DISCUSSION PAPER SERIES

DP14447

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Pasquale Schiraldi and Matthew Levy

INDUSTRIAL ORGANIZATION

ISSN 0265-8003

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Pasquale Schiraldi and Matthew Levy

Discussion Paper DP14447
Published 26 February 2020
Submitted 25 February 2020

Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

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Abstract

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JEL Classification: N/A

Keywords: dynamic discrete choice, identification, Time preferences, quasi-hyperbolic discounting

Pasquale Schiraldi - p.schiraldi@lse.ac.uk
LSE and CEPR

Matthew Levy - M.R.Levy@lse.ac.uk
LSE

Identification of intertemporal preferences in history-dependent dynamic discrete choice models

Matthew R. Levy* and Pasquale Schiraldi†

February 2020

Abstract

We study the identification of intertemporal preferences in a stationary dynamic discrete decision model. We propose a new approach which focuses on problems which are intrinsically dynamic: either there is endogenous variation in the choice set, or preferences depend directly on the history. History dependence links the choices of the decision-maker across periods in a more fundamental sense standard dynamic discrete choice models typically assume. We consider both exponential discounting as well as the quasi-hyperbolic discounting models of time preferences. We show that if the utility function or the choice set depends on the current states as well as the past choices and/or states, then time preferences are non-parametrically point-identified separately from the utility function under mild conditions on the data and we may also recover the instantaneous utility function without imposing any normalization on the utility across states.

*London School of Economics, m.r.levy@lse.ac.uk
†London School of Economics and CEPR, p.schiraldi@lse.ac.uk

Study the past if you would define the future.

– Confucius

1 Introduction

We study the identification of intertemporal preferences in a stationary dynamic discrete decision model. We propose a new approach which focuses on problems which are intrinsically dynamic: either there is endogenous variation in the choice-sets, governed by previous choices and previous realizations of the state, or preferences depend directly on the history. In this environment, the history-dependence feature links the choices of the decision-maker across periods in a more fundamental sense than standard dynamic discrete choice model typically assume. We consider both the common environment where agents exhibit exponential discounting as well the environment where agents are quasi-hyperbolic discounters.

Time preferences play an essential role in understanding inter-temporal economic behavior. However, a well-known characteristic of dynamic discrete-choice decision models is that the discount factor is not non-parametrically identified.[1] Most empirical papers have usually fixed the discount factor at an arbitrarily-chosen value, or imposed ad-hoc functional form restrictions.[2] We show that if the utility function or the choice set depends on the current states as well as the history of choices, then time preferences are in fact non-parametrically point-identified separately from the utility function. Moreover, we may also recover the instantaneous utility without imposing the standard, though arbitrary, normalization of the utility of one reference alternative across all states. Specifically, in the exponential discount environment, the discount factor and utility function are identified as long as one choice is not available in some states, even with arbitrarily small probability, provided this probability varies across histories.[3]

Rust (1994) showed that the discount factor in standard dynamic discrete choice models is generically not identified. As dynamic discrete-choice approaches have grown in empirical applications, a literature has developed to characterize the degree of under-identification and has proposed some solutions. Most relevant for this paper, Magnac and Thesmar (2002) and Abbring and Daljord (2019b) expanded Rust's non-identification results and proposed exclusion

[1] Komarova et al. (2018) discusses the identification of the time preference in a parametric context.

[2] See for example Hendel and Nevo (2006); Schiraldi (2011); Gowrisankaran and Rysman (2012); Lee (2013); Ho (2015); Shcherbakov (2016); Bayer et al. (2016); De Groote and Verboven (2019); Agarwal et al. (2019) and for the quasi-hyperbolic discount see for example Fang and Wang (2015); Chan (2017); Dalton et al. (2019)

[3] Whereas more requirement are needed in the quasi-hyperbolic discount case as discussed below.

restrictions on the utility or value function that lead to the identification of the standard discount factor. However, as pointed out by Abbring and Daljord (2019b), even these exclusion restrictions do not usually provide point identification but only set identification. In contrast, our setup achieves point identification using variation in the data rather than utility restrictions.

Why are time preferences not identified in this environment? One intuition lies in the fact that, given a sequence of observed states and choices, the typical approach would not distinguish between any permutations on the ordering, provided that the same state-choice pairs were maintained.[4] Indeed, the only value of having panel data at all is to estimate the state transition probabilities: the choices themselves are considered only in the cross-section. In a cross section, however, the discrete choice probabilities identify only the differences in the conditional value functions, which in turn are functions of the instantaneous utility and the time preferences, but these objects cannot be separately identified without further restrictions.

A typical approach to identification in the exponential discounting model adds exclusion restrictions on utility (conditional value function) across states, the presence of an absorbing choice, (e.g. Magnac and Thesmar, 2002; Abbring and Daljord, 2019b), or restricts attention to finite horizon model (e.g. Yao et al., 2012; Chung et al., 2014; Bajari et al., 2016; Chou, 2016), usually coupled with a strong normalization on the utility of the reference alternative. Some of these restrictions may be thought of as extreme forms of history-dependent preferences. For example, an absorbing choice imposes that after choosing the absorbing alternative k, the subsequent utility of any other choice is infinitely negative. In a finite-horizon model, the decision-maker's instantaneous utility set to zero in all histories after period T. The utility is thus immediately revealed by period-T choices, and the discount factor identified by explicitly calculating the expected continuation value in the period-$(T-1)$ choice. In the present paper, we formalize and generalize this intuition and show conditions under which allowing the instantaneous utility to depend on past choice either directly[5] or indirectly through the availability of different choice sets,[6] identification is achieved.

The key advantage of this approach is that the precise sequence of decisions is now used

[4]That is, a decision-maker choosing alternative j in state s at time t and then alternative j' in state s' at time $(t+1)$ has exactly the same implications for the utility model as a decision-maker choosing alternative j' in state s' at time t and then alternative j in state s at time $(t+1)$.

[5]See for example Eckstein and Wolpin (1989) and Keane and Wolpin (1997) where the history of choices enter in the instantaneous utility

[6]The choice set may be viewed as a form of exclusion restriction on utility, holding utility fixed when an alternative is available and setting it to negative infinity when unavailable. In contrast to the restrictions used in the literature, though, availability may be observed in the data.

longitudinally, and it is this variation which identifies the decision-makers' time preferences separately from the utility function. We show that the necessary (and sufficient in the exponential discounting model) condition for the identification of the time preference is that the history dependence should be of length equal to at least 2. A consequence is that dynamic discrete-choice models with simple switching costs, transaction costs, reference dependent preferences, etc. are often not identified as the typical application imposes history-dependence of length one.[7] This non-identification follows from the fact that in the standard Markovian world, the period-$(t+1)$ state is a sufficient condition for the expected period-$(t+1)$ continuation value conditional on the period-t state and choice. This holds equally true in both the case of no-history dependence as well as history dependence of length 1, and thus no additional identification is provided. In contrast, applications which allow the costs or preferences to depend more flexibly on the history are non-parametrically identified.

Without further restrictions, in the standard dynamic discrete-choice model discussed in the literature not only the discount factor, but also the utility is not identified nonparametrically without a priori restrictions such as prespecifying the utility of one reference choice in all states (see for example Rust, 1994; Magnac and Thesmar, 2002). Clearly such a restriction on the utility across states is not without loss of generality. Importantly, counterfactual choice probabilities, which are often the objects of interest in dynamic discrete-choice analysis, are generally not invariant to the choice of reference utility (see for example Aguirregabiria, 2010; Aguirregabiria and Suzuki, 2014; Arcidiacono and Miller, 2019; Norets and Tang, 2014; Kaloupsidi et al., 2019). In contrast, endogenous variation in the choice set allows us to recover not only the discount factor δ, but also the instantaneous utility for all alternative in all states.[8] That is, the reference alternative is no longer a free parameter but is pinned down state-by-state from data. Intuitively in a framework where past choices and states affect future utility through the availability of alternatives, the choice and therefore the utility of different alternatives are linked across states which breaks the indeterminacy above and allow us to identify the instantaneous utility without any restriction.

Endogenous choice sets may be thought of as a special case of history-dependent preferences more generally. In the case of history-dependent choice sets, the utility of an alternative is held constant across all histories except those in which it is unavailable, in which case it is

[7] See for example Schiraldi (2011) and Shcherbakov (2016)

[8] Formally, the utility is identified up to a constant, as a single normalization on the felicity is without loss of generality.

equivalent to an infinitely negative utility.[9] We show that our identification results extend to arbitrary history-dependent preferences under a generalization of these features. Namely, that there exist at least two alternatives for which the difference in utilities across the subset of histories involving just one another is either zero or observed. This is a common feature of many applications such as those involving switching costs or storable goods, which we show are identified provided the history-dependence is extended beyond a single lag.

Building on the history-dependent choice set approach, we extend the class of preferences beyond the exponential discounting model to consider dynamically inconsistent time preferences, namely the $\beta - \delta$ quasi-hyperbolic discounting model (Phelps and Pollak, 1968; Laibson, 1997; O'Donoghue and Rabin, 1999). An endogenous choice set is a common feature in most applications, but especially so in the context of hyperbolic discounting where the notion of pre-commitment has played a central role. Demand for commitment devices has been taken as evidence of sophisticated present-bias, e.g. DellaVigna and Malmendier (2006). A sophisticated agent may want to lock in her savings to avoid spending by future present-biased selves, that is, to restrict her future choice sets without receiving a current period pay-off. A history-dependent choice set is in general a form of partial commitment, while the presence of an absorbing choice is a form of full commitment. We show that a natural generalization of the absorbing choice to apply over a longer history, i.e. a form of delayed commitment, is able to separate the short-run from the long-run discount factor. We may also contrast our result with the existing literature in behavioural economics which has already made use of *exogenous* variation in the choice set, i.e. commitment contracts. The typical field experiment essentially offers only a single choice in a single period; our results show why this is insufficient and provide a blueprint for how experiments could be extended to solve this shortcoming. Related to our paper is Abbring et al. (2019), where the authors study the identification of the β and δ in a finite horizon model and show that the time preferences are set identified under some exclusion restrictions including the normalization of the utility function of a reference alternative to zero across states. Fang and Wang (2015) claim a proof of identification of the quasi-hyperbolic discount factors, however Abbring and Daljord (2019a) have highlighted that this proof in fact has no implications for identification.

The rest of the paper is organized as follows. Section 2 describes the general model. Section 3 provides our main identification results in the case of history-dependent choice sets. Section

[9] Note that as we allow the choice set to be stochastic conditional on a history and realized state, this interpretation requires an expansion of the state space.

4 provides results in the broader class of history-dependent preferences. Section 5 extends the results of section 3 to the quasi-hyperbolic discounting model. Section 6 extends the results to the case of unobserved heterogeneity. Section 7 concludes.

2 Set-up and Assumptions

We consider a decision-maker facing an infinite sequence of choices d_t over a finite discrete set of alternatives $d_t \in \mathcal{J} = \{0, \ldots, J\}$. The agent's utility from these choices is additively time-separable, with instantaneous flows of utility $\bar{u}_t$ exponentially discounted[10] by an unknown factor $\delta \in (0,1)$: $U(u_1, \ldots) = \sum_{t=1}^{\infty} \delta^t \bar{u}_t$.

The agent's consumption utility depends on state variables $(\tilde{s}_t, \varepsilon_t)$. As is common in the discrete choice literature, these include a random preference shock to the utility of each alternative, $\varepsilon_t \equiv \{\varepsilon_{0,t}, \ldots, \varepsilon_{J,t}\}$. The agent observes the realization of $(\tilde{s}_t, \varepsilon_t)$ at the beginning of each period prior to making their choice, but the econometrician only observes the subset of the state $\tilde{s}_t$. Without any restriction on the unobserved shocks, it is of course possible to rationalize any observed behavior by a series of appropriate shocks for any utility function. We therefore make the common assumption that the random utility component is stationary. We further assume independence in order to highlight the main features of the model, though this assumption may be easily relaxed and has been studied elsewhere.

Assumption 1 (Independent discrete shock)**.** *The unobservable state variable ε_t is i.i.d. distributed, with a cumulative distribution function $G(\varepsilon)$ that is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^J$.*

Notice that while we assume that the cumulative distribution function $G(\varepsilon)$ is known, existing methods show when it may non-parametrically identified.[11] We next assume that the decision-maker's utility is additively separable in the error term. This can be relaxed to an assumption of monotonicity (see, Matzkin, 2007, for an overview), but separability is a common assumption which greatly simplifies the exposition.

Assumption 2 (Additive separability)**.** *Instantaneous utility functions $u_j(\tilde{s}, \varepsilon)$ are given by:*

$$\forall j \in \mathcal{J}, \forall t \quad \bar{u}_{jt}(\tilde{s}, \varepsilon) = u_j(\tilde{s}) + \varepsilon_j \tag{1}$$

[10] We relax this assumption in Section 5.

[11] See for example Blevins (2014), Chen (2017), and Buchholz et al. (2019)

Moreover, as is common in the discrete-choice literature we also assume that the states have finite support:

Assumption 3 (Discrete support). *The support of s_t is finite: $s_t \in \mathcal{S} = \{s_1, \dots s_{|\mathcal{S}|}\}$ with $|\mathcal{S}| < \infty$*

Assumption 3 may of course be relaxed, though the benefit of a continuous state space often lies in recovering the preference shock nonparametrically rather than the utility function. As the shock is not our focus here, we focus on the simpler case of a finite state-space.

We now introduce our key innovation to make the problem dynamic in a more meaningful sense, by considering the role of different histories. We decompose $\tilde{s}_t$ into two components, $\tilde{s}_t = (s_t, h_t)$, where $s_t \in \mathcal{S}$ denotes observed state variables that enter into the instantaneous utility function in the standard way, and $h_t \in \mathcal{H}_t = (\mathcal{S} \times \mathcal{J})^H$ represents the history of lagged choices and states. We allow an arbitrary, albeit finite, history dependence where H denotes the length of this dependence.

We first describe the law of motion of the observed state variables. The history evolves deterministically given the current state and choice. Let $h_t^n = (\{s_{t-n}, d_{t-n}\}, \dots, \{s_{t-1}, d_{t-1}\})$ denote the length-n history in period-t. The evolution of the history is then, as usual, given by $h_{t+1} = (h_t^{H-1}, s_t, d_t) = h(h_t, s_t, d_t)$. We then make the following conditional independence assumption (see Rust, 1987) about the transition of the remaining state variables and the distribution of the shocks:

Assumption 4. *(Conditional Independence): The transition distribution of the states has the following factorization:*

$$\Pr(s_{t+1}, h_{t+1}, \varepsilon_{t+1} | s_t, h_t, \varepsilon_t, d_t) = \pi_{d_t}(s_{t+1}|s_t, h_t) \cdot G(\varepsilon_{t+1}) \cdot \mathbb{1}\{h_{t+1} = h(h_t, s_t, d_t)\} \qquad (2)$$

Suppose the agent faces in each period t a choice set $m_t \subseteq \mathcal{J}$, which depends on the current state as well as the current history where $u_k(s_t, h_t) = -\infty$ if $k \notin m_t$. The set of possible choice sets, $\mathcal{Q}$, is a subset of the power set of $\mathcal{J}$ excluding the empty set, and excluding any sets which do not include the default alternative 0.[12] Let $Q = |\mathcal{Q}|$. In fact, our identification results only require $Q = 2$ and this will be guaranteed simply if one choice is not always available with some probability which varies with the history.

[12] While we can allow the default alternative not to be always available, this seems a reasonable feature to have which fits most of the applications usually studied and greatly simplify the notation in the proofs.

Assumption 5. *The current choice set is observed, and distributed according to* $M(h_t, s)$: $\{\mathcal{S} \times \mathcal{J}\}^H \times S \to \Delta\mathcal{Q}$ *with* $|\mathcal{Q}| \geq 2$.

We assume individuals discount the future at rate δ and maximize the present discounted value of their lifetime utilities. Under the assumption above, the value function from the perspective of the beginning of the period can be expressed recursively as

$$V(s_t, h_t, m_t, \varepsilon_t) = \max_{j \in m_t}\{u_j(s_t, h_t) + \varepsilon_{jt} + \delta E[V(s_{t+1}, h_{t+1}, m_{t+1}, \varepsilon_{t+1})|j, s_t, h_t, m_t]\} \tag{3}$$

The ex-ante value function (or integrated value function), $\bar{V}(s, h, m)$ — the continuation value of being in state s, in history h, in choice set m — is obtained by integrating $V(s, h, m, \varepsilon)$ over ε:

$$\bar{V}(s_t, h_t, m_t) = \int V(s_t, h_t, m_t) dG(\varepsilon_t) \tag{4}$$

We may also consider the expectation over all possible choice sets:

$$\bar{V}(s_t, h_t) = \int \bar{V}(s_t, h_t, m) dM(m|s_t, h_t) \tag{5}$$

We now define the conditional value function $v_j(s_t, h_t, m_t)$ as the present discounted value (net of ε_t only) of choosing $j \in m_t$ where m_t is the realized choice set and behaving optimally from period $(t + 1)$ onwards:

$$v_j(s_t, h_t, m_t) \equiv \begin{cases} u_j(s_t, h_t) + \delta E_{s,m}\left[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|j_t, s_t, h_t, m_t\right] & \text{if } j \in m_t \\ -\infty & \text{if } j \notin m_t \end{cases} \tag{6}$$

To simplify the notation, we will hereafter always refer to the case where $j \in m_t$ unless it is explicitly stated otherwise. First, note that we can write the discrete choice probabilities in terms of the choice-specific value function as:

$$\Pr(d_t = j|s_t, h_t, m_t) = E\left[\mathbb{1}\left\{j = \arg\max_{k \in m_t} v_k(s_t, h_t, m_t, \varepsilon_{kt})|s_t, h_t, m_t\right\}\right]$$

Under Assumptions 1–4, there exists a one-to-one mapping from the conditional choice probabilities to differences in the choice-specific value function given the vector of states (see

Hotz and Miller (1993)):

$$\Delta v_j(s_t, h_t, m_t) = v_j(s_t, h_t, m_t) - v_0(s_t, h_t, m_t) = \bar{\varphi}_j\left(\Pr\left(d=0|s_t, h_t, m_t\right), \ldots, \Pr\left(d=J|s_t, h_t, , m_t\right)\right) \equiv \varphi_j(s_t, h_t, m_t) \tag{7}$$

Note that $\Pr(d = k|s_t, h_t, m_t) = 0$ for any $k \notin m_t$ in equation (2) above. However, as $\varphi_j(s_t, h_t, m_t) = \varphi_j(s_t, h_t, m_t')$ if j is an element of both choice sets (and infinitely negative if j is unavailable), we will suppress the m_t notation from this point and consider only cases where j is available.

Moreover, we can write the surplus function for any alternative k as follows:

$$\bar{V}(s, h, m) = \Psi(v_0(s, h, m), ..., v_J(s, h, m)) \tag{8}$$

which has the additivity property (Rust, 1994; Arcidiacono and Miller, 2011):

$$\begin{aligned}\Psi(v_0(s, h, m), ..., v_k(s, h, m), ..., v_J(s, h, m)) &= \Psi(\Delta v_0(s, h, m), ..., 0, ...\Delta v_J(s, h, m)) + v_k(s, h, m) \\ &= q_k(\tilde{s}, m) + v_k(\tilde{s}, m)\end{aligned} \tag{9}$$

for any alternative $k \in m$. Notice that (7) implies that $q_k(s, h, m)$ is a unique function of the choice probabilities $\vec{p}(s, h, m)$. We may therefore re-write equation (6) as:

$$v_j(s_t, h_t, m_t) = u_j(s_t, h_t, m_t) + \delta E_{s_{t+1}, m_{t+1}}\left[q_0(s_{t+1}, h_{t+1}, m_{t+1}) + v_0(s_{t+1}, h_{t+1}, m_{t+1})|j_t, s_t, h_t, m_t\right] \tag{10}$$

For the remainder of the paper, for the sake of expositional clarity, we focus on the case where the history of choices is a sufficient statistic for the complete history, i.e. $\mathcal{H} = \mathcal{J}^H$. All results extend to the general case, albeit with (significant) additional notational complexity.

3 History-Dependent Choice Sets

In this section, we focus our attention on the identification of the dynamic discrete choice model only allowing history-dependent in the choice set. Endogenous choice sets are a common feature of many economic problems, for example in the presence of an underlying budget constraint whereby the current choice of consumption determines which alternatives are affordable in the next period; the choice of available products at the supermarket depends on past realized states, such as the introduction on new products, and past choices, such as the decision of moving in a different location; the choice of utility-services depends on past realized states, such as the entry on new firms and products, and past choices, such as the length of the subscribed contract; the availability of cars may also depend on past introduction/replacement of different model as well as past purchases through the presence of transaction costs or budget constraint etc.

To simplify the analysis and to emphasis the role of endogenously variation in the choice set, we consider utility functions which are not themselves history-dependent, deferring their analysis to Section 4.

Given the history-dependent choice sets, the decision-maker's optimal strategy must state a distribution over choices in each state, in each history, for each realized choice set. Note that if the distribution of choice sets is history-dependent of degree H, then the choice probabilities conditional on a realized choice set are history-dependent of degree $(H-1)$, as the current utility is independent of the history and the continuation value depends on $(H-1)$ lagged choices and the current choice.

Thus the distribution of choice sets given history h_t and state s is given by $\boldsymbol{M}(h_t, s)$, a the $Q \times 1$ probability in $\Delta\mathcal{Q}$. Let

$$\boldsymbol{M}(h_t) = \begin{bmatrix} \boldsymbol{M}(h_t, s_1)' & & \\ & \ddots & \\ & & \boldsymbol{M}(h_t, s_S)' \end{bmatrix}$$

be the $S \times QS$ block-diagonal matrix giving the distribution of choice sets for all states in history h_t.

Let $\tilde{\boldsymbol{q}}_0(h^{H-1}, s) = \left[q_0(h^{H-1}, s, m_1), \ldots, q_0(h^{H-1}, s, m_Q)\right]'$ be the $Q \times 1$ vector of expected surpluses in state s and length-$(H-1)$ history h^{H-1} for all possible choice sets.

Finally, given the (ex-ante) value function in state s with history h, $\bar{V}(s,h)$, let $\boldsymbol{V}(h)$ denote the $S \times 1$ vector across all states. Similarly, define $\boldsymbol{\varphi}_j(h)$ as $\boldsymbol{\varphi}_j(s,h)$ stacked across all states.[13]

[13]Recall that we consider j only when it is available, and thus may omit m as an argument.

3.1 Identification of the discount factor

We first show that history-dependent choice sets lead to the identification of the discount factor under very mild conditions. Specifically, the identifying variation is how the choice probabilities for the same alternative in the same state differ across histories. In order to make such a comparison, there must be some alternative which is indeed available (with positive probability) in different histories.[14] Moreover, conditional on j being available, the only way these two histories may be informative is that they differ in their implications for the future distribution of chocie sets. Assumption 6 formalizes these two requirements.

Assumption 6. *Define $A_j(s) = \{h : j \in supp \bigcup_{m \in M(h,s)} m\}$. There exists $j \neq 0$, s, and $x, y \in A_j(s)$ such that $x^{H-1} \neq y^{H-1}$, $x^{H-2} = y^{H-2}$, and $M(h(x, j, s), \cdot) \neq M(h(y, j, s), \cdot)$.*

The set $A_j(s)$ consists of the set of all histories where alternative j is sometimes available. Note that this does not imply that they are all available at once, or with the same probability, or in all histories. We require that for at least one alternative there exist two histories, which differ only in period $t - (H - 1)$, and possibly also in period $t - H$, such that there is a state where the alternative is available with positive probability. One sufficient condition likely to be satisfied in many applications would be that the full choice set $\mathcal{J}$ is always available with some probability. Note that as $M(\cdot, s)$ is a point in the Q-simplex, it will generically be in the interior and assign positive probability to all choice sets, thus satisfying Assumption 6 for all j and for all s. More importantly, this assumption may be readily verified in the data.

Treating history-independent choice sets as the $H = 0$ case, one might imagine that moving to the $H = 1$ case may suffice. Under our timing assumptions, however, this provides too little variation as choices are only revealed conditional on a choice set, which renders the history irrelevant for $H = 1$. Instead, we show that under our assumptions, $H = 2$ is a necessary and sufficient condition to identify the discount factor.

Theorem 1. *Under assumptions 1–6, the discount factor δ is identified if and only if $H \geq 2$.*

Proof. If

[14]Consider, for instance, the case where $M(\cdot, j)$ is degenerate on $m_j = \{j\}$ for all j. This would violate our assumption by locking the decision-maker in to an infinite stream of the same alternative repeated forever. The discount factor would obviously not be separable from the felicity in this example.

The difference in choice-specific value functions between alternatives j and 0 in history (x, j) can now be written as:

$$\begin{aligned}\boldsymbol{\varphi}_j(x,j) = \boldsymbol{u_j} - \boldsymbol{u_0} + \delta\boldsymbol{\pi}_j\left(\boldsymbol{u_0} + \delta\boldsymbol{\pi}_0\boldsymbol{V}(j,0) + \boldsymbol{M}(j,j)\boldsymbol{q_0}(j)\right)\\ - \delta\boldsymbol{\pi}_0(\boldsymbol{u_0} + \delta\boldsymbol{\pi}_0\boldsymbol{V}(0,0) + \boldsymbol{M}(j,0)\boldsymbol{q_0}(0))\end{aligned} \tag{11}$$

Next, consider some other history (x, k). Equation (11) becomes:

$$\begin{aligned}\boldsymbol{\varphi}_j(x,k) = \boldsymbol{u_j} - \boldsymbol{u_0} + \delta\boldsymbol{\pi}_j\left(\boldsymbol{u_0} + \delta\boldsymbol{\pi}_0\boldsymbol{V}(j,0) + \boldsymbol{M}(k,j)\boldsymbol{q_0}(j)\right)\\ - \delta\boldsymbol{\pi}_0(\boldsymbol{u_0} + \delta\boldsymbol{\pi}_0\boldsymbol{V}(0,0) + \boldsymbol{M}(k,0)\boldsymbol{q_0}(0))\end{aligned} \tag{12}$$

Differencing these two equations yields:

$$\begin{aligned}\boldsymbol{\varphi}_j(x,j) - \boldsymbol{\varphi}_j(x,k) = \delta\left(\pi_j\left[\boldsymbol{M}(j,j)\boldsymbol{q}_0(j) - \boldsymbol{M}(k,j)\boldsymbol{q}_0(j)\right]\right)\\ -\pi_0\left[\boldsymbol{M}(j,0)\boldsymbol{q}_0(0) - \boldsymbol{M}(k,0)\boldsymbol{q}_0(0)\right])\end{aligned} \tag{13}$$

Equation (13) is a system of S equations in one unknown, namely δ. The system has a unique solution if either $\boldsymbol{M}(j,j) \neq \boldsymbol{M}(k,j)$ or $\boldsymbol{M}(j,0) \neq \boldsymbol{M}(k,0)$. Finally, as one can let $k = 0$, δ is identified if $|J| \geq 2$.

Only If

Suppose the decision-maker's true preferences are given by $\{u_j(s)\}_{j\in\mathcal{J}}$ and δ, and that $H = 1$.

Define $\tilde{u}_j(s_t) = u_j(s_t) + \delta E[V(s_{t+1}, j))|j, s_t]$, and let $\tilde{\delta} = 0$. Then $\tilde{v}_j(s) = v_j(s)$ for all j, s, which is independent both of the history and the realized choice set (provided j is available). Under assumptions 1–4 the decision-maker's choices are completely characterized by the choice-specific value functions $v_j(s, h)$. A decision-maker with preferences (u, δ) is therefore observationally equivalent to a decision-maker with preferences $(\tilde{u}, 0)$. To see this, note that if $H = 1$ then:

$$\begin{aligned}\tilde{u}_j(s_t, h_t) &= u_j(s_t, h_t) + \delta E[V(s_{t+1}, j)|s_t, j]\\ &= u_j(s_t, h'_t) + \delta E[V(s_{t+1}, j)|s_t, j] = \tilde{u}_j(s_t, h'_t)\\ &\equiv \tilde{u}_j(s_t)\end{aligned}$$

and thus a history-independent felicity $\tilde{u}_j(s)$ exists. If instead $H \geq 2$, the value function will differ between the second and third expressions above given the dependence of $M(\cdot,\cdot)$ on h_t and thus $\tilde{u}$ does not exist. As the preferences given by (δ, u) and $(0, \tilde{u})$ are observationally equivalent and both satisfy assumptions 1–4, the discount factor is not identified.

$\square$

In the proof of Theorem 1, we obtain more equations than unknowns, and it is therefore natural to consider what may be relaxed in the model. One compelling case is that of a state-dependent discount factor.[15] In addition to macroeconomic examples, health applications with variable mortality chances for instance naturally feature a stochastic discount factor, which is identified under the same condition as Theorem 1:

Corollary 1. *Under assumptions 1–6, the state-dependent discount factor $\delta(s)$ is identified if and only if $H \geq 2$ for all $s \in \mathcal{S}$*

Proof. The proof immediately follows from theorem 1 as we can write equation (13) for state s as follows

$$\begin{aligned}\boldsymbol{\varphi}_j(x,j,s) - \boldsymbol{\varphi}_j(x,k,s) = \delta(s)\,(\pi_j(s)\,[\boldsymbol{M}(j,j)\boldsymbol{q}_0(j) - \boldsymbol{M}(k,j)\boldsymbol{q}_0(j)] \\ -\pi_0(s)\,[\boldsymbol{M}(j,0)\boldsymbol{q}_0(0) - \boldsymbol{M}(k,0)\boldsymbol{q}_0(0)]) \qquad (14)\end{aligned}$$

$\delta(s)$ is therefore identified as $(\pi_j(s)[\boldsymbol{M}(j,j)\boldsymbol{q}_0(j) - \boldsymbol{M}(k,j)\boldsymbol{q}_0(j)] - \pi_0(s)[\boldsymbol{M}(j,0)\boldsymbol{q}_0(0) - \boldsymbol{M}(k,0)\boldsymbol{q}_0(0)]) \neq 0$. Repeating the analysis for all s identifies the vector $\boldsymbol{\delta}$. $\square$

3.2 Identification of utility

Rust (1994) and Magnac and Thesmar (2002) have established that the non-identifiability of instantaneous utility in standard dynamic discrete choice framework discussed in the literature. They show that utility can be only identify after imposing $|\mathcal{S}|$ restrictions, i.e. by setting the utility of a reference choice to 0. These restrictions are clearly made with loss of generality and affect counterfactual predictions[16]. This non-identification follows from the fact that in the

[15]Note, we consider the case in which the decision-maker at date t treats the unknown future discount factors as welfare-relevant, thus maximizing the expected discounted value for all possible discount factors. This contrasts with the case where the decision-maker has a dynamically inconsistent preference, such as the quasi-hyperbolic discounting model which we consider in Section 5.

[16]see e.g. Kaloupsidi et al. (2019)

standard Markovian world, the period-(t + 1) state is a sufficient condition for the expected period-(t + 1) continuation value conditional on the period-t state and choice. In contrast, we show below that in applications which allow the costs or preferences to depend more flexibly on the history are non-parametrically identified. Specifically, we show that in a world where past choices and states affect future utility through the availability of alternatives, the choice of different alternatives are linked across states which breaks the indeterminacy above and allow us to identify the instantaneous utility without any restriction (but for a single normalization of the reference instantaneous utility in one state which is without loss of generality).

It is worth to emphasis that both the discount factor and the utility function are identified under very weak conditions. In particular, as $|\mathcal{Q}| = 2$ this implies that the intertemporal preferences are identified as long as one choice is not available in some states, even with arbitrarily small probability, provided this probability varies across histories of length $H \geq 2$.

Theorem 2. *Under assumptions 1–6, utility is non-parametrically identified if $H \geq 2$.*

Proof. Assumptions 1–5 and $H \geq 2$ immediately imply that the discount factor δ is identified by Theorem 1. Given δ, it is possible to write the value function for history $(0,0)$ recursively, as:

$$V(0,0) = u_0 + M(0,0)\tilde{q}_0 + \delta\boldsymbol{\pi}_0 V(0,0) = (I - \delta\boldsymbol{\pi}_0)^{-1}\left[M(0,0)\tilde{\boldsymbol{q}}_0 + \boldsymbol{u_0}\right] \tag{15}$$

Moreover, in histories ending with 0, we have:

$$V(k,0) = u_0 + M(k,0)\tilde{q}_0 + \delta\boldsymbol{\pi}_0\,(I - \delta\boldsymbol{\pi}_0)^{-1}\left[M(0,0)\tilde{\boldsymbol{q}}_0 + \boldsymbol{u_0}\right] \tag{16}$$

Rather than writing the differences in choice-specific value functions in terms of a single period-$(t+1)$ alternative as we do in (11), which would subsume all information about utility into $\boldsymbol{q}_0$, we may instead consider two choices and the probabilities with which they are chosen under the optimal strategy, conditional on choosing one of those alternatives. That is, let $\boldsymbol{P}_{0j}(h^{H-1})$ be the $QS \times 2QS$ matrix (where $Q = 2$) formed by diagonally stacking the probabilities $p_0(h^{H-1}|s_t m_t d_t \in \{0,j\})$ and $p_j(h^{H-1}|s_t, m_t, d_t \in \{0,j\})$. Note that these weights are constrained by the probability that alternative j is available; this variation will be key in identifying the model. We thus rewrite (11) as:

$$\boldsymbol{\varphi}_j(x,k) = \boldsymbol{u_j} + \delta\boldsymbol{\pi}_j \left[\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\left(\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{q}}(j) + \delta\tilde{\boldsymbol{V}}(j)\right)\right] \tag{17}$$

$$- \delta\boldsymbol{\pi}_0(\boldsymbol{u_0} + \delta\boldsymbol{\pi}_0\boldsymbol{V}(0,0) + \boldsymbol{M}(j,0)\boldsymbol{q_0}(0)) \tag{18}$$

where $\tilde{\boldsymbol{V}}(j)$ is the expectation over period-$(t+2)$ continuation values in all states for both period-$(t+1)$ choices $\boldsymbol{V}(j, d_{t+1})$.[17]. By equations (16) and (15), $\boldsymbol{\pi}_j\boldsymbol{M}(k,j)\boldsymbol{P}(j)\tilde{\boldsymbol{V}}(j)$ may be written in terms of the value of consuming u_0 in periods $t+2$ and $t+3$, the surpluses in period $(t+2)$ and $(t+3)$, and $V(0,0)$ beginning in period $(t+4)$ all given the induced distributions of states, choice sets, and period $t+2$ and $t+3$ histories.

The period-$(t+2)$ surpluses depend on the joint distribution of period-$(t+2)$ states, choice sets, and histories induced by the choice probabilities in period-$(t+1)$. We collect these terms as $\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j)\hat{\boldsymbol{q}}_{\boldsymbol{0}}^{t+2}$.[18] Similarly, the period-$(t+3)$ surpluses may be collected as $\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\check{\boldsymbol{\pi}}\check{\boldsymbol{M}}(j)\check{\boldsymbol{q}}_{\boldsymbol{0}}^{t+3}$. The period-$t+4$ surpluses onwards all follow a history $(0,0)$ and thus may be written in terms of the induced distribution of s_{t+4}, which can in turn be written as the induced $t+2$ distribution iterated forward two periods by $\boldsymbol{\pi}_0$. That is, as: $\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_s^{2QS}\boldsymbol{\pi}_{\boldsymbol{0}}^2(I-\delta\boldsymbol{\pi}_0)^{-1}\left[\boldsymbol{M}(0,0)\boldsymbol{q}_0(0)\right]$.

The flow utilities received in period-$(t+1)$ depend on the period-$(t+1)$ states, choice sets, and choices. We can collect these terms as $\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}(j)\boldsymbol{\chi}\boldsymbol{u}$, where, importantly, $\boldsymbol{u} = \begin{bmatrix}\boldsymbol{u}_0\\ \boldsymbol{u}_j\end{bmatrix}$ is the unknown vector of utilities of these two choices across all states and $\boldsymbol{\chi}$ is a permutation matrix which expands $\boldsymbol{u}$ to be conformable.

Beginning in period-$(t+2)$, the value function is again written in terms of u_0. We thus require the conditional distribution of period-$(t+2)$ states, which we have defined above. The utilities beginning in period-$(t+2)$ onwards may thus be collected as $\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{2QS}(I-\delta\boldsymbol{\pi}_{\boldsymbol{0}})^{-1}\boldsymbol{Z}_0\boldsymbol{u}$ where $\boldsymbol{Z}_0$ is a $S\times 2S$ matrix consisting of an identity matrix followed by zeros such that $\boldsymbol{Z}_j\boldsymbol{u} = \boldsymbol{u}_0$; similarly, define $\boldsymbol{Z}_j$ such that $\boldsymbol{Z}_j\boldsymbol{u} = \boldsymbol{u}_j$.

With this notation, equation (17) may now be written as:

[17] Specifically, let $\boldsymbol{V}(j,s) = \begin{bmatrix}\pi_0(s)'\boldsymbol{V}(j,0)\\ \pi_j(s)'\boldsymbol{V}(j,j)\end{bmatrix}$ be the 2×1 vector yielding the expectation of the value function following each choice, taken from state s in history $(\cdot, j)$.

Let $\tilde{\boldsymbol{V}}(j,s) = [\boldsymbol{V}(j,s)', \ldots, \boldsymbol{V}(j,s)']'$ be the $2Q\times 1$ vector formed repeating $\boldsymbol{V}(j,s)$ Q times.

Then $\tilde{\boldsymbol{V}}(j) = \left[\tilde{\boldsymbol{V}}(j,s_1)', \ldots, \tilde{\boldsymbol{V}}(j,s_S)'\right]'$ be the $2QS\times 1$ vector formed by stacking $\tilde{\boldsymbol{V}}(k,s)$ across all states.

[18] Full definitions of these matrices are relegated to the appendix to preserve the sanity of readers.

$$
\begin{aligned}
\boldsymbol{\varphi}_j(x,k) &= \boldsymbol{u_j} + \delta\boldsymbol{\pi}_j\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\tilde{\boldsymbol{q}}(j) \\
&+ \delta^2\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j)\hat{\boldsymbol{q}}_{\mathbf{0}}^{t+2} + \delta^3\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\check{\boldsymbol{\pi}}\check{\boldsymbol{M}}(j)\check{\boldsymbol{q}}_{\mathbf{0}}^{t+3} \\
&+ \delta^4\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_s^{2QS}\boldsymbol{\pi}_{\mathbf{0}}^2(I-\delta\boldsymbol{\pi}_0)^{-1}\left[\boldsymbol{M}(0,0)\boldsymbol{q}_0(0)\right] \\
&+ \left[\delta\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\boldsymbol{\chi} + \delta^2\boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{2QS}(I-\delta\boldsymbol{\pi}_{\mathbf{0}})^{-1}\boldsymbol{Z}_0\right]\boldsymbol{u} \\
&- \boldsymbol{u}_0 - \delta\boldsymbol{\pi}_{\mathbf{0}}(\boldsymbol{u}_0 + M(k,0)\boldsymbol{q}_{\mathbf{0}}(0) + \delta(I-\delta\boldsymbol{\pi}_{\mathbf{0}})^{-1}(\boldsymbol{u}_0 + \boldsymbol{M}(0,0)\boldsymbol{q}_0(0)) \qquad (19) \\
&= \boldsymbol{\theta}_j(k,l) + \left[\boldsymbol{Z}_j - (I-\delta\boldsymbol{\pi}_{\mathbf{0}})^{-1}\boldsymbol{Z}_0 + \boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\boldsymbol{\xi}\right]\boldsymbol{u} \qquad (20)
\end{aligned}
$$

where equation (20) simplifies notation by using $\boldsymbol{\theta}_j(k,l)$ to collect the $\boldsymbol{q}$ terms, and $\boldsymbol{\xi} = \delta\boldsymbol{\chi} + \delta^2\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{2QS}(I-\delta\boldsymbol{\pi}_0)^{-1}\boldsymbol{Z}_0$ simplifies the terms multiplying $\boldsymbol{u}$.

Next, note that as (20) has row-rank S, the system is not invertible. We may, however, stack across an additional history $l \neq k$, as both j and 0 are known to sometimes be available in at least two histories by Assumption 6, to obtain a total of $2S$ equations:

$$
\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\varphi}_j(x,k) - \boldsymbol{\theta}_j(k,0) \\ \boldsymbol{\varphi}_j(x,l) - \boldsymbol{\theta}_j(l,0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_j - (I-\delta\boldsymbol{\pi}_{\mathbf{0}})^{-1}\boldsymbol{Z}_0 + \boldsymbol{\pi_j}\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\boldsymbol{\xi} \\ \boldsymbol{Z}_j - (I-\delta\boldsymbol{\pi}_{\mathbf{0}})^{-1}\boldsymbol{Z}_0 + \boldsymbol{\pi_j}\boldsymbol{M}(l,j)\boldsymbol{P}_{0j}(j)\boldsymbol{\xi} \end{bmatrix}\boldsymbol{u} = \boldsymbol{\lambda}\boldsymbol{u} \qquad (21)
$$

Note that the system defined by (20) is homogeneous of degree zero in $\boldsymbol{u}$[19], or equivalently $\boldsymbol{\lambda}$ is only column-rank $(2S-1)$. We may at this point normalize a single value of $\boldsymbol{u}$ without loss of generality, say, $u_0(s_1) = 0$. Let $\boldsymbol{\lambda_A}$ be the $2S+1 \times 2S$ matrix formed by adding a row $[0,\ldots,0,1]$ to $\boldsymbol{\lambda}$, which now has full column rank and is thus left-invertible. We obtain:

$$
\begin{bmatrix} \boldsymbol{u_0} \\ \boldsymbol{u_j} \end{bmatrix} = \left[\left(\boldsymbol{\lambda}_{\boldsymbol{A}}^T\boldsymbol{\lambda_A}\right)^{-1}\right]\boldsymbol{\lambda}_{\boldsymbol{A}}^T \begin{bmatrix} \boldsymbol{\varphi}_j(x,k) - \boldsymbol{\theta}_j(k,0) \\ \boldsymbol{\varphi}_j(x,l) - \boldsymbol{\theta}_j(l,0) \\ 0 \end{bmatrix}
$$

Once $\boldsymbol{u_0}$ has been recovered, the utility of any remaining alternative may be recovered in any history by evaluating equation (11) directly.

$\square$

[19] To see this, note that the columns of $\boldsymbol{Z}_j$ and $\boldsymbol{Z}_0$ sum to one, as do the columns of $\boldsymbol{\pi_j}, \boldsymbol{M}(k,j), \boldsymbol{P}_{0j}, \boldsymbol{\chi}$, and $\hat{\boldsymbol{\pi}}$. The columns of $\boldsymbol{\pi}_j\boldsymbol{M}(k,j)\boldsymbol{P}_{0j}(j)\boldsymbol{\xi}$ therefore sum to $\delta/(1-\delta)$, and the columns of $\boldsymbol{\lambda}$ thus sum to zero. This equates to the standard result that utility is identified only up to a constant.

4 History-dependent preferences

History-dependent choice sets may be thought of as a special case of history-dependent preferences, wherein utilities are held constant across a subset of histories (i.e. those in which an alternative is "available") and infinitely negative in the complementary histories (i.e. those when it is "unavailable"). In this section, we consider the case in which the decision-maker's felicity, conditional on the state, depends directly on the history of choices in an arbitrary way.

Assumption 7 (History-dependent preference)**.** *There exists some j and histories h, h' such that $u_j(s, h) \neq u_j(s, h')$ for some s.*

Assumption 7 implies that the decision-maker's instantaneous consumption utility depends directly on both the current state and the history . We allow for arbitrary history-dependence to an arbitrary length $H < \infty$. This assumption encompasses and generalizes several economically intuitive cases, including:

Switching costs With additive switching costs, $u_j(s_t, h_t) = u_j^*(s_t) - c_j(s_t, h_t)$ for some state-dependent cost function c_j. Typically it is assumed that for all j, $c_j(s_t, h_t) = 0$ if $d_{t-1} = j$. See for example Ho (2015); Shcherbakov (2016).

Habit formation and addiction The addiction model of Becker and Murphy (1988) imposes adjacent complementarity in consumption of an addictive good. Assume that $\mathcal{J}$ is ordered such that $j < k$ means that k represents greater consumption of the good. Then adjacent complementarity implies that the marginal utility is increasing in lagged consumption, i.e. that $u_k(s_t, j) - u_j(s_t, j) < u_k(s_t, k) - u_j(s_t, k)$.

Reference-dependent preferences Models of reference-dependent utility explicitly impose history-dependence on the instantaneous utility. For example, Kaheman and Tversky's (1979) prospect theory would imply that that the decision-maker evaluates alternatives according to differences relative to a reference point. Letting the reference point be status quo, i.e. d_{t-1} and letting alternatives represent levels of consumption, this implies $u_j(s_t, k) = u^*(s_t, j - k)$. See Gentry and Pesendorfer (2018) for an empirical application.

Finite horizon models Perhaps the strongest form of history-dependence occurs when the environment has a finite decision horizon of T periods. In this case, the decision-maker's instantaneous utility is immediately revealed by period-T choices, and the discount factor identified by explicitly calculating the expected continuation value in the period-$(T-1)$ choice. Such models may be though of as imposing history-dependence in an infinite-horizon model by setting utility to zero after period-T. Denoting a "null" alternative $\emptyset$ to allow histories before the problem has "begun", this implies : $u_j(s_t, h_t) = 0$ for all j if $d_{t-\tau} \neq \emptyset$ for any $\tau \geq T$. See for example Bajari et al. (2016) and the literature cited therein.

Storable/durable goods In models of storable goods with unit consumption, the decision-maker's utility depends on the stock (for storable goods, see for example Hendel and Nevo (2006)) or the value (possibly net of transaction costs for durable goods, for example Schiraldi (2011)) of goods on hand and the current-period purchase decision or the utility.

Given a history-dependent preference, the choice-specific value function v_j is given by:

$$v_j(\tilde{s}_t) = u_j(s_t, h_t) + \delta E_{\tilde{s}_{t+1}} \left[u_k(\tilde{s}_{t+1}) + q_k(\tilde{s}_{t+1}) + \delta E_{\tilde{s}_{t+2}} \left[V(h_{t+2}) | k, h_{t+1} \right] | j, \tilde{s}_t \right] \tag{22}$$

Differencing 22 for alternatives j and 0 in state $\tilde{s}_t = (s_t, h_t)$ yields:

$$\begin{aligned} \boldsymbol{\varphi}_j(s_t, h_t) = u_j(\tilde{s}_t) - u_0(\tilde{s}_t) + \delta E_{\tilde{s}_{t+1}} \left[u_k(\tilde{s}_{t+1}) + q_k(\tilde{s}_{t+1}) + \delta E_{\tilde{s}_{t+2}} \left[V(h_{t+2}) | k, h_{t+1} \right] | j, s_t, h_t \right] \\ - \delta E_{\tilde{s}_{t+1}} \left[u_k(\tilde{s}_{t+1}) + q_k(\tilde{s}_{t+1}) + \delta E_{\tilde{s}_{t+2}} \left[V(h_{t+2}) | k, h_{t+1} \right] | 0, s_t, h_t \right] \end{aligned} \tag{23}$$

Without further restrictions, equation (22) admits too many degrees of freedom to identify a dynamic choice model. In contrast with the history-dependent choice set model of Section 3, now not only the current utility but also the future utilities and value functions vary with the history. Rather than imposing the history-invariant utility assumed in the case of history-dependent choice sets, we may impose a much weaker assumption that will play a similar role in identifying the model. Namely, that this history-invariance applies only to a single alternative, and for a restricted subset of histories:

Assumption 8 (Limited history independence)**.** *Define $\mathcal{I}_{mn}$ such that if $j \in \mathcal{I}_{mn}$, then for all $h, h' \in \{m, n\}^H$ and for all s, $u_j(s, h) = u_j(s, h')$. Then there exists j such that $j \in \mathcal{I}_{jk} \subset \mathcal{J}$.*

Assumption 8 requires that there is at least one alternative for which the decision-maker's utility is history-independent over histories made up of itself and some other alternative. For example, if $H = 1$, then $u_j(s, j) = u_j(s, k)$ for some $k \neq j$. For example, in a durable goods application j could be no consumption and k an alternative which fully depreciates in one period, while other alternatives depreciate over longer horizons. A much stronger, but perhaps easier to interpret, assumption would be that j is globally history-independent across all histories, and perhaps even across states. This is far stronger than we will need to assume, but nevertheless may be a very natural assumption in some applications in particular with respect to the reference alternative 0. Finally, we note that the subset in assumption 8 is strict, as otherwise it could be satisfied trivially by all alternatives being globally history-independent, which would then violate assumption 7.

Just as assumption 8 relaxes assumption that the utility of an alternative is constant whenever it is available, we next generalize the analogous assumption regarding the utility difference when the alternative is "unavailable". Rather than assuming that the difference is known to be negatively infinite, we allow this difference to vary arbitrarily across histories, but instead require that it is known for one alternative in one history.

Assumption 9 (Normalization of history dependence)**.** *$u_j(s,h) - u_j(s,h')$ is known for some $s \in \mathcal{S}$, $h, h' \in \{j,k\}^H$ and $k \in \mathcal{I}_{jk}$*

Assumption 9 states that for some alternative j, for which there exists a k which satisfies assumption 8 with respect to j, the difference in utilities across two histories involving j and k is known. Note that this difference may be zero. We state the assumption in this form as it is the necessary condition for Theorem 3, but in fact most economic applications will imply something much stronger. For example, if there are two globally history-independent alternatives alongside some history-dependent ones, then assumptions 8 and 9 are implied. We relax this by only requiring a condition on certain histories, and indeed allowing the utility to vary by a known amount rather than zero. This latter requirement may be satisfied, for example, by switching cost models in which at least some switching costs are either observed or known to be zero.

Theorem 3. *Under assumptions 1–4 and 7–9 the discount factor δ is identified if and only if $H \geq 2$*

Proof. If

We present the $H = 2$ case for simplicity. The proof extends naturally for $H > 2$ by iterating the expectation in (22) to include $H-1$ flow utilities and surpluses and end with the expectation of $V(s_{t+H}, d)$. By Assumption 8, let $j, 0 \in \mathcal{I}^s_{j0}$ for all s.

Differencing (22) for alternatives j and 0 in state $\tilde{s}_t = (s_t, (j, j))$ yields:

$$\begin{aligned}\boldsymbol{\varphi}_j(s_t, (j, j)) = u_j(s_t, (j, j)) - u_0(s_t, (j, j)) \\ + \delta E_{s_{t+1}} \left[u_j(s_{t+1}, (j, j)) + q_j(s_{t+1}, (j, j)) + \delta E_s \left[V(s, (j, j)) | j, s_{t+1}, h_{t+1} \right] | j, s_t \right] \\ - \delta E_{s_{t+1}} \left[u_j(s_{t+1}, (j, 0)) + q_j(s_{t+1}, (j, 0)) + \delta E_s \left[V(s, (0, j)) | j, s_{t+1}, h_{t+1} \right] | 0, s_t \right]\end{aligned} \tag{24}$$

where we set the arbitrary period-$(t + 1)$ choice equal to j. Evaluating this same equation in the same state but history $(j, 0)$ yields:

$$\begin{aligned}\boldsymbol{\varphi}_j(s_t, (j, 0)) = u_j(s_t, (j, 0)) - u_0(s_t, (j, 0)) \\ + \delta E_{s_{t+1}} \left[u_j(s_{t+1}, (0, j)) + q_j(s_{t+1}, (0, j)) + \delta E_s \left[V(s, (j, j)) | j, s_{t+1}, h_{t+1} \right] | j, s_t \right] \\ - \delta E_{s_{t+1}} \left[u_j(s_{t+1}, (0, 0)) + q_j(s_{t+1}, (0, 0)) + \delta E_s \left[V(s, (0, j)) | j, s_{t+1}, h_{t+1} \right] | 0, s_t \right]\end{aligned} \tag{25}$$

We then take the difference between (24) and (25) to obtain:

$$\begin{aligned}\boldsymbol{\varphi}_j(s_t, (j, j)) - \boldsymbol{\varphi}_j(s_t, (j, 0))) = \kappa + \delta \big[E_{s_{t+1}} \left[q_j(s_{t+1}, (j, j)) - q_j(s_{t+1}, (0, j)) | j, s_t \right] \\ - E_{s_{t+1}} \left[q_j(s_{t+1}, (j, 0)) - q_j(s_{t+1}, (0, 0)) | 0, s_t \right] \big]\end{aligned} \tag{26}$$

Under assumption 9, the constant κ is known. Equation (26) is thus a linear equation in one unknown, namely δ. Moreover, as both the distribution of states s_{t+1} differs across j and 0 and the surpluses differ across length-2 histories, the term in brackets on the right-hand side will generically be nonzero. There is therefore a unique value of δ which satisfies this equation, providing identification.

Only If

Suppose the decision-maker's true preferences are given by $\{u_j(s, h)\}_{s \in \mathcal{S}, h \in \mathcal{J}^H}$ and δ, and that $H = 1$. Under assumption 6, there exists at least one alternative k that yields history-independent utility over histories involving itself and some other j. Without loss, let $k = 0$.

Next define $\tilde{u}_j(s_t, h_t) = u_j(s_t, h_t) + \delta E[V(s_{t+1}, h_{t+1}) | j, s_t, h_t]$, and let $\tilde{\delta} = 0$. Then $\tilde{v}_j(s, h) = v_j(s, h)$ for all j, s, h. Under assumptions 1–7 the decision-maker's choices are completely characterized by the choice-specific value functions $v_j(s, h)$. A decision-maker with preferences (u, δ) is therefore observationally equivalent to a decision-maker with preferences $(\tilde{u}, 0)$.

Next, note that if $H = 1$ then:

$$\begin{aligned}\tilde{u}_0(s,0) &= u_0(s,0) + \delta E[V(s_{t+1},0)|s,0] \\ &= u_0(s,j) + \delta E[V(s_{t+1},0)|s,0] = \tilde{u}_0(s,j)\end{aligned}$$

where the second inequality follows by the history-independence of alternative 0 over histories involving itself and j. More generally, if $u_0(s,h)$ is history-independent over some set of histories, $\tilde{u}_0(s,h)$ will also be history-independent of the same set. This clearly does not hold if $H \geq 2$, as the value function will differ between the second and third expressions above given the presence of alternatives whose utility does vary across these histories. The preference $\tilde{u}$ therefore also satisfies assumption 8. As the two models are observationally equivalent and both satisfy assumptions 1–6, the discount factor is not identified.

□

Corollary 2. *Under assumptions 1–4 and 7–9 the state-dependent discount factor $\delta(s)$ is identified in any state for which assumption 9 holds if $H \geq 2$.*

Unlike the history-dependent choice set case studied in the previous section, we cannot identify the utility function without further restrictions (e.g. $u_0(s) = 0$ for all s) when it exhibits arbitrary history-dependence. Going across histories increases the number of equations but also increases the number of unknowns commensurately. We could, however, add a history-dependent choice set to provide additional identification — particularly under an assumption that the length of history-dependence in the choice set exceeded that in the utility function.

5 $\beta - \delta$ discounting

We now extend our results beyond the simple exponential discounting model, and assume that the agent's intertemporal preferences are represented by the now commonly used formulation of time-inconsistent preferences: quasi-hyperbolic discounting (Phelps and Pollak, 1968; Laibson, 1997; O'Donoghue and Rabin, 1999):

Definition 1. *((β,δ)-preferences). The (β,δ)-preferences are intertemporal preferences represented by*

$$U_{it}(\bar{u}_{it}, \bar{u}_{it+1}, ...) \equiv \bar{u}_{it} + \beta \sum_{\tau=t+1}^{+\infty} \delta^{\tau-t} \bar{u}_{\tau}$$

where $\beta \in (0,1]$ *and* $\delta \in (0,1]$

The parameter δ is referred to as the standard discount factor, which captures long-run, time-consistent discounting; and the parameter β is called the present-bias factor, which captures short-term impatience. It is β that generates the dynamic inconsistency in the agent's preferences. The standard model is nested as a special case of (β, δ)-preferences when $\beta = 1$. When $\beta \in (0,1)$, (β, δ)-preferences capture the "quasi-hyperbolic" time discounting (Laibson, 1997). We say that an agent's preferences are time-consistent if $\beta = 1$, and are present-biased if $\beta \in (0,1)$.

Since its creation and subsequent refinement, a quite large empirical literature has developed to estimate the parameters of the quasi-hyperbolic discounting model. Most methods have focused on continuous choices, for example over time-dated monetary rewards (e.g. Thaler, 1981; Andersen et al., 2008) or over the allocation effort over time (e.g. Augenblick et al., 2015; Augenblick and Rabin, 2019). At the same time, a large field-experimental literature has purported to show evidence of present-biased preferences as revealed by "commitment contracts" which constrain the decision-maker's future choices in some manner (for a review, see, e.g. DellaVigna (2009)). However, as Carerra et al. (2019) have recently pointed out, commitment contracts may be thought of as a discrete choice problem, in which case choosing them reveals only the utility shock and not the decision-maker's time-inconsistent preference. That paper and others (e.g. Acland and Levy, 2015; Heidhues and Strack, 2019; Bracke, Levy and Schiraldi, 2019) have proposed combining the discrete choice with a continuous choice to provide identification of both the degree of inconsistency as well as beliefs about future inconsistency (i.e. "naivete"). In this section, we provide conditions under which present-bias is identified only by discrete choices, using the history-dependent choice sets already introduced.

Behavior of agents with (β, δ)-preferences Following previous studies of time-inconsistent preferences, we analyze the behavior of an agent by thinking of the single individual as consisting of many autonomous selves, one for each period. Each period-t self chooses her current behavior to maximize her current utility $U_{it}(\bar{u}_{it}, \bar{u}_{it+1}, ...)$, while her future selves control their subsequent decisions. While O'Donoghue and Rabin (1999, 2001) allow agents to be naive about their future selves' behavior, we focus on the fully sophisticated case as in Laibson (1997) in which the preferences and behavior of future selves are correctly forecasted.

A (Markovian) strategy profile for all selves is $\sigma \equiv \{\sigma_t\}_{t=1}^{\infty}$ where $\sigma_t : \mathcal{S} \times (|\mathcal{J}|)^H \times \mathbb{R}^{J+1} \rightarrow \mathcal{J}$ for all t and it specifies for each self her action in all possible states and under all possible realizations of shock vectors. For any strategy profile σ, let $\sigma_t^+ \equiv \{\sigma_k\}_{k=t}^{\infty}$ be the continuation strategy profile from period t on.

To characterize the equilibrium of the intra-personal game of an agent with potentially time-inconsistent preferences, we first write $\bar{V}_t(s_t, h_t, m_t; \sigma_t^+)$ as the agent's period-t expected continuation utility when the state variable is s_t and history h_t under her long-run time preference for a given continuation strategy profile σ_{it}^+.[20] We can think of $\bar{V}_{it}(s_{it}, h_t, m_t; \sigma_{it}^+)$ as representing, hypothetically, her intertemporal preferences from some prior perspective when her own present-bias is irrelevant.

The solution concept we use to solve this game is subgame-perfection.[21] Each self takes the strategy of all future selves $\sigma_{t+1}^+ = \{\sigma_{t+1}, \sigma_{t+2}, ...\}$ as given, and best-responds to these beliefs.

We define the subgame-perfect strategy profile a strategy profile $\sigma^* \equiv \{\sigma_t^*\}_{t=1}^{\infty}$ with $\sigma_t^* = (j^*)$ such that for all t, s_t, and h_t

$$\sigma_t^*(s_t, h_t, m_t, \varepsilon_t) = \arg\max_{j \in m_t} \left\{ u_j(s_t, h_t) + \varepsilon_{jt} + \beta\delta E[\bar{V}_{t+1}(s_{t+1}, h_{t+1}, m_{t+1}; \sigma_{t+1}^+)|s_t, h_t, m_t, \sigma_t(s_t, h_t, m_t)] \right\} \tag{27}$$

Under the assumption above, we rewrite the long-run value function, given the agent's strategy σ^* is:

$$\bar{V}(s_t, h_t, m_t) = E_\varepsilon \left[u_{\sigma^*}(s_t, h_t) + \varepsilon_\sigma + \delta E[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|\sigma^*, s_t, h_t, m_t \right] \tag{28}$$

$$= \sum_k \Pr(d_t = k|s_t, h_t, m_t) \left[q_k(s_t, h_t, m_t) + u_k(s_t, h_t) + \delta E\left[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|k, s_t, h_t, m_t \right] \right] \tag{29}$$

and

$$\bar{V}(s_t, h_t, m_t) = E_\varepsilon \left[u_{\sigma^*}(s_t, h_t) + \varepsilon_\sigma + \delta E[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|\sigma^*, s_t, h_t, m_t \right] \tag{30}$$

The decision-maker's choices are governed by the choice-specific current value function, which reflects her degree of present bias. The choice-specific current value function is given by:

[20] As before, we define the ex-ante value function (or integrated value function), $\bar{V}(s_t, h_t, m_t; \sigma_t^+)$ given by integrating $V(s_t, h_t, m_t, \varepsilon_t; \sigma_t^+)$ over ε_t.

[21] We are assuming that agents are perfectly sophisticated with respect to their future preferences. Subgame perfection coincides with perception-perfection in the case of perfectly sophisticated agents (O'Donoghue and Rabin, 1999)

$$w_j(s_t, h_t, m_t) \equiv u_j(s_t, h_t) + \beta\delta E\left[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|j, s_t, h_t, m_t\right] \tag{31}$$

We can then rewrite (28) as

$$\begin{aligned}
\bar{V}(s_t, h_t, m_t) &= E\left[w_\sigma(s_t, h_t, m_t) + \varepsilon_{\sigma t} + (1-\beta)\delta E\left[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|\sigma^*, s_t, h_t, m_t\right]\right] \\
&= E\left[\max_{j \in m_t} w_j(s_t, h_t, m_t) + \varepsilon_{jt}\right] + (1-\beta)\delta \sum_k \Pr(d_t = k|s_t, h_t, m_t) E\left[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|k, s_t, h_t, m_t\right] \\
&= q_0(s_t, h_t, m_t) + w_0(s_t, h_t, m_t) + (1-\beta)\delta \sum_k \Pr(d_t = k|s_t, h_t, m_t) E\left[\bar{V}(s_{t+1}, h_{t+1}, m_{t+1})|k, s_t, h_t, m_t\right]
\end{aligned} \tag{32}$$

Assumption 10 (Final commitment)**.** *$M(h, 0)$ is degenerate on choice set $\{0\} \in \mathcal{Q}$ for all $h \in \{\mathcal{J} \times \mathcal{S}\}^{H-1}$.*

While the assumption of an absorbing choice has appeared previously in the discrete choice literature, it has often been accompanied by a strong restriction that $u_0(\cdot)$ is constant across all s. In contrast, we invoke an absorbing choice but still allow the utility of all alternatives to depend on the realized state. As before under sophistication, the agent correctly perceives their future selves' strategies.

By substituting (32) into (31), we write the choice probabilities of j and 0 in terms of $\boldsymbol{w_0}$, adjusted by the perception-perfect strategy $\boldsymbol{P}$ and for all possible period-$(t+1)$ choice sets:

$$\begin{aligned}
\boldsymbol{\varphi}_j(x, j) = \boldsymbol{u_j} - \boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_j\Big(&\boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0 + \boldsymbol{M}(j,j)\left[\tilde{\boldsymbol{q}}_0(j) + (1-\beta)\delta\boldsymbol{P}(j)\tilde{\boldsymbol{V}}(j)\right]\Big) \\
&- \beta\delta\boldsymbol{\pi}_0\left(\boldsymbol{u}_0 + \delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right)
\end{aligned} \tag{33}$$

where $\Upsilon_{0,s_{t+1}} \equiv E\left[\sum_{\tau=t+2}^{\infty} \delta^{\tau-(t+2)} u_0(s_\tau, (j, 0))|0, s_{t+1}\right]$ and $\boldsymbol{\Upsilon_0} \equiv (I - \delta\boldsymbol{\pi}_0)^{-1}\boldsymbol{u_0}$ is the vector of $\Upsilon_{0,s_{t+1}}$ across different states.

Note that by assumption 5, alternative 0 is always available and thus the only uncertainty in the $\boldsymbol{u_0}$ and $\boldsymbol{\Upsilon}$ terms are the realizations of s_{t+1} and s_{t+2}. The probability of choosing any alternative, however, does depend on the realized choice set, and thus $\tilde{\boldsymbol{q}}_{\boldsymbol{0}}$ and $\boldsymbol{V}(j)$ require an iterated expectation over choice sets conditional on each state.

Finally, we make an assumption that there exists an alternative which provides delayed commitment. The absorbing choice in assumption 10 can be thought of as committing the decision-maker to choose alternative 0 in all future periods, beginning with period-$(t+1)$. A natural generalization of this would be an alternative which commits the decision-maker to choose alternative 0 in all future periods beginning with some strictly later date. That is, alternative 0 is an absorbing choice in all histories $(h, 0)$; delayed commitment will do the same in the reverse situation of (d, h).

Assumption 11 (Delayed Commitment)**.** *There exists an alternative $d \neq 0$ such that $M(d, h)$ is degenerate on choice set $\{0\} \in \mathcal{Q}$ for all $h \in \{\mathcal{J} \times \mathcal{S}\}^{H-1}$.*

Unlike the case of exponential discounting, the continuation value of the quasi-hyperbolic discounter must take into account the choice probabilities given by the perception-perfect strategy. Having delayed commitment available, however, means that at some point those probabilities are greatly simplified. Our identification proof relies on observing how the probability of choosing delayed commitment varies across histories which may change the distribution of available alternatives before the delayed commitment is binding. We therefore modify Assumption 6 to guarantee that delayed commitment is indeed available in at least two histories, and that these two histories indeed vary in their implications for the choice set during the intervening periods.

Assumption 6′. *For all s there exist $x, y \in A_d(s)$ such that $M(h(x, d, s), \cdot) \neq M(h(y, d, s), \cdot)$.*

With the presence of a delayed commitment alternative, we now show that the quasi-hyperbolic discount factors and utility function are identified.

Theorem 4. *If $|\mathcal{J}| \geq 4$ and under assumptions 1–5, 6′ and 10–11: $\beta\delta$ is identified iff $H \geq 2$ and β, δ and the utility function are identified if $H \geq 3$*

Proof. **Part 1: $\beta\delta$ is identified iff $H \geq 2$.** We may consider equation (33) in history (x, k), for $k \in \mathcal{J} \setminus \{0, d\}$:

$$\begin{aligned}\boldsymbol{\varphi}_d(x,k) = \boldsymbol{u_d} - \boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_d\left(\boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right) - \beta\delta\boldsymbol{\pi}_0\left(\boldsymbol{u}_0 + \delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right) \\ + \beta\delta\boldsymbol{\pi}_d\boldsymbol{M}(x,k)\left[\tilde{\boldsymbol{q}}_0(j) + (1-\beta)\delta\boldsymbol{P}(d)\tilde{\boldsymbol{V}}(d)\right]\end{aligned} \tag{34}$$

where $\boldsymbol{M}(x, k)$ is $S \times Q$. We take a difference with respect to $\boldsymbol{\varphi}$ in one alternative history (x, l):

$$\boldsymbol{\varphi}_d(x,k) - \boldsymbol{\varphi}_d(x,l) = \beta\delta\boldsymbol{\Lambda}_d(x,k,l)\left[\tilde{\boldsymbol{q}}_0(d) + (1-\beta)\delta\boldsymbol{P}(d)\tilde{\boldsymbol{V}}(d)\right] \tag{35}$$

where $\boldsymbol{\Lambda}_d(x,k,l) = (\boldsymbol{\pi}_d\boldsymbol{M}(x,k) - \boldsymbol{\pi}_d\boldsymbol{M}(x,l))$. By assumption 11 and $H \geq 2$, there exists d such that the distribution of period-$(t+2)$ choice sets is identical to the distribution induced by choosing 0. Under assumption 10, this distribution is degenerate on the singleton $\{0\}$. Thus $V(d,\cdot) = \boldsymbol{\Upsilon}_0$ regardless of the period-$(t+1)$ choice, and we may therefore write $\tilde{\boldsymbol{V}}(j) = \tilde{\boldsymbol{\pi}}\boldsymbol{\Upsilon}_0$ and Equation (35) as

$$\boldsymbol{\varphi}_d(x,k) - \boldsymbol{\varphi}_d(x,l) = \begin{bmatrix}\boldsymbol{\Lambda}_d(x,k,l)\tilde{\boldsymbol{q}}_0(d) & \boldsymbol{\Lambda}_d(x,k,l)\boldsymbol{P}(d)\tilde{\boldsymbol{\pi}}\end{bmatrix}\begin{bmatrix}\beta\delta \\ \beta\delta^2(1-\beta)\boldsymbol{\Upsilon}_0\end{bmatrix} \tag{36}$$

Equation (36) is therefore a system of S linearly independent equations in $S+1$ unknowns, namely $\beta\delta$ and $\beta\delta^2(1-\beta)\boldsymbol{\Upsilon}_0$. Moreover, the column rank of $\left(\begin{bmatrix}\boldsymbol{\Lambda}_d(x,k,l)\tilde{\boldsymbol{q}}_0(d) & \boldsymbol{\Lambda}_d(x,k,l)\boldsymbol{P}(d)\tilde{\boldsymbol{\pi}}\end{bmatrix}\right)$ is equal to $rank\left(\begin{bmatrix}\boldsymbol{\Lambda}_d(x,k,l)\tilde{\boldsymbol{q}}_0(d) & \boldsymbol{\Lambda}_d(x,k,l)\boldsymbol{P}(d)\tilde{\boldsymbol{\pi}}\end{bmatrix}\right) = S$. Thus $\beta\delta$ is identified if we normalize $\Upsilon_0(s') = 0$ for one arbitrary s' WLOG[22] and $|\mathcal{J}| \geq 4$.
The only if part follows from a similar argument as in Theorem 3.

Part 2: β, δ and the utility function are identified if $H \geq 3$. We start by defining the following induced distribution over histories and states. As in the proof of Theorem 2, the joint distribution over $s_{t+2}, d_{t+1}, m_{t+1}$, and s_{t+1} conditional on choosing $d_t = d$ in history (x,k,j) is given by $\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}$. The joint distribution over $m_{t+2}, s_{t+2}, d_{t+1}, m_{t+1}$, and s_{t+1} is given by $\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d)$. We must further specify a distribution over period-$(t+2)$ choices, which is given by $\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d)\hat{\boldsymbol{P}}(d)$, where $\hat{\boldsymbol{P}}(d)$ is defined in the appendix. Finally, we must specify the distribution over s_{t+3}, when the delayed commitment forces the decision-maker to begin consuming alternative 0. This is given by $\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d)\hat{\boldsymbol{P}}(d)\breve{\boldsymbol{\pi}}$, where $\breve{\boldsymbol{\pi}}$ is defined in the appendix.

[22] Note that Υ is a linear function of u_0 with all (weakly) positive coefficients, and hence for an arbitrary u_0 there exists a constant κ which may be added to the flow utility in all states such that $\Upsilon_0(s') = 0$. As the preference is uniquely determined only up to a constant shift, this is without loss of generality.

For $H = 3$, equation (34) is written as:

$$\begin{aligned}\boldsymbol{\varphi}_d(x,k,j) = \boldsymbol{u_d} - \boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_d\left(\boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right) - \beta\delta\boldsymbol{\pi}_0\left(\boldsymbol{u}_0 + \delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right)\\ + \beta\delta\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\left[\tilde{\boldsymbol{q}}_0(j,d) + (1-\beta)\delta\boldsymbol{P}(j,d)\tilde{\boldsymbol{V}}(j,d)\right]\end{aligned} \tag{37}$$

Next, we substitute for $\tilde{\boldsymbol{V}}(j,d)$ using equation (32). By assumption 11, however, the period-$(t+2)$ continuation values are functions only of $\boldsymbol{\Upsilon}_0$ given that $d_t = d$. We therefore have

$$\begin{aligned}\boldsymbol{\varphi}_d(x,k,j) = \boldsymbol{u_d} - \boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_d\left(\boldsymbol{u_0} + \beta\delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right) - \beta\delta\boldsymbol{\pi}_0\left(\boldsymbol{u}_0 + \delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right) + \beta\delta\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\tilde{\boldsymbol{q}}_0(j,d)\\ + (1-\beta)\beta\delta^2\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\left[\left(\tilde{\boldsymbol{I}}_S^{(J+1)QS}\left(\boldsymbol{u}_0 + \beta\delta\boldsymbol{\pi}_0\boldsymbol{\Upsilon}_0\right) + \hat{\boldsymbol{M}}(j,d)\hat{\boldsymbol{P}}(d)\breve{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{(J+1)^2Q^2S}\boldsymbol{\Upsilon}_0\right)\right.\\ \left. + \hat{\boldsymbol{M}}(j,d)\hat{\boldsymbol{q}}_0^{t+2}(d)\right]\end{aligned} \tag{38}$$

where $\hat{\boldsymbol{q}}_0^{t+2}(d)$ considers surpluses for all possible $(t+1)$ choices as in Theorem 2, though as we consider $H = 3$ we now explicitly condition on $d_t = d$. Without loss, let Assumption 6′ apply to $h = (x, k.j)$ and $\hat{h} = (x, j, j)$. We then take the difference in equation (34) evaluated in these two histories to obtain:

$$\begin{aligned}\boldsymbol{\varphi}_d(x,k,j) - \boldsymbol{\varphi}_d(x,j,j) = \beta\delta\left[\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\tilde{\boldsymbol{q}}_{\boldsymbol{0}}(j,d) - \boldsymbol{\pi}_d\boldsymbol{M}(j,j,d)\tilde{\boldsymbol{q}}_{\boldsymbol{0}}(k,d)\right]\\ + (1-\beta)\beta\delta^2\left[\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{M}}(j,d) - \boldsymbol{\pi}_d\boldsymbol{M}(j,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{M}}(j,d)\right]\hat{\boldsymbol{P}}(d)\breve{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{(J+1)^2Q^2S}\boldsymbol{\Upsilon}_{\boldsymbol{0}}\\ + (1-\beta)\beta\delta^2\left[\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d) - \boldsymbol{\pi}_d\boldsymbol{M}(j,j,d)\boldsymbol{P}(j,d)\right]\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{(J+1)QS}\boldsymbol{\pi}_{\boldsymbol{0}}\boldsymbol{\Upsilon}_0\\ + (1-\beta)\beta\delta^2\left[\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d) - \boldsymbol{\pi}_d\boldsymbol{M}(j,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d)\right]\hat{\boldsymbol{q}}_0^{t+2}(d)\\ + (1-\beta)\beta\delta^2\left[\boldsymbol{\pi}_d\boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d) - \boldsymbol{\pi}_d\boldsymbol{M}(j,j,d)\boldsymbol{P}(j,d)\right]\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{(J+1)QS}\boldsymbol{u}_{\boldsymbol{0}}\end{aligned} \tag{39}$$

Note that the conditions of Part 1 are satisfied, and we may treat $\beta\delta$ and $(1-\beta)\beta\delta^2\boldsymbol{\Upsilon}_{\boldsymbol{0}}$ as known. We therefore collect the first three lines of the right-hand side of equation (39), which are either data or functions of these known quantities, into a single term $\boldsymbol{\theta}((x,j),j,k)$. Equation (39) is a system of S equations in $S+1$ unknowns:

$$\boldsymbol{\varphi}_d(x,k,j) - \boldsymbol{\varphi}_d(x,j,j) - \boldsymbol{\theta}((x,j),j,k) = \tag{40}$$

$$\underbrace{\begin{bmatrix} \left[\left[\boldsymbol{\pi}_d \boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d) - \boldsymbol{\pi}_d \boldsymbol{M}(j,j,d)\boldsymbol{P}(j,d)\hat{\boldsymbol{\pi}}\hat{\boldsymbol{M}}(j,d)\right]\hat{\boldsymbol{q}}_0^{t+2}(d)\right]^T \\ \left[\left[\boldsymbol{\pi}_d \boldsymbol{M}(k,j,d)\boldsymbol{P}(j,d) - \boldsymbol{\pi}_d \boldsymbol{M}(j,j,d)\boldsymbol{P}(j,d)\right]\hat{\boldsymbol{\pi}}\tilde{\boldsymbol{I}}_S^{(J+1)QS}\right]^T \end{bmatrix}^T}_{\equiv \boldsymbol{\Gamma}((h,k),j,k)} \begin{bmatrix} \beta\delta^2(1-\beta) \\ \beta\delta^2(1-\beta)\boldsymbol{u}_0 \end{bmatrix}$$

The matrix $\boldsymbol{\Gamma}((h,k),j,k)$ has dimension $S \times (S+1)$ with rank S, and is therefore not invertible. Importantly, it has column rank S due to the differences in $\boldsymbol{MP}$ and therefore adding additional rows by repeating the exercise across additional histories will provide no additional identification. Instead, we may add a row $[0,\ldots,0,1]$ to $\boldsymbol{\Gamma}((h,k),j,k)$ to form an $(S+1)\times(S+1)$ matrix $\boldsymbol{\Gamma}$ and an arbitrary constant x to the left-hand side of (40), i.e. we normalize $(1-\beta)\beta\delta^2 u_0(s_S) = x$. Given the normalization of $\boldsymbol{\Upsilon_0}$ in the first step of the proof, this extra normalization is no longer without loss of generality. We therefore apply a fixed-point argument to complete the proof.

Given an arbitrarily chosen x, and given that the product $\beta\delta$ is known from step 1, we obtain a unique value for the long-run discount factor:

$$\hat{\delta}(x) = (\beta\delta)^{-1}\left(\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}\left[\left(\boldsymbol{\Gamma}^T\boldsymbol{\Gamma}\right)^{-1}\boldsymbol{\Gamma}\right]\begin{bmatrix}\boldsymbol{\varphi}_d(h,k,j) - \boldsymbol{\varphi}_d(h,k,k) - \boldsymbol{\theta}((h,k),j,k) \\ x\end{bmatrix} + (\beta\delta)^2\right) \tag{41}$$

Note that $\hat{\delta}(x)$ is linear in x, i.e. $\hat{\delta}(x) = a + bx$ for some a, b. As (40) is not identified, it must be that $b \neq 0$ or else the choice of normalization would not affect the inverse which is a contradiction. Given $\hat{\delta}(x)$, and the previously-identified quantity $(1-\beta)\beta\delta^2\boldsymbol{\Upsilon_0}$, we may obtain an estimated $\hat{\boldsymbol{u}}_0 = (I - \hat{\delta}\boldsymbol{\pi_0})\boldsymbol{\Upsilon_0}$, and define a function:

$$f(x) = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}\left(I - \hat{\delta}(x)\boldsymbol{\pi}_0\right)\left[(1-\beta)\beta\delta^2\boldsymbol{\Upsilon_0}\right] \tag{42}$$

Notice that $f(x)$ is linear in x, with a nonzero coefficient. Thus there exists a unique fixed point x^* such that $f(x^*) = x^*$. That is, there is only one value of $u_0(s_S)$ in equation (40) that is consistent with the normalization of $\boldsymbol{\Upsilon_0}(s')$ in equation (36).

We thus obtain:

$$\begin{bmatrix} \beta\delta^2(1-\beta) \\ \boldsymbol{u}_0 \end{bmatrix} = \left(\boldsymbol{\Gamma}^T\boldsymbol{\Gamma}\right)^{-1}\boldsymbol{\Gamma}^T \begin{bmatrix} \boldsymbol{\varphi}_d(x,k,j) - \boldsymbol{\varphi}_d(x,j,j) - \boldsymbol{\theta}((x,j),j,k) \\ x^* \end{bmatrix} \tag{43}$$

This identifies $\boldsymbol{u}_0$, and given $\beta\delta$ from the first step, identifies β and δ separately.

The utilities of any other choices then follow from $\boldsymbol{\varphi}_j(o,l,k)$ observed for all $j \in \mathcal{J} \setminus \{0\}$ and any arbitrary history (o,l,k) with $o \neq d$. □

It is worth noting that the conditions imposed by Theorem 1 to identify the exponential discounting model are a strict relaxation of those imposed by Theorem 4 to identify the quasi-hyperbolic discounting model. The most apparent difference is the requirement in the latter of assumptions 10 and 11. However, in many ways the more fundamental difference is the requirement of the former that $|\mathcal{J}| \geq 2$ and of the latter that $|\mathcal{J}| \geq 4$, which has a very natural interpretation. Consider the simplest problem in which the decision-maker's intertemporal preferences may be revealed, namely, a finite horizon optimal stopping problem in which the decision-maker must complete a single task in one of T periods as in O'Donoghue and Rabin (1999). We normalize the utility of task completion to be zero in all periods, and allow the utility of waiting to be stochastic with a period-specific mean.

Note that the stationary infinite-horizon model which has been considered is sufficiently flexible to capture this simple finite-horizon problem. First, note that the absorbing choice maps onto task completion as once the task is completed there are no other choices to be made by the decision-maker. Next, because we have assumed history-independent utility, the time variation in the utility of waiting must be captured by different alternatives. It is clear that the finite-horizon model would have no implications for intertemporal preferences if $T = 1$. If $T = 2$, the decision-maker's $t = 1$ choices involve δ and $\beta\delta$, respectively, in the two models, and so of course a quasi-hyperbolic discounter is not distinguished from an exponential discounter with a discount factor equal to the product $\beta\delta$. A $T = 3$ model is necessary to distinguish the two, which is precisely what is provided by adding an additional alternative to the choice set. Even this is not sufficient to separate the discount factors from the time-varying utility, however. This task is provided by the commitment, as there must be some histories where self-1 can commit self-2 to complete the task with some probability, a requirement which is imposed by our conditions on M and the introduction of the delayed commitment alternative.

6 Extensions

6.1 Unobserved heterogeneity

In dynamic discrete choice model, controlling for unobserved heterogeneity is an important issue. Finite mixture models, which are commonly used in empirical analyses, provide flexible ways to account for it. Assume that each individual belongs to one of $\mathcal{Z}$ types, and his/her type attribute is unknown.[23] The probability of belonging to type z is r_z, where the r_z's are positive and sum to 1. Let the initial (unobservable) distribution of choices be $p(s_1, d_1, m_1|z)$. We assume that the aggregate sequence of choice probabilities, i.e. $Pr(\{d_t, s_t, m_t\}_{t=1}^T)$ (where T could be infinity), are observable but type-specific choice probability, i.e. $Pr(d_t|s_t, h_t, m_t, z)$, is not. The utility functions and the discount factors differ by types, as potentially do the state transition probabilities. In contrast, we assume that the distribution over $\mathcal{Q}$ is common across all types. The identification relies on the results in Kasahara and Shimotsu (2009)

Assumption 12. *Assume that the utility function, discount factor and the transition probability are type-specific: $u(s; z)$, $\delta(s; z)$ and $\pi(s_{t+1}|s_t, h_t, m_t, z)$ respectively.*

With assumption 12, the mixing probabilities are identified in all three cases studied in sections 3–5. Formally:

Corollary 3. *Under assumptions 1–6 and 12 and for known $|\mathcal{Z}|$ if $|\mathcal{J}| \geq 3$, $H = 2$, $T \geq 9$, then $u(s; z)$, $\delta(s; z)$, r_z and $p(s_1, d_1, m_1|z)$ are identified.*

Proof. The proof is articulated in two steps. We first identify r_z, $p(s_1, d_1, m_1, z)$ and $Pr(d_t = j|s_t, h_t, m_t, z)$. The second step we identify $u(s; z)$ and $\delta(s; z)$

Step 1: Notice that we can write the observable choice probability sequence as general stationary finite mixture model of dynamic discrete choices:

$$
\begin{aligned}
&Pr(\{d_t, s_t, m_t\}_{t=1}^T) = \\
&\sum_{z=1}^{Z} r_z \cdot p(s_1, d_1, m_1|z) \prod_{t=2}^{T} \pi(s_{t+1}|s_t, d_t; z) \cdot M(m_t|h_t) \cdot Pr(d_t|s_t, h_t, m_t, z)
\end{aligned} \tag{44}
$$

As $M(m_t|h_t)$ does not depend on z and therefore is known we can divide both side by $\prod_{t=2}^{T} M(m_t|h_t)$

[23] Kasahara and Shimotsu (2009) study nonparametric identifiability of the number of types, $|\mathcal{Z}|$

so to get

$$\tilde{Pr}(\{d_t, s_t, m_t\}_{t=1}^T) = \sum_{z=1}^{Z} r_z \cdot p(s_1, d_1, m_1|z) \prod_{t=2}^{T} \pi(s_{t+1}|s_t, d_t; z) \cdot Pr(d_t|s_t, h_t, m_t, z) \tag{45}$$

where $\tilde{Pr}(\{d_t, s_t, m_t\}_{t=1}^T) \equiv \frac{Pr(\{d_t, s_t, m_t\}_{t=1}^T)}{\prod_{t=2}^T M(m_t|h_t)}$. The identification problem in equation 45 has been studied in Kasahara and Shimotsu (2009) and therefore the identification of r_z, $p(s_1, d_1|m_t, z)$, $\pi(s_{t+1}|s_t, h_t, z)$ and $Pr(d_t|s_t, h_t, m_t, z)$ follows for their proposition 6.

Step 2: Given $Pr(d_t = j|s_t, h_t, m_t, z)$, we obtain $\varphi_{d_t}(s_t, h_t, m_t|z)$ then the identification of $u(s; z)$, $\delta(s; z)$ follows from theorem 1 and corollary 1. $\square$

Corollary 4. *Under assumptions 1–5, 6′ and 10–11 and 12 and for known* $|\mathcal{Z}|$ *if* $\mathcal{J} \geq 4$, $H = 3$, $T \geq 12$, *then* $u(s; z)$, δ, β, r_z *and* $p(s_1, d_1, m_1|z)$ *are identified.*

Proof. The first step of the proof is identical to the first step of the proof in corollary 3 and the second step follows immediately from theorem 4. $\square$

7 Conclusion

Dynamic discrete choice models are in general underidentified, and the degree of underidentification and potential solutions in the form of exclusion or parametric restrictions has been studied in a large literature. We propose a new approach which focuses on problems which are intrinsically dynamic. History-dependent choice sets and preferences link the decision-maker across periods in a more fundamental sense than is achieved in history-independent problems typically studied which lead to the identification of the exponential discount factor.

The power of this approach is greatly strengthened when the class of history-dependence can be narrowed to history-dependent choice set. An endogenous choice set is a common feature in most applications, and in some sense plays a similar role to the intertemporal budget constraint in a dynamic continuous choice model. In this environment, we show that also the utility function is identify without any normalization which is usually imposed in the literature. Moreover, when the history-dependence of the choice set also includes an absorbing choice, the time preferences and utility of a quasi-hyperbolic discounter may be identified. A vast literature in behavioural economics has made use of exogenous variation in the choice set, i.e. commitment contracts, and we provide conditions under which the choice of commitment could provide identification. The typical field experiment offers only a single choice in a single period, essentially $|\mathcal{J}| = 2$; our results show why this is insufficient and provide a blueprint for how experiments could be extended to solve this shortcoming. Finally, we note that our examination of hyperbolic discounting focused on the case of fully sophisticated agents. Our results may be extended to the naive or partially-naive case, though the latter will require additional assumptions regarding the length of history-dependence and the available variation.

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Appendix A Additional notation

The following definitions apply for the proof of Theorem 4. For the proof of Theorem 2, let $J = 1$ such that $\mathcal{J} = \{0, j\}$, as only two alternatives are considered for period $(t+1)$ consumption.

Let $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_0} \\ \vdots \\ \boldsymbol{u_J} \end{bmatrix}$ be the $(J+1)S \times 1$ vector of all utilities across all states.

Define $\tilde{\boldsymbol{I}}^{\boldsymbol{n}}_{\boldsymbol{m}} = \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}$ to be $nm \times n$ matrix of n vertically stacked $m \times m$ identity matrices.

Define

$$\boldsymbol{\chi}(s) = \begin{bmatrix} \overbrace{0, \ldots, 0}^{s-1}, 1, 0, \ldots, 0, 0 \ldots 0 \\ \vdots \\ \underbrace{0, \ldots, 0, \ldots, 0}_{JS+s-1}, 1, 0 \ldots, 0 \end{bmatrix}$$

be the $(J+1) \times (J+1)S$ matrix where row i has a one in column $(i-1)S + s$ and zeros elsewhere.

Finally, let

$$\boldsymbol{\chi} = \begin{bmatrix} \tilde{\boldsymbol{I}}^{Q}_{(J+1)} \boldsymbol{\chi}(s_1) \\ \vdots \\ \tilde{\boldsymbol{I}}^{Q}_{(J+1)} \boldsymbol{\chi}(s_S) \end{bmatrix}$$

be the $Q(J+1)S \times (J+1)S$ matrix formed first by stacking each $\boldsymbol{\chi}(s)$ Q times vertically, and then stacking these across s. The expected consumption utility in period-$(t+1)$ given d_t and d_{t-1} is then: $\pi_{d_t} M(d_{t-1}, d_t) P(d_t) \boldsymbol{\chi} \boldsymbol{u}$

Let

$$\hat{\boldsymbol{\pi}}(s, m) = \begin{bmatrix} \boldsymbol{\pi}'_0(s) & & \\ & \ddots & \\ & & \boldsymbol{\pi}'_J(s) \end{bmatrix}$$

be the $(J+1) \times (J+1)S$ transition for each possible choice, conditional on state s and choice set m. As m does not affect the transition directly, it is ignored when we stack across choice sets to define the $(J+1)Q \times (J+1)QS$ matrix:

$$\hat{\boldsymbol{\pi}}(s) = \begin{bmatrix} \hat{\boldsymbol{\pi}}(s,\cdot) & & \\ & \ddots & \\ & & \hat{\boldsymbol{\pi}}(s,\cdot) \end{bmatrix}$$

Finally, we define the $(J+1)QS \times (J+1)QS^2$ matrix:

$$\hat{\boldsymbol{\pi}} = \begin{bmatrix} \hat{\boldsymbol{\pi}}(s_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{\pi}}(s_S) \end{bmatrix}$$

Next, define the distribution of period-$(t+2)$ choice sets conditional on $d_t, s_{t+1}, m_{t+1}, d_{t+1}$ for each possible state s_{t+2} as the $S \times QS$ matrix:

$$\hat{\boldsymbol{M}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}) = \begin{bmatrix} M(d_t, d_{t+1}, s_1) & & \\ & \ddots & \\ & & M(d_t, d_{t+1}, s_S) \end{bmatrix}$$

Define the distribution conditional on d_t, s_{t+1}, and m_{t+1} as the $(J+1)S \times (J+1)QS$ matrix:

$$\hat{\boldsymbol{M}}(d_t, s_{t+1}, m_{t+1}) = \begin{bmatrix} M(d_t, s_{t+1}, m_{t+1}, 0) & & \\ & \ddots & \\ & & M(d_t, s_{t+1}, m_{t+1}) \end{bmatrix}$$

Define the distribution conditional on d_t and s_t as the $(J+1)QS \times (J+1)Q^2S$ matrix:

$$\hat{\boldsymbol{M}}(d_t, s_{t+1}) = \begin{bmatrix} \hat{\boldsymbol{M}}(d_t, s_{t+1}, m_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{M}}(d_t, s_{t+1}, m_Q) \end{bmatrix}$$

Finally, define the distribution conditional only on d_t as the $(J+1)QS^2 \times (J+1)Q^2S^2$ matrix:

$$\hat{\boldsymbol{M}}(d_t) = \begin{bmatrix} \hat{\boldsymbol{M}}(d_t, s_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{M}}(d_t, s_S) \end{bmatrix}$$

The expected period-$(t+2)$ flow utility from consuming alternative 0 is thus given by: $\boldsymbol{\Pi}_{t+2}\boldsymbol{u_0} = \boldsymbol{\pi}_j M(k,j) P(j) \hat{\boldsymbol{\pi}} \tilde{\boldsymbol{I}}_S^{(J+1)QS} \boldsymbol{u_0}$.

Now define the period-(t+2) surpluses from alternative 0 conditional on $s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}$ for all m_{t+2} as the $Q \times 1$ vector: $\hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}) = [q_0(d_{t+1}, s_{t+2}, m_1), \ldots, q_0(d_{t+1}, s_{t+2}, m_Q)]'$

Define the period-(t+2) surpluses from alternative 0 conditional on $s_{t+1}, m_{t+1}, d_{t+1}$ as the $QS \times 1$ vector:

$$\hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}, d_{t+1}) = \begin{bmatrix} \hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}, d_{t+1}, s_1) \\ \vdots \\ \hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}, d_{t+1}, s_S) \end{bmatrix}$$

Define the period-(t+2) surpluses from alternative 0 conditional on s_{t+1}, m_{t+1} for all d_{t+1} as the $(J+1)QS \times 1$ vector:

$$\hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}) = \begin{bmatrix} \hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}, 0) \\ \vdots \\ \hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1}, J) \end{bmatrix}$$

Because d_{t+1} is a sufficient statistic for m_{t+1}, we define the period-(t+2) surpluses from alternative 0 conditional on s_{t+1} for all m_{t+1} as the $(J+1)Q^2S \times 1$ vector: $\hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}) = \tilde{\boldsymbol{I}}_{(J+1)QS}^{Q} \hat{\boldsymbol{q}}_0^{t+2}(s_{t+1}, m_{t+1})$

Similarly as s_{t+2} is a sufficient statistic for s_{t+1}, we define the period-(t+2) surpluses as the $(J+1)Q^2S^2 \times 1$ vector: $\hat{\boldsymbol{q}}_0^{t+2} = \tilde{\boldsymbol{I}}_{(J+1)Q^2S}^{S} \hat{\boldsymbol{q}}_0^{t+2}(s_{t+1})$.

We will also need the period-$(t+2)$ choices for every $s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}, m_{t+2}$ realization, given $h_t = (d_{t-3}, d_{t-2}, d_{t-1})$ and an arbitrary d_t. Define the period-(t+2) choice probabilities conditional on $s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}, m_{t+2}$ as the $1 \times (J+1)$ vector:

$$\hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}, m_{t+2}) = P((d_t, d_{t+1}), s_{t+2}, m_{t+2})$$

Aggregating across m_{t+2} to create the $Q \times (J+1)Q$ matrix:

$$\hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}) = \begin{bmatrix} \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}, m_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}, m_1 \end{bmatrix}$$

Aggregating across s_{t+2} to create the $QS \times (J+1)QS$ matrix:

$$\hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}) = \begin{bmatrix} \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}, s_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, d_{t+1}, s_S) \end{bmatrix}$$

Aggregating across d_{t+1} to create the $(J+1)QS \times (J+1)^2QS$ matrix:

$$\hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}) = \begin{bmatrix} \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, 0) & & \\ & \ddots & \\ & & \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_{t+1}, J) \end{bmatrix}$$

Aggregating across m_{t+1} to create the $Q^2(J+1)S \times Q^2(J+1)^2S$ matrix:

$$\hat{\boldsymbol{P}}(d_t, s_{t+1}) = \begin{bmatrix} \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{P}}(d_t, s_{t+1}, m_Q) \end{bmatrix}$$

And finally aggregating across s_{t+1} to create the $(J+1)Q^2S^2 \times (J+1)^2Q^2S^2$ matrix:

$$\hat{\boldsymbol{P}}(d_t) = \begin{bmatrix} \hat{\boldsymbol{P}}(d_t, s_1) & & \\ & \ddots & \\ & & \hat{\boldsymbol{P}}(d_t, s_s) \end{bmatrix}$$

Looking to $(t+3)$ conditional on choosing $d_{t+2} = 0$, note that s_{t+2} is irrelevant but s_{t+1} and m_{t+1} affect d_{t+1} and so will affect m_{t+3} and q_0. Still, it is convenient to at least omit s_{t+2}. Thus the distribution of s_{t+3} conditional on s_{t+1}, m_{t+1} for all d_{t+1} is the $(J+1) \times (J+1)S$ matrix:

$$\check{\boldsymbol{\pi}}(s_{t+1}, m_{t+1}) = \begin{bmatrix} \boldsymbol{\pi_0}(s_{t+1})'\boldsymbol{\pi_0} & & \\ & \ddots & \\ & & \boldsymbol{\pi_J}(s_{t+1})'\boldsymbol{\pi_0} \end{bmatrix}$$

Aggregating across m_{t+1} contains no new information when creating the $(J+1)Q \times (J+1)QS$ matrix:

$$\check{\boldsymbol{\pi}}(s_{t+1}) = \begin{bmatrix} \check{\boldsymbol{\pi}}(s_{t+1}, \cdot) & & \\ & \ddots & \\ & & \check{\boldsymbol{\pi}}(s_{t+1}, \cdot) \end{bmatrix}$$

Finally we aggregate across s_{t+1} to create the $(J+1)QS \times (J+1)QS^2$ matrix:

$$\check{\boldsymbol{\pi}} = \begin{bmatrix} \check{\boldsymbol{\pi}}(s_1) & & \\ & \ddots & \\ & & \check{\boldsymbol{\pi}}(s_S) \end{bmatrix}$$

Next, define the distribution of period-$(t+3)$ choice sets conditional on $s_{t+1}, m_{t+1}, d_{t+1}$ and $d_{t+2} = 0$ for each possible state s_{t+3} as the $S \times QS$ matrix:

$$\check{\boldsymbol{M}}(s_{t+1}, m_{t+1}, d_{t+1}) = \begin{bmatrix} M(d_{t+1}, 0, s_1) & & \\ & \ddots & \\ & & M(d_{t+1}, 0, s_S) \end{bmatrix}$$

Aggregate across all d_{t+1} to obtain the $(J+1)S \times (J+1)QS$ matrix:

$$\check{\boldsymbol{M}}(s_{t+1}, m_{t+1}) = \begin{bmatrix} \check{\boldsymbol{M}}(s_{t+1}, m_{t+1}, 0) & & \\ & \ddots & \\ & & \check{\boldsymbol{M}}(s_{t+1}, m_{t+1}, J) \end{bmatrix}$$

Aggregate across all m_{t+1} to obtain the $(J+1)QS \times (J+1)Q^2S$ matrix:

$$\check{\boldsymbol{M}}(s_{t+1}) = \begin{bmatrix} \check{\boldsymbol{M}}(s_{t+1}, m_1) & & \\ & \ddots & \\ & & \check{\boldsymbol{M}}(s_{t+1}, m_Q) \end{bmatrix}$$

And aggregate across all s_{t+1} to obtain the $(J+1)QS^2 \times (J+1)Q^2S^2$ matrix:

$$\check{\boldsymbol{M}} = \begin{bmatrix} \check{\boldsymbol{M}}(s_1) & & \\ & \ddots & \\ & & \check{\boldsymbol{M}}(s_S) \end{bmatrix}$$

Finally, define the period-(t+3) surpluses from alternative 0 conditional on $s_{t+1}, m_{t+1}, d_{t+1}, s_{t+3}$ and $d_{t+2} = 0$ for all m_{t+3} as the $Q \times 1$ vector: $\hat{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}, d_{t+1}, s_{t+2}) = [q_0(0, s_{t+2}, m_1), \ldots, q_0(0, s_{t+2}, m_Q)]'$

Aggregate across all s_{t+3} to obtain the $QS \times 1$ vector:

$$\check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}, d_{t+1}) = \begin{bmatrix} \check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}, d_{t+1}, s_1) \\ \vdots \\ \check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}, d_{t+1}, s_S) \end{bmatrix}$$

Aggregate across all d_{t+1} to obtain the $(J+1)QS \times 1$ vector:

$$\check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}) = \begin{bmatrix} \check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}, 0) \\ \vdots \\ \check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, m_{t+1}, J) \end{bmatrix}$$

Repeat Q times across all m_{t+1} to obtain the $(J+1)Q^2S \times 1$ vector:

$$\check{\boldsymbol{q}}_0^{t+3}(s_{t+1}) = \begin{bmatrix} \check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, \cdot) \\ \vdots \\ \check{\boldsymbol{q}}_0^{t+3}(s_{t+1}, \cdot) \end{bmatrix}$$

Repeat S times across all s_{t+1} to obtain the $(J+1)Q^2S^2 \times 1$ vector:

$$\check{\boldsymbol{q}}_0^{t+3} = \begin{bmatrix} \check{\boldsymbol{q}}_0^{t+3}(\cdot) \\ \vdots \\ \check{\boldsymbol{q}}_0^{t+3}(\cdot) \end{bmatrix}$$