DISCUSSION PAPER SERIES

DP14722

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LABOUR ECONOMICS

ISSN 0265-8003

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Discussion Paper DP14722
Published 06 May 2020
Submitted 04 May 2020

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## Abstract

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JEL Classification: C73, J30

Keywords: Relational Contracts, private savings



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# Relational Contracts: Public versus Private Savings*

**Francesc Dilmé**[†] **Daniel Garrett**[‡]

May 4, 2020

**Abstract**

We study relational contracting with an agent who has consumption-smoothing preferences as well as the ability to save to defer consumption (or to borrow). Our focus is the comparison of principal-optimal relational contracts in two settings. The first where the agent's consumption and savings decisions are private, and the second where these decisions are publicly observed. In the first case, the agent smooths his consumption over time, the agent's effort and payments eventually decrease with time, and the balances on his savings account eventually increase. In the second, the agent consumes earlier than he would like, consumption and the balance on savings fall over time, and effort and payments to the agent increase. Our results suggest a possible explanation for low savings rates in certain industries where compensation often comes in the form of discretionary payments.

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*We are grateful for helpful comments from Nageeb Ali, Daniel Bird, Mathias Fahn, William Fuchs, Drew Fudenberg, Chris Shannon, Andy Skrzypacz, Rani Spiegler, Philipp Strack, Balazs Szentes, Jean Tirole, Marta Troya Martinez, Mark Voorneveld, and seminar participants at Barcelona Graduate School of Economics, Berkeley, Boston University, Hebrew University of Jerusalem, Higher School of Economics, Stockholm School of Economics, Tel Aviv University, University of Bonn, University Pompeu Fabra, the 2nd Japanese-German Workshop on Contracts and Incentives in Munich, the 5th Workshop on Relational Contracts in Madrid, the Kent Bristol City Workshop in Economic Theory, and the Stony Brook International Conference on Game Theory. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 714147). Dilmé thanks the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) for research support through grant CRC TR 224.

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# 1 Introduction

Early literature on repeated moral hazard with a risk-averse agent — for instance Rogerson (1985) and Fudenberg, Holmstrom, and Milgrom (1990) — highlighted the value to the principal of controlling the agent's consumption or savings through formal contracts. Optimal dynamic contracts as studied by Rogerson force the agent to consume more than he would like early in the relationship (the agent would gain by secretly saving and deferring some consumption to a later date).[1] Fudenberg, Holmstrom, and Milgrom show that a sequence of short-term contracts can often implement the outcome of a long-term contract if the short-term contracts can stipulate agent consumption or savings. Yet, in modern employment relationships, workers' consumption expenditures are largely at their discretion and free from formal agreements.

Nonetheless, some employers may be able to monitor, at least to a degree, the consumption and savings decisions of workers. "Conspicuous consumption" decisions by workers include choices of clothes, car, or leisure activities.[2] Some savings decisions (such as pension contributions, student loan repayments or repayments of employer-offered mortgages) might be observed as direct deductions from employee paychecks.

In this paper, we ask how the evolution of employment relationships can be expected to depend on the observability of consumption and savings decisions when formal agreements on these decisions are ruled out, but where *relational* incentives might nonetheless exist. The idea that many productive relationships rest on goodwill and implicit agreements, together with the implicit threat of future punishments for deviations, has been developed following work such as Bull (1987), MacLeod and Malcomson (1989) and Levin (2003). While such agreements often concern both the level of output delivered to the principal, as well as agent pay (or bonuses), we introduce the possibility that they also concern the level of agent consumption, to the extent this is jointly observed. Implicit understandings on appropriate consumption levels could reflect part of a workplace's culture, sustained through repeated interaction with the employer. For instance, one possibility is that an employee with an insufficiently frivolous lifestyle (say in terms of car, dress, and leisure activities) is dismissed, having been deemed a "poor fit" with the firm's culture.

We consider a simple relational contracting model in which the agent exerts costly effort and has concave utility from consumption in each period. Concave utility implies a preference for smooth consumption. We study two polar opposite cases: one where consumption is

[1] Other papers where the principal controls the level of agent consumption include Lambert (1983), Spear and Srivastava (1987), Rey and Salanie (1990), Phelan and Townsend (1991), Sannikov (2008) and Garrett and Pavan (2015).

[2] The term "conspicuous consumption" was introduced by Veblen (1899).

perfectly jointly observed, and the other where consumption is unobservable to the principal. Although reality might often lie between these two extremes, studying each separately is illuminative and simplifies the analysis. Note that both of these cases are new to the literature: existing literature has not sought to characterize optimal *relational* contracts when agents have consumption-smoothing preferences and can save.

Our simple setting is deterministic, with the output enjoyed by the principal equal to the agent's effort. In each period, first the agent chooses effort and consumption, then the principal makes a (bonus) payment to the agent. The initial balance on the agent's savings account is common knowledge, and the evolution of these savings are determined by pay and consumption. A relational contract can be understood as an agreed stream of effort, consumption and pay.

As elsewhere in the relational contracting literature, we focus on agreements that are "self-enforcing". We thus impose two key sets of constraints that ensure neither party wishes to publicly breach the contract, given that any such breach results in termination of a productive relationship (i.e., no further effort by the agent, and no further payments from the principal).[3] One set of constraints ensures willingness of the agent never to quit the relationship by deviating from the agreed effort and consumption, if observed, and then continuing to optimally consume his savings. The other ensures credibility of the principal's payments to the agent. That is, it ensures that the principal's discounted future profits from continuing the relationship always exceed the payments promised to the agent. We investigate self-enforcing agreements that maximize the principal's discounted profits.

We begin by studying the problem with unobservable consumption. In this case, any relational contract must involve constant consumption over time, as constant consumption maximizes the agent's payoff. If the agent plans to deviate in effort at a given date, he optimally smooths his reduced lifetime income from the beginning by choosing constant consumption. Since lifetime earnings accumulate the longer the agent obediently exerts effort, and since agent utility is concave, the pay needed to keep the agent from deviating from a given level of effort increases with time.

When the principal and agent are sufficiently patient, the effort in an optimal contract is constant over time. For lower levels of patience, the agent's effort is eventually decreasing with time. The reason is that the payment required to compensate a given level of effort gradually increases, for the reason explained above. This means that the contract becomes gradually less profitable, making it more difficult to sustain agreement at later dates. We thus show that the payments that the principal can credibly promise eventually decline, and so effort

[3]Termination of a productive relationship represents an optimal punishment in the sense that it is as severe on both players as possible.

decreases as well.

The above dynamics stand in contrast to the case when consumption is instead jointly observed. Note that the possibility for the informal agreement to condition on consumption enlarges the set of sustainable outcomes for two reasons. One is that non-constant consumption paths become possible, due to relational incentives on the level of consumption. The other is that, if the agent plans to deviate from the specified effort at a given date, he cannot lower his consumption at earlier dates without this being detected, triggering premature termination of his pay. This effectively reduces the profitability for the agent of deviations from the agreed effort (compared to what he could obtain were he able to reduce consumption at earlier dates and continue being paid).

In terms of the dynamics of an optimal relational contract when consumption is jointly observed, we find the following. When the players are sufficiently patient, the optimal policy again involves constant effort. For lower discount factors, agent effort strictly increases with time. The reason can be understood by noting that the optimal contract stipulates high consumption for the agent early in the relationship, which drives down the balance on his account. Since the balance on his savings decreases, the agent needs to be paid less at later dates to keep him willing to work. Since the principal's profits in the relationship grow with time, the pay she can credibly offer increases, and so the contract can call for higher effort.

The key to understanding why such dynamics are optimal in the setting with observed consumption lies in recognizing the value of increasing the profitability of the relationship at later dates, which in turn relaxes the principal's credibility constraint and permits high pay early on. As noted, the channel for increasing future profits is to induce high consumption by the agent, reducing the balance on his account and making him more willing to work. We show that this process continues indefinitely, with the contract approaching stationarity in the long run. While these dynamics might be thought of as a form of "immiseration" (the agent's payoff falls throughout the relationship), they are borne out of the principal's inability to commit, rather than being sourced in information frictions as with the immiseration results of Thomas and Worrall (1990) and Atkeson and Lucas Jr (1992).[4]

As mentioned above, in real-world relationships, not all of an agent's consumption expenditures are likely to be observable to the principal. Nonetheless, our analysis of fully observable consumption suggests a force that could perhaps be relevant in some settings. In particular, our results suggest a theory of high consumption and low savings that favors the profits of the principal. The idea could be relevant in certain industries such as banking, where high remuneration (often through discretionary bonuses) is accompanied by a propensity for high spending. Such a propensity seems widely acknowledged: Former British banker Geraint An-

[4] See also Williams (2011).

derson has commented that bankers are essentially trapped in a culture of high consumption, adding that life as a banker is "like a gilded cage".[5] Gary Goldstein, co-founder of executive search firm Whitney Partners, has commented in the context of high-spending high earners that:[6]

> "It's really not that unusual to find Wall Street bankers who are close to declaring themselves bankrupt.... Some people are really struggling."

An article at *Huffpost Business* explains that bankers "are under constant social pressure to spend and spend some more".[7] This kind of behavior would increase the willingness of bankers to continue devoting long hours, which seems advantagenous to firms and might even be encouraged by them. The idea that the behavior reflects cultural norms, that may be sustained through repeated interaction, offers an alternative to the possibility that high consumption is simply about signaling.

A further source of evidence on pressure to spend may come from supply arrangements, especially those managed by large firms. One possibility is that larger firms push for expenditures by suppliers that harm their financial positions, perhaps making them more financially dependent. A possible example comes from the US poultry industry, where chicken farmers supply to a few large firms that dominate the industry. An article at *The Guardian* documented the situation of a farmer that contracted with the chicken producer Tyson Foods.[8] The farmer in question entered an exclusive agreement with Tyson. After some time, Tyson began demanding additional expenditures on equipment such as extra feed bins and chicken houses the farmer believed unnecessary. The farmer commented: "If we are independent contractors, then why does the company have the right to tell us what equipment to use?" Having failed to comply with the demands, the relationship deteriorated and, in the end, it was terminated. Our theory seems consistent with this kind of story, although in our model the agent's expenditures are on consumption rather than improving productivity.

The rest of this paper is as follows. Next, in Section 1.1, we provide a summary of relevant literature. Section 2 introduces a model, Section 3 considers first-best contracts, Section 4 considers relational contracts when consumption is unobservable to the principal, and Section

[5] 'The perils of earning a £100,000 salary' by Jon Kelly, *BBC News Magazine*, 22 September 2010, https://www.bbc.com/news/magazine-11382591.

[6] 'When a million isn't enough: why top bankers are struggling to get by' by Sarah Butcher, *efinancialcareers*, 29 April 2013, https://news.efinancialcareers.com/uk-en/140070/when-a-million-isnt-enough-why-top-bankers-are-struggling-to-get-by/.

[7] 'Bankers Explain How They Cannot Possibly Live On $1 Million Pay' by Mark Gongloff, *Huffpost Business*, 1 May 2013, https://www.huffpost.com/entry/bankers-1-million-pay_n_3188177.

[8] 'Fowl play: the chicken farmers being bullied by big poultry', by Alison Moodie, *The Guardian*, 22 April 2017, https://www.theguardian.com/sustainable-business/2017/apr/22/chicken-farmers-big-poultry-rules.

5 considers relational contracts where consumption is observable. While the results in Sections 3 to 5 suppose the agent can save, Section 6 determines the optimal relational contract when the agent cannot save and so consumes what he is paid in every period. Section 7 concludes.

## 1.1 Other literature

This paper contributes to the literature on relational contracts, reviewed in MacLeod (2007) and Malcomson (2015). While much of this literature is interested in settings with moral hazard, the players most often have linear preferences over money and savings plays no role. Some examples include Levin (2003), Fuchs (2007), Chassang (2010), Halac (2012), Li and Matouschek (2013), Yang (2013), Malcomson (2016) and Fong and Li (2017). Also unlike our paper, much of the interest in these works lies in the role of exogenous uncertainty, which is often a source of private information for one of the parties.

Our model features concave agent utility over payments, giving the agent a preference for smooth consumption. We also allow the agent to save and dissave. The role of consumption-smoothing preferences and saving has been given little attention in the relational contracting literature to date. Pearce and Stacchetti (1998) consider a relational contracting model with a risk-averse agent, but there is no scope for the agent to save. Bull (1987) considers a model with overlapping generations of agents with concave utility of consumption and a long-lived principal where the agents can privately save. Yet his concern is only with the conditions under which the first-best outcome can be sustained.

Apart from models with productive effort, there are settings with lending and insurance. Thomas and Worrall (1990) consider a model where the principal insures a risk-averse agent. The paper devotes some attention to self-enforcing insurance arrangements (thus relaxing insurer commitment in the spirit of relational contracting); however, the agent is not permitted to save by himself. Thomas and Worrall (1994) study the problem of an investor in an agent (country) that can steal the invested capital and walk away from the relationship. The setting is relational (neither party can commit), and they consider separately cases where the capital of the agent can accumulate (arguably akin to saving), and where the agent is risk averse.

An important driver of dynamics in our model is evolution of the agent's outside option of ceasing productive effort and "living off" the balance on his account. The role of the value of the agent's outside option in repeated relationships has been emphasized in work such as Baker, Gibbons, and Murphy (1994) and McAdams (2011). In Baker, Gibbons, and Murphy, the outside option is endogenously determined by the possibility of contracting on objective performance measures, while in McAdams (2011) it is endogenously determined by the opportunities of partners to a relationship to rematch. In general, the higher the outside options of the parties to a relationship, the harder it is for a productive relationship to be sustained. In some papers, as in ours, the outside option evolves dynamically. One example is Thomas and Worrall (1994) model of capital accumulation mentioned above, where the value to the agent of departing with the accumulated capital increases with time. Another is Garicano and Rayo (2017), where the agent is paid by increments in productive knowledge which increases the value of his outside option.[9]

Our paper is also related to work on dynamic moral hazard where the agent can privately save. Examples include Edmans, Gabaix, Sadzik, and Sannikov (2012), Abraham, Koehne, and Pavoni (2011), He (2012), and Di Tella and Sannikov (2016). In this work, the principal has full commitment power. While these papers often focus on contracts that give the agent no incentive to privately save, there are many ways to time payments in an optimal contract. The timing of payments plays a more important role in our model where (given that the principal cannot commit) it affects whether the principal is willing to continue abiding by the contract. For this reason, our theory more readily generates predictions on how payments are optimally spread across time (this is true in both the public and private savings versions of our model).

# 2 Setting

**Environment and preferences.** A principal and agent meet in discrete time at dates $t = 1, 2, \ldots$. Letting $r > 0$ be the interest rate that will apply to the balance on the agent's savings account, we suppose the players have a common discount factor $\delta = \frac{1}{1+r}$. In every period $t$, first the agent exerts an effort $e_t$ and consumes an amount $c_t$. Then, the principal makes a discretionary payment $w_t$ to the agent. These variables are all restricted to be non-negative.

The agent has initial savings balance $b_1 > 0$ as well as access to a savings technology (with the interest rate $r$ as specified above). The initial balance will be common knowledge between the principal and agent, including in our model of private savings in Section 4. The agent's balance at time $t + 1 > 1$ then satisfies

$$b_{t+1} = \frac{b_t + w_t - c_t}{\delta} = b_1 \delta^{-t} + \sum_{s=1}^{t} \delta^{s-t-1} (w_s - c_s) . \quad (1)$$

Balances can, in principle, be negative (i.e., the agent can borrow). We say that the agent's

[9] Also related is Fudenberg and Rayo (2019); the agent's outside option improves over time also due to the accumulation of knowledge, but the paper focuses on the case where the principal can commit to a long-term contract.

intertemporal budget constraint is satisfied in case

$$\sum_{t=1}^{\infty} \delta^{t-1} c_t \leq b_1 + \sum_{t=1}^{\infty} \delta^{t-1} w_t. \tag{2}$$

The agent's felicity from consumption $c_t$ in any period $t$ is denoted $v(c_t)$, where $v : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. We assume that $v(c)$ is real-valued for $c > 0$, and takes value $-\infty$ at $c = 0$. We further assume that $v$, when evaluated on positive consumption values, is twice continuously differentiable, strictly increasing and strictly concave. In addition, $v$ is onto all of $\mathbb{R}$, implying $\lim_{c \searrow 0} v(c) = -\infty$.

The agent's disutility of effort $e_t$ is $\psi(e_t)$. We assume that $\psi$ is continuously differentiable, strictly increasing, strictly convex, and such that $\psi(0) = \psi'(0) = 0$, and that $\lim_{e \to \infty} \psi'(e) = \infty$.

The agent's period-$t$ payoff will be $v(c_t) - \psi(e_t)$, while the principal's will be $e_t - w_t$; hence, we interpret effort as equal to the output enjoyed by the principal.

**Relational contracts.** We focus for tractability on *deterministic* relational contracts.[10] We identify relational contracts with their outcomes; denote them $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t \geq 1}$. We restrict attention to contracts that satisfy the following feasibility constraints.

**Definition 2.1.** A *feasible relational contract* is a sequence $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t \geq 1}$ satisfying the following conditions:

1. **Positivity:** $\tilde{e}_t, \tilde{w}_t, \tilde{c}_t \geq 0$ for all $t$.

2. **Balance dynamics and constraint:** Conditions (1) and (2) hold.

3. **Bounded consumption:** The sequences of consumption, pay and effort $((\tilde{c}_t)_{t \geq 1}, (\tilde{w}_t)_{t \geq 1},$ and $(\tilde{e}_t)_{t \geq 1})$ are bounded.

While the first and second conditions reflect features of the environment introduced above, the third condition guarantees that the players' payoffs are well-defined in a feasible contract.

# 3 First best and principal full commitment

Consider first the problem of maximizing the principal's payoff by choice of a feasible relational contract subject only to the constraint that the agent is initially willing to participate. If the

[10] This assumption is in common with some other work such as Ray (2002). In examining contracts that are optimal for the principal, whether random contracts can improve on deterministic ones might be expected to depend on the nature of risk aversion (e.g., whether $v$ exhibits increasing or decreasing risk aversion). Our results below, however, will hold irrespective of how risk preferences change with the level of consumption.

agent does not participate, a possibility we describe as "autarky", we stipulate that he consumes $(1-\delta)b_1$ per period. This is the optimal consumption for the agent among consumption streams satisfying the intertemporal budget constraint in Equation (2) given that all payments are set to zero. Therefore, we consider maximizing the principal's payoff over feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ such that the payoff of the agent

$$\sum_{t=1}^{\infty} \delta^{t-1} \left( v(\tilde{c}_t) - \psi\left(\tilde{e}_t\right)\right) \tag{3}$$

is no lower than his autarky value, $\frac{1}{1-\delta} v((1-\delta)b_1)$.

**Proposition 3.1.** *Consider maximizing the principal's discounted payoff by choice of feasible contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$, subject to ensuring the agent a payoff at least his autarky value $\frac{1}{1-\delta} v((1-\delta)b_1)$. In any optimal feasible contract, effort and consumption are constant at $e^{FB}(b_1) > 0$ and $c^{FB}(b_1) > (1-\delta)\, b_1$, respectively, being the unique solutions to:*

1. ***First order condition:*** *$\psi'(e^{FB}(b_1)) = v'(c^{FB}(b_1))$, and*
2. ***Agent's indifference condition:*** *$v(c^{FB}(b_1)) - \psi(e^{FB}(b_1)) = v((1-\delta)b_1)$.*

*Furthermore, the payoff of the principal is $V^{FB}(b_1) \equiv \frac{1}{1-\delta}(e^{FB}(b_1) - (c^{FB}(b_1) - (1-\delta)b_1))$, which is a strictly decreasing function of $b_1$.*

The results in the proposition are easily anticipated. Given that $v$ is concave, it is optimal to prescribe constant consumption. Similarly, the convexity of the disutility of effort implies the optimality of constant effort. At an optimum, the agent is indifferent between participating in the contract and autarky.

It is worth observing that the first-best policies depend on both $b_1$ and $\delta$, although we reduce the notational burden by making dependence only on $b_1$ explicit. More specifically, they depend on the value of autarky consumption $(1-\delta)b_1$, as is clear from Condition 2 of Proposition 3.1.

Note that the proposition does not specify the timing of payments. These payments must be chosen to be feasible and to satisfy the agent's budget constraint (2) with equality, but these are the only requirements. Payments may be constant over time, in which case they are equal to $c^{FB}(b_1) - (1-\delta)b_1$ in each period.

# 4 Unobservable consumption

This section studies the case where, at each date $t$, the principal can observe the previous and current effort choices of the agent $(e_s)_{s=1}^{t}$, but not the consumption choices nor the agent's balance. Given that players cannot commit to future actions, the relational contracting literature has studied agreements termed "self-enforcing". In our setting, we consider feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ which coincide with the outcomes of a *perfect Bayesian equilibrium* (PBE) of a dynamic game. Our aim will be to characterize the relational contracts among these that maximize the principal's payoff.

We begin by defining the histories in our game. For $t \geq 0$, a $t$-history for the agent is $h_t^A = (e_s, w_s, c_s)_{1\leq s<t}$, which gives the observed effort, payments and consumption up until time $t-1$. The set of such histories at date $t \geq 1$ is $\mathcal{H}_t^A = \mathbb{R}_+^{3(t-1)}$ (with the convention that $\mathbb{R}_+^0 = \emptyset$). Note that, given $h_t^A$ and the agent's initial balance $b_1$, we can completely determine the evolution of the balance up to date $t$ using Equation (1). We denote the date-$t$ balance by $b(h_t^A)$. A $t$-history for the principal is $h_t^P = (e_s, w_s)_{1\leq s<t}$. The set of such histories at date $t \geq 1$ is $\mathcal{H}_t^P = \mathbb{R}_+^{2(t-1)}$.

A strategy for the agent is then a collection of functions

$$\alpha_t : \mathcal{H}_t^A \to \mathbb{R}_+^2,\ t \geq 1,$$

and a strategy for the principal is a collection of functions

$$\sigma_t : \mathcal{H}_t^P \times \mathbb{R}_+ \to \mathbb{R}_+,\ t \geq 1.$$

Here, $\alpha_t$ maps the $t$-history of the agent to a pair $(e_t, c_t)$ of effort and consumption. Also, $\sigma_t$ maps the $t$-history of the principal, together with the agent's effort choice $e_t$, to a payment $w_t$.

As noted above, we will restrict attention to equilibria whose outcomes coincide with a feasible relational contract. However, we do not restrict the strategies that are available to the players. Certain strategies imply, for instance, the violation of the agent's intertemporal budget constraint in Equation (2). To ensure that the agent finds it optimal to satisfy this constraint, we make the following assumption on payoffs. While the principal's payoff is as specified above (and so given by $\sum_{t=1}^{\infty} \delta^{t-1} (e_t - w_t)$), the agent obtains the payoff $\sum_{t=1}^{\infty} \delta^{t-1} (v(c_t) - \psi(e_t))$ if the constraint in Equation (2) is satisfied, and obtains payoff $-\infty$ otherwise.[11]

To obtain the set of feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that are PBE outcomes, we consider PBE where publicly observed deviations from the agreed outcomes are punished by

[11] Alternative assumptions can be made which yield the same results as documented below. For instance, another possibility involves permitting negative consumption (assigning it a value $-\infty$ in the agent's payoff), but limiting the extent the agent can draw down the balance on his account (i.e., imposing a hard lower bound on $b_t - c_t$). Allowing the agent to violate his intertemporal budget constraint, but then assigning payoff $-\infty$, appears the simplest assumption that ensures the agent chooses to satisfy the constraint (both on the equilibrium path as well as after a deviation).

"autarky". This means that, if the agent deviates from the agreed effort $\tilde{e}_t$, or if the principal deviates from the agreed payment $\tilde{w}_t$, the principal makes no payments and the agent exerts no effort from then on; the agent perfectly smoothing the balance of his account over the infinite future.[12] If the agent's balance is negative when autarky begins, the intertemporal budget constraint in Equation (2) is necessarily violated (as the agent receives no further payments), the agent must earn payoff $-\infty$, and so we can specify for instance that the agent consumes zero in every period. Note that deviations by the agent from the specified consumption, provided they are not accompanied by any deviation in effort, go unpunished (i.e., the principal continues to adhere to the payments specified by the agreement).

Note that, if the agent plans to always choose effort in accordance with the contract, he optimally consumes

$$\bar{c}_\infty \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1}\tilde{w}_s\right)$$

in every period. Clearly, any contract to which the agent is willing to adhere must then specify $\tilde{c}_t = \bar{c}_\infty$ for all $t$. To conclude that the agent does not want to deviate from the contract, it is then enough to show that he does not gain by planning to shirk on effort for the first time at any given date $t$, while making all other choices optimally. Suppose then that the agent plans to shirk for the first time at some date $t$, and so puts effort equal to $\tilde{e}_s$ for all $s < t$, and then optimally sets it equal to zero at all later dates. Then the agent optimally sets consumption equal to

$$\bar{c}_{t-1} \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{t-1} \delta^{s-1}\tilde{w}_s\right) \tag{4}$$

at all dates, so as to completely smooth consumption and exhaust lifetime earnings.

Given the above, the maximum payoff the agent achieves when deviating in choice of effort for the first time at date $t$ is

$$\frac{1}{1-\delta}v(\bar{c}_{t-1}) - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s).$$

Hence, the agent does not want to deviate from the agreement if and only if, for all $t \geq 1$,

$$\frac{1}{1-\delta}v(\bar{c}_{t-1}) - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s) \leq \frac{1}{1-\delta}v(\bar{c}_\infty) - \sum_{s=1}^{\infty} \delta^{s-1}\psi(\tilde{e}_s). \tag{$\mathrm{AC}_t^{\mathrm{un}}$}$$

[12] The reason we can consider autarky punishments is that they deliver the lowest possible individually rational payoffs for the players. For instance, if the agent has a positive balance on his account, then he can always secure the autarky payoff simply by exerting no effort and consuming a constant amount in each period that exhausts his budget, i.e. which ensures that (2) holds with equality. The principal can always obtain a continuation payoff zero simply by making no payments.

The principal remains willing to continue abiding by the agreement if and only if, at each time $t$, the payment $\tilde{w}_t$ that is due is less than her continuation payoff in the agreement. The exact requirement is that, for all $t \geq 1$,

$$\tilde{w}_t \leq \sum_{s=t+1}^{\infty} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s). \tag{PC$_t$}$$

The following result states that the above constraints determine whether a feasible relational contract is the outcome of a PBE.

**Proposition 4.1.** *Fix a feasible contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. *It is the outcome of a PBE if and only if, for all* $t \geq 1$, *Conditions* ($\mathrm{AC}_t^{\mathrm{un}}$) *and* ($\mathrm{PC}_t$) *are satisfied, and* $\tilde{c}_t = \bar{c}_\infty$.

From now on, inspired by the terminology of the literature, we refer to a contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that satisfies the conditions of Proposition 4.1 as *self-enforceable.*[13] Our task reduces to characterizing feasible contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that maximize the principal's payoff subject to the requirement of being self-enforceable. We term such contracts *optimal.*

To determine the properties of optimal contracts, we first show that we can restrict attention to contracts with a particular pattern of payments over time. This pattern involves paying the agent as early as possible, subject to satisfying the agent's incentive constraints. This requires that the agent's obedience constraints in Condition ($\mathrm{AC}_t^{\mathrm{un}}$) hold with equality for all $t \geq 1$. Inspired by the terminology of Board (2011), we refer to this condition as "fastest payments". We show the following.

**Lemma 4.1.** *If there is an optimal relational contract, then there is another optimal contract* $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ *with the same sequence of efforts and consumption such that, for all* $t \geq 1$,

$$\frac{v\left(\bar{c}_{t-1}\right)}{1-\delta} - \sum_{s=1}^{t-1} \delta^{s-1}\psi(\tilde{e}_s) = \frac{v\left((1-\delta)\, b_1\right)}{1-\delta}. \tag{FP$_t^{\mathrm{un}}$}$$

An explanation for the result is as follows. First, note that it is optimal to hold the agent to his outside option, and hence

$$\frac{v\left(\bar{c}_{\infty}\right)}{1-\delta} - \sum_{t=1}^{\infty} \delta^{t-1}\psi(\tilde{e}_t) = \frac{v\left((1-\delta)\, b_1\right)}{1-\delta}. \tag{5}$$

If Condition (5) does not hold, $\tilde{e}_1$ can be slightly increased (keeping the rest of the contract the same) so that the constraints ($\mathrm{AC}_t^{\mathrm{un}}$) and ($\mathrm{PC}_t$) continue to hold for all $t$. Second, when

[13] We think this terminology is appropriate since we identify a contract with the equilibrium outcomes, rather than the complete specification of strategies (and beliefs). Outcomes being self-enforceable simply means that they can be supported in equilibrium.

($\text{FP}_t^{\text{un}}$) holds for all $t$, the agent is paid as early as possible while preserving the constraints ($\text{AC}_t^{\text{un}}$). The agent cannot be paid earlier, otherwise he will prefer to work obediently for a certain number of periods, save his income at a higher rate than specified in the agreement, and then quit by exerting no effort. It is easily seen that moving payments earlier in time only relaxes the "principal's constraints" ($\text{PC}_t$).

Concerning "fastest payments", we have the following result.

**Lemma 4.2.** *Consider a feasible relational contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ that satisfies Condition ($\text{FP}_t^{\text{un}}$) at all dates. For any $t$, if $\tilde{e}_t > 0$, then*

$$\tilde{w}_t \in \left( \frac{\psi\left(\tilde{e}_t\right)}{v'\left(\bar{c}_{t-1}\right)}, \frac{\psi\left(\tilde{e}_t\right)}{v'\left(\bar{c}_t\right)} \right). \tag{6}$$

This lemma implies that (assuming effort remains strictly positive) the ratio $\frac{\tilde{w}_t}{\psi(\tilde{e}_t)}$ increases with $t$. The implication is immediate after noticing that $\bar{c}_t$ increases with $t$. One way to intuitively explain the observation is that, the longer the agent obediently works, the more he is paid in total. Since he can smooth his consumption of these payments over his entire lifetime, and since he has concave utility of consumption, he values additional payments less. Therefore, the payments needed to keep the agent obediently in the relationship, relative to the disutility of effort incurred, increase with time. This observation will be useful for understanding the shape of optimal relational contracts below.

Apart from the observation in Lemma 4.2, the usefulness of Lemma 4.1 is that it permits the design of the relational contract to be reduced to the choice of an effort sequence $(\tilde{e}_t)_{t\geq 1}$. From $(\tilde{e}_t)_{t\geq 1}$ we can obtain $(\bar{c}_t)_{t\geq 1}$ using ($\text{FP}_t^{\text{un}}$) (so the corresponding consumption $\tilde{c}_t = \bar{c}_\infty$ is also pinned down). Then $(\tilde{w}_t)_{t\geq 1}$ is obtained from Equation (4), and $(\tilde{b}_t)_{t\geq 1}$ from Equation (1). We next discuss the implementation of first-best contracts (Section 4.1), before moving to consider optimal contracts when there is no first-best contract that is self-enforceable (Section 4.2).

## 4.1 Implementation a first-best contract

Lemma 4.1 is also useful for understanding the conditions under which the principal obtains the first-best payoff. For instance, we can observe that the first-best effort and consumption, which is constant over time and equal to $e^{FB}(b_1)$ and $c^{FB}(b_1)$, can be implemented when the principal can commit to the agreement, but the agent cannot commit. For this, we simply suppose the principal agrees to payments satisfying the conditions in Equation ($\text{FP}_t^{\text{un}}$), provided the agent chooses effort obediently. Any deviation by the agent from the required effort is met with zero payments from then on.

Now consider whether the principal can attain the first-best payoff when neither the principal nor agent can commit; i.e., whether there is a first-best contract that is self-enforceable. According to Lemma 4.2, payments to the agent increase over time. In the long run, payments approach

$$\frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}.$$

Because the principal's constraints ($\text{PC}_t$) tighten over time, verifying they are always satisfied amounts to verifying that

$$\frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)} \leq \frac{\delta}{1-\delta}\left(e^{FB}(b_1) - \frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}\right). \tag{7}$$

The right-hand side is the limiting value of the principal's future discounted profits in the agreement, while the left-hand side is the limiting value of the payment to the agent. Because there is no loss in restricting attention to "fastest payments" (due to Lemma 4.1), this condition is also necessary, and so we have the following result.

**Proposition 4.2.** *Suppose that neither the principal nor agent can commit to the terms of the agreement and that consumption is unobservable. Then the principal attains the first-best payoff in an optimal contract if and only if Condition* (7) *is satisfied.*

While understanding the parameter range for which Condition (7) holds is clearly important for understanding optimal contracts, this is complicated by the dependence of the first-best policy on both $b_1$ and $\delta$. Nonetheless, if we vary $\delta$ while allowing $b_1$ to adjust, holding $b_1(1-\delta)$ constant, then the first-best consumption and effort remain constant. There is then a threshold value of $\delta$ above which Condition (7) is satisfied, and below which it fails.

## 4.2 Main characterization for unobservable consumption

We now state our main result for the unobservable consumption case, which is a characterization of optimal effort when the first-best effort cannot be sustained.

**Proposition 4.3.** *An optimal relational contract exists. Suppose the principal cannot attain the first-best payoff in a self-enforceable contract (i.e., Condition* (7) *is not satisfied). Then, for any optimal contract, there is a date $\bar{t} \geq 1$ such that effort is constant up to this date, and is subsequently strictly decreasing. Effort converges to a value $\tilde{e}_\infty > 0$ in the long run. There exist $b_1$ and $\delta$ such that, for any optimal contract, the value $\bar{t}$ is strictly greater than one; in particular, effort can indeed be constant in the initial periods.*[14]

[14]The proof shows that effort is initially constant in an optimal contract for values of $b_1$ and $\delta$ close to those for which the principal can attain the first-best payoff.

The dynamics of optimal effort when the principal cannot attain the first-best payoff can be explained as follows. There may be some initial periods when the effort is constant. This occurs if the principal's constraint ($\text{PC}_t$) is initially slack. Given that we consider "fastest payments", the payments rise over these periods for the reasons discussed in relation to Lemma 4.2. Given the principal cannot achieve the first-best payoff, it turns out that the principal's constraint eventually binds, and so payments must be reduced. This is only possible by reducing the level of effort. Note that how much effort can be asked without violating the principal's constraint depends on the future profitability of the relationship. Profitability declines over time, both because higher payments must be made relative to the agent's disutility of effort (see Lemma 4.2), and because the effort that can be requested is less. The fact that profitability declines contributes to the decline in effort, which creates a feedback loop.

Our approach to proving Proposition 4.3 relies on variational arguments. For contracts that fail to exhibit the dynamics described in the proposition, we are able to construct more profitable contracts satisfying all the constraints in Proposition 4.1. We demonstrate some of these arguments below.

One central result links the dynamics of effort to the dates at which the principal's constraint ($\text{PC}_t$) is slack (rather than holding with equality). We can show the following.

**Lemma 4.3.** *Suppose that $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is an optimal relational contract. Suppose that the principal's constraint is slack at $t^*$, i.e. $\tilde{w}_{t^*} < \sum_{s=t^*+1}^{\infty} \delta^{s-t^*} (\tilde{e}_s - \tilde{w}_s)$. Then, $\tilde{e}_{t^*+1} \leq \tilde{e}_{t^*}$; also, if $t^* > 1$, then $\tilde{e}_{t^*-1} \leq \tilde{e}_{t^*}$.*

*Proof.* **Proof that $\tilde{e}_{t^*+1} \leq \tilde{e}_{t^*}$.** Suppose, for the sake of contradiction, that $\tilde{e}_{t^*+1} > \tilde{e}_{t^*}$. We can choose a new contract with efforts $(\tilde{e}'_t)_{t\geq 1}$, and payments $(\tilde{w}'_t)_{t\geq 1}$ chosen to satisfy Equation ($\text{FP}_t^{\text{un}}$), such that they coincide with the original policy except in periods $t^*$ and $t^*+1$. In these periods, $\tilde{e}'_{t^*}$ and $\tilde{e}'_{t^*+1}$ are such that $\tilde{e}_{t^*} < \tilde{e}'_{t^*} \leq \tilde{e}'_{t^*+1} < \tilde{e}_{t^*+1}$ and

$$\psi\left(\tilde{e}'_{t^*}\right) + \delta\psi\left(\tilde{e}'_{t^*+1}\right) = \psi\left(\tilde{e}_{t^*}\right) + \delta\psi\left(\tilde{e}_{t^*+1}\right),$$

which implies (by convexity of $\psi$) that $\tilde{e}'_{t^*} + \delta\tilde{e}'_{t^*+1} > \tilde{e}_{t^*} + \delta\tilde{e}_{t^*+1}$. We then have also that $\tilde{w}_{t^*} < \tilde{w}'_{t^*}$ and $\tilde{w}'_{t^*} + \delta\tilde{w}'_{t^*+1} = \tilde{w}_{t^*} + \delta\tilde{w}_{t^*+1}$ (since the NPV of payments does not change, equilibrium consumption does not change in any period $t$; so the balance at date $t^*+1$ is larger under the new contract). Provided the changes are small, the principal's constraint ($\text{PC}_t$) at $t^*$ remains satisfied. The above observations regarding payments imply that $\tilde{w}'_{t^*+1} < \tilde{w}_{t^*+1}$, so the principal's constraint is relaxed at date $t^*+1$. Since the NPV of output goes up, the principal's constraint is relaxed at all periods before $t^*$.[15] The contract after date $t^*+1$ is

[15] Note that, for the new contract, the principal's constraint at any date $\hat{t}$ may be written as $\sum_{t=\hat{t}}^{\infty} \delta^{t-\hat{t}} \tilde{w}'_t \leq \sum_{t=\hat{t}+1}^{\infty} \delta^{t-\hat{t}} \tilde{e}'_t$. For $\hat{t} < t^*$ this inequality is satisfied strictly since $\sum_{t=\hat{t}}^{\infty} \delta^{t-\hat{t}} \tilde{w}'_t = \sum_{t=\hat{t}}^{\infty} \delta^{t-\hat{t}} \tilde{w}_t$, while $\sum_{t=\hat{t}+1}^{\infty} \delta^{t-\hat{t}} \tilde{e}'_t > \sum_{t=\hat{t}+1}^{\infty} \delta^{t-\hat{t}} \tilde{e}_t$.

unaffected. The modified contract is thus self-enforceable, and it is strictly more profitable than the original, establishing a contradiction.

**Proof that $\tilde{e}_{t^*-1} \leq \tilde{e}_{t^*}$.** Consider now $t^* > 1$, and suppose for the sake of contradiction that $\tilde{e}_{t^*-1} > \tilde{e}_{t^*}$. We can choose again a new contract, this time with efforts $(\tilde{e}'_t)_{t\geq 1}$ that coincide with $(\tilde{e}_t)_{t\geq 1}$ except in periods $t^*-1$ and $t^*$. We specify $\tilde{e}'_{t^*-1}$ and $\tilde{e}'_{t^*}$ so that $\tilde{e}_{t^*} < \tilde{e}'_{t^*} \leq \tilde{e}'_{t^*-1} < \tilde{e}_{t^*-1}$ and

$$\psi\left(\tilde{e}'_{t^*-1}\right) + \delta\psi\left(\tilde{e}'_{t^*}\right) = \psi\left(\tilde{e}_{t^*-1}\right) + \delta\psi\left(\tilde{e}_{t^*}\right),$$

which implies that $\tilde{e}'_{t^*-1} + \delta\tilde{e}'_{t^*} > \tilde{e}_{t^*-1} + \delta\tilde{e}_{t^*}$. We take the new payments $(\tilde{w}'_t)_{t\geq 1}$ to satisfy Equation ($\mathrm{FP}_t^{\mathrm{un}}$).

Note then that

$$\tilde{w}'_{t^*-1} + \delta\tilde{w}'_{t^*} = \tilde{w}_{t^*-1} + \delta\tilde{w}_{t^*}.$$

Also, $\tilde{w}'_{t^*-1} < \tilde{w}_{t^*-1}$ and $\tilde{w}'_{t^*} > \tilde{w}_{t^*}$. Provided the changes are small, the principal's constraint ($\mathrm{PC}_t$) at $t^*$ remains satisfied. Moreover, the principal's constraints are relaxed at date $t^*-1$, and because the NPV of effort increases, also at all earlier dates. Therefore, the principal's constraints are satisfied at all dates and the principal's payoff strictly increases. Again, this establishes a contradiction. □

An immediate implication is that effort is constant over any sequence of periods for which the principal's constraint is slack. To see why, suppose that the constraint ($\mathrm{PC}_t$) is slack on adjacent dates $t^*$ and $t^*+1$ say. Because the constraint is slack at $t^*$, $\tilde{e}_{t^*+1} \leq \tilde{e}_{t^*}$. Because the constraint is slack at $t^*+1$, $\tilde{e}_{t^*} \leq \tilde{e}_{t^*+1}$. Therefore, $\tilde{e}_{t^*} = \tilde{e}_{t^*+1}$.

The explanation for constant effort over the initial periods is thus as follows. If the principal's constraints ($\mathrm{PC}_t$) are slack in the initial periods, and effort is *not* constant over these periods, then effort can be smoothed over time, yielding a more profitable contract that still satisfies the constraints in Proposition 4.1. Such smoothing is profitable given that the disutility of effort is strictly convex (so that differences in effort across periods are inefficient).

A further key part of the proof of Proposition 4.3 is to show that effort strictly decreases from a finite date $\bar{t}$ onwards. The main steps of this argument can be explained as follows. Building on Lemma 4.3, we are able to show (in Lemma A.5 in the Appendix) that effort is weakly decreasing with time. Lemma A.6 then establishes that, if the principal's constraint ($\mathrm{PC}_t$) holds with equality at some date $\hat{t}$, then $\tilde{e}_{\hat{t}+1} < \tilde{e}_{\hat{t}}$ and the constraint holds with equality also at $\hat{t}+1$. Hence effort strictly decreases from $\hat{t}$ onwards.

The argument for Lemma A.6 can be summarized as follows. By assumption, the principal's constraint ($\mathrm{PC}_t$) at date $\hat{t}$ holds as an equality, i.e.

$$\tilde{w}_{\hat{t}} = \sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}} \left(\tilde{e}_s - \tilde{w}_s\right).$$

We are able to show that $\tilde{e}_{\hat{t}+1} - \tilde{w}_{\hat{t}+1} > \tilde{e}_s - \tilde{w}_s$ for all $s > \hat{t}+1$. This follows because $\psi'\left(\tilde{e}_t\right) \leq v'\left(\bar{c}_\infty\right)$ for all $t$ (as established in Lemma A.1), because effort is weakly decreasing over time (as noted above), and making use of Lemma 4.2 (which recall implies that the ratio of payments to disutility of effort increases with time). Therefore,

$$\tilde{w}_{\hat{t}} = \sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}} \left(\tilde{e}_s - \tilde{w}_s\right) > \sum_{s=\hat{t}+2}^{\infty} \delta^{s-\hat{t}-1} \left(\tilde{e}_s - \tilde{w}_s\right) \geq \tilde{w}_{\hat{t}+1}$$

where the second inequality is the principal's constraint ($\mathrm{PC}_t$) at date $\hat{t}+1$. Hence, (again using Lemma 4.2) effort is *strictly* lower in period $\hat{t}+1$ (i.e., $\tilde{e}_{\hat{t}+1} < \tilde{e}_{\hat{t}}$). In turn, using Lemma 4.3, the principal's constraint must hold again with equality at $\hat{t}+1$. So we have shown that, if the principal's constraint holds with equality at a given date, it must hold with equality from then on, and so effort strictly decreases with time.

The above argument assumes that the principal's constraint ($\mathrm{PC}_t$) holds with equality at some date. To show this must in fact be the case when the principal cannot attain the first-best payoff, we assume the contrary. Then Lemma 4.3 implies that optimal effort is constant at all dates, say at a value $\tilde{e}_\infty$ (using the notation of the proposition). Letting the payments and the equilibrium consumption $\bar{c}_\infty$ be determined by Equation ($\mathrm{FP}_t^{\mathrm{un}}$), payments increase over time, and converge to $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}$. The principal's constraint ($\mathrm{PC}_t$) is then satisfied at all dates if and only if

$$\frac{\psi\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\right)} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\right)}\right),$$

where the left-hand side can be read as the limiting payment to the agent, while the right-hand side is the limiting NPV of future profits to the principal. For the most profitable choice of a constant effort $\tilde{e}^*_\infty$, this inequality holds as equality. The principal's constraints ($\mathrm{PC}_t$) tighten over time, but never hold with equality.

Because effort is below the first-best level, we have $\psi'\left(\tilde{e}^*_\infty\right) < v'\left(\bar{c}^*_\infty\right)$, with $\bar{c}^*_\infty$ the level of agent consumption that corresponds to a contract with constant effort $\tilde{e}^*_\infty$. It follows that any sufficiently small adjustment to the effort policy that raises the NPV of effort, together with a change in payments that leaves the agent's payoff in the contract unchanged, raises profits. We therefore suggest a perturbation to the constant-effort contract (see Lemma A.7 in the Appendix) that increases the NPV of effort, but (assuming that payments continue to satisfy ($\mathrm{FP}_t^{\mathrm{un}}$)) leaves the principal's constraints ($\mathrm{PC}_t$) intact. To be more precise, we consider increasing effort by a little at date one and lowering it by a constant amount in future periods. If we only raise effort at date one, leaving other effort values unchanged and assuming that payments are adjusted to satisfy ($\mathrm{FP}_t^{\mathrm{un}}$) at all dates, the principal's constraint ($\mathrm{PC}_t$) is eventually violated (since $v$ is strictly concave and total pay increases, it becomes more costly to compensate the agent for his effort; in particular, payments must increase in all periods). Therefore the reduction in effort at future dates is a "correction" intended to relax the principal's constraint ($\mathrm{PC}_t$) when it is tightest. This part of the proof is illuminative, for it highlights the value in reducing effort at later dates when the principal's constraint is tightest and increasing effort early on when the principal's constraint is most slack.

We have established then that, when the first-best payoff is not attainable, the principal's constraint ($\mathrm{PC}_t$) holds with equality from some date onwards. At these dates, the principal is indifferent between paying the agent and reneging. This feature is the same as in the optimal contracts of Ray (2002) who provides, in a quite general (though distinct) relational contracting environment, a sense in which agent payoffs are backloaded.[16]

We would like to translate the findings of Proposition 4.3 into predictions for payments and the agent's balance. This is complicated by a potential multiplicity of optimal payment paths $(\tilde{w}_t)_{t\geq 1}$. In particular, while Lemma 4.1 tells us that it is optimal for Condition ($\mathrm{FP}_t^{\mathrm{un}}$) to hold at all dates, this is not the only possibility. We therefore provide a partial converse for Lemma 4.1. The following result has the implication that, when the principal cannot obtain her first-best payoff, payments in any optimal contract are uniquely determined after enough time.

**Proposition 4.4.** *Suppose the principal cannot attain the first-best payoff in a self-enforceable contract. Fix any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ and let $\bar{t}$ be the date from which effort is strictly decreasing (see Proposition 4.3). Then, Condition ($\mathrm{FP}_t^{\mathrm{un}}$) holds for all $t > \bar{t}$. Payments to the agent strictly decrease from date $\bar{t}+1$ onwards, while the agent's balances strictly increase.*

The intuitive reason why payments are strictly decreasing from date $\bar{t}+1$ is explained above. The fact that the agent's balance increases over time then follows straightforwardly from Equation (1) and from Equation (2) taken to hold with equality. These observations, together with the fact that the agent consumes a constant $\bar{c}_\infty$ per period, yield in particular that

$$\tilde{b}_t = \frac{\bar{c}_\infty}{1-\delta} - \sum_{\tau\geq t} \delta^{\tau-t}\tilde{w}_\tau$$

which strictly increases with $t$ when payments to the agent fall over time.

[16] Ray considers settings where enforceability constraints prevent the implementation of an efficient contract. His main result is that, after enough time, continuation contracts maximize agent payoffs over constrained efficient continuation contracts that satisfy enforceability constraints.

Note that, when $\bar{t} > 1$, the principal's constraint ($\text{PC}_t$) is initially slack. In this case, Condition ($\text{FP}_t^{\text{un}}$) need not hold at $t < \bar{t}$, and so payments before date $\bar{t}$ are not uniquely determined. When this "fastest payments" condition is nonetheless taken to hold, payments in fact increase over time up to date $\bar{t}$ (as was mentioned above).

# 5 Observed consumption

We now study the case where, at each time $t$, before making the payment $w_t$, the principal can observe the agent's past and present-period effort choices $(e_s)_{s=1}^t$ as well as past and present-period consumption choices $(c_s)_{s=1}^t$. Since payments and consumption are commonly observed, the balance $b_t$ at the beginning of each period $t$ is also commonly known (as deduced from Equation (1)).

The game is now one of complete information, and the relevant solution concept *subgame perfect Nash equilibrium* (SPNE). Both players observe at date $t$ the history $h_t = (e_s, w_s, c_s)_{1 \le s < t}$. The set of such histories at each date $t$ is $\mathcal{H}_t = \mathbb{R}_+^{3(t-1)}$. Re-using notation from Section 4 introduces no confusion, so we describe a strategy for the agent as a collection of functions

$$\alpha_t : \mathcal{H}_t \to \mathbb{R}_+^2, \ t \ge 1,$$

and a strategy for the principal as a collection of functions

$$\sigma_t : \mathcal{H}_t \times \mathbb{R}_+^2 \to \mathbb{R}_+, \ t \ge 1.$$

Here, $\alpha_t$ maps the public $t$-history to a pair $(e_t, c_t)$ of effort and consumption. Also, $\sigma_t$ maps the public $t$-history, together with the observed effort and consumption choices $(e_t, c_t)$ of the agent, to a payment $w_t$. We assume that payoffs are as specified in Section 4 (i.e., the agent earns a payoff $-\infty$ in case his intertemporal budget constraint (2) is violated).

Again we identify a relational contract with the equilibrium outcomes, and we want to characterize contracts that maximize the principal's payoff. A first step is then to determine equilibrium outcomes $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t \ge 1}$ that are feasible relational contracts. Analogous to the arguments made in the previous section, we begin supposing deviations from the agreed outcomes are punished by "autarky". That is, when either player deviates from the contract, all future effort and payments cease, and the agent perfectly smooths his balance over time. In autarky, the agent consumes $b_t(1-\delta)$ when his balance is $b_t > 0$, and we specify zero consumption in case the balance is $b_t \le 0$ (in the latter case, the agent can only obtain a payoff of $-\infty$ since violating the intertemporal budget constraint in Equation (2) implies this payoff; hence we might as well set consumption to zero). Now, autarky follows not only deviations in effort and payments, but also in consumption.

Suppose that the agreed contract is $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, and deviations are punished by autarky. The agent's payoff, if complying until date $t-1$ and optimally failing to comply from $t$ onwards, is now

$$\sum_{s=1}^{t-1} \delta^{s-1}(v(\tilde{c}_s) - \psi(\tilde{e}_s)) + \delta^{t-1}\frac{v(\max\{0, (1-\delta)\tilde{b}_t\})}{1-\delta}.$$

This takes into account that the agent who deviates at date $t$ optimally exerts no effort from then on, and consumes $\max\left\{0, (1-\delta)\tilde{b}_t\right\}$ per period as explained above. Thus, the agent is willing to follow the prescription of the contract if and only if, at all dates $t$,

$$\frac{v(\{0, (1-\delta)\tilde{b}_t\})}{1-\delta} \leq \sum_{s=t}^{\infty} \delta^{s-t}(v(\tilde{c}_s) - \psi(\tilde{e}_s)). \tag{$\mathrm{AC}_t^{\mathrm{ob}}$}$$

The reason for the difference to Condition ($\mathrm{AC}_t^{\mathrm{un}}$) is that publicly honoring the agreement up to date $t-1$ ensures that the agent begins period $t$ with the specified balance $\tilde{b}_t$, which in turn determines the wealth he has available to spend in autarky. Condition ($\mathrm{AC}_t^{\mathrm{un}}$), on the other hand, takes into account that the agent who plans to publicly deviate at date $t$ (by shirking on effort) can save in advance for this event, because consumption is not observed.

We can characterize equilibrium outcomes as follows.

**Proposition 5.1.** *Fix a feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$. It is the outcome of an SPNE in the environment where consumption is observed if and only if, for all $t \geq 1$, Conditions* ($\mathrm{AC}_t^{\mathrm{ob}}$) *and* ($\mathrm{PC}_t$) *are satisfied.*

Notice here that the principal's constraint ($\mathrm{PC}_t$) is the one in Section 4. A feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the conditions in the proposition is again termed *self-enforceable* and a self-enforceable contract that maximizes the principal's payoff is *optimal.* We can now state a result similar to Lemma 4.1: it is without loss of optimality to restrict attention to relational contracts where the agent is indifferent to quitting the relationship at any period (i.e., to consider contracts with the "fastest payments").

**Lemma 5.1.** *For any optimal contract, there exists another optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ with the same effort and consumption, with $\tilde{b}_t > 0$ for all $t$, and where the timing of payments ensures that agent constraints hold with equality in all periods; that is, for all $t \geq 1$,*

$$\frac{v(\tilde{b}_t(1-\delta))}{1-\delta} = \sum_{s=t}^{\infty} \delta^{s-t}\left(v\left(\tilde{c}_s\right) - \psi\left(\tilde{e}_s\right)\right). \tag{8}$$

Lemma 5.1 implies that we can focus on relational contracts where, for all $t \geq 1$,

$$\tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t) = v(\tilde{c}_t) - \psi(\tilde{e}_t) + \tfrac{\delta}{1-\delta} v((1-\delta)\tilde{b}_{t+1}). \tag{$\mathrm{FP}_t^{\mathrm{ob}}$}$$

This says that the agent is indifferent between quitting at date $t$ (i.e., ceasing to exert effort) and smoothing the balance $\tilde{b}_t$ optimally over the infinite future, and instead working one more period, exerting effort $\tilde{e}_t$ and consuming $\tilde{c}_t$, before quitting at date $t+1$ and smoothing the balance $\tilde{b}_{t+1}$ over the infinite future.

## 5.1 Implementing a first-best contract

Let us turn now to the question of when the principal can attain the first-best payoff in a self-enforceable relational contract. By Lemma 5.1, we can focus on payments such that Equation (8) is satisfied at all dates. Given effort and consumption constant at the first-best levels $e^{FB}(b_1)$ and $c^{FB}(b_1)$, the agent's balance is constant and equal to $b_1$. Therefore, the payment is constant over time and equal to $w^{FB}(b_1) \equiv c^{FB}(b_1) - (1-\delta)b_1$. The principal's continuation payoff in the contract is constant and equal to

$$V^{FB}(b_1) = \frac{e^{FB}(b_1) - w^{FB}(b_1)}{1-\delta}.$$

Using these observations, we have the following result.

**Proposition 5.2.** *Suppose that consumption is observable. Then the principal attains the first-best payoff in an optimal relational contract if and only if*

$$w^{FB}(b_1) \leq \frac{\delta}{1-\delta}\big(e^{FB}(b_1) - w^{FB}(b_1)\big). \tag{9}$$

Condition (9) is more easily satisfied than Condition (7) (the condition for the unobservable consumption case); i.e., if the first-best effort and consumption are sustained in the unobservable consumption case, then they are sustained when consumption is observed. This will follow immediately from showing that

$$w^{FB}(b_1) < \frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}. \tag{10}$$

Here, $w^{FB}(b_1)$ is the constant payment to the agent in the observed-consumption case, as specified above. On the other hand, $\frac{\psi\left(e^{FB}(b_1)\right)}{v'\left(c^{FB}(b_1)\right)}$ is the limiting payment for the unobserved-consumption case (assuming that payments satisfy the "fastest payments" condition in Equation ($\mathrm{FP}_t^{\mathrm{un}}$)).

The key insight is that, in the observed-consumption case, the principal's constraints ($\mathrm{PC}_t$) are identical in every period, since payments remain constant. In contrast, in the unobserved-consumption case, we saw that they tighten over time. After enough time, the payments in the unobserved-consumption case exceed the constant payments in the observed-consumption case (note that the NPV of payments in both cases is the same), which makes the principal's

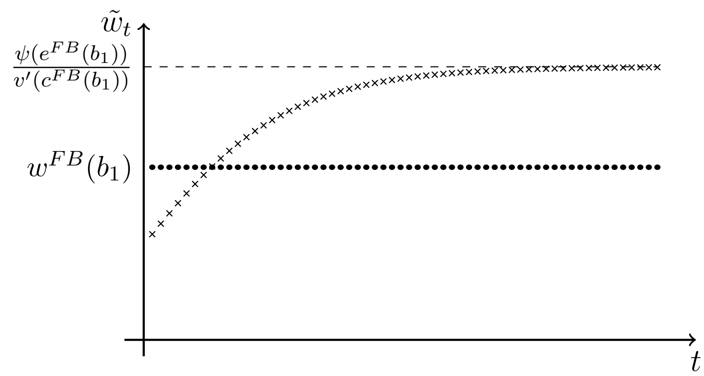


Figure 1: Payments for optimal relational contracts satisfying fastest payments when Equation (7) holds (and so the the principal obtains his first-best payoff), in the unobservable case (crosses) and observable case (circles).

constraints more difficult to satisfy. Figure 1 illustrates the payments in optimal contracts achieving the first-best payoff for the principal in the unobserved and observed consumption cases.

The reason the fastest payments are later in the unobserved consumption case can be understood as the agent having available more attractive deviations: in particular, the agent can plan to deviate in effort at a given date, reducing consumption from the initial date without this being detected by the principal. In the observed-consumption case, deviations in consumption instead trigger autarky immediately.

To derive the inequality (10) formally, observe that by concavity of $v$, and because $c^{FB}(b_1) > (1-\delta) b_1$, we have

$$v\big(c^{FB}(b_1)\big) - v\left((1-\delta) b_1\right) > v'\big(c^{FB}(b_1)\big)\big(c^{FB}(b_1) - (1-\delta) b_1\big) = v'\big(c^{FB}(b_1)\big) w^{FB}(b_1).$$

The result then follows because the first-best effort and consumption satisfy $v\big(c^{FB}(b_1)\big) - v\left((1-\delta) b_1\right) = \psi\big(e^{FB}\left(b_1\right)\big)$ by Condition 2 of Proposition 3.1.

## 5.2 Optimal contract with observed consumption

Now consider the problem of characterizing an optimal contract when the principal's first-best payoff is not attainable. We can consider the "fastest payments" as given in Lemma 5.1. The principal's problem can now be written recursively, with the balance $\tilde{b}_t$ a state variable for the relationship, applying the principle of optimality.[17] Indeed, suppose for some date $t$ that

[17] We did not make use of a recursive formulation of the principal's problem in characterizing optimal contracts for the unobserved-consumption case in Section 4. A recursive statement of the problem is not possible for that case with a single state variable (convenient state variables for such a formulation are both (i)

the continuation contract $(\tilde{e}_s, \tilde{w}_s, \tilde{c}_s, \tilde{b}_s)_{s\geq t}$ does not maximize the continuation value of the relationship to the principal $\sum_{s=t}^{\infty} \delta^{s-t}(e_s - w_s)$, subject to it being self-enforceable; i.e., there is some more profitable self-enforceable continuation contract $(\tilde{e}'_s, \tilde{w}'_s, \tilde{c}'_s, \tilde{b}'_s)_{s\geq t}$ with $\tilde{b}'_t = \tilde{b}_t$, which can be taken to satisfy the agent's indifference conditions (8) at all dates. Then this contract can be substituted, increasing the continuation value $\sum_{s=t}^{\infty} \delta^{s-t}(e_s - w_s)$ (and hence the principal's payoff in the contract overall), maintaining the agent indifference conditions (8) at all dates, and continuing to satisfy the principal's constraints ($\mathrm{PC}_t$).

Since an optimal contract maximizes the principal's continuation profits, an optimal contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ must solve a sequence of sub-problems with value $V(\tilde{b}_t)$ given by

$$V(\tilde{b}_t) = \max_{e_t, b_{t+1}, c_t \in \mathbb{R}_+} \left( e_t - (\delta b_{t+1} - \tilde{b}_t + c_t) + \delta V(b_{t+1}) \right) \tag{11}$$

subject to the agent's indifference condition

$$v(c_t) - \psi(e_t) + \tfrac{\delta}{1-\delta} v((1-\delta) b_{t+1}) = \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t). \tag{12}$$

and to the principal's constraint

$$\delta b_{t+1} - \tilde{b}_t + c_t \leq \delta V(b_{t+1}). \tag{13}$$

The left-hand side of (13) can be understood as the date-$t$ payment $w_t$, which is divided into date-$t$ consumption $c_t \in \mathbb{R}_+$ and savings $\delta b_{t+1} - b_t \in \mathbb{R}$. Note that non-negativity of the payment $\delta b_{t+1} - \tilde{b}_t + c_t$ is assured by the equality (12) and the concavity of $v$: if the payment is negative, the agent is better off going to autarky. The same equality ensures that, given $\tilde{b}_t$ is strictly positive, the choice of $c_t$ and $b_{t+1}$ must be strictly positive as well.

We show that any optimal policy for the principal can be characterized as follows.

**Proposition 5.3.** *An optimal contract exists. Suppose that, given the balance $b_1$, an optimal contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ fails to obtain the first-best payoff $V^{FB}(b_1)$. Then the agent's balance $\tilde{b}_t$ and consumption $\tilde{c}_t$ decline strictly over time, with $\tilde{b}_t \to \tilde{b}_\infty$ for some $\tilde{b}_\infty > 0$. Effort $\tilde{e}_t$ and the payments $\tilde{w}_t$ determined by the Conditions* (8) *increase strictly over time. We have $V(\tilde{b}_t) \to V^{FB}(\tilde{b}_\infty)$ as $t \to \infty$, and effort and consumption converge to first-best levels for the balance $\tilde{b}_\infty$.*

A heuristic account of the forces behind this result is as follows. When the principal's constraint (($\mathrm{PC}_t$) or equivalently (13)) binds, effort is suppressed. That is, if the principal

discounted past pay, and (ii) the date of the relationship). In the present problem with observed consumption, many of our arguments in characterizing optimal contracts do not directly exploit the recursive formulation, although it seems helpful to describe the principal's problem in this way.

could increase effort and credibly increase payments to keep the agent as well off, she would gain by doing so. However, the principal's value function $V(\cdot)$ is strictly decreasing; intuitively, because a lower balance makes the agent cheaper to compensate to keep him in the agreement. Therefore, for any date $t$, reducing the balance $b_{t+1}$ increases the principal's continuation payoff $V(b_{t+1})$ and relaxes the principal's date-$t$ constraint (13). Therefore, the principal asks the agent to consume earlier than he would like, driving the balance down over time. This continues to a point where, given the revised balance, the contract is close to first best, and so the value of continuing to distort consumption vanishes.

It is worth pointing out here that the dynamics of $V(\tilde{b}_t)$ are determinative of both the dynamics of effort and payments. When there is no self-enforceable first-best contract, $V(\tilde{b}_t)$ strictly increases with $t$, and moreover the principal's constraint (13) binds. The latter implies that, for all $t$, both $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$ and $V(\tilde{b}_t) = \tilde{e}_t - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) = \tilde{e}_t$.[18]

It is also worth mentioning how we show $V(\tilde{b}_t)$ is strictly increasing with $t$. As noted above, that $V(\cdot)$ is a strictly decreasing function perhaps seems intuitive given that the agent should be easier to motivate when his balance is low. To understand this in more detail (following the logic of Lemma A.14 in the Appendix), consider a contract that is optimal from date $t$, when the balance is $\tilde{b}_t$. Notice that, when the agent's indifference condition ($\mathrm{FP}_t^{\mathrm{ob}}$) holds, we have

$$\tilde{e}_t = \psi^{-1}\big(v(\tilde{c}_t) - \tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_t) + \tfrac{\delta}{1-\delta}v((1-\delta)\tilde{b}_{t+1})\big). \tag{14}$$

Consider a small decrease in the date-$t$ balance $\tilde{b}_t$, while all payments remain unchanged. If date-$t$ consumption is reduced by the same amount, the date-$t+1$ balance $\tilde{b}_{t+1}$ remains unchanged and we can keep the date-$t+1$ continuation contract unchanged. Determining date-$t$ effort by Equation (14) ensures that the condition ($\mathrm{FP}_t^{\mathrm{ob}}$) is satisfied; hence all of the agent's constraints (see ($\mathrm{AC}_t^{\mathrm{ob}}$)) continue to be satisfied. Note then that (as we establish in Lemma A.12), $\tilde{c}_t > (1-\delta)\tilde{b}_t$, so that $v'(\tilde{c}_t) < v'((1-\delta)\tilde{b}_t)$, showing that date-$t$ effort *increases*. This shows that profits to the principal increase, and also guarantees that the principal's constraints (($\mathrm{PC}_t$) or equivalently (13)) are satisfied. The Appendix argues that these observations imply $V(\cdot)$ is strictly decreasing. Finally, we have to show that $\tilde{b}_t$ strictly decreases with $t$. This argument is more involved and makes use of a variational argument: if $\tilde{b}_t$ fails to be strictly decreasing in $t$, it is possible to specify a strictly more profitable self-enforceable contract (this is established in Lemma A.13 in the Appendix).

[18]Note that the conclusion the the principal's constraint (13) binds is obtained under the assumption that Conditions ($\mathrm{FP}_t^{\mathrm{ob}}$) hold at all dates; but we establish in Proposition 5.4 below that the satisfaction of Conditions ($\mathrm{FP}_t^{\mathrm{ob}}$) is necessary for optimality.

Another part of our analysis that is worth mentioning is an Euler equation

$$1-\frac{v'((1-\delta)\tilde{b}_{t+1})}{v'(\tilde{c}_t)}=\frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})}\left(1-\frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'(\tilde{c}_{t+1})}\right)$$

which must hold for an optimal contract at all dates $t$, and which we use to derive several key properties. This condition is derived (in Lemma A.12) by fixing the contract at and before $t-1$, and from date $t+2$ onwards, and then requiring the contractual variables at $t$ and $t+1$ to be chosen optimally. The equation captures the relationship between a static distortion in effort and a dynamic distortion in consumption. In particular, when the principal's first-best payoff cannot be attained, we are able to show that $\psi'(\tilde{e}_{t+1}) < v'(\tilde{c}_{t+1})$ for all $t$ (reflecting a static (downward) distortion in effort), and correspondingly $(1-\delta)\,\tilde{b}_{t+1} < \tilde{c}_{t+1} < \tilde{c}_t$ (i.e., consumption strictly decreases over time, which is a dynamic distortion). A trade-off between the static and dynamic distortions should be anticipated, since asking the agent to consume excessively early in the relationship increases the agent's marginal utility of consumption later on, which makes him easier to motivate and permits higher effort and profits at later dates. In turn, this relaxes the principal's credibility constraint ($\mathrm{PC}_t$), permitting higher payments and therefore effort also early in the relationship. As $\tilde{b}_t \to \tilde{b}_\infty$, consumption falls to its lower bound, becoming almost constant, so $\frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})} \to 1$, which accords with convergence of effort and consumption to first-best levels.

Finally, analogous to Proposition 4.4, we provide a result on the uniqueness of the timing of payments.

**Proposition 5.4.** *Suppose the principal cannot attain the first-best payoff in a self-enforceable relational contract. Then, in any contract that is optimal for the principal, Condition* ($\mathrm{FP}_t^{\mathrm{ob}}$) *holds at all dates. Hence payments to the agent strictly increase over time.*

The logic of this result is that, if the Condition ($\mathrm{FP}_t^{\mathrm{ob}}$) fails, then payments can be made earlier in time, while maintaining the agent constraints ($\mathrm{AC}_t^{\mathrm{ob}}$). This induces slack in the principal's constraint ($\mathrm{PC}_t$), which can then be exploited by increasing payments, consumption and effort, increasing profits. As noted for the case of unobservable consumption, such an observation is related to arguments in Ray (2002).

# 6 Optimal relational contract in the absence of savings

In this section we aim at understanding optimal relational contracts when the agent cannot save, and so has consumption determined directly by pay. We make the same assumptions on the agent's utility of consumption $v$ and disutility of effort $\psi$ as in the model set-up, but now suppose the agent receives a base level of consumption $\underline{c} > 0$ per period. The reason for this assumption is to ensure that the agent does not obtain a payoff $-\infty$ when he is not paid, since this would allow the principal to obtain unbounded profits. We assume that the base consumption $\underline{c}$ is not paid by the principal (for instance, it could be a "living wage" paid by the government).[19]

In each period, the agent publicly chooses effort $e_t \geq 0$ and the principal then makes the public payment $w_t \geq 0$, with the agent consuming $c_t = \underline{c} + w_t$. A deterministic relational contract can now be described as a sequence of efforts and payments, $(\tilde{e}_t, \tilde{w}_t)_{t\geq 1}$, and we impose the feasibility requirement that these are non-negative and bounded. In this setting with complete information, we consider feasible relational contracts to be *self-enforceable* if they are outcomes of an SPNE. We can consider without loss deviations from the relational contract being punished by autarky, where effort and pay is set to zero, and the agent therefore consumes $\underline{c}$ per period.

A feasible relational contract $(\tilde{e}_t, \tilde{w}_t)_{t\geq 1}$ is therefore self-enforceable if, for all $t$, the principal's constraint ($\text{PC}_t$) holds, and in addition the agent's constraint

$$\sum_{s=t}^{\infty} \delta^{s-t}(v(\underline{c} + \tilde{w}_s) - \psi(\tilde{e}_s)) \geq \frac{1}{1-\delta} v\left(\underline{c}\right) \tag{15}$$

is satisfied. Here, the left-hand side of Equation (15) is the continuation utility of the agent at time $t$. A relational contract is *optimal* if it is self-enforceable and maximizes the principal's payoff $\sum_{t=1}^{\infty} \delta^{t-1}\left(\tilde{e}_t - \tilde{w}_t\right)$. We characterize optimal relational contracts as follows.

**Proposition 6.1.** *There is a unique optimal contract, and it is stationary. Letting $(e^{FB}, w^{FB})$ be the unique pair satisfying $\psi'(e^{FB}) = v'(\underline{c} + w^{FB})$ and $v(\underline{c} + w^{FB}) - \psi(e^{FB}) = v(\underline{c})$, we have*

1. *If $w^{FB} \leq \delta e^{FB}$, then the optimal contract satisfies $(\tilde{e}_t, \tilde{w}_t) = (e^{FB}, w^{FB})$ for all $t \geq 1$.*
2. *If $w^{FB} > \delta e^{FB}$, then the optimal contract satisfies $(\tilde{e}_t, \tilde{w}_t) = (e^*, \delta e^*)$ for all $t \geq 1$, where $e^*$ is the unique strictly positive value satisfying $v(\underline{c} + \delta e^*) - \psi(e^*) = v(\underline{c})$.*

The result confirms a sense in which non-trivial dynamics arise in our models with concave utility from consumption due to the possibility of agent savings. It turns out that, when the agent cannot save, the agent is compensated for effort only through the current-period payment; in particular, the agent's continuation payoff is always equal to his autarky payoff. As shown in the proof of the proposition, this insight markedly simplifies the search for an optimal contract.

[19] Note that we could give the same interpretation to the value $(1-\delta)\, b_1$ in the previous sections, taking the agent's initial wealth then to equal zero. Because there are no restrictions on the agent's saving or dissaving in these previous sections, the model in those sections is equivalent to one where the agent has zero initial wealth and is guaranteed to receive amounts $(1-\delta)\, b_1$ in each period, in addition to what is paid by the principal.

# 7 Conclusions

This paper has studied optimal relational contracts in a simple deterministic setting where the agent has consumption-smoothing preferences and is able to save. We contrasted the case where the agent's consumption is unobservable to the principal (the case of "private savings") and where consumption is observed. In the case where consumption is unobservable, we found that the relationship eventually becomes less profitable with time, implying that the payments the principal can credibly offer must decline. Hence effort eventually declines with time. When consumption is instead observable, the agent consumes inefficiently early (i.e., saves too little), the balance on his savings account gradually declines, the relationship becomes more profitable as the agent grows easier to incentivize, payments to the agent gradually increase, and the agent's effort increases. It is worth remarking that the contract when the principal observes the agent's consumption is a Pareto improvement on the one when it is not observed. This is in spite of the fact that there is an additional source of distortion, namely in the timing of consumption. This distortion is more than offset by an improvement in the provision of incentives.

We conclude with some further remarks concerning applications and some open questions. First, the paper has focused on characterizing the dynamics of optimal relational contracts. As we argued in the Introduction, the dynamics of consumption when it is observable to the principal (see Section 5) suggests a theory of inefficiently high consumption spending by workers in industries which rely on discretionary incentive pay (banking may be an example). But the model can also be used to address other questions of applied interest. For instance, one observation (common to both the case with unobservable consumption in Section 4 and observable consumption in Section 5) is that the principal's profits in an optimal relational contract decrease with the agent's initial wealth $b_1$.[20] The model therefore suggests a preference of employers for employees who are known to have low savings. Although this seems at odds with US employers using credit scores to vet employees (see, e.g., Gallagher, 2005), a firm could perhaps profit nonetheless from employees known to have high student debt (for instance, Hancock, 2009, has suggested such individuals might be more willing to work overtime).

Related to the above ideas, future theoretical research might examine the effects of turnover and competition among employers. For instance, the savings of a worker when switching to a new employer (say as the result of an exogenous shock) would depend on the contract with

[20] For the model in Section 5, this is simply the observation that the principal's profits $V(b_1)$ are strictly decreasing in $b_1$, as we already noted. In Section 4, the result follows because, if a relational contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ is optimal, then for $\varepsilon \in (0, b_1)$, and for $\eta > 0$ and sufficiently small, the contract $(\tilde{e}_t + \eta, \tilde{w}_t, \tilde{c}_t - \varepsilon(1-\delta), \tilde{b}_t - \varepsilon)_{t\geq 1}$ is self-enforceable and generates strictly higher profits for the principal.

the previous employer. Consider the case when consumption is observable and the agent can save (as in Section 5). In this case, while we have documented how an employer can benefit from a contract that calls for high consumption and impoverishes the worker over time, such a distortion in consumption might increase profits for the worker's next employer in a setting with turnover. This suggests a positive externality from relational contracts that call for high consumption, one that employers might be expected not to internalize.

As mentioned in the Introduction, a further possibility that we have left unexplored is where only part of an agent's consumption expenditures are observable to the principal. For instance, the agent might have preferences for both "conspicuous" and "inconspicuous" consumption, with diminishing marginal utility for each. An optimal relational contract might then call for inefficient amounts of the former to impoverish the agent over time, similar to the case of observable consumption studied in Section 5. As mentioned, however, such a model seems less tractable than the cases of fully observable or fully unobservable consumption.

Another question left for future research is the role of exogenous uncertainty. This could take a range of forms. For instance, the agent's initial balance could be random and the agent's private information, the agent could receive taste or income shocks over time, or there could be shocks to the principal's willingness or ability to pay compensation. These possibilities again would substantially complicate the analysis.

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# A Appendix: Omitted proofs

## A.1 Proofs of the results in Section 3

### Proof of Proposition 3.1

*Proof.* Consider the problem of maximizing the principal's payoff

$$\sum_{t=1}^{\infty} \delta^{t-1} \left(\tilde{e}_t - \tilde{w}_t\right)$$

over the set of feasible relational contracts $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying

$$\sum_{t=1}^{\infty} \delta^{t-1} \left(v\left(\tilde{c}_t\right) - \psi\left(\tilde{e}_t\right)\right) \geq \frac{v\left(b_1\left(1-\delta\right)\right)}{1-\delta}. \tag{16}$$

For any feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$, define

$$\tilde{c} = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \tilde{c}_t,$$

$$\tilde{e} = \psi^{-1}\left((1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \psi\left(\tilde{e}_t\right)\right),$$

and

$$\tilde{w} \equiv (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t.$$

Note these values are all finite by feasibility.

We now show there is no loss in considering "stationary" contracts where effort, payments and consumption are constant at the values $\tilde{e}$, $\tilde{w}$ and $\tilde{c}$, and where balances are constant at $b_1$. To this end, we modify the arbitrary (not necessarily stationary) feasible contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the inequality in Equation (16) considered above. In particular, we modify it to a contract $(\tilde{e}'_t, \tilde{w}'_t, \tilde{c}'_t, \tilde{b}'_t)_{t\geq 1}$ with, for all $t$, $\tilde{e}'_t = \tilde{e}$, $\tilde{w}'_t = \tilde{w}$, $\tilde{c}'_t = \tilde{c}$ and $\tilde{b}'_t = b_1$.

First, note the contract $(\tilde{e}'_t, \tilde{w}'_t, \tilde{c}'_t, \tilde{b}'_t)_{t\geq 1}$ satisfies Equation (1) for all $t$. This follows because, for any $t \geq 1$, if $\tilde{b}'_t = b_1$, then

$$\begin{aligned}
\frac{\tilde{b}'_t + \tilde{w}'_t - \tilde{c}'_t}{\delta} &= \frac{b_1 + \tilde{w} - \tilde{c}}{\delta} \\
&= \frac{b_1 + (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1}\tilde{w}_t - (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\tilde{c}_t}{\delta} \\
&= b_1.
\end{aligned}$$

The contract $(\tilde{e}_t', \tilde{w}_t', \tilde{c}_t', \tilde{b}_t')_{t\geq 1}$ also satisfies the inequality in Equation (2) with equality because contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ does so. Feasibility of the contract $(\tilde{e}_t', \tilde{w}_t', \tilde{c}_t', \tilde{b}_t')_{t\geq 1}$ is then immediate. In addition, the inequality in Equation (16) continues to be satisfied, given the concavity of $v$. Also, the principal's payoff is (weakly) higher in the contract $(\tilde{e}_t', \tilde{w}_t', \tilde{c}_t', \tilde{b}_t')_{t\geq 1}$, because the NPV of output is (weakly) higher, by convexity of $\psi$. This confirms that stationary contracts are optimal.

The above argument can also be used to show that effort and consumption are constant in *any* optimal contract. First, we see that $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ is not optimal unless $\tilde{e}_t = \tilde{e}$ for all $t$. This follows by strict convexity of $\psi$. Further, optimality of $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t\geq 1}$ requires $\tilde{c}_t = \tilde{c}$ for all $t$. If this is not the case, then the inequality in Equation (16) holds strictly for the contract $(\tilde{e}_t', \tilde{w}_t', \tilde{c}_t', \tilde{b}_t')_{t\geq 1}$. Hence, further modifying this contract by raising effort $\tilde{e}_1'$ by a small enough amount yields a feasible contract that satisfies Equation (16) and gives a strictly higher payoff for the principal.

Now consider the optimal specification of the stationary contract $(\tilde{e}_t', \tilde{w}_t', \tilde{c}_t', \tilde{b}_t')_{t\geq 1}$. Optimality requires $\tilde{w} = \tilde{c} - (1-\delta)b_1$. Also, the constraint in Equation (16) can be written as $v(\tilde{c}) - \psi(\tilde{e}) \geq v(b_1\,(1-\delta))$, which must hold with equality for an optimum, yielding the second condition in the proposition. Using this, we may write

$$\tilde{c} = v^{-1}\left(\psi(\tilde{e}) + v(b_1\,(1-\delta))\right). \tag{17}$$

The principal's payoff can then be written as

$$\tilde{e} - (\tilde{c} - (1-\delta)b_1) = \tilde{e} - v^{-1}\left(\psi(\tilde{e}) + v(b_1\,(1-\delta))\right) + (1-\delta)b_1.$$

This is strictly concave in $\tilde{e}$ and an optimum is attained when the first-order condition $1 - \frac{\psi'(\tilde{e})}{v'(\tilde{c})} = 0$ is satisfied (with $\tilde{c}$ given by Equation (17)). This is the first condition in the proposition. By the assumption that $\psi'(0) = 0$, we have $\tilde{e} > 0$ and hence $\tilde{c} > b_1\,(1-\delta)$.

It remains to show that the principal's payoff is strictly decreasing in $b_1$. Consider the optimal (stationary) contract specified above for an initial balance $b_1$, and consider any reduction $\varepsilon \in (0, b_1)$ in this balance. If consumption is also reduced in each period by $\varepsilon\,(1-\delta)$, while payments remain unchanged, then the agent's balance remains constant at $b_1 - \varepsilon$. Equation (2) continues to hold with equality. Hence, the adjusted contract is feasible. The inequality in Equation (16) holds strictly. Therefore, a further adjustment to the contract that comprises increasing date-1 effort by a small amount yields a feasible contract that satisfies Equation (16) and generates a strictly higher profit for the principal than the optimal contract when the agent's initial balance is $b_1$.

$\square$

## A.2 Proofs of the results in Section 4

### Proof of Proposition 4.1

*Proof.* **Necessity.** Fix a PBE with outcome given by some feasible contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. Suppose first it is not true that $\tilde{c}_t = \bar{c}_\infty$ for all $t$. Then the agent can choose a strategy which specifies obedient (on-path) effort and specifies consumption given in each period by $\bar{c}_\infty$. Note that

$$b_1 + \sum_{s=1}^{\infty} \delta^{s-1} \left(\tilde{w}_s - \bar{c}_\infty\right) = 0.$$

Hence, the feasibility constraint (2) is satisfied on the path of play under the new strategy, and by strict concavity of $v$ the agent obtains a strictly higher payoff.

If ($\mathrm{AC}_t^{\mathrm{un}}$) is not satisfied for some $t$, then the agent can consume $\bar{c}_{t-1} \leq \bar{c}_\infty$ in every period and exert the specified effort $e_s = \tilde{e}_s$ up to date $t-1$, and zero effort from then on, attaining a higher payoff.

If ($\mathrm{PC}_t$) is not satisfied at date $t$, the principal can pay zero from then on. The principal then attains a non-negative continuation payoff, while by continuing to make the payments $(\tilde{w}_s)_{s\geq t}$, the principal obtains a negative continuation payoff

$$\sum_{s=t+1}^{\infty} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s) - \tilde{w}_t.$$

Therefore, the principal has a profitable deviation at date $t$.

**Sufficiency.** We fix some feasible contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ satisfying the conditions of the proposition and we provide strategies and beliefs that constitute a PBE with outcome $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$.

We specify a PBE as follows. On the principal's side, put $\sigma_t\left(h_t^P, e_t\right) = \tilde{w}_t$ if $(e_s, w_s) = (\tilde{e}_s, \tilde{w}_s)$ for all $s \leq t-1$ and $e_t = \tilde{e}_t$, and put $\sigma_t\left(h_t^P, \tilde{e}_t\right) = 0$ otherwise.

For the agent's strategy, put $\alpha_t(h_t^A) = (\tilde{e}_t, \tilde{c}_t)$ if $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$. Put $\alpha_t(h_t^A) = \left(0, \max\{0, (1-\delta)b(h_t^A)\}\right)$ whenever $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$ for some $s \leq t-1$. Note that this specification of the agent's strategy will guarantee its sequential optimality at date-$t$ histories where $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$ for some $s < t$. This is because, at time $t$ after a public deviation, there are two possibilities. If $b(h_t^A) < 0$ then the agent's payoff is $-\infty$ independently of the strategy he uses, because the feasibility constraint (2) is violated (as he no receives any further payment by the principal). If, instead, $b(h_t^A) \geq 0$, the agent smooths consumption by consuming $(1-\delta)b(h_t^A)$ in all subsequent periods (given that he does not expect the principal to make a positive payment in the future).

Determining the agent's equilibrium strategy for the remaining possible histories, where $(e_s, w_s) = (\tilde{e}_s, \tilde{w}_s)$ for all $s \leq t-1$, and yet $c_s \neq \tilde{c}_s$ for some values $s \leq t-1$, is then more involved. Consider a history $h_t^A$ for the agent with $(e_s, w_s) = (\tilde{e}_s, \tilde{w}_s)$ for all $s \leq t-1$, and yet $c_s \neq \tilde{c}_s$ for some values. First note that, if

$$b\left(h_t^A\right) + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau \tag{18}$$

is non-positive, then the continuation payoff of the agent at history $h_t^A$ is $-\infty$ under any continuation strategy. In this case we might as well specify that $e_t = \tilde{e}_t$ and $c_t = 0$. Assume then that the expression (18) is strictly positive. In this case, an optimal continuation strategy for the agent, given the principal's strategy, should induce a continuation outcome of the following form. There should be some $t' \geq t$ (possibly $+\infty$) so that effort is $e_s = \tilde{e}_s$ for all $s \in \{t, t+1, \ldots, t'-1\}$, and so that effort is $e_s = 0$ for $s \geq t'$. Consumption should be specified optimally. Given the concavity of $v$, the highest continuation payoff the agent can achieve at date $t$, given that he works obediently until date $t'-1$, is that obtained by putting

$$c_s = \max\left\{0, (1-\delta)\left(b\left(h_t^A\right) + \sum_{\tau=t}^{t'-1} \delta^{\tau-t} \tilde{w}_\tau\right)\right\}$$

for all $s \geq t$.

We can now consider the problem of choosing the optimal "public deviation" time $t'$, given optimal consumption as specified above. The existence of a solution to this problem follows from "continuity at infinity" of the agent's payoff in the public deviation date $t'$; i.e., because, for all $t$, all histories $h_t^A$ for which there is, as yet, no public deviation ($e_s = \tilde{e}_s$ and $w_s = \tilde{w}_s$ for all $s < t$),

$$\begin{aligned} &\frac{v\Big(\max\Big\{0, (1-\delta)\big(b(h_t^A) + \sum_{\tau=t}^{t'-1} \delta^{\tau-t} \tilde{w}_\tau\big)\Big\}\Big)}{1-\delta} - \sum_{\tau=t}^{t'-1} \delta^{\tau-t} \psi(\tilde{e}_\tau) \\ &\longrightarrow \frac{v\big((1-\delta)\big(b(h_t^A) + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau\big)\big)}{1-\delta} - \sum_{\tau=t}^{\infty} \delta^{\tau-t} \psi(\tilde{e}_\tau) \end{aligned}$$

as $t' \to \infty$. This follows by continuity of $v$.

In determining the agent's continuation strategy at date $t$ and private history $h_t^A$, we take $t'$ to be the largest value that attains the optimal payoff for the agent (again, continuity at infinity implies that such a largest value exists, and it could be $+\infty$). Hence, the strategy specifies that, at private history $h_t^A$ for the agent, effort is $e_t = \tilde{e}_t$ if $t' > t$ and $e_t = 0$ if $t' = t$, and consumption is $c_t = (1-\delta)\left(b(h_t^A) + \sum_{\tau=t}^{t'-1} \delta^{\tau-t} \tilde{w}_\tau\right)$.

Finally, let us specify the principal's beliefs on the agent's previous consumption choices. Let these beliefs be degenerate sequences, and denote the believed consumption up to date $t$ by $(\hat{c}_s)_{s=1}^t$. If date $t$ is such that there has not been a public deviation (i.e., $(e_s, w_s) = (\tilde{e}_s, \tilde{w}_s)$

for all $s \le t-1$ and if $e_t = \tilde{e}_t$), then the principal believes that the agent has consumed as the strategy specifies; that is, $\hat{c}_s = \tilde{c}_s$ at each date $s \le t$. If instead $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$ for some $s \le t-1$, or if $e_t \neq \tilde{e}_t$, let $t'$ be the first date of such a public deviation (i.e. the first date $s \le t-1$ at which $(e_s, w_s) \neq (\tilde{e}_s, \tilde{w}_s)$, or if there is no such date, then date $t$). If $e_{t'} \neq \tilde{e}_{t'}$ (so the agent is first to publicly deviate), we let the principal's belief be given by

$$\hat{c}_s = (1-\delta)\left(b_1 + \sum_{\tau=1}^{t'-1} \delta^{\tau-1}\tilde{w}_\tau\right)$$

for all $s \in \{1, \ldots, t'-1\}$ (i.e., the principal believes that the agent optimally consumed from time 1 given the deviation and the principal's specified strategy), while, for all $s \in \{t', \ldots, t\}$, the principal believes that the agent consumes

$$\hat{c}_s = (1-\delta)\hat{b}_s,$$

where $\hat{b}_s$ are beliefs on the agent's balance determined recursively from the principal's payments and the agent's believed consumption (i.e., $\hat{b}_s = (\hat{b}_{s-1} + w_{s-1} - \hat{c}_{s-1})/\delta$, with $\hat{b}_1 = b_1$). If $e_{t'} = \tilde{e}_{t'}$ (so the principal is first to publicly deviate), then the principal believes that the agent consumes $\hat{c}_s = \tilde{c}_s$ for all $s \le t'$ and $\hat{c}_s = \max\left\{0, (1-\delta)\hat{b}_s\right\}$ for all $s = t'+1, ..., t$ (again, the values of $\hat{b}_s$ are determined recursively by $\hat{b}_s = (\hat{b}_{s-1} + w_{s-1} - \hat{c}_{s-1})/\delta$, with $\hat{b}_1 = b_1$). These beliefs are consistent with updating of the principal's prior beliefs according to the specified strategy of the agent whenever there is no public evidence the agent's strategy has not been followed.

We now want to verify the sequential optimality of the above strategies, given the beliefs. First, note that at any information set at which the principal has not yet observed a deviation, the fact that $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\ge1}$ satisfies Condition (PC$_t$) implies that the principal optimally sets $w_t = \tilde{w}_t$ (if $w_t \neq \tilde{w}_t$, then the principal's continuation profits are no greater than zero). If instead the principal has observed a deviation, then the principal can obtain at most zero, since the agent exerts no effort, and hence paying $w_t = 0$ is optimal.

Finally observe that the agent's strategy is constructed to be sequentially optimal. □

**Proof of Lemma 4.1**

*Proof.* Fix an optimal relational contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\ge1}$; that is, a feasible contract that maximizes the principal's discounted payoff subject to the conditions of Proposition 4.1. We show first that

$$\frac{v\left(\bar{c}_\infty\right)}{1-\delta} - \sum_{s=1}^{\infty} \delta^{s-1}\psi(\tilde{e}_s) \tag{19}$$

is equal to $\frac{v(b_1(1-\delta))}{1-\delta}$. Clearly the only way this can fail in a self-enforceable relational contract is if (19) strictly exceeds $\frac{v(b_1(1-\delta))}{1-\delta}$. However, in this case, there is a more profitable feasible contract for which the conditions of Proposition 4.1 still hold, and in which $\tilde{e}_1$ increases by a small amount.

Next, the previous observation implies that, if

$$\frac{v\left(\bar{c}_{t-1}\right)}{1-\delta} - \sum_{s=1}^{t-1} \delta^{s-1} \psi(\tilde{e}_s) \tag{20}$$

exceeds $\frac{v(b_1(1-\delta))}{1-\delta}$ at any date $t$, then the inequality ($\mathrm{AC}_t^{\mathrm{un}}$) must not be satisfied; i.e., the conditions of Proposition 4.1 are not satisfied (there can be no PBE with contractual outcomes $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$).

Finally, suppose that the expression (20) is strictly less than $\frac{v(b_1(1-\delta))}{1-\delta}$ at some increasing sequence of dates $(t_n)_{n=1}^{N}$, where $N$ may be finite or infinite. For each $n$, there is $\varepsilon_n > 0$ such that

$$\frac{1}{1-\delta} v(\bar{c}_{t_n-1} + \delta^{t_n-2}\varepsilon_n(1-\delta)) - \sum_{s=1}^{t_n-1} \delta^{s-1} \psi(\tilde{e}_s) = \frac{v\left(b_1\left(1-\delta\right)\right)}{1-\delta}.$$

Increase $\tilde{w}_{t_n-1}$ by $\varepsilon_n$, and reduce $\tilde{w}_{t_n}$ by $\frac{\varepsilon_n}{\delta}$; note that this leads to a change in $\bar{c}_{t_n-1}$, but does not affect $\bar{c}_t$ for $t \neq t_n$. After this adjustment has been made for each $n$, we have a relational contract for which the expression (20) is equal to $\frac{v(b_1(1-\delta))}{1-\delta}$ at all dates $t$. Also, because $\psi$ is non-negative, $\bar{c}_t$ must be a non-decreasing sequence, and hence all payments $\tilde{w}_t$ in the new relational contract are non-negative. Hence, the new contract is feasible, and we have observed that the agent's constraints ($\mathrm{AC}_t^{\mathrm{un}}$) are satisfied. Also, the principal's constraints ($\mathrm{PC}_t$) are satisfied. To see the latter, note that these constraints are affected by the adjustments to the original contract only at dates satisfying $t = t_n$ for some $n$. At such dates the principal's constraint is *slackened* by the amount $\frac{\varepsilon_n}{\delta}$. □

### Proof of Lemma 4.2

*Proof.* Observe from Condition ($\mathrm{FP}_t^{\mathrm{un}}$) evaluated at consecutive dates, we have

$$\frac{v\left(\bar{c}_{t-1} + \left(1-\delta\right)\delta^{t-1}\tilde{w}_t\right) - v\left(\bar{c}_{t-1}\right)}{1-\delta} = \delta^{t-1}\psi\left(\tilde{e}_t\right).$$

By the Fundamental Theorem of Calculus, we have

$$\int_0^{\tilde{w}_t} v'\left(\bar{c}_{t-1} + \left(1-\delta\right)\delta^{t-1}x\right) dx = \psi\left(\tilde{e}_t\right)$$

and hence

$$k\tilde{w}_t = \psi\left(\tilde{e}_t\right)$$

for $k \in \left(v'\left(\bar{c}_t\right), v'\left(\bar{c}_{t-1}\right)\right)$, which proves the result. □

**Proof of Propositions 4.2 and 4.3**

*Proof.* The remaining steps in the proof of the Proposition 4.3 are divided into nine lemmas. The proof of Proposition 4.2 is provided in the process, in Lemma A.7. Throughout, we restrict attention to payments determined under the restriction to "fastest payments", i.e. satisfying Condition ($\mathrm{FP}_t^{\mathrm{un}}$).

1. Lemma A.1 bounds effort and hence payments.
2. Lemma A.2 shows that an optimal relational contract exists.
3. Lemma A.3 shows that the principal achieves a strictly positive payoff and that effort remains strictly positive in any optimal relational contract.
4. Lemma A.4 shows that the contract becomes (approximately) stationary in the long run.
5. Lemma A.5 shows that effort is weakly decreasing.
6. Lemma A.6 establishes that, if the principal's constraint ($\mathrm{PC}_t$) binds at date $t$, then it continues to bind at all future dates. Also effort strictly decreases over these dates.
7. Lemma A.7 establishes the condition for a first-best policy to be self-enforceable. Also, when this condition is not satisfied, for any optimal contract, there exists a date $\bar{t}$ satisfying the properties in the proposition (i.e., effort is constant up to date $\bar{t}$, and subsequently strictly decreasing).
8. Lemma A.8 shows that the limiting effort is strictly positive.
9. Lemma A.9 establishes that the date $\bar{t}$ in the proposition can be strictly greater than one for some specification of model primitives.

The following lemma argues that we can restrict attention to contracts such that the marginal disutility of effort is bounded by the marginal utility of consumption.

**Lemma A.1.** *There is no loss of optimality in restricting to self-enforceable contracts such that $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ for all $t$.*

*Proof.* Take a contract satisfying Condition ($\mathrm{FP}_t^{\mathrm{un}}$) for all $t$, and let $t^*$ be the first date at which $\psi'(\tilde{e}_{t^*}) > v'(\bar{c}_\infty)$. We can adjust such a contract by reducing date $t^*$ effort by some $\eta \in (0, \tilde{e}_{t^*})$ (holding effort at other dates fixed). This determines a new contract, with adjusted consumption and payments, again satisfying ($\mathrm{FP}_t^{\mathrm{un}}$) for all $t$. Let us index the revised effort policy by the date-$t^*$ adjustment $\eta$, writing $\tilde{e}_t(\eta)$ for all $t$. Correspondingly, write

$$\bar{c}_\infty(\eta) \equiv (1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1}\tilde{w}_s(\eta)\right)$$

where $(\tilde{w}_s(\eta))_{s\geq 1}$ are the payments determined from the adjusted effort policy. Then

$$\frac{v(\bar{c}_\infty(0)) - v(\bar{c}_\infty(\eta))}{1-\delta} = \delta^{t^*-1}\big(\psi(\tilde{e}_{t^*}(0)) - \psi(\tilde{e}_{t^*}(\eta))\big).$$

Differentiating with respect to $\eta$,

$$\frac{\bar{c}'_\infty(\eta)}{1-\delta} = \frac{-\delta^{t^*-1}\psi'(\tilde{e}_{t^*}(\eta))}{v'(\bar{c}_\infty(\eta))}.$$

This expression coincides with the derivative of the NPV of payments to the agent with respect to $\eta$. The derivative of the principal's profits is therefore

$$-\delta^{t^*-1} + \frac{\delta^{t^*-1}\psi'(\tilde{e}_{t^*}(\eta))}{v'(\bar{c}_\infty(\eta))}$$

which is strictly positive for $\eta \in [0, \bar{\eta})$, with $\bar{\eta}$ satisfying $\psi'(\tilde{e}_{t^*}(\bar{\eta})) = v'(\bar{c}_\infty(\bar{\eta}))$. The effect on profit from reducing date $t^*$ effort by $\bar{\eta}$ is therefore to increase it by

$$\int_0^{\bar{\eta}} \left(-\delta^{t^*-1} + \frac{\delta^{t^*-1}\psi'(\tilde{e}_{t^*}(\eta))}{v'(\bar{c}_\infty(\eta))}\right) d\eta > 0.$$

Note (from ($\mathrm{FP}_t^{\mathrm{un}}$)) that payments $\tilde{w}_t(\bar{\eta})$ are reduced for all $t \geq t^*$, with the implication that the principal's constraints ($\mathrm{PC}_t$) are relaxed for all $t$ (note that the principal's constraints are relaxed for $t < t^*$). Hence, the new contract is self-enforceable. Note then that $\psi'(\tilde{e}_t(\bar{\eta})) < v'(\bar{c}_\infty(\bar{\eta}))$ for all $t < t^*$. We can therefore continue iteratively, by proceeding to the next date $t > t^*$ at which $v'(\bar{c}_\infty(\bar{\eta})) < \psi'(\tilde{e}_t(\bar{\eta}))$, if any, and reducing effort precisely as for at $t^*$. Proceeding sequentially, we obtain a self-enforceable contract for which $\psi'(\tilde{e}_t) \leq v'(\bar{c}_\infty)$ at all dates $t$, and which is strictly more profitable than the original. □

The above result establishes that the marginal disutility of effort $\psi'(\tilde{e}_t)$ in an optimal contract is bounded by $v'(\bar{c}_\infty)$, which is certainly no greater than $v'(b_1(1-\delta))$, given payments $\tilde{w}_t$ are non-negative. Also, given any bounded effort policy, payments are also bounded, which can be seen from Lemma 4.2.

**Lemma A.2.** *An optimal relational contract exists.*

*Proof.* As we already observed, under the condition "fastest payments" given in ($\mathrm{FP}_t^{\mathrm{un}}$), the relational contract is determined solely by the effort policy $(\tilde{e}_t)_{t\geq 1}$. Hence, the payoff obtained by the principal can be written

$$W\left((\tilde{e}_t)_{t=1}^{\infty}\right) = \sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t - \sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t$$

where each $\tilde{w}_t$ is recursively obtained from ($\mathrm{FP}_t^{\mathrm{un}}$). Note that, from Lemma A.1, we can restrict attention to effort policies in $[0, z\left(v'\left(b_1\left(1-\delta\right)\right)\right)]^{\infty}$, where $z$ denotes the inverse of $\psi'$.

Now, let $W^{\sup}$ be the supremum of $W(\cdot)$ over effort policies $(\tilde{e}_t)_{t\geq 1}$ in $[0, z\left(v'\left(b_1\left(1-\delta\right)\right)\right)]^{\infty}$ for which the implied contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ (i.e., the one implied by ($\mathrm{FP}_t^{\mathrm{un}}$)) satisfies the principal's constraints ($\mathrm{PC}_t$) (such contracts are feasible and satisfy all the conditions of Proposition 4.1). Note the set is non-empty; for instance, because effort constant at zero is in the set.

Consider then a sequence of policies $((\tilde{e}_t^n)_{t=1}^{\infty})_{n=1}^{\infty}$ in $[0, z\left(v'\left(b_1\left(1-\delta\right)\right)\right)]^{\infty}$ and with

$$W\left((\tilde{e}_t^n)_{t=1}^{\infty}\right) > W^{\sup} - 1/n$$

for all $n$, and for which the contract defined by each effort policy (using ($\mathrm{FP}_t^{\mathrm{un}}$)) satisfies the principal's constraints ($\mathrm{PC}_t$). There then exists a sequence $(\tilde{e}_t^{\infty})_{t\geq 1} \in [0, z\left(v'\left(b_1\left(1-\delta\right)\right)\right)]^{\infty}$ and a subsequence $\left((\tilde{e}_t^{n_k})_{t\geq 1}\right)_{k\geq 1}$ convergent pointwise to $(\tilde{e}_t^{\infty})_{t\geq 1}$. Let $(\tilde{w}_t^{\infty})_{t\geq 1}$ be the payments corresponding to $(\tilde{e}_t^{\infty})_{t\geq 1}$ determined using ($\mathrm{FP}_t^{\mathrm{un}}$).

Note that, for any policy $(\tilde{e}_t)_{t\geq 1}$ in $[0, z\left(v'\left(b_1\left(1-\delta\right)\right)\right)]^{\infty}$, using that the payments $(\tilde{w}_t)_{t\geq 1}$ determined by ($\mathrm{FP}_t^{\mathrm{un}}$) are bounded, as well as continuity of $v$,

$$\frac{v\left(\left(1-\delta\right) b_1 + \left(1-\delta\right) \sum_{s=1}^{\infty} \delta^{s-1} \tilde{w}_s\right)}{1-\delta} - \sum_{t=1}^{\infty} \delta^{t-1} \psi\left(\tilde{e}_t\right) = \frac{v\left(b_1\left(1-\delta\right)\right)}{1-\delta}. \tag{21}$$

Notice also that

$$\sum_{t=1}^{\infty} \delta^{t-1} \psi\left(\tilde{e}_t^{n_k}\right) \to \sum_{t=1}^{\infty} \delta^{t-1} \psi\left(\tilde{e}_t^{\infty}\right)$$

as $k \to \infty$ (by continuity of $\psi$ and discounting). Therefore, we have (by Equation (21), using that $v$ is strictly increasing) that

$$\sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t^{n_k} \to \sum_{t=1}^{\infty} \delta^{t-1} \tilde{w}_t^{\infty}.$$

Since, also,

$$\sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t^{n_k} \to \sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t^{\infty},$$

we can conclude that $W\left((\tilde{e}_t^{\infty})_{t\geq 1}\right) = W^{\sup}$.

Our result will then follow if we can show that the contract defined by $(\tilde{e}_t^{\infty})_{t\geq 1}$ satisfies the principal's constraints ($\text{PC}_t$). Suppose with a view to contradiction that there is some $t^*$ at which the principal's constraint does not hold, and so

$$\tilde{w}_{t^*}^{\infty} > \sum_{s=t^*+1}^{\infty} \delta^{s-t^*}(\tilde{e}_s^{\infty} - \tilde{w}_s^{\infty}).$$

It is easily verified, from ($\text{FP}_t^{\text{un}}$) and the pointwise convergence of $(\tilde{e}_t^{n_k})_{t\geq 1}$ to $(\tilde{e}_t^{\infty})_{t\geq 1}$ and $(\tilde{w}_t^{n_k})_{t\geq 1}$ to $(\tilde{w}_t^{\infty})_{t\geq 1}$, that for large enough $k$

$$\tilde{w}_{t^*}^{n_k} > \sum_{s=t^*+1}^{\infty} \delta^{s-t^{n_k}}(\tilde{e}_s^{n_k} - \tilde{w}_s^{n_k})$$

contradicting that the contract determined by $(\tilde{e}_t^{n_k})_{t\geq 1}$ satisfies the principal's constraints ($\text{PC}_t$). □

We now establish the following regarding the non-degeneracy of optimal contracts.

**Lemma A.3.** *The principal obtains a strictly positive payoff in any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. *Moreover,* $\tilde{e}_t$ *and* $\tilde{w}_t$ *are strictly positive at all dates* $t$.

*Proof.* Consider effort set constant to some level $\tilde{e} > 0$. Let $g(\tilde{e}) = \sum_{t=1}^{\infty} \delta^{t-1}\tilde{w}_t$ be the NPV of payments that must be made to the agent when satisfying the indifference conditions ($\text{FP}_t^{\text{un}}$), given that effort is constant at $\tilde{e}$. Recall from Equation (21) that this satisfies

$$v\left((1-\delta)\, b_1 + (1-\delta)\, g(\tilde{e})\right) = \psi(\tilde{e}) + v\left((1-\delta)\, b_1\right).$$

Differentiating with respect to $\tilde{e}$ yields

$$g'(\tilde{e}) = \frac{\psi'(\tilde{e})}{(1-\delta)\, v'\left((1-\delta)\, b_1 + (1-\delta)\, g(\tilde{e})\right)}.$$

Since the principal's payoff is $\frac{1}{1-\delta}\tilde{e} - g(\tilde{e})$, and since $\psi'(0) = 0$, it follows that the principal's payoff is strictly positive for small positive $\tilde{e}$. Moreover, by Lemma 4.2, payments determined by the conditions ($\text{FP}_t^{\text{un}}$) rise over time approaching

$$\frac{\psi(\tilde{e})}{v'\left((1-\delta)\, b_1 + (1-\delta)\, g(\tilde{e})\right)}$$

which is $o(\tilde{e})$ as $\tilde{e} \to 0$ (i.e., vanishes much faster than $\tilde{e}$). It is then easy to see that, when $\tilde{e}$ is small enough, all the principal constraints ($\text{PC}_t$) are satisfied. Hence the contract determined from specifying constant effort $\tilde{e}$, for small $\tilde{e}$, is self-enforceable and generates a strictly positive payoff.

Now we show that, in an optimal contract, effort is strictly positive in every period. Suppose that payments $\tilde{w}_t$ are determined from effort using ($\mathrm{FP}_t^{\mathrm{un}}$). First note that the principal's continuation profits

$$\sum_{s=t}^{\infty} \delta^{s-t} \left(\tilde{e}_s - \tilde{w}_s\right)$$

must be strictly positive at all dates. Otherwise, this expression is zero at some date $t$, and so $\tilde{w}_{t-1} = 0$. Condition ($\mathrm{FP}_t^{\mathrm{un}}$) then implies that $\tilde{e}_{t-1} = 0$. Iterating backwards, we find the optimal profit is zero in contradiction with the previous claim. Suppose then that effort is zero at some date, and consider a date $t$ such that effort is zero at this date but strictly positive at the subsequent date. Then $\tilde{w}_t = 0$ and so the principal's constraint ($\mathrm{PC}_t$) is slack at date $t$. However, this contradicts Lemma 4.3. It then follows immediately from the restriction to "fastest payments" that all payments are also strictly positive, as stated in the lemma. $\square$

We now establish an important property of relational contracts: they become (approximately) stationary in the long run.

**Lemma A.4.** *Suppose that $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is an optimal relational contract satisfying* ($\mathrm{FP}_t^{\mathrm{un}}$). *Then, there exists an effort/payment pair $(\tilde{e}_\infty, \tilde{w}_\infty)$ such that $\lim_{t\to\infty} (\tilde{e}_t, \tilde{w}_t) = (\tilde{e}_\infty, \tilde{w}_\infty)$.*

*Proof.* **Step 0.** In this step we observe that, for an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ satisfying ($\mathrm{FP}_t^{\mathrm{un}}$),

$$\lim_{t\to\infty} \left(\tilde{w}_t - \frac{\psi\left(\tilde{e}_t\right)}{v'\left(\bar{c}_\infty\right)}\right) = 0.$$

This follows from Lemma 4.2, after noticing that $\left(\tilde{e}_t\right)_{t\geq 1}$ is bounded in an optimal contract.

**Step 1.** Define $\bar{e} \equiv \limsup_{t\to\infty} \tilde{e}_t$, which we know from Lemma A.1 is no greater than $z\left(v'\left(\bar{c}_\infty\right)\right)$ (recall that $z$ is the inverse of $\psi'$). We now show that, for any $e \in [0, \bar{e}]$,

$$\frac{\psi\left(e\right)}{v'\left(\bar{c}_\infty\right)} \leq \frac{\delta}{1-\delta}\left(e - \frac{\psi\left(e\right)}{v'\left(\bar{c}_\infty\right)}\right). \tag{22}$$

That is, the principal's constraints ($\mathrm{PC}_t$) would be satisfied if paying a constant wage $\frac{\psi(e)}{v'(\bar{c}_\infty)}$ per period, in return for effort $e \leq \bar{e}$. Note that, if the inequality (22) is satisfied at $\bar{e}$, then it is satisfied for all $e \in [0, \bar{e}]$; this follows because the left-hand side of (22) is convex, and equal to zero at zero, while the right hand side is concave, and also equal to zero at zero.
Assume now for the sake of contradiction that the inequality (22) is not satisfied for some $e \in [0, \bar{e}]$. Then we must have

$$\frac{\psi\left(\bar{e}\right)}{v'\left(\bar{c}_\infty\right)} > \frac{\delta}{1-\delta}\left(\bar{e} - \frac{\psi\left(\bar{e}\right)}{v'\left(\bar{c}_\infty\right)}\right). \tag{23}$$

Observe then that there is a sequence $(\varepsilon_t)_{t=1}^{\infty}$ convergent to zero such that, for all $t \geq 1$,

$$\tilde{e}_t - \tilde{w}_t \leq \bar{e} - \frac{\psi\left(\bar{e}\right)}{v'\left(\bar{c}_\infty\right)} + \varepsilon_t.$$

This follows because $\tilde{w}_t - \frac{\psi(\tilde{e}_t)}{v'(\bar{c}_\infty)} \to 0$ as $t \to \infty$ (by Step 0), because $e - \frac{\psi(e)}{v'(\bar{c}_\infty)}$ increases over effort levels $e$ in $[0, \bar{e}]$ (since $\psi'\left(\bar{e}\right) \leq v'\left(\bar{c}_\infty\right)$ by Lemma A.1), and by definition of $\bar{e}$ as $\limsup_{t\to\infty} \tilde{e}_t$.

We therefore have that

$$\limsup_{t\to\infty} \sum_{s=t+1}^{\infty} \delta^{s-t}\left(\tilde{e}_s - \tilde{w}_s\right) \leq \frac{\delta}{1-\delta}\left(\bar{e} - \frac{\psi\left(\bar{e}\right)}{v'\left(\bar{c}_\infty\right)}\right) < \frac{\psi\left(\bar{e}\right)}{v'\left(\bar{c}_\infty\right)},$$

where the last inequality holds by Equation (23). However, Step 0 implies that the superior limit of payments to the agent must be $\frac{\psi(\bar{e})}{v'(\bar{c}_\infty)}$, which implies that the principal's constraint ($\text{PC}_t$) is not satisfied at some time $t$. This contradicts the definition of $\bar{e}$ as $\limsup_{t\to\infty} \tilde{e}_t$ (with $\left(\tilde{e}_t\right)_{t\geq 1}$ the effort profile in a self-enforceable relational contract).

**Step 2.** We complete the proof by showing that $\liminf_{t\to\infty} \tilde{e}_t = \bar{e}$. This is immediate if $\bar{e} = 0$, so assume $\bar{e} > 0$. Assume, for the sake of contradiction, that $\liminf_{t\to\infty} \tilde{e}_t < \bar{e}$. In this case, there exists some $t' > 1$ such that $\tilde{e}_{t'} < \min\left\{\bar{e}, \tilde{e}_{t'+1}\right\}$.

**Step 2a.** We have

$$\tilde{w}_{t'} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_{t'+1} - \frac{\psi\left(\tilde{e}_{t'+1}\right)}{v'\left(\bar{c}_\infty\right)}\right). \tag{24}$$

This follows because (i) $\tilde{w}_{t'} \leq \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)}$ by Lemma 4.2 and the assumption that payments satisfy condition ($\text{FP}_t^{\text{un}}$); (ii) $\frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(\tilde{e}_{t'} - \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)}\right)$, by assumption that $\tilde{e}_{t'} < \bar{e}$ and by Step 1; and (iii) $\tilde{e}_{t'} - \frac{\psi(\tilde{e}_{t'})}{v'(\bar{c}_\infty)} \leq e_{t'+1} - \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_\infty)}$ because $z\left(v'\left(\bar{c}_\infty\right)\right) \geq \tilde{e}_{t'+1} > \tilde{e}_{t'}$ (recall that the inequality $z\left(v'\left(\bar{c}_\infty\right)\right) \geq \tilde{e}_{t'+1}$ is established in Lemma A.1).

**Step 2b.** We now show that the principal's constraint ($\text{PC}_t$) is slack at $t'$. Note first that, for any $t \geq 1$, we have

$$\begin{aligned}
\tilde{w}_{t+1} - \tilde{w}_t &= \frac{\bar{c}_{t+1} - \bar{c}_t}{\delta^t(1-\delta)} - \frac{\bar{c}_t - \bar{c}_{t-1}}{\delta^{t-1}(1-\delta)} \\
&\geq \frac{v(\bar{c}_{t+1}) - v(\bar{c}_t)}{\delta^t(1-\delta)v'(\bar{c}_t)} - \frac{v(\bar{c}_t) - v(\bar{c}_{t-1})}{\delta^{t-1}(1-\delta)v'(\bar{c}_t)} \\
&= \frac{\psi(\tilde{e}_{t+1}) - \psi(\tilde{e}_t)}{v'(\bar{c}_t)},
\end{aligned}$$

where we used that $v$ is concave and Lemma 4.2. Hence, we have that $\tilde{e}_{t+1} > \tilde{e}_t$ implies $\tilde{w}_{t+1} > \tilde{w}_t$.

Since $t'$ was chosen so that $\tilde{e}_{t'+1} > \tilde{e}_{t'}$, we have $\tilde{w}_{t'+1} > \tilde{w}_{t'}$. Hence,

$$\begin{aligned}
\tilde{w}_{t'} &< (1-\delta)\,\tilde{w}_{t'} + \delta \tilde{w}_{t'+1} \\
&\leq \delta \left( \tilde{e}_{t'+1} - \frac{\psi\left(\tilde{e}_{t'+1}\right)}{v'\left(\bar{c}_{\infty}\right)} \right) + \delta \sum_{s=t'+2}^{\infty} \delta^{s-t'-1} \left(\tilde{e}_s - \tilde{w}_s\right) \\
&\leq \sum_{s=t'+1}^{\infty} \delta^{s-t'} \left(\tilde{e}_s - \tilde{w}_s\right),
\end{aligned}$$

where the second inequality uses (i) Equation (24) from Step 2a, and (ii) the principal's constraint ($\text{PC}_t$) in period $t'+1$. The third inequality uses that $\tilde{w}_{t'+1} \leq \frac{\psi(\tilde{e}_{t'+1})}{v'(\bar{c}_{\infty})}$, which follows from Lemma 4.2.

**Step 2c.** We finish the proof with the following observation. The fact the principal's constraint ($\text{PC}_t$) is slack at time $t'$ (proven in Step 2b) contradicts Lemma 4.3, since effort is strictly higher at $t'+1$ than at $t'$. □

The following lemma determines that, in an optimal contract, effort is weakly decreasing.

**Lemma A.5.** *In an optimal contract, the effort policy $(\tilde{e}_t)_{t\geq 1}$ is a weakly decreasing sequence. Therefore, for all t, $\tilde{e}_t \geq \tilde{e}_{\infty} \equiv \lim_{s\to\infty} \tilde{e}_s$.*

*Proof.* By Lemma A.4, $(\tilde{e}_t)_{t=1}^{\infty}$ is a convergent sequence, so using the notation in its proof, we have $\tilde{e}_{\infty} = \bar{e}$. Step 2 in the proof of Lemma A.4 proves that there is no time $t'$ such that $\tilde{e}_{t'} < \min\{\bar{e}, \tilde{e}_{t'+1}\}$. Hence, there is no $t'$ such that $\tilde{e}_{t'} < \tilde{e}_{\infty}$.

Now suppose, for the sake of contradiction, that $(\tilde{e}_t)_{t=1}^{\infty}$ is not a weakly decreasing sequence. Thus, there exists a date $t'$ where $\max_{t>t'} \tilde{e}_t > \tilde{e}_{t'}$ (the maximum exists by the first part of this proof, and because $\lim_{t\to\infty} \tilde{e}_t = \tilde{e}_{\infty}$ by Lemma A.4). Let $t^*(t')$ be the smallest value $t > t'$ where the maximum is attained, that is, $\tilde{e}_{t^*(t')} = \max_{t>t'} \tilde{e}_t$.

For any $s > t^*(t')$,

$$\tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} > \tilde{e}_{t^*(t')} - \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{t^*(t')})} \geq \tilde{e}_{t^*(t')} - \frac{\psi(\tilde{e}_{t^*(t')})}{v'(\bar{c}_{s-1})} \geq \tilde{e}_s - \frac{\psi(\tilde{e}_s)}{v'(\bar{c}_{s-1})} > \tilde{e}_s - \tilde{w}_s. \tag{25}$$

The first inequality follows from Lemma 4.2; the second inequality follows because $\bar{c}_{s-1} \geq \bar{c}_{t^*(t')}$. The third inequality follows because $e - \frac{\psi(e)}{v'(\bar{c}_{s-1})}$ is increasing in $e$ over $[0, z\left(v'\left(\bar{c}_{\infty}\right)\right)]$, and because $\tilde{e}_s \leq \tilde{e}_{t^*(t')}$ for $s > t^*\left(t'\right)$ by definition of $t^*\left(t'\right)$. The fourth inequality follows because $\tilde{w}_s > \frac{\psi(\tilde{e}_s)}{v'(\bar{c}_{s-1})}$ by Lemma 4.2 and because $\tilde{e}_s > 0$ by Lemma A.3.

Equation (25) implies that

$$\tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} > (1-\delta) \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right),$$

so that

$$\begin{aligned}
\sum_{s=t^*(t')}^{\infty} \delta^{s-t^*(t')} \left(\tilde{e}_s - \tilde{w}_s\right) &= \tilde{e}_{t^*(t')} - \tilde{w}_{t^*(t')} + \delta \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right) \\
&> (1-\delta) \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right) + \delta \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right) \\
&= \sum_{s=t^*(t')+1}^{\infty} \delta^{s-t^*(t')-1} \left(\tilde{e}_s - \tilde{w}_s\right). \qquad (26)
\end{aligned}$$

Recall from Lemma 4.3 that the principal's constraint must hold with equality at $t^*(t') - 1$ (since $\tilde{e}_{t^*(t')} > \tilde{e}_{t^*(t')-1}$ by the definition of $t^*(t')$). The inequality (26), then implies (given satisfaction of the principal's constraint ($\text{PC}_t$)) that $\tilde{w}_{t^*(t')-1} > \tilde{w}_{t^*(t')}$. But then, recalling Lemma 4.2 (as well as Lemma A.3), we have

$$\frac{\psi\left(\tilde{e}_{t^*(t')-1}\right)}{v'\left(\bar{c}_{t^*(t')-1}\right)} > \tilde{w}_{t^*(t')-1} > \tilde{w}_{t^*(t')} > \frac{\psi(\tilde{e}_{t^*(t')})}{v'\left(\bar{c}_{t^*(t')-1}\right)}.$$

Hence, $\tilde{e}_{t^*(t')-1} > \tilde{e}_{t^*(t')}$, contradicting the definition of $t^*(t')$. □

Having shown that the effort is weakly decreasing in an optimal relational contract (Lemma A.5) we now show that, in fact, it is strictly decreasing when the principal's constraint holds with equality.

**Lemma A.6.** *If the principal's constraint* ($\text{PC}_t$) *holds with equality at some date* $t^*$*, then* $\tilde{e}_{t^*} > \tilde{e}_{t^*+1}$. *Hence, by Lemma 4.3, the principal's constraint also holds with equality at* $t^*+1$.

*Proof.* The same arguments we used in Lemma A.5 to establish the inequalities in (25) imply that $\tilde{e}_{t^*+1} - \tilde{w}_{t^*+1} > \tilde{e}_s - \tilde{w}_s$ for all $s > t^* + 1$ (the only requirement there was that, relative to the date under consideration, the future dates $s$ have weakly lower effort; that indeed $\tilde{e}_s \leq \tilde{e}_{t^*+1}$ for all $s > t^* + 1$ follows from Lemma A.5). In turn, this means that, if the principal's constraint ($\text{PC}_t$) holds with equality at $t^*$, then $\tilde{w}_{t^*} > \tilde{w}_{t^*+1}$. Indeed, because the principal's constraint holds with equality at $t^*$,

$$\begin{aligned}
\tilde{w}_{t^*} &= \delta\Bigg(\tilde{e}_{t^*+1} - \tilde{w}_{t^*+1} + \delta \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s)\Bigg) \\
&> \delta\Bigg((1-\delta) \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s) + \delta \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-2}(\tilde{e}_s - \tilde{w}_s)\Bigg) \\
&= \sum_{s=t^*+2}^{\infty} \delta^{s-t^*-1}(\tilde{e}_s - \tilde{w}_s) \\
&\geq \tilde{w}_{t^*+1}.
\end{aligned}$$

The final inequality follows from the principal's constraint ($\mathrm{PC}_t$) at date $t^*+1$. Using Lemma 4.2, we have $\tilde{e}_{t^*+1} < \tilde{e}_{t^*}$. Hence, by Lemma 4.3, the principal's constraint holds with equality at $t^*+1$. $\square$

Lemma A.6 implies that, given payments satisfy condition ($\mathrm{FP}_t^{\mathrm{un}}$), if the principal's constraint ($\mathrm{PC}_t$) holds with equality at some date, then effort is strictly decreasing forever after (and the principal's constraints ($\mathrm{PC}_t$) hold with equality forever after). Our next goal is therefore to establish the condition under which the principal attains the first-best payoff, and, when this condition fails, establish that there is necessarily a date at which the principal's constraint is satisfied with equality.

**Lemma A.7.** *An optimal contract achieves the first-best payoff of the principal if and only if Condition (7) holds. If this condition is not satisfied, then there is a time $t^* \in \mathbb{N}$ such that the principal's constraint is slack if and only if $t < t^*$. Hence, effort is constant up to date $t^*-1$ and strictly decreases from date $t^*$.*[21]

*Proof.* Consider payments satisfying ($\mathrm{FP}_t^{\mathrm{un}}$), for all $t$, and determined given the first-best effort (this is $e^{FB}(b_1)$ in Proposition 3.1). Lemma 4.2 shows that the payments increase over time and tend to $\frac{\psi(e^{FB}(b_1))}{v'(c^{FB}(b_1))}$. Hence, the upper limit of payments is given by $\frac{\psi(e^{FB}(b_1))}{v'(c^{FB}(b_1))}$ while the lower limit of per-period profits is given by $e^{FB}(b_1) - \frac{\psi(e^{FB}(b_1))}{v'(c^{FB}(b_1))}$, which establishes Condition (7) is both necessary and sufficent for implementation of the first best.

Assume now that Condition (7) fails, and fix an optimal contract that is not first best. Lemma A.6 established that there are two possibilities. First, we might have a finite date $t^* \in \mathbb{N}$, with the principal's constraint ($\mathrm{PC}_t$) holding with equality at $t^*$, and every subsequent date, but slack at dates $t^*-1$ and earlier. In this case, effort is constant from the initial date up to $t^*-1$ (by Lemma 4.3) and strictly decreases from date $t^*$. Second, we might have that the principal's constraint ($\mathrm{PC}_t$) is slack at all dates. Effort is then constant over all periods (by Lemma 4.3), but not first-best. The result in the lemma is established if we can show this second case does not occur; so assume for a contradiction that it does. Letting $\tilde{e}_\infty$ be the constant effort level and $\bar{c}_\infty$ equilibrium consumption, Proposition 3.1 then implies that $v'(\bar{c}_\infty) \neq \psi'(\tilde{e}_\infty)$. By Lemma A.1, we have $v'(\bar{c}_\infty) > \psi'(\tilde{e}_\infty)$. By Lemma A.3, we have $\tilde{e}_\infty > 0$.

Assuming that payments to the agent satisfy Condition ($\mathrm{FP}_t^{\mathrm{un}}$) for all $t$, we have $\tilde{w}_t$ increasing over time and converging to $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}$ from below (from Lemma 4.2). We claim then that

$$\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} = \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right). \tag{27}$$

[21] We leave open the possibility that effort could either be constant up to date $t^*$, or instead only up to $t^*-1$ (i.e., that $\tilde{e}_{t^*} < \tilde{e}_{t^*-1}$).

If instead $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} > \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right)$, then, for large enough $t$ we must have

$$\tilde{w}_t > \sum_{s=t+1}^{\infty} \delta^{s-t}\left(\tilde{e}_\infty - \tilde{w}_t\right),$$

so the principal's constraint is violated at $t$. If instead $\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} < \frac{\delta}{1-\delta}\left(\tilde{e}_\infty - \frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)}\right)$, we have $\tilde{w}_t$ remains bounded below $\sum_{s=t+1}^{\infty}\delta^{s-t}\left(\tilde{e}_\infty - \tilde{w}_t\right)$. Without violating ($\mathrm{PC}_t$), effort can then be increased by a small constant amount across all periods, with payments determined via condition ($\mathrm{FP}_t^{\mathrm{un}}$). This increases profits (see the proof of Lemma A.3 for a similar argument).

Note then that Condition (27) can be written as

$$\frac{\psi(\tilde{e}_\infty)}{v'(\bar{c}_\infty)} = \delta\tilde{e}_\infty.$$

Because $\psi$ is strictly convex, we have

$$\frac{\psi'\left(\tilde{e}_\infty\right)}{v'\left(\bar{c}_\infty\right)} > \delta.$$

We will now consider an adjusted contract in which effort increases at date 1 by $\varepsilon > 0$, raising the disutility of effort at date 1 by $\psi(\tilde{e}_\infty + \varepsilon) - \psi(\tilde{e}_\infty)$. Because payments to the agent increase at all dates under condition ($\mathrm{FP}_t^{\mathrm{un}}$), the new policy will not satisfy the principal's constraint ($\mathrm{PC}_t$) if this is the only adjustment. We therefore simultaneously reduce effort from some fixed date 2 onwards by $\kappa(\varepsilon) > 0$ to be determined (i.e., effort is given by $e_t = \tilde{e}_\infty - \kappa(\varepsilon)$ for $t \geq 2$).

We let $\bar{c}_\infty(\varepsilon, \kappa(\varepsilon))$ denote equilibrium consumption under the new plan (naturally, $\bar{c}_\infty(0,0)$ is consumption under the original plan). The new consumption satisfies

$$\begin{aligned}\frac{v\left(\bar{c}_\infty\left(\varepsilon,\kappa\left(\varepsilon\right)\right)\right)}{1-\delta} - \frac{v\left(\bar{c}_\infty\left(0,0\right)\right)}{1-\delta} =& \psi\left(\tilde{e}_\infty + \varepsilon\right) - \psi\left(\tilde{e}_\infty\right) \\ &- \frac{\delta}{1-\delta}\left(\psi\left(\tilde{e}_\infty\right) - \psi\left(\tilde{e}_\infty - \kappa\left(\varepsilon\right)\right)\right)\end{aligned}$$

or

$$\bar{c}_\infty(\varepsilon,\kappa(\varepsilon)) = v^{-1}\left(\begin{array}{c}(1-\delta)(\psi(\tilde{e}_\infty+\varepsilon)-\psi(\tilde{e}_\infty)) \\ -\delta(\psi(\tilde{e}_\infty)-\psi(\tilde{e}_\infty-\kappa(\varepsilon))) + v(\bar{c}_\infty(0,0))\end{array}\right)$$

To determine the value for $\kappa(\varepsilon)$, define the following function

$$f(\varepsilon,k) \equiv \frac{\psi\left(\tilde{e}_\infty - k\right)}{v'(\bar{c}_\infty(\varepsilon,k))} - \delta\left(\tilde{e}_\infty - k\right). \tag{28}$$

We then define $\kappa(\varepsilon)$ by $f(\varepsilon,\kappa(\varepsilon)) = 0$ for positive $\varepsilon$ in a neighborhood of 0. We will use the implicit function theorem to show that such a local solution $\kappa(\varepsilon)$ exists.

To apply the implicit function theorem, note that $f(\varepsilon, k)$ is continuously differentiable in a neighborhood of $(\varepsilon, k) = (0, 0)$. The derivative of $f(\varepsilon, k)$ with respect to $k$, evaluated at $(\varepsilon, k) = (0, 0)$, is

$$f_2(0,0) = \delta - \frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} + v''(\bar{c}_\infty(0,0)) \left( \frac{\delta \psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3} \right) \psi(\tilde{e}_\infty).$$

This is strictly negative, using that $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} > \delta$. The derivative $f(\varepsilon, k)$ instead with respect to $\varepsilon$, evaluated at $(\varepsilon, k) = (0, 0)$, is

$$f_1(0,0) = -v''(\bar{c}_\infty(0,0)) \left( \frac{(1-\delta)\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3} \right) \psi(\tilde{e}_\infty).$$

The implicit function theorem then gives us that $\kappa$ is locally well-defined by $f(\varepsilon, \kappa(\varepsilon)) = 0$ on some interval around 0, unique, and continuously differentiable, with derivative approaching

$$\begin{aligned}
\kappa'(0) &= -\frac{f_1(0,0)}{f_2(0,0)} \\
&= \frac{v''(\bar{c}_\infty(0,0)) \left( \frac{(1-\delta)\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3} \right) \psi(\tilde{e}_\infty)}{\delta - \frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} + v''(\bar{c}_\infty(0,0)) \left( \frac{\delta\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))^3} \right) \psi(\tilde{e}_\infty)} \\
&< \frac{1-\delta}{\delta}
\end{aligned} \tag{29}$$

as $\varepsilon \to 0$ (the strict inequality follows because $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))} > \delta$).

For small enough $\varepsilon$, the new effort policy and payments defined by condition ($\mathrm{FP}_t^{\mathrm{un}}$) satisfy the principal's constraints ($\mathrm{PC}_t$). This can be seen, for instance, by observing that, when $\varepsilon$ is small, the constraint ($\mathrm{PC}_t$) remains slack at date $t = 1$. For all other dates, the satisfaction of the constraint ($\mathrm{PC}_t$) follows from $f(\varepsilon, \kappa(\varepsilon)) = 0$, and by Lemma 4.2.

It remains to show that, for small enough positive $\varepsilon$, the principal's profits strictly increase. The NPV of effort increases by

$$\varepsilon - \frac{\delta}{1-\delta}\kappa(\varepsilon) = \left(1 - \frac{\delta}{1-\delta}\kappa'(0)\right)\varepsilon + o(\varepsilon).$$

From the inequality (29) we have $1 - \frac{\delta}{1-\delta}\kappa'(0) > 0$, and so the increase in effort is strictly positive for $\varepsilon$ small enough. Using that payments continue to satisfy Condition ($\mathrm{FP}_t^{\mathrm{un}}$), a marginal increase in the NPV of effort is compensated by an increase in the NPV of payments to the agent by $\frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))}$. Therefore, the principal's payoff under the new policy increases by

$$\left(1 - \frac{\psi'(\tilde{e}_\infty)}{v'(\bar{c}_\infty(0,0))}\right)\left(1 - \frac{\delta}{1-\delta}\kappa'(0)\right)\varepsilon + o(\varepsilon)$$

which is strictly positive for small enough $\varepsilon$, recalling that $v'(\bar{c}_\infty(0,0)) > \psi'(\tilde{e}_\infty)$. □

We have established that, for any optimal contract that does not attain the first-best payoff of the principal, there is a date $\bar{t} \geq 1$ such that effort is constant up to this date, and subsequently strictly decreasing to a value $\tilde{e}_\infty$, as stated in the proposition. Let us now show that $\tilde{e}_\infty > 0$, which requires only ruling out $\tilde{e}_\infty = 0$.

**Lemma A.8.** *Suppose the principal cannot attain the first-best payoff. In any optimal contract, the limiting value of effort $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ is strictly positive.*

*Proof.* Consider any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ that does not achieve the principal's first-best payoff, and suppose for a contradiction that $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ is equal to zero.

Because $\psi'(0) = 0$, there exists a value $\hat{e} \in (0, \tilde{e}_1)$ satisfying $\frac{\psi(\hat{e})}{v'(\bar{c}_\infty)} \leq \delta\hat{e}$. Note that $\lim_{t\to\infty} (\tilde{e}_t - \tilde{w}_t) = 0$. Hence, there is $T$ satisfying that (i) $\frac{\delta^{T-1}}{1-\delta}\psi(\hat{e}) < \psi(\tilde{e}_1) - \psi(\hat{e})$, (ii) $\tilde{e}_t < \hat{e}$ for all $t \geq T$, and (iii) $\tilde{e}_t - \tilde{w}_t < \hat{e} - \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)}$ for all $t \geq T$.

Now, let $m \equiv \sum_{t=T}^{\infty} \delta^{t-1} (\psi(\hat{e}) - \psi(\tilde{e}_t)) > 0$, and define a new contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$ by specifying $\tilde{e}'_1$ to satisfy

$$\psi(\tilde{e}'_1) = \psi(\tilde{e}_1) - m,$$

$\tilde{e}'_t = \tilde{e}_t$ for all $t \in \{2, ..., T-1\}$, as well as $\tilde{e}'_t = \hat{e}$ for all $t \geq T$. Then let this effort specification determine the other variables, assuming the condition $(\mathrm{FP}_t^{\mathrm{un}})$ is satisfied.

Note we have

$$\sum_{t=1}^{\infty} \delta^{t-1}\psi(\tilde{e}'_t) = \sum_{t=1}^{\infty} \delta^{t-1}\psi(\tilde{e}_t)$$

and hence the value of $\bar{c}_\infty$ and the NPV of the payments are the same in the new contract. In addition, $\psi(\tilde{e}'_1) = \psi(\tilde{e}_1) - m > \psi(\tilde{e}_1) - \frac{\delta^{T-1}}{1-\delta}\psi(\hat{e}) > \psi(\hat{e})$. Hence, $\tilde{e}'_1 > \hat{e}$. Using the strict convexity of $\psi$, the principal's payoff is strictly higher in the new contract. To obtain a contradiction, we therefore need only show that the new contract is self-enforceable, which requires showing that it satisfies the principal's constraints $(\mathrm{PC}_t)$.

Consider now the per-period profits $\tilde{e}'_t - \tilde{w}'_t$ at each date $t$ under the new contract. Note that $\tilde{w}'_t < \tilde{w}_t$ for all $t < T$ by concavity of $v$. This shows that per-period profits satisfy $\tilde{e}'_t - \tilde{w}'_t > \tilde{e}_t - \tilde{w}_t$ for $t \in \{2, ..., T-1\}$. In addition, per-period profits at each date $t \geq T$ are greater than $\hat{e} - \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)} > e_t - w_t$, since $\tilde{w}'_t < \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)}$ for $t \geq T$ (this follows by Lemma 4.2). This immediately implies the satisfaction of the principal's constraint $(\mathrm{PC}_t)$ at all dates $t < T$. For $t \geq T$, we have

$$\tilde{w}'_t < \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)} \leq \frac{\delta}{1-\delta}\left(\hat{e} - \frac{\psi(\hat{e})}{v'(\bar{c}_\infty)}\right) < \sum_{s=t+1}^{\infty} \delta^{s-t}(\tilde{e}'_s - \tilde{w}'_s).$$

Hence, the principal's constraint is satisfied at these dates as well. □

We now establish that, for some configuations of the problem, $\bar{t} > 1$. In this case effort is constant in the initial periods, before strictly decreasing.

**Lemma A.9.** *For any $v$ and $\psi$ admitted in the model set-up, there exists a discount factor $\delta$ and initial balance $b_1$ such that (i) the principal's payoff in an optimal contract is less than the first-best payoff, and (ii) for any optimal contract, the principal's constraint* (PC$_t$) *is slack for at least $t = 1, 2$.*

*Proof.* Fix $v$ and $\psi$ satisfying the properties in the model set-up, and fix a scalar $\gamma > 0$. Define the function $b_1(\delta) = \frac{\gamma}{1-\delta}$. As explained in Section 4.1, there is then a threshold value $\delta^* \in (0, 1)$ such that $\delta \geq \delta^*$ and $b_1 = b_1(\delta)$ implies the principal can attain the first-best payoff in a self-enforceable contract, while $\delta < \delta^*$ and $b_1 = b_1(\delta)$ implies this is not the case. We aim to show that the principal's constraint (PC$_t$) is slack over some initial periods when $\delta$ is below, but close enough to, $\delta^*$, and when $b_1 = b_1(\delta)$. We do so in three steps. In these steps, we let $\delta$ parameterize the environment, leaving $b_1 = b_1(\delta)$ implicit.

**Step 1.** First, by considering constant effort policies, it is easily seen that the principal's payoff in an optimal contract approaches that for parameter $\delta^*$ as $\delta \to \delta^*$ from below.

**Step 2.** Next, let $e^*$ be the first-best effort for parameter $\delta^*$. We show that, for any $\varepsilon > 0$ and period $T$, there exists $\hat{\delta}(T, \varepsilon)$ such that, for $\delta \in (\hat{\delta}(T, \varepsilon), \delta^*)$, $\max_{t \leq T} \left|\tilde{e}_t - e^*\right| < \varepsilon$, where $(\tilde{e}_t)_{t\geq 1}$ is any optimal effort policy for parameter $\delta$.

By Lemma A.1, any optimal effort policy is contained in $[0, z(v'(\gamma))]^\infty$. The principal's payoff under a self-enforceable relational contract with arbitrary effort policy $(\tilde{e}_t)_{t=1}^\infty$ (and satisfying Condition (FP$_t^{\text{un}}$)) is

$$\sum_{t=1}^{\infty} \delta^{t-1} \tilde{e}_t - \frac{v^{-1}\left((1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \psi(\tilde{e}_t) + v(\gamma)\right)}{1-\delta} + b_1(\delta)$$

which varies continuously in $\delta$, with the continuity uniform over $\delta \leq \delta^*$ and effort policies contained in $[0, z(v'(\gamma))]^\infty$.

Fix $\delta = \delta^*$, and fix any $\varepsilon > 0$ and $T \in \mathbb{N}$. There is then $\nu > 0$ such that the following is true. For any effort policy $(\tilde{e}_t)_{t\geq 1}$ contained in $[0, z(v'(\gamma))]^\infty$ and satisfying $\max_{t\leq T} \left|\tilde{e}_t - e^{FB}(b_1)\right| \geq \varepsilon$, and for payments satisfying Condition (FP$_t^{\text{un}}$), the principal's payoff is less than that sustained by the first-best contract by at least $\nu$. This follows from uniqueness of the first-best policy and continuity of the principal's objective in the effort policy $(\tilde{e}_t)_{t\geq 1}$. However, the aforementioned continuity of the principal's payoff in $\delta$, together with Step 1, implies that, when $\delta$ is close enough to (but below) $\delta^*$, any effort policy satisfying $\max_{t\leq T} \left|\tilde{e}_t - e^{FB}(b_1)\right| \geq \varepsilon$ cannot be optimal.

**Step 3.** Notice that, for $\delta = \delta^*$, under the first-best policy, the principal's constraint ($\text{PC}_t$) is slack at both dates $t = 1$ and $t = 2$. It then follows that, provided $\varepsilon$ is taken small enough, and $T$ large enough, these constraints must also be slack under an optimal policy when $\delta \in (\hat{\delta}(T, \varepsilon), \delta^*)$. □

*(End of the proof of Proposition 4.3.)* □

**Proof of Proposition 4.4**

*Proof.* Fix an optimal relational contract $(\tilde{e}_t, \tilde{w}_t, \tilde{c}_t, \tilde{b}_t)_{t \geq 1}$. We want to show that Condition ($\text{FP}^{\text{un}}_t$) holds at all dates $t > \bar{t}$ (with $\bar{t}$ identified in the proposition). Suppose for a contradiction this is not the case, and so the condition fails at some $t' > \bar{t}$. Since the contract is optimal, the condition in Equation (5) holds (by the arguments in the main text). We therefore have

$$\begin{aligned}
&\frac{v\left((1-\delta)\left(b_1 + \sum_{s=1}^{t'-1} \delta^{s-1} \tilde{w}_s\right)\right)}{1-\delta} - \sum_{s=1}^{t'-1} \delta^{s-1} \psi(\tilde{e}_s) \\
<&\frac{v\left((1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1} \tilde{w}_s\right)\right)}{1-\delta} - \sum_{t=1}^{\infty} \delta^{t-1} \psi(\tilde{e}_t) \\
=&\frac{v\left((1-\delta) b_1\right)}{1-\delta}.
\end{aligned}$$

Now consider the relational contract with the same effort and consumption but where payments ensure the satisfaction of Condition ($\text{FP}^{\text{un}}_t$) at all dates. Denote the payments by $\tilde{w}'_t$ for all $t$. We have

$$\begin{aligned}
&\frac{v\left((1-\delta)\left(b_1 + \sum_{s=1}^{t'-1} \delta^{s-1} \tilde{w}'_s\right)\right)}{1-\delta} - \sum_{s=1}^{t'-1} \delta^{s-1} \psi(\tilde{e}_s) \\
=&\frac{v\left((1-\delta)\left(b_1 + \sum_{s=1}^{\infty} \delta^{s-1} \tilde{w}'_s\right)\right)}{1-\delta} - \sum_{t=1}^{\infty} \delta^{t-1} \psi(\tilde{e}_t). \\
=&\frac{v\left((1-\delta) b_1\right)}{1-\delta}.
\end{aligned}$$

Therefore,

$$\sum_{s=t'}^{\infty} \delta^{s-t'} \tilde{w}'_s < \sum_{s=t'}^{\infty} \delta^{s-t'} \tilde{w}_s$$

and hence the principal's constraint ($\text{PC}_t$) is slack at $t'$. Because $t' > \bar{t}$, we have $\tilde{e}_{t'} < \tilde{e}_{t'-1}$. This contradicts Lemma 4.3.

The final part of the proposition concerns the observation that payments are strictly decreasing from $\bar{t} + 1$ onwards. Considering the principal's constraint ($\text{PC}_t$) holds with equality at these dates, it is enough to observe that profits $\tilde{e}_t - \tilde{w}_t$ are strictly decreasing in $t$ after date $\bar{t}+1$. This follows by the same arguments that establish the inequalities in Equation (25). That the agent's balances are strictly increasing from date $\bar{t}+1$ is established in the main text.

□

## A.3 Proofs of the results in Section 5

### Proof of Proposition 5.1

*Proof.* **Necessity**. Consider supposed equilibrium strategies $(\alpha_t)_{t\geq 1}$ and $(\sigma_t)_{t\geq 1}$ inducing outcomes that coincide with a feasible relational contract. Suppose Condition ($\text{AC}_t^{\text{ob}}$) fails at some date, and let $t$ be the earliest such date. Then the agent obtains a strictly higher payoff by abiding by the contract to date $t-1$ and then consuming $\tilde{b}_t(1-\delta)$ from date $t$ onwards, as opposed to following the putative equilibrium strategy. That is, the agent has a profitable deviation. If instead ($\text{PC}_t$) fails at some date, then there is a first date $t$ at which it is violated. The principal has a profitable deviation by ceasing all payments from then on.

**Sufficiency**. Let $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ be a feasible contract satisfying conditions ($\text{AC}_t^{\text{ob}}$) and ($\text{PC}_t$). Specify strategies $(\alpha_t)_{t\geq 1}$ and $(\sigma_t)_{t\geq 1}$ for the agent and principal as follows. Provided that $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$, the agent consumes $\tilde{c}_t$ and chooses effort $\tilde{e}_t$. Otherwise, the agent consumes $\max\{0, (1-\delta)\, b(h_t)\}$, with $b(h_t)$ the balance determined recursively from $b_1$, given the history $h_t = (e_s, c_s, w_s)_{s=1}^{t-1}$ and chooses effort zero (if $b(h_t) < 0$, the agent violates his intertemporal budget constraint and earns payoff $-\infty$; but this is the highest payoff that can be obtained). Provided that $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$, and $(e_t, c_t) = (\tilde{e}_t, \tilde{c}_t)$, the principal pays $\tilde{w}_t$. Otherwise, she pays zero.

Now, we can check that the players do not want to deviate at any history. Consider a date-$t$ history at which $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$. Given the principal's strategy, the agent obtains continuation payoff $\sum_{s=t}^{\infty} \delta^{s-t}(v(\tilde{c}_s) - \psi(\tilde{e}_s))$ by remaining faithful to the specified strategy. If the agent deviates, then he is paid zero from $t$ onwards. The agent's optimal per-period consumption if he deviates is $\max\{0, (1-\delta)\tilde{b}_t\}$. Therefore, the highest payoff the agent can obtain under a deviation is $\frac{v(\max\{0,(1-\delta)\tilde{b}_t\})}{1-\delta} \in [-\infty, +\infty)$. The inequality ($\text{AC}_t^{\text{ob}}$) states that this is less than $\sum_{s=t}^{\infty} \delta^{s-t}\left(v\left(\tilde{c}_s\right) - \psi(\tilde{e}_s)\right)$. At any other history (i.e., if $(e_s, c_s, w_s) \neq (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for some $s < t$) the agent will never be paid again and so consuming $\max\{0, (1-\delta)b\,(h_t)\}$ and putting no effort is optimal.

On the principal's side, for a date-$t$ history at which $(e_s, c_s, w_s) = (\tilde{e}_s, \tilde{c}_s, \tilde{w}_s)$ for all $s < t$, and given $(e_t, c_t) = (\tilde{e}_t, \tilde{c}_t)$, the principal's continuation payoff from following the specified strategy is $\sum_{s=t+1}^{\infty} \delta^{s-t}(\tilde{e}_s - \tilde{w}_s)$ which is larger than the date-$t$ payment $\tilde{w}_t$ by Condition

(PC$_t$). Hence, the principal prefers to follow the specified strategy. Following any deviation, given the agent's strategy, the principal clearly finds paying zero optimal. $\square$

### Proof of Lemma 5.1

*Proof.* Fix an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ and suppose that Condition (8) is not satisfied for some $t$. Since the contract is self-enforceable, we have, for all $t$, the inequality in Equation (AC$_t^{\text{ob}}$) holds. First note that, because the contract is optimal, the inequality in Equation (AC$_t^{\text{ob}}$) cannot hold as a strict inequality at date $t = 1$. Otherwise, both $\tilde{c}_1$ and $\tilde{w}_1$ can be reduced by the same small amount $\varepsilon > 0$, leaving $\tilde{b}_2$ unchanged. The rest of the relational contract can also be kept the same. This adjustment leaves unchanged the constraints of the principal (PC$_t$) in all periods $t > 1$, and slackens the principal's constraint at $t = 1$. It also leaves the constraints of the agent (AC$_t^{\text{ob}}$) unaffected in all periods $t > 1$, and if $\varepsilon > 0$ is small enough, the agent's date-1 constraint is still satisfied. The principal's payoff strictly increases, showing that the original contract was not optimal.

Now suppose that the inequality in Equation (AC$_t^{\text{ob}}$) is strict at any $t > 1$. Then we can consider a new contract with payment reduced at date $t$ by $\varepsilon > 0$, and with payment increased at date $t-1$ by $\delta\varepsilon$ (we show below that the new payment at time $t$ is non-negative). We can also hold the consumption and effort profile the same. The change increases $\tilde{b}_t$ by $\varepsilon$ and, for appropriately chosen $\varepsilon$, the constraint (AC$_t^{\text{ob}}$) holds with equality. All the other agent constraints (AC$_t^{\text{ob}}$) are unaffected. The principal's constraint (PC$_t$) is slackened at date $t$, and the principal's constraints for all other dates are unaffected. The principal's payoff in the contract remains unchanged. Therefore, the adjusted contract remains optimal.

Now note that the above adjustments can be applied sequentially at the dates for which (AC$_t^{\text{ob}}$) holds as a strict inequality starting with the earliest one (which is at least $t = 2$ by the argument in the first paragraph), yielding a contract for which (AC$_t^{\text{ob}}$) holds at all dates.

Finally, note that when Condition (FP$_t^{\text{ob}}$) is satisfied for all $t$, all payments to the agent are non-negative given that the disutility of effort is non-negative. This ensures that the above adjustments also yield a contract that is feasible. Also, balances $\tilde{b}_t$ remain strictly positive at all $t$, as the agent's constraint (AC$_t^{\text{ob}}$) for date 1 guarantees the agent earns a strictly positive payoff. $\square$

### Proof of Proposition 5.2

*Proof.* Suppose that condition (9) is satisfied and recall that, by Lemma 5.1, we can restrict the focus to contracts satisfying Equation (FP$_t^{\text{ob}}$). As in the main text, consider the contract where the agent puts effort $e^{FB}(b_1)$ per period, is paid $w^{FB}(b_1)$ in each period, and where the agent consumes $c^{FB}(b_1)$ in each period.

The agent's balance remains constant at $b_1$, and by the agent's indifference condition in Proposition 3.1, the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) is satisfied. Condition (9) is simply the principal's constraint ($\mathrm{PC}_t$). Hence, the first-best contract is self-enforceable.

The converse result is proved analogously: if the principal attains the first-best payoff in an optimal relational contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, such contract should satisfy $\tilde{e}_t = e^{FB}(b_1)$ and $\tilde{c}_t = c^{FB}(b_1)$ for all $t$ (by the uniqueness of the first-best effort and consumption established in Proposition 3.1). By the arguments in the main text, the agent is paid $w^{FB}(b_1)$ in each period in a contract satisfying Equation (8) for all dates $t$. The principal's constraint is then satisfied only if the inequality in Equation (9) holds. □

### Proof of Proposition 5.3

*Proof.* It will be useful to write the recursive problem in the main text by substituting out agent effort. To this end, define a function $\hat{e}$ by

$$\hat{e}(c_t, b_t, b_{t+1}) \equiv \psi^{-1}\Big(v(c_t) + \tfrac{\delta}{1-\delta} v\left((1-\delta)\, b_{t+1}\right) - \tfrac{1}{1-\delta} v((1-\delta) b_t)\Big) \tag{30}$$

for $c_t, b_t, b_{t+1} > 0$ and $v(c_t) + \frac{\delta}{1-\delta} v\left((1-\delta)\, b_{t+1}\right) - \frac{1}{1-\delta} v((1-\delta) b_t) \geq 0$. We will focus throughout on relational contracts that satisfy the "fastest payments" condition ($\mathrm{FP}_t^{\mathrm{ob}}$). Hence, given contractual variables $\tilde{c}_t$, $\tilde{b}_t$ and $\tilde{b}_{t+1}$, the date-$t$ effort must be given by $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$.

We can then write the principal's optimal payoff given balance $\tilde{b}_t > 0$ (which we establish below can be attained by a self-enforceable contract) as follows:

$$V(\tilde{b}_t) = \max_{c_t, b_{t+1} > 0} \left(\hat{e}(c_t, \tilde{b}_t, b_{t+1}) - (\delta b_{t+1} - \tilde{b}_t + c_t) + \delta V(b_{t+1})\right) \tag{31}$$

subject to the principal's constraint

$$\delta b_{t+1} - \tilde{b}_t + c_t \leq \delta V\left(b_{t+1}\right) \tag{32}$$

and to the requirement that the implied effort is non-negative, i.e.

$$v(c_t) + \tfrac{\delta}{1-\delta} v\left((1-\delta)\, b_{t+1}\right) - \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t) \geq 0. \tag{33}$$

The proof of Proposition 5.3 will now consist of eight lemmas.

1. Lemma A.10 shows that the principal's payoff is strictly positive for all $b_1 > 0$.
2. Lemma A.11 shows that all contractual variables remain strictly positive. Also, analogously to Lemma A.1, it bounds the marginal disutility of effort by the marginal utility of consumption in any optimal contract.
3. Lemma A.12 proves the validity of the Euler equation for any optimal contract and shows that consumption in an optimal contract is weakly decreasing in time.

4. Lemma A.13 shows that any optimal contract either gives the principal his first-best payoff and the balance is constant over time, or the balance is strictly decreasing towards some $\tilde{b}_\infty > 0$.
5. Lemma A.14 shows that if an optimal contract does not achieve the first best, then the continuation payoff of the principal strictly increases over time.
6. Lemma A.15 shows that if an optimal contract does not achieve the first best, then the principal's constraint ($\text{PC}_t$) holds with equality in every period.
7. Lemma A.16 shows that if an optimal contract does not achieve the first best, effort and payments strictly increase over time, while consumption strictly decreases.
8. Lemma A.17 shows that an optimal contract exists.

**Lemma A.10.** *Fix an optimal relational contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. For all $t$ and $\tilde{b}_t > 0$, $V(\tilde{b}_t) \in (0, V^{FB}(\tilde{b}_t)]$.*

*Proof.* The principal can never do better than offering the first-best contract: i.e., $V(\tilde{b}) \leq V^{FB}(\tilde{b})$ for all $\tilde{b} > 0$. Let us therefore show that $V(\tilde{b}) > 0$ irrespective of the value $\tilde{b} > 0$. For this, let us state the conditions for a stationary contract (i.e., $(\tilde{e}_\tau, \tilde{c}_\tau, \tilde{w}_\tau, \tilde{b}_\tau)_{\tau\geq t}$ with $(\tilde{e}_\tau, \tilde{c}_\tau, \tilde{w}_\tau, \tilde{b}_\tau) = (\tilde{e}, \tilde{c}, \tilde{w}, \tilde{b})$ for all $\tau \geq t$) to be self-enforceable. The principal's constraint ($\text{PC}_t$) at any date may be written

$$\delta\tilde{e} \geq \tilde{w} = \tilde{c} - (1-\delta)\tilde{b}.$$

The agent's constraint ($\text{AC}_t^{\text{ob}}$) is

$$v(\tilde{c}) - \psi(\tilde{e}) \geq v((1-\delta)\tilde{b}).$$

Taking the latter to hold with equality, we have $\tilde{e} = \check{e}(\tilde{c}, \tilde{b})$, where

$$\check{e}(c, b) \equiv \psi^{-1}(v(c) - v((1-\delta)b)).$$

From $\check{e}((1-\delta)\tilde{b}, \tilde{b}) = 0$ and $\lim_{c\downarrow(1-\delta)\tilde{b}} \check{e}_1(c, \tilde{b}) = +\infty$,[22] it follows that the principal's constraint above is satisfied when $\tilde{c}$ is above but close enough to $(1-\delta)\tilde{b}$. Hence, if $\tilde{c}$ is above but close enough to $(1-\delta)\tilde{b}$, and if $\tilde{e} = \check{e}(\tilde{c}, \tilde{b})$, we have $\delta\tilde{e} > \tilde{w} > 0$, which confirms that the principal's payoff $\frac{\tilde{e}-\tilde{w}}{1-\delta}$ is strictly positive. That is, $V(\tilde{b}) > 0$. □

[22] Here, $\check{e}_1$ denotes the derivative with respect to the first argument of $\check{e}$.

**Lemma A.11.** *In any optimal contract* $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ *that satisfies the conditions* $(\mathrm{FP}_t^{\mathrm{ob}})$*, we have* $\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t > 0$ *for all* $t$*. Furthermore,* $\psi'(\tilde{e}_t) \leq v'(\tilde{c}_t)$ *for all* $t$*, and* $\psi'(\tilde{e}_t) < v'(\tilde{c}_t)$ *only if* $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$*.*

*Proof.* **Proof that $\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t > 0$ for all $t$.** We first prove that $\tilde{w}_t > 0$ for all $t$. To show this, assume for the sake of contradiction, that $\tilde{w}_t = 0$ for some $t$. Then we have $\delta\tilde{b}_{t+1} - \tilde{b}_t + \tilde{c}_t = 0$. This implies that $\tilde{c}_t = (1-\delta)\tilde{b}_t$ and $\tilde{b}_{t+1} = \tilde{b}_t$ (this is the only possibility for Condition (33) to be satisfied), and so $\tilde{e}_t = \hat{e}((1-\delta)\tilde{b}_t, \tilde{b}_t, \tilde{b}_t) = 0$. Hence, $V(\tilde{b}_t) = \delta V(\tilde{b}_t)$; that is, $V(\tilde{b}_t) = 0$. But this contradicts Lemma A.10.

To prove that $\tilde{e}_t > 0$ for all $t$, suppose to the contrary that $\tilde{e}_t = 0$ for some $t$. If $\tilde{w}_t < \delta V(\tilde{b}_{t+1})$, we can raise effort to $\check{e}_t = \varepsilon$ at date $t$ for $\varepsilon > 0$; raise date-$t$ consumption to

$$\check{c}_t = v^{-1}\big(\psi(\varepsilon) - \tfrac{\delta}{1-\delta}v((1-\delta)\tilde{b}_{t+1}) + \tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_t)\big);$$

and raise the date-$t$ payment to $\check{w}_t = \tilde{w}_t + \check{c}_t - \tilde{c}_t$. Thus, the agent's balance at $t+1$ remains unchanged, and the only adjustments to the contract are at date $t$. For $\varepsilon$ sufficiently small, we have $\check{w}_t < \delta V(\tilde{b}_{t+1})$, so the principal's constraints $(\mathrm{PC}_t)$ are satisfied. By construction, the agent's constraints $(\mathrm{AC}_t^{\mathrm{ob}})$ are satisfied. The principal's payoff strictly increases, so the original contract with effort $\tilde{e}_t = 0$ was not optimal, a contradiction. The remaining case is where $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$. In this case, we have $V(\tilde{b}_t) = 0$, but this contradicts Lemma A.10.

That $\tilde{c}_t, \tilde{b}_t > 0$ for all $t$ follows immediately from our assumption that the Conditions $(\mathrm{FP}_t^{\mathrm{ob}})$ hold at all dates $t$, and because $b_1 > 0$.

**Proof that $\psi'(\tilde{e}_t) \leq v'(\tilde{c}_t)$ for all $t$, and $\psi'(\tilde{e}_t) < v'(\tilde{c}_t)$ only if $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$.** Define

$$\underline{c}(\tilde{b}_t, \tilde{b}_{t+1}) \equiv v^{-1}\big(\tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_t) - \tfrac{\delta}{1-\delta}v\left((1-\delta)\,\tilde{b}_{t+1}\right)\big),$$

interpreted as the lowest consumption level that permits the constraint (33) to be satisfied, for fixed values of $\tilde{b}_t$ and $\tilde{b}_{t+1}$. Consider the problem of maximizing

$$\hat{e}(c_t, \tilde{b}_t, \tilde{b}_{t+1}) - (\delta\tilde{b}_{t+1} - \tilde{b}_t + c_t) + \delta V(\tilde{b}_{t+1}) \tag{34}$$

with respect to $c_t$ on $[\underline{c}(\tilde{b}_t, \tilde{b}_{t+1}), +\infty)$. Given that $\hat{e}(\cdot, \tilde{b}_t, \tilde{b}_{t+1})$ is a continuous and strictly concave function, and that $\lim_{c\to+\infty} \hat{e}_1(c, \tilde{b}_t, \tilde{b}_{t+1}) = 0$,[23] there is a unique solution of the maximization problem, denoted $c^*(\tilde{b}_t, \tilde{b}_{t+1})$. Furthermore, since $\psi'(0) = 0$, we have that $c^*(\tilde{b}_t, \tilde{b}_{t+1}) > \underline{c}(\tilde{b}_t, \tilde{b}_{t+1})$, and the first-order condition establishes

$$\psi'(\hat{e}(c^*(\tilde{b}_t, \tilde{b}_{t+1}), \tilde{b}_t, \tilde{b}_{t+1})) = v'(c^*(\tilde{b}_t, \tilde{b}_{t+1})).$$

[23] Here, $\hat{e}_1$ denotes the derivative of $\hat{e}$ with respect to the first argument.

If we have $\delta\tilde{b}_{t+1} - \tilde{b}_t + c^*(\tilde{b}_t, \tilde{b}_{t+1}) \leq \delta V(\tilde{b}_{t+1})$ then it is clear that optimality requires $\tilde{c}_t = c^*(\tilde{b}_t, \tilde{b}_{t+1})$, and so $\tilde{e}_t = \hat{e}(c^*(\tilde{b}_t, \tilde{b}_{t+1}), \tilde{b}_t, \tilde{b}_{t+1})$. In this case, we have $\tilde{w}_t = \delta\tilde{b}_{t+1} - \tilde{b}_t + \tilde{c}_t \leq \delta V(\tilde{b}_{t+1})$ and $\psi'(\tilde{e}_t) = v'(\tilde{c}_t)$.

If instead $\delta\tilde{b}_{t+1} - \tilde{b}_t + c^*(\tilde{b}_t, \tilde{b}_{t+1}) > \delta V(\tilde{b}_{t+1})$, given the concavity of (34) in $c_t$, we must have

$$\tilde{c}_t = \delta V(\tilde{b}_{t+1}) - \delta\tilde{b}_{t+1} + \tilde{b}_t < c^*(\tilde{b}_t, \tilde{b}_{t+1}).$$

Note that in this case, $\tilde{w}_t = \delta\tilde{b}_{t+1} - \tilde{b}_t + \tilde{c}_t = \delta V\left(\tilde{b}_{t+1}\right)$. Moreover,

$$\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) < \hat{e}(c^*(\tilde{b}_t, \tilde{b}_{t+1}), \tilde{b}_t, \tilde{b}_{t+1}),$$

and so we have $\psi'\left(\tilde{e}_t\right) < v'(c^*(\tilde{b}_t, \tilde{b}_{t+1})) < v'\left(\tilde{c}_t\right)$. $\square$

The following result establishes the Euler equation given in the main text as well as the monotonicity of the consumption plan.

**Lemma A.12.** *Any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ satisfies the Euler equation*

$$1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'\left(\tilde{c}_t\right)} = \frac{v'\left(\tilde{c}_{t+1}\right)}{\psi'\left(\tilde{e}_{t+1}\right)}\left(1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'\left(\tilde{c}_{t+1}\right)}\right) \tag{35}$$

*in all periods. Furthermore, $\tilde{c}_t \geq \tilde{c}_{t+1} > (1-\delta)\tilde{b}_{t+1}$ for all $t$.*

*Proof.* We divide the proof in 3 steps:

**Step 1:** Fix an optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$. Consider a contract $(\check{e}_t, \check{c}_t, \check{w}_t, \check{b}_t)_{t\geq 1}$, coinciding with the original contract in all periods except for periods $t$ and $t+1$ (so, also, $\check{b}_t = \tilde{b}_t$). We specify that the new contract keeps the agent indifferent between being obedient and optimally deviating in all periods. This requires

$$v(\check{c}_t) - \psi(\check{e}_t) + \tfrac{\delta}{1-\delta} v\left(\tfrac{1-\delta}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t)\right) = \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t), \tag{36}$$

$$v\left(\tfrac{1}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t) + \check{w}_{t+1} - \delta\tilde{b}_{t+2}\right) - \psi(\check{e}_{t+1}) + \tfrac{\delta}{1-\delta} v((1-\delta)\tilde{b}_{t+2}) = \tfrac{1}{1-\delta} v\left(\tfrac{1-\delta}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t)\right), \tag{37}$$

which uses that consumption in period $t+1$ under the new contract is $\check{c}_{t+1} = \frac{1}{\delta}(\tilde{b}_t + \check{w}_t - \check{c}_t) + \check{w}_{t+1} - \delta\tilde{b}_{t+2}$ (guaranteeing the agent has savings $\tilde{b}_{t+2}$ at date $t+2$).

Fix $\check{e}_t = \tilde{e}_t$ and $\check{w}_{t+1} = \tilde{w}_{t+1}$. Equations (36) and (37) implicitly define $\check{e}_{t+1}$ and $\check{w}_t$ as functions of $\check{c}_t$. Let these functions be denoted $\hat{e}_{t+1}(\cdot)$ and $\hat{w}_t(\cdot)$, respectively. We can use the implicit function theorem to compute the derivatives at $\check{c}_t = \tilde{c}_t$:

$$\hat{e}'_{t+1}(\tilde{c}_t) = \frac{v'(\tilde{c}_t)\big(v'((1-\delta)\tilde{b}_{t+1}) - v'(\tilde{c}_{t+1})\big)}{\delta\psi'\left(\hat{e}_{t+1}(\tilde{c}_t)\right) v'((1-\delta)\tilde{b}_{t+1})} \quad \text{and} \quad \hat{w}'_t(\tilde{c}_t) = 1 - \frac{v'(\tilde{c}_t)}{v'((1-\delta)\tilde{b}_{t+1})}.$$

Note that the original contract is obtained by setting $\check{c}_t = \tilde{c}_t$. If $\check{c}_t$ is changed from $\tilde{c}_t$ to $\tilde{c}_t + \varepsilon$, for some (positive or negative) $\varepsilon$ small, the total effect on the continuation payoff of the principal at time $t$ is $(-\hat{w}'_t(\tilde{c}_t) + \delta \hat{e}'_{t+1}(\tilde{c}_t))\varepsilon + o(\varepsilon)$ (where $o(\varepsilon)$ represents terms that vanish faster than $\varepsilon$ as $\varepsilon \to 0$). Hence, a necessary condition for optimality is that $-\hat{w}'_t(\tilde{c}_t) + \delta \hat{e}'_{t+1}(\tilde{c}_t) = 0$, which is equivalent the Euler equation (35).

The Euler equation implies that if $v'(\tilde{c}_{t+1}) = \psi'(\tilde{e}_{t+1})$ we have $\tilde{c}_t = \tilde{c}_{t+1}$. From Lemma A.11 we have that, if instead $v'(\tilde{c}_{t+1}) \neq \psi'(\tilde{e}_{t+1})$, then $v'(\tilde{c}_{t+1}) > \psi'(\tilde{e}_{t+1})$. In this second case, there are three possibilities:

1. If both sides of the Euler equation are strictly positive, then $\tilde{c}_t < \tilde{c}_{t+1} < (1-\delta)\tilde{b}_{t+1}$.

2. If both sides of the Euler equation are zero, then $\tilde{c}_t = \tilde{c}_{t+1} = (1-\delta)\tilde{b}_{t+1}$.

3. If both sides of the Euler equation are strictly negative, then $\tilde{c}_t > \tilde{c}_{t+1} > (1-\delta)\tilde{b}_{t+1}$.

**Step 2:** We now prove that if $\tilde{c}_t \leq (1-\delta)\tilde{b}_t$ then $\tilde{c}_s \leq \tilde{c}_{s+1} < (1-\delta)\tilde{b}_{s+1}$ for all $s \geq t$. Assume first that there is a period $t$ such that $\tilde{c}_t \leq (1-\delta)\tilde{b}_t$. Hence, since $\tilde{e}_t = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) > 0$ (recall Lemma A.11) we have $\tilde{b}_{t+1} > \tilde{b}_t$. This shows that each side of the Euler equation is strictly positive, i.e.

$$1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'\left(\tilde{c}_t\right)} = \frac{v'\left(\tilde{c}_{t+1}\right)}{\psi'\left(\tilde{e}_{t+1}\right)} \left(1 - \frac{v'((1-\delta)\,\tilde{b}_{t+1})}{v'\left(\tilde{c}_{t+1}\right)}\right) > 0.$$

Since $v'\left(\tilde{c}_{t+1}\right)/\psi'\left(\tilde{e}_{t+1}\right) \geq 1$ (from Lemma A.11), $(1-\delta)\tilde{b}_{t+1} > \tilde{c}_{t+1} \geq \tilde{c}_t$. The result then follows by induction.

**Step 3:** We prove that $\tilde{c}_t > (1-\delta)\tilde{b}_t$ for all $t > 1$; it then follows immediately from Step 1 that consumption is (weakly) decreasing in $t$. Assume then, for the sake of contradiction, that there is a $t' > 1$ such that $\tilde{c}_{t'} \leq (1-\delta)\tilde{b}_{t'}$. We will construct a self-enforceable contract that is strictly more profitable than the original, contradicting the optimality of the original.

We first make some preliminary observations. From Step 2, we have that $\tilde{c}_s \leq \tilde{c}_{s+1} < (1-\delta)\tilde{b}_{s+1}$ for all $s \geq t'$. Also, since effort is strictly positive at all times (from Lemma A.11), we have

$$\sum_{s=t'}^{\infty} \delta^{s-t'} v(\tilde{c}_s) > \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_{t'}).$$

Hence, there must be a period $s \geq t'$ where $\tilde{c}_{s+1} > \tilde{c}_{t'}$. Let $t''$ be the earliest such period, and note that it satisfies $\tilde{c}_{t''+1} > \tilde{c}_{t''}$. Additionally, we can observe that, for all $t$,

$$\tilde{b}_t + \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{w}_\tau = \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{c}_\tau. \tag{38}$$

If this is not the case (the right-hand side is strictly smaller), then, applying Equation (1) repeatedly, we have $\tilde{b}_t \to \infty$ and so the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) must be violated for large $t$ (given that $(\tilde{c}_t)_{t=1}^{\infty}$ is bounded, as the contract is feasible).

Now let us construct the more profitable contract for the principal, given our assumption that $\tilde{c}_{t'} \leq (1-\delta)\tilde{b}_{t'}$. We first construct a self-enforceable contract $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$ in which the agent obtains a strictly higher payoff than in the original, while the principal obtains the same payoff. We then show how that contract can be further adjusted to obtain one which is strictly better for the principal. In the new contract that is better for the agent, we maintain $\tilde{w}_t^{new} = \tilde{w}_t$ and $\tilde{e}_t^{new} = \tilde{e}_t$ for all $t$, but specify a different agreed consumption sequence $\tilde{c}_t^{new}$ (and hence different balances $\tilde{b}_t^{new}$).

The change in agent consumption is to specify *constant* consumption $\bar{c}$ in each period from $t''$ onwards, where

$$\bar{c} = (1-\delta)\sum_{\tau=t''}^{\infty} \delta^{\tau-t''}\tilde{c}_\tau. \tag{39}$$

That is, $\tilde{c}_t^{new} = \bar{c}$ for all $t \geq t''$, while $\tilde{c}_t^{new} = \tilde{c}_t$ for $t < t''$. Notice that, $\bar{c} < (1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\tilde{c}_\tau$ for all $t > t''$.

Balances are determined recursively by Equation (1). That is, they are given by $\tilde{b}_t^{new} = \tilde{b}_t$ for $t \leq t''$, and by

$$\tilde{b}_t^{new} = \delta^{t''-t}\tilde{b}_{t''} + \sum_{\tau=t''}^{t-1}\delta^{\tau-t}\left(\tilde{w}_\tau - \bar{c}\right)$$

for all $t > t''$. Observe that

$$\begin{aligned}
\tilde{b}_t^{new} + \sum_{\tau=t}^{\infty}\delta^{\tau-t}\tilde{w}_\tau &= \delta^{t''-t}\tilde{b}_{t''} + \sum_{\tau=t''}^{\infty}\delta^{\tau-t}\tilde{w}_\tau - \sum_{\tau=t''}^{t-1}\delta^{\tau-t}\bar{c} \\
&= \sum_{\tau=t''}^{\infty}\delta^{\tau-t}\tilde{c}_\tau - \sum_{\tau=t''}^{t-1}\delta^{\tau-t}\bar{c} \\
&= \frac{\bar{c}}{1-\delta},
\end{aligned}$$

where the second equality uses Equation (38) and the third equality uses Equation (39). Therefore, for all $t > t''$,

$$\tilde{b}_t^{new} + \sum_{\tau=t}^{\infty}\delta^{\tau-t}\tilde{w}_\tau = \frac{\bar{c}}{1-\delta} < \sum_{\tau=t}^{\infty}\delta^{\tau-t}\tilde{c}_\tau = \tilde{b}_t + \sum_{\tau=t}^{\infty}\delta^{\tau-t}\tilde{w}_\tau,$$

where the second equality follows from Equation (38). This implies that $\tilde{b}_t^{new} < \tilde{b}_t$ for all $t > t''$.

Now, we want to show that the contract $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$ is self-enforceable. Because effort and payments are unchanged relative to the original contract, the principal's constraints ($\mathrm{PC}_t$) remain intact. Consider then the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) for each period $t \geq 1$. For all $t \leq t''$, the agent anticipates a strictly higher continuation payoff under the new contract, i.e.

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(v\left(\tilde{c}_{\tau}^{new}\right) - \psi\left(\tilde{e}_{\tau}^{new}\right)\right) > \sum_{\tau=t}^{\infty} \delta^{s-t} \left(v\left(\tilde{c}_{\tau}\right) - \psi\left(\tilde{e}_{\tau}\right)\right).$$

The strict inequality is immediate from the strict concavity of $v$, and because consumption from date $t''$ onwards is constant in the new contract, but the NPV of this consumption is the same as in the original. Since, in addition, $v(\tilde{b}_t^{new}(1-\delta)) = v(\tilde{b}_t(1-\delta))$, the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied at dates $t \leq t''$ as strict inequalities.

To understand how the agent's constraints change at each $t > t''$, define

$$\bar{c}^{(t)} \equiv (1-\delta) \sum_{\tau=t}^{\infty} \delta^{\tau-t} \tilde{c}_{\tau}.$$

Consider the original contract, and suppose that the agent's consumption is changed from date $t$ onwards, being set equal to $\bar{c}^{(t)}$ in all such periods. The agent's payoff increases from the smoothing of consumption, and so

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t}(v(\bar{c}^{(t)}) - \psi(\tilde{e}_{\tau})) \geq \sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(v\left(\tilde{c}_{\tau}\right) - \psi\left(\tilde{e}_{\tau}\right)\right) \geq \frac{v(\tilde{b}_t(1-\delta))}{1-\delta}, \tag{40}$$

where the second inequality follows because the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied in the original contract.

Because $\psi$ is non-negative, the inequalities in Equation (40) imply $\bar{c}^{(t)} \geq \tilde{b}_t(1-\delta)$. Therefore, since $v$ is concave, we have

$$v(\bar{c}^{(t)}) - v(\bar{c}^{(t)} - (1-\delta)(\tilde{b}_t - \tilde{b}_t^{new})) \leq v(\tilde{b}_t(1-\delta)) - v(\tilde{b}_t(1-\delta) - (1-\delta)(\tilde{b}_t - \tilde{b}_t^{new})). \tag{41}$$

Note that $\bar{c} = \bar{c}^{(t)} - (1-\delta)\left(\tilde{b}_t - \tilde{b}_t^{new}\right)$. Combining Equations (40) and (41), we therefore have that, for all $t > t''$,

$$\sum_{\tau=t}^{\infty} \delta^{\tau-t} \left(v\left(\bar{c}\right) - \psi\left(\tilde{e}_{\tau}\right)\right) \geq \frac{v\left(\tilde{b}_t^{new}(1-\delta)\right)}{1-\delta}.$$

This shows that, for the contract $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$, the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied also at dates $t > t''$.

We have thus shown that $\left(\tilde{e}_t^{new}, \tilde{c}_t^{new}, \tilde{w}_t^{new}, \tilde{b}_t^{new}\right)_{t\geq 1}$ is a self-enforceable contract (in particular, it satisfies all the constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) and ($\mathrm{PC}_t$). Moreover, we saw that the constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) are satisfied strictly at all $t \leq t''$. We can therefore further adjust the contract by raising effort at date $t''$ by a small amount $\varepsilon > 0$ such that, without any other changes to the contract, all the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) remain intact. The adjusted contract then satisfies all the constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) and ($\mathrm{PC}_t$), and the principal obtains a strictly higher payoff than in the original contract, contradicting the optimality of the original. $\square$

**Lemma A.13.** *In any optimal contract, $(\tilde{b}_t)_{t\geq 1}$ is a weakly decreasing sequence. It is constant if it attains the first-best payoff, and strictly decreasing towards some $\tilde{b}_\infty > 0$ otherwise. Also, $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$.*

*Proof.* **Step 0.** If the first-best payoff is achievable at $b_1$, then equilibrium consumption and effort is uniquely determined by the conditions in Proposition 3.1. Because we restrict attention to payments timed to satisfy Equation (8), the balance is constant as claimed in the statement of this lemma (and explained in the main text). Suppose from now on that $V(b_1) < V^{FB}(b_1)$.

**Step 1. Proof that $(\tilde{b}_t)_{t\geq 1}$ is weakly decreasing.** Consider an optimal contract $\left(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t\right)_{t\geq 1}$. To show that the balance $\tilde{b}_t$ is weakly decreasing, we suppose for a contradiction that $\tilde{b}_{\hat{t}+1} > \tilde{b}_{\hat{t}}$ for some date $\hat{t}$. We construct a self-enforceable contract that achieves strictly higher profits for the principal.

**Step 1a.** First, denote a new contract by $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$, which we will choose to coincide with the original contract until $\hat{t}-1$, and with $\tilde{e}'_{\hat{t}} = \tilde{e}_{\hat{t}}$. For dates $t \geq \hat{t}$, let

$$\begin{aligned}
\tilde{c}'_t = \bar{c} &\equiv (1-\delta)\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{w}_\tau + (1-\delta)\tilde{b}_{\hat{t}} \\
&= (1-\delta)\sum_{\tau\geq\hat{t}} \delta^{\tau-\hat{t}}\tilde{c}_\tau
\end{aligned}$$

where the last equality is for the same reason as Equation (38). For dates $t \geq \hat{t}+1$, let $\tilde{e}'_t = \bar{e}$, where $\bar{e}$ is defined by

$$\psi(\bar{e}) = (1-\delta)\sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\psi(\tilde{e}_\tau).$$

Let also, for all $t \geq \hat{t}$, $\tilde{w}'_t = \bar{w}$, where $\bar{w} = (1-\delta)\sum_{\tau\geq\hat{t}}\delta^{\tau-\hat{t}}\tilde{w}_\tau$. Thus, we must have $\tilde{b}'_t = \bar{b} \equiv \tilde{b}_{\hat{t}}$ for all $t \geq \hat{t}$.

**Step 1b.** We now want to show that, for the new contract, the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) is satisfied at all dates. Note that the new contract is stationary from date $\hat{t}+1$ onwards. Let's then consider the agent's constraint for these dates. Note first that, by the previous lemma, we must have $\tilde{c}_{\hat{t}} \geq \bar{c}$. Therefore,

$$\sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\bar{c} \geq \sum_{\tau\geq\hat{t}+1} \delta^{\tau-\hat{t}-1}\tilde{c}_\tau.$$

Also, the NPV of disutility of effort from date $\hat{t}+1$ onwards is the same for both the original contract and the new contract. The fact that the original contract satisfies the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) at date $\hat{t}+1$, plus the observation that $\bar{b} < b_{\hat{t}+1}$, then implies

$$\sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} v(\bar{c}) - \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} \psi(\bar{e}) > \frac{v\left((1-\delta)\,\bar{b}\right)}{1-\delta}, \tag{42}$$

which means that the agent's constraint is satisfied as a *strict inequality* from $\hat{t}+1$ onwards.

Note then that

$$\sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} v(\bar{c}) \geq \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} v(\tilde{c}_\tau)$$

(with a strict inequality if the consumption levels $\tilde{c}_\tau$ for $\tau \geq \hat{t}$ are non-constant). Also, the NPV of the disutility of effort is the same from $\hat{t}$ onwards under both policies. Therefore, the agent's constraint continues to be satisfied at $\hat{t}$, and by the same logic all earlier periods.

**Step 1c.** Now we show that the principal's constraint ($\mathrm{PC}_t$) is satisfied in all periods. Because the NPV of disutility of effort from date $\hat{t}+1$ onwards is the same under both contracts; and because $\psi$ is convex, we have $\bar{e} \geq (1-\delta)\sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}-1} \tilde{e}_\tau$. Therefore,

$$\begin{aligned} \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}} \tilde{e}'_\tau - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}'_\tau &= \frac{\delta \bar{e}}{1-\delta} - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}_\tau \\ &\geq \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}} \tilde{e}_\tau - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}_\tau \\ &\geq 0 \end{aligned} \tag{43}$$

where the second inequality holds because the principal's constraint is satisfied at date $\hat{t}$ under the original contract. Hence the principal's constraint is satisfied under the new contract at date $\hat{t}$. Because $\tilde{e}'_t$ is constant for $t \geq \hat{t}+1$, and because $\tilde{w}'_t$ is constant for $t \geq \hat{t}$, the same inequality implies the satisfaction of the principal's constraint also from $\hat{t}+1$ onwards. Checking that the principal's constraint is satisfied also at dates before $\hat{t}$ follows the same logic. For $t < \hat{t}$, the principal's constraint is

$$\sum_{\tau=t+1}^{\hat{t}} \delta^{\tau-t} \tilde{e}'_\tau - \sum_{\tau=t}^{\hat{t}-1} \delta^{\tau-t} \tilde{w}'_\tau + \delta^{\hat{t}-t} \left( \sum_{\tau \geq \hat{t}+1} \delta^{\tau-\hat{t}} \tilde{e}'_\tau - \sum_{\tau \geq \hat{t}} \delta^{\tau-\hat{t}} \tilde{w}'_\tau \right) \geq 0,$$

which is satisfied because (i) $\tilde{e}'_\tau = \tilde{e}_\tau$ for $\tau \leq \hat{t}$, and $\tilde{w}'_\tau = \tilde{w}_\tau$ for $\tau < \hat{t}$, (ii) the first inequality in Equation (43) holds, and (iii) the principal's constraint is satisfied at date $t$ under the original policy.

**Step 1d.** Finally, we show that the contract can be further (slightly) adjusted to a self-enforceable contract with a strictly higher payoff for the principal. The original contract was taken to satisfy

$$v\left(\tilde{c}_{\hat{t}}\right)-\psi\left(\tilde{e}_{\hat{t}}\right)=\frac{v((1-\delta)\,\bar{b})-\delta v((1-\delta)\,\tilde{b}_{\hat{t}+1})}{1-\delta}<v\left((1-\delta)\,\bar{b}\right).$$

Hence,

$$\psi\left(\tilde{e}_{\hat{t}}\right)>v\left(\tilde{c}_{\hat{t}}\right)-v\left((1-\delta)\,\bar{b}\right)\geq v\left(\bar{c}\right)-v\left((1-\delta)\,\bar{b}\right)>\psi\left(\bar{e}\right)$$

where the final inequality follows from (42). Hence $\tilde{e}_{\hat{t}}>\bar{e}$. Recall that $\tilde{e}'_{\hat{t}}=\tilde{e}_{\hat{t}}$, and $\tilde{e}'_{\tau}=\bar{e}$ for $\tau>\hat{t}$; so we have $\tilde{e}'_{\hat{t}}>\tilde{e}'_{\tau}$ for all $\tau>\hat{t}$.

Now, pick $\tilde{e}''_{\hat{t}}$ and $\tilde{e}''_{\hat{t}+1}$, with

$$\tilde{e}'_{\hat{t}+1}<\tilde{e}''_{\hat{t}+1}<\tilde{e}''_{\hat{t}}<\tilde{e}'_{\hat{t}}$$

and such that

$$\psi\big(\tilde{e}''_{\hat{t}}\big)+\frac{\delta}{1-\delta}\psi\big(\tilde{e}''_{\hat{t}+1}\big)=\psi\big(\tilde{e}'_{\hat{t}}\big)+\frac{\delta}{1-\delta}\psi\big(\tilde{e}'_{\hat{t}+1}\big).$$

Substitute $\tilde{e}''_{\hat{t}}$ for $\tilde{e}'_{\hat{t}}$ and $\tilde{e}''_{\hat{t}+1}$ for $\tilde{e}'_{\tau}$, for all $\tau\geq\hat{t}+1$, in the contract defined in Step 1a. The agent's value from remaining in the contract from $\hat{t}$ onwards remains unchanged, so the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) remains satisfied at $\hat{t}$, and at all earlier dates. Note that, due to (42), the agent's constraints ($\mathrm{AC}_t^{\mathrm{ob}}$) at dates $\hat{t}+1$ onwards are slack under the contract defined in Step 1a, and hence continue to be satisfied under the contract with the further modification, provided the adjustment in effort is small. Moreover, because $\psi$ is strictly convex, the NPV of effort from date $\hat{t}$ onwards increases; so the principal's payoff strictly increases. Also, the principal's constraints ($\mathrm{PC}_t$) clearly continue to be satisfied. Thus, we have constructed a self-enforceable contract that is strictly more profitable for the principal than the original, which completes Step 1.

**Step 2. Proof that if $V(\tilde{b}_1)<V^{FB}(\tilde{b}_1)$ then $(\tilde{b}_t)_{t\geq1}$ is a strictly decreasing sequence.**

**Step 2a.** Consider an optimal contract. We first prove that if $\tilde{b}_{\hat{t}}=\tilde{b}_{\hat{t}+1}$ for some $\hat{t}\geq1$, then $V(\tilde{b}_{\hat{t}})=V^{FB}(\tilde{b}_{\hat{t}})$. To do this, note that if $\tilde{b}_{\hat{t}}=\tilde{b}_{\hat{t}+1}$ for some $\hat{t}$, then it is optimal to specify $\tilde{c}_\tau=\tilde{c}_{\hat{t}}$, $\tilde{w}_\tau=\tilde{w}_{\hat{t}}$, and $\tilde{e}_\tau=\tilde{e}_{\hat{t}}$ for all $\tau>\hat{t}$; that is, it must be optimal for the contract to be stationary from period $\hat{t}$ onwards. The Euler equation (35) then requires that $\psi'(\tilde{e}_\tau)=v'(\tilde{c}_\tau)$ for all $\tau\geq\hat{t}+1$,[24] and by stationarity also $\psi'(\tilde{e}_{\hat{t}})=v'(\tilde{c}_{\hat{t}})$. Then, $\tilde{e}_\tau$ and $\tilde{c}_\tau$ satisfy, for all $\tau\geq\hat{t}$, the first-order and agent's indifference conditions in Proposition 3.1, given initial balance $\tilde{b}_{\hat{t}}$. Therefore they are the first-best effort and consumption given balance $\tilde{b}_{\hat{t}}$. This shows that $V(\tilde{b}_{\hat{t}})=V^{FB}(\tilde{b}_{\hat{t}})$, as desired.

[24] To see this, recall from Lemma A.12 that $\tilde{c}_\tau>\tilde{b}_\tau$.

**Step 2b.** Now we consider any optimal contract, and show the following. If $V(\tilde{b}_1) < V^{FB}(\tilde{b}_1)$, then $V(\tilde{b}_t) < V^{FB}(\tilde{b}_t)$ for all $t \geq 1$, and in addition, $(\tilde{b}_t)_{t\geq 1}$ is strictly decreasing.

Suppose that, for some $\hat{t}$, $V(\tilde{b}_{\hat{t}}) < V^{FB}(\tilde{b}_{\hat{t}})$, which by Step 1 and Step 2a implies $\tilde{b}_{\hat{t}+1} < \tilde{b}_{\hat{t}}$. The result will follow by induction if we can show that $V(\tilde{b}_{\hat{t}+1}) < V^{FB}(\tilde{b}_{\hat{t}+1})$. Hence, suppose for a contradiction that the contract achieves the first-best continuation payoff for the principal at date $\hat{t}+1$, given the balance is $\tilde{b}_{\hat{t}+1}$ (that is, suppose $V(\tilde{b}_{\hat{t}+1}) = V^{FB}(\tilde{b}_{\hat{t}+1})$). This implies that $\tilde{e}_\tau = e^{FB}(\tilde{b}_{\hat{t}+1})$ and $\tilde{c}_\tau = c^{FB}(\tilde{b}_{\hat{t}+1})$ for all $\tau > \hat{t}$. By assumption that Equation (8) holds in all periods, we then have $\tilde{b}_\tau = \tilde{b}_{\hat{t}+1}$ for all $\tau > \hat{t}+1$. Hence, the contract is stationary from $\hat{t}+1$ onwards; in particular, the payment is constant at $\tilde{w}_\tau = \bar{w}$ for $\tau \geq \hat{t}+1$, for some value $\bar{w}$.

From the Euler equation (35) and the fact that $v'(\tilde{c}_{\hat{t}+1}) = \psi'(\tilde{e}_{\hat{t}+1})$, we have $\tilde{c}_{\hat{t}} = \tilde{c}_{\hat{t}+1}$. Hence, using $\tilde{b}_{\hat{t}+2} = \tilde{b}_{\hat{t}+1} < \tilde{b}_{\hat{t}}$, we have (using ($\mathrm{FP}_t^{\mathrm{ob}}$))

$$\begin{aligned}\psi(\tilde{e}_{\hat{t}}) &= v(\tilde{c}_{\hat{t}}) + \tfrac{\delta}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}+1}) - \tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}}) \\ &< v(\tilde{c}_{\hat{t}+1}) + \tfrac{\delta}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}+2}) - \tfrac{1}{1-\delta}v((1-\delta)\tilde{b}_{\hat{t}+1}) = \psi(\tilde{e}_{\hat{t}+1}).\end{aligned}$$

Consequently, $\tilde{e}_{\hat{t}} < \tilde{e}_{\hat{t}+1}$, and so $\frac{\psi'(\tilde{e}_{\hat{t}})}{v'(\tilde{c}_{\hat{t}})} < \frac{\psi'(\tilde{e}_{\hat{t}+1})}{v'(\tilde{c}_{\hat{t}+1})} = 1$. We then know (from Lemma A.11) that the principal's constraint ($\mathrm{PC}_t$) binds at $\hat{t}$, and so

$$\sum_{s=\hat{t}+1}^{\infty} \delta^{s-\hat{t}}\tilde{e}_s = \sum_{s=\hat{t}}^{\infty} \delta^{s-\hat{t}}\tilde{w}_s = \sum_{s=\hat{t}}^{\infty} \delta^{s-\hat{t}}\tilde{c}_s - \tilde{b}_{\hat{t}},$$

where the second equality follows for the same reason as for Equation (38). Using that $\tilde{e}_\tau = e^{FB}(\tilde{b}_{\hat{t}+1})$ for all $\tau \geq \hat{t}+1$, and $\tilde{c}_\tau = c^{FB}(\tilde{b}_{\hat{t}+1})$ for all $\tau \geq \hat{t}$, we have

$$\delta e^{FB}(\tilde{b}_{\hat{t}+1}) = c^{FB}(\tilde{b}_{\hat{t}+1}) - (1-\delta)\tilde{b}_{\hat{t}} < c^{FB}(\tilde{b}_{\hat{t}+1}) - (1-\delta)\tilde{b}_{\hat{t}+1} = \bar{w} = \tilde{w}_{\hat{t}+1}.$$

That $\delta e^{FB}(\tilde{b}_{\hat{t}+1}) < \tilde{w}_{\hat{t}+1}$ means the principal's constraint ($\mathrm{PC}_t$) in period $\hat{t}+1$ (as well as at future dates) is violated, so we reach our contradiction. This completes Step 2.

**Step 3. Proof that $\tilde{b}_\infty > 0$ and $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$.**

Consider an optimal contract, and suppose that $V(\tilde{b}_1) < V^{FB}(\tilde{b}_1)$. Then $(\tilde{b}_t)_{t\geq 1}$ is a strictly decreasing sequence, as we saw in the previous step. By Lemma 5.1, $\tilde{b}_t > 0$ for all $t$, so the limit $\lim_{t\to\infty}\tilde{b}_t$ exists and is non-negative. We want to show this limit, call it $\tilde{b}_\infty$, is strictly positive and $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$.

**Step 3a.** We first show that the function $V$ is continuous at any $b > 0$. Suppose, for the sake of contradiction, that there is a point of discontinuity $\check{b} > 0$. Then there is $\varepsilon > 0$ and a sequence $(b_n)_{n=1}^{\infty}$ convergent to $\check{b}$ with $|V(b_n) - V(\check{b})| \geq \varepsilon$ for all $n$. Denote $\check{c}$ and $\check{b}'$ the optimal consumption and next-period balance when the balance is $\check{b}$. Then (given our restriction to "fastest payments"), present-period effort is given by $\hat{e}(\check{c}, \check{b}, \check{b}')$. The present-period payment is $\check{w} = \check{c} + \delta\check{b}' - \check{b}$.

Suppose now there is a subsequence $(b_{n_k})$ along which $V(b_{n_k}) \le V(\check{b}) - \varepsilon$ for all $k$. If there is no such subsequence, then there is a subsequence $(b_{n_k})$ for which $V(\check{b}) \le V(b_{n_k}) - \varepsilon$; the argument will then be symmetric, and hence is omitted. Consider then a balance $b_{n_k}$, and choose present-period consumption equal to $c_{n_k} \equiv \check{c} + b_{n_k} - \check{b}$, and next-period balance equal to $b'_{n_k} \equiv \check{b}'$. Note this implies present-period effort is $e_{n_k} \equiv \hat{e}(c_{n_k}, b_{n_k}, b'_{n_k})$, and the payment is $w_{n_k} \equiv c_{n_k} + \delta\check{b}' - b_{n_k} = \check{w}$. Because the principal's next-period continuation payoff is $V(\check{b}')$, the same as in an optimal contract following balance $\check{b}$, and because the payment is the same (i.e., $\check{w}$), the principal's constraint in Equation (32) is satisfied. By continuity of $\hat{e}(\cdot,\cdot,\cdot)$, for large enough $k$, we have $V(b_{n_k}) > V(\check{b}) - \varepsilon$. This contradicts the assumption that $V(b_{n_k}) \le V(\check{b}) - \varepsilon$.

**Step 3b.** We now prove that $\tilde{b}_\infty > 0$. For this, we first show $\lim_{b\searrow 0} \frac{c^{FB}(b)-(1-\delta)b}{e^{FB}(b)} = 0$ and so, by Proposition 5.2, there exists some $\bar{b} > 0$ such that an optimal contract achieves the first-best payoff of the principal for all $b \le \bar{b}$. This follows after noting that $v(c^{FB}(b)) - v((1-\delta)b) = \psi(e^{FB}(b)) > 0$, so we have that either $\lim_{b\searrow 0} c^{FB}(b) = 0$ or $\lim_{b\searrow 0} e^{FB}(b) = +\infty$. Since $\psi'(e^{FB}(b)) = v'(c^{FB}(b))$ we have, in fact, that both $\lim_{b\searrow 0} c^{FB}(b) = 0$ and $\lim_{b\searrow 0} e^{FB}(b) = +\infty$, which establishes the result. Next, recall from Step 2 that, given $V(b_1) < V^{FB}(b_1)$, the sequence $(\tilde{b}_t)_{t\ge 1}$ of balances in the optimal contract is strictly decreasing and such that $V(\tilde{b}_t) < V^{FB}(\tilde{b}_t)$ for all $t$. That is, $\tilde{b}_t$ remains above $\bar{b}$, and so converges to some value $\tilde{b}_\infty \ge \bar{b}$.

**Step 3c.** Finally, we prove that $V(\tilde{b}_\infty) = V^{FB}(\tilde{b}_\infty)$. Recall we assumed that $V(b_1) < V^{FB}(b_1)$. By the continuity of $V$ established in Step 3a, we have that $\lim_{t\to\infty} V(\tilde{b}_t) = V(\tilde{b}_\infty)$. Because the principal's constraint ($\text{PC}_t$) binds for all $t$, we have $V(\tilde{b}_t) = \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1})$ for all $t$. By continuity of $\hat{e}(\cdot,\cdot,\cdot)$, we have $\lim_{t\to\infty} \hat{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}) = \hat{e}(\tilde{c}_\infty, \tilde{b}_\infty, \tilde{b}_\infty)$, where $\tilde{c}_\infty \equiv \lim_{t\to\infty} \tilde{c}_t$, which exists because $\tilde{c}_t$ is decreasing and remains above $\tilde{b}_\infty$ by Lemma A.12. Therefore,

$$V(\tilde{b}_\infty) = \hat{e}(\tilde{c}_\infty, \tilde{b}_\infty, \tilde{b}_\infty) = \psi^{-1}(v(\tilde{c}_\infty) - v((1-\delta)\tilde{b}_\infty)).$$

Since $V(\tilde{b}_\infty) > 0$ (recall Lemma A.10), $\tilde{c}_\infty > (1-\delta)\tilde{b}_\infty$. Therefore, the Euler equation (35) implies that, necessarily, $\lim_{t\to\infty} \frac{v'(\tilde{c}_{t+1})}{\psi'(\tilde{e}_{t+1})} = 1$, and therefore $\tilde{e}_\infty \equiv \lim_{t\to\infty} \tilde{e}_t$ exists. It is then clear that both Conditions 1 and 2 of Proposition 3.1 hold for $\tilde{e}_\infty$, $\tilde{c}_\infty$, and $\tilde{b}_\infty$ (instead of $e^{FB}(b_1)$, $c^{FB}(b_1)$, and $b_1$). This establishes the result. $\square$

**Lemma A.14.** *Assume $V(b_1) < V^{FB}(b_1)$. Then $(V(\tilde{b}_t))_{t\geq 1}$ is a strictly increasing sequence.*

*Proof.* Recall from Lemma A.13 we have that, if $V(b_1) < V^{FB}(b_1)$, then $(\tilde{b}_t)_{t\geq 1}$ is strictly decreasing. Therefore, the result will follow if we can show $V(\cdot)$ is strictly decreasing.

**Step 1.** We show that if $V(\cdot)$ fails to be strictly decreasing, then there exists a value $b^* > 0$ such that, for every $\varepsilon > 0$, there is a $\check{b} \in (b^* - \varepsilon, b^*)$ satisfying $V(\check{b}) \leq V(b^*)$.

First, by Step 3a of the proof of the previous lemma, $V(\cdot)$ is continuous on strictly positive values. Suppose $V(\cdot)$ fails to be strictly decreasing, which means that there are values $b', b''$ with $0 < b' < b''$, and with $V(b') \leq V(b'')$. Consider maximizing $V$ on $[b', b'']$, and note that a maximum exists by continuity of $V$. Because $V(b') \leq V(b'')$, there is at least one maximizer in $(b', b'']$. We can take any such value to be $b^*$.

**Step 2.** Consider an optimal continuation contract when $\tilde{b}_t = b^*$, and consider a change to $\tilde{b}_t = b^* - \nu$ for $\nu > 0$ small enough such that $V(b^* - \nu) \leq V(b^*)$. Then we can reduce $\tilde{c}_t$ by the same amount $\nu$, holding $\tilde{b}_{t+1}$ and $\tilde{w}_t$, as well as all other variables, constant. Note then that, provided $\nu$ is small enough,

$$v(\tilde{c}_t - \nu) - \tfrac{1}{1-\delta} v((1-\delta)(\tilde{b}_t - \nu)) > v(\tilde{c}_t) - \tfrac{1}{1-\delta} v((1-\delta)\tilde{b}_t),$$

which follows again because $\tilde{c}_t > (1-\delta)\tilde{b}_t$ (by Lemma A.12) and by concavity of $v$. Hence, we have

$$\hat{e}(\tilde{c}_t - \nu, \tilde{b}_t - \nu, \tilde{b}_{t+1}) > \tilde{e}(\tilde{c}_t, \tilde{b}_t, \tilde{b}_{t+1}).$$

By construction, the agent remains indifferent to continuing in the contract at all dates (we have that ($\mathrm{AC}_t^{\mathrm{ob}}$) holds as an equality at all dates). The continuation of the relationship from $t+1$ onwards is precisely as before, and therefore the principal's constraint at date $t$ is satisfied (since $\tilde{w}_t$ is unchanged). Hence,

$$V(\tilde{b}_t) = \tilde{e}_t - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) < \hat{e}(\tilde{c}_t - \nu, \tilde{b}_t - \nu, \tilde{b}_{t+1}) - \tilde{w}_t + \delta V(\tilde{b}_{t+1}) \leq V(\tilde{b}_t - \nu).$$

However, this contradicts $V(b^* - \nu) \leq V(b^*)$. $\square$

**Lemma A.15.** *Assume $V(b_1) < V^{FB}(b_1)$. Then, in any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, $v'(\tilde{c}_t) > \psi'(\tilde{e}_t)$ for all $t$.*

*Proof.* We first show that, if there is a date $\check{t}$ with $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$, then $v'(\tilde{c}_{\check{t}+1}) = \psi'(\tilde{e}_{\check{t}+1})$. To do so, assume for a contradiction that $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ and $v'(\tilde{c}_{\check{t}+1}) \neq \psi'(\tilde{e}_{\check{t}+1})$ for some date $\check{t}$. Then, by Lemma A.11, we have $v'(\tilde{c}_{\check{t}+1}) > \psi'(\tilde{e}_{\check{t}+1})$, and therefore $\tilde{w}_{\check{t}+1} = \delta V(\tilde{b}_{\check{t}+2})$. In turn, this implies

$$\tilde{e}_{\check{t}+1} = \tilde{e}_{\check{t}+1} - \tilde{w}_{\check{t}+1} + \delta V(\tilde{b}_{\check{t}+2}) = V(\tilde{b}_{\check{t}+1}) > V(\tilde{b}_{\check{t}}) = \tilde{e}_{\check{t}} - \tilde{w}_{\check{t}} + \delta V(\tilde{b}_{\check{t}+1}) \geq \tilde{e}_{\check{t}}$$

where the strict inequality follows by the previous lemma, and the weak inequality follows because the principal's constraint is satisfied in an optimal contract at $\check{t}$.

There are two cases: either $\tilde{w}_{\check{t}} < \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}} (\tilde{e}_s - \tilde{w}_s)$ or $\tilde{w}_{\check{t}} = \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}} (\tilde{e}_s - \tilde{w}_s)$. Consider the first. Define a new contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$, which is identical to the original, except that $\tilde{e}'_{\check{t}} = \tilde{e}_{\check{t}} + \varepsilon$ and $\tilde{e}'_{\check{t}+1} = \tilde{e}_{\check{t}+1} - \nu(\varepsilon)$, with $\nu(\varepsilon)$ defined by

$$\psi(\tilde{e}_{\check{t}} + \varepsilon) + \delta\psi(\tilde{e}_{\check{t}+1} - \nu(\varepsilon)) = \psi(\tilde{e}_{\check{t}}) + \delta\psi(\tilde{e}_{\check{t}+1}).$$

Thus

$$\nu'(0) = \frac{\psi'(\tilde{e}_{\check{t}})}{\delta\psi'(\tilde{e}_{\check{t}+1})}$$

and so the change in the NPV of effort is

$$\varepsilon - \delta\nu(\varepsilon) = \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon + o(\varepsilon)$$

which is strictly positive for $\varepsilon$ sufficiently small. It is easy to see that the agent's constraint ($\mathrm{AC}_t^{\mathrm{ob}}$) is unchanged at all dates except $\check{t}+1$, when the constraint is relaxed. The principal's constraint ($\mathrm{PC}_t$) is unchanged from date $\check{t}+1$ onwards, relaxed at date $\check{t}-1$ and earlier (because the NPV of effort increases), but is tightened at date $\check{t}$. Provided $\varepsilon$ is small enough, the date-$\check{t}$ constraint remains intact. Profits increase, contradicting the optimality of the original contract.

Now suppose that $\tilde{w}_{\check{t}} = \sum_{s=\check{t}+1}^{\infty} \delta^{s-\check{t}} (\tilde{e}_s - \tilde{w}_s)$, and note the above adjustment now leads to a violation of the principal's constraint ($\mathrm{PC}_t$) at date $\check{t}$. In this case, we reduce slightly the payment, effort and consumption at date $\check{t}$, keeping the agent's payoff unchanged, but ensuring the principal's constraint ($\mathrm{PC}_t$) is satisfied. This has a negligible effect on profits since $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$. Hence, we again contradict the optimality of the original contract.

Now let us demonstrate precisely an adjustment that yields a self-enforceable contract. We further adjust the modified contract $(\tilde{e}'_t, \tilde{c}'_t, \tilde{w}'_t, \tilde{b}'_t)_{t\geq 1}$ by reducing the date-$\check{t}$ payment and consumption by an amount $\gamma(\varepsilon)$, and reducing date-$\check{t}$ effort by an amount $\eta(\varepsilon)$ to leave agent payoffs unchanged. The date-$\check{t}$ principal constraint ($\mathrm{PC}_t$) will then hold as equality by setting $\gamma(\varepsilon) = \delta\nu(\varepsilon)$. The requirement on $\eta(\varepsilon)$, that the agent's payoff is unaffected by the adjustment, is

$$v(\tilde{c}_{\check{t}} - \gamma(\varepsilon)) - \psi(\tilde{e}_{\check{t}} + \varepsilon - \eta(\varepsilon)) = v(\tilde{c}_{\check{t}}) - \psi(\tilde{e}_{\check{t}} + \varepsilon).$$

We then have

$$v'(\tilde{c}_{\check{t}})\gamma'(0) = \psi'(\tilde{e}_{\check{t}})\eta'(0).$$

Therefore, the overall increase in date-$\check{t}$ profits from all changes to the contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$

is

$$\varepsilon - \delta\nu(\varepsilon) - (\eta(\varepsilon) - \gamma(\varepsilon)) = \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon - \delta\nu'(0)\left(\frac{v'(\tilde{c}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}})} - 1\right)\varepsilon + o(\varepsilon)$$
$$= \left(1 - \frac{\psi'(\tilde{e}_{\check{t}})}{\psi'(\tilde{e}_{\check{t}+1})}\right)\varepsilon + o(\varepsilon)$$

which is strictly positive for $\varepsilon$ sufficiently small.

Hence, for small enough $\varepsilon$, the overall effect on profits of all changes to the original contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$ is positive, with the continuation profits from date $\check{t}$ increasing. The principal's constraint ($\text{PC}_t$) is relaxed at dates $\check{t}-1$ and earlier, it is satisfied by construction at $\check{t}$, and it is unchanged from date $\check{t}+1$ onwards. Again, the fact profits strictly increase contradicts the optimality of $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$.

From the above, and by induction, we have that $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ at some $\check{t}$ implies $v'(\tilde{c}_t) = \psi'(\tilde{e}_t)$ for all $t \geq \check{t}$. By the Euler equation (35), consumption and effort remain constant from $\check{t}$ onwards. Moreover, the agent's indifference condition in Equation (8) is presumed to hold at all dates. This shows that the agent's balances $\tilde{b}_t$ remains constant from date $\check{t}$ onwards, which given the assumption $V(b_1) < V^{FB}(b_1)$, contradicts the finding of Lemma A.13 that balances strictly decrease. Hence, we cannot have $v'(\tilde{c}_{\check{t}}) = \psi'(\tilde{e}_{\check{t}})$ at any $\check{t}$. □

Lemma A.15 implies by Lemma A.11 that, if $V(b_1) < V^{FB}(b_1)$, the principal's constraint ($\text{PC}_t$) holds with equality in every period. We use this to show the following.

**Lemma A.16.** *If $V(b_1) < V^{FB}(b_1)$, then, in any optimal contract $(\tilde{e}_t, \tilde{c}_t, \tilde{w}_t, \tilde{b}_t)_{t\geq 1}$, effort $\tilde{e}_t$ and payments $\tilde{w}_t$ strictly increase over time, while consumption $\tilde{c}_t$ strictly declines over time.*

*Proof.* The fact that the principal's constraint binds at every date, as argued above, can be stated as $\tilde{w}_t = \delta V(\tilde{b}_{t+1})$ for all $t$; hence payments are strictly increasing in $t$ by Lemma A.14. We also have $V(\tilde{b}_t) = \tilde{e}_t$ for all $t$, so effort is strictly increasing as well.

Now consider consumption. By Lemma A.12, we know that $\tilde{c}_{t-1} \geq \tilde{c}_t$ for all $t \geq 2$. Hence, if consumption fails to be strictly decreasing, we must have $\tilde{c}_{t-1} = \tilde{c}_t$ for some $t$. We then have, by Equation (35) (and noting that $\tilde{c}_t > (1-\delta)\tilde{b}_t$, also by Lemma A.12), that $\psi'(\tilde{e}_t) = v'(\tilde{c}_t)$. However, this contradicts Lemma A.15. □

**Lemma A.17.** *An optimal contract exists.*

*Proof.* If $\delta \geq \frac{c^{FB}(b_1)-(1-\delta)b_1}{e^{FB}(b_1)} \in (0,1)$ then there is a self-enforceable efficient contract (by Proposition 5.2), and so existence is established. The remainder of the proof is needed for the values $b_1$ such that there is no self-enforceable first-best contract.

We denote by $\Pi(b_t)$ the sequences $(c_s, b_{s+1})_{s=t}^{\infty}$ which are part of self-enforceable contracts beginning with balance $b_t$. Assuming "fastest payments", such a sequence completely defines the continuation contract from date $t$, with effort given at each date $s \geq t$ by $\hat{e}(c_s, b_s, b_{s+1})$, and the payment given by $\delta b_{s+1} - b_s + c_s$. Note that these sequences satisfy, for all $s \geq t$,

$$\delta b_{s+1} - b_s + c_s \leq \sum_{\tau=s+1}^{\infty} \delta^{\tau-s} \left(\hat{e}(c_\tau, b_\tau, b_{\tau+1}) - (\delta b_{\tau+1} - b_\tau + c_\tau)\right)$$

as well as

$$v(c_s) + \tfrac{\delta}{1-\delta} v\left((1-\delta)\, b_{s+1}\right) - \tfrac{1}{1-\delta} v((1-\delta) b_s) \geq 0.$$

Note also that $\Pi(b_t)$ is not empty: for instance, it contains the "autarky" continuation contract, where $c_s = (1-\delta)b_t$ and $b_s = b_t$ for all $s \geq t$ (recall that $\hat{e}((1-\delta)b_t, b_t, b_t) = 0$).

Given any $b_t > 0$, let

$$V(b_t) \equiv \sup_{(c_s, b_{s+1})_{s=t}^{\infty} \in \Pi(b_t)} \sum_{s=t}^{\infty} \delta^{s-t} \left(\hat{e}(c_s, b_s, b_{s+1}) - (\delta b_{s+1} - b_s + c_s)\right).$$

Note that $V(b_t)$ is no greater than the first-best value $V^{FB}(b_t)$. Usual arguments imply that the continuation payoff of the principal in an optimal contract (if it exists) is a fixed point of the following operator

$$TW(b_t) \equiv \sup_{c_t > 0, b_{t+1} > 0} \big(\hat{e}(c_t, b_t, b_{t+1}) - (\delta b_{t+1} - b_t + c_t) + \delta W(b_{t+1})\big) \tag{44}$$

subject to the principal's constraint

$$\delta b_{t+1} - b_t + c_t \leq \delta W(b_{t+1}) \tag{45}$$

and to

$$v(c_t) + \tfrac{\delta}{1-\delta} v\left((1-\delta)\, b_{t+1}\right) - \tfrac{1}{1-\delta} v((1-\delta) b_t) \geq 0. \tag{46}$$

Note that the operator $T$ is monotone: if $W_1 \geq W_2$, then $TW_1 \geq TW_2$. Also, we have $TV^{FB} \leq V^{FB}$. Applying $T$ to both sides, we have that $(T^n V^{FB}(b_t))_{n \geq 1}$ is a decreasing sequence for all $b_t > 0$. Therefore there is some pointwise limit of $T^n V^{FB}$, call it $\bar{V}$. Straightforward continuity arguments show that $\bar{V}$ is a fixed point of $T$.

**Outline of Proof.** We want to show that $\bar{V}(b_t) = V(b_t)$ and that this payoff is attained by a feasible contract $(\tilde{e}_s, \tilde{c}_s, \tilde{w}_s, \tilde{b}_s)_{s \geq t}$, with $\tilde{b}_t = b_t$, and which respects the principal and agent constraints, (PC$_t$) and (AC$_t^{\text{ob}}$). Note that $\bar{V}(b_t)$ is an upper bound on the principal's continuation payoff at a date $t$ when the balance is $b_t$. Hence, we only need to find a contract $(\tilde{e}_s, \tilde{c}_s, \tilde{w}_s, \tilde{b}_s)_{s \geq t}$ satisfying the aforementioned constraints and attaining a payoff $\bar{V}(b_t)$ for the principal.

There are two steps. In Step 1, we argue that it is possible to determine, given any date-$t$ balance $b_t > 0$, a sequence $(c_s, b_{s+1})_{s=t}^{\infty}$ by solving the maximization in the functional equation given by Equations (44) to (46) for $W = \bar{V}$. This sequence defines a contract that gives the principal a continuation payoff equal to $\bar{V}(b_t)$. In Step 2 we argue that this contract satisfies the principal and agent constraints, ($\mathrm{PC}_t$) and ($\mathrm{AC}_t^{\mathrm{ob}}$), as well as the feasibility conditions in Definition 2.1, and hence is self-enforceable.

**Step 1. Determining a policy from $\bar{V}$:** We want to show that the supremum in the problem defined by Equations (44) to (46) for $W = \bar{V}$ is attained by some values $c_t$ and $b_{t+1}$ at each $b_t > 0$. By analogous arguments to Step 3a of the proof of Lemma A.13, we have that $\bar{V}$ is continuous. Therefore, our supremum will be attained if (a) the values of $b_{t+1}$ that satisfy the constraints of the functional equation are contained in a bounded interval $[l^b(b_t), u^b(b_t)]$ with $l^b(b_t) > 0$, and (b) we can additionally restrict attention to consumption in a bounded interval $[l^c(b_t), u^c(b_t)]$ with $l^c(b_t) > 0$.

We begin with Part (a). Observe that

$$\lim_{b_{t+1}\to\infty} \delta(\bar{V}(b_{t+1}) - b_{t+1}) \leq \lim_{b_{t+1}\to\infty} \delta(V^{FB}(b_{t+1}) - b_{t+1}) = -\infty \ .$$

Hence, the satisfaction of the principal's constraint (45) implies $b_{t+1}$ must be bounded above by some $u^b(b_t)$.

We now show that, given $b_t$, satisfaction of the constraints in Equations (45) and (46) implies that $b_{t+1}$ must be no less than some $l^b(b_t) > 0$. Assume for the sake of contradiction that, given some $b_t > 0$, $b_{t+1}$ can be taken arbitrarily close to zero, without violating either of these constraints. In particular, consider $b_{t+1} < \bar{b}$, where $\bar{b} > 0$ is such that $\bar{V}(b) = V^{FB}(b)$ for all $b \in (0, \bar{b}]$ (note that it exists by Step 3b of the proof of Lemma A.13). These constraints may be written

$$v(c_t) \geq \tfrac{1}{1-\delta} v((1-\delta)b_t) - \tfrac{\delta}{1-\delta} v((1-\delta)b_{t+1}) \ \text{ and } \ c_t \leq b_t + \delta(V^{FB}(b_{t+1}) - b_{t+1}).$$

Combining these two equations we have

$$V^{FB}(b_{t+1}) \geq \tilde{V}(b_{t+1}) \equiv \frac{v^{-1}\left(\tfrac{1}{1-\delta} v((1-\delta)b_t) - \tfrac{\delta}{1-\delta} v((1-\delta)b_{t+1})\right) - b_t}{\delta} + b_{t+1}. \tag{47}$$

Now, notice that the right-hand side of Equation (47) tends to $+\infty$ as $b_{t+1} \to 0$. Hence, if the constraints are satisfied, we must have $\lim_{b_{t+1}\to 0} V^{FB}(b_{t+1}) = +\infty$ and

$$\lim_{b_{t+1}\to 0} \frac{\tilde{V}(b_{t+1})}{V^{FB}(b_{t+1})} \leq 1.$$

However, we now show that the value of this limit is instead $+\infty$.

First, notice that

$$V^{FB}(b_{t+1}) = \tfrac{1}{1-\delta} \max_{w} \left\{ \psi^{-1} \left( v \left( b_{t+1} \left( 1-\delta \right) + w \right) - v \left( b_{t+1} \left( 1-\delta \right) \right) \right) - w \right\}.$$

At the optimal choice of $w$, we have $c^{FB} \left( b_{t+1} \right) = b_{t+1} \left( 1-\delta \right) + w$, and

$$e^{FB} \left( b_{t+1} \right) = \psi^{-1} \left( v \left( b_{t+1} \left( 1-\delta \right) + w \right) - v \left( b_{t+1} \left( 1-\delta \right) \right) \right).$$

Therefore, by the envelope theorem,

$$\frac{d}{db_{t+1}} V^{FB} \left( b_{t+1} \right) = \frac{v' \left( c^{FB} \left( b_{t+1} \right) \right) - v' \left( b_{t+1} \left( 1-\delta \right) \right)}{\psi' \left( e^{FB} \left( b_{t+1} \right) \right)} = 1 - \frac{v' \left( b_{t+1} \left( 1-\delta \right) \right)}{\psi' \left( e^{FB} \left( b_{t+1} \right) \right)}.$$

On the other hand, the derivative of $\tilde{V}(b_{t+1})$ is given by

$$1 - \frac{v'((1-\delta)b_{t+1})}{v'\Big(v^{-1}\Big(\tfrac{1}{1-\delta}v((1-\delta)b_t) - \tfrac{\delta}{1-\delta}v((1-\delta)b_{t+1})\Big)\Big)}. \tag{48}$$

From l'Hôpital's rule, we have that

$$\begin{aligned}
\lim_{b_{t+1}\to 0} \frac{\tilde{V}(b_{t+1})}{V^{FB}(b_{t+1})} &= \lim_{b_{t+1}\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}b_{t+1}}\tilde{V}(b_{t+1})}{\frac{\mathrm{d}}{\mathrm{d}b_{t+1}}V^{FB}(b_{t+1})} \\
&= \lim_{b_{t+1}\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}b_{t+1}}\tilde{V}(b_{t+1}) - 1}{\frac{\mathrm{d}}{\mathrm{d}b_{t+1}}V^{FB}(b_{t+1}) - 1} \\
&= \lim_{b_{t+1}\to 0} \frac{-\frac{v'((1-\delta)b_{t+1})}{v'\left(v^{-1}\left(\frac{1}{1-\delta}v((1-\delta)b_t) - \frac{\delta}{1-\delta}v((1-\delta)b_{t+1})\right)\right)}}{-\frac{v'((1-\delta)b_{t+1})}{v'(c^{FB}(b_{t+1}))}} \\
&= \lim_{b_{t+1}\to 0} \frac{v'(c^{FB}(b_{t+1}))}{v'\Big(v^{-1}\Big(\frac{1}{1-\delta}v((1-\delta)b_t) - \frac{\delta}{1-\delta}v((1-\delta)b_{t+1})\Big)\Big)} \\
&= +\infty.
\end{aligned}$$

The second equality holds because both the numerator and the denominator tend to $-\infty$. The final equality holds because $c^{FB}(b_{t+1}) \to 0$ as $b_{t+1} \to 0$, by Step 3b of Lemma A.13.

We have therefore shown that, given a date-$t$ balance $b_t$, the choices of $b_{t+1}$ that are available in the above program while satisfying the constraints Equations (45) and (46) come from some bounded set $\left[l^b(b_t), u^b(b_t)\right]$ with $l^b(b_t) > 0$. It is then immediate that consumption $c_t$ must be chosen from some bounded interval $[l^c(b_t), u^c(b_t)]$ as well. Hence, given the continuity of $\bar{V}$, the problem defined by Equations (44) to (46) and $W = \bar{V}$ has a solution. We can then solve iteratively to determine an optimal (date-$t$) continuation policy $(c_s, b_{s+1})_{s=t}^{\infty}$. Such a sequence defines (assuming "fastest payments", as noted above) a contract that satisfies the principal's constraints ($\text{PC}_t$) and agent's constraints ($\text{AC}_t^{\text{ob}}$), as well as the first two feasibility conditions in Definition 2.1.

**Step 2. Showing the contract is self-enforceable.** To establish that the value $\bar{V}(b_t)$ can be attained in the problem of interest, we need to verify that the third feasibility condition in Definition 2.1 (bounded consumption, payments and effort) is satisfied by the aforementioned contract. Note first that the principal's first-best payoff continues to be given by $V^{FB}(b_t)$, given any agent balance $b_t > 0$, if we replace Condition 3 of Definition 2.1 with the mere requirement that the NPV of consumption is finite (i.e., $\sum_{s \geq t} \delta^{s-t} c_s < +\infty$). This follows from the arguments in the proof of Proposition 3.1.[25] Therefore $\bar{V}(b_t)$ continues to be an upper bound on profits in a contract satisfying the principal's constraints ($\text{PC}_t$) and agent's constraints ($\text{AC}_t^{\text{ob}}$), and the modified feasibility conditions.

In addition, if $b_s < \bar{b}$ (with $\bar{b}$ as defined above and $b_s$ the balance at date $s$), then $\bar{V}(b_s) = V^{FB}(b_s)$, as this value is attained in the maximization problem associated with $TV^{FB}(b_s)$ in Equations (44) to (46) (taking $W = V^{FB}$) by setting $c_s = c^{FB}(b_s)$ and $b_{s+1} = b_s$ (so that $\hat{e}(c_s, b_s, b_{s+1}) = e^{FB}(b_s)$). Therefore, in the continuation policy $(c_s, b_{s+1})_{s=t}^{\infty}$ generated by $\bar{V}$ starting from a balance $b_t$ at date $t$, if $b_s < \bar{b}$ for any $s \geq t$, then $b_{s'} = b_s$ for all $s' \geq s$ and consumption remains constant at value $c^{FB}(b_s)$ (i.e., $c_{s'} = c^{FB}(b_s)$ for all $s' \geq s$). Because $\bar{V} \leq V^{FB}$, and because $V^{FB}(\cdot)$ is decreasing (by Proposition 3.1), $\bar{V}(b_{s+1})$ is bounded, say by a value $B > 0$, over all $s \geq t$. Therefore, for all $s \geq t$,

$$c_s + \delta b_{s+1} - b_s \leq \delta B.$$

Therefore, for any $\tau \geq t$, we have

$$\begin{aligned}\sum_{s=t}^{\tau} \delta^{s-t} c_s &= b_t - \delta^{\tau-t+1} b_{\tau+1} + \sum_{s=t}^{\tau} \delta^{s-t} \left(c_s + \delta b_{s+1} - b_s\right) \\ &\leq b_t - \delta^{\tau-t+1} b_{\tau+1} + \frac{\delta\left(1 - \delta^{\tau+1-t}\right)}{1-\delta} B.\end{aligned}$$

Because balances remain positive, we can conclude that $\sum_{s=t}^{\infty} \delta^{s-t} c_s$ is no greater than $b_t + \frac{\delta}{1-\delta} B < +\infty$.

The above argument shows that the contract determined from $\bar{V}$ starting at balance $b_t$ at date $t$ is optimal given the requirement that the contract satisfies the principal's constraints ($\text{PC}_t$) and agent's constraints ($\text{AC}_t^{\text{ob}}$), the first two feasibility requirements of Defintion 2.1,

[25] Consider any contract for which the NPV of consumption is finite. Then the NPV of the disutility of effort must also be finite if the inequality in Equation (16) is to be satisfied. The NPV of output is therefore finite. It is then clear that the NPV of payments must be finite in a contract that maximizes the principal's payoff subject to the condition in Equation (16) and the requirement that the NPV of consumption is finite. The arguments in the proof of Proposition 3.1 then apply directly.

and the requirement that the NPV of consumption is finite. Note that all of the arguments in Lemmas A.10 to A.16 apply also to contracts that are optimal with the weakened feasibility condition. Hence, consumption, pay and effort are bounded, and so the contract is also optimal in the more restrictive class of contracts satisfying all conditions of Definition 2.1. This concludes Step 2. □

*(End of the proof of Proposition 5.3.)* □

**Proof of Proposition 5.4**

*Proof.* If the result does not hold, then there is a date $t$ such that

$$\frac{v(\tilde{b}_t\,(1-\delta))}{1-\delta} < \sum_{s=t}^{\infty} \delta^{s-t}\,(v\,(\tilde{c}_s) - \psi(\tilde{e}_s))\,.$$

If this date is $t = 1$, then date-$t$ effort can be increased to obtain another self-enforceable contract that is more profitable for the principal, so we may assume $t > 1$. We can then increase the payment to the agent at date $t-1$ by $\varepsilon\delta$ for $\varepsilon > 0$, and reduce the date-$t$ payment by $\varepsilon$. All other variables are unchanged. Provided $\varepsilon$ is small enough, all constraints are preserved. Because the date-$t$ payment is reduced, the principal's constraint ($\text{PC}_t$) is then slack at date $t$.

Because the contract is optimal, but not first best, we have that effort strictly increases over time. We can then change the date-$t$ effort to a value $\tilde{e}'_t$, and the date-$t+1$ effort to $\tilde{e}'_{t+1}$, with $\tilde{e}_t < \tilde{e}'_t < \tilde{e}'_{t+1} < \tilde{e}_{t+1}$, and with

$$\psi\,(\tilde{e}'_t) + \delta\psi\left(\tilde{e}'_{t+1}\right) = \psi\,(\tilde{e}_t) + \delta\psi\,(\tilde{e}_{t+1})\,.$$

All other variables remain unchanged. This affects the agent constraints ($\text{AC}_t^{\text{ob}}$) by increasing the profitability of remaining in the contract from date $t+1$ onwards (i.e., the date-$t+1$ constraint is slackened). It relaxes the principal's constraint at date $t-1$ and earlier, because the NPV of effort increases (by convexity of $\psi$). It tightens the principal's constraint at date $t$, but provided the changes are small, it remains slack. The principal's constraints are unaffected from date $t+1$ onwards. Because the NPV of effort increases, profits strictly increase. This contradicts the optimality of the original contract, which establishes the result. □

## A.4 Proofs of the results in Section 6

**Proof of Proposition 6.1**

*Proof.* Let $(\tilde{e}_t, \tilde{w}_t)_{t\geq 1}$ be an optimal contract. Define, for all $t$, $U_t \equiv \sum_{s=t}^{\infty} \delta^{s-t}(v(\underline{c} + \tilde{w}_s) - \psi(\tilde{e}_s))$. Also, for the rest of the proof, normalize $v(\underline{c}) \equiv 0$.

**Step 1.** We first note that $U_1 = 0$. Otherwise, the effort at time 1 can be increased, so the payoff of the principal increases and, if the effort increase is small enough, all constraints are satisfied.

**Step 2.** We now prove that, for all $t \geq 1$, if $U_{t+1} > 0$ then $\tilde{w}_{t+1} \leq \tilde{w}_t$ and $\tilde{e}_{t+1} \geq \tilde{e}_t$. To see the first implication we assume, for the sake of contradiction, that $U_{t'+1} > 0$ and $\tilde{w}_{t'+1} > \tilde{w}_{t'}$ for some $t' \geq 1$. Consider a contract $(\tilde{e}'_t, \tilde{w}'_t)_{t\geq 1}$ where payments at times different from $t'$ and $t'+1$ remain the same, and all efforts remain the same. Slightly increase the date-$t'$ payment so that $\tilde{w}'_{t'} > \tilde{w}_{t'}$ and slightly reduce date-$t'+1$ payment so that $\tilde{w}'_{t'+1} < \tilde{w}_{t'+1}$, maintaining $\tilde{w}'_{t'+1} > \tilde{w}'_{t'}$, as well as

$$v(\underline{c} + \tilde{w}'_{t'}) + \delta v(\underline{c} + \tilde{w}'_{t'+1}) = v(\underline{c} + \tilde{w}_{t'}) + \delta v(\underline{c} + \tilde{w}_{t'+1}) \ .$$

The agent's constraint (15) is affected only at date $t'+1$: $U_{t'+1}$ decreases, but if the modification to the contract is small enough, we still have $U_{t'+1} > 0$ and so the agent's constraint is satisfied.

Since $\tilde{w}'_{t'+1} < \tilde{w}_{t'+1}$ (while all consumption and efforts after $t'+1$ remain the same), the principal's constraint ($\text{PC}_t$) is relaxed at time $t'+1$. Given that $v$ is strictly concave, the time-$t'$ present value of the payments decreases, so the principal's constraint is relaxed at all dates $t'$ and earlier. Hence, the contract $(\tilde{e}'_t, \tilde{w}'_t)_{t\geq 1}$ is self-enforceable and it is strictly more profitable than the original, contradicting the optimality of the original.

We now prove that, for all $t \geq 1$, if $U_{t+1} > 0$, then $\tilde{e}_{t+1} \geq \tilde{e}_t$. To see this we assume, for the sake of contradiction, that $U_{t'+1} > 0$ and $\tilde{e}_{t'+1} < \tilde{e}_{t'}$ for some $t' \geq 1$. Consider a contract $(\tilde{e}'_t, \tilde{w}'_t)_{t\geq 1}$ where payments remain the same at all times, and where efforts differ only at $t'$ and $t'+1$. In particular, we set $\tilde{e}'_{t'+1}$ slightly higher than $\tilde{e}_{t'+1}$, and choose $\tilde{e}'_{t'}$ so that

$$\psi(\tilde{e}'_{t'}) + \delta\psi(\tilde{e}'_{t'+1}) = \psi(\tilde{e}_{t'}) + \delta\psi(\tilde{e}_{t'+1}) \ .$$

The agent's constraints (15) are then unaffected, except at $t'+1$ where the constraint is tightened, but for a small enough adjustment to the contract we still have $U_{t'+1} > 0$. Principal constraints ($\text{PC}_t$) remain unaffected from $t'+1$ onwards. Since $\tilde{e}'_{t'+1} > \tilde{e}_{t'+1}$ (while all payments and efforts after $t'+1$ remain the same), the principal's constraint is relaxed at time $t'$. Given that $\psi$ is convex, the date-$t'$ present value of the effort increases (provided the changes in effort are small enough), so the principal's constraint is relaxed at all times before $t'$ as well. Again, the modified contract is more profitable, which gives our contradiction.

**Step 3.** We now prove that $U_t = 0$ for all $t$. We already established in Step 1 that $U_1 = 0$. Now suppose for a contradiction that the result is not true, so there is a $t$ satisfying $U_t = 0$ and $U_{t+1} > 0$ (it could be that $t = 1$). Let then $t+T$ be the first date after $t$ at which $U_{t+T} = 0$ ($T = +\infty$ means that $U_s > 0$ for all $s > t$), so we have assumed $T > 1$. We have

$$U_t = \sum_{s=t}^{t+T-1} \delta^{s-t}(v(\underline{c} + \tilde{w}_s) - \psi(\tilde{e}_s)) \geq \sum_{s=t}^{t+T-2} \delta^{s-t}(v(\underline{c} + \tilde{w}_s) - \psi(\tilde{e}_s)) \ ,$$

where the inequality follows from the fact that (1) when $T < +\infty$, since $U_{t+T-1} > 0$, we have $v(\underline{c} + \tilde{w}_{t+T-1}) - \psi(\tilde{e}_{t+T-1}) > 0$, and (2) when $T = +\infty$, the inequality holds as equality. Note then that, since $\tilde{w}_{s+1} \leq \tilde{w}_s$ and $\tilde{e}_{s+1} \geq \tilde{e}_s$ for all $s \in \{t, ..., t+T-2\}$ we have

$$U_t \geq \sum_{s=t}^{t+T-2} \delta^{s-t}(v(\underline{c} + \tilde{w}_{s+1}) - \psi(\tilde{e}_{s+1})) = \sum_{s=t+1}^{t+T-1} \delta^{s-t-1}(v(\underline{c} + \tilde{w}_s) - \psi(\tilde{e}_s)) = U_{t+1} > 0 \ .$$

This is a contradiction because $U_t = 0$.

Note that the conclusion $U_t = 0$ for all $t$ implies that the optimal contract is stationary. For any date $t$ the continuation of an optimal contract $(\tilde{e}_s, \tilde{w}_s)_{s \geq t}$ must maximize continuation profits

$$\sum_{s=t}^{\infty} \delta^{s-t} \left(e_s - w_s\right)$$

subject to $v\left(\underline{c} + w_s\right) = \psi\left(e_s\right)$ and to principal constraints $w_s \leq \sum_{s'=s+1}^{\infty} \delta^{s'-s} \left(e_{s'} - w_{s'}\right)$ for all $s \geq t$. In particular, note that maximizing continuation profits at date $t$ only relaxes the principal's constraints ($\text{PC}_t$) at dates $t-1$ and earlier.

**Step 4.** Given that any optimal relational contract is stationary, we can now consider the problem of maximizing by choice of effort $e$ and payment $w$ the per period profit $e - w$ subject to the agent earning zero payoff (that this must hold is established above), i.e. $v(\underline{c} + w) - \psi(e) = 0$, and subject to the principal's constraint ($\text{PC}_t$) which takes the form $w \leq \delta e$. Case 1 in the proposition then corresponds to the case where the principal's constraint does not bind, while Case 2 corresponds to the one where it binds. $\square$