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JEL Classification: D4, J6, L1, L8, R3

Keywords: price posting, ordered search, capacity- and selling-constrained firms

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Selling Constraints*

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May 5, 2020

Abstract

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1 Introduction

Despite the fact that every organization tries to optimize its marketing and sales team,[1] *selling constraints*, which refer to the inability of firms to attend to all interested buyers, are ubiquitous when customer visits are stochastic. Queuing at shops, waiting at the phone to be attended to and ultimately getting rationed are features of everyday life. How do consumers and firms factor selling constrains into their decision-making? To the best of our knowledge, this paper is the first to study how selling constraints affect the functioning of search markets.[2] In particular, we ask:

- How do selling constraints affect the price equilibrium? Do prices increase or decrease as selling constraints weaken?
- How do selling constraints impact the efficiency of the market? Are selling constraints a source of marker power?

We consider a search market where there is a large number of sellers, each of them selling one unit of a differentiated good, and a large number of buyers, each of them interested in buying one unit of a satisfactory product. Sellers compete for buyers by posting prices. Buyers, after observing the posted prices, choose which seller to visit. Once an individual buyer is at a seller, depending on how many other buyers queue at the same seller and the ability of the seller to interact with various buyers, she may be offered the opportunity to inspect the seller's product, in which case she learns her valuation and decides whether to buy it or not.

With selling constraints, depending on how many buyers show up at their premises, firms may encounter themselves in a situation where they have to neglect buyers. This has a bearing on equilibrium pricing. A laxer selling constraint operates on the equilibrium price in two ways. On the one hand, because a firm can attend to more buyers, it makes attracting them to its premises more valuable; this gives firms incentives to decrease the equilibrium price. On the other hand, because a firm may offer the opportunity to inspect the product to more buyers, it increases the chance that a higher price is accepted by one of them; this gives firms incentives to increase the equilibrium price. When there are few buyers per seller, the second effect plays a weak role and the equilibrium price decreases as selling constraints become softer. By contrast, when the number of buyers per seller is significant, the first effect plays little role and the equilibrium price increases as selling constraints weaken.

[1] Sales force management, which constitutes a typical course in business programs worldwide (e.g. Chicago Booth, INSEAD and MIS Singapore), has been a research topic in management and marketing for various decades. For a textbook approach to the topic, see for example Johnston and Marshall (2013) and Rich (2017).

[2] To be sure, selling constraints may restrict sellers over and above what capacity constraints do. For example, a homeowner, who typically has only one house to rent, may very well have limited ability to let prospective tenants view the house.

The limiting case in which the number of buyers per seller is infinitely large is of special interest in that it reflects monopoly. We show that the notion of monopoly price is intimately linked to the extent to which selling constraints bind sellers. In the limiting case of extreme selling constraints in which firms can only offer their products to one buyer picked at random, the monopoly price is the standard textbook one. But with laxer selling constraints, the demand of a typical seller becomes more inelastic because failing to sell its product to a first consumer, the seller can offer it to a second consumer and so on till the selling constraint binds. In the limiting case in which firms do not face selling constraints whatsoever, the seller can continue to offer its product without limit so the equilibrium price converges to the upper bound of the distribution of match values.

Social welfare is typically non-monotonic in price, which implies that the social optimum involves firms charging prices above the marginal cost. In standard markets, marginal cost pricing is a cornerstone of efficiency and thus the shortcut definition of market power is the ability of a firm to sustain prices about marginal cost. The reason for this is that as the price increases away from the marginal cost, low-valuation buyers are excluded without affecting the probability with which high-valuation buyers trade, which clearly generates a dead-weight loss. In our model, by contrast, because firms are capacity constrained and just have one unit to sell, a higher price excludes low-valuation buyers in benefit of high-valuation ones, which has a positive effect on surplus. The planner, thus, faces a trade-off. By raising the price, the chance that the product sells goes down but in case of a sale the surplus generated goes up. These two forces operate on welfare in a way that welfare first increases in price and then decreases. Welfare maximization consists of balancing these two effects, which drives a wedge between the optimal price and marginal cost. Hence, the shortcut notion of market power as the ability of firms to sustain prices about marginal cost is of no use here. Instead, market power has to be assessed as the capacity of firms to sustain prices above the efficient level, which differs from the marginal cost.

We show that the efficiency of the pricing equilibrium turns out to hinge upon the firms' ability to show their products to the buyers. In choosing the equilibrium price, an individual firm faces a similar trade-off as the social planner. By raising the price, the firm incurs the risk that all the buyers that the firm can attend to happen to choose to not buy the product, but it gets a higher profit if just one of the buyers acquires it. We show that when each seller can only attend to a finite number of buyers, equilibrium markups turn out to be inefficiently high. The intuition is that firms have too weak incentives to attract buyers; the fewer the buyers they can attend to, the weaker the incentives to attract them. As the number of buyers an individual firm can attend to goes up, the incentives to attract buyers increase and firms correspondingly lower the price. In the limiting case in which all sellers can continue to offer its product to all the buyers that show up at their premises no matter how many, then markups are at the efficient level. We thus conclude that selling constraints constitute a source of market power.

To the best of our knowledge, our paper is the first to study the impact of selling constraints on the efficiency of the market equilibrium. In doing so, we have modelled a one-shot search market where products are horizontally differentiated in the tradition of Wolinsky (1986) and prices are observable by consumers before choosing which sellers to visit. Our model is thus connected to two strands of the economics literature. The first line of work is the literature on consumer search for differentiated products initiated by Wolinsky (1986) and Anderson and Renault (1999). More recent contributions include e.g. Armstrong et al. (2009), Bar-Isaac et al. (2012) and Haan and Moraga-González (2011). Following the tradition commenced by Diamond (1971), this literature has typically modelled markets in which prices are non-observable before search. However, in recent years, perhaps motivated by the availability of data from the Internet, there has been a surge of interest in the modelling of search markets in which consumers observe prices before search. Our model in which firms are capacity constrained contributes to this line of work by putting forward a tractable way to model price competition in search markets. In the absence of capacity constraints, price observability does not combine well with product differentiation to yield a tractable model. The problem is that an equilibrium in pure-strategies does not exist and the mixed-strategy equilibrium is difficult to characterize. To model price-directed search, thus, several authors have modified the standard setting in alternative ways (see Armstrong and Zhou (2011), Choi et al. (2018), Ding and Zhang (2018) and Haan et al. (2019)).

The second line of work our paper relates to is the search and matching literature (for a recent survey, see Wright et al. (2019)). In this literature, it is standard to model capacity-constrained firms but typically products (to be sure, jobs in most of the articles) are assumed to be homogeneous. When products are homogeneous, however, selling constraints are inconsequential because the first buyer to whom the firm offers the product will buy it. Because the quality of trade does not matter, the equilibrium price has no bearing on welfare, hence the literature's focus on other aspects of efficiency, in particular the efficiency of entry.[3] With product differentiation, by contrast, both the firms and the social planner face a trade-off between the quantity and the quality of trade. As a result, both the equilibrium price and the socially optimal price involve positive markups. With selling constraints, however, firms put more weight on the quality of trade than the planner and therefore the equilibrium price is inefficiently high. This misalignment between the private and the social incentives vanishes in the limit when selling constraints become negligible.

[3] An exception is when firms are asymmetric for example because some sellers have more units to sell than others (see Watanabe (2010, 2018, 2019), Tan (2012), and Godenhielm and Kultti (2015)).

2 The model

There is a measure B of buyers, and a measure S of sellers. Each buyer has unit demand. Each seller has one unit to sell.[4] Let x be the buyer-seller ratio, i.e. $x \equiv B/S$. The limiting case $x \to 0$ represents a case in which there are infinitely many firms per buyer so that the market is extremely competitive. The other limiting case $x \to \infty$ refers to a situation in which there are infinitely many buyers per firm so that each firm enjoys a monopoly position. For simplicity, we normalize unit production costs to zero.

Products are horizontally differentiated. We model product differentiation as in the random utility framework of Perloff and Salop (1985).[5] The exact value a buyer ℓ places on the product of a seller i, denoted $\varepsilon_{i\ell}$, depends on how well the product matches the tastes of the buyer. Such a match value can only be learnt upon inspection of the product. We assume that match values are identically and independently distributed across buyers and sellers. Let F be the distribution of match values, with density f and support $[0,1]$. From now on, we drop the sub-index of $\varepsilon_{i\ell}$.

With differentiated products, it is important to pay attention to the ability of a seller to offer its product to the next buyer in line after the earlier buyer has decided to not buy it because her value falls short of the price. We refer to this (lack of) ability as *selling capacity*. We assume that each seller can (sequentially) offer its product to a total of k buyers, $k = 1, 2, ..., \infty$.[6]

The interaction between buyers and sellers is modeled as a one-shot game.[7] First, firms post prices; then, buyers, after observing all the prices posted, independently pick a seller to visit. Once buyers are allocated across sellers, each seller (with customers) offers the good to a first buyer; after inspecting the good, the buyer decides whether to buy it or not. If the buyer buys the good, she gets utility $\varepsilon_i - p_i$ while the seller gets p_i. If the buyer decides to not buy the good, in which case she gets zero utility, the seller offers the good to the next buyer (if there is one in his queue). The process continues in the same fashion until either there are no more buyers in the queue of the seller or the seller can no longer attend to buyers because his selling capacity k is exhausted.

We are interested in the characterization of a symmetric equilibrium price. An individual firm,

[4] The insights of our paper should carry over to situations where firms have more units to sell but still face selling constraints. For an elaboration of this point, see Section 5.

[5] This assumption is central to our model. Actually, the fact that products are horizontally differentiated makes selling constraints relevant; with homogeneous products, on the contrary, the first buyer contacted by the firm acquires the product and selling constraints thus become inconsequential.

[6] The case $k = 1$ represents the extreme case in which each seller can only offer its product to a single buyer. The case $k \to \infty$ represents the also extreme case in which a seller can continue to offer its product to all the buyers in its queue, even if there are infinitely many of them. In most settings, a small k will reflect better the reality than a large k, for example when firms do not have sufficient salespeople or when products are highly complex so that each consumer takes quite a bit of time to evaluate it (houses, campers, boats, etc.).

[7] The main results of our paper should extend to situations where consumers are allowed to search sequentially till they find a satisfactory match, as it is standard in the consumer search literature. In such a case, the consumers' reservation value plays the role of the equilibrium price. For further details, see Section 5.

taking as given the price of the other sellers and anticipating buyers' behavior, picks a price to maximize profits. As in the directed search literature (see e.g. Peters (1991)), we assume that buyers pick which firms to visit so as to make their expected utilities equal across sellers.[8]

3 Analysis

We start the analysis by deriving the probability with which a buyer queuing at a seller gets an opportunity to inspect the seller's product and decide whether to buy it or not. Later in Section 3.2 we study the pricing game.

3.1 Inspection probability

It is well-known that in large markets in which buyers cannot coordinate their visiting strategies, the number of buyers n that visit a seller follows a Poisson distribution with the Poisson parameter equal to the queue of buyers the seller expects, in this case x. Thus, the probability that $\ell = 0, 1, 2, ...$ buyers show up at a given seller is equal to:

$$\Pr(n = \ell) = \frac{x^{\ell} e^{-x}}{\ell!},$$

where the symbol Pr stands for probability.

Let us denote by $\eta(x, p; k)$ the probability with which a given buyer visiting a seller charging a price p with an expected queue x gets an opportunity to inspect the product when the seller can offer its product to a maximum of k buyers. This probability will in general depend on the ratio of buyers to sellers x because the more buyers showing up at a seller, the less likely it is that a particular buyer gets an opportunity to inspect and possibly buy the product. This likelihood also depends on the price p the firm charges because a higher price makes it more probable that other buyers contacted earlier than the buyer in question choose to not buy the product. Finally, this probability is also affected by the selling capacity k of the seller because the seller will not be able to attend to the given buyer if she has k other buyers queuing in front of her. Our first contribution is to derive the inspection function for an arbitrary selling capacity k:

Proposition 1 *The probability with which a buyer queuing at a firm charging price p and with an*

[8] It is implicitly assumed that buyers cannot coordinate themselves to visit firms in a way that minimizes the probability of being rationed. This assumption is standard in that literature and is quite reasonable in large markets with many buyers and sellers.

expected queue x gets an opportunity to inspect and buy the product of the firm is equal to:

$$\eta(x,p;k) = \frac{1}{x(1-F(p))} m(x,p;k) \tag{1}$$

where

$$m(x,p;k) = 1 - F(p)^k + \left(\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) F(p)^k - \frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)} e^{-x(1-F(p))},$$

and $\Gamma(k+1) = k! = \int_0^\infty t^k e^{-t} dt$ *and* $\Gamma(k+1,x) = \int_x^\infty t^k e^{-t} dt$.

Proof. See the Appendix. ■

In deriving this probability, one needs to take into account situations in which the number of buyers arriving at a given seller exceeds the number of buyers that the seller can attend to ($n > k$), as well as cases in which this is not the case ($n \leq k$).

It is now didactic to consider two special cases of interest. The first is the $k = 1$ case, which represents a situation of extreme selling constraints. Setting $k = 1$ in (1) gives the well-known matching probability (see e.g. Butters (1977)):

$$\eta(x,p;1) = \frac{1-e^{-x}}{x}.$$

Notice that this probability is decreasing in x but independent of the price p. A higher buyer-seller ratio x makes it less likely that a particular buyer is offered the product. The price does not matter because if the first buyer rejects the product of the seller, then the seller cannot offer it to anyone else.

The second special case is that in which $k \to \infty$, which represents a situation in which firms do not have selling constraints whatsoever. Taking the limit of the probability in (1) when $k \to \infty$ gives:

$$\eta(x,p;\infty) = \lim_{k\to\infty} \eta(x,p;k) = \frac{1-e^{-x(1-F(p))}}{x(1-F(p))}$$

This expression is similar in spirit to the one that obtains when $k = 1$ when we interpret the quantity $x(1 - F(p))$ as an “effective” queue, that is, the queue of buyers with a match value above the price. Because F is a distribution function, it is straightforward to verify the intuitive result that $\eta(x,p;\infty) > \eta(x,p;1)$.

In general, it is more realistic to consider environments where selling constraints are neither extreme nor non-existent. Our next result gives some general properties of the inspection probability $\eta(x,p;k)$ for arbitrary k.

Proposition 2 *The probability $\eta(x, p; k)$ (with which a buyer gets an opportunity to inspect and buy a product) is increasing and concave in k, decreasing in x, and increasing in $F(p)$ for $k \geq 2$.*

Proof. See the Appendix. ■

3.2 Pricing

We now move to the characterization of a symmetric equilibrium price. Let p be the symmetric equilibrium price. In order to derive p, consider a deviation by an individual seller i to a price $p_i \neq p$. Because buyers observe the prices posted by all the sellers before they decide which individual seller to visit, sellers' prices influence the expected number of buyers they receive. Let $x_i \equiv x(p_i; p)$ denote the expected number of buyers showing up at the deviating seller i when the rest of the sellers charge price p. The deviant's profit function is:

$$\Pi(p_i; p) = p_i \left(\sum_{\ell=1}^{k} \Pr[n_i = \ell](1 - F(p_i)^{\ell}) + \sum_{\ell=k+1}^{\infty} \Pr[n_i = \ell](1 - F(p_i)^{k}) \right)$$

This profit function reflects the fact that the actual queue of buyers at the deviant firm, n_i, may be smaller or larger than the maximum number of buyers the firm can attend to, k. In the Appendix we show that this payoff can be written more conveniently as:

$$\Pi(p_i; p) = p_i(1 - F(p_i))x(p_i; p)\eta(x_i, p_i; k). \tag{2}$$

To determine the expected queue of buyers $x(p_i; p)$, we assume that the buyer-seller ratio that the deviant expects is such that buyers are indifferent between the utility they expect to get at the deviant and the utility they expect to get at any other seller in the market. The expected utility of a buyer who chooses to visit the deviant seller, denoted by $V(p_i; p)$, is given by:

$$\begin{aligned} V(p_i; p) &= \eta(x_i, p_i; k)[1 - F(p_i)][E(\varepsilon \mid \varepsilon \geq p_i) - p_i] \\ &= \eta(x_i, p_i; k)I(p_i), \end{aligned}$$

where

$$I(p_i) \equiv \int_{p_i}^{1} (\varepsilon - p_i) f(\varepsilon) d\varepsilon$$

is the consumer's expected utility conditional on meeting with the seller. For later use, note that $\partial I / \partial p_i = -(1 - F(p_i))$. Meanwhile the expected utility of a buyer who picks any other seller charging p is:

$$V(p;p) = \eta(x,p;k)[1-F(p)][E(\varepsilon \mid \varepsilon \geq p) - p] = \quad \eta(x,p;k)I(p).$$

Solving the equation $V(p_i;p) = V(p;p)$ for $x(p_i;p)$ gives the expected number of buyers who will visit a deviant seller charging price p_i. Unfortunately, the function $x(p_i;p)$ cannot be obtained in closed form. Nevertheless, in order to study equilibrium pricing we can apply the implicit function theorem to the equation $V(p_i;p) - V(p;p) = 0$ to obtain the sensitiveness of the deviant seller i's queue to its own price:

$$\frac{\partial x(p_i;p)}{\partial p_i} = -\frac{\frac{\partial \eta_i}{\partial p_i} I(p_i) - \eta_i(1-F(p_i))}{\frac{\partial \eta_i}{\partial x_i} I(p_i)}, \tag{3}$$

where, to shorten notation, we have written η_i instead of $\eta(x_i,p_i;k)$.

The first order condition (FOC) for profits maximization, $\Pi'(p_i;p) = 0$, is:

$$[1-F(p_i) - p_i f(p_i)]\, x_i \eta_i + p_i(1-F(p_i))x_i \frac{\partial \eta_i}{\partial p_i} + p_i(1-F(p_i)) \left(\eta_i + x_i \frac{\partial \eta_i}{\partial x_i}\right) \frac{\partial x_i}{\partial p_i} = 0. \tag{4}$$

After imposing symmetry, i.e. $p_i = p$, which also implies that $x_i = x$ and $\eta_i = \eta$, and using (3), we can rewrite the FOC (4) as follows:

$$\begin{aligned}
& x\eta\,[1-F(p) - pf(p)] + xp(1-F(p)) \left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p} \\
& + p(1-F(p)) \left(\eta + x \left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}\right) \frac{\eta(1-F(p)) - I(p) \left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p}}{I(p) \left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}} \\
& = x\eta\,[1-F(p) - pf(p)] + \eta p(1-F(p)) \frac{\eta(1-F(p)) - I(p) \left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p}}{I(p) \left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}} \\
& + x\eta p \frac{(1-F(p))^2}{I(p)} = 0.
\end{aligned}$$

Rearranging terms, this expression can be rewritten as:

$$\frac{\partial \eta}{\partial x} x \left\{[1-F(p) - pf(p)]\, I(p) + p(1-F(p))^2\right\} + p\eta(1-F(p))^2 - \frac{\partial \eta}{\partial p} p(1-F(p))I(p) = 0, \tag{5}$$

where we have replaced $\left.\frac{\partial \eta_i}{\partial p_i}\right|_{p_i=p}$ and $\left.\frac{\partial \eta_i}{\partial x_i}\right|_{p_i=p}$ by $\frac{\partial \eta}{\partial p}$ and $\frac{\partial \eta}{\partial x}$, respectively, for simplicity of notation.

In the Appendix, we establish the following useful relationship between $\frac{\partial \eta}{\partial p}$ and $\frac{\partial \eta}{\partial x}$:

$$\frac{(1-F(p))}{f(p)}\frac{\partial \eta}{\partial p} = -x\frac{\partial \eta}{\partial x} + d(p), \tag{6}$$

where

$$d(p) \equiv -\frac{kF(p)^{k-1}}{x}\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right) \leq 0.$$

Using relationship (6) we can state that:

Lemma 1 *If there exists a symmetric equilibrium price, it is given by the solution to:*

$$\begin{aligned}\frac{(1-F(p))^2}{f(p)}&\left[\frac{\partial \eta}{\partial p}I(p) + \frac{\partial \eta}{\partial p}p(1-F(p)) - p\eta f(p)\right] \\ &= d(p)\left[(1-F(p)-pf(p))\,I(p) + p(1-F(p))^2\right].\end{aligned} \tag{7}$$

Proof. See the Appendix. ■

In regard to the existence of a symmetric equilibrium price, we first observe that equation (7) has at least one solution, which means that a price satisfying the necessary condition always exists. Let us refer to a solution to equation (7) as a candidate equilibrium price. A sufficient condition for a candidate equilibrium price to indeed be a symmetric equilibrium price is that the function $m(x_i, p_i; k)$ is log-concave in p_i. If this is the case, then the demand in (2) is log-concave in p_i and by implication the corresponding payoff is quasi-concave in p_i. Verification of this condition is extremely hard because the function $x(p_i; p)$ cannot be computed in closed form. Nevertheless, we can prove the existence of equilibrium when $k = 1$ and the distribution of match values is uniform. These observations are collected in the following result.

Proposition 3 *A price that satisfies the necessary condition (7) always exists. When $k = 1$ and F is the uniform distribution, a solution to (7) is certainly a symmetric equilibrium price.*

Proof. See the Appendix. ■

A candidate equilibrium price always exists and, though it is very difficult in general to analytically verify that such a candidate is surely an equilibrium, for fixed primitives x, k and F, it is straightforward to numerically check the (quasi-)concavity of the payoff. To illustrate, we provide a couple of instances in the next section.[9]

[9]Note also that the payoff in (2) is known to be strictly concave in p_i when firms sell homogeneous products (see e.g. Peters, 1984). By continuity, we can pick densities f arbitrarily close to the degenerate density at e.g. $\varepsilon = 1$ that the payoff will remain strictly concave.

3.3 Comparative statics

In this section we examine how the equilibrium price depends on the buyer-seller ratio x and on the selling constraint k.

We start by looking at how the equilibrium price depends on the buyer-seller ratio. Recall that the limiting case $x \to 0$ represents a case in which there are infinitely many firms per buyer so that the market is extremely competitive. Meanwhile, the limiting case $x \to \infty$ refers to a situation in which there are infinitely many buyers per firm so that each firm enjoys a monopoly position. The dependence of the equilibrium price on the buyer-seller ratio is also sensitive to the extent to which firms face selling constraints. This is easy to understand after recognizing that a firm that has many buyers in line may not be able to take full advantage of this richness when its ability to attend to buyers is limited.

Consider first the case in which the firms' selling constraint is extremely severe so that $k = 1$. Equation (7) simplifies to:

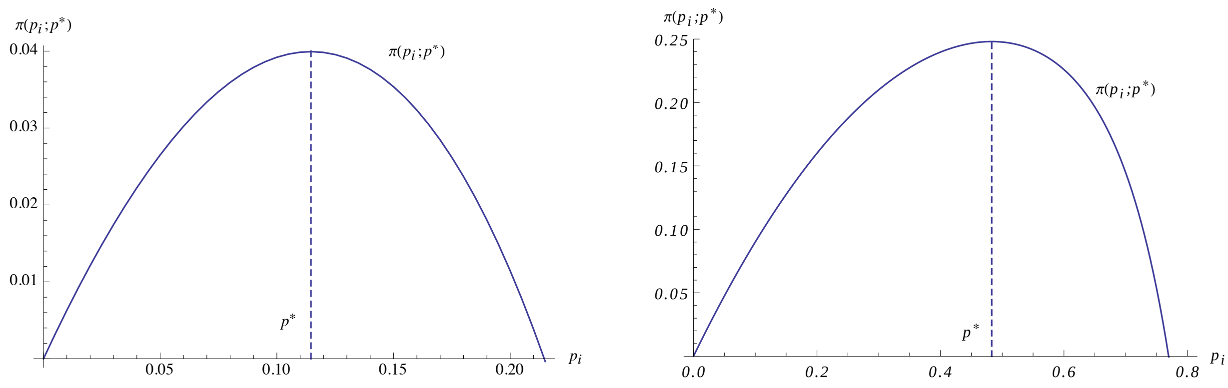
$$p = \frac{1 - F(p)}{f(p) + \frac{(1-F(p))^2}{I(p)} \frac{xe^{-x}}{1-e^{-x}-xe^{-x}}}. \tag{8}$$

Note that the expression $\frac{xe^{-x}}{1-e^{-x}-xe^{-x}}$ is decreasing in x, with

$$\lim_{x \to 0} \frac{xe^{-x}}{1 - e^{-x} - xe^{-x}} = \infty \text{ and } \lim_{x \to \infty} \frac{xe^{-x}}{1 - e^{-x} - xe^{-x}} = 0.$$

This implies that the equilibrium price that solves equation (8) converges to the "standard" monopoly price $p = \frac{1-F(p)}{f(p)}$ as the buyer-seller ratio x goes to infinity and approaches marginal cost when the buyer-seller ratio x goes to zero. In the former case, firms do not really compete with one another and the "standard" monopoly price is the price that maximizes the payoff $\pi(p) = p(1 - F(p))$. In the latter case, firms operate in an extremely competitive environment and we get marginal cost pricing.

For a given buyer-seller ratio x and a distribution of match values F, equation (8) can be solved numerically for the equilibrium price. In Figure 1 we plot the payoff of a firm when selling capacity is $k = 1$ assuming match values are uniformly distributed on $[0, 1]$. On the left panel, Figure 1(a) represents a case in which the market is relatively tight, with one buyer per two sellers. The equilibrium price is relatively low, approximately $p = 0.1146$ and correspondingly firms obtain very low profits, $\Pi = 0.039$. On the right panel, Figure 1(b) represents a case in which the market is relatively loose, with 5 buyers per seller. In this case the price is much higher, approximately $p = 0.4830$, now much closer to the monopoly price of 0.5, and profits reach $\Pi = 0.2480$. The graph also shows the (quasi-)concavity of the payoff, as per Proposition 3.



(a) Tight market: $x = 0.5$.

(b) Loose market: $x = 5$.

Figure 1: (Quasi-)concavity of the payoff when $k = 1$ and price equilibrium.

Consider now the case in which firms' selling capacity is not restricted whatsoever. When $k \to \infty$, equation (7) simplifies to:

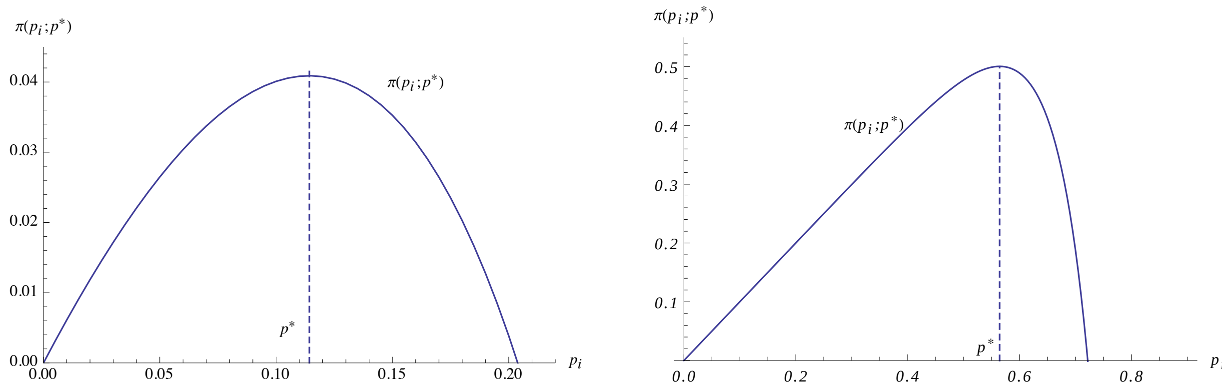
$$p = \frac{1 - e^{-x(1-F(p))} - x(1-F(p))e^{-x(1-F(p))}}{1 - e^{-x(1-F(p))}} \frac{\int_p^1 \varepsilon f(\varepsilon) d\varepsilon}{1 - F(p)}.$$

Note that the expression $\frac{1-e^{-z}-ze^{-z}}{1-e^{-z}}$ is increasing in z, with

$$\lim_{z \to 0} \frac{1 - e^{-z} - ze^{-z}}{1 - e^{-z}} = 0 \text{ and } \lim_{z \to \infty} \frac{1 - e^{-z} - ze^{-z}}{1 - e^{-z}} = 1.$$

This implies that, exactly like in the case in which $k = 1$, the price that solves this equation converges to the marginal cost as the buyer-seller ratio goes to zero. When the buyer-seller ratio goes to infinity, things are quite different though. The equilibrium price converges to a monopoly price that is higher than before, namely, $p = 1$. In fact, using the limit result above, when $x \to \infty$ the equilibrium price solves the equation $p(1 - F(p)) - \int_p^1 \varepsilon f(\varepsilon) d\varepsilon = 0$ which, using the integration by parts formula, can be rewritten as $1 - p - \int_p^1 F(\varepsilon) d\varepsilon = 0$. Because the LHS of this expression is decreasing in p the solution is $p = 1$. In the absence of any selling constraint, an individual firm can charge a price as high as the highest match value; this is because facing a queue of infinitely many buyers and being able to offer its product to as many buyers as it likes, the seller can "pick" a buyer with a match value equal to 1. The notion of market power, here reflected in the monopoly price, is thus clearly linked to the selling capacity of a firm.

In Figure 2 we plot the payoff (2) for the case in which firms do not have selling constraints, again assuming that match values are uniformly distributed on $[0, 1]$. The left graph, Figure 2(a), represents the case in which the market is relatively tight, with $x = 0.5$. The equilibrium price is approximately $p = 0.1142$, which is slightly lower than in the case where $k = 1$, but firms obtain higher profits, $\Pi = 0.0408$. The right graph, Figure 2(b), represents the case in which $x = 5$. In this case, the price is approximately $p = 0.5645$, clearly higher than before and higher than the "standard" monopoly price of 1/2. Profits reach $\pi = 0.5005$, significantly higher than when firms do not have selling constraints.



(a) Tight market: $x = 0.5$.

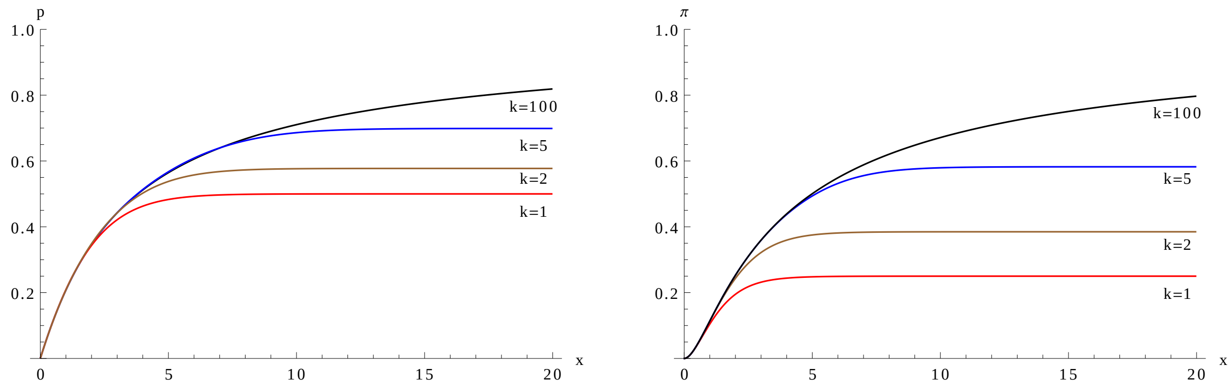
(b) Loose market: $x = 5$.

Figure 2: (Quasi-)concavity of the payoff when $k \to \infty$.

The previous results are summarized in the next proposition.

Proposition 4 *(a) Suppose firms' selling constraints are maximal, that is, $k = 1$. Then, the equilibrium price is equal to the marginal cost if $x = 0$ and approaches the standard monopoly price $p = \frac{1-F(p)}{f(p)}$ if $x \to \infty$.*
(b) Suppose firms do not have selling constraints, that is, $k \to \infty$. Then, the equilibrium price is equal to the marginal cost if $x = 0$ and approaches the monopoly price $p = 1$ when $x \to \infty$.

For intermediate levels of k, it is quite difficult to derive analytical results on the relationship between the equilibrium price, firms' profits and the buyer-seller ratio. We thus proceed to compute the equilibrium numerically. Figure 3 represents the equilibrium price and firms' profits as a function of x for various levels of the selling constraint. We observe that the no matter how severe the selling constraint is, both the equilibrium price and the profits of the firms increase in the buyer-seller ratio, thus reflecting how sellers take advantage of the buyers as market competitiveness loosens.



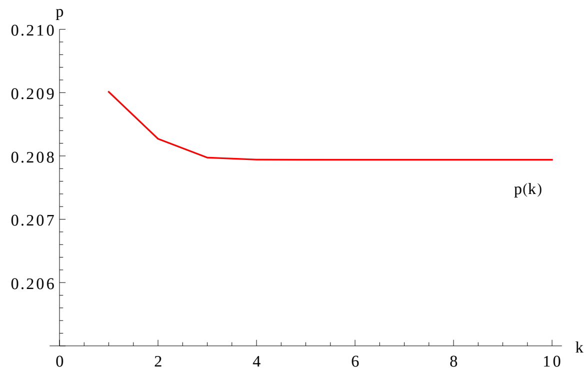
(a) Price

(b) Profits

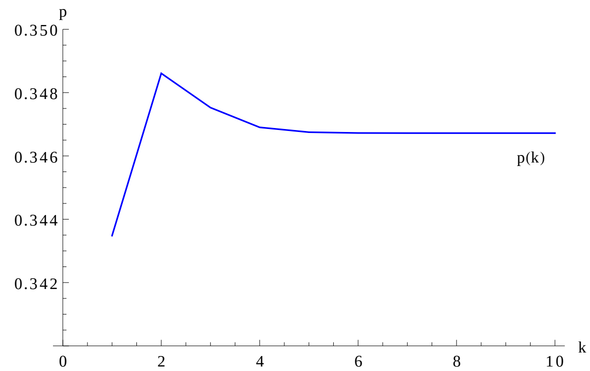
Figure 3: Equilibrium price, profits and the buyer-seller ratio.

The graph also shows that the equilibrium price and profits typically increase in k. To be sure, this is clearly visible for large x. When x is small the different curves are too cluttered to be able to distinguish between price levels for different k's. We now analyze this relationship in more detail. In general, an increase in k affects the equilibrium price in two ways that operate in opposite directions. On the one hand, a softer selling constraint makes attracting buyers to its premises more valuable to a firm because it can offer its product to more consumers. By this effect, an increase in k tends to reduce the equilibrium price. On the other hand, a laxer selling constraint increases the maximum of the willingness to pay of the k consumers a firm can offer its product to. By this effect, an increase in k tends to increase the equilibrium price. We have already seen analytically that when $x \to 0$, the equilibrium price is equal to the marginal cost no matter whether $k = 1$ or $k \to \infty$. This is true for any k as a matter of fact. The reason is that when $x \to 0$, the second effect plays no role. When $x \to \infty$, we have the opposite case in which the first effect plays no role. In such a case, the equilibrium price increases in k. In fact, we have already seen that the equilibrium price when $k \to \infty$ is higher than when $k = 1$. When the buyer-seller ratio x takes on an intermediate value, the equilibrium price may increase or decrease as the selling constraint becomes laxer.

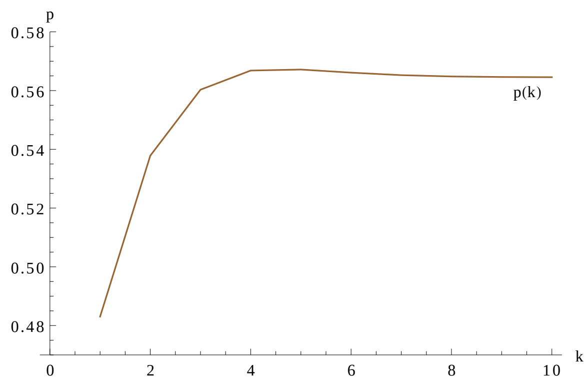
We illustrate these results in Figure 4, where we represent the equilibrium price as a function of the selling constraint for various levels of the buyer-seller ratio. The graphs illustrate that the equilibrium price is decreasing in relatively tight markets where there are very few buyers per seller (Figure 4(a)), non-monotonic in markets where the buyer-seller ratio is intermediate (Figures 4(a) and 4(b)) and increasing in relatively loose markets where there are many buyers per seller (Figure 4(d)).



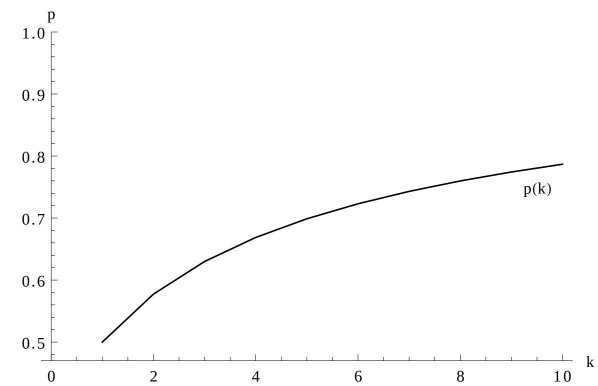
(a) Very tight market: $x = 1$.



(b) Tight market: $x = 2$.



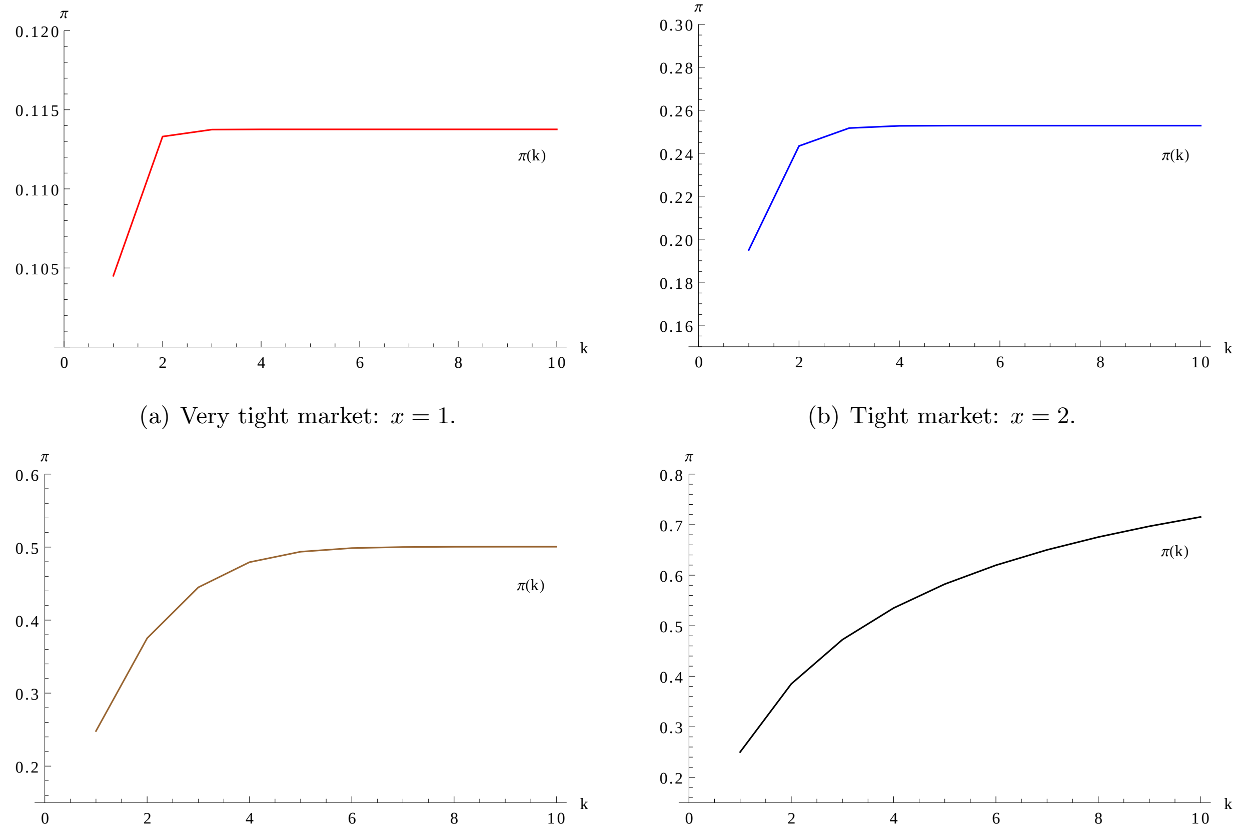
(c) Loose market: $x = 5$.



(d) Very loose market: $x = 100$.

Figure 4: Equilibrium price and selling constraints.

Despite the fact that the equilibrium price may decrease in k when there are few buyers per seller, equilibrium profits are increasing in k. This can be seen in Figure 5 where we have plotted the equilibrium profits corresponding to the equilibrium prices depicted in Figure 4.



(a) Very tight market: $x = 1$.

(b) Tight market: $x = 2$.

(c) Loose market: $x = 5$.

(d) Very loose market: $x = 100$.

Figure 5: Equilibrium profits and selling constraints.

We conclude this section by summarizing our numerical findings.

Numerical result 1 *(a) The equilibrium price and firms' profits are monotonically increasing in x for any k. (b) For low x, the equilibrium price is decreasing in k; for intermediate x, the equilibrium price is first increasing in k and then decreasing; for large x, the equilibrium price is increasing in k. Firms' profits increase in k.*

4 Welfare

In this section we study the efficiency of the market equilibrium price. To do this, we first characterize the efficient price and then we compare it to the market equilibrium.

Social welfare, as usual, equals the sum of buyers' utility and sellers' profits:

$$\begin{aligned} W &= BV + S\Pi \\ &= B\eta(x,p;k)(1-F(p))\left[E(\varepsilon \mid \varepsilon \geq p) - p\right] + Spx\eta(x,p;k)(1-F(p)) \end{aligned}$$

Using the fact that $B = xS$, we can simplify the welfare expression:

$$W = B \underbrace{\eta(x, p; k)(1 - F(p))}_{\text{prob. of trade}} \underbrace{E(\varepsilon \mid \varepsilon \geq p)}_{\text{value of trade}}.$$

Inspection of this expression reveals that an increase in the price has both a positive and a negative effect on social welfare. The positive effect is to increase the value of a transaction, that is, the value of $E(\varepsilon \mid \varepsilon \geq p)$. A higher price serves as a selection mechanism: only a consumer with a sufficiently high match value will accept the trade, which generates a higher social surplus. The negative effect is to decrease the quantity of trade, that is, the probability $\eta(x, p; k)(1 - F(p))$ with which buyers buy. This is because a higher price makes it less likely that anyone queuing at a seller happens to have a sufficiently high match utility for the product of the seller. Formally, this observation follows from the fact that:

$$\begin{aligned}
\frac{x}{f(p)} \frac{\partial \eta(1 - F(p))}{\partial p} &= kF(p)^{k-1} \frac{\Gamma(k+1, x) - \Gamma(k+1)}{\Gamma(k+1)} \\
&\quad + \frac{xe^{-x(1-F(p))}}{\Gamma(k+1)} \left(x^k F(p)^k e^{-xF(p)} - \Gamma(k+1, xF(p)) \right) \\
&= kF(p)^{k-1} \frac{\Gamma(k+1, x) - \Gamma(k+1)}{\Gamma(k+1)} - \frac{xe^{-x(1-F(p))}}{\Gamma(k+1)} \int_{xF(p)}^{\infty} kt^{k-1} e^{-t} dt < 0,
\end{aligned}$$

where the second equality follows from integrating by parts the Γ function. Hence, in choosing a price the planner faces a trade-off between the quantity and the quality of trade.

Taking the FOC for welfare maximization we can state that:

Proposition 5 *The socially optimal price, denoted p^o, satisfies the FOC:*

$$\frac{\partial \eta}{\partial p} I(p^o) + \frac{\partial \eta}{\partial p} p^o (1 - F(p^o)) - p^o \eta f(p^o) = 0. \tag{9}$$

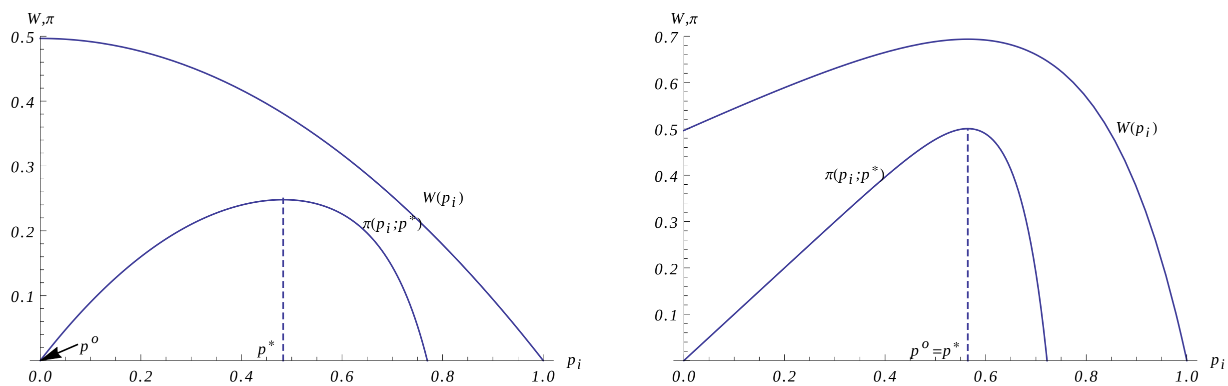
When $k = 1$, η does not depend on price so the socially optimal price is equal to the marginal cost. For $k \geq 2$ the socially optimal price is strictly greater than the marginal cost.

This result is at odds with the standard view in economics that marginal cost pricing is efficient. Except in the case in which $k = 1$, in our model the welfare function is typically non-monotonic in price, which implies that *efficient pricing involves positive markups.* What distinguishes our model from the standard model is that firms are capacity constrained and sell differentiated products. Note that it is these two features together that create a trade-off for the planner: a higher price lowers the chance a transaction occurs, but increases its value if it occurs. Welfare maximization consists of balancing these two effects, which drives a wedge between the optimal price and the marginal cost. Hence, the shortcut notion of market power as the ability of firms to sustain prices about marginal cost is not valid in our setting. Instead, market power has to be assessed as the capacity of firms to sustain prices above the efficient level, which differs from the marginal cost.

Note that equation (9) has surely a solution. This is because equation (9) is exactly the same as the LHS of equation (7) and in the proof of Proposition 3 we show that this expression is strictly positive at $p = 0$ and negative at $p = 1$. Comparing the socially optimal price with the market equilibrium price leads to the following insight:

Proposition 6 *Equilibrium prices are inefficiently high except in the limit when $k \to \infty$, in which case they are efficient (and greater than the marginal cost).*

Figure 6 illustrates this result by plotting together the payoff function and social welfare. The left graph, Figure 6(a), shows the case in which firms face extreme selling constraints so that $k = 1$. In this case, the planner just wishes to maximize the probability of trade and sets a price equal to the marginal cost. Equilibrium price is clearly excessive. The right graph, Figure 6(b), represents the case in which firms do not face any selling constraint, $k \to \infty$. In this case, welfare is non-monotonic and is maximized at the same price as the equilibrium price. In both these graphs, we have set $x = 5$ so the equilibrium prices correspond to those in Figures 1(b) and 2(b).



(a) Tight selling constraint, $k = 1$.

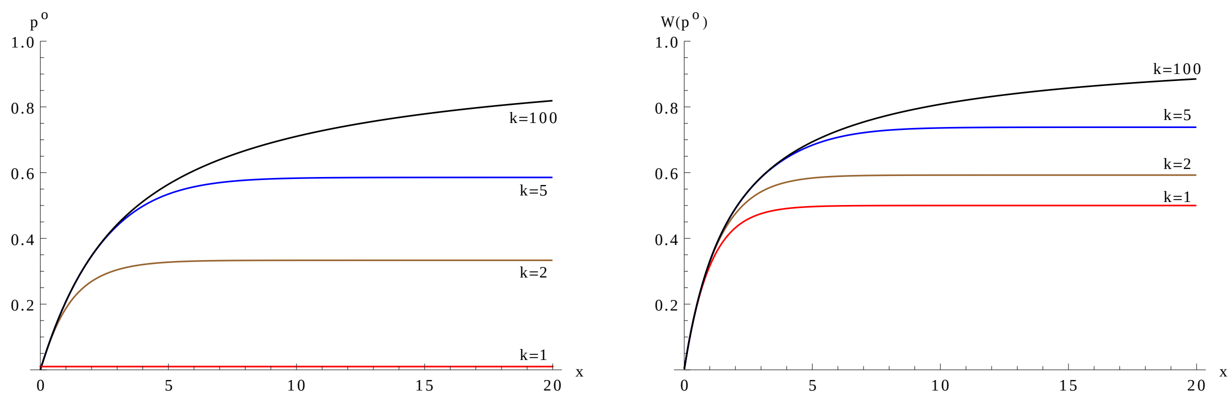
(b) No selling constraints, $k \to \infty$.

Figure 6: Social optimum and equilibrium.

The equilibrium price is efficient only in the limiting case in which $k \to \infty$ and thus selling constraints are completely absent. In such a case, at the efficient price, the social trade-off between the quantity and the quality (or price) of trade is exactly identical to its private counterpart. Comparing the equilibrium condition (7) and the efficiency condition (9), we observe that they become exactly identical if and only if the factor $d(p) = 0$, which implies that firms should be able to continue to show their products to all the buyers who show up at their premises. In fact, only when $k \to \infty$ is the probability of having more buyers than their selling capacity equal to zero $(1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)} \to 0$ as $k \to \infty)$ and therefore the factor $d(p) = 0$.

When firms face non-trivial selling constraints ($k < \infty$), the probability that a firm has more buyers queuing at its doors than it can attend to is strictly positive. That is, $1 - \frac{\Gamma(k+1,x)}{\Gamma(k+1)} > 0$ and thus $d(p) < 0$. This means that an individual firm may find itself in a situation where it fails to sell its product while there still are buyers interested in the product who cannot however be approached. This reduces the firms' incentive to attract buyers by lowering price. Hence, the private benefit of increasing price becomes higher than its social counterpart, leading to an inefficiently high equilibrium price.

The socially optimal price that solves (9), and hence the level of welfare attained in the economy, depends on the buyer-seller ratio and the selling constraint. In Figure 7 we plot the efficient price and the corresponding welfare level per seller as a function of the buyer-seller ratio for various levels of the selling constraint. It can be seen tat both the efficient price and the maximum welfare level attained are increasing in x and k.



(a) Efficient price

(b) Welfare

Figure 7: Efficient price and welfare.

We conclude this section by summarizing our numerical findings.

Numerical result 2 *The socially optimal price and the level of welfare are monotonically increasing in x and k.*

5 Concluding remarks

Despite the fact that selling constrains are ubiquitous and often firms cannot attend to all the buyers who are interested in inspecting their products, as far as we know, the literature has not paid attention to their impact on the functioning of search markets. This paper has started to close this gap. Our main conclusion has been that selling constraints are a source of market power.

In reaching this conclusion we have used a model with some specific features. One of the assumptions of the model has been that firms face two types of constraints. First, firms are capacity-constrained and have at their disposal just one unit of a differentiated product. Second, firms face selling constraints and can only attend to a maximum of k buyers. The insights of our paper should carry over to a situation where firms' capacity constraints are not so stringent. In fact, suppose that sellers have at their disposal ℓ units of the differentiated product but can only attend to k buyers. It is clear that sellers will only be able to sell a maximum number of units lower than or equal to $\min\{\ell, k\}$. As a result, situations in which $\ell < k$ will be similar to the one studied in this paper because sellers continue to face a trade-off between the quantity of trade and the quality of trade. If $\ell > k$, by contrast, sellers would not face such a trade-off and the equilibrium pricing would be similar to the $k = 1$ case.

Another simplifying assumption of the model has been that sellers and buyers interact for just a single period. This implies that in our search model search costs do not play any role and the only source of search frictions is the potential rationing that buyers may suffer due to the firms' capacity and selling constraints. A more complete depiction of search frictions in markets for differentiated products ought to include both demand- and supply-side frictions. Assuming that in every period the previously matched buyers and sellers are replaced by new agents in the economy, it is not very hard to extend our model to allow for consumers' sequential search as it is standard in the consumer search literature. In that case, the search cost becomes the key factor that influences the trade-off between the quantity and the quality of trade. A higher search cost makes consumers less picky, which increases the probability of trade but reduces the value of trade. Welfare is thus non-monotonic in search costs and the social welfare maximizing search cost is typically bounded away from zero. This is akin to our positive efficient markups result.

Finally, we have assumed that firms' selling constraints are exogenous. However, as mentioned in the Introduction, firms not only choose their prices but try to optimize its marketing and sales team to maximize their profits. Though extending our work to allow for the possibility that firms choose k is quite challenging, it would be very interesting to know how a factor such as market tightness influences the firm size. We leave the full development of these extensions as topic for further research.

Appendix

Proof of Proposition 1.

Note that the number of buyers visiting a seller n follows a Poisson distribution, Prob.$(n = i) = \frac{x^i e^{-x}}{i!}$. We consider the offer probability to a buyer who visits a seller. Let an index i count the number of the *other* buyers arriving at a seller. Then, we have

$$\begin{aligned}\eta &= \sum_{i=0}^{k-1} \frac{x^i e^{-x}}{i!} \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} F(p)^j (1-F(p))^{i-j} \frac{1}{i+1-j} \\ &\quad + \sum_{i=k}^{\infty} \frac{x^i e^{-x}}{i!} \frac{k}{i+1} \sum_{j=0}^{k-1} \frac{k-1!}{j!(k-1-j)!} F(p)^j (1-F(p))^{k-1-j} \frac{1}{k-j}\end{aligned}$$

The first summation represents cases in which the number of the other buyers is less than the number of buyers that the seller can handle, i.e., $i \leq k-1$. With $j \leq i$ of the other buyers turning out *not* to like the seller's product, which comes in $\frac{i!}{j!(i-j)!}$ ways and occurs with probability $F(p)^j(1-F(p))^{i-j}$, the given buyer will be offered with probability $\frac{1}{i-j+1}$. The second summation represents cases in which $i \geq k$. Note that the seller has to randomly select $k \leq i$ buyers, and the given buyer is selected with probability $\frac{k}{i+1}$. With $j \leq k-1$ of the other *selected* buyers turning out not to like the seller's product, which comes in $\frac{k-1!}{j!(k-1-j)!}$ ways and occurs with probability $F(p)^j(1-F(p))^{k-1-j}$, the given buyer will be offered with probability $\frac{1}{k-j}$.

Note that we can simplify the terms in the first summation,

$$\begin{aligned}& \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} F(p)^j (1-F(p))^{i-j} \frac{1}{i+1-j} \\ =& \frac{1}{(i+1)(1-F(p))} \sum_{j=0}^{i} \frac{(i+1)!}{j!(i+1-j)!} F(p)^j (1-F(p))^{i+1-j} \\ =& \frac{1}{(i+1)(1-F(p))} \left[\sum_{j=0}^{i+1} \frac{(i+1)!}{j!(i+1-j)!} F(p)^j (1-F(p))^{i+1-j} - F(p)^{i+1} \right] \\ =& \frac{1-F(p)^{i+1}}{(i+1)(1-F(p))},\end{aligned}$$

and the terms in the second summation,

$$
\begin{aligned}
& \frac{k}{i+1}\sum_{j=0}^{k-1}\frac{k-1!}{j!(k-1-j)!}F(p)^j(1-F(p))^{k-1-j}\frac{1}{k-j} \\
= \quad & \frac{1}{(i+1)(1-F(p))}\sum_{j=0}^{k-1}\frac{k!}{j!(k-j)!}F(p)^j(1-F(p))^{k-j} \\
= \quad & \frac{1}{(i+1)(1-F(p))}\left[\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}F(p)^j(1-F(p))^{k-j}-F(p)^k\right] \\
= \quad & \frac{1-F(p)^k}{(i+1)(1-F(p))}.
\end{aligned}
$$

Using these simplifications, we have

$$
\begin{aligned}
\eta \quad = \quad & \sum_{i=0}^{k-1}\frac{x^ie^{-x}}{i!}\frac{1-F(p)^{i+1}}{(i+1)(1-F(p))}+\sum_{i=k}^{\infty}\frac{x^ie^{-x}}{i!}\frac{1-F(p)^k}{(i+1)(1-F(p))} \\
= \quad & \frac{1}{x(1-F(p))}\sum_{i=0}^{k-1}\frac{x^{i+1}e^{-x}(1-F(p)^{i+1})}{(i+1)!}+\frac{1-F(p)^k}{x(1-F(p))}\sum_{i=k}^{\infty}\frac{x^{i+1}e^{-x}}{(i+1)!}.
\end{aligned}
$$

Setting $h \equiv i+1$, it is further simplified to

$$
\begin{aligned}
\eta \quad = \quad & \frac{1}{x(1-F(p))}\sum_{h=0}^{k}\left[\frac{x^he^{-x}}{h!}-\frac{[xF(p)]^he^{-x}}{h!}\right]+\frac{1-F(p)^k}{x(1-F(p))}\sum_{h=k+1}^{\infty}\frac{x^he^{-x}}{h!} \\
= \quad & \frac{1}{x(1-F(p))}\left[\sum_{h=0}^{k}\frac{x^he^{-x}}{h!}-e^{-x(1-F(p))}\sum_{h=0}^{k}\frac{[xF(p)]^he^{-xF(p)}}{h!}\right]+\frac{1-F(p)^k}{x(1-F(p))}\left[1-\sum_{h=0}^{k}\frac{x^he^{-x}}{h!}\right] \\
= \quad & \frac{1}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)}{\Gamma(k+1)}-e^{-x(1-F(p))}\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}\right]+\frac{1-F(p)^k}{x(1-F(p))}\left[1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right],
\end{aligned}
$$

where we used $\sum_{h=0}^{k}\frac{x^he^{-x}}{h!}=\frac{\Gamma(k+1,x)}{\Gamma(k+1)}$ (i.e., the series definition of the cumulative gamma function), with $\Gamma(k+1)=k!=\int_0^\infty t^ke^{-t}dt$ and $\Gamma(k+1,x)=\int_x^\infty t^ke^{-t}dt$. Rearranging terms, we obtain the expression in (1). ■

Proof of Proposition 2.

(a) In order to show that η is increasing in k, it suffices to show that m is increasing in k. For this, we compute the difference:

$$\begin{aligned}
m(k+1)-m(k) &= 1-F(p)^{k+1}+\frac{\Gamma(k+2,x)}{\Gamma(k+2)}F(p)^{k+1}-\frac{\Gamma(k+2,xF(p))}{\Gamma(k+2)}e^{-x(1-F(p))} \\
&\quad -\left(1-F(p)^{k}+\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^{k}-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))}\right) \\
&= F(p)^k-F(p)^{k+1}+\frac{(k+1)\Gamma(k+1,x)+x^{k+1}e^{-x}}{\Gamma(k+2)}F(p)^{k+1} \\
&\quad -\frac{(k+1)\Gamma(k+1,xF(p))+(xF(p))^{k+1}e^{-xF(p)}}{\Gamma(k+2)}e^{-x(1-F(p))}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^k \\
&\quad +\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))},
\end{aligned}$$

where we have used the property of the Gamma function (see Jameson, 2016):

$$\Gamma(k+2,x)=(k+1)\Gamma(k+1,x)+x^{k+1}e^{-x}. \tag{10}$$

Because $\Gamma(k+1)=k\Gamma(k)$, we can rewrite the previous expression as follows:

$$\begin{aligned}
m(k+1)-m(k) &= F(p)^k-F(p)^{k+1}+\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^{k+1}+\frac{x^{k+1}e^{-x}}{\Gamma(k+2)}F(p)^{k+1}-\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))} \\
&\quad -\frac{(xF(p))^{k+1}e^{-xF(p)}}{\Gamma(k+2)}e^{-x(1-F(p))}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^k+\frac{\Gamma(k+1,xF(p))}{\Gamma(k+1)}e^{-x(1-F(p))} \\
&= F(p)^k-F(p)^{k+1}+\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^{k+1}-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}F(p)^k \\
&= F(p)^k(1-F(p))\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right).
\end{aligned}$$

The last expression is positive because $\Gamma(k+1,x)=\int_x^\infty t^k e^{-t}dt<\int_0^\infty t^k e^{-t}dt=\Gamma(k+1)$, which completes the proof that m is increasing in k.

To demonstrate that η is concave in k, we compute the difference:

$$[m(k+2)-m(k+1)]-[m(k+1)-m(k)]=(1-F(p))F(p)^k\left[F(p)\left(1-\frac{\Gamma(k+2,x)}{\Gamma(k+2)}\right)-\left(1-\frac{\Gamma(k+1,x)}{\Gamma(k+1)}\right)\right].$$

For concavity, this expression must be negative. Becasue $F(p)<1$, it is sufficient that

$$\frac{\Gamma(k+2,x)}{\Gamma(k+2)}>\frac{\Gamma(k+1,x)}{\Gamma(k+1)},$$

or that

$$\frac{\Gamma(k+2,x)}{k+1}>\Gamma(k+1,x),$$

which is true because of the property (10).

We now use differentiation to show that η is decreasing in x. For this result, notice that it suffices to show that m is decreasing in x. Taking the derivative of m with respect to x, and putting common factors together, gives:

$$\frac{\partial m}{\partial x} = -\frac{e^{-x}(1-F)\left((xF)^k - e^{xF}\Gamma(k+1,xF)\right)}{\Gamma(k+1)}$$

The sign of this expression depends on the sign of $(xF)^k - e^{xF}\Gamma(k+1,xF)$. Using (10), we have:

$$(xF)^k - e^{xF}\Gamma(k+1,xF) = (xF)^k - e^{xF}\left(k\Gamma(k,xF) + (xF)^k e^{-xF}\right) = -e^{xF}k\Gamma(k,xF) < 0.$$

Therefore, m decreases in x and so does η.

We finally show that η is increasing in $F(p)$. For this, we first note that

$$x\frac{\partial \eta}{\partial F(p)} = \frac{\frac{\partial m}{\partial F(p)}(1-F) + m}{(1-F)^2}$$

The sign of $\partial\eta/\partial F(p)$ depends on the sign of the numerator. We note that

$$\frac{\partial m}{\partial F(p)} = \frac{kx\left(-e^{-x(1-F)}\right)\Gamma(k,xF) - kF^{k-1}(\Gamma(k+1) - \Gamma(k+1,x))}{\Gamma(k+1)}.$$

Using this, we calculate:

$$\begin{aligned}\Gamma(k+1)\left(\frac{\partial m}{\partial F(p)}(1-F) + m\right) &= \Gamma(k+1) - (F + k(1-F))F^{k-1}(\Gamma(k+1) - \Gamma(k+1,x)) \\ &\quad - e^{-x(1-F)}((1-F)kx\Gamma(k,xF) + \Gamma(k+1,xF)) \qquad (11)\end{aligned}$$

The RHS of this expression is decreasing in F because its derivative with respect to F can be written as

$$-k(1-F)\left[(k-1)F^{k-2}\left(\Gamma(k+1) - \Gamma(k+1,x)\right) + x^2 e^{-x(1-F)}\left(\Gamma(k,xF) - (xF)^{k-1}e^{-xF}\right)\right],$$

which is negative because the term $\Gamma(k,xF) - (xF)^{k-1}e^{-xF} = (k-1)\Gamma(k-1,xF) > 0$ for $k \geq 2$.

It is straightforward to see that when we set $F = 1$ in the RHS of equation (11) we obtain 0. This means that (11) is positive for all F, which completes the proof that η is increasing in $F(p)$ for $k \geq 2$. ■

Derivation of the profit function in (2)

The expected profit of seller i is given by:

$$\Pi(p_i;p) = p_i \left(\sum_{l=1}^{k} \Pr[n_i = \ell] \left(1 - F(p_i)^{\ell}\right) + \sum_{\ell=k+1}^{\infty} \Pr[n_i = \ell] \left(1 - F(p_i)^{k}\right) \right). \tag{12}$$

Because the expected number of buyers visiting a seller n follows a Poisson distribution, $\Pr(n_i = \ell) = \frac{x_i^{\ell} e^{-x_i}}{\ell!}$.

To obtain the expression in (2), observe that the first term in the bracket of (12) can be simplified as follows:

$$\begin{aligned}
\sum_{\ell=1}^{k} \Pr(n_i = \ell) \left(1 - F(p_i)^{\ell}\right) &= \sum_{\ell=0}^{k} \frac{x_i^{\ell} e^{-x_i}}{\ell!} (1 - F(p_i)^{\ell}) \\
&= \sum_{\ell=0}^{k} \left[\frac{x_i^{\ell} e^{-x_i}}{\ell!} - \frac{[x_i F(p_i)]^{\ell} e^{-x_i}}{\ell!} \right] \\
&= \sum_{\ell=0}^{k} \frac{x_i^{\ell} e^{-x_i}}{\ell!} - e^{-x_i(1-F(p_i))} \sum_{\ell=0}^{k} \frac{[x_i F(p_i)]^{\ell} e^{-x_i F(p_i)}}{\ell!} \\
&= \frac{\Gamma(k+1, x_i)}{\Gamma(k+1)} - e^{-x_i(1-F(p_i))} \frac{\Gamma(k+1, x_i F(p_i))}{\Gamma(k+1)}
\end{aligned}$$

where we have used the series definition of the cumulative gamma function: $\sum_{h=0}^{k} \frac{x^h e^{-x}}{h!} = \frac{\Gamma(k+1,x)}{\Gamma(k+1)}$.

Likewise, the second term in the bracket of (12) can be simplified as follows:

$$\begin{aligned}
\sum_{l=k+1}^{\infty} \Pr(n_i = \ell) \left(1 - F(p_i)^{k}\right) &= \left(1 - F(p_i)^{k}\right) \sum_{\ell=k+1}^{\infty} \frac{x_i^{l} e^{-x_i}}{\ell!} \\
&= \left(1 - F(p_i)^{k}\right) \left[1 - \sum_{\ell=0}^{k} \frac{x_i^{\ell} e^{-x_i}}{\ell!} \right] \\
&= \left(1 - F(p_i)^{k}\right) \left[1 - \frac{\Gamma(k+1, x_i)}{\Gamma(k+1)} \right].
\end{aligned}$$

Hence, the payoff expression in (12) can be written as

$$\Pi(p_i;p) = p_i \left(\tfrac{\Gamma(k+1,x_i)}{\Gamma(k+1)} - e^{-x_i(1-F(p_i))} \tfrac{\Gamma(k+1,x_i F(p_i))}{\Gamma(k+1)} + \left(1 - F(p_i)^{k}\right) \left[1 - \tfrac{\Gamma(k+1,x_i)}{\Gamma(k+1)} \right] \right)$$

Using the expression for η in equation (1), it is now straightforward to obtain the payoff in expression (2).

Proof of relationship in equation (6).

Computing the derivatives involved in this relationship gives:

$$\begin{aligned}\frac{\partial \eta}{\partial x} &= -\frac{m(x,p;k)}{x^2(1-F(p))} \\ &\quad + \frac{1}{x(1-F(p))}\left[\frac{F(p)^k}{\Gamma(k+1)}\frac{\partial \Gamma(k+1,x)}{\partial x} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(\frac{\partial \Gamma(k+1,xF(p)}{\partial x} - (1-F(p))\Gamma(k+1,xF(p)\right)\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \eta}{\partial p} &= \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{1}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}f(p)\right] \\ &\quad + \frac{1}{x(1-F(p))}\left[-\frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(\frac{\partial \Gamma(k+1,xF(p)}{\partial p} + xf(p)\Gamma(k+1,xF(p)\right)\right].\end{aligned}$$

Notice that:

$$\begin{aligned}\frac{\partial \Gamma(k+1,x)}{\partial x} &= -x^k e^{-x} \\ \frac{\partial \Gamma(k+1,xF(p)}{\partial x} &= -x^k F(p)^{k+1} e^{-xF(p)} \\ \frac{\partial \Gamma(k+1,xF(p)}{\partial p} &= -x^{k+1}F(p)^k f(p) e^{-xF(p)}.\end{aligned}$$

Using these, we can rewrite $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial p}$ as follows:

$$\begin{aligned}\frac{\partial \eta}{\partial x} &= -\frac{m(x,p;k)}{x^2(1-F(p))} \\ &\quad - \frac{1}{x(1-F(p))}\left[\frac{F(p)^k}{\Gamma(k+1)}x^k e^{-x} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^k F(p)^{k+1} e^{-xF(p)} + (1-F(p))\Gamma(k+1,xF(p)\right)\right],\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \eta}{\partial p} &= \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{f(p)}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)}kF(p)^{k-1}\right] \\ &\quad + \frac{f(p)}{x(1-F(p))}\left[\frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^{k+1}F(p)^k e^{-xF(p)} - x\Gamma(k+1,xF(p)\right)\right].\end{aligned}$$

It is convenient to multiply and divide the squared bracket of $\frac{\partial \eta}{\partial x}$ by x, and that of $\frac{\partial \eta}{\partial p}$ by $1-F(p)$.

This gives:

$$
\begin{aligned}
\frac{\partial \eta}{\partial x} &= -\frac{m(x,p;k)}{x^2(1-F(p))} \\
&\quad - \frac{1}{x^2(1-F(p))}\left[\frac{F(p)^k x^{k+1} e^{-x}}{\Gamma(k+1)} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^{k+1}F(p)^{k+1}e^{-xF(p)} + x(1-F(p))\Gamma(k+1, xF(p))\right)\right],
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\frac{\partial \eta}{\partial p} &= \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{f(p)}{x(1-F(p))^2}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)} kF(p)^{k-1}(1-F(p)\right] \\
&\quad + \frac{f(p)}{x(1-F(p))^2}\frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left[x^{k+1}F(p)^k e^{-xF(p)}(1-F(p) - x(1-F(p)\Gamma(k+1,xF(p))\right] \\
&= \frac{f(p)m(x,p;k)}{x(1-F(p))^2} + \frac{f(p)}{x(1-F(p))^2}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)} kF(p)^{k-1}(1-F(p)\right] \\
&\quad + \frac{f(p)}{x(1-F(p))^2}\left[\frac{F(p)^k x^{k+1} e^{-x}}{\Gamma(k+1)} - \frac{e^{-x(1-F(p))}}{\Gamma(k+1)}\left(x^{k+1}F(p)^{k+1}e^{-xF(p)} + x(1-F(p)\Gamma(k+1, xF(p))\right)\right]
\end{aligned} \tag{14}
$$

where, to establish the second equality, we have rewritten the term $x^{k+1}F(p)^k e^{-xF(p)}(1-F(p)$ as a sum.

Finally, close inspection of (13) and (14) reveals that:

$$
\frac{1-F(p)}{f(p)}\frac{\partial \eta}{\partial p} = -x\frac{\partial \eta}{\partial x} + \frac{1}{x(1-F(p))}\left[\frac{\Gamma(k+1,x)-\Gamma(k+1)}{\Gamma(k+1)} kF(p)^{k-1}(1-F(p)\right],
$$

which is relationship (6). ■

Proof of Lemma 1.

Using (6), the FOC for profits maximization in expression (5) can be rewritten as follows

$$
\begin{aligned}
\left(-\frac{(1-F(p))}{f(p)}\frac{\partial \eta}{\partial p} + d(p)\right)&\left[(1-F(p)-pf(p))\,I(p) + p(1-F(p))^2\right] \\
&+p\eta(1-F(p))^2 - \frac{\partial \eta}{\partial p}p(1-F(p))I(p) = 0.
\end{aligned}
$$

Rearranging terms gives:

$$\left(\frac{(1-F(p))}{f(p)}\frac{\partial\eta}{\partial p}\right)\left[(1-F(p)-pf(p))\,I(p)+p(1-F(p))^2\right]-p\eta(1-F(p))^2+\frac{\partial\eta}{\partial p}p(1-F(p))I(p)$$
$$=d(p)\left[(1-F(p)-pf(p))\,I(p)+p(1-F(p))^2\right],$$

which can be simplified to:

$$\left(\frac{\partial\eta}{\partial p}I(p)\right)\frac{(1-F(p))^2}{f(p)}+\left(\frac{\partial\eta}{\partial p}\right)\frac{(1-F(p))}{f(p)}p(1-F(p))^2-p\eta(1-F(p))^2$$
$$=d(p)\left[(1-F(p)-pf(p))\,I(p)+p(1-F(p))^2\right].$$

This can be rewritten as in the Proposition:

$$\frac{(1-F(p))^2}{f(p)}\left[\frac{\partial\eta}{\partial p}I(p)+\frac{\partial\eta}{\partial p}p(1-F(p))-p\eta f(p)\right]$$
$$=d(p)\left[(1-F(p)-pf(p))\,I(p)+p(1-F(p))^2\right],$$

which completes the proof. ■

Proof of Proposition 3

We first show that (7) has at least one solution. For this, it is convenient to rewrite (7) as follows:

$$\frac{\partial\eta}{\partial p}I(p)+\frac{\partial\eta}{\partial p}p(1-F(p))-p\eta f(p)$$
$$=d(p)\frac{f(p)}{(1-F(p))^2}\left[(1-F(p)-pf(p))\,I(p)+p(1-F(p))^2\right]. \tag{15}$$

Consider the LHS of (15). Observe that it is strictly positive at $p=0$. This follows easily from using (6) and noting that $d(0)=0$ and

$$\frac{\partial\eta}{\partial x}=\frac{1}{x(1-F(p))}\left(\frac{\partial m}{\partial x}-\frac{m}{x}\right),$$

where

$$\frac{\partial m}{\partial x}=\frac{e^{-x}(1-F(p))e^{xF(p)}\Gamma(k,xF(p))}{\Gamma(k)}.$$

To evaluate the LHS of (15) at $p=1$, note that $F(1)=1$, $I(1)=0$ and $\eta(1)=1$. Observe also that, again using (6) and the fact that, by the L'Hopital rule,

$$\lim_{p\to 1} \frac{I(p)}{1-F(p)} = \lim_{p\to 1} \frac{-(1-F(p))}{-f(x)} = 0,$$

we have:

$$\lim_{p\to 1} \frac{\partial \eta}{\partial p} I(p) = \lim_{p\to 1} \frac{f(p)I(p)}{1-F(p)} \left(\frac{-\frac{\partial m}{\partial x}}{1-F(p)} + \eta + d(p) \right) = 0$$

$$\lim_{p\to 1} \frac{\partial \eta}{\partial p} p(1-F(p)) = f(1) \left(1 - \frac{\Gamma(k,x)}{\Gamma(k)} - d(1) \right)$$

Hence, altogether the LHS of (15) takes on value $-f(1)\left(\frac{\Gamma(k,x)}{\Gamma(k)} - d(1)\right) < 0$ at $p = 1$.

Consider now the RHS of (15). Note that it is equal to 0 at $p = 0$. Therefore, for the existence of a candidate equilibrium it suffices to show that at $p = 1$ the LHS of (15) is lower than the RHS of (15). Taking the limit of the RHS of (15) when $p \to 1$ gives

$$d(1)f(1) \left(-f(1) \lim_{p\to 1} \frac{I(p)}{(1-F(p))^2} + 1 \right) = \frac{1}{2} d(1)f(1) \tag{16}$$

because by the L'Hopital rule,

$$\lim_{p\to 1} \frac{I(p)}{(1-F(p))^2} = \frac{1}{2f(1)}.$$

Therefore, the existence of a candidate equilibrium is guaranteed if

$$-f(1) \left(\frac{\Gamma(k,x)}{\Gamma(k)} - d(1) \right) < \frac{1}{2} d(1)f(1),$$

or

$$-f(1) \left(\frac{\Gamma(k,x)}{\Gamma(k)} - \frac{d(1)}{2} \right) < 0,$$

which is always true because $d(1) < 0$.

We now show that the equilibrium exists when $k = 1$. In such a case, the payoff in (2) simplifies to:

$$\pi(p_i; p) = p_i(1 - F(p_i))(1 - e^{-x(p_i,p)})$$

Because $1 - F$ is log-concave, it suffices to show that $1 - e^{-x(p_i,p)}$ is concave in p_i. The second derivative of $1 - e^{-x(p_i,p)}$ with respect to p_i is:

$$-e^{-x(p_i,p)} \left(\frac{\partial x_i}{\partial p_i} \right)^2 + e^{-x(p_i,p)} \frac{\partial^2 x_i}{\partial p_i^2}.$$

Inspection of this derivative reveals that concavity of the function $x(p_i, p)$ in p_i suffices for the result.

Rewrite equation (3) as follows:

$$\left(\frac{\partial \eta_i}{\partial x_i}\frac{\partial x_i}{\partial p_i}+\frac{\partial \eta_i}{\partial p_i}\right) I(p_i)-\eta_i(1-F(p_i))=0. \tag{17}$$

and notice that in this case of $k=1$ the function η_i does not directly depend on p_i, that is, $\partial \eta_i/\partial p_i = 0$.

Taking the derivative with respect to p_i gives:

$$\left(\frac{\partial^2 \eta_i}{\partial x_i^2}\frac{\partial x_i}{\partial p_i}+\frac{\partial \eta_i}{\partial x_i}\frac{\partial^2 x_i}{\partial p_i^2}\right) I(p_i)-2\frac{\partial \eta_i}{\partial x_i}\frac{\partial x_i}{\partial p_i}(1-F(p_i))+\eta_i f(p_i)=0. \tag{18}$$

Isolating $\partial^2 x_i/\partial p_i^2$ yields:

$$\frac{\partial^2 x_i}{\partial p_i^2}=-\frac{\frac{\partial^2 \eta_i}{\partial x_i^2}\frac{\partial x_i}{\partial p_i}I(p_i)-2\frac{\partial \eta_i}{\partial x_i}\frac{\partial x_i}{\partial p_i}(1-F(p_i))+\eta_i f(p_i)}{\frac{\partial \eta_i}{\partial x_i}I(p_i)}.$$

The denominator of this expression is negative. Therefore, for concavity of the function x_i, the numerator must be negative. Observe that the first term of the numerator is negative because for the $k=1$ case η_i is convex in x_i and x_i is decreasing in p_i. The second term is also negative because η_i is decreasing in x_i. The last term is clearly positive.

We now show that the last two terms together are negative for the uniform distribution, which concludes the proof:

$$\begin{aligned}
&-2\frac{\partial \eta_i}{\partial x_i}\frac{\partial x_i}{\partial p_i}(1-F(p_i))+\eta_i f(p_i)\\
&=-2\left(\frac{e^{-x_i}\left(1-e^{x_i}+x_i\right)}{x_i^2}\right)\left(\frac{(1-F(p_i))\left(1-e^{-x_i}\right)}{\frac{e^{-x_i}(1-e^{x_i}+x_i)}{x_i}I(p_i)}\right)(1-F(p_i))+\frac{1-e^{-x_i}}{x_i}f(p_i)\\
&=-2\frac{(1-F(p_i))^2\left(1-e^{-x_i}\right)}{x_i I(p_i)}+\frac{1-e^{-x_i}}{x_i}f(p_i)=\frac{1-e^{-x_i}}{x_i}\left(-2\frac{(1-F(p_i))^2}{I(p_i)}+f(p_i)\right)\\
&=\frac{1-e^{-x_i}}{x_i}\left(-2\frac{(1-p_i)^2}{\frac{1}{2}(1-p_i)^2}+1\right)=-\frac{1-e^{-x_i}}{x_i}3<0.
\end{aligned}$$

The proof is now complete. ■

Proof of Proposition 5.

Welfare is given by

$$W=B\eta(x,p;k)\int_p^1 \varepsilon f(\varepsilon)d\varepsilon.$$

Taking the FOC gives

$$\frac{1}{B}\frac{\partial W}{\partial p} = \frac{\partial \eta}{\partial p}\int_p^1 \varepsilon f(\varepsilon) f\varepsilon - \eta p f(p) = 0.$$

Using the expression for $I(p) = \int_p^1 (\varepsilon - p) f(\varepsilon) f\varepsilon$, this can be rewritten as

$$\frac{1}{B}\frac{\partial W}{\partial p} = \frac{\partial \eta}{\partial p} I(p) + \frac{\partial \eta}{\partial p} p(1 - F(p)) - \eta p f(p) = 0,$$

which is the expression given in the proposition.

When $p = 0$, this expression is positive. When $p = 1$, it is negative. This ensures that p^o exists.

■

Proof of Proposition 6.

Recall that the SNE price p is given by the solution to:

$$\begin{aligned} &\frac{(1 - F(p))^2}{f(p)} \left[\frac{\partial \eta}{\partial p} I(p) + \frac{\partial \eta}{\partial p} p(1 - F(p)) - p\eta f(p) \right] \\ &\qquad - d(p) \left[(1 - F(p) - pf(p))\, I(p) + p(1 - F(p))^2 \right] = 0. \end{aligned} \tag{19}$$

while the socially optimal price p^o satisfies the FOC:

$$\frac{\partial \eta}{\partial p} I(p^o) + \frac{\partial \eta}{\partial p} p^o(1 - F(p^o)) - p^o \eta f(p^o) = 0. \tag{20}$$

Comparing (19) and (20), we immediately see that the equilibrium and the optimum coincide when $d(p) = 0$. This occurs in the limit when $k \to \infty$.

To prove the claim for finite k, we show that

$$\left.\frac{\partial \Pi_i}{\partial p_i}\right|_{p_i = p^o} > 0,$$

which implies that the payoff of a firm increases at $p_i = p^o$ so that $p > p^o$. For this, define

$$\Psi(p) \equiv (1 - F(p) - pf(p))\, I(p) + p(1 - F(p))^2$$

for $p \in [0, 1]$. In what follows, we show that $\Psi(p) > 0$ for all $p \in (0, 1)$, which implies that, since $d(p) < 0$ for finite k, the RHS of (7) is negative.

Observe that: $\Psi(0) = I(0) > 0$; $\Psi(1) = 0$;

$$\Psi'(p) = -\left(2f(p) + pf'(p)\right) I(p) - p(1 - F(p))f(p). \tag{21}$$

and notice that

$$\lim_{p\to 1}\frac{\Psi'(p)}{1-F(p)} = -f(p) < 0.$$

To establish a contradiction, suppose there exists a region of prices $p \in (0,1)$ for which $\Psi(p) \leq 0$. Then, because $\Psi(p)$ is decreasing at $p = 1$, there must exist some $\tilde{p} \in (0,1)$ such that $\Psi'(\tilde{p}) = 0$ and $\Psi(\tilde{p}) > 0$. Using (21), the condition $\Psi'(\tilde{p}) = 0$ gives:

$$\tilde{p}(1-F(\tilde{p}) = -\frac{(2f(\tilde{p}) + \tilde{p}f'(\tilde{p}))\, I(\tilde{p})}{f(\tilde{p})}$$

Using this relation in the condition $\Psi(\tilde{p}) > 0$ gives

$$\begin{aligned}\Psi(\tilde{p}) &= (1-F(\tilde{p}) - \tilde{p}f(\tilde{p}))\, I(\tilde{p}) + \tilde{p}(1-F(\tilde{p}))^2 \\ &= -\left(1-F(\tilde{p}) + \tilde{p}\frac{f(\tilde{p})^2 + f'(\tilde{p})(1-F(\tilde{p}))}{f(\tilde{p})}\right) I(\tilde{p}) > 0.\end{aligned}$$

But this is impossible because, by the log-concavity of $f(p)$, the hazard rate $\frac{f(p)}{1-F(p)}$ is increasing in p and the expression in brackets is positive. We therefore reach the desired contradiction and so we must have $\Psi(p) > 0$ for all $p \in (0,1)$. This completes the proof of Proposition 5. ■

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