$ is stable if all the eigenvalues have negative real parts.

2.4 Invariant equilibria

A pattern $\mathbf{h}$ is said to be invariant if it is a steady-state of (10) for all $(\phi, \theta) \in S$. A distinctive feature of geographical economics is that an invariant pattern may satisfy the equilibrium conditions (8) and (9) over a non-zero measure subset of S. Such an equilibrium is said to be an *invariant equilibrium*. The *principle of path-dependency* states that, once the economy is at an invariant equilibrium $\mathbf{h}^*$ for some $(\phi, \theta) \in S(\mathbf{h}^*) \subset S$, this equilibrium still prevails when ϕ and/or θ vary within $S(\mathbf{h}^*)$.

In this paper, most of the analysis is conducted in terms of commuting costs. Therefore, we define the *sustain point* $\theta^s(\phi, \mathbf{h}^*)$ as the threshold at which a pattern $\mathbf{h}^*$ becomes a spatial equilibrium. The *break point* $\theta^b(\phi, \mathbf{h}^*)$ is a threshold at which $\mathbf{h}^*$ ceases to be stable. We will show that these two thresholds are such that the invariant pattern $\mathbf{h}^*$ is a stable spatial equilibrium over the interval $(\theta^b(\phi, \mathbf{h}^*); \theta^s(\phi, \mathbf{h}^*)]$ whenever this interval is non-empty. On the other hand, a spatial equilibrium $\mathbf{h}^*(\phi, \theta)$ that varies with $(\phi, \theta) \in S$ is called *non-invariant*. At such an equilibrium, the size of cities changes continuously over S.

Let $\mathbf{h}_m$ be a pattern with $1 \leq m \leq n$ cities. Since the equilibrium always involves at least one city, we may assume without loss of generality that there is a city at $x_0 = 0$ ($h_0 > 0$).

Set

$$\Phi_i \equiv \sum_{j \in \mathcal{I}_m} \phi_{ij} \in [0, m]$$

for any location $i \in \mathcal{I}$. This index measures the accessibility of city i to the other cities and varies with their locations. When $\Phi_i = 0$, city i has no access to the others while it has the highest accessibility when Φ_i is equal to m.

The following proposition summarizes the main properties of invariant patterns.

Proposition 2.

(a) *The pattern $\mathbf{h}_m$ is invariant if and only if the following two conditions hold:*

$$h_i = \frac{1}{m} \quad \forall i \in \mathcal{I}_m, \qquad h_i = 0 \quad \forall i \in \mathcal{I}_0 \equiv \mathcal{I} \backslash \mathcal{I}_m, \tag{11}$$

$$\Phi_0 = \Phi_i \quad \forall i \in \mathcal{I}_m. \tag{12}$$

(b) *An invariant pattern $\mathbf{h}_m$ is a spatial equilibrium if and only if the following inequality is satisfied:*

$$\Phi_0 \geq \left(\frac{m}{m-\theta} \right)^{\frac{\sigma(\sigma-1)}{2\sigma-1}} \Phi_i, \quad \forall i \in \mathcal{I}_0. \tag{13}$$

(c) *An invariant pattern $\mathbf{h}_m$ is a stable spatial equilibrium if the following two conditions hold:*

$$\text{(i) } \Phi_0 > \left(\frac{m}{m-\theta} \right)^{\frac{\sigma(\sigma-1)}{2\sigma-1}} \Phi_i, \quad \forall i \in \mathcal{I}_0, \tag{14}$$

(ii) *all the eigenvalues of the Jacobian* $\left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j} \right)_{i,j \in \mathcal{I}_m}$ *have negative real parts.* (15)

Proof: See Appendix A.

Proposition 2(a) shows that *cities have the same size* and the same accessibility index. In this case, *cities have the same total volume of trade* because they have the same global accessibility to their trading partners. However, $v_i(\mathbf{h}_m)$ and $v_j(\mathbf{h}_m)$ need not be equal when location $j \in \mathcal{I}_0$ because the accessibility from j to $i \in \mathcal{I}_m$ may vary with i. Proposition 2(b) implies that an unpopulated location has a lower accessibility than a city when $\theta > 0$. In other words, the symmetric invariant pattern $\mathbf{h}_m$ with $m < n$ satisfies (13) if and only if $v_0(\mathbf{h}_m) \geq v_i(\mathbf{h}_m)$. Proposition 2(c) states that $v_i(\mathbf{h}_m) > v_j(\mathbf{h}_m)$ for $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_0$ must hold for $\mathbf{h}_m$ to be stable. As a result, the unpopulated locations can be ignored in the stability analyses.

To illustrate, let us describe what the above concepts are in the core-periphery model developed by Krugman (1991). Denoting by λ the share of people in location 0, $\lambda = 1/2$ is always an invariant spatial equilibrium, which is stable when transportation costs are sufficiently high. On the other hand, $\lambda = 1$ ($\lambda = 0$) is always an invariant pattern, but it becomes a stable spatial equilibrium only if transportation costs are small enough. In Murata and Thisse (2005), $\lambda = 1/2$ is always an invariant spatial equilibrium, which is stable when transportation costs are sufficiently low and/or commuting costs are high enough. By contrast, $\lambda = 1$ is always an invariant pattern, but it is a stable spatial equilibrium only if transportation costs are high and/or commuting costs are low.

Proposition 2 does not say anything about the relative position of cities along $\mathbf{C}$. Using (12), it can be shown that there are two classes of patterns that satisfy the conditions (11) and (12) (Ikeda et al., 2019). The former, which has been extensively studied in the literature, is defined by the *symmetric* patterns in which cities are equally distributed over $\mathbf{C}$. In the latter, patterns are *pairwise-symmetric*, which means that the distribution of cities is symmetric about one oblique axis that goes through the center of the circle. In other words, any city has a mirror image located symmetrically about the oblique axis. The critical difference with the symmetric pattern is that *two cities may be sufficiently close to each other to form an urban cluster*, which is isolated from the other cities.

3 Sustainable and stable patterns with 4 locations

Our setting may exhibit several stable equilibria that differ in nature. In order to gain insights about these equilibria, we consider a racetrack economy with 4 locations ($n = 4$), prior to the general analysis undertaken in Section 4. We first consider the *invariant* patterns in which cities have the same size. Then, we turn our attention to *non-invariant* patterns in which cities have different sizes.

3.1 The candidate spatial equilibria

We first consider the invariant patterns. The class of *symmetric* patterns comprises the uniform distribution $\mathbf{h}_4 = (1/4, 1/4, 1/4, 1/4)$, the 2-city symmetric pattern $\mathbf{h}_2 = (1/2, 0, 1/2, 0)$, and full agglomeration $\mathbf{h}_1 = (1, 0, 0, 0)$. Less expected (at least to us), another type of invariant pattern, which we call *pairwise-symmetric*, may also emerge as an equilibrium outcome. It is given by $\mathbf{h}_2^p = (1/2, 1/2, 0, 0)$ where the two cities are now located at $x_0 = 0$ and $x_1 = C/4$. The cities are symmetric about the oblique axis passing through the center of the circle and the point $x = C/8$. Clearly, a pairwise-symmetric pattern displays a proclivity toward agglomeration that is absent in the symmetric pattern. Figure 1(a) depicts these four invariant patterns.

We now turn our attention to non-invariant patterns, which are depicted in Figure 1(b). First, the *hierarchy* $\mathbf{h}_4^{\text{non}}$ such that $h_0 > h_{C/4} = h_{3C/4} > h_{C/2} > 0$. The largest city located at $x_0 = 0$ is flanked by two medium-size cities at $x_1 = C/4$ and $x_3 = 3C/4$, while the smallest city is established at $x_2 = C/2$. The pattern $\mathbf{h}_4^{\text{non}}$ may be viewed as the hierarchical counterpart of $\mathbf{h}_4$ as the city size decreases with the distance to the largest city. A related hierarchal pattern appears in $\mathbf{h}_3^{\text{non}}$ with $h_0 > h_{C/4} = h_{3C/4} > h_{C/2} = 0$. Note that $\mathbf{h}_3^{\text{non}}$ is more concentrated than $\mathbf{h}_4^{\text{non}}$ as the city at $x_2 = C/2$ vanishes.

This is not yet the end of the story. A still different hierarchical pattern is given by $\mathbf{h}_4^{p\text{non}}$, which is formed by two large cities established at $x_0 = 0$ and $x_1 = C/4$, while two small cities are located at $x_1 = C/2$ and $x_3 = 3C/4$, with $h_0 = h_{C/4} > h_{C/2} = h_{3C/4} > 0$. The two large cities form the core of the economy while the two small cities constitute its periphery. This configuration corresponds to a hierarchical version of the pairwise-symmetric pattern $\mathbf{h}_2^p$.

[Figure 1 about here.]

Since our numerical analysis reveals that $\mathbf{h}_4^{\text{non}}$ and $\mathbf{h}_4^{p\text{non}}$ never emerge as stable equilibria, we will focus on the symmetric patterns $\mathbf{h}_4$, $\mathbf{h}_2$, and $\mathbf{h}_1$, the pairwise-symmetric pattern $\mathbf{h}_2^p$, and the hierarchical pattern $\mathbf{h}_3^{\text{non}}$. However, it seems impossible to obtain analytically the sustain and break points for $\mathbf{h}_3^{\text{non}}$. Therefore, we will study numerically its properties in Section 3.2.

Proposition 3. *There exist non-negligible subsets of parameters over which the patterns $\mathbf{h}_m$, $\mathbf{h}_2^p$ and $\mathbf{h}_m^{\text{non}}$ are spatial equilibria.*

Proof: See Appendix B.

While $\mathbf{h}_4$ is always a spatial equilibrium, $\mathbf{h}_m$ with $m < 4$ is a spatial equilibrium if and only if $\theta \le \min\{\theta^s(\phi, \mathbf{h}_m), 1/2\}$. Furthermore, the following results are proven in Appendix E. First, $\mathbf{h}_2^p$ has a larger sustain point than $\mathbf{h}_1$ and $\mathbf{h}_2$:

$$\theta^s(\phi, \mathbf{h}_2) < \theta^s(\phi, \mathbf{h}_2^p), \qquad \theta^s(\phi, \mathbf{h}_1) < \theta^s(\phi, \mathbf{h}_2^p), \qquad \text{for } 0 < \phi < 1. \tag{16}$$

As a result, if $\mathbf{h}_4, \mathbf{h}_2, \mathbf{h}_2^p$ are spatial equilibria, they are stable when the following condition holds:

$$\max\{\theta^b(\phi, \mathbf{h}_m), 0\} < \theta \leq 1/2, \qquad \text{for } \mathbf{h}_m = \mathbf{h}_4, \mathbf{h}_2, \mathbf{h}_2^p.$$

Second, the break points can be ranked as follows (see (E.2)):

$$\theta^b(\phi, \mathbf{h}_2^p) < \theta^b(\phi, \mathbf{h}_2) < \theta^b(\phi, \mathbf{h}_4), \qquad \text{for } 0 < \phi < 1. \tag{17}$$

Finally, it follows from (E.1) and (E.2) that

$$\theta^b(\phi, \mathbf{h}_2^p) < \theta^s(\phi, \mathbf{h}_1) < \theta^s(\phi, \mathbf{h}_2^p)$$

holds for any $0 < \phi < 1$.

We may thus conclude as follows: *when θ steadily decreases, h_2^p emerges before h_1, while h_4 ceases to be stable before h_2, which becomes unstable before h_2^p.*

3.2 Stability areas

The non-negligible subset of S over which $\mathbf{h}_m$ is a stable spatial equilibrium (if any) is called the *stability area* of $\mathbf{h}_m$. We have determined the stability areas of the above patterns for $\sigma = 6$, i.e., a value that is in accordance with various estimations of the elasticity of substitution (Bergstrand et al., 2013). In Figure 2, the stability areas corresponding to the five stable equilibria are described by the shaded areas of the parameter space S. The stability areas for the four invariant patterns have been obtained analytically, while that associated with the non-invariant pattern is obtained by carrying out a series of computational analyses with respect to ϕ by changing the value of θ through fine intervals.

[Figure 2 about here.]

Consider first $\mathbf{h}_1$ and $\mathbf{h}_4$. Full agglomeration ($\mathbf{h}_1$) is a stable equilibrium in the area situated at the lower-left corner of Figure 2(a), which is colored in red. The dispersed distribution ($\mathbf{h}_4$) is a stable equilibrium in the area situated at the upper-right corner colored in grey. These two areas do not cover S. In other words, $\mathbf{h}_1$ and $\mathbf{h}_4$ are not stable equilibria when the parameters ϕ and θ belong to the white area of Figure 2(a). This area can be covered by the stability areas of the other three patterns, that is, the pattern $\mathbf{h}_2$ in Figure 2(b), the pattern $\mathbf{h}_2^p$ in Figure 2(c), and the pattern $\mathbf{h}_3^{\text{non}}$ in Figure 2(d). Since these stability areas overlap, there are *multiple stable equilibria* over some parameter domains.

Figure 3 describes (i) the subset of S associated with a single stable equilibrium (panel (a)) and (ii) the subset of S that generate multiple stable equilibria (panel (b)).

[Figure 3 about here.]

Assume that ϕ and θ are such that $\mathbf{h}_1$ and $\mathbf{h}_4$ are not stable. Using (16) and (17) shows that the interval $(\theta^b(\phi, \mathbf{h}_2), \theta^s(\phi, \mathbf{h}_2)]$ over which $\mathbf{h}_2$ is a stable spatial equilibrium is a subset of $(\theta^b(\phi, \mathbf{h}_2^p), \theta^s(\phi, \mathbf{h}_2^p)]$ for which $\mathbf{h}_2^p$ is a stable spatial equilibrium. Consequently, $\mathbf{h}_2^p$ is a stable equilibrium for a wider range of commuting cost values than $\mathbf{h}_2$. Similarly, the stability area of $\mathbf{h}_2^p$ covers the stability areas of $\mathbf{h}_3^{\text{non}}$.

Putting these results together strongly suggests that *the pairwise-symmetric pattern* $\mathbf{h}_2^p$ *is the most likely equilibrium outcome for intermediate commuting cost values.*

It is also worth noting that the equilibrium utility level at $\mathbf{h}_2^p$ is higher than that at $\mathbf{h}_2$ because transportation costs are lower in the former than in the latter, while commuting costs and the mass of supplied varieties are the same because cities host the same population. In other words, $\mathbf{h}_2^p$ Pareto-dominates $\mathbf{h}_2$. This makes the above claim even more likely.

3.3 The impact of decreasing commuting costs

To show how an equilibrium path emerges in response to a change in spatial frictions, we first consider the case of decreasing commuting costs. Since cities belong to the same country, we expect ϕ to be significantly larger than in the case of international trade costs. More specifically, we assume that the economy is characterized by low transportation costs ($\phi = 0.8$) and high commuting costs ($\theta = 0.35$). At these values, Figure 2 shows that there exists a unique stable equilibrium given by $\mathbf{h}_4$. When commuting costs start decreasing, the principle of path dependency implies that the economy remains at $\mathbf{h}_4$. When point **A** of Figure 2(a), where θ is about 0.14, is reached, $\mathbf{h}_4$ ceases to be stable. Since the neighboring stability area is $\mathbf{h}_2^p$, the economy shifts to this new pattern. In other words, the population located in cities 2 and 3 migrate to cities 0 and 1 whose size doubles. When θ decreases further, the economy remains pairwise-symmetric until point **B** of Figure 2(c) where $\theta \simeq 0.07$. When commuting costs take on a value slightly smaller than 0.07, $\mathbf{h}_2^p$ is no longer stable. As a result, the economy enters the stability area of $\mathbf{h}_1$. The whole population is now concentrated in city 0 located at $x_0 = 0$ (see Figure 2(a)).

In sum, *the equilibrium path*, which is obtained by combining the principle of path dependency and the contiguity of stability areas, *is formed by the concatenation of three invariant stable equilibria*. Note that the economy follows a similar path when transportation costs are high ($\phi = 0.2$), but the θ-domain for which $\mathbf{h}_4$ ($\mathbf{h}_1$) prevails is much narrower (wider).

We may thus conclude that firms and consumers get more and more agglomerated as commuting costs steadily fall. During this process, firms and consumers are gathered in two identical cities situated at $x_0 = 0$ and $x_1 = C/4$ that trade along the shorter route linking them.

To summarize, as commuting costs steadily decrease, the equilibrium path is as follows: for any given $\phi \in [0, 1]$,

$$\textit{Dispersion}\ (\mathbf{h}_4) \longrightarrow \textit{Pairwise-symmetric pattern}\ (\mathbf{h}_2^p) \longrightarrow \textit{Agglomeration}\ (\mathbf{h}_1).$$

To the best of our knowledge, it is the first time that the spatial equilibrium $\mathbf{h}_2^p$ is mentioned in the literature.

3.4 The impact of decreasing transportation costs

We now consider the standard thought experiment of geographical economics, i.e., the effect of decreasing transportation costs for a given value of commuting costs. Inspecting the stability areas of Figure 2 shows that agglomeration, then the pairwise-symmetric pattern and, finally, dispersion are stable equilibria as transportation costs fall from prohibitive to negligible values (ϕ increases from 0 to 1). Hence, a growing number of smaller cities become the stable outcome as shipping goods gets less and less inexpensive.

Consequently, as transportation costs steadily decrease, the equilibrium path is as follows:

$$\textit{Agglomeration}\ (\mathbf{h}_1) \longrightarrow \textit{Pairwise-symmetric pattern}\ (\mathbf{h}_2^p) \longrightarrow \textit{Dispersion}\ (\mathbf{h}_4).$$

The intuition behind this finding is as follows. When transportation costs are prohibitive, individuals consume mainly the locally produced varieties. However, they are willing to bear high commuting costs generated by a large population because they have a preference for variety. Therefore, as transportation costs steadily decrease, importing varieties from other cities become cheaper, so that the market solves the congestion problem by spreading the production over a growing number of smaller cities in which the individual labor supply rises. In other words, lowering transportation costs weakens the agglomeration force stressed in geographical economics. By contrast, as long as the population distribution remains the same, the intensity of the dispersion force is unaffected. It is no surprise that, eventually, the latter overcomes the former.

To conclude, when urban costs, rather than the existence of a rural sector, are the dispersion force, decreasing transportation costs leads to a conclusion that runs against the main prediction of geographical economics: *decreasing commuting costs within cities, rather than transportation costs between cities, foster the agglomeration of activities.* Furthermore, there is a whole domain of parameters in which the economy involves only two neighboring cities. That $\mathbf{h}_2^p$ may be a stable spatial equilibrium may come as a surprise because the general belief holds that a spatial equilibrium does not involve two large cities situated in close proximity. Yet, the empirical evidence is less conclusive. For example, Cuberes et al. (2019) use data on U.S. counties and metro areas to show that proximity to large urban centers need not prevent the growth of neighboring places.

4 Sustainability and stability of invariant patterns

4.1 Symmetric patterns

Assume that $n = 2^k$. We study below the sustainability and stability of symmetric patterns $h_m \equiv (h_i)_{i\in\mathcal{I}_m}$ which involve $m = 2^{k_1}$ — with $k_1 = 1, 2, .., k-1$ — cities hosting $h_i = 1/m$ workers and located at x_i where $i \in \mathcal{I}_m \equiv \{0, \frac{C}{m}, 2\frac{C}{m}, ..., (m-1)\frac{C}{m}\}$, while $h_i = 0$ for $i \in \mathcal{I}_0$.

We show in Appendix D that there exist a unique sustain point $\theta^s(\phi, \mathbf{h}_m)$, i.e., the threshold at which $\mathbf{h}_m$ becomes or ceases to be a spatial equilibrium (see (14)) and a unique break point $\theta^b(\phi, \mathbf{h}_m)$, i.e., the threshold at which $\mathbf{h}_m$ becomes or ceases to be stable (see (15)). Note that $\mathbf{h}_m$ need not be stable nor a spatial equilibrium because $\theta^b(\phi, \mathbf{h}_m)$ is not necessarily smaller than $\theta^s(\phi, \mathbf{h}_m)$.

By appealing to the symmetry of $\mathbf{h}_m$, we have the following lemmas, which provide a necessary and sufficient condition for $\mathbf{h}_m$ to satisfy the conditions (14) and (15), respectively.

Lemma 1. Consider a symmetric invariant pattern $\mathbf{h}_m$ with $m < n$. Then, $\mathbf{h}_m$ satisfies (14) if and only if $v_0(\mathbf{h}_m) > v_{C/n}(\mathbf{h}_m)$.

Proof: See Appendix C.

Lemma 2. Suppose m is even. Then, a symmetric invariant pattern $\mathbf{h}_m$ satisfies (15) if and only if the commuting rate θ is such that

$$K(\phi, \mathbf{h}_m) \cdot \left[\frac{2\sigma - 1}{\sigma - 1}(1 - 2\theta h_0) - (\sigma - 1)\theta h_0\right] - \sigma\theta h_0 < 0,$$

where $K(\phi, h_m)$ is defined by

$$K(\phi, \mathbf{h}_m) \equiv \frac{(1 - \phi^{\frac{n}{m}})^2}{1 - 2\cos\left(\frac{2\pi}{m}\right)\phi^{\frac{n}{m}} + \phi^{\frac{2n}{m}}} \frac{1 + \phi^{\frac{n}{2}}}{1 - \phi^{\frac{n}{2}}} > 0 \tag{18}$$

for all $0 \leq \phi < 1$.

Proof: See Appendix C.

The sustain and break points are given by the following expressions (see Appendix D):

$$\theta^s(\phi, \mathbf{h}_m) = m\left[1 - \left(\frac{\Phi_{C/n}}{\Phi_0}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \qquad \theta^b(\phi, \mathbf{h}_m) = \frac{m}{2}\frac{\frac{2\sigma-1}{\sigma-1}K(\phi, \mathbf{h}_m)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right)K(\phi, \mathbf{h}_m) + \frac{\sigma}{2}}, \tag{19}$$

It can be shown that the right-hand side of (18) is decreasing in ϕ. Therefore, the break point $\theta^b(\phi, \mathbf{h}_m)$ decreases when transportation costs fall, which means that lowering transportation costs allows $\mathbf{h}_m$ to remain a stable equilibrium for lower commuting costs. Consequently, we have the following proposition:

Proposition 4. *If* $m_1 < m_2$, *then,* $\theta^b(\phi, \mathbf{h}_{m_2}) > \theta^b(\phi, \mathbf{h}_{m_1})$ *for any* $0 < \phi < 1$.

[Figure 4 about here]

Thus, we may rank the break points by decreasing values of m (see Figure 4(a)). As θ decreases and crosses the break point $\theta^b(\phi, \mathbf{h}_n)$ from above, Proposition 4 implies that the next stable equilibrium necessarily involves a smaller number of equidistant cities. Furthermore, since we focus on stable equilibria, this proposition is sufficient for the following result to hold: the smaller the number of cities at a stable equilibrium, the lower the value of the commuting rate θ at which the corresponding pattern ceases to be stable. In sum, for a given value of ϕ, *when commuting costs steadily fall, the market outcome may involve the step-wise agglomeration of activities in a decreasing number of larger cities.*

Unlike the break points, the sustain points cannot be ranked by decreasing values of m because the ranking may change with ϕ (see Figure 4(b)). Nevertheless, Proposition 4 has two important implications. First, the symmetric pattern with

m cities is stable and sustainable over the interval $(\theta^b(\phi, \mathbf{h}_m), \theta^s(\phi, \mathbf{h}_m)]$ if and only if $\theta^b(\phi, \mathbf{h}_m) < \theta^s(\phi, \mathbf{h}_m)$. When $\theta^b(\phi, \mathbf{h}_m)$ is not smaller than $\theta^s(\phi, \mathbf{h}_m)$, the symmetric pattern with m cities is a spatial equilibrium for $\theta \leq \theta^s(\phi, \mathbf{h}_m)$, but this equilibrium is unstable. Second, since the symmetric pattern with m cities is never an equilibrium when $\theta > \theta^s(\phi, \mathbf{h}_m)$, the equilibrium path of the economy bypasses the symmetric pattern with m cities when the inequality $\theta^b(\phi, \mathbf{h}_m) \geq \theta^s(\phi, \mathbf{h}_m)$ holds. By implication, $\mathbf{h}_{m_1}$ and $\mathbf{h}_{m_2}$ may be stable equilibria while $\mathbf{h}_m$ such that $m_1 = 2^{k_1} < m < m_2 = 2^{k_2}$ may not be a stable equilibrium. To put differently, *when θ decreases, the transition from m_2 to m_1 cities need not go through all the values of m belonging to the interval (m_1, m_2).*

The next proposition determines the values of m and the necessary and sufficient conditions on θ for $\mathbf{h}_m$ to be a stable equilibrium.

Proposition 5.

(a) *The dispersed pattern $\mathbf{h}_n$ is a stable equilibrium if and only if θ is larger than $\theta^b(\phi, \mathbf{h}_n)$.*

(b) *For any even number $m < n$, the symmetric pattern $\mathbf{h}_m$ is a stable equilibrium for $\phi \geq 0$ if and only if $\theta^b(\phi, \mathbf{h}_m) < \theta \leq \theta^s(\phi, \mathbf{h}_m)$.*

(c) *The agglomerated pattern $\mathbf{h}_1$ is a stable equilibrium for $\phi \geq 0$ if and only if θ is smaller than $\theta^s(\phi, \mathbf{h}_1)$.*

To summarize, three cases may arise when θ crosses $\theta^b(\phi, \mathbf{h}_m)$ from above: (i) there is multiplicity of stable invariant equilibria that have less than m cities; (ii) there is a unique stable symmetric or pairwise-symmetric invariant equilibrium with a number of cities smaller than m; and (iii) the economy may display a path of non-invariant equilibria in which the size of cities changes with the level of commuting and transportation costs.

4.2 Pairwise-symmetric invariant patterns

We can follow the same procedure as in Appendix D to show that the sustain and break points of a pairwise symmetric invariant $\mathbf{h}_m^p$ are still given by (19), where $\Phi_{C/n}$ is replaced by $\max_{i \in \mathcal{I}_0} \Phi_i$ while $K(\phi, \mathbf{h}_m^p)$ is the largest eigenvalue of the matrix $(\phi_{ij}/\Phi_0)_{i,j \in \mathcal{I}_m}$. Furthermore, Proposition 5(b) still holds for pairwise-symmetric patterns. Unfortunately, we have not been able to obtain the analytical expression of $K(\phi, \mathbf{h}_m^p)$, which prevents us to rank the break points for pairwise-symmetric patterns. However, this can be done numerically as shown in Sections 3 and 5.

5 Stable and sustainable patterns with 8 locations

To gain further insights about the emergence of pairwise-symmetric and non-invariant patterns, we consider a racetrack economy with 8 locations ($n = 8$). We have checked that results are qualitatively similar when $n = 16$.

5.1 The candidate spatial equilibria

Figure 5(a) depicts the invariant patterns for $n = 8$. There are 4 symmetric configurations given by full dispersion $\mathbf{h}_8$, $\mathbf{h}_4 = (1/4, 0, 1/4, 0, 1/4, 0, 1/4, 0)$, $\mathbf{h}_2 = (1/2, 0, 0, 0, 1/2, 0, 0, 0)$, and full agglomeration $\mathbf{h}_1$. There are 4 pairwise-symmetric ones for $m = 4$ and 2. The pairwise-symmetric pattern for $m = 4$ is such that $\mathbf{h}_4^p = (1/4, 1/4, 0, 0, 1/4, 1/4, 0, 0)$. In this configuration, one city is located at $x_0 = 0$ and another one at $x_4 = C/2$ like in the symmetric pattern, but the two remaining cities are located at $x_1 = C/8$ and $x_5 = 5C/8$. When $m = 2$, there are three pairwise-symmetric 2-city patterns where the second city is located at $x_1 = C/8$, $x_2 = C/4$, and $x_3 = 3C/8$, respectively (see Figure 5(a)). In these configurations, denoted $\mathbf{h}_2^{C/8}$, $\mathbf{h}_2^{C/4}$, and $\mathbf{h}_2^{3C/8}$, the population is more concentrated than in the symmetric one $\mathbf{h}_2$.

[Figure 5 about here.]

Patterns other than those shown in Figure 5(a) are all non-invariant (Ikeda et al., 2019). Among them, we have the hierarchical patterns $\mathbf{h}_m^{\mathrm{non}}$ ($m = 6, 5, 4, 3$) displayed in Figure 5(b) where a bigger circle means a city hosting a larger population, while cities having the same rank have the same size.

The proof of Proposition 3 and the Supplementary Material show that all those patterns are spatial equilibria over non-negligible subsets of S, while the expressions (E.3) in Appendix E imply that the following inequalities

$$\theta^b(\phi, \mathbf{h}_8) > \theta^b(\phi, \mathbf{h}_4) > \theta^b(\phi, \mathbf{h}_2), \qquad \text{for } 0 < \phi < 1,$$

hold for the symmetric patterns. As for the pairwise-symmetric patterns, the inequalities are as follows:

$$\theta^b(\phi, \mathbf{h}_2^{3C/8}) > \theta^b(\phi, \mathbf{h}_2^{C/4}) > \theta^b(\phi, \mathbf{h}_2^{C/8}), \qquad \text{for } 0 < \phi < 1, \tag{20}$$

$$\theta^s(\phi, \mathbf{h}_2^{C/8}) > \theta^s(\phi, \mathbf{h}_2^{C/4}), \qquad \theta^s(\phi, \mathbf{h}_2^{C/8}) > \theta^s(\phi, \mathbf{h}_2^{3C/8}), \qquad \text{for } 0 < \phi < 1. \tag{21}$$

5.2 Stability areas

We are now equipped to investigate the stability of the spatial equilibria discussed above, that is, the eight invariant patterns and the two types of non-invariant patterns depicted in Figure 5. Like in Section 3, we use the principle of path dependency and the contiguity of stability areas to select a path of stable spatial equilibria. The stability areas for the invariant patterns were obtained analytically by using the same approach as in Section 3 and those for the non-invariant patterns by carrying out a series of numerical analyses. The stability areas are drawn in Figure 6 for $\sigma = 6$. Simulations show that the stability areas are similar for $\sigma = 4$ and $\sigma = 8$.

[Figure 6 about here.]

The shaded areas in Figure 6(a) describe the stability areas of the symmetric invariant patterns for $m = 8$, $m = 4$, $m = 2$, and $m = 1$ cities. The flat distribution $\mathbf{h}_8$ has a relatively small stability area at the upper-right corner of the parameter space, while agglomeration $\mathbf{h}_1$ has a large stability area at the lower-left corner, which encompasses more than half of the square. However, the stability areas of $\mathbf{h}_1$ and $\mathbf{h}_8$ do not cover the white domain. The upper-left of this domain is covered by the stability areas of $\mathbf{h}_2$ and $\mathbf{h}_4$ in Figure 6(a).[4] The stability areas of $\mathbf{h}_2^{C/8}$, $\mathbf{h}_2^{C/4}$, and $\mathbf{h}_2^{3C/8}$ are depicted in Figure 6(b). Note that $\mathbf{h}_4^{p}$ is never a stable equilibrium.

The inequalities (20) and (21) imply that the stability area of $\mathbf{h}_2^{C/8}$ includes those of the other two pairwise-symmetric 2-city patterns. Furthermore, Figures 6(a) and 6(b) show that the stability area of $\mathbf{h}_2$ is included in that of $\mathbf{h}_2^{C/8}$. Note also that the equilibrium utility level at $\mathbf{h}_2^{C/8}$ is higher than that achieved at $\mathbf{h}_2$, $\mathbf{h}_2^{C/4}$ and $\mathbf{h}_2^{3C/8}$ because varieties are shipped over a shorter distance while urban costs take on the same value. In other words, $\mathbf{h}_2^{C/8}$ Pareto-dominates $\mathbf{h}_2$, $\mathbf{h}_2^{C/4}$, and $\mathbf{h}_2^{3C/8}$. Combing these results strongly suggests that, among the 2-city patterns, $\mathbf{h}_2^{C/8}$ is the most natural candidate.

What about the white area in Figure 6(c) which includes no stable invariant equilibrium? This domain is covered by the stability areas of the two types of non-invariant hierarchical patterns defined in Section 5.1 for $m = 6, 5, 4, 3$ cities and depicted in Figures 6(d), 6(e), and 6(f). Thus, there exists at least one stable equilibrium for any $(\phi, \theta) \in S$. More importantly, *non-invariant patterns may emerge as the only stable equilibria over a non-negligible set of parameters.* This is to be contrasted with what we saw in Section 3.3 where such equilibria always

[4] The stability area for $\mathbf{h}_4$ is extremely small.

coexist with invariant equilibria. Indeed, for parameters that belong to the white area of Figure 6(c), the urban system involves a hierarchy of cities which involves different numbers of city-types. Consequently, *the number of locations matters for the nature of the equilibrium urban system.*

5.3 The impact of decreasing commuting costs

As in Section 3, we assume that $\phi = 0.8$. Figure 7 shows that *a steady decrease in commuting costs leads to the gradual concentration of firms and consumers.* Between full dispersion and agglomeration, the economy obeys the hierarchical principle with one (or two) primate city whose size grows when θ falls, while the size of the other cities decreases as the distance to the primate city rises. Furthermore, the migration of consumers toward the primate city implies that the small cities gradually disappear from the urban system. Unexpectedly perhaps, some cities, such as those at $x_1 = C/8$ and $x_7 = 7C/8$, grow during the first phases of the agglomeration process before declining at the benefit of the biggest cities. Note also that, during this process, an expanding arc of the racetrack, which used to host small cities, ends up being empty. This goes together with *the hollowing-out of an expanding circular arc.*

[Figure 7 about here.]

In sum, as θ steadily decreases, the equilibrium path is as follows: for any given $\phi \in [0, 1]$,

$$\begin{aligned}&\textit{Dispersion }(\mathbf{h}_8) \longrightarrow \textit{Hierarchical patterns }(\mathbf{h}_m^{\text{non}};\ m = 5, 4, 3)\\ &\quad\longrightarrow \textit{Megalopolis }(\mathbf{h}_2^{C/8}) \longrightarrow \textit{Agglomeration }(\mathbf{h}_1).\end{aligned}$$

The comparison of the cases $n = 4$ and $n = 8$ shows that *a larger number of potential sites lead to the emergence of non-invariant patterns that do not appear when n is smaller.* Indeed, the spatial equilibria that arise between the extreme cases of agglomeration and dispersion are hierarchical. These equilibria vastly differ from $\mathbf{h}_4$ and $\mathbf{h}_2$ that are a priori the natural candidate equilibria. This stresses the importance of the characteristics of "first nature" for the characteristics of "second nature," a relationship that is too often overlooked in the literature despite its empirical relevance. What is more, the sequence of bifurcations also differs from what Akamatsu et al. (2012) and Ikeda et al. (2012) obtained in the core-periphery model. This shows once more that accounting for urban costs leads to very different and richer conclusions than those obtained in Krugman's setting where land and commuting are ignored.

6 Concluding remarks

In this paper, we have proposed a new approach to determine the path of stable spatial equilibria in a multi-location setting that involves migration, transportation and commuting costs. In other words, we recognize that workers/consumers use land, which anchor cities to specific locations, while shipping goods between cities remains costly. Furthermore, even though working with several rather than two locations renders the analysis more complex, modeling space as a racetrack has led to new results. First, there exist new and empirically relevant equilibria that cannot emerge in a two-location setting. Second, the multiplicity of stable equilibria is not an exotica. Third, by combining the concepts of stability areas and path dependency, we have been able to select plausible equilibrium paths that display urban patterns that are either pairwise-symmetric or hierarchical. By contrast, symmetric patterns, which seem a priori the most natural candidate equilibria in our setting, seldom emerge when the economy is subject to various shocks. Our analysis also confirms the idea that changing the dispersion force (from a rural sector to city commuting) may reverse results.

Our analysis remains incomplete in several respects. First, our model disregards several general equilibrium effects that shape the actual space-economy. For example, we assume that exogenous markups and homogeneous firms. Yet, it has been shown that shocks to transportation and commuting costs foster tougher competition and firm selection when preferences are no longer modeled by the CES (Behrens et al., 2017). Second, although we have identified two types of asymmetric equilibria that have been overlooked in geographical economics, we acknowledge that these equilibria do not replicate the richness of real-world urban hierarchies. In particular, our cities have the same size or form a hierarchy in which cities get smaller as the distance to the biggest city rises. Instead, we would like to obtain patterns in which cities having different sizes alternate as in Akamatsu et al. (2019). This is something we hope to accomplish in the future.

Third, assuming that cities are endowed with several employment centers rather than one amounts to lowering the aggregate land rent and total commuting costs. As a result, for the same population, the level of urban costs in each city is lower (Gaigné and Thisse, 2019). This weakens the intensity of the dispersion force, but does not affect the nature of our results. Things are more complex when consumers are free to choose their lot size. Indeed, the values of the sustain and break points change because the equilibrium utility level now depends on the substitutability between land and the consumption good. However, we expect our results to remain qualitatively the same when consumers have Cobb-Dougals preferences. On the other hand, it is unclear what the paths of stable spatial equilibria become.

Fourth, our paper relies on internal increasing returns. Yet, empirical evidence shows the existence of significant agglomeration economies that take the form of external increasing returns. Accounting for such effects makes the analysis much more difficult. However, in the case of invariant patterns we can show that agglomeration economies slow down the process of dispersion associated with the decrease of transportation costs. We find it reasonable to expect the same to hold for other stable equilibria

Finally, and unfortunately, the stability analysis of non-invariant patterns remains so far out of reach from the analytical point of view.

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Appendix

A. Proof of Proposition 2

Proof of Proposition 2(a) We first consider the case where $\phi = 0$ to obtain a necessary condition for a pattern $\mathbf{h}_m$ to be an invariant steady-state. When $\phi = 0$, the indirect utility $v_i(\mathbf{h}_m)$ is given by

$$v_i(\mathbf{h}_m) = \zeta\,(1-\theta h_i)^{\frac{\sigma}{\sigma-1}}\,h_i^{\frac{1}{\sigma-1}}.$$

Therefore, for $\mathbf{h}_m$ to be a steady-state for any θ, that is, $v_i(\mathbf{h}_m) = v_0(\mathbf{h}_m)$ for all $i \in \mathcal{I}_m \backslash \{0\}$, it must be that

$$h_i = \frac{1}{m}, \quad \forall i \in \mathcal{I}_m. \tag{A.1}$$

In other words, $\mathbf{h}_m$ is an invariant steady-state only if all cities have the same size.

Next, we consider the general case of Proposition 2(a). Using (A.1), it is readily verified that the wage equation (5) implies that the wage bill is the same across all cities:

$$w_i L_i = \begin{cases} w_0 L_0 & \text{if} \quad i \in \mathcal{I}_m, \\ 0 & \text{if} \quad i \in \mathcal{I}_0. \end{cases}$$

Substituting this expression into (5), we obtain the equilibrium wage at the potential city i:

$$w_i^\sigma = \sum_{j\in\mathcal{I}_m} \frac{\phi_{ij}}{w_0^{-\sigma}\sum_{k\in\mathcal{I}_m}\phi_{kj}} = \frac{\Phi_i}{\Phi_0}w_0^\sigma.$$

Using the above expressions, we can rewrite the indirect utility (6) as follows:

$$v_i(\mathbf{h}_m) = \zeta\,(1-\theta h_i)\,L_0^{\frac{1}{\sigma-1}}\left(\frac{\Phi_i}{\Phi_0}\right)^{\frac{1}{\sigma}}\Phi_i^{\frac{1}{\sigma-1}}, \quad \forall i \in \mathcal{I}, \tag{A.2}$$

where $L_0 \equiv h_0(1-\theta h_0)$ is the labor supply in a city where $h_i = 1/m$ for $i \in \mathcal{I}_m$, while $h_i = 0$ otherwise.

It follows from (A.2) that $\mathbf{h}_m$ is a steady-state if and only if

$$\frac{v_0(\mathbf{h}_m)}{v_i(\mathbf{h}_m)} = \left(\frac{\Phi_0}{\Phi_i}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}} = 1 \quad \forall i \in \mathcal{I}_m.$$

As a result, $\mathbf{h}_m$ is a steady-state for all $(\phi,\theta) \in S$ if and only if (A.1) and the following condition hold:

$$\Phi_0 = \Phi_i \quad \forall i \in \mathcal{I}_m.$$

Proof of Proposition 2(b) It follows from (8) and (A.2) that a necessary and sufficient condition for an invariant steady-state $\mathbf{h}_m$ $(m < n)$ to be a spatial equilibrium is given by the following inequality:

$$\frac{v_0(\mathbf{h}_m)}{v_i(\mathbf{h}_m)} = \left(\frac{m-\theta}{m}\right)\left(\frac{\Phi_0}{\Phi_i}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}} \geq 1.$$

Proof of Proposition 2(c) By permuting appropriately the components of $\mathbf{h}_m$, we obtain:

$$\hat{\mathbf{h}}_m = (\mathbf{h}_{m+}, \mathbf{h}_{m0}),$$

where $\mathbf{h}_{m+} = (h_i)_{i\in\mathcal{I}_m}$ and $\mathbf{h}_{m0} = (h_i)_{i\in\mathcal{I}_0}$. In line with Ikeda et al. (2012), we may rearrange the Jacobian matrix given in Appendix F as follows:

$$\begin{aligned}
\hat{\mathbf{J}} &= \begin{bmatrix} \mathbf{J}_+ & \mathbf{J}_{+0} \\ \mathbf{0} & \mathbf{J}_0 \end{bmatrix} \\
\mathbf{J}_+ &= \left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = h_0\left(\mathbf{I} - h_0\mathbf{E}\right)\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - h_0 v_0(\mathbf{h}_m)\mathbf{E}, \\
\mathbf{J}_{+0} &= \left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i\in\mathcal{I}_m,\ j\in\mathcal{I}_0} = h_0\left(\mathbf{I} - h_0\mathbf{E}\right)\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i\in\mathcal{I}_m,\ j\in\mathcal{I}_0} \\
\mathbf{J}_0 &= \left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_0} = \mathrm{diag}\left[(v_i(\mathbf{h}_m) - v_0(\mathbf{h}_m))_{i\in\mathcal{I}_0}\right].
\end{aligned}$$

This implies that the eigenvalues of $\nabla\mathbf{F}(\mathbf{h}_m)$ are given by the eigenvalues of $\mathbf{J}_+$ and $\mathbf{J}_0$. Since the eigenvalues of $\mathbf{J}_0$ are $(v_i(\mathbf{h}_m) - v_0(\mathbf{h}_m))_{i\in\mathcal{I}_0}$, all the eigenvalues of $\hat{\mathbf{J}}$ have the negative real part if and only if

(i) $v_i(\mathbf{h}_m) < v_0(\mathbf{h}_m) \quad \forall i \in \mathcal{I}_0$,

(ii) all the eigenvalues of $\mathbf{J}_+$ have negative real part.

B. Proof of Proposition 3

We show that the invariant and non-invariant patterns are spatial equilibria over some subsets of $[0,1] \times [0,1/2]$.

(i) Proposition 2(b) implies that there exist values of σ, ϕ, and θ such that the invariant patterns of Figures 1(a) and 5(a) are spatial equilibria.

(ii) The wage equation (5) implies

$$\Delta_i w_i^\sigma = \Delta_i \sum_{j\in\mathcal{I}_m} \frac{L_j w_j \phi_{ij}}{\Delta_j} \qquad \text{for all } i \in \mathcal{I}, \tag{B.1}$$

since $L_i > 0$ for any $i \in \mathcal{I}_m$ and $L_i = 0$ for any $i \in \mathcal{I}_0$. Substituting $\Delta_i = \sum_{j\in\mathcal{I}_m} L_j w_j^{1-\sigma}\phi_{ji}$ and $\phi_{ji} = \phi_{ij}$ into the left-hand side of (B.1) yields

$$\sum_{j\in\mathcal{I}_m}\left[\left(\frac{w_i}{w_j}\right)^{\sigma} - \frac{\Delta_i}{\Delta_j}\right] L_j w_j \phi_{ij} = 0 \qquad \text{for all } i \in \mathcal{I}.$$

Since the market equilibrium is unique, the relative wage $(w_i/w_0)_{i\in\mathcal{I}}$ is the unique solution of the wage equation (5) if there exists $(w_i/w_0)_{i\in\mathcal{I}}$ satisfying

$$\frac{w_i}{w_0} = \left(\frac{\Delta_i}{\Delta_0}\right)^{\frac{1}{\sigma}} \qquad \text{for any } i \in \mathcal{I}. \tag{B.2}$$

When $\mathbf{h}$ is a spatial equilibrium, the indirect utilities satisfy the following conditions:

$$v_0(\mathbf{h}) \begin{cases} = v_i(\mathbf{h}) & \text{for any } i \in \mathcal{I}_m, \\ \geq v_i(\mathbf{h}) & \text{for any } i \in \mathcal{I}_0. \end{cases}$$

Since $v_0(\mathbf{h})/v_i(\mathbf{h}) = (y_0/y_i)(\Delta_0/\Delta_i)^{1/(\sigma-1)}$, these conditions are equivalent to

$$\frac{1-\theta h_0}{1-\theta h_i}\frac{w_0}{w_i}\left(\frac{\Delta_0}{\Delta_i}\right)^{\frac{1}{\sigma-1}} \begin{cases} = 1 & \text{for any } i \in \mathcal{I}_m, \\ \geq 1 & \text{for any } i \in \mathcal{I}_0. \end{cases} \tag{B.3}$$

Using (B.3), we can rewrite (B.2) as follows:

$$\frac{w_i}{w_0} = \begin{cases} \left(\frac{1-\theta h_0}{1-\theta h_i}\right)^{\frac{\sigma-1}{2\sigma-1}} & \text{for all } i \in \mathcal{I}_m, \\ \left(\frac{\Delta_i}{\Delta_0}\right)^{\frac{1}{\sigma}} = \left[\frac{\sum_{j\in\mathcal{I}_m} h_j(1-\theta h_j)^{\sigma^2/(2\sigma-1)}\phi_{ji}}{\sum_{j\in\mathcal{I}_m} h_j(1-\theta h_j)^{\sigma^2/(2\sigma-1)}\phi_{j0}}\right]^{\frac{1}{\sigma}} & \text{for all } i \in \mathcal{I}_0. \end{cases} \tag{B.4}$$

This shows the existence of $(w_i/w_0)_{i\in\mathcal{I}}$ satisfying (B.2) at the spatial equilibrium.

It follows from (B.3) and (B.4) that a non-invariant pattern is a spatial equilibrium if and only if the following conditions hold:

$$\Gamma_i \begin{cases} = 0 & \text{for any } i \in \mathcal{I}_m, \\ \geq 0 & \text{for any } i \in \mathcal{I}_0, \end{cases} \tag{B.5}$$
$$\Gamma_i \equiv H_0\Delta_0 - H_i\Delta_i = \sum_{j\in\mathcal{I}_m} h_j (H_j)^{\frac{\sigma}{\sigma-1}}\left[\phi_{j0}H_0 - \phi_{ji}H_i\right],$$

where $H_i \equiv (1-\theta h_i)^{\frac{\sigma(\sigma-1)}{2\sigma-1}}$.

When $n = 4$, (B.5) becomes:

$$\begin{cases} h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi H_{C/4}\right) + h_{C/4}(H_{C/4})^{\frac{\sigma}{\sigma-1}}\left(2\phi H_0 - (1+\phi^2)H_{C/4}\right) = 0 \\ h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi^2\right) + h_{C/4}(H_{C/4})^{\frac{\sigma}{\sigma-1}} 2\phi\left(H_0 - 1\right) \geq 0 \end{cases} \quad \text{for } \mathbf{h}_3^{\text{non}},$$

$$h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi H_{C/2}\right) + h_{C/2}(H_{C/2})^{\frac{\sigma}{\sigma-1}}\left(\phi H_0 - H_{C/2}\right) = 0 \qquad \text{for } \mathbf{h}_4^{p\text{non}},$$

$$\begin{cases} h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi H_{C/4}\right) + h_{C/4}(H_{C/4})^{\frac{\sigma}{\sigma-1}}\left(2\phi H_0 - (1+\phi^2)H_{C/4}\right) \\ \qquad +h_{C/2}(H_{C/2})^{\frac{\sigma}{\sigma-1}}\left(\phi^2 H_0 - \phi H_{C/4}\right) = 0 \\ h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi^2 H_{C/2}\right) + h_{C/4}(H_{C/4})^{\frac{\sigma}{\sigma-1}} 2\phi\left(H_0 - H_{C/2}\right) \\ \qquad +h_{C/2}(H_{C/2})^{\frac{\sigma}{\sigma-1}}\left(\phi^2 H_0 - H_{C/2}\right) = 0 \end{cases} \quad \text{for } \mathbf{h}_4^{\text{non}},$$

Since there exists a non-negligible domain of ϕ, θ, σ and $\mathbf{h} = (h_0, h_{C/4}, h_{C/2}, h_{3C/4})$ satisfying these conditions and (9), the non-invariant patterns $\mathbf{h}_3^{\text{non}}$, $\mathbf{h}_4^{p\text{non}}$, and $\mathbf{h}_4^{\text{non}}$ are spatial equilibria over this domain.

When $n = 8$, (B.5) for the non-invariant pattern $\mathbf{h}_3^{\text{non}} = (h_0, h_{C/8}, 0, 0, 0, 0, 0, h_{C/8})$ becomes:

$$\begin{aligned} & h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi H_{C/8}\right) + h_{C/8}(H_{C/8})^{\frac{\sigma}{\sigma-1}}\left[2\phi H_0 - (1+\phi^2)H_{C/8}\right] = 0, \\ & h_0(H_0)^{\frac{\sigma}{\sigma-1}}\left(H_0 - \phi^2\right) + h_{C/8}(H_{C/8})^{\frac{\sigma}{\sigma-1}}\left[2\phi H_0 - \phi(1+\phi^2)\right] \geq 0, \end{aligned} \tag{B.6}$$

because $\Gamma_{C/4} = \Gamma_{3C/4} \leq \Gamma_{3C/8} = \Gamma_{5C/8} \leq \Gamma_{C/2}$. Hence, $\mathbf{h}_3^{\text{non}}$ is a spatial equilibrium if and only if (B.6) and $H_{C/8} \geq \phi$ hold. Since there are four parameters, these conditions are satisfied over a non-negligible domain of σ, ϕ, θ, and $h_{C/8} < 1/3$ ($h_0 = 1 - 2h_{C/8} > 1/3$).

Finally, we specify in the Supplementary Material the non-negligible domains over which the non-invariant patterns ($\mathbf{h}_4^{\text{non}}$, $\mathbf{h}_5^{\text{non}}$, $\mathbf{h}_6^{\text{non}}$, $\mathbf{h}_7^{\text{non}}$, $\mathbf{h}_8^{p\text{non}}$, and $\mathbf{h}_8^{\text{non}}$) are also spatial equilibria.

C. Proof of Lemmas 1 and 2

(i) **Lemma 1.** We have

$$\begin{aligned} \mathcal{I}_m &= \left\{0, \tfrac{C}{m}, 2\tfrac{C}{m}, ..., (m-1)\tfrac{C}{m}\right\}, \\ \mathcal{I}_0 &= \left\{\tfrac{C}{n}, 2\tfrac{C}{n}, 3\tfrac{C}{n}, \ldots, \tfrac{C}{m} - \tfrac{C}{n}, \tfrac{C}{m} + \tfrac{C}{n}, \tfrac{C}{m} + 2\tfrac{C}{n}, \ldots, 2\tfrac{C}{m} - \tfrac{C}{n}, \ldots\right\}. \end{aligned}$$

By symmetry, it is sufficient to consider $i \in \{\frac{C}{n}, 2\frac{C}{n}, \dots, \frac{C}{2m}\} \subset \mathcal{I}_0$. Since

$$\Phi_0 = 1 + 2\phi^{\frac{n}{m}} + 2\phi^{2\frac{n}{m}} + \cdots + 2\phi^{\frac{n}{2}-\frac{n}{m}} + \phi^{\frac{n}{2}},$$
$$\Phi_i = (\phi^{\frac{n}{C}i} + \phi^{\frac{n}{m}-\frac{n}{C}i}) + (\phi^{\frac{n}{m}+\frac{n}{C}i} + \phi^{2\frac{n}{m}-\frac{n}{C}i}) + \cdots + (\phi^{\frac{n}{2}-\frac{n}{m}+\frac{n}{C}i} + \phi^{\frac{n}{2}-\frac{n}{C}i}) \quad \text{for } i = \frac{C}{n}, 2\frac{C}{n}, \dots, \frac{C}{2m},$$

we have:

$$\Phi_i - \Phi_{i+C/n} = (1-\phi)(1-\phi^{\frac{n}{m}-2\frac{n}{C}i-1})\phi^{\frac{n}{C}i}(1+\phi^{\frac{n}{m}}+\cdots\phi^{\frac{n}{2}-\frac{n}{m}}) > 0 \quad \text{for } i = 0, \frac{C}{n}, 2\frac{C}{n}, \dots, \frac{C}{2m} - \frac{C}{n},$$

This implies

$$\Phi_0 > \Phi_{C/n} > \Phi_{2C/n} > \cdots > \Phi_{C/(2m)}, \tag{C.1}$$

thereby showing that $\Phi_{C/n} = \max_{i\in\mathcal{I}_0} \Phi_i$ and $v_{C/n}(\mathbf{h}_m) = \max_{i\in\mathcal{I}_0} v_i(\mathbf{h}_m)$. As a result, $\mathbf{h}_m$ satisfies (14) if and only if $v_0(\mathbf{h}_m) > v_{C/n}(\mathbf{h}_m)$.

Lemma 2. The eigenvalues of the Jacobian matrix $(\partial F_i(\mathbf{h}_m)/\partial h_j)_{i,j\in\mathcal{I}_m}$ are given by (D.2). Hence, $\mathbf{h}_m$ satisfies (15) if and only if the following condition holds:

$$K(\phi, \mathbf{h}_m) \cdot \left[\frac{2\sigma-1}{\sigma-1}(1-2\theta h_0) - (\sigma-1)\theta h_0\right] - \sigma\theta h_0 < 0,$$

where $K(\phi, \mathbf{h}_m) \equiv \max_{i\in\mathcal{I}_m\setminus\{0\}} f_i$.

Since f_i is an eigenvalue of $\mathbf{D}_m \equiv (\phi_{ij}/\Phi_0)_{i,j\in\mathcal{I}_m}$, it follows from Akamatsu et al. (2019, Lemma C.1) that f_i is given by

$$f_i = \frac{1}{\Phi_0}\frac{(1-\phi^{\frac{2n}{m}})[1-(-1)^i\phi^{\frac{n}{2}}]}{1-2\phi^{\frac{n}{m}}\cos\left(\frac{2\pi}{m}\right)+\phi^{\frac{2n}{m}}}, \tag{C.2}$$

where

$$\Phi_0 = 1 + \phi^{\frac{n}{2}} + 2\sum_{k=1}^{m/2-1}\phi^{\frac{n}{m}k} = \frac{(1+\phi^{\frac{n}{m}})(1-\phi^{\frac{n}{2}})}{1-\phi^{\frac{n}{m}}} > 0.$$

Furthermore, it follows from Akamatsu et al. (2019, Lemma C.1) that $f_1 = \max_{i\in\mathcal{I}_m\setminus\{0\}} f_i$.

D. Sustain and break points for invariant patterns

(i) It follows from (13) that the sustain point $\theta^s(\mathbf{h}_m, \phi)$ with $m < n$ cities is uniquely determined by

$$\theta^s(\phi, \mathbf{h}_m) \equiv m \left[1 - \left(\frac{\max_{i \in \mathcal{I}_0} \Phi_i}{\Phi_0} \right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}} \right]. \tag{D.1}$$

Note that (C.1) holds (i.e., $\Phi_{C/n} = \max_{i \in \mathcal{I}_0} \Phi_i$) when $\mathbf{h}_m$ is a symmetric invariant pattern. Therefore, the sustain point for the symmetric invariant patterns is given by (19).

(ii) We now determine the break point $\theta^b(\phi, \mathbf{h}_m)$ for $m > 1$. The symmetry of invariant patterns of a racetrack economy implies that the matrix $\mathbf{D}_m$ is a block-circulant matrix with circulant blocks (BCCB). Since the matrices $\mathbf{I}$ and $\mathbf{E}$ are also BCCB, the three matrices have the same eigenvectors (Davis, 1979). Therefore, the eigenvalues $\mathbf{g}_m = (g_i(\mathbf{h}_m))_{i \in \mathcal{I}_m}$ of the Jacobian $(\partial F_i(\mathbf{h}_m)/\partial h_j)_{i,j \in \mathcal{I}_m}$ are given by

$$g_i(\mathbf{h}_m) = \begin{cases} -v_0(\mathbf{h}_m) < 0 & \text{if} \quad i = 0, \\ h_0 e_i(\mathbf{h}_m) & \text{if} \quad i \in \mathcal{I}_m \backslash \{0\}, \end{cases}$$

where the eigenvector for $g_0(\mathbf{h}_m)$ is $\mathbf{1}$. Furthermore,

$$\begin{aligned} e_i(\mathbf{h}_m) = &\frac{v_0(\mathbf{h}_m)}{L_0 \{\sigma + (\sigma-1) f_i(\phi, \mathbf{h}_m)\}} \\ &\cdot \left\{ \left[\frac{2\sigma-1}{\sigma-1}(1 - 2\theta h_0) - (\sigma-1)\theta h_0 \right] f_i(\phi, \mathbf{h}_m) - \sigma\theta h_0 \right\} \end{aligned} \tag{D.2}$$

are the eigenvalues of the Jacobian $(\partial v_i(\mathbf{h}_m)/\partial h_j)_{i,j \in \mathcal{I}_m}$, while $f_i(\phi, \mathbf{h}_m)$ is the i-th eigenvalue of the matrix $\mathbf{D}_m$. Applying Gershgorin circle theorem implies that $-1 \leq f_i(\phi, \mathbf{h}_m) \leq 1$ for $\phi \in [0, 1]$ and $i \in \mathcal{I}_m$ (Horn and Johnson, 2013). As a result, if $\mathbf{h}_1$ is a spatial equilibrium, it is always stable.

Consider an invariant pattern $\mathbf{h}_m$ with $m > 1$. Proposition 2(c) implies that this pattern is stable only if $e_i(\mathbf{h}_m) < 0$ for all $i \in \mathcal{I}_m \backslash \{0\}$. Let $\theta_i(\phi, \mathbf{h}_m)$ be the solution of $e_i(\mathbf{h}_m) = 0$. Therefore, $e_i(\mathbf{h}_m) < 0$ if and only if

$$\theta > \theta_i(\phi, \mathbf{h}_m) \equiv \frac{m}{2} \frac{\frac{2\sigma-1}{\sigma-1} f_i(\phi, \mathbf{h}_m)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2} \right) f_i(\phi, \mathbf{h}_m) + \frac{\sigma}{2}}.$$

Since θ_i is an increasing for f_i, the break point $\theta^b(\phi, \mathbf{h}_m)$ is uniquely determined by

$$\theta^b(\phi, \mathbf{h}_m) \equiv \frac{m}{2} \frac{\frac{2\sigma-1}{\sigma-1} K(\phi, \mathbf{h}_m)}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) K(\phi, \mathbf{h}_m) + \frac{\sigma}{2}}, \tag{D.3}$$

where $K(\phi, \mathbf{h}_m) \equiv \max_{i \in \mathcal{I}_m \setminus \{0\}} f_i(\phi, \mathbf{h}_m)$. Consequently, a necessary and sufficient condition for $\mathbf{h}_m$ to be stable is given by $\theta > \theta^b(\phi, \mathbf{h}_m)$.

When $\mathbf{h}_m$ is a symmetric invariant pattern, the matrix $\mathbf{D}_m$ is a circulant matrix (a special case of the BCCB) and its eigenvalue is given by (C.2). Furthermore, the break point $\theta^b(\phi, \mathbf{h}_m)$ for the symmetric invariant patterns decreases with ϕ. Indeed, we have

$$\frac{\partial \theta_i(\phi, \mathbf{h}_m)}{\partial \phi} = \frac{m}{2} \frac{\frac{2\sigma-1}{\sigma-1} \frac{\sigma}{2}}{\left[\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) f_i(\mathbf{h}_m, \phi) + \frac{\sigma}{2}\right]^2} \frac{\partial f_i(\mathbf{h}_m, \phi)}{\partial \phi} < 0,$$

because $\frac{\partial f_i(\mathbf{h}_m, \phi)}{\partial \phi} < 0$ for all $i \in \mathcal{I}$ when m is even (Akamatsu et al., 2019, Lemma A.1).

E. Sustain and break points when $n = 4$ and $n = 8$

(i) When $n = 4$, (D.1) yields the sustain points for $\mathbf{h}_2, \mathbf{h}_2^p$ and $\mathbf{h}_1$:

$$\begin{cases} \theta^s(\phi, \mathbf{h}_2) = 2\left[1 - \left(\frac{2\phi}{1+\phi^2}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \\ \theta^s(\phi, \mathbf{h}_2^p) = 2\left(1 - \phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right), \\ \theta^s(\phi, \mathbf{h}_1) = \left(1 - \phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right). \end{cases} \tag{E.1}$$

Clearly, we have $\theta^s(0, \cdot) = 2$. In this case, the patterns $\mathbf{h}_4$, $\mathbf{h}_2$, $\mathbf{h}_2^p$ and $\mathbf{h}_1$ are all spatial equilibria. Furthermore, since $\theta^s(\phi, \cdot)$ is a decreasing function of ϕ, $\mathbf{h}_2^p$, $\mathbf{h}_2$ and $\mathbf{h}_1$ are spatial equilibria over intervals whose right bound increases with ϕ.

As for the break points, it follows from (D.3) that they are given by

$$\theta^b(\phi, \mathbf{h}_m) = \frac{m}{2} \frac{\frac{2\sigma-1}{\sigma-1} K}{\left(\frac{2\sigma-1}{\sigma-1} + \frac{\sigma-1}{2}\right) K + \frac{\sigma}{2}} \geq 0, \tag{E.2}$$

where

$$K \equiv \begin{cases} \frac{1-\phi}{1+\phi} & \text{for } \mathbf{h}_4 \text{ and } \mathbf{h}_2^p, \\ \frac{1-\phi^2}{1+\phi^2} & \text{for } \mathbf{h}_2. \end{cases}$$

Since $\phi = 1$ implies $K = 0$, we have $\theta^b(1, \mathbf{h}_m) = 0$, which means that $\mathbf{h}_4, \mathbf{h}_2, \mathbf{h}_2^p$

are stable equilibria when transportation costs are negligible. By implication of (E.2), the break point θ^b is a decreasing positive function of ϕ. As a consequence, $\mathbf{h}_4$, $\mathbf{h}_2$, $\mathbf{h}_2^p$ are stable over intervals whose left bound decreases with ϕ.

(ii) When $n = 8$, we obtain from (D.1) and (D.3) the sustain and break points for the symmetric configurations:

$$\begin{cases} \theta^b(\phi, \mathbf{h}_8) = 4\frac{\frac{2\sigma-1}{\sigma-1}K_8}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)K_8+\frac{\sigma}{2}}, & \\ \theta^b(\phi, \mathbf{h}_4) = 2\frac{\frac{2\sigma-1}{\sigma-1}K_4}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)K_4+\frac{\sigma}{2}}, & \theta^s(\phi, \mathbf{h}_4) = 4\left[1-\left(\frac{2\phi}{1+\phi^2}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \\ \theta^b(\phi, \mathbf{h}_2) = \frac{\frac{2\sigma-1}{\sigma-1}K_2}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)K_2+\frac{\sigma}{2}}, & \theta^s(\phi, \mathbf{h}_2) = 2\left[1-\left(\frac{\phi(1+\phi^2)}{1+\phi^4}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \\ & \theta^s(\phi, \mathbf{h}_1) = \left(1-\phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right). \end{cases} \tag{E.3}$$

The expression (18) yields the values of K_i:

$$\begin{aligned} K_8 &\equiv K(\phi, \mathbf{h}_8) = \Psi(\phi^2)\frac{1+\sqrt{2}\phi+\phi^2}{(1+\phi)^2}, \\ K_4 &\equiv K(\phi, \mathbf{h}_4) = \Psi(\phi^2), \\ K_2 &\equiv K(\phi, \mathbf{h}_2) = \Psi(\phi^4), \end{aligned}$$

with

$$\Psi(\phi) \equiv \frac{1-\phi}{1+\phi}.$$

By implication of Proposition 4, we have:

$$\theta^b(\phi, \mathbf{h}_8) > \theta^b(\phi, \mathbf{h}_4) > \theta^b(\phi, \mathbf{h}_2) \qquad \text{for } 0 < \phi < 1.$$

As for the pairwise-symmetric patterns, we also use the expressions (D.1) and (D.3) to determine the corresponding break and sustain points:

$$\begin{aligned} &\theta^b(\phi, \mathbf{h}_2^{C/8}) = \frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi)+\frac{\sigma}{2}}, && \theta^s(\phi, \mathbf{h}_2^{C/8}) = 2\left(1-\phi^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right), \\ &\theta^b(\phi, \mathbf{h}_2^{C/4}) = \frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi^2)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi^2)+\frac{\sigma}{2}}, && \theta^s(\phi, \mathbf{h}_2^{C/4}) = 2\left[1-\left(\frac{2\phi}{1+\phi^2}\right)^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right], \\ &\theta^b(\phi, \mathbf{h}_2^{3C/8}) = \frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi^3)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi^3)+\frac{\sigma}{2}}, && \theta^s(\phi, \mathbf{h}_2^{3C/8}) = 2\left\{1-\left[\frac{\phi(1+\phi)}{1+\phi^3}\right]^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right\}, \\ &\theta^b(\phi, \mathbf{h}_4^p) = 2\frac{\frac{2\sigma-1}{\sigma-1}\Psi(\phi^3)}{\left(\frac{2\sigma-1}{\sigma-1}+\frac{\sigma-1}{2}\right)\Psi(\phi^3)+\frac{\sigma}{2}}, && \theta^s(\phi, \mathbf{h}_4^p) = 4\left\{1-\left[\frac{\phi(1+\phi)}{1+\phi^3}\right]^{\frac{2\sigma-1}{\sigma(\sigma-1)}}\right\}. \end{aligned}$$

F. Jacobian matrix of the adjustment process

For any (invariant or non-invariant) pattern $\mathbf{h}$, the Jacobian $\nabla\mathbf{F}(\mathbf{h})$ of the adjustment process is given by

$$\nabla\mathbf{F}(\mathbf{h}) = \mathbf{R}(\mathbf{h}) \cdot \nabla\mathbf{v}(\mathbf{h}) + \mathbf{J}(\mathbf{h}),$$

where the matrices $\mathbf{R}(\mathbf{h})$ and $\mathbf{J}(\mathbf{h})$ are defined as follows:

$$\begin{aligned}\mathbf{R}(\mathbf{h}) &\equiv \mathrm{diag}[\mathbf{h}] \cdot (\mathbf{I} - \mathbf{E}\cdot\mathrm{diag}[\mathbf{h}]),\\ \mathbf{J}(\mathbf{h}) &\equiv \mathrm{diag}[\mathbf{v}(\mathbf{h}) - \bar{v}(\mathbf{h})\mathbf{1}] - \mathbf{h}\mathbf{v}(\mathbf{h})^\top.\end{aligned}$$

(i) Note that the Jacobian $\nabla\mathbf{v}(\mathbf{h})$ of the indirect utility vector is given by

$$\begin{aligned}\nabla\mathbf{v}(\mathbf{h}) = \mathrm{diag}[\mathbf{v}(\mathbf{h})] \cdot \Bigg\{ &\frac{1}{\sigma - 1}\mathbf{M}^\top \cdot \mathrm{diag}[\mathbf{L}]^{-1} \cdot \mathrm{diag}[\mathbf{1} - \theta\mathbf{h}] \\ &+ (\mathbf{I} - \mathbf{M}) \cdot \mathrm{diag}[\mathbf{w}]^{-1} \cdot \nabla\mathbf{w}(\mathbf{h}) - \frac{\theta}{2}\mathrm{diag}\left[\mathbf{1} - \frac{\theta}{2}\mathbf{h}\right]^{-1}\Bigg\},\end{aligned}$$

where $\mathbf{1}$ is the vector whose elements equal 1, $\mathbf{E}$ is the $n \times n$ matrix whose elements equal to 1, $\mathbf{I}$ is the identity matrix, $\mathbf{D} \equiv (\phi_{ij})_{i,j\in\mathcal{I}}$, while

$$\begin{aligned}\mathbf{\Delta} &\equiv \mathbf{D} \cdot \mathrm{diag}[\mathbf{w}]^{1-\sigma} \cdot \mathbf{L},\\ \mathbf{M} &\equiv \mathrm{diag}[\mathbf{L}] \cdot \mathrm{diag}[\mathbf{w}]^{1-\sigma} \cdot \mathbf{D}\cdot\mathrm{diag}[\mathbf{\Delta}]^{-1};\end{aligned}$$

(ii) As for the Jacobian $\nabla\mathbf{w}(\mathbf{h})$ of the wage vector, it is given by

$$\nabla\mathbf{w}(\mathbf{h}) = -\left(\frac{\partial W_i(\mathbf{h})}{\partial w_j}\right)^{-1}_{i,j\in\mathcal{I}} \left(\frac{\partial W_i(\mathbf{h})}{\partial h_j}\right)_{i,j\in\mathcal{I}},$$

with

$$\begin{aligned}\left(\frac{\partial W_i(\mathbf{h})}{\partial w_j}\right)_{i,j\in\mathcal{I}} &= (\mathbf{I} - \mathbf{M}) \cdot \mathrm{diag}[\mathbf{L}] + (\sigma - 1)\left(\mathrm{diag}[\mathbf{MY}] - \mathbf{M}\cdot\mathrm{diag}[\mathbf{Y}] \cdot \mathbf{M}^\top\right) \cdot \mathrm{diag}[\mathbf{w}]^{-1})\\ \left(\frac{\partial W_i(\mathbf{h})}{\partial h_j}\right)_{i,j\in\mathcal{I}} &= \mathrm{diag}[\mathbf{1} - \theta\mathbf{h}] \cdot (\mathrm{diag}[\mathbf{w}] - \mathrm{diag}[\mathbf{L}]^{-1} \cdot \mathrm{diag}[\mathbf{MY}]),\\ &\quad + (\mathbf{M}\cdot\mathrm{diag}[\mathbf{Y}] \cdot \mathbf{M}^\top \cdot \mathrm{diag}[\mathbf{L}]^{-1} - \mathbf{M}\cdot\mathrm{diag}[\mathbf{w}]) \cdot \mathrm{diag}[\mathbf{1} - \theta\mathbf{h}],\end{aligned}$$

where $\mathbf{Y} \equiv (L_i w_i)_{i\in\mathcal{I}}$ while $W_i(\mathbf{h})$ is defined as follows:

$$W_i(\mathbf{h}) = L_i w_i - \sum_{j\in\mathcal{I}} \frac{L_i w_i^{1-\sigma}\phi_{ij}}{\sum_{k\in\mathcal{I}} L_k w_k^{1-\sigma}\phi_{kj}} L_j w_j.$$

Note that the wage equation (5) is equivalent to $W_i(\mathbf{h}) = 0$ for all $i \in \mathcal{I}$.

When $\mathbf{h} = \mathbf{h}_m$, the Jacobian matrix $\nabla\mathbf{F}(\mathbf{h})$ can be written as follows:

$$\left(\frac{\partial F_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = h_0\left(\mathbf{I} - h_0\mathbf{E}\right)\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - h_0 v_0(\mathbf{h}_m)\mathbf{E},$$

where

$$\begin{aligned}\left(\frac{\partial v_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = \operatorname{diag}\left[(v_i(\mathbf{h}_m))_{i\in\mathcal{I}_m}\right]\cdot\Bigg\{&\frac{1}{\sigma-1}\frac{1-\theta h_0}{L_0}\mathbf{D}_m \\ &+\frac{1}{w_0}(\mathbf{I}-\mathbf{D}_m)\left(\frac{\partial w_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} - \frac{\theta h_0}{2L_0}\mathbf{I}\Bigg\}\end{aligned}$$

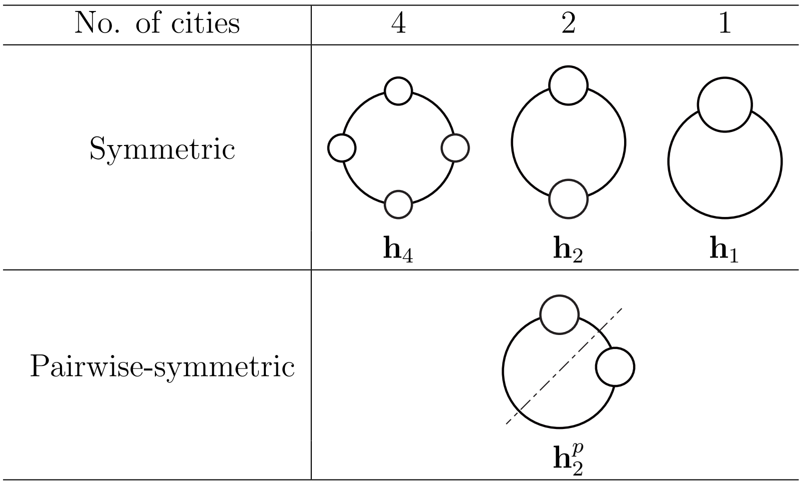
and $\mathbf{D}_m \equiv (\phi_{ij}/\Phi_0)_{i,j\in\mathcal{I}_m}$.

Since $\partial W_i(\mathbf{h}_m)/\partial h_j = 0$ for all $j \in \mathcal{I}_0$, we have:

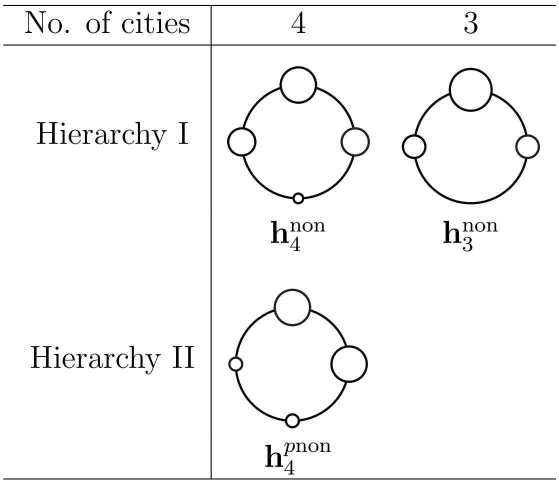
$$\left(\frac{\partial w_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} = -\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial w_j}\right)_{i,j\in\mathcal{I}_m}^{-1}\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m},$$

where

$$\begin{aligned}\left(\frac{\partial W_i(\mathbf{h}_m)}{\partial w_j}\right)_{i,j\in\mathcal{I}_m} &= L_0(\mathbf{I}-\mathbf{D}_m)\left\{\sigma\mathbf{I} + (\sigma-1)\mathbf{D}_m\right\}, \\ \left(\frac{\partial W_i(\mathbf{h}_m)}{\partial h_j}\right)_{i,j\in\mathcal{I}_m} &= -w_0(1-\theta h_0)(\mathbf{I}-\mathbf{D}_m)\mathbf{D}_m.\end{aligned}$$

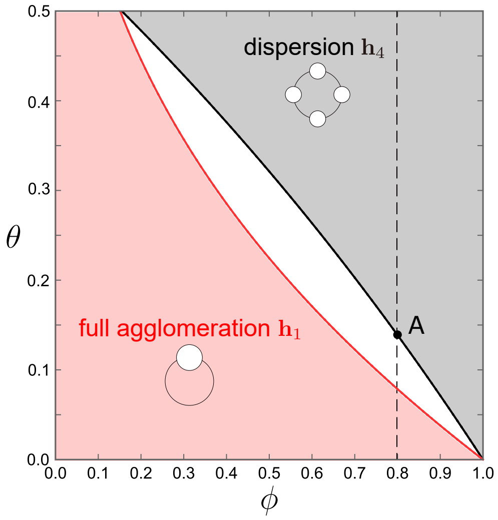


(a) Invariant patterns

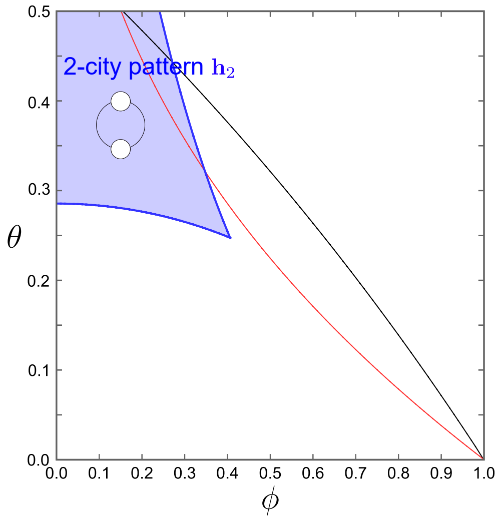


(b) Non-invariant patterns

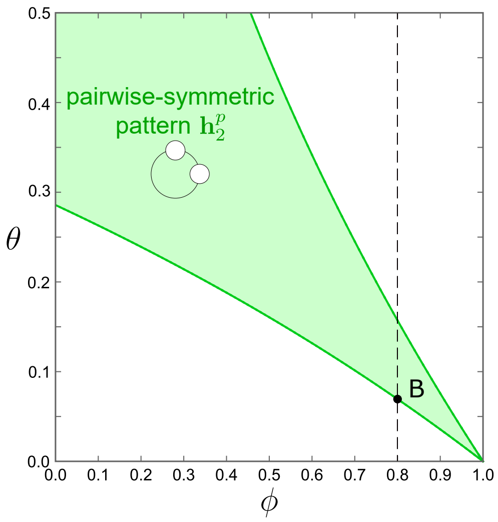
Figure 1: Invariant patterns for the racetrack economy with 4 locations
(A larger circle expresses a city with larger population.)



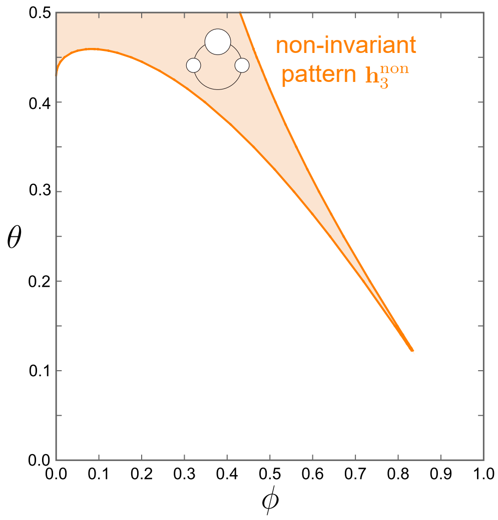
(a) Symmetric invariant patterns I



(b) Symmetric invariant patterns II

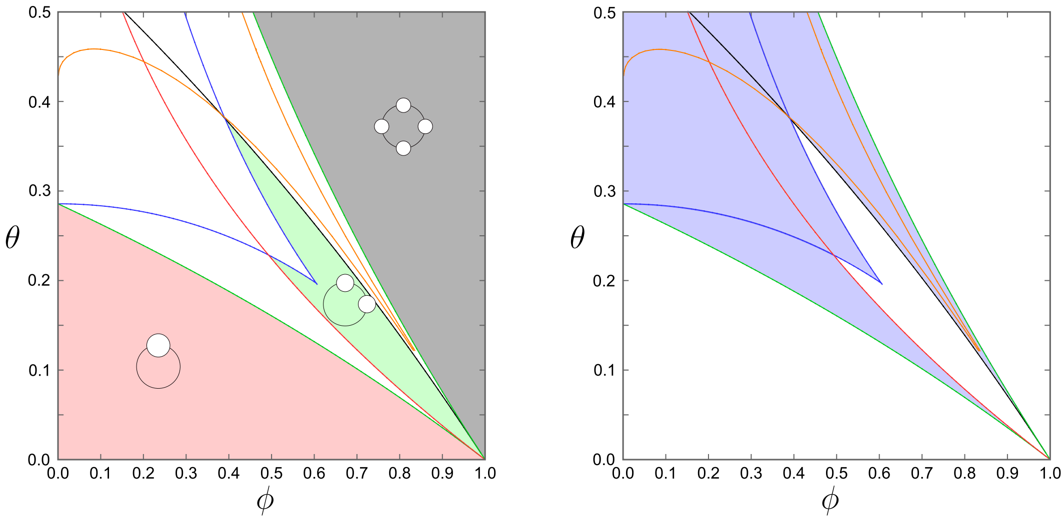


(c) Pairwise-symmetric invariant pattern



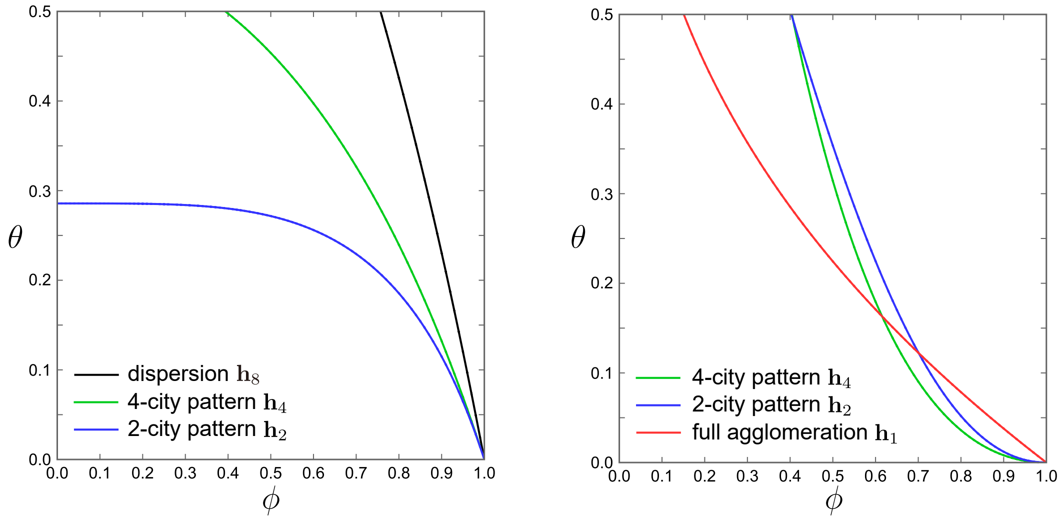
(d) Non-invariant pattern

Figure 2: Stability areas of (ϕ, θ) for the five patterns for 4 locations ($\sigma = 6.0$)



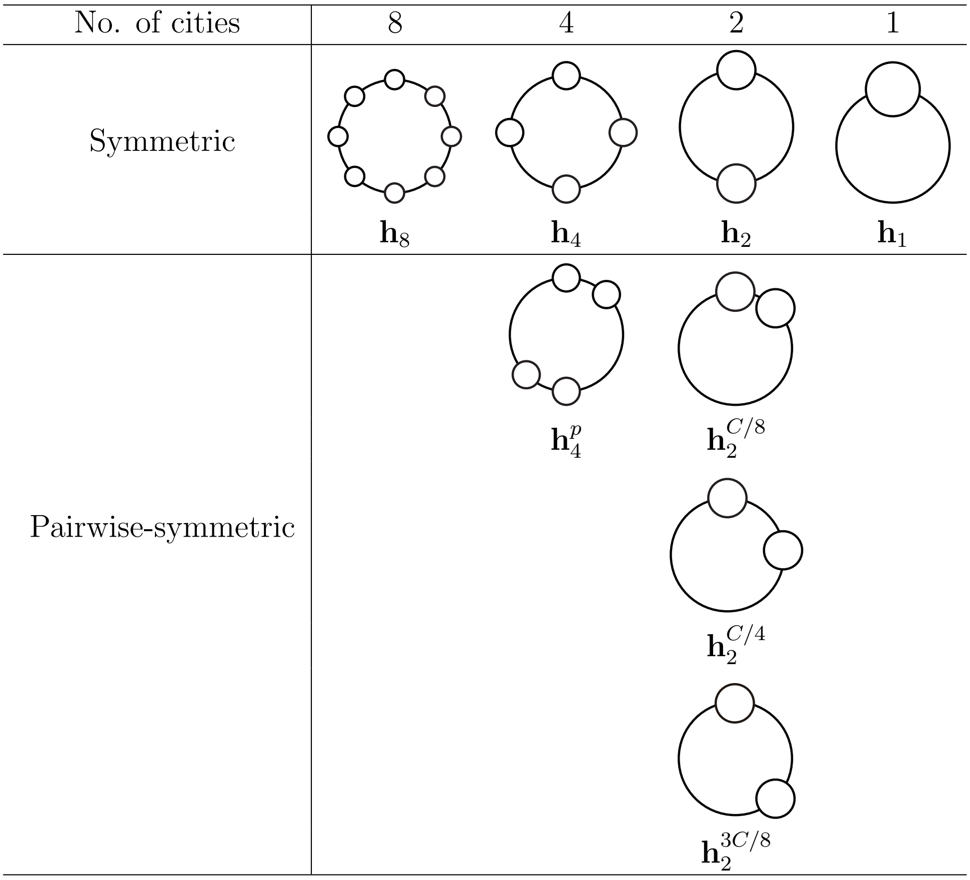
(a) Unique stable equilibrium (b) Multiple stable equilibria

Figure 3: Zoning of (ϕ, θ) based on the multiplicity of stable equilibria for 4 locations ($\sigma = 6.0$)

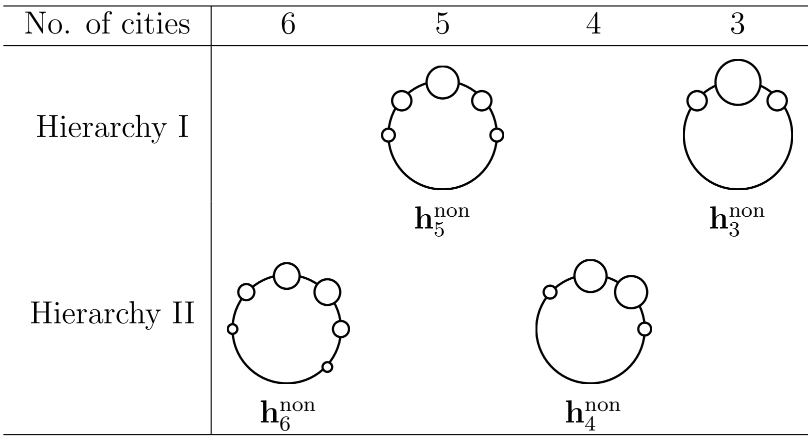


(a) Break points (b) Sustain points

Figure 4: Break and sustain points for symmetric invariant patterns for 8 locations ($\sigma = 6.0$)

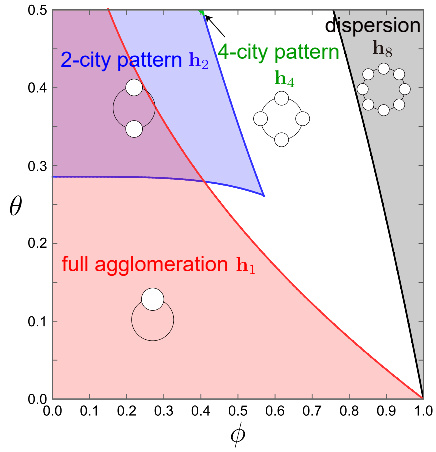


(a) Invariant patterns

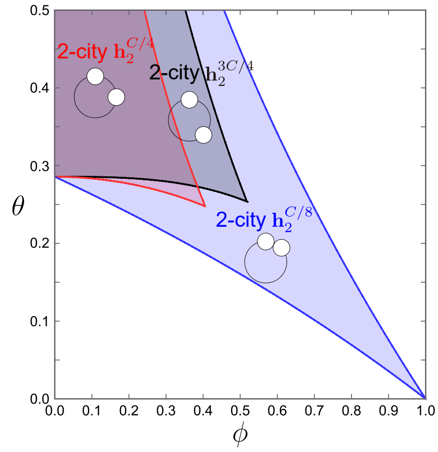


(b) Non-invariant patterns (hierarchical patterns)

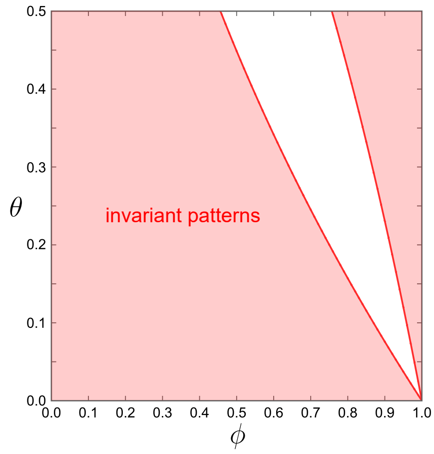
Figure 5: Invariant and non-invariant patterns for the racetrack economy with 8 locations (A larger circle expresses a city with larger population.)



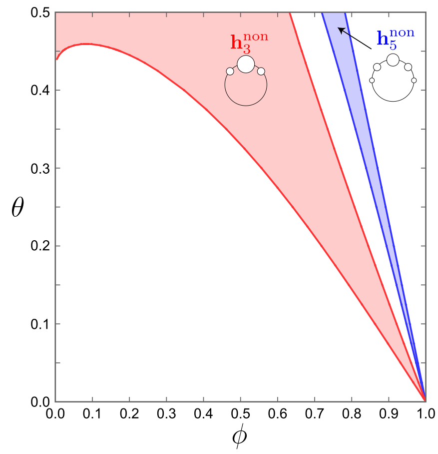
(a) Symmetric invariant patterns



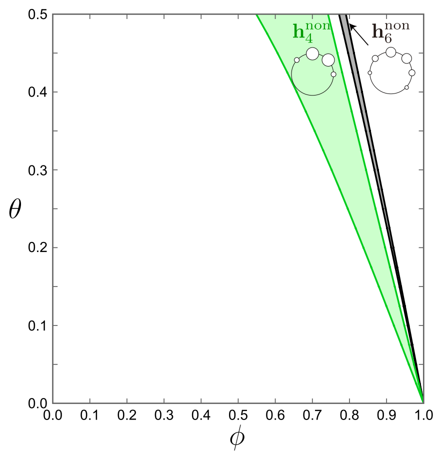
(b) Pairwise-symmetric invariant patterns



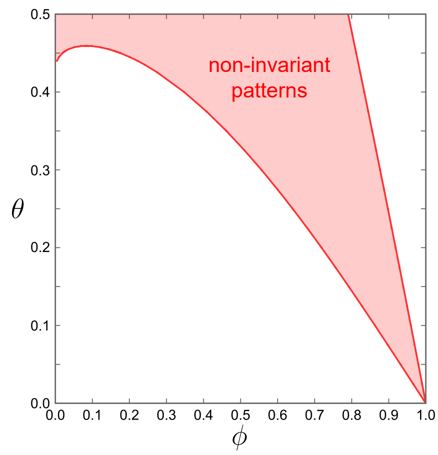
(c) Area with at least one stable invariant pattern



(d) Hierarchy I patterns

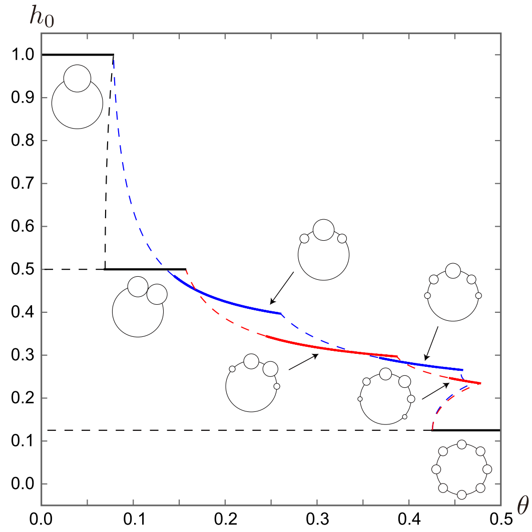


(e) Hierarchy II patterns

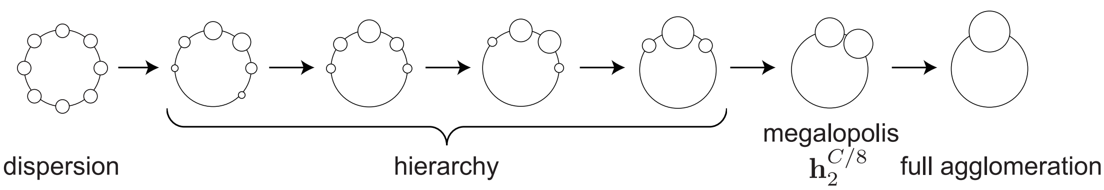


(f) Area with at least one stable non-invariant pattern

Figure 6: Stability areas of patterns of interest in the parameter space (ϕ, θ) for 8 locations ($\sigma = 6.0$)



(a) Equilibrium curves



(b) Transition of stable agglomeration patterns when θ decreases

Figure 7: Equilibrium path for $0 < \theta < 1$ ($\phi = 0.8, \sigma = 6.0$)
(solid lines: stable equilibria, dashed lines: unstable equilibria)