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JEL Classification: D43, D85, L13, L14

Keywords: competitive pricing, Entry, market structure, optimal network structure

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Network Topology and Market Structure*

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March 13, 2020

Abstract

We develop a two-stage oligopolistic network competition model where, first, firms simultaneously determine their prices and, then, users connected through a network determine their product's consumption. We show that denser networks (network topology) reduce prices and that a higher number of firms (market structure) reduces prices only when competition is weak. However, the price for the most influential users can increase with the number of firms when competition is very fierce and when there are enough network externalities. We also show that increasing competition always leads to a lower firm's profit while increasing network density leads to a clockwise rotation of the profit curve as a function of the number of firms. Finally, we study the effect of network topology and market structure on price dispersion and determine the optimal network structure from the perspective of both firms and users.

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1 Introduction

People around the world use social network in its various forms (e.g., news feeds on Facebook and Twitter, private messaging on WhatsApp and WeChat, and discussion forums on Reddit) for a number of purposes. These can generally be categorized as (i) digitally communicating and socializing with known others, such as family and friends, (ii) doing the same but with unknown others who share common interests, and (iii) accessing and contributing to digital content such as news, gossips, and user-generated product reviews. This means that the way agents influence each other in their product consumption is key to understand how product prices are determined and how consumption choices are made. Indeed, firms are well aware of how social interactions between agents affect their consumption decisions and, therefore, do take into account these network externalities between consumers in their pricing decisions. For instance, Nintendo, Sony and Microsoft compete for gaming console dominance knowing that people playing games have to choose among the Switch (Nintendo), the PlayStation (Sony), or the Xbox (Microsoft). Key for this choice are the console price and how people play these games with each other and how their interact in the different online platforms.

In this paper, we examine these issues by modeling the interplay between the market structure regarding the firms that provide differentiated products to consumers and the underlying network structure among users who make consumption decisions based on the consumption's decisions of their friends and other connections. To this end, we provide a general network oligopolistic competition model with an arbitrary number of firms, (local) network effects, and an arbitrary degree of product differentiation. To the best of our knowledge, this is the first attempt to study how general market structure and general network effects have a non-trivial impact on prices, firms' profits and consumers' utility.

To be more precise, each firm sells a differentiated product to users connected in a network of social interactions. Users derive utility from their own consumption as well as from other users' consumptions who are directly linked to them in the network (who could be friends, neighbors, but also people who are connected online and sharing common interests). The latter captures the local network effects from interacting with other users in the network. The firms simultaneously determine the (potentially discriminatory) prices in the first stage; while, in the second stage, users determine their consumption for each product, given the prices set by firms.

Each firm faces the following trade-off when setting prices: lowering prices enhances the firm's demands but, at the same time, it is costly for firms because it also reduces the price-cost margins for existing demands. These effects are modulated by the degree of network externalities and the number of firms in the market. We show that the price set by each firm is a decreasing function of the position in the network (in terms of Katz-Bonacich centrality) of each user, which means that firms network-price discriminate users by charging lower prices to more central agents. This implies that influencers pay a lower price and can even be subsidized for consuming a product.[1]

We also show that prices are lower when either the network becomes denser (changing network structure/ topology) or the intensity of network effects becomes stronger. Indeed, increasing network density or network externalities intensifies the price competition between firms for the central nodes in the network, who in equilibrium are compensated by being charged lower prices. We then study how a change in market structure (number of firms L) affects prices. For regular networks (those in which all nodes have the same degree), prices always decrease with the number of firms but this is not necessarily true for non-regular networks. In fact, for the latter, there is a non-monotonic relationship between prices and L. In particular, we show that, for very connected individuals (the influencers), whose price are negative (they are subsidized by firms to consume their goods), when L is large and network externalities are important, increasing further L increases rather than decreases prices. In other words, for these agents only, in a very competitive market, *increasing competition leads to an increase in prices.* Indeed, when competition is very fierce, it becomes less profitable for firms to subsidize these influential agents if competition further increases and, in the limit, when L goes to infinity, firms will set a price (mark-up in our setting) of zero for the influencers.

We then measure the degree of price discrimination by the *price dispersion* in the market, i.e., the maximal price differential among users. Price dispersion turns out to be small for the monopoly case and the very competitive case and attains a maximum value for intermediate number of firms. Some illustrative examples suggest that the extent of price dispersion can be significant, and the maximal value is attained when there are two to four firms in the market.

Next, we examine the impact of market structure and network topology on firm's profitability. Intuitively, a firm's profit decreases with fiercer competition. What is less obvious, however, is that increasing the intensity of network effects does not necessarily increases firms' profits because it depends on the degree of competition. We show that, instead, it leads to a *clockwise rotation* of the profit curve, which means that, under low competition, the firm's profit is higher when there are more network externalities, while, under fierce competition, the opposite is true. This unintended outcome arises from two effects. On the one hand, a strong network intensity amplifies the benefits of network externalities, thereby leading to more users' consumption. On the other hand, it pushes the firms to lower their prices due to stronger competition, and this price effect can drive down the firm's profitability. When there are only a few firms, the first effect dominates and improvement in network technology (more network externalities) generates a higher firm's profit. When there

[1] The idea of using celebrities in consumer markets, who have a high social value, to influence others is a well-known marketing strategy (Knoll and Matthes (2017)).

are many firms, however, the price effect takes off and improvement in network technology dampens the firms' profitability.

The above result also implies that if free entry is allowed, the improvement in network technology increases the equilibrium number of participating firms only when the entry cost is high. This is because a higher entry cost deters firms from participating in the market. Thus, the number of firms tends to be small, and it is precisely in this regime that the improvement in network technology can boost the firm's profitability.

Finally, we demonstrate how our analysis can be used to characterize the optimal network structures from the perspectives of firms and users. We find that their ranking of network structures are consistent when the number of firms is small and the products are sufficiently differentiated. However, when there is a large number of firms or products are sufficiently homogeneous, firms and users hold completely opposite views of what the optimal network structure is.

1.1 Related literature

Our paper is related to the game-on-network literature (for overviews, see Jackson (2008); Jackson and Zenou (2015); Bramoullé and Kranton (2016); Jackson, Rogers, and Zenou (2017)), in particular, the part of this literature that deals with pricing under imperfect competition in networks (see Bloch (2016) for a survey on targeting and pricing in networks).[2]

In this literature, different aspects of imperfect competition with network effects have been addressed. In terms of pricing issues, Bloch and Quérou (2013); Belhaj and Deroïan (2016); Fainmesser and Galeotti (2016); Leduc, Jackson, and Johari (2017); Candogan, Bimpikis, and Ozdaglar (2012) deal with the monopoly case while Chen et al. (2018a); Aoyagi (2018); Fainmesser and Galeotti (2020) examine the duopoly framework.[3] A paper that studies a general market structure (oligopoly with N firms) with network effects is that of Amir and Lazzati (2011). However, in their model, the network structure is not modeled since network effects are captured by the fact that users' willingness to pay is increasing in the number of agents acquiring the same good.

To the best of our knowledge, our paper is the first that studies the interaction between

[2]There is an early literature on network externalities with imperfect competition which either focuses on the aggregate level of network externalities (e.g. Farrell and Saloner (1985); Katz and Shapiro (1985); for an overview, see Economides (1996)) or on the competitive pricing problem in the context of two-sided networks in which players on one side care about the aggregate contributions of those on the other side (see e.g. Caillaud and Jullien (2003); Armstrong (2006); Rochet and Tirole (2006)), which corresponds to a very specific network structure: the complete bipartite network. In our model, we deal with any possible network structure.

[3]See also Ushchev and Zenou (2018) who model the substituability between differentiated goods as a network and determine the equilibrium prices.

market structure and network structure/topology in a general network oligopolistic competition model with an arbitrary number of firms, an arbitrary network structure, and a flexible degree of product differentiation.

2 Model

2.1 Setting

Consider L firms that sell differentiated products to N users in a social network, where $L \geq 1$, $N \geq 2$. Let $\mathcal{L} := \{1, 2, \cdots, L\}$ and $\mathcal{N} := \{1, 2, \cdots, N\}$ denote the set of firms and users, respectively. Each firm $l \in \mathcal{L}$ sells variety l to all users (consumers have a love for variety so that each of them consumes all goods)[4] and sets prices $\mathbf{p}^l = (p_1^l, \cdots, p_N^l)'$, since prices may differ across users, depending possibly on their network positions and other characteristics such as a_i (defined below). Each user $i \in \mathcal{N}$ has a quasi-linear utility, $x_i^0 + u_i(\mathbf{x}_i, \mathbf{x}_{-i})$, where x_i^0 is i's consumption of the numeraire good, $\mathbf{x}_i = (x_i^1, \cdots, x_i^L)' \in \mathbf{R}_+^L$ is i's consumption bundle of products offered by the L different firms, and $\mathbf{x}_{-i}$ is the consumption profile for users other than i.

Denote by $\mathbf{G} = (g_{ij})_{n \times n}$ the adjacency matrix representing the network structure among these users. In other words, $g_{ij} = 1$ if and only if i and j are directly connected, and $g_{ij} = 0$, otherwise. We also assume that $g_{ii} = 0$ (no self-loops) and $g_{ij} = g_{ji} \in \{0, 1\}$ (undirected and unweighted network).[5]

We adopt the following explicit functional form for the utility function:

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) := \underbrace{\left(\sum_{l=1}^{L} a_i^l x_i^l - \frac{1}{2} \sum_{l=1}^{L} (x_i^l)^2 - \frac{1}{2} \sum_{l=1}^{L} \sum_{s \neq l} \beta x_i^s x_i^l \right)}_{:= v_i(\mathbf{x}_i)} + \underbrace{\delta \left(\sum_{l=1}^{L} \left(\sum_{j=1}^{N} g_{ij} x_i^l x_j^l \right) \right)}_{:= \eta_i(\mathbf{x}_i, \mathbf{x}_{-i})}. \tag{1}$$

The utility of user i's consists of two terms. The first term $v_i(\mathbf{x}_i)$ represents i's own consumption utility, and it only depends on $\mathbf{x}_i$ with β measuring the curvature of $v_i(\cdot)$ and the degree of substitution between the L products consumed. For each i, $s \neq l$, $\frac{\partial^2 u_i}{\partial x_i^l \partial x_i^s} = -\beta$, which is negative (positive) when β is positive (negative). Therefore, products are substitutable (complementary) when $\beta > (<) 0$.

[4] Indeed, in the utility function (1), each consumer i consumes all goods (*love for variety*). A necessary and sufficient condition for consumers to have love for variety, or, more formally, to have strictly convex preferences over the space of differentiated products, is Assumption 2 below, which we assume throughout the paper.

[5] This is without loss of generality. Our model is flexible enough to allow for arbitrary network structure $\mathbf{G}$, and obtain the same main results.

The second term in (1), $\eta_i(\mathbf{x}_i, \mathbf{x}_{-i})$, captures the *network effects* (peer effects) enjoyed by user i from interacting with other users in network $\mathbf{G}$, and these effects are scaled by the parameter δ, so that $\frac{\partial^2 \eta_i}{\partial x_i^l \partial x_j^l} = \delta g_{ij}$ for any $l \in \mathcal{L}$.[6] Therefore, i's utility depends on j's consumption when i and j are directly connected, which reflects the *local* network effects among users in the network. In other words, the value of consuming a product for a given user increases when others directly connected to this user consume this product. The budget equation for user i is $x_i^0 + \sum_{l=1}^{L} p_i^l x_i^l = Y_i$, where p_i^l is the price user i pays to firm l for the consumption of x_i^l. As usual, we assume the income Y_i is sufficiently large so that the nonnegativity constraints on consumptions $\{x_i^l\}$'s and x_i^0 are not binding.[7]

Given the price profile $\mathbf{p} = (\mathbf{p}^1, \cdots, \mathbf{p}^L)$, and consumption profile $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$,[8] firm l's profit is given by:

$$\Pi^l(\mathbf{p}^l, \mathbf{p}^{-l}; \mathbf{x}) := \sum_{i \in \mathcal{N}} (p_i^l - c_i) x_i^l,$$

where c_i is the marginal cost of serving user i.

Our first assumption is to focus on ex ante symmetric firms.

Assumption 1. *For each user i, $a_i^l = a_i^s = a_i > c_i \geq 0$ for any l and s.*

Since each firm l is defined by its product l, this assumption is equivalent to having symmetric products. Let $\mathbf{a} = (a_1, \cdots, a_N)'$ denote the marginal utility vector. For the equilibrium analysis in Section 3, we consider arbitrary $\mathbf{a}$. For some analysis in Sections 4 and 5, we further restrict our analysis to the case when $a_i = a, c_i = 0, \forall i \in \mathcal{N}$ (see Assumption 3). While this is not without loss of generality, it best serves our purpose in this paper because our main focus is on the network topology (see footnote 16 for further discussions).

Third, for ease of the exposition, we focus on *substitutable* products ($\beta \geq 0$) with *positive network* effects ($\delta \geq 0$). We assume $\beta \in [0, 1)$, under which i's utility $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ in (1) is strictly concave in $\mathbf{x}_i$ (see Lemma A1 in the Online Appendix A for the proof). Let $\lambda_1(\mathbf{G})$ denote the largest eigenvalue of $\mathbf{G}$.

Assumption 2. *$\delta \geq 0$, $0 \leq \beta < 1$ and*

$$1 - \beta - \delta \lambda_1(\mathbf{G}) > 0, \tag{2}$$

Condition (2) guarantees the uniqueness of consumption equilibrium for any price profile, and the concavity of the firm's profit in prices. This assumption imposes conditions on both

[6] Also, note that, for $l \neq s$, $i \neq j$, $\frac{\partial^2 \eta_i}{\partial x_i^l \partial x_j^s} = 0$, i.e., we assume away cross-product network effects.

[7] Effectively, user i chooses $\mathbf{x}_i$ to maximize $u_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \sum_{l=1}^{L} p_i^l x_i^l$.

[8] Note that the dimension of $\mathbf{p}^l$ is N, while the dimension of $\mathbf{x}_i$ is L so that both $\mathbf{p}$ and $\mathbf{x}$ have dimension NL.

the product differentiation parameter β, the network effect parameter δ, and the spectral radius of $\mathbf{G}$ (which, by the Perron-Frobenius theorem, is $\lambda_1(\mathbf{G})$, the largest eigenvalue of $\mathbf{G}$ since $\mathbf{G}$ is a matrix with non-negative entries) but does not depend on L, the number of firms. This condition is satisfied when δ is not too large, i.e., $\delta < (1-\beta)/\lambda_1(\mathbf{G})$. Analogous conditions to (2) are imposed in many network papers.[9]

We study the subgame perfect equilibrium of the following two-stage game: first, firms simultaneously choose their prices and, second, users simultaneously choose their consumption bundles. The network structure $\mathbf{G}$, together with other model parameters $\beta, \delta, \mathbf{a}$, are known and common knowledge for firms and users. We take as given the market structure L and the network structure $\mathbf{G}$ in most of the analysis, with the exception of Section 6 where we study the free-entry of firms (so that the market structure L is endogenized) and the optimal network structure. Throughout the paper, we maintain Assumptions 1 and 2 in all propositions, corollaries, and lemmas. Thus, we will not explicitly spell them out.[10]

2.2 Discussions of model and assumptions

We now provide several discussions of our model and give some justifications of some of our assumptions.

The first one concerns the functional form of users' utility given by (1). The quadratic form in the private part of $v_i(\cdot)$ has been widely adopted in industrial organization, trade, and macroeconomics (see for example Singh and Vives (1984); Vives (2001); Ottaviano, Tabuchi, and Thisse (2002); Asplund and Nocke (2006); Foster, Haltiwanger, and Syverson (2008); Melitz and Ottaviano (2008); Syverson (2019)).[11] Also, quasi-linear utility function such as (1) is commonly adopted in many network models (e.g. Ballester et al. (2006); Bramoullé and Kranton (2007); Bramoullé et al. (2014)), especially IO network models (e.g. Candogan et al. (2012); Bloch and Quérou (2013); Fainmesser and Galeotti (2016, 2020); Chen et al. (2018a); Ushchev and Zenou (2018)). Moreover, the quadratic interaction terms in the network benefit part $\eta_i(\mathbf{x}_i, \mathbf{x}_{-i})$, together with the quadratic term $v_i(\mathbf{x}_i)$, gives us great tractability of the model. This enables us to conduct extensive comparative statics with respect to model parameters (such as the network structure $\mathbf{G}$ and market structure L) in the equilibrium and welfare analysis.

Second, our model considers general market structure L, and it naturally incorporates

[9]See, for instance, Ballester, Calvó-Armengol, and Zenou (2006); Candogan, Bimpikis, and Ozdaglar (2012); Bloch and Quérou (2013); Bramoullé, Kranton, and d'Amours (2014); Fainmesser and Galeotti (2016); Chen, Zenou, and Zhou (2018a); Galeotti, Golub, and Goyal (2020).

[10]In Appendix A, we provide the matrix notations used in this paper.

[11]In fact, without network externalities (i.e., $\delta = 0$), our model reduces to the standard IO and trade models where the utility function is given by: $u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = v_i(\mathbf{x}_i)$.

and generalizes several existing models in the literature about pricing network effects. For instance, it generalizes the monopoly model ($L = 1$) of Bloch and Quérou (2013) and Candogan, Bimpikis, and Ozdaglar (2012), and the duopolistic model ($L = 2$) of Chen et al. (2018a). As shown in Sections 4 and 5, under different market structures, we obtain qualitatively different results, and in several cases even the opposite results. These observations highlight the importance of allowing for general market structure L.

Third, we focus on substitutable products ($\beta \geq 0$) with positive network effects ($\delta \geq 0$).[12] This is mainly for the ease of exposition, and our analytical results can directly carry over to the settings with negative network effects ($\delta < 0$) and/or complementary products ($\beta < 0$), after corresponding modification of Assumption 2 and model interpretation.

Fourth, we can give an alternative interpretation of our model: each node i in the network represents a market (for instance, a region, a city, or a country) with a representative consumer in each market with the utility function specified in (1). Firms offer differentiated products to these markets, and δg_{ij} represents the degree of demand spillovers across two markets i and j. Our model and results will have the same implications under this alternative interpretation.

Lastly, the common knowledge and perfect information about the network structure merit additional justifications. Note that we allow for arbitrary but exogenously given network structure and no additional restriction on the network structure is imposed. As an initial analysis with general market structures, it is natural to impose this common knowledge assumption on $\mathbf{G}$. Further analysis can be developed to relax this assumption.[13] Moreover, under the alternative interpretation in which each node in the network represents a market, it seems plausible to assume that the spillovers between two markets, δg_{ij}, are fully known to firms. This would justify our assumption of common knowledge on $\mathbf{G}$.

3 Equilibrium analysis

3.1 Consumption equilibrium

We first characterize the users' simultaneous consumption decisions. Given a price profile $\mathbf{p} = (\mathbf{p}^1, \cdots, \mathbf{p}^L)$, each user i chooses $\mathbf{x}_i$ that maximizes $u_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \sum_{l=1}^{L} p_i^l x_i^l$, while taking as given the consumption decisions of all other users in the network. A consumption equilibrium

[12] When $\beta = 0$, our model reduces to L independent products. When $\delta = 0$, there are no externalities between users.

[13] See, among others, Fainmesser and Galeotti (2016, 2020) for pricing with imperfect network knowledge under monopolistic and duopolistic competition, respectively. It would be interesting to extend the models of Fainmesser and Galeotti (2016, 2020) to an oligopolistic setting and to consider the interaction of network knowledge and market structure. This is beyond the scope of this paper.

(CE) $\mathbf{x}(\mathbf{p})$ is a Nash equilibrium of the second stage consumption game among users in the network. In our setting, this consumption game belongs to the family of network games with multi-dimensional strategy space studied in Chen et al. (2018b). Here we sketch the main results regarding the consumption equilibrium; see Appendix C for details.

When each user i makes her consumption decisions, she takes into account the following factors: her own preference a_i^l for each product l, the degree of substitution β between the different products, the number of firms L in the market, which determines the number of products consumed, the price p_i^l of each good l, and the local network effects δ (what her friends consume and how important they decisions are for i). Lemma C1 in Appendix C shows that, under Assumptions 1 and 2, there exists a unique consumption equilibrium (CE) $\mathbf{x}(\mathbf{p})$ and the consumption of each product for each user is determined by the sum and difference of the marginal utility of consumption and prices of each product multiplied by two matrices $\mathbf{M}^+$ and $\mathbf{M}^-$, defined as:

$$\mathbf{M}^+ := [(1+(L-1)\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}, \quad \mathbf{M}^- := [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}. \tag{3}$$

As shown in Appendix C, we interpret $\mathbf{M}^+$ as the *market expansion social multiplier*, and $\mathbf{M}^-$ as the *business-stealing social multiplier*, because the former measures the marginal reduction of the average demand of firms following a marginal increment of average price (see equations (C7) or (C6)), and the latter measures (the negative value of) the marginal reduction of the demand difference between two firms following a marginal increment of the price difference between these two firms (see equation (C8)).

We show that the business-stealing social multiplier is stronger than the market expansion social multiplier, i.e., $\mathbf{M}^- \succeq \mathbf{M}^+$. We also show that, as the number of firm L increases, the market expansion effect of social network is diminished ($\mathbf{M}^+$ is weaker), while the business stealing effect of social network $\mathbf{M}^-$ remains the same.

Corollary C1 shows that linear combinations of $\mathbf{M}^+$ and $\mathbf{M}^-$ collectively determine the own-(cross-) price elasticities of a firm's demands. Since products are substitutable, raising firm l's prices reduces l's demands and increases competitor s's demands (i.e., $\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l} \preceq \mathbf{0}$ and $\frac{\partial \mathbf{x}^s}{\partial \mathbf{p}^l} \succeq \mathbf{0}$). We also show that the "own-price" sensitivity $-\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l}$ is a convex combination of $\mathbf{M}^+$ and $\mathbf{M}^-$. And it puts more weight on $\mathbf{M}^{-1}$ and, therefore, smaller weight on $\mathbf{M}^+$, for a larger number of firms. These observations will play a key role in shaping the incentives of firms regarding prices in the first stage.

3.2 Pricing equilibrium

We now characterize the pricing decisions of firms in the first stage. Suppose there exists a symmetric equilibrium in the pricing stage in which all firms charge the same prices $\mathbf{p}^*$

(i.e., the price is the same for all firms but may be different for each user). We now derive the necessary conditions to sustain $\mathbf{p}^*$ as a symmetric pricing equilibrium. As such, the equilibrium demand for each firm would be $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$ as in part (ii) in Lemma C1. If a firm, say firm 1, deviates and lowers her price vector by $\Delta\mathbf{p}^1$, it must then satisfy the following *no-deviating* condition:

$$\overbrace{\langle \Delta\mathbf{p}^1, \underbrace{\mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)}_{=\mathbf{x}^* \text{ by Lemma C1}} \rangle}^{\text{marginal loss of lowering prices}} = \underbrace{\langle \overbrace{\frac{\mathbf{M}^+ + (L-1)\mathbf{M}^-}{L}\Delta\mathbf{p}^1}^{\Delta\mathbf{x}^1 \text{ by Corollary C1}}, (\mathbf{p}^* - \mathbf{c})\rangle}_{\text{marginal benefit of lowering prices}}.$$

Indeed, when firm 1 lowers her prices for all her users by $\Delta\mathbf{p}^1$, she increases her profits because all users will consume more of product 1, but this increase will depend on the network effects δ, the degree of substitution β between the different goods, and how it propagates through the network; all these aspects are captured by both $\mathbf{M}^+$, the market expansion social multiplier, and $\mathbf{M}^-$, the business-stealing social multiplier. By lowering her prices by $\Delta\mathbf{p}^1$, she also reduces her profits because she loses money for each product sold to users and this depends on the same parameters, which are captured by $\mathbf{M}^+$ but not by $\mathbf{M}^-$ since the latter measures the business-stealing effect.

The above equation must hold for any $\Delta\mathbf{p}^1$ in $\mathbf{R}^N$, implying the following identity:

$$\mathbf{M}^+(\mathbf{a} - \mathbf{p}^*) = \frac{\mathbf{M}^+ + (L-1)\mathbf{M}^-}{L}(\mathbf{p}^* - \mathbf{c}). \tag{4}$$

Solving (4) yields the $\mathbf{p}^*$ stated in Proposition 1.[14]

Proposition 1. *There exists a unique equilibrium in the pricing stage in which all firms charge the same price* $\mathbf{p}^*$ *defined as follows:*

$$\mathbf{p}^* = \frac{\mathbf{a} + \mathbf{c}}{2} - \frac{(L-1)\beta}{2}[(2 + (L-3)\beta)\mathbf{I}_n - 2\delta\mathbf{G}]^{-1}(\mathbf{a} - \mathbf{c}). \tag{5}$$

In equilibrium, each firm's profit is $\Pi^* = \langle \mathbf{x}^*, (\mathbf{p}^* - \mathbf{c})\rangle$, *where* $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$.

The pricing formula (5) in Proposition 1 is determined by the trade-off specified in (4): lowering prices enhances the firm's demands but, at the same time, it is costly for the firm because it also reduces the price-cost margins for existing demands. The degree of demand enhancing effects and the size of existing demands are captured by Corollary C1 and Lemma C1, respectively. The relative strength of these two forces pins down the equilibrium prices in Proposition 1.

[14] All proofs can be found in Appendix B.

It is easily verified that the pricing formula in (5) exhibits the following decomposition:

$$\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{(L-1)\beta}{2\left((2+(L-3)\beta)\right)} \underbrace{\left[\mathbf{I}_n - \frac{2\delta}{(2+(L-3)\beta)}\mathbf{G}\right]^{-1}(\mathbf{a}-\mathbf{c})}_{=\mathbf{b}\left(\mathbf{G},\frac{2\delta}{(2+(L-3)\beta)},\mathbf{a}-\mathbf{c}\right)} \tag{6}$$

The first term does not depend on the network, while the second term is proportional to the Katz-Bonacich centralities[15] of nodes with the discount factor δ adjusted by a factor of $\frac{2}{(2+(L-3)\beta)}$. Unless $L = 1$ (the monopoly case) and/or $\beta = 0$ (independent products), the equilibrium prices exhibit network-based discrimination, i.e., the price is a function of the user's position in the network space. In particular, it shows that the more central a user is (in terms of Katz-Bonacich centrality), the lower the price she will be charged for consuming a good. This is because a high-central user generates network externalities to her friends who are more likely to consume the good and this is taken into account by each firm. As shown in (6), this price discount depends on the intensity of the network effects δ, the degree of substitution between goods β, the network structure $\mathbf{G}$ and the market structure L.

Our general model extends several existing models and the equilibrium prices in (5) are consistent with existing papers when we restrict L. For instance, the prices reduce to $\frac{\mathbf{a}+\mathbf{c}}{2}$ in the case of a monopoly firm ($L = 1$, see Bloch and Quérou (2013); Candogan et al. (2012)). Similarly, for the duopoly case ($L = 2$), the prices are $\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{\beta}{2}[(2-\beta)\mathbf{I}_n - 2\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{c})$ (see Chen et al. (2018a)). Observe that the monopoly prices ($L = 1$) are equal to $\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2}$, so the prices in (5) are always below the monopoly prices.

We investigate the impact of network structure and market structure on equilibrium prices in Section 4, and on firms' profits in Section 5. In what follows, we impose an additional assumption.

Assumption 3. $\mathbf{c} = \mathbf{0}$ *and* $\mathbf{a} = a\mathbf{1}_N$.

First, we normalize c_i to zero so that, in the analysis below, the mark-up $(p_i^t - c_i)$ is the same as the price. Such a normalization is without any loss of generality. Also, to simplify the notation and without much loss of generality, we impose $a_i = a$ for every i, so that users have the same marginal utility for the products. Nonetheless, the network structure is arbitrary, and hence users are not necessarily located symmetrically.[16]

[15] See Definition A1 in Appendix A for a formal definition of Katz-Bonacich centralities.

[16] Under Assumption 3, the only heterogeneity between users is their network positions, which is our main focus. Adding further heterogeneity on marginal utilities a_i does not offer any new additional economic insights.

4 The effects of market and network structure on equilibrium prices

We use the equilibrium characterization in Proposition 1 to study the impact of network topology and market structure on the equilibrium prices. Here is an overview of our findings. First, we show that, by fixing the market structure (i.e. the number of firms L), the network density has a negative impact on prices. Second, by fixing the network structure, we demonstrate that the equilibrium prices decreases with the number of firms when L is small. However, the prices for the most influential users can increase with L for sufficiently large L when the network effect δ is strong enough. This is a surprising result since it shows that more competition can be associated with higher prices. Finally, we show that the degree of price dispersion in the network has a non-monotonic relationship with the market structure L: it is low when the number of firms L is either small or large and has its maximal value when L takes intermediate values.

We first investigate the effects of network structure and the strength of network effects on equilibrium prices, while fixing the number of firms L.

Proposition 2. *Suppose that Assumption 3 holds and the number of firms L is fixed. Then, increasing network density* $\mathbf{G}$ *or the strength of network effects* δ *decreases the equilibrium price for all users. Formally, suppose* $\mathbf{G}' \succeq \mathbf{G}''$*, and* $\delta' \geq \delta''$*. Then we have*

$$\mathbf{p}^*(\mathbf{G}', \delta') \preceq \mathbf{p}^*(\mathbf{G}'', \delta'').$$

Since the equilibrium price vector in Proposition 1 is decreasing in the Katz-Bonacich centrality measures of users in the network (see (6)), the result in Proposition 2 is straightforward. Indeed, increasing network density or network externalities intensifies the price competition among firms for the central nodes in the network, who in equilibrium are compensated for lower prices.

Example 1. *Consider the kite network in Figure 1. Set* $\beta = 0.4$*.*[17] *We plot the equilibrium prices for node 1 (the blue curve), node* 2 *(the green curve), and node* 4 *(the red curve) while varying the number of firms L (the x-axis): the left panel (Figure 2) is for* $\delta = 0.17$*, and the right panel (Figure 3) is for* $\delta = 0.27$*. The solid black line is for* $\delta = 0$*.*[18]

In both Figures, we see that, by fixing δ *and* L *and consistent with Proposition 1, the most central node (node 1 in the kite) is charged the lowest price while the least central node (node 4) obtains the highest price. Comparing both Figures, we show that indeed all the*

[17] The largest eigenvalue λ_1 is about 2.17. Assumption 2 holds when $\delta < (1-\beta)/\lambda_1 \approx 0.276$.

[18] When $\delta = 0$, there is no price difference across different nodes: every node obtained the same price equal to $a\frac{1-\beta}{2+(l-3)\beta}$.

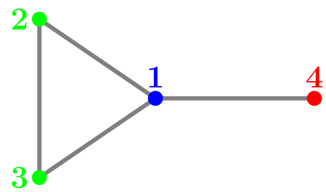

Figure 1: A kite with four nodes

curves are shifted downward, i.e., the price for each node is lower when δ *increases from* 0.17 *to* 0.27. *In addition, the black curve* ($\delta = 0$) *is higher than all the other colored curves, which correspond to prices with strictly positive values of* δ.

Next, we explore the impact of competition L on prices while fixing $\mathbf{G}$. In fact, Example 1 already reveals several interesting observations on the effects of competition. In Figure 2 with a smaller δ, the price for every node decreases with competition. In Figure 3 with a larger δ, interestingly, the price for node 1 is not monotone in L. In both figures, the prices eventually converge to zero. The following proposition demonstrates that these patterns hold in general settings.

Proposition 3. *Suppose that Assumption 3 holds and let* d_i *denote the degree of node* i.

(i) If, $\forall i \in \mathcal{N}$, $1 - \beta - \delta d_i > 0$, *then* $\mathbf{p}^*$ *decreases with* L *for any* L.

(ii) Suppose that, for some i, $1 - \beta - \delta d_i < 0$. *Then* p_i^* *has a non-monotonic relationship with the number of firms* L. *In particular,* $\partial p_i^* / \partial L < 0$ *when* L *is small, and* $\partial p_i^* / \partial L > 0$ *for sufficiently large* L.

In Appendix D.2, we showed that, for *regular networks*, prices always decrease with the number of firms L.[19] Proposition 3 demonstrates that this is not necessarily true for non-regular networks. Indeed, there exists a value of δ such that there is a non-monotonic relationship between prices and L for at least some individuals (part (ii) of Proposition 3). To be more precise, when $\frac{1-\beta}{\max_j d_i} < \delta < \frac{1-\beta}{\lambda_1}$,[20] the price of at least one user, say i, has a non-monotonic relationship with L, generating an interesting and very counter-intuitive

[19] Indeed, under Assumption 3 and setting $a = 1$, we obtain the following common equilibrium price for regular networks with degree d:

$$p^*_{reg} = \frac{(1 - \beta - \delta d)}{2 + (L-3)\beta - 2\delta d}, \tag{7}$$

where $p^*_{reg} \in (0, \frac{1}{2}]$ (see (D9) in Appendix D). Clearly, $\partial p^*_{reg} / \partial L < 0$ as $1 - \beta - \delta d > 0$ by Assumption 2 for regular networks.

[20] For any non-regular network, $\lambda_1 < (\max_i d_i)$ and thus this range is always not empty.

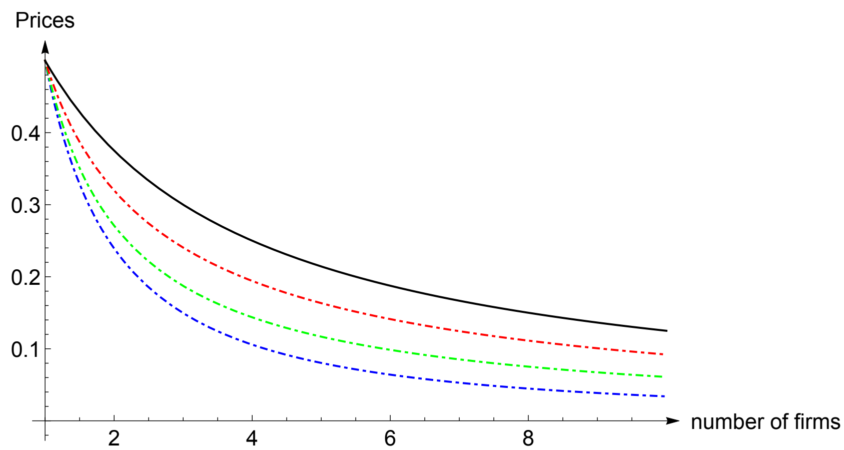

Figure 2: $\delta = 0.17$

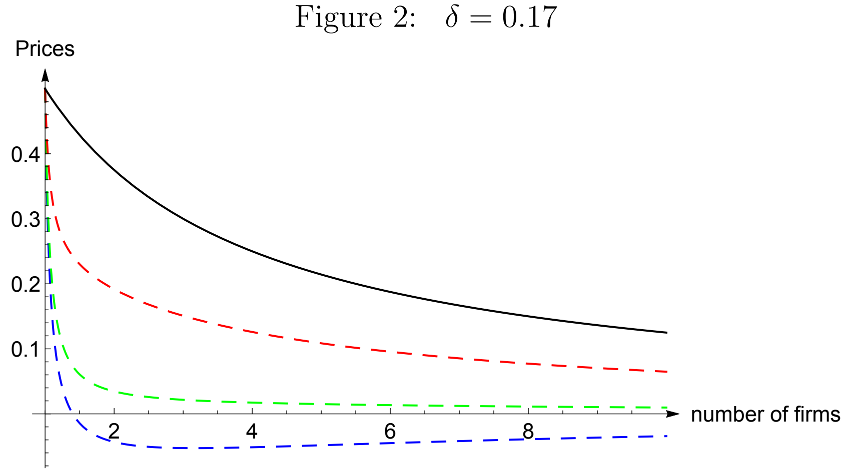

Figure 3: $\delta = 0.27$

relationship between competition and prices, i.e. *more competition (more firms) may increase prices for highly connected users if there are already many firms in the market.*

Basically, for any node i with $1-\beta-\delta d_i < 0$ (case (ii) of Proposition 3), for large L, it can be shown that:[21]

$$p_i^* = \frac{1-\beta-\delta d_i}{\beta L} + \mathcal{O}(L^{-2}),$$

which means that this price (or mark-up in our setting) is below zero. In other words, when competition is very fierce (large L), firms find it profitable to subsidize (negative prices) the very central users for consuming their products while, when there is very little competition, firms charge a positive price to these high-central users. Of course, as shown in Figures 2 and 3, this also depends on the degree of network externalities δ. If, as in Figure 2, δ is low, then firms always charge a positive price to all users because they don't find it profitable to "subsidize" highly-central users since they do not generate enough positive externalities on their friends. On the contrary, when δ is high enough as in Figure 3, then firms do subsidize some of the very influential users. Moreover, in both cases, when there is not enough competition (low L), prices do decrease with increased competition. When L is very large and δ is high enough, increasing competition (more firms) increases the prices for the highly users because firms want to subsidize them less and less; eventually prices converge to zero in the limit. See the price curve of node 1 in blue color in Figure 3.

In equilibrium, firms charge prices depending on the positions of nodes in the network. To evaluate the extent of such network-based price discrimination, we define $Disp(\cdot)$ as the maximal difference in equilibrium prices among users in the network.

Definition 1.

$$Disp(L) := \max_{i \neq j} |p_i^*(L) - p_j^*(L)|.$$

We have the following result:

Proposition 4. *$Disp(1) = 0$ (monopoly), and $\lim_{L\to\infty} Disp(L) = 0$ (perfect competition). Suppose that $\beta \neq 0$. For any non-regular network,[22] there exists an intermediate value L^* such that $Disp(L)$ is maximized. In particular, L^* is strictly greater than one.*

Proposition 4 demonstrates that the degree of price dispersion is hump-shaped in the number of firms. When there is only one firm, there is no dispersion. When the number of firms is infinite, all prices converge to zero and, again, there is no dispersion. The maximal dispersion occurs when L^* takes an intermediate value, generating a non-monotonic relationship between price dispersion and competition.

[21] See equation (E16) in Proposition E4 in Appendix E. Given a real-valued function f, we write $f(L) = \mathcal{O}(L^{-2})$ if $\limsup_{L\to\infty} \left|\frac{f(L)}{L^{-2}}\right| < \infty$.

[22] For regular networks, $Disp(L) = 0, \forall L$ as all firms charge the same price to all users; see Appendix D.1.

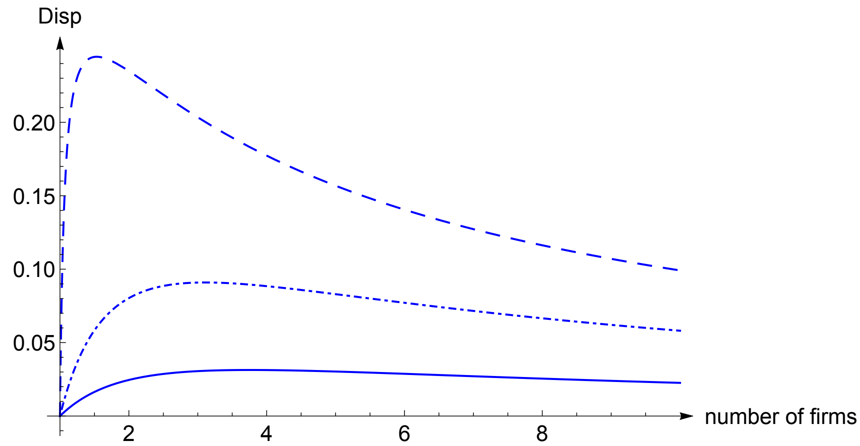

Figure 4: Price Dispersion curve (y-axis) as a function of l (x-axis): Dashed line ($\delta = 0.27$), Dot-dashed line ($\delta = 0.17$) and Solid line ($\delta = 0.07$).

Example 2. *Consider the kite network with four nodes displayed in Figure 1. In Figure 4, we plot $Disp(L)$ using different values of δ. We see that the maximal degree of dispersion may occur when the number of firms is reasonably small. In Figure 4, $L^* \approx 2$ for the dashed line ($\delta = 0.27$), $L^* \approx 3$ for the dot-dashed line ($\delta = 0.17$), $L^* \approx 4$ for the solid line ($\delta = 0.07$). We also see that the price differences in the network can be significant. In the dashed curve, the maximal dispersion is about 0.25. (Note that the monopoly price is 0.5 as we set $a = 1, c = 0$ in this example. See also Figure 3.)*

When the strength of network effects is not very large, we have the following simple characterization of the price dispersion.

Remark 1. *For small δ,*

$$Disp(L) \approx \delta \frac{(L-1)\beta(d_{max} - d_{min})}{[2 + (L-3)\beta]^2}.$$

The maximal dispersion is about $\frac{(d_{max}-d_{min})}{8(1-\beta)}\delta$, obtained when $L^* = -1 + 2/\beta > 1$.

When δ is small, the Katz-Bonacich centrality reduces to the simple counting of degrees of nodes in the network. In Remark 1, we show that the maximal dispersion increases with the difference of the maximal degree and the minimal degree, i.e., the degree dispersion in the network. Also, the optimal L^* takes the simple form of $-1 + 2/\beta$, which decreases with the degree of product differentiation β. In Example 2, $\beta = 0.4$, so $L^* = 4$, which is consistent with the solid blue line for $\delta = 0.07$ in Figure 4.

In sum, the results in this section highlight the importance of considering general market structure. The value of L qualitatively matters for the analysis of both the price trend and the price dispersion. When the number of firms increases from $L = 1$ to $L = 2$, the equilibrium prices always go down. One naive conjecture would be that prices always go down with competition. However, such an observation regarding the relationship between prices and competition does not globally extend to the general setting with any L. As shown in Proposition 3, prices can easily increase with competition, especially when δ and L are large. This suggests that we need to be cautious when drawing empirical implications for price trend in the presence of network effects.

Moreover, the degree of price dispersion in non-regular networks can be significant, and the market structure generating the largest degree of dispersion, L^*, although not equal to monopoly, is not necessarily the duopoly case with $L = 2$. This further validates that local network effects crucially craft the tactical pricing strategies and interact with the market competitiveness. Finally, for markets with a large number of competing firms, the structure of the network has little impact on equilibrium prices.

5 The effects of market structure and network structure on firms' profits

In this section, we characterize the effects of the network structure $\mathbf{G}$ and the number of firms L on firms' profits. As expected, we show that the firm's profit curve, as a function of L, is downward sloping because of the intensified competition. However, increasing network density does not lead to an overall upward shift of the firm's profit curve. This is surprising because, other things being equal, an increased network density leads to a higher gross utility for each user since a user now benefits more from other users' consumption. Instead, it leads to a *clockwise rotation* of the profit curve. In other words, as the network density increases, the firm's profit increases when L is relatively small, and decreases when L is sufficiently large. The market structure at which the profit curve rotates depends negatively on the product differentiation parameter β and the strength of network effects δ. Regarding the firms' profits, increasing δ has similar effects as increasing network density.

5.1 Regular networks

We first start with regular networks and then proceed to general network structures. We set $a = 1$ to simplify notation and assume $\beta > 0$.

In Appendix D.3, we show that increasing competition (i.e., higher L) always leads to a lower profit per firm Π^*_{reg}.[23] We then study the impact of the network effect δ on Π^*_{reg}.

[23] Under Assumption 3 and setting $a = 1$, in equation (D11) in Appendix D, we show that each firm's

There are two main economic forces. On the one hand, increasing δ decreases the equilibrium price p^*_{reg} because the firms are competing more intensively in the pricing game, which leads to uniformly lower prices (*price effect*). This negatively affects the profit. On the other hand, a rise in δ increases the social multiplier effect, which positively affects profits (*social multiplier effect*). In Proposition D1, we show that the impact of δ on Π^*_{reg} is non-monotonic. More precisely, when the number of firms L is sufficiently small (large), increasing δ increases (decreases) Π^*_{reg}. Indeed, when L is relatively small, the equilibrium prices p^*_{reg} are very close to the monopoly price $1/2$ and, accordingly, $1-2p^*_{reg}$ is close to zero. Consequently, the price effect is negligible compared to the social multiplier effect.[24] As a result, the firm's profit increases with δ. On the contrary, when L is sufficiently large, competition is so intense that the equilibrium price is close to zero and the price effect of δ dominates the social multiplier effect. This implies that the firm's profit decreases with δ.

We now discuss some immediate consequences of Proposition D1. First, an improvement in the network technology (caused, for example, by an increase in δ so that peer effects are stronger) does not always generate a higher firm's profit. In fact, whether firms can profit from such a technology improvement critically depends on the degree of competition they face. As the extreme, a monopoly firm can extract some of the additional network benefits enjoyed by the users and, hence, obtains a higher profit. With competition, the extent of such value extraction by firms is jeopardized by the intensified price competition. In fact, when competition is too strong, the firm's profit is reduced. This suggests some potential incentives for firms not to improve the network technology. Second, when δ increases marginally, the profit curve, as a function of L, *rotates*, instead of *shifting* globally. Furthermore, Proposition D1 explicitly identifies the critical $\bar{L} := \chi\left(\frac{\beta}{1-\delta d}\right)$ at which the profit curve rotates, where $\chi(.)$ is the function defined in Lemma D1 in Appendix D.3. $\chi(\cdot)$ is continuously differentiable and strictly decreasing. The graph of χ is plotted in Figure D1. Interestingly, the threshold $\bar{L}$ decreases with δ and β as $\chi(\cdot)$ is decreasing.

Example 3. *In Figure 5, we plot Π^*_{reg} as a function of L for a regular network with $d = 2$ (for instance, a ring network). We set $\delta = 0.2$ for the blue curve, and $\delta = 0.3$ for the red curve. We see that the red curve is a clockwise rotation of the blue curve, which means that, under low competition (i.e., $L \leq 3.5$), the firm's profit is higher when δ is higher (more network externalities) while, under fierce competition (i.e., $L > 3.5$), the opposite is true.*

In Appendix D.3, we show that one implication of Proposition D1 is Corollary D2:

equilibrium profit for a regular network with degree d is given by:

$$\Pi^*_{reg} := \frac{n(1-p^*_{reg})p^*_{reg}}{1+(L-1)\beta-\delta d}, \tag{8}$$

where p^*_{reg} is given by (7) in footnote 19. In Section D.3, we show that: $\partial\Pi^*_{reg}/\partial L < 0$.

[24] in Appendix D.3, we explicitly decompose the net effect of δ on profit into these two effects.

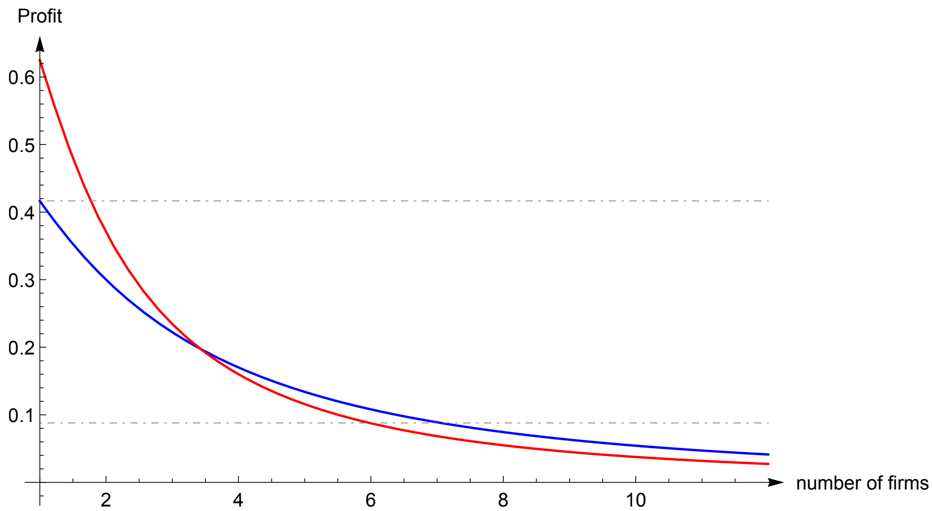

Figure 5: Π^*_{reg} curve, as a function of L, for small δ (blue curve; $\delta = 0.2$) and large δ (red curve; $\delta = 0.3$)

Π^*_{reg} has a non-monotonic relationship with δ depending of the market competition L. In particular, when there is a monopolist, increasing δ always increases profit while, when L is large enough, increasing δ always decreases profits. For intermediary values of L, there is an inverted U-shaped curve between δ and Π^*_{reg}. The following example illustrates this result:

Example 4. *In Figure 6, we plot the curve of Π^*_{reg} as a function of δ under different market structures. For $L = 1$ (monopoly, blue dashed curve), the firm's profit always increases with δ. For $L = 2$ (duopoly, black dashed curve), the firm's profit first increases and then decreases with δ.*[25] *For $L = 5$ (oligopoly, red dashed curve), the profit always decreases with δ.*

In Proposition D2 in Appendix D.3, we show that increasing the degree d of a regular network (denser networks) does not always increase the firm's profit. In fact, increasing d leads to a similar clockwise rotation to the one we observed when increasing δ. As a result, we can obtain the same Figure 5 in Example 3 if we fix $\delta = 0.2$ but increase d from 2 to 3. (Note that, in Example 3, d is fixed at 2, but δ increases from 0.2 to 0.3.)

5.2 General network structures

The results for regular networks in the previous subsection qualitatively extend to the settings with general network structures.

[25] $\partial\Pi^*_{reg}/\partial\delta$ vanishes only when $L = \chi(\beta/(1-\delta d))$, or equivalently, $\delta = \frac{1-\chi^{-1}(L)/\beta}{d}$.

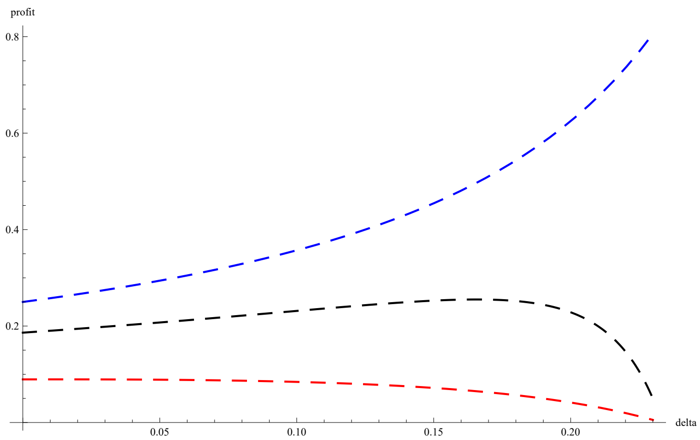

Figure 6: The profit Π^*_{reg} curve as a function of δ under different market structures: Blue $L = 1$; Black $L = 2$; Red $L = 5$

Proposition 5. *For any network structure* $\mathbf{G}$*, let* $Spec(\mathbf{G}) = \{\lambda_1, \cdots, \lambda_N\}$ *denote the set of eigenvalues of* $\mathbf{G}$*.*

(i) The firm's profit Π^* *decreases with* L*, and it converges to zero as* $L \to \infty$*.*

(ii) Increasing δ *leads to higher (lower) profit when the market is sufficiently concentrated (competitive). More precisely,*

(a) $\partial\Pi^*/\partial\delta > 0$ *if* $L < \chi(\beta/(1-\delta\lambda_i))$ *for any* $\lambda_i \in Spec(\mathbf{G})$*;*[26]

(b) $\partial\Pi^*/\partial\delta < 0$ *if* $L > \chi(\beta/(1-\delta\lambda_i))$ *for any* $\lambda_i \in Spec(\mathbf{G})$*.*[27]

The results in Proposition 5 are similar to those derived under regular networks and have the same intuition. Indeed, Proposition 5 (i) is the counterpart of equation (D12) and shows that competition drives down firm's profit. Proposition 5 (ii) generalizes Proposition D1 by showing that increasing δ leads to a rotation of the profit curve.

Proposition 6. *Given two network structures* $\mathbf{G}'$ *and* $\mathbf{G}''$ *with* $\mathbf{G}' \succeq \mathbf{G}''$*, there exist cutoffs* $\bar{L}$ *and* $\underline{L}$ *such that*[28]

(i) For any $L < \underline{L}$*,* $\quad \Pi^*(\mathbf{G}'; \beta, \delta, l) \geq \Pi^*(\mathbf{G}''; \beta, \delta, L)$;

(ii) For any $L > \bar{L}$*,* $\quad \Pi^*(\mathbf{G}'; \beta, \delta, L) \leq \Pi^*(\mathbf{G}''; \beta, \delta, L)$.

[26] Equivalently, $L < \min_i \chi(\beta/(1-\delta\lambda_i)) = \chi(\beta/(1-\delta\lambda_1))$ as $\chi(\cdot)$ is decreasing.

[27] Equivalently, $L > \max_i \chi(\beta/(1-\delta\lambda_i)) = \chi(\beta/(1-\delta\lambda_N))$ as $\chi(\cdot)$ is decreasing, where λ_N is the smallest eigenvalue.

[28] These thresholds depend on $\mathbf{G}'$, $\mathbf{G}''$, β and δ.

Increasing network density generates a similar rotation of the profit curve, as shown by Proposition D2 (for regular networks) and Propositions 5 and 6 (for general networks). As above, these results are due to the fact that there is a trade-off between the price effect and the social multiplier effect. When L, the number of firms, is small, the latter dominates the former and firms benefit from an increase in network externalities δ and in network density. When competition becomes very fierce because L is large, increasing δ or network density decreases profit because the negative impact on prices is stronger than the positive network effect.

5.3 A technical contribution

We discuss several technical aspects behind our results.

The underlying driving forces behind the profit results (regular and non-regular networks) are similar, i.e., the trade-off between the *interaction of price effects of competition* and the *market expansion social multiplier effects*. For regular networks, these two effects take simpler form due to the symmetry of the degree. The proofs are more involved for non-regular network structures as both effects operate in the space of matrices due to the heterogeneity of the nodes. Essentially, we reduce the complex problem of a general network structure into a series of sub-problems with regular networks, in which each sub-problem is easy to solve. Formally, we demonstrate that the equilibrium firm's profit in any non-regular network $\mathbf{G}$ can be decomposed into a weighted aggregation of several terms, where each term corresponds to a regular network with d replaced by the corresponding eigenvalue of $\mathbf{G}$, and where the positive weight is equal to the square of the inner product of the vector $a\mathbf{1}/\sqrt{N}$ and the associated eigenvector of $\mathbf{G}$.

Mathematically, we show the following identity for the equilibrium firm's profit:[29]

$$\Pi^*(\mathbf{G};\beta,\delta,L) := \sum_{\lambda_i \in Spec(\mathbf{G})} \Pi^*_{reg}(\lambda_i,\beta,\delta,L) \times (\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}}\rangle)^2 \tag{9}$$

where $Spec(\mathbf{G}) = \{\lambda_1, \cdots, \lambda_N\}$ is the set of eigenvalues of $\mathbf{G}$, and $\mathbf{u}_i$ is the corresponding normalized eigenvector[30] associated with λ_i. The proof of this identity uses the key Lemma B1 in Appendix B, which is an application of the spectral decomposition theorem of the network matrix $\mathbf{G}$. The coefficient $(\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}}\rangle)^2$ is positive, unless $\mathbf{u}_i$ is orthogonal to $\mathbf{1}_N$.[31]

[29] We can adopt similar techniques to study comparative statics of other welfare measures. The details are available upon request.

[30] In other words, each $\mathbf{u}_i$ is of unit length, and $\mathbf{G}\mathbf{u}_i = \lambda_i \mathbf{u}_i$.

[31] An eigenvalue λ_i satisfying $\langle \mathbf{u}_i, \mathbf{1}\rangle \neq 0$ is called a main eigenvalue of the network. The set of main eigenvalues is called the "main part of the spectrum", see Cvetković (1970).

Consider a regular network $\mathbf{G}$ with degree d, we have $\lambda_1 = d$ and $\mathbf{u}_1 = \mathbf{1}_N/\sqrt{N}$, and only one term in the summation in (9) is non-zero. The reason is that all the eigenvectors $\mathbf{u}_i$ for $i \neq 1$ is orthogonal to $\mathbf{u}_1$, thus $(\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}} \rangle)^2 = 0$. For a non-regular network, there are usually more non-zero summation terms in (9), as multiple coefficients can be positive.[32]

This identity greatly simplifies our analysis of the comparative statics results of the profit in a general network structure, as we can exploit the results for the case of regular networks, which are simpler and fully studied in Appendix D. For example, here we illustrate how one can apply the above identity to prove Proposition 5. First, note that the coefficients $(\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{n}} \rangle)^2$ are nonnegative. Analogous to equation (D12), we can first show that $\partial \Pi^*_{reg}(\lambda_i, \beta, \delta, L)/\partial L < 0$ for any $\lambda_i \in Spec(\mathbf{G})$. Therefore, $\partial \Pi^*(\mathbf{G}; \beta, \delta, L)/\partial L < 0$, i.e., we prove Proposition 5 (i). Second, similar to Proposition D1, we prove that $\partial \Pi^*_{reg}(\lambda_i, \beta, \delta, L)/\partial \delta > (<) 0$ if and only if $L < (>) \chi(\beta/(1 - d\lambda_i))$. Therefore, suppose $L < \chi(\beta/(1 - \delta\lambda_i))$ for any $\lambda_i \in Spec(\mathbf{G})$, we have $\partial \Pi^*(\mathbf{G}; \beta, \delta, L)/\partial \delta > 0$; this corresponds to case (a) in Proposition 5 (ii). Case (b) can be shown in a similar way.

These techniques are very similar to those used in a recent paper by Galeotti, Golub, and Goyal (2020), which also use spectral decomposition theorem to simplify the optimal targeted interventions in networks. Here, we use them in a very different context of comparative statics in oligopoly networks. Both papers, though focusing on different economic issues, highlight the analytical advantages of looking at the problems through the angle of eigen-decomposition and eigenvector space of the network matrix.

6 Free entry and optimal network structures

We now study how the market structure and network structure are determined. First, we endogeneize the *market structure* by allowing for free entry of firms. Second, we endogeneize the *network structure* by determining the firm's and the user's optimal network structure.

6.1 Equilibrium market structure with free entry

Suppose that the number of firms in the market is determined by a free-entry condition. We assume that each firm pays a fixed cost for entering the market. We show that, when the fixed cost is sufficiently large, the equilibrium market structure is relatively concentrated. In this region, increasing network density or network technology level leads to more firms in the free-entry equilibrium. On the contrary, when the fixed cost is sufficiently low, the equilibrium

[32] This identity should be viewed as a useful mathematical result rather than an economic identity since, when the eigenvalue λ_i is negative (or not integer-valued), it does not make much sense to say that $\Pi^*_{reg}(\lambda_i, \beta, \delta, L)$ corresponds to a regular graph with degree λ_i.

market structure is relatively competitive. We reach a different region: increasing network density or network technology level leads to fewer firms in the free-entry equilibrium. The results are mainly driven by the rotation of profit curve studied in the previous Section.

Given a fixed entry fee/cost $f > 0$, define L^{FE*} as the number of firms in the free-entry equilibrium such that $\Pi^*(L^{FE*}) = f$. Since $\Pi^*(\cdot)$ is strictly decreasing in L, L^{FE*} is uniquely determined.[33] For ease of notation, we omit the other dependent variables (such as $\mathbf{G}, \beta, \delta$) in the definition of L^{FE*}.

Proposition D3 in Appendix D.4 considers a regular network of degree d and shows that the equilibrium number of firms L^{FE*} decreases (increases) with δ when the entry cost f is sufficiently small (large). It is a direct consequence of Proposition D1. For illustration, consider Figure 5 in Example 3. When f is higher than the profit at the rotation point ($L \approx 3.5$), the free-entry number of firms is higher for the red curve with higher δ than for the blue curve with lower δ. The reverse happens when f is lower than the profit at the intersection point. Indeed, for a marginal increase in δ, the rotation point occurs exactly when $L = L^* = \chi\left(\frac{\beta}{1-\delta d}\right)$ by Proposition D1. Therefore, the threshold $\bar{f}$ in Proposition D3 is, in fact, equal to $\bar{f} = \Pi^*_{reg}(d; \beta, \delta, \chi\left(\frac{\beta}{1-\delta d}\right))$. In reality, the size of f depends on the institutional context, which varies from one industry to another. The above Proposition provides several possible implications of the impact of technology or network improvement on market concentration with endogenous firm entry.

Since the observation that increasing network density or δ leads to similar rotations of the profit curve (see Proposition 5 (ii) and Proposition 6), a similar result (available upon request) as in Proposition D3 holds in general network settings. What is new here is the fact that the equilibrium number of firms not only depends on the entry cost f and the degree of product substitution β, as it is usually the case, but also on the network structure and on the intensity of the network effects δ.

6.2 User-optimal and firm-optimal network structures

So far, we have treated the network structure as given. We now fix the market structure L and discuss what is the optimal network structure from the perspective of firms and from that of the users.[34] We show that, regardless of the market structure, users' equilibrium utilities always increase with network density. As a result, the optimal user-optimal network is always the complete network. In contrast, firms' preferences over network structure critically depend on the market structure: the optimal network is the complete network when the number of

[33] We treat L as a continuous variable. To make our problem interesting, we assume f is below the monopoly profit, i.e., $f < \Pi^*(L)|_{L=1}$.

[34] Very few papers have examined optimal network design in network games. Exceptions include Belhaj et al. (2016); Hiller (2017); König et al. (2019).

firms is sufficiently small, while it is the empty network when competition is very fierce (i.e., when there is a sufficiently large number of firms). Consequently, when the market is relatively concentrated, i.e., L is small, firms' and users' preferences over network structures are fully aligned. However, in highly competitive markets, i.e., L is large, their preferences are exactly the opposite. These results highlight the importance of considering market structure in determining network structures for various users.

In Proposition E5 in Appendix E.2, we show that the *user*-optimal network is the *complete network*. Indeed, increasing network density benefits users in two different ways. First, the equilibrium prices are lower, due to intensified price competition among firms (see Proposition 2). Second, the network multiplier is stronger, which generates a greater level of network benefits for users. Formally, in the proof of Proposition E5, we show that the equilibrium user's surplus is monotone in the equilibrium consumption of users, i.e, $u_i^* \propto (x_i^*)^2$. Recall that, in equilibrium, the consumption is given by the multiplication of market expansion and the difference of marginal utility vector and price vector, i.e., $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$. With a denser network, the multiplier $\mathbf{M}^+$ is stronger and prices are lower. The result then immediately follows.

From the firms' perspective, increasing network density again generates two forces, which work in the opposite directions. On the one hand, higher density leads to lower equilibrium prices due to more competition, which reduces the profit's margins and, hus, hurts firm's profitability. On the other hand, the demand enhancing effects due to stronger network effects can benefit the firms. Which force dominates the other depends on L, the number of existing competitors in the market.

Proposition E6 in Appendix E.3 shows that, when L is low enough, the firm-optimal network is the complete network while, when L is high enough, the firm-optimal network is the empty network. Moreover, part (ii) of Proposition E6 highlights the fact that adding a link between any pair of nodes can result in an intensified price competition among firms and, therefore, may drive down their profits. This phenomenon appears when the market is relatively concentrated, and, in this case, users and firms hold opposite views of the optimal network structure. Note that even if the network is empty, firms' profits can still be strictly positive because each user's willingness to pay is positive from the stand-alone consumption utility.

It is worth mentioning that, in reality, neither firms nor users have full flexibility in adjusting the network structures. Thus, our results in Proposition E6 shall be regarded as benchmarks of optimal networks. Nonetheless, the proofs of Propositions E5 and E6 reveal a deeper principle: we provide unambiguous answers to the directions of the firm profit's and the user surplus' changes if we add one additional link to an existing network.

7 Conclusion

In this paper, we examine the interplay between the market structure (as measured by the number of firms in the market that sell differentiated products) and the network structure (as measured by its topology) among users who make consumption decisions. We show that prices are set lower when either the network becomes denser (by adding links) or the intensity of the network effects is stronger. This suggests an intensified competition among firms due to network effects among users. Moreover, we show that price dispersion in the market turns out to be small for the monopoly case and the very competitive case (i.e., when there are many firms), and attains a maximum value in the intermediate range. We also show that, when we increase the number of firms, initially prices are always driven down (due to competition), but can revert to an uprising trend when the number of firms becomes large.

We also find that increasing the intensity of network effects does not shift the firms' profit curves but, instead, leads to a clockwise rotation. Improvement in network technology generates a higher firm's profit when there are only a few firms, but it dampens the firms' profitability when there are many firms. This implies that, if free entry is allowed, the improvement in network technology increases the equilibrium number of participating firms only when the entry cost is high. Finally, we characterize the optimal network structures from the perspectives of firms and users. Intriguingly, their rankings of network structures are consistent when the number of firms is small and the products are sufficiently differentiated. However, when there is a large number of firms or products are sufficiently homogeneous, firms and users hold completely opposite views of the optimal network structures.

Our model could be extended in several possible directions. First, we mostly focus on symmetric firms. Extending the analysis to asymmetric firms' competition can address new research questions. In addition, this symmetry prohibits us to study the impact of mergers, which could be an interesting research avenue. Second, our analysis relies on the common and perfect knowledge regarding the network structure among users and firms. While this assumption is standard and holds for relatively small networks, relaxing it will lead to asymmetric network information and would allow us to study the interactions between network knowledge and market structure. We leave these exciting topics for future research.

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Appendix

A Matrix notation, Katz-Bonacich centrality and some preliminary results

Matrix notation. Let $\mathbf{A}'$ denote the transpose of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbf{J}_{mn}$ is the $m \times n$ matrix with 1's, and $\mathbf{1}_n = \mathbf{J}_{n1}$ is a column vector with 1s:

$$\mathbf{I}_n = \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{bmatrix}_{n\times n}, \quad \mathbf{J}_{mn} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{m\times n}, \quad \mathbf{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}.$$

The inner product of two column vectors $\mathbf{x} = (x_1, \cdots, x_n)'$ and $\mathbf{y} = (y_1, \cdots, y_n)'$ in $\mathbf{R}^n$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}'\mathbf{y} = \sum_i x_i y_i$. We use $\mathbf{0}$ to denote the zero matrix with suitable dimensions. For any two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \preceq (\succeq)\mathbf{B}$ if component-wise $a_{ij} \leq (\geq) b_{ij}$ for all i, j. Consequently, $\mathbf{A}$ is a positive matrix if $\mathbf{A} \succeq \mathbf{0}$. A square symmetric matrix $\mathbf{A}$ is called positive definite if all of its eigenvalues are strictly positive.

Katz-Bonacich centrality. Let us define the Katz-Bonacich centrality. Denote by $\lambda_1(\mathbf{G})$ the spectral radius of matrix $\mathbf{G}$. Since $\mathbf{G}$ is a nonnegative matrix, by the Perron-Frobenius Theorem it is also equal to its largest eigenvalue.

Definition A1. *Assume $0 \leq \delta < 1/\lambda_1(\mathbf{G})$. Then, for any vector $\mathbf{a} = (a_1, \cdots, a_n)' \in \mathbf{R}^n$, the Katz-Bonacich centrality vector with weight $\mathbf{a}$ is defined as:*

$$\mathbf{b}(\mathbf{G}, \delta, \mathbf{a}) := \mathbf{M}(\mathbf{G}, \delta)\mathbf{a}, \tag{A1}$$

where

$$\mathbf{M}(\mathbf{G}, \delta) = [\mathbf{I} - \delta\mathbf{G}]^{-1} = \mathbf{I} + \sum_{k\geq 1} \delta^k \mathbf{G}^k. \tag{A2}$$

Let $b_i(\mathbf{G}, \delta, \mathbf{a})$ be the ith entry of $\mathbf{b}(\mathbf{G}, \delta, \mathbf{a})$. Let $m_{ij}(\mathbf{G}, \delta)$ be the ij entry of $\mathbf{M}(\mathbf{G}, \delta)$. Then,

$$b_i(\mathbf{G}, \delta, \mathbf{a}) = \sum_j m_{ij}(\mathbf{G}, \delta) a_j.$$

Some preliminary results. We would like now to present some results: Lemma A1 and Lemma A2, which will be used in the proofs in Appendix B.

Lemma A1. *Suppose $\beta \in [0,1)$ and $L \geq 1$. Define the $L \times L$ matrix $\mathbf{\Psi}$ as:*

$$\mathbf{\Psi} = \begin{bmatrix} 1 & \beta & \cdots & \beta \\ \beta & 1 & \cdots & \beta \\ \vdots & \ddots & \ddots & \vdots \\ \beta & \cdots & \beta & 1 \end{bmatrix}_{L\times L}. \tag{A3}$$

Then (i) the matrix $\mathbf{\Psi}$ is positive definite. (ii) For $a > 0$, the function

$$v(\mathbf{x}) = a\mathbf{x}'\mathbf{1}_L - \frac{1}{2}\mathbf{x}'\mathbf{\Psi}\mathbf{x} = a(\sum_{t=1}^{L} x^t) - \frac{1}{2}\sum_{t=1}^{L}(x^t)^2 - \frac{\beta}{2}\sum_{t=1}^{L}\sum_{s\neq t} x^s x^t, \quad \mathbf{x} = (x^1, \cdots, x^L) \in \mathbf{R}^L$$

has a unique maximizer at $\mathbf{x}^* = \hat{x}\mathbf{1}_L$ with the maximum value $v(\mathbf{x}^*) = \frac{L(1+(L-1)\beta)}{2}(\hat{x})^2$, where $\hat{x} = \frac{a}{1+(L-1)\beta}$.

Proof of Lemma A1: The eigenvalues of $\mathbf{\Psi}$: $1-\beta$ (with multiplicity $L-1$), and $1+(L-1)\beta$ (with multiplicity 1), are strictly positive, and hence $\mathbf{\Psi}$ is positive definite. The FOC of maximizing v is just $a\mathbf{1}_L = \mathbf{\Psi}\mathbf{x}^*$ which, by symmetry, leads to:

$$\mathbf{x}^* = a\mathbf{\Psi}^{-1}\mathbf{1}_L = \frac{a}{1+(L-1)\beta}\mathbf{1}_L$$

Since $v(\cdot)$ is strictly concave by (i), $\mathbf{x}^*$ is the unique global maximizer. Substituting $\mathbf{x}^*$ into $v(\cdot)$ yields the maximum value. □

Lemma A2. *Under Assumption 2.*

(i) For any $\lambda_i \in Spec(\mathbf{G})$,

$$1-\beta-\lambda_i\delta > 0, \;\; 1+(L-1)\beta-\lambda_i\delta > 0, \;\; 1+(L-2)\beta-\lambda_i\delta > 0, \;\; 2+(L-3)\beta-2\lambda_i\delta > 0.$$

(ii) The following matrices are symmetric and positive definite:

$$[(1-\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}, \;\; [(1+(L-1)\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}, \;\; [(1+(L-2)\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}, \;\; [(2+(L-3)\beta)\mathbf{I}_n-2\delta\mathbf{G}]^{-1}.$$

Moreover, for each matrix above, every entry is nonnegative.

Proof of Lemma A2: (i) Note that λ_1 is the largest eigenvalue in $Spec(\mathbf{G})$. So $\lambda_i \leq \lambda_1$, implying $1 - \beta - \lambda_i \delta \geq 1 - \beta - \lambda_1 \delta > 0$ by Assumption 2. Moreover,

$$1 + (L-1)\beta - \lambda_i \delta \geq 1 + (L-2)\beta - \lambda_i \delta = \underbrace{(1 - \beta - \lambda_i \delta)}_{>0} + \underbrace{(L-1)\beta}_{\geq 0} > 0$$

and

$$2 + (L-3)\beta - 2\lambda_i \delta = (1 - \beta - \lambda_i \delta) + (1 + (L-2)\beta - \lambda_i \delta) > 0.$$

(ii) The eigenvalues of $[(1-\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1}$ are precisely $\frac{1}{1-\beta-\lambda_i \delta}$, where $\lambda_i \in Spec(\mathbf{G})$. Since $\frac{1}{1-\beta-\lambda_i \delta} > 0$ by part (i), $[(1-\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1}$ is positive definite. The nonnegativeness of the matrix follows from

$$[(1-\beta)\mathbf{I}_n - \delta \mathbf{G}]^{-1} = \frac{1}{1-\beta}\mathbf{M}(\mathbf{G}, \frac{\delta}{1-\beta}) = \frac{1}{1-\beta}\sum_{j\geq 0}(\frac{\delta}{1-\beta})^j \mathbf{G}^j \succeq \mathbf{0}.$$

The proofs for the other three matrices are similar. □

B Proofs

Proof of Proposition 1: The FOC at the symmetric prices $\mathbf{p}^*$ is derived in the main text:

$$[(1+(L-1)\beta)\mathbf{I}_N-\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{p}^*)=\frac{[(1+(L-1)\beta)\mathbf{I}_N-\delta\mathbf{G}]^{-1}+(L-1)[(1-\beta)\mathbf{I}_N-\delta\mathbf{G}]^{-1}}{L}(\mathbf{p}^*-\mathbf{c}).$$

Solving this linear equation of $\mathbf{p}^*$ yields the symmetric prices in Proposition 1. We provide two equivalent expressions for the equilibrium prices $\mathbf{p}^*$ in (B4). We will adopt the most convenient form in the analysis later.

$$\begin{aligned}\mathbf{p}^* &= [(2+(L-3)\beta)\mathbf{I}_N-2\delta\mathbf{G}]^{-1}\left[(1-\beta)\mathbf{I}_N-\delta\mathbf{G}\right]\mathbf{a}+\left[(1+(L-2)\beta)\,\mathbf{I}_N-\delta\mathbf{G}\right]\mathbf{c}\\ &= \frac{\mathbf{a}+\mathbf{c}}{2}-\frac{(L-1)\beta}{2}[(2+(L-3)\beta)\mathbf{I}_N-2\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{c}).\end{aligned}\tag{B4}$$

Next we check the second-order conditions of firms' optimization problem. Recall that $\Pi^1=\langle\mathbf{x}^1,\mathbf{p}^1-\mathbf{c}\rangle$. Since $\mathbf{x}^1$ is linear in $\mathbf{p}^1$, the Hessian matrix of Π^1 with respect to $\mathbf{p}^1$ is just $-2\frac{\mathbf{M}^++(L-1)\mathbf{M}^-}{L}$ by Corollary C1, which is negative definite by Lemma A2. As a result, the profit function Π^1 is strictly concave in $\mathbf{p}^1$, and FOCs are sufficient for optimality.

Under this symmetric pricing equilibrium, for each firm the consumption vector is

$$\begin{aligned}\mathbf{x}^* &= \mathbf{M}^+(\mathbf{a}-\mathbf{p}^*)=[(1+(L-1)\beta)\mathbf{I}_N-\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{p}^*)\\ &= [(1+(L-1)\beta)\mathbf{I}_N-\delta\mathbf{G}]^{-1}[(2+(L-3)\beta)\mathbf{I}_N-2\delta\mathbf{G}]^{-1}[(1+(L-2)\beta)\mathbf{I}_N-\delta\mathbf{G}](\mathbf{a}-\mathbf{c}).\end{aligned}$$

and the equilibrium profit is $\Pi=\langle\mathbf{x}^*,(\ \mathbf{p}^*-\mathbf{c})\rangle$, which can be simplified to $\langle(\mathbf{a}-\mathbf{c}),\Phi(\mathbf{G})(\mathbf{a}-\mathbf{c})\rangle$, where

$$\Phi(z):=\frac{(1+(L-2)\beta-\delta z)(1-\beta-\delta z)}{(1+(L-1)\beta-\delta z)(2+(L-3)\beta-2\delta z)^2}.$$

□

Remark B1. *In the symmetric equilibrium, the consumption vector $\mathbf{x}^*=\mathbf{M}^+(\mathbf{a}-\mathbf{p}^*)$ is positive as $\mathbf{M}^+\succeq\mathbf{0}$ and $\mathbf{a}-\mathbf{p}^*\succeq(\mathbf{a}-\mathbf{c})/2$. Moreover we can show that there is no asymmetric pricing equilibrium. Hence, the symmetric pricing equilibrium stated in Proposition 1 is unique.*

Proof of Proposition 2: With Assumption 3, which assumes that $c=0$ and $a_i=a$ for all i, the equilibrium prices are given by:

$$\mathbf{p}^*=\frac{1}{2}\mathbf{1}_N-\frac{(L-1)\beta}{2}[(2+(L-3)\beta)\mathbf{I}_N-2\delta\mathbf{G}]^{-1}\mathbf{1}_N,$$

by using the formula in the first row of (B4). Note that

$$[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1} = \frac{1}{(2+(L-3)\beta)}\sum_{j\geq 0}\left(\frac{2\delta}{2+(L-3)\beta}\right)^j \mathbf{G}^j.$$

When $\mathbf{G}' \succeq \mathbf{G}''$, and $\delta' \geq \delta''$, clearly,

$$\begin{aligned}
& [(2+(L-3)\beta)\mathbf{I}_N - 2\delta'\mathbf{G}']^{-1} \\
= \; & \frac{1}{(2+(L-3)\beta)}\sum_{j\geq 0}\left(\frac{2\delta'}{2+(L-3)\beta}\right)^j \mathbf{G}'^j \\
\succeq \; & \frac{1}{(2+(L-3)\beta)}\sum_{j\geq 0}\left(\frac{2\delta''}{2+(L-3)\beta}\right)^j \mathbf{G}''^j \\
= \; & [(2+(L-3)\beta)\mathbf{I}_N - 2\delta''\mathbf{G}'']^{-1}.
\end{aligned}$$

Consequently, $\mathbf{p}^*(\mathbf{G}', \delta') \preceq \mathbf{p}^*(\mathbf{G}'', \delta'')$. □

Proof of Proposition 3: With Assumption 3, which assumes that $c = 0$ and $a_i = a$ for all i, the equilibrium prices are given by:

$$\mathbf{p}^* = a[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1}_N.$$

by using the formula in the first row of (B4). We have

$$\frac{\partial \mathbf{p}^*}{\partial L} = -a\beta[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1}_N.$$

For case (i), we have $1-\beta-\delta d_i > 0$ for every i, so $\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1}_N \succeq \mathbf{0}$. Furthermore, $\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1} \succeq \mathbf{0}$ and $[(2+(L-3)\beta)\mathbf{I}_n - 2\delta\mathbf{G}]^{-1} \succeq \mathbf{0}$. Thus, $\frac{\partial \mathbf{p}^*}{\partial L} \preceq \mathbf{0}$.

For case (ii), for any node i with $1-\beta-\delta d_i < 0$, for large L, by equation (E16) in Proposition E4 in Appendix E, we have:

$$p_i^* = \frac{1-\beta-\delta d_i}{\beta L} + \mathcal{O}(L^{-2})$$

Clearly, $\partial p_i^*/\partial L > 0$ when L is sufficiently large (see (E17)). Moreover, the fact that $\partial p_i^*/\partial L < 0$ for small L follows from the observation that

$$\frac{\partial \mathbf{p}^*}{\partial L}|_{L=1} = -\frac{a\beta}{4}\underbrace{[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}\mathbf{1}}_{\succeq \mathbf{0}} \preceq \mathbf{0}.$$

This proves the results. $\square$

Proof of Proposition 4 : When $L = 1$, $p_i^* = p_j^* = \frac{a}{2}$, so the dispersion is zero. As $L \to \infty$, the equilibrium price p_i^* converges to zero for any $i \in \mathcal{N}$ by Proposition E4; again the dispersion is zero. Consequently, the maximal dispersion must occur when L takes some intermediate value l^*. Obviously, such l^* is strictly greater than one. $\square$

Proof of Remark 1 : The equilibrium price, by taking the Taylor series expansion of (5) with respect to δ, takes the following forms when δ is small:

$$\mathbf{p}^* = \frac{(1-\beta)\mathbf{a} + (1+(L-2)\beta)\mathbf{c}}{(2+(L-3)\beta)} - \delta\frac{(L-1)\beta}{(2+(L-3)\beta)^2}\mathbf{G}(\mathbf{a}-\mathbf{c}) + \mathcal{O}(\delta^2).$$

Under Assumptions 3, $p_i^* - p_j^* \approx \delta\frac{(L-1)\beta}{(2+(L-3)\beta)^2}(d_i - d_j)$ (we omit the higher-order terms of δ), implying $Disp(l) = \delta\frac{(L-1)\beta}{(2+(L-3)\beta)^2}(d_{max} - d_{min})$. The result about L^* just follows from the observation that:

$$\frac{(L-1)\beta}{(2+(L-3)\beta)^2} = \frac{(L-1)\beta}{((2-2\beta)+(L-1)\beta)^2} \leq \frac{(L-1)\beta}{4(2-2\beta)(L-1)\beta} = \frac{1}{8(1-\beta)}$$

Here we use a simple fact: for positive numbers a and b, $(a+b)^2 \geq 4ab$ with equality when $a = b$. $\square$

Proof of Proposition 5: Before proving this proposition, we state and prove the following Lemma:

Lemma B1 (Localization Lemma)**.** *Suppose $f(z)$ is an analytical function on an interval which contains $Spec(\mathbf{G})$, so that $f(\mathbf{G})$ is well-defined. Then, we have the following*

$$\mathbf{v}'f(G)\mathbf{v} = \sum_{\lambda_i \in Spec(\mathbf{G})} f(\lambda_i)(\mathbf{v}'\mathbf{u}_i)^2.$$

In particular, if $f(z)$ is positive (nonnegative) at any $\lambda_i \in Spec(\mathbf{G})$, then $\mathbf{v}'f(\mathbf{G})\mathbf{v} > (\geq)0$ for any $\mathbf{v} \in \mathbf{R}^n \backslash \{\mathbf{0}\}$.

Proof of Lemma B1: The first part of this Lemma is a direct application of the spectral theorem adapted to the symmetric matrix $\mathbf{G}$, so we omit the proof, which is standard. For the second part, when $f(\lambda_i) \geq 0$ for every λ_i, clearly $\mathbf{v}'f(G)\mathbf{v} \geq 0$. When $f(\lambda_i) > 0$, and $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v}'\mathbf{u}_i$ is non-zero for at least one i as these eigenvectors $\{\mathbf{u}_i\}$ form a basis of $\mathbf{R}^n$. As a result, $\mathbf{v}'f(G)\mathbf{v} > 0$. $\square$

We can now prove Proposition 5. We first show identity (9). In the proof of Proposition 1, we show that the equilibrium profit, which equals $\langle \mathbf{M}^{+}(\mathbf{a}-\mathbf{p}^*), \mathbf{p}^* \rangle$, can be rewritten as

$$\Pi^*(\mathbf{G}; \beta, \delta, L) = \langle \mathbf{a}, \Phi^{PT}(\mathbf{G})\mathbf{a} \rangle,$$

where

$$\Phi^{PT}(z; \beta, \delta, L) := \frac{(1+(L-2)\beta-\delta z)(1-\beta-\delta z)}{(1+(L-1)\beta-\delta z)(2+(L-3)\beta-2\delta z)^2}.$$

Note that when $z = d$, $\Phi^{PT}(d; \beta, \delta, L)$ exactly equals $\frac{1}{n} \times \Pi_{reg}(d; \delta, \beta, L)$ in (D11). The identity then just follows from Lemma B1. The rest of the proof follows from the discussion in the main text. □

Proof of Proposition 6: We assume $\mathbf{G}' \succ \mathbf{G}''$. When $L = 1$, $\Pi^*(\mathbf{G}; \beta, \delta, L = 1) = \frac{1}{4}\mathbf{a}'[\mathbf{I}_N - \delta\mathbf{G}]^{-1}\mathbf{a}$ is monotone in $\mathbf{G}$; in other words, $\Pi^*(\mathbf{G}'; \beta, \delta, L = 1) > \Pi^*(\mathbf{G}''; \beta, \delta, L = 1)$ as $\mathbf{G}' \succ \mathbf{G}''$. By continuity, we obtain part (i). Using the following tarylor series expansion of Φ^{PT} at $L = \infty$:

$$\Phi^{PT}(z; \beta, \delta, L) = \frac{(1-\beta-\delta z)}{\beta^2}\frac{1}{L^2} + \mathcal{O}(L^{-3}),$$

we obtain

$$\Pi^*(\mathbf{G}; \beta, \delta, L) = \langle \mathbf{a}, \frac{(1-\beta)\mathbf{I}_N - \delta\mathbf{G})}{\beta^2}\frac{1}{L^2}\mathbf{a} \rangle + \mathcal{O}(L^{-3}).$$

Therefore,

$$\Pi^*(\mathbf{G}'; \beta, \delta, L) - \Pi^*(\mathbf{G}''; \beta, \delta, L) = \underbrace{\langle \mathbf{a}, \frac{\delta(\mathbf{G}''-\mathbf{G}')}{\beta^2}\mathbf{a} \rangle}_{<0, \text{ as } \mathbf{G}' \succ \mathbf{G}''} \times \frac{1}{L^2} + \mathcal{O}(L^{-3}).$$

So, for sufficiently large L, $\Pi^*(\mathbf{G}'; \beta, \delta, L) < \Pi^*(\mathbf{G}''; \beta, \delta, L)$,which proves part (ii). □

C Consumption equilibrium

Given the price profile, each user i maximizes

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \sum_{l=1}^{L} p_i^l x_i^l = \sum_{l=1}^{L}(a_i - p_i^l)x_i^l - \frac{1}{2}\sum_{l=1}^{L}(x_i^l)^2 - \frac{1}{2}\sum_{l=1}^{L}\sum_{s\neq l}\beta x_i^s x_i^l + \delta \sum_{l=1}^{L}\left(\sum_{j=1}^{N} g_{ij} x_i^l x_j^l\right).$$

With the presence of prices, the marginal utility a_i is reduced exactly by p_i^t for each product t. The following Lemma regarding the consumption equilibrium directly follows from Theorem 3 of Chen et al. (2018b), after taking into account the prices.

Recall

$$\mathbf{M}^+ := [(1 + (L-1)\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}, \quad \mathbf{M}^- := [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}.$$

Lemma C1 (Consumption Equilibrium)**.** *In the consumption game:*

(i) Given any price profile $\mathbf{p}$, there exists a unique consumption equilibrium $\mathbf{x}(\mathbf{p}) = (\mathbf{x}^l(\mathbf{p}), l \in \mathcal{L})$.[1] *Furthermore, the demand vector of each firm l, $\mathbf{x}^l(\mathbf{p})$, is linear in $\mathbf{p}$ and given by:*

$$\mathbf{x}^l(\mathbf{p}) = \mathbf{M}^+\left(\mathbf{a} - \frac{\sum_{s\in\mathcal{L}}\mathbf{p}^s}{L}\right) - \mathbf{M}^-\left(\mathbf{p}^l - \frac{\sum_{s\in\mathcal{L}}\mathbf{p}^s}{L}\right). \tag{C5}$$

(ii) When every firm charges the same prices $\mathbf{p}$, the unique CE is symmetric, and, for any $t \in \mathcal{L}$, is equal to:

$$\mathbf{x}^l = \mathbf{x}^{sym}(\mathbf{p}) = \mathbf{M}^+(\mathbf{a} - \mathbf{p}) = [(1 + (L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}(\mathbf{a} - \mathbf{p}). \tag{C6}$$

Proof of Lemma C1: Part (i) of this Lemma directly follows from Theorem 3 of Chen et al. (2018b). Given the functional forms of user utilities, the system determined by the first-order conditions for the underlying consumption equilibrium is linear both in $\mathbf{x}$ and $\mathbf{p}$. Under Assumption 2, the system has a unique solution $\mathbf{x}(\mathbf{p})$ (i.e., the CE), which linearly changes with prices as given in (C5). Part (ii) directly follows from (i), as when $\mathbf{p}^t = \mathbf{p}$ for any $l \in \mathcal{L}$, we have $\mathbf{p}^t = \frac{\sum_{s\in\mathcal{L}}\mathbf{p}^s}{L} = \mathbf{p}$ for any l. □

To obtain some intuition, we first consider the average demands of firms by summing

[1] Here $\mathbf{x}^l = (x_1^l, \cdots, x_N^l)'$ is the demand vector of firm l (which is different from $\mathbf{x}_i$, the consumption bundle of user i). So $(\mathbf{x}^l, l \in \mathcal{L})$ and $(\mathbf{x}_i, i \in \mathcal{N})$ are two equivalent representations of $\mathbf{x}$.

up (C5) over L:

$$\frac{1}{L}\sum_{s\in\mathcal{L}}\mathbf{x}^s = \mathbf{M}^+\left(\mathbf{a} - \frac{1}{L}\sum_{s\in\mathcal{L}}\mathbf{p}^s\right). \tag{C7}$$

As a result, $\mathbf{M}^+$ measures the marginal reduction of the *average demands* of firms for marginal increment of *average prices*.[2] Similarly, the difference of demands between two different firms l and s, by taking the difference of (C5) for l and s, satisfies

$$\mathbf{x}^l - \mathbf{x}^s = -\mathbf{M}^-(\mathbf{p}^l - \mathbf{p}^s). \tag{C8}$$

In other words, $\mathbf{M}^-$ measures (the negative value of) the marginal reduction of the *demand difference* between two firms for marginal increment of *price differences* between these two firms.[3] Accordingly, two firms charging the same prices must obtain equal demands; this is consistent with Lemma C1 (ii). Since $\mathbf{M}^+$ and $\mathbf{M}^-$ measure the marginal reductions, we hereby call them the *sensitivity matrices*.

Next, we link these sensitivity matrices $\mathbf{M}^+$ and $\mathbf{M}^-$ to the underlying network structure and market structure. We introduce the following inverse Leontief matrix in the network literature

$$\mathbf{M}(\mathbf{G},\delta) = [\mathbf{I}_N - \delta\mathbf{G}]^{-1} = \mathbf{I}_N + \delta\mathbf{G} + \delta^2\mathbf{G}^2 + \cdots +,$$

where each entry m_{ij} of $\mathbf{M}$ represents the total number of walks from i to j in network $\mathbf{G}$ with each walk of length k discounted by δ^k (the infinite sum converges when $\delta < 1/\lambda_1(G)$, see, for instance, Ballester et al. (2006)). In fact, both matrices are proportional to the inverse Leontief matrix $\mathbf{M}$ with some adjustments. The discount factor δ is adjusted by a factor of $\frac{1}{1+(L-1)\beta}$ for $\mathbf{M}^+$ and a factor of $\frac{1}{1-\beta}$ for $\mathbf{M}^-$:[4]

$$\begin{aligned}
\mathbf{M}^+ &= \frac{1}{1+(L-1)\beta}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{1+(L-1)\beta}\right) = \frac{1}{1+(L-1)\beta}\sum_{j\geq 0}\left(\frac{\delta}{1+(L-1)\beta}\right)^j\mathbf{G}^j,\\
\mathbf{M}^- &= \frac{1}{(1-\beta)}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{1-\beta}\right) = \frac{1}{(1-\beta)}\sum_{j\geq 0}\left(\frac{\delta}{1-\beta}\right)^j\mathbf{G}^j.
\end{aligned}$$

Since $\beta \in [0,1)$, $\frac{1}{1+(L-1)\beta}$ is less than one, and it decreases with the number of firms L (and product differentiation parameter β). The latter implies that the network structure is less pronounced in shaping the sensitivity of the average demands with respect to prices, when there are more firms or when products are more homogeneous. However, $\frac{1}{1-\beta}$ is greater

[2] It also implies that $-\frac{\partial\{\mathbf{x}^1+\cdots+\mathbf{x}^L\}}{\partial\mathbf{p}^s} = [(1+(L-1)\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \mathbf{M}^+$.

[3] It also implies that $-\frac{\partial\{\mathbf{x}^l-\mathbf{x}^s\}}{\partial\mathbf{p}^l} = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \mathbf{M}^-$.

[4] Both $\mathbf{M}^+$ and $\mathbf{M}^-$ have the similar path-counting interpretations as $\mathbf{M}$. Both infinite sums converge by Assumption 2.

than one, which implies that competition among substitutable products amplifies the role of network structure in shaping the sensitivity of relative demand of firms with respect to price changes. Notably, the adjustment factor $1/(1-\beta)$ for $\mathbf{M}^-$ does not change with L.

In view of the interpretations in equations (C7)-(C8), we call $\mathbf{M}^+$ the *market expansion social multiplier*, and $\mathbf{M}^-$ the *business-stealing social multiplier*, as the former reflects the impact of network structure on the aggregate demand of all firms (see (C7) or (C6)), and the latter refers to the impact of network structure on the demand differences of two competing firms (see (C8)). Interestingly, because $\frac{1}{1+(L-1)\beta} \leq 1 \leq \frac{1}{1-\beta}$, the business-stealing social multiplier is stronger than the market expansion social multiplier, i.e.,[5] $\mathbf{M}^- \succeq \mathbf{M}^+$. Moreover, as the number of firm L increases, the market expansion effect of social network is diminished ($\mathbf{M}^+$ is weaker), while the business stealing effect of social network $\mathbf{M}^-$ remains the same.[6]

The following Corollary directly follows from Lemma C1 and our discussions above.

Corollary C1. *For $l \neq s \in \mathcal{L}$, we have:*

$$\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l} = -\frac{\mathbf{M}^+ + (L-1)\mathbf{M}^{-1}}{L} \preceq \mathbf{0}$$

and

$$\frac{\partial \mathbf{x}^s}{\partial \mathbf{p}^l} = -\frac{\mathbf{M}^+ - \mathbf{M}^{-1}}{L} \succeq \mathbf{0}.$$

Corollary C1 shows that linear combinations of $\mathbf{M}^+$ and $\mathbf{M}^-$ determine the sensitivity of a firm's demands with respect to prices charged by herself and her competitors. Since products are substitutable, raising firm l's prices reduces l's demands and increases competitor s's demands: $\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l} \preceq \mathbf{0}$ and $\frac{\partial \mathbf{x}^s}{\partial \mathbf{p}^l} \succeq \mathbf{0}$. Interestingly, the "own-price" sensitivity $-\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l}$ is a convex combination of $\mathbf{M}^+$ and $\mathbf{M}^-$ but with more weight on $\mathbf{M}^{-1}$ and, therefore, smaller weight on $\mathbf{M}^+$, for a larger number of firms.

[5]Formally, $\mathbf{M}^- - \mathbf{M}^+ = \mathbf{M}^- L\beta\mathbf{M}^+ \succeq \mathbf{0}$.
[6]Formally, $\partial\mathbf{M}^+/\partial L = -\mathbf{M}^+\beta\mathbf{M}^+ \preceq \mathbf{0}$ and $\partial\mathbf{M}^-/\partial L = \mathbf{0}$.

D Regular networks

In this Appendix, we focus on regular graphs.

Definition D1. *A network* $\mathbf{G}$ *is called regular with degree* d *if every node has the same degree* d*, i.e.,* $\sum_{j\in\mathcal{N}} g_{ij} = d, \forall i \in \mathcal{N}$*. Equivalently,* $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$*.*

The complete and the circle network are both regular with degree $(N-1)$ and 2, respectively.

D.1 Equilibrium prices

For a regular network, $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$. Using (B4), we obtain the following equilibrium prices:

$$\begin{aligned} \mathbf{p}^* &= a[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}[((1-\beta)\mathbf{I}_N - \delta\mathbf{G})\mathbf{1}_N \\ &= a\frac{(1-\beta-\delta d)}{2+(L-3)\beta-2\delta d}\mathbf{1}_N \\ &= a\frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}}\mathbf{1}_N. \end{aligned} \tag{D9}$$

Set $a=1$ in (D9). Then, we obtain the following common equilibrium price (mark-up) for regular networks with degree d:

$$p_i^* = p_j^* = p_{reg}^* := \frac{(1-\beta-\delta d)}{2+(L-3)\beta-2\delta d} = \frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}} \in (0, \frac{1}{2}]. \tag{D10}$$

Indeed, since the Katz-Bonacich centrality measures are the same for all nodes in a regular network, every user gets the same equilibrium price. Note that $\frac{1}{1+(L-1)\beta-\delta d}\times(1-p_{reg}^*)$ is each user's equilibrium demand while $\frac{1}{1+(L-1)\beta-\delta d}$ is the corresponding market expansion social multiplier. Indeed, for a regular network, $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$, and therefore

$$\mathbf{M}^+\mathbf{1}_N = \frac{1}{1+(L-1)\beta-\delta d}\mathbf{1}_N.$$

This term decreases with L and β, and increases with δ and d. For a regular network with degree d, each firm's equilibrium profit is then equal to:

$$\Pi_{reg}^* := \frac{n(1-p_{reg}^*)p_{reg}^*}{1+(L-1)\beta-\delta d}. \tag{D11}$$

D.2 The effects of market and network structure on equilibrium prices

By differentiating the equilibrium price in (D10), we easily obtain:[7]

$$\frac{\partial p^*_{reg}}{\partial \delta} < 0, \quad \frac{\partial p^*_{reg}}{\partial d} < 0, \quad \frac{\partial p^*_{reg}}{\partial L} < 0.$$

Thus, for a regular network with degree d, only Proposition 3 (i) occurs and prices decrease with L for all nodes, i.e., $\frac{\partial p^*_{reg}}{\partial L} < 0$. Moreover, by Proposition E4, equilibrium prices (markups in our setting) for a regular network always lie above zero.[8]

D.3 The effects of market and network structure on equilibrium profits

The equilibrium firm's profit is given by (D11), which is the product of $(1 - p^*_{reg})p^*_{reg}$ and $\frac{1}{1+(L-1)\beta-\delta d}$. We study the impact of the number of firms L on the firm's profit. First, note that the price effect of L is negative:

$$\frac{\partial\{(1 - p^*_{reg})p^*_{reg}\}}{\partial L} = (1 - 2p^*_{reg})\frac{\partial p^*_{reg}}{\partial L} < 0$$

as p^*_{reg} decreases with L and p^*_{reg} lies between 0 and 1/2 (see (D10)). Moreover, the social multiplier $\frac{1}{1+(L-1)\beta-\delta d}$ is weaker for larger L. Since both effects move in the same direction, we have:[9]

$$\frac{\partial \Pi^*_{reg}}{\partial L} < 0. \tag{D12}$$

Next, we characterize the impact of δ on the firm's profit. Similarly, the price effect of δ is negative:

$$\frac{\partial\{(1 - p^*_{reg}) \times p^*_{reg}\}}{\partial \delta} = (1 - 2p^*_{reg})\frac{\partial p^*_{reg}}{\partial \delta} < 0,$$

but the social multiplier effect is positive: $\frac{1}{1+(L-1)\beta-\delta d}$ is stronger with larger δ. Therefore, the net effect of δ on the firm's profit is determined by the battle of these two effects. The following Proposition characterizes the exact necessary and sufficient conditions under which one force dominates the other and the net effect can be determined.

[7] Indeed, since $\frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}}$ strictly decreases with δ, d and L, the results immediately follow.

[8] For a regular network with (common) degree d, $\lambda_1 = d$ and Assumption 2 reduces to $1 - \beta - \delta d > 0$. So Assumption 2 rules out case (ii) in Proposition 3.

[9] It is easy to show that Π^*_{reg} converges to zero as L goes to infinity.

Proposition D1. *Consider a regular network with degree d. The sign of* $\frac{\partial \Pi^*_{reg}}{\partial \delta}$ *is positive when L is sufficiently small and negative when L is sufficiently large. More precisely, there exists a threshold* $\bar{L}$ *(depending on* β *and* δ*) such that*

$$\frac{\partial \Pi^*_{reg}}{\partial \delta} > (<) 0 \textit{ if and only if } L < (>) \bar{L} := \chi\left(\frac{\beta}{1-\delta d}\right),$$

where $\chi(.)$ *is the function defined in Lemma D1 in Appendix D.3.*[10]

Before proving Proposition D1, we need to state and prove the following Lemma:

Lemma D1. *Define*

$$h(\beta, l) := 2 + 3(l-3)\beta - 6(l-2)\beta^2 - (l^3 - 2l^2 - 2l + 5)\beta^3. \tag{D13}$$

on the domain $\mathcal{O} = \{(\beta, l) \in R^2 | \beta \in [0,1], l \geq 1\}$*. There exists a continuously differentiable and strictly decreasing function* $\chi() : [0,1] \to [1, \infty)$ *of* β *with* $\chi(1) = 1$*,* $\lim_{\beta \to 0^+} \chi(\beta) = \infty$ *such that*

$$h(\beta, l) > (<) 0 \quad \textit{if and only if} \quad l < (>) \chi(\beta). \tag{D14}$$

Proof of Lemma D1: We complete the proof in several steps.

First, we show that for each $l \geq 1$, there exists a unique number $\beta^*(l) \in [0,1]$ with $g(\beta^*(l), l) = 0$. The existence of such a root β^* follows from Mean Value Theorem, as $g(0,l) = 2 > 0$, and $g(1,l) = -l(l-1)^2 \leq 0$. To show the uniqueness, we need to check the first and second derivatives of h:

$$h_\beta(\beta, l) = 3(l-3) - 12(l-2)\beta - 3(l^3 - 2l^2 - 2l + 5)\beta^2,$$

with $h_\beta(0, l) = 3(l-3)$, and

$$h_{\beta\beta}(\beta, l) = -12(l-2) - 6(l^3 - 2l^2 - 2l + 5)\beta.$$

Also note that for $l \geq 1$, the coefficient of β^3, $-(l^3 - 2l^2 - 2l + 5)$, of g is negative as the minimum value of $l^3 - 2l^2 - 2l + 5$ on $l \in [1, \infty)$ is about $0.73 > 0$ at $l^* \approx 1.72076$.

(i) When $l > 3$, $h_{\beta\beta}(\beta, l) < 0$ for any $\beta \in [0,1]$, so h is concave in β. Moreover, $h_\beta(0,l) = 3(l-3) > 0$, so h first increases , then decrease with β. Since $h(0,l) = 2 > 0$, h has a unique root on $[0,1]$.

(ii) When $2 \leq l \leq 3$, h_β is negative for any $\beta \in [0,1]$. Thus, h is strictly decreasing in β, which implies uniqueness.

[10] χ is continuously differentiable and strictly decreases with β. See the proof of Lemma D1. The graph of χ is plotted in Figure D1.

(iii) When $1 < l < 2$, recall that $-h_\beta/3 = ((l^3 - 2l^2 - 2l + 5)\beta^2 - 4(2-l)\beta + (3-l))$, since the leading coefficient $(l^3 - 2l^2 - 2l + 5) > 0$, and the discriminant

$$\mathcal{D} := (-4(2-l))^2 - 4(l^3 - 2l^2 - 2l + 5)(3-l) = 4(l-1)^2 \underbrace{(1 + l^2 - 3l)}_{<0,\text{ given } l\in(1,2)} < 0,$$

So $((l^3 - 2l^2 - 2l + 5)\beta^2 - 4(2-l)\beta + (3-l)) > 0$ for any $\beta \in [0,1]$, and therefore h is strictly decreasing in β. This implies uniqueness.

(iv) When $l = 1$, $h(\beta, 1) = 2(1-\beta)^3$, so the unique root is $\beta^*(1) = 1$.

The analysis above also shows that

$$g(\beta, l) > (< 0) \text{ if and only if } \beta < (>)\beta^*(l).$$

Clearly the unique $\beta^*(l)$ is continuously differentiable in l. Moreover $\beta^*(1) = 1$ and $\beta^* \to 0^+$ as $l \to \infty$.

Second, we show that for each fixed $\beta \in (0,1]$, there exists a unique $\chi^*(\beta) \geq 1$ such that $h(\beta, l^*(\beta)) = 0$. To this end, we write h as

$$h(\beta, l) = 2(1-\beta)^3 + 3\beta(1-\beta)^2(l-1) - \beta^3(l-1)^2 - \beta^3(l-1)^3.$$

Again the existence of such as $l^*(\beta)$ follows from mean value theorem, as $g(\beta, 1) = 2(1-\beta)^3 \geq 0$ and $g(\beta, \infty) = -\infty$. To show uniqueness, we check that

$$h_{ll}(\beta, l) = -2\beta^3 - 6\beta^3(l-1) < 0 \quad (\text{ as} l \geq 1)$$

and

$$h_l(\beta, 1) = 3\beta(1-\beta)^2 > 0.$$

Therefore, $h(\beta, l)$ is concave in l, and it is positive for small l, first increases with l, and then decreases with l. So the solution to $\{l : h(\beta, l) = 0\}$ is unique. Clearly, $\chi^*(\beta)$ is continuously differentiable in β. The analysis above also shows that

$$g(\beta, l) > (< 0) \text{ if and only if } l < (>)\chi^*(\beta).$$

Third, combining with both observations about $\beta^*(l)$ and $\chi^*(\beta)$, it is immediate that $\beta^*(l)$ is the inverse function of $\chi^*(\beta)$, and vice versa. In particular, both curves are injective, and hence monotone. In Figure D1, we plot the cutoff curve $\beta^*(\cdot)$, which is indeed decreasing with l. The curve $\chi(\cdot)$ is just the inverse function of such $\beta^*(\cdot)$, and hence it decreases with β.[11]

□

[11] For $l = 1, 2, \cdots, 10$, $\beta^*(l)$ is about 1.0, 0.596072, 0.45541, 0.369902, 0.311759, 0.269522, 0.237411,

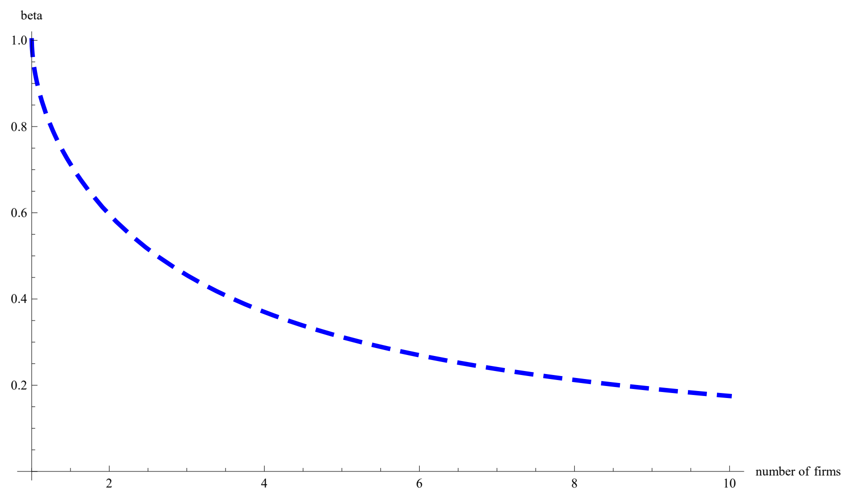

Figure D1: The cutoff curve $\beta^*(l)$: below (above) the curve, the function h in Lemma D1 is positive (negative).

Proof of Proposition D1: Direct differentiation shows that

$$\begin{aligned}\frac{\partial \Pi^*_{reg}}{\partial \delta} &= \frac{dn\left[(-(5-2L-2L^2+L^3)\beta^3 - 6(L-2)\beta^2(1-d\delta) + 3(L-3)\beta(1-d\delta)^2 + 2(1-d\delta)^3)\right]}{(1+(L-1)\beta-\delta d)^2(2+(L-3)\beta-2\delta d)^3} \\ &= \underbrace{\left\{\frac{dn(1-d\delta)^3}{(1+(L-1)\beta-\delta d)^2(2+(L-3)\beta-2\delta d)^3}\right\}}_{>0} h\left(\left(\frac{\beta}{1-\delta d}\right), L\right)\end{aligned}$$

where $h(\cdot,\cdot)$ is defined in (D13) in Lemma D1. Therefore, $\frac{\partial \Pi^*_{reg}}{\partial \delta}$ has the same sign as that of $h(\left(\frac{\beta}{1-\delta d}\right), L)$. The rest just follows from Lemma D1. □

The economic intuition behind Proposition D1 is very simple. When the number of firms L is relatively small (or β is very small), the equilibrium prices are very close to 1/2 (the monopoly price), and accordingly $1-2p^*_{reg}$ is close to zero. Consequently, the effect of $\frac{\partial\{(1-p^*_{reg})\times p^*_{reg}\}}{\partial \delta} = (1-2p^*_{reg})\frac{\partial p^*_{reg}}{\partial \delta}$ is not very strong compared to the social multiplier effect. As a result, the firm's profit increases with δ. On the contrary, when L is sufficiently large, competition is so intense that the equilibrium price is close to zero (see Proposition E4), and the price effect of δ dominates the social multiplier effect. This implies that the firm's profit decreases with δ. Moreover, the threshold $\chi\left(\frac{\beta}{1-\delta d}\right)$ is strictly above 1. Thus, for very small

0.212159, 0.191773, 0.174969, respectively. Conversely, for $\beta = 0.1, 0.2, \cdots, 1.0$, the threshold $\chi^*(\beta)$ is about 18.5628, 8.57185, 5.24991, 3.59808, 2.61803, 1.97933, 1.54419, 1.25, 1.07078, 1., respectively.

L, say $L = 1$, we have $\frac{\partial \Pi^*_{reg}}{\partial \delta} > 0$. Also, $\chi\left(\frac{\beta}{1-\delta d}\right)$ is finite by Lemma D1; eventually, $\frac{\partial \Pi^*_{reg}}{\partial \delta}$ is negative for sufficiently large L. Moreover, since $\chi(\cdot)$ is decreasing, the cutoff $\bar{L}$ is smaller with higher β, higher d and larger L.

Proposition D1 have several additional implications.

Corollary D2. *Suppose Assumption 3 holds. Consider a regular network with degree d, as a function of $\delta \in [0, \frac{1-\beta}{d})$.*[12] *Then,*

$$\Pi^*_{reg} \text{ is } \begin{cases} \text{monotonically increasing in } \delta, & \text{if} \quad L = 1; \\ \text{an inverted U-shaped curve of } \delta, & \text{if} \quad 1 < L < \chi(\beta); \\ \text{monotonically decreasing in } \delta, & \text{if} \quad L \geq \chi(\beta). \end{cases}$$

Proof of Corollary D2: By Proposition D1,

$$\frac{\partial \Pi^*_{reg}}{\partial \delta} > (<) 0 \text{ if and only if } L < (>) \chi\left(\frac{\beta}{1-\delta d}\right) \text{ if and only if } \frac{\beta}{1-\delta d} < (>) \chi^{-1}(L).$$

The rest follows immediately. □

We now study the impact of varying the degree d. Since $d\delta$ is a sufficient statistics of Π^*_{reg}, the next result immediately follows from Proposition D1.

Proposition D2. *Consider a regular network with degree d. Then, $Sign\{\partial \Pi^*_{reg}/\partial d\} = sign\{\partial \Pi^*_{reg}/\partial \delta\}$.*

Proof of Proposition D2: Note that Π^*_{reg} depends on d and δ only through the product $d\delta$. Therefore, $\partial \Pi^*_{reg}/\partial d = \frac{\delta}{d} \partial \Pi^*_{reg}/\partial \delta$. The rest follows immediately. □

D.4 Equilibrium market structure with free entry

Proposition D3. *Consider a regular network with degree d. The number of firms L^{FE*} in the free-entry equilibrium decreases (increases) with δ when the entry fee f is sufficiently small (large). More precisely, there exists a threshold $\bar{f}$ such that*

$$\frac{\partial L^{FE*}}{\partial \delta} > (<) 0 \textit{ if and only if } f > (<) \bar{f}.$$

This result is a direct consequence of Corollary D2. Note that $\bar{f} = \Pi^*_{reg}(d; \beta, \delta, \chi\left(\frac{\beta}{1-\delta d}\right))$.

[12]The upper bound on δ makes sure that Assumption 2 holds.

E Additional results

E.1 Perfect competition

Regardless of the network structure, the equilibrium price must go to zero in the perfect competition limit.

Proposition E4. *For large L, we have:*

$$\mathbf{p}^* = \frac{(1-\beta)\mathbf{1}_n - \delta \mathbf{G}\mathbf{1}_n}{\beta L} + \mathcal{O}(L^{-2}) \tag{E15}$$

or equivalently, for each $i \in \mathcal{N}$,

$$p_i^* = \frac{1-\beta-\delta d_i}{\beta L} + \mathcal{O}(L^{-2}) \tag{E16}$$

As $L \to \infty$, p_i^* converges to 0, for any $\mathbf{G}, \delta, \beta$.

Proof of Proposition E4: Using the Taylor expansion of the equilibrium prices (5) in Proposition 1 at $L = \infty$, we obtain the following result

$$\mathbf{p}^* = \frac{(1-\beta)\mathbf{1}_n - \delta \mathbf{G}\mathbf{1}_n}{\beta L} + \mathcal{O}(L^{-2}),$$

which proves (E16). In particular, as $L \to \infty$, clearly we have $\mathbf{p}^* \to \mathbf{0}$. □

Recall that we normalize the marginal cost to be zero, and prices here are mark-ups. Also, for large L,

$$\frac{\partial p_i^*}{\partial L} \approx -\frac{1-\beta-\delta d_i}{\beta L^2} \tag{E17}$$

E.2 User-optimal network structure

Let $\mathcal{G}$ denote the set of networks with n nodes: $\mathcal{G} = \{\mathbf{G} \in \{0,1\}^{n\times n} : g_{ii} = 0, g_{ij} = g_{ji}\}$. Let $\mathbf{K}_n$ denote the complete network, and $\emptyset$ denote the empty network.

Proposition E5. *Under any market structure, among networks in $\mathcal{G}$, the user-optimal network is the complete network.*

Proof of Proposition E5: By Lemma A1, each user's equilibrium utility is equal to: $\frac{L(1+(L-1)\beta)}{2}(x_i^*)^2$, for each i, while the equilibrium consumption $\mathbf{x}^* = (x_1^*, \cdots, x_n^*)$ is given by

$\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$. The rest just follows from the discussions in the main text.[13] □

E.3 Firm-optimal network structure

Proposition E6. *Fix the number of firms L. Then, the firm-optimal network is given by:*

(i) When $L < \chi(\beta)$, there exists $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$, the firm-optimal network is the complete network.

(ii) When $L > \chi(\beta)$, there exists $\bar{\bar{\delta}} > 0$ such that, for any $0 < \delta < \bar{\bar{\delta}}$, the firm-optimal network is the empty network.

Proof of Proposition E6: When δ is small, each firm's equilibrium profit is equal to:

$$\begin{aligned}\Pi^*(\mathbf{G}) &= \frac{(1+(L-2)\beta)(1-\beta)}{(1+(L-1)\beta)(2+(L-3)\beta)^2}\langle \mathbf{a}, \mathbf{a}\rangle \\ &\quad + \delta\frac{2+3(L-3)\beta-6(L-2)\beta^2-(L^3-2L^2-2L+5)\beta^3}{(1+(L-1)\beta)^2(2+(L-3)\beta)^3}\langle \mathbf{a}, \mathbf{G}\mathbf{a}\rangle + \mathcal{O}(\delta^2)\end{aligned}$$

which follows from the following Taylor series expansion of Φ^{PT} at $\delta = 0$:

Consider two networks $\mathbf{G}'$ and $\mathbf{G}''$,

$$\Pi^*(\mathbf{G}')-\Pi^*(\mathbf{G}'') = \delta\overbrace{\frac{2+3(L-3)\beta-6(L-2)\beta^2-(L^3-2L^2-2L+5)\beta^3}{\underbrace{(1+(L-1)\beta)^2(2+(L-3)\beta)^3}_{>0}}}^{:=h(\beta,L)}\langle \mathbf{a}, (\mathbf{G}'-\mathbf{G}'')(\mathbf{a})\rangle+\mathcal{O}(\delta^2)$$

Note that $\langle \mathbf{a}, (\mathbf{G}'-\mathbf{G}'')(\mathbf{a})\rangle = \sum_{i,j} a_i a_j (g'_{ij} - g''_{ij})$ is positive whenever $\mathbf{G}' \succ \mathbf{G}''$ (i.e., $g'_{ij} \geq g''_{ij}$ for any i, j, with strict inequality for some i, j). The sign of $h(\beta, L)$ is given by Lemma D1. When $L < \chi(\beta)$, $h(\beta, L)$ is positive, and therefore increasing network density generates a higher firm profit: $\Pi^*(\mathbf{G}') - \Pi^*(\mathbf{G}'') > 0$ for small δ, whenever $\mathbf{G}' \succ \mathbf{G}''$. This shows case (i). The proof of case (ii) is similar. □

[13] By the same argument, user surpluses increase with δ for any $\mathbf{G}$ and L.