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Abstract

We study two reverse contests, A and B, with two agents, each of whom has both a linear reward function that increases in the agent's effort and an effort constraint. However, since the effort (output) of the agents has a negative effect on society, if the agents' effort constraints are relatively high, the designer in reverse contest A imposes a punishment such that the agent with the highest effort who caused the greatest damage is punished. Conversely, if the agents' effort constraints are relatively low, in reverse contest B, the designer awards a prize to the agent with the lowest effort who caused the smallest damage. We analyze the behavior of both symmetric and asymmetric agents in both contests A and B. In equilibrium, independent of the levels of the agents' effort constraints, both agents are active and they have positive expected payoffs. Furthermore, the agents might have the same expected payoff regardless of their asymmetric values of the prize/punishment or their asymmetric effort constraints.

JEL Classification: N/A

Keywords: N/A

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Reverse Contests*

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January 7, 2019

Abstract

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Keywords: Contests, prizes, punishments

JEL classification: D44, D72, D82

1 Introduction

We consider two firms that produce a homogenous product for which the production process yields some damage (for example, pollution). To deal with this situation, the regulator would want to reduce the firms'

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total production.[1] One way to do this could be to place a cap on the firms' production or to impose a tax that is relative to the level of production. In this paper, however, to attain this goal we assume that the regulator imposes a punishment on the firm with the highest production or offers a prize to the firm with the lowest production. For both options, each firm has an incentive to increase its production in order to increase its profit, but, on the other hand, it also has an incentive to reduce production in order to either win the prize or not be punished. As such, the firms' behavior in such an environment is not straightforward and our aim is to study how firms react to the regulator's punishment/prize policy. For this purpose, we consider a model with two agents, each of whom has a linear reward function that is a combination of his production and cost functions, and which increases in the agent's effort (output). In addition, each agent has an effort constraint which restricts his ability to produce an effort larger than this constraint. However, since the agents' production has a negative effect on society, the regulator either imposes a punishment on the agent with the highest effort or, alternatively, offers a prize to the agent with the lowest effort. The agents are not necessarily symmetric but may have asymmetric reward functions, asymmetric values of the prize/punishment, and asymmetric effort constraints. We refer to our model when the designer imposes a punishment on the agent with the highest effort as *reverse contest A,* and when the designer awards a prize to the agent with the lowest effort as *reverse contest B.*

The difference between contests A and B is that instead of awarding a prize to the agent with the lowest effort as in contest B, the agent with the highest effort has to pay a punishment in contest A. When there are n agents, contest A with one punishment for the agent with the highest effort is equivalent to a contest with $n-1$ identical prizes for all the agents except the one with the highest effort, while in contest B only the agent with the lowest effort wins a prize. Alternatively, contest B with one prize for the agent with the lowest effort is equivalent to a contest with $n-1$ identical punishments for all the agents except the one with the lowest effort, while in contest A only the agent with the highest effort is punished. Thus, if the number of agents is larger than two, there is no equivalence between contests A and B and this difference is derived from the fact that prizes and punishments do not have the same effect on agents' efforts (see Moldovanu et al. 2012 and Sela 2019). However, by the above argument, reverse contests A and B with only two agents are

[1] This situation is similar to rent seeking contests in which the designer wishes to reduce the contestants' expenditure.

equivalent, namely, they have the same mixed strategies given that the agents have to participate whether they have positive or negative expected payoffs. Note that if the effort constraints are relatively low agents may have negative expected payoffs in reverse contest A when they are punished, while in reverse contest B the agents always have positive expected payoffs in equilibrium since they are rewarded. Thus, as we assume that agents are rational and they participate only if they have positive expected payoffs, we study reverse contest A when the effort constraints are relatively high, and reverse contest B when the effort constraints are relatively low. Then, in both contests forms the agents have an incentive to participate since they have ex-ante positive expected payoffs.

We begin by analyzing reverse contest A with two agents when the agents are symmetric, namely, they have the same reward function, the same value for the punishment, and the same effort constraint, and afterwards when they are asymmetric. Then since each kind of asymmetry has a different effect on the agents' strategies as well as on their expected payoffs we study three sub-cases: 1) there are asymmetric reward functions; 2) there are asymmetric values for the punishment/prize; and 3) there are asymmetric effort constraints. Last, we study the case that combines all these three sub-cases. Similarly, we analyze the agents' behavior in reverse contest B when they are either symmetric or asymmetric.

In both contests A and B with two agents, if the agents are either symmetric or asymmetric they both use mixed strategies in equilibrium. While in contest A the agents randomly choose an effort from an interval between an effort larger than zero and their effort constraint, in contest B the agents randomly choose an effort from an interval between zero and their effort constraint. In contest A, the minimal effort of the agents (left point of the support of the agents' mixed strategy) is determined such that if an agent chooses this effort he is certainly not punished, and he has the same expected payoff as he chooses the effort constraint. In contest B, on the other hand, the minimal effort of the agents is zero since the value of the prize for the agent with the lowest effort is relatively high and then each agent has an incentive to choose the lowest effort.

Our results demonstrate that when the agents have asymmetric reward functions, they have different expected payoffs in both contest A and B. On the other hand, when they have asymmetric values for the punishment in contest A or asymmetric values for the prize in contest B, regardless of the agents' asymmetry, they have the same expected payoff. Furthermore, when the agents have asymmetric effort constraints, in contest A, regardless of the agents' asymmetry, they have the same expected payoff, while in contest B they have different expected payoffs. The differences between contests A and B are derived from the fact that each contest deals with different levels of effort constraints.

We also consider the reverse contests A and B with more than two agents. When the agents are symmetric, in both contests there is an equilibrium in which all the agents use the same mixed strategy. When the agents are asymmetric in reverse contest B, where the levels of the effort constraints are relatively low, the equilibrium is quite simple if we assume that only one agent obtains the prize, but if there is more than one agent with the lowest effort, no one wins the prize. Then, the two agents use the same mixed strategies as in the two agent contest, and all the other agents choose efforts that are equal to their effort constraints. On the other hand, when the agents are asymmetric in reverse contest A, and the levels of the efforts constraints are relatively high, the equilibrium is not simple and we are not able to provide a complete characterization of the equilibrium strategies.

In order to understand the uniqueness of our reverse contests, note that in reverse contest A all the agents except the one with the highest effort win the same prize. Similarly, in reverse contest B all the $n-1$ agents with the highest efforts are punished except the one with the lowest effort. Furthermore, although the equilibrium analyses of our reverse contests have features in common with the standard models of the all-pay auction under complete information (see, for example, Baye et al. 1986, Hillman and Samet 1987, Hillman and Reily 1989, Sela 2012, and Siegel 1989), especially those with multiple prizes (see, Barut and Kovenock 1998, and Clark and Riis 1998) and those with prizes for winning and losing (see, Baye et al. 2012), these contests are not strategically equivalent to our reverse contests.[2] To see this, in our contest when the agents have the same effort constraint, the distribution of the agents' mixed strategies is always strictly increasing while in the all-pay auction there is a gap between the effort constraint and the other possible efforts chosen by the agents (see Che and Gale 1998 and Hart 2016 and Cohen et al. 2019). As such, one of the unique results of our reverse contests and not of the all-pay auction, is that both agents, even when they are completely asymmetric, might have the same positive expected payoff.

[2] Contests are strategically equivalent if they generate the same best response functions, and as a result, the same equilibrium efforts (see Chowdhury and Sheremeta 2015)

2 Reverse contest A

Consider two agents, each of whom has both a production function $\beta_i(x_i) = \beta_i x_i, \beta_i > 1$ (where x_i is contestant i's effort) and an effort cost function $c(x_i) = x_i, i = 1, 2$. The designer imposes a punishment on the agent with the highest effort, and in a case that both agents have the highest effort then both are punished . Let $P_i, i = 1, 2$, be agent i's value for this punishment. We define agent i's reward function as $\alpha_i(x_i) = \beta_i(x_i) - c(x_i) = (\beta_i - 1)x_i = \alpha_i x_i, i = 1, 2$. Then, agent i's expected payoff is

$$u_i(x_i) = \begin{cases} \alpha_i x_i - P_i & \text{if } x_i \geq x_j \\ \alpha_i x_i & \text{if } x_i < x_j \end{cases}$$

In addition, agent i has an effort constraint of d_i such that $x_i \leq d_i, i = 1, 2$. Each agent chooses his effort in order to maximize his expected payoff given the effort of the other agent.

2.1 Symmetric contests

We begin by analyzing the agents' behavior in reverse contest A when the agents are symmetric, namely, they have the same reward function, $\alpha_i = \alpha, i = 1, 2$, the same value for the punishment if they win, $P_i = P$, $i = 1, 2$, and the same effort constraint $d_i = d, i = 1, 2$. The next result shows that if the agents' effort constraint is relatively high such that $d > \frac{P}{\alpha}$, both agents use the same mixed strategy equilibrium where the minimal effort (the left point of the support of the agents' mixed strategy) of the agents is positive while the maximal effort (the right point of the support of the agents' mixed strategy) is equal to their effort constraint. Furthermore, the expected payoff of both agents is the same and positive.

Proposition 1 *In reverse contest A with two symmetric agents, if $\alpha d > P$, there is a mixed strategy equilibrium in which agents $1, 2$ randomize on the interval $[d - \frac{P}{\alpha}, d]$ according to their cumulative distribution function $F(x)$ which is given by*

$$-PF(x) + \alpha x = -P + \alpha d$$

Thus, each agent's equilibrium effort is distributed according to the cumulative distribution function

$$F(x) = \frac{-P + \alpha(d - x)}{-P} \;, \;\; d - \frac{P}{\alpha} \leq x \leq d \tag{1}$$

Proof. See Appendix A. ■

The expected payoff of each agent is then

$$\pi_i = -P + \alpha d \ , \ i = 1, 2$$

The condition $\alpha d > P$ in Proposition 1 implies that all the agents have positive expected payoffs. Otherwise, they will stay out of the contest. The agents' expected total effort is

$$\begin{aligned} TE_A &= 2\int_{d-\frac{P}{\alpha}}^{d} xF'(x)dx = 2\int_{d-\frac{P}{\alpha}}^{d} x\frac{\alpha}{P}dx \\ &= 2d - \frac{P}{\alpha} \end{aligned} \tag{2}$$

As we could expect, the total effort increases in the values of the effort constraint and the marginal reward, but, on the other hand, it decreases in the value of the punishment.

2.2 Contests with asymmetric reward functions

We assume here that the agents are asymmetric such that they have different reward functions. Without loss of generality, $\alpha_1 \geq \alpha_2$. We also assume that agents have the same value for the punishment if they win, $P_i = P$, and the same effort constraint $d_i = d \ , \ i = 1, 2$. The next result shows that if the effort constraint is relatively high such that $d > \frac{P}{\alpha_1}$ then both agents use mixed strategies and compete not to be punished when they have different expected payoffs.

Proposition 2 *In reverse contest A with two asymmetric agents such that $\alpha_1 \geq \alpha_2$, if $\alpha_1 d > P$, there is a mixed strategy equilibrium in which both agents randomize on the intervals $[d - \frac{P}{\alpha_1}, d]$ according to their cumulative distribution functions $F_1(x), F_2(x)$ which are given by*

$$\begin{aligned} -PF_2(x) + \alpha_1 x &= -P + \alpha_1 d \\ -PF_1(x) + \alpha_2 x &= -\frac{\alpha_2}{\alpha_1}P + \alpha_2 d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-\frac{\alpha_2}{\alpha_1}P + \alpha_2(d-x)}{-P} & , \ d - \frac{P}{\alpha_1} \leq \ x < d \\ 1 & , \quad x \geq d \end{cases} \tag{3}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{-P + \alpha_1(d-x)}{-P} \ , \ \ d - \frac{P}{\alpha_1} \leq x \leq d \tag{4}$$

Proof. See Appendix A. ■

Then, the agents' expected payoffs are given by

$$\begin{aligned} \pi_1 &= -P + \alpha_1 d \\ \pi_2 &= -\frac{\alpha_2}{\alpha_1}P + \alpha_2 d \end{aligned}$$

Note that $\alpha_1 > \alpha_2$ implies that $\pi_1 > \pi_2$. The condition $\alpha_1 d > P$ implies that $\pi_i > 0$ for $i = 1, 2$, otherwise the agents will stay out of the contest. The expected total effort is

$$TE_A = \int_{d-\frac{P}{\alpha_1}}^{d} xF_1'(x)dx + \int_{d-\frac{P}{\alpha_1}}^{d} xF_2'(x)dx = 2d - P\frac{\alpha_1 + \alpha_2}{2\alpha_1^2} \tag{5}$$

When comparing the expected total effort in the symmetric (2) and asymmetric (5) contests with two agents, the expected total effort in the asymmetric contest with reward functions of $\alpha_i x$, $i = 1, 2$ is higher than the expected total effort in the symmetric contest in which the common reward function is either $\alpha_1 x$ or $\alpha_2 x$.

2.3 Contests with asymmetric punishments

We assume now that the agents are asymmetric such that they have different values for the punishment if they win where $P_1 \leq P_2$,. We also assume that agents have the same reward function, $\alpha_i = \alpha$, and the same effort constraint $d_i = d$ for $i = 1, 2$. The next result shows that if the agents' effort constraint is relatively high such that $d > \frac{P_1}{\alpha}$, then both agents use mixed strategies and compete not to be punished .In this case the equilibrium strategies are similar to when there are asymmetric reward functions except for the fact that both agents have the same expected payoff.

Proposition 3 *In reverse contest A with two asymmetric agents where $P_1 \leq P_2$, if $\alpha d > P_1$, there is a mixed-strategy equilibrium in which agents 1, 2, randomize on the intervals $[d - \frac{P_1}{\alpha}, d]$ according to their cumulative distribution functions $F_1(x), F_2(x)$ which are given by*

$$\begin{aligned} -P_1F_2(x) + \alpha x &= -P_1 + \alpha d \\ -P_2F_1(x) + \alpha x &= -P_1 + \alpha d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-P_1+\alpha(d-x)}{-P_2} \ , \ d - \frac{P_1}{\alpha} \leq \ x < d \\ \\ 1 \qquad , \qquad\qquad x \geq d \end{cases} \tag{6}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \frac{-P_1 + \alpha(d-x)}{-P_1} \ , \ d - \frac{P_1}{\alpha} \leq x \leq d \tag{7}$$

Proof. See Appendix A. ■

In that case, although the agents have asymmetric values for the punishment they have the same expected payoff

$$\pi_i = -P_1 + \alpha d \ , \ i = 1, 2$$

The condition $\alpha d > P_1$ implies that both agents have a positive expected payoff. Then, the agents' expected total effort is

$$TE_A = 2d - \frac{P_1(P_1 + P_2)}{2\alpha P_2} \tag{8}$$

When comparing the expected total effort in the symmetric (2) and asymmetric (8) contests with two agents, the expected total effort in the asymmetric contest with values of punishments P_i, $i = 1, 2$ is higher than the expected total effort in the symmetric contest with a common value of punishment of either P_1 or P_2.

2.4 Contests with asymmetric effort constraints

We assume here that agents have asymmetric effort constraints where $d_1 \geq d_2$. We also assume that agents have the same value of punishment $P_i = P$, and the same reward function $\alpha_i = \alpha$ for $i = 1, 2$. The next result shows that if the effort constraints are relatively high such that the highest effort constraint satisfies $d_1 > \frac{P}{\alpha}$ and $P > \alpha(d_1 - d_2)$, then the two agents use mixed strategies in which each of them chooses the effort that is equal to his effort constraint with a positive probability, namely the supports of the agents are not the same as in the previous cases. Similar to when there are asymmetric values for the punishments, regardless of the asymmetry, both agents have the same expected payoff.

Proposition 4 *In reverse contest A with two asymmetric agents where $d_1 \geq d_2$, if $\alpha d_1 > P$ and $P > \alpha(d_1 - d_2)$, there is a mixed-strategy equilibrium in which agent $i, i = 1, 2$ randomizes on the interval $[d_1 - \frac{P}{\alpha}, d_i]$ according to his cumulative distribution function $F_i(x)$ which is given by*

$$\begin{aligned} -PF_2(x) + \alpha x &= -P + \alpha d_1 \\ -PF_1(x) + \alpha x &= -P + \alpha d_1 \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = \begin{cases} \frac{-P+\alpha(d_1-x)}{-P} \quad , & x \leq d_2 \\ \frac{-P+\alpha(d_1-d_2)}{-P} \quad , & d_2 < x < d_1 \\ 1 \quad , & x \geq d_1 \end{cases} \tag{9}$$

agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \begin{cases} \frac{-P+\alpha(d_1-x)}{-P} \quad , & x < d_2 \\ 1 \quad , & x \geq d_2 \end{cases} \tag{10}$$

Proof. See Appendix A. ■

In that case, although the agents have asymmetric effort constraints, they have the same expected payoff

$$\pi_i = -P + \alpha d_1 \ , \ i = 1, 2$$

The condition $\alpha d_1 > P$ implies that the highest effort constraint is relatively high, and the condition $P > \alpha(d_1 - d_2)$ implies that the difference between the levels of both agents' effort constraints are relatively small. These conditions imply that both agents have a positive expected payoff, otherwise the agents will stay out of the contest. The condition $P > \alpha(d_1 - d_2)$ also implies that the interval $[d_1 - \frac{P}{\alpha}, d_2]$ exists, and if this condition does not hold then each agent will choose an effort that is equal to his effort constraint.

In that case, the agents' expected total effort is

$$TE_A = \int_{d_1 - \frac{P}{\alpha}}^{d_1} xF_1'(x)dx + \int_{d_1 - \frac{P}{\alpha}}^{d_2} xF_2'(x)dx2d_1 = 2d_1 - \frac{P}{\alpha} \tag{11}$$

When comparing the expected total effort in the symmetric (2) and asymmetric (11) contests with two agents, the expected total effort in the asymmetric contest with effort constraints d_i, $i = 1, 2$ is higher than the expected total effort in the symmetric contest with a common effort constraint of d_2 but is the same as the expected total effort in the symmetric contest with a common effort constraint of d_1.

2.5 Contests with asymmetric punishments, asymmetric reward functions and asymmetric effort constraints

We assume here that agents have asymmetric effort constraints where $d_1 \geq d_2$.The next result shows that if the agents' effort constraints are relatively high such that $\alpha_1 d_1 > P_1$ and $P_1 > \alpha_1(d_1 - d_2)$ then the two agents use mixed strategies and they have different expected payoffs.

Proposition 5 *In reverse contest A with two asymmetric agents where*

1) $d_1 \geq d_2$

2) $\alpha_1 d_1 > P_1$ *and* $P_1 > \alpha_1(d_1 - d_2)$,

3) $\frac{P_1}{\alpha_1} \leq \frac{P_2}{\alpha_2}$

there is a mixed-strategy equilibrium in which both agents randomize on the intervals $[d_1 - \frac{P_1}{\alpha_1}, d_i]$ *according to their cumulative distribution functions* $F_1(x), F_2(x)$ *which are given by*

$$\begin{aligned} -P_1 F_2(x) + \alpha_1 x &= -P_1 + \alpha_1 d_1 \\ -P_2 F_1(x) + \alpha_2 x &= -\frac{\alpha_2}{\alpha_1} P_1 + \alpha_2 d_1 \end{aligned}$$

Thus, agent 1*'s equilibrium effort is distributed according to the cumulative distribution function*

$$F_1(x) = \begin{cases} \frac{-\frac{\alpha_2}{\alpha_1} P_1 + \alpha_2(d_1 - x)}{-P_2} \quad , & x \leq d_2 \\ \frac{-\frac{\alpha_2}{\alpha_1} P_1 + \alpha_2(d_1 - d_2)}{-P_2} \quad , & d_2 < x < d_1 \\ 1 \quad , & x \geq d_1 \end{cases} \tag{12}$$

and agent 2*'s equilibrium effort is distributed according to the cumulative distribution function*

$$F_2(x) = \begin{cases} \frac{-P_1 + \alpha_1(d_1 - x)}{-P_1} \quad , & x < d_2 \\ 1 \quad , & x \geq d_2 \end{cases} \tag{13}$$

Proof. See Appendix. ■

In that case, the agents' expected payoffs are given by

$$\begin{aligned} \pi_1 &= -P_1 + \alpha_1 d_1 \\ \pi_2 &= -\frac{\alpha_2}{\alpha_1} P_1 + \alpha_2 d_1 \end{aligned}$$

Note that $\alpha_1 d_1 > P_1$ implies that $\pi_1 > \pi_2$. It also implies that $F_2(x)$ is well defined and the condition $\frac{P_1}{\alpha_1} \leq \frac{P_2}{\alpha_2}$ implies that $F_1(x)$ is well defined. The expected total effort is then

$$TE = \int_{d-\frac{P_1}{\alpha_1}}^{d_1} xF_1'(x)dx + \int_{d-\frac{P_1}{\alpha_1}}^{d_2} xF_2'(x)dx = 2d_1 - P\frac{\alpha_1 + \alpha_2}{2\alpha_1^2} \tag{14}$$

3 Reverse contest B

Consider now two agents, each of whom has both a production function $\beta_i(x_i) = \beta_i x_i, \beta_i > 1$ (where x_i is contestant i's effort) and an effort cost function $c(x_i) = x_i, i = 1, 2$. We define agent i's reward function as $\alpha_i(x_i) = \beta_i(x_i) - c(x_i) = (\beta_i - 1)x_i = \alpha_i x_i, i = 1, 2$. The designer allocates a prize to the agent with the lowest effort. Let $V_i, i = 1, 2,$ be agent i's value for this prize. Agent i's expected payoff is

$$u_i(x_i) = \begin{cases} \alpha_i x_i & \text{if } \; x_i \geq x_j \\ \alpha_i x_i + V_i & \text{if } \; x_i < x_j \end{cases}$$

In addition, agent i has an effort constraint of d_i such that $x_i \leq d_i, i = 1, 2$. Each agent chooses his effort in order to maximize his expected payoff given the efforts of the other agent.

3.1 Symmetric contests

Now we assume that the agents are symmetric, namely, they have the same reward function, $\alpha_i = \alpha, i = 1, 2$; the same value for the prize, $V_i = V,\ i = 1, 2$; and the same effort constraint $d_i = d, i = 1, 2$. Then, if the effort constraint is relatively low such that $d < \frac{V}{\alpha}$, the agents' mixed strategy equilibrium is given by

Proposition 6 *In reverse contest B with two symmetric agents, if $\alpha d < V$, there is a mixed strategy equilibrium in which agents $1, 2$ randomize on the interval $[0, d]$ according to their cumulative distribution function $F(x)$ which is given by*

$$V(1 - F(x)) + \alpha x = \alpha d$$

Thus, each agent's equilibrium effort is distributed according to the cumulative distribution function

$$F(x) = 1 - \frac{\alpha(d - x)}{V} \tag{15}$$

Proof. See Appendix B. ■

The agents have the same expected payoff

$$\pi_i = \alpha d \ , \ i = 1, 2$$

and their expected total effort is

$$TE_B = 2\int_0^d xF'(x)dx = 2\int_0^d \frac{ax}{V}dx = \frac{ad^2}{V} \tag{16}$$

Note that the symmetric equilibrium in reverse contest B and A are similar although they deal with different levels of the effort constraint.

3.2 Contests with asymmetric reward functions

We assume here that the agents are asymmetric in that they have different reward functions. Without loss of generality, $\alpha_1 \geq \alpha_2$. We also assume that agents have the same value for the prize, $V_i = V$, and the same effort constraint $d_i = d \ , \ i = 1, 2$. The next result shows that if the agents' effort constraint is relatively low such that $d < \frac{V}{\alpha_1}$ then the two agents use mixed strategies and they have different expected payoffs.

Proposition 7 *In reverse contest B with two asymmetric agents such that $\alpha_1 \geq \alpha_2$, if $\alpha_1 d < V$, there is an equilibrium in which both agents randomize on the intervals $[0, d]$ according to their cumulative distribution functions $F_i(x), i = 1, 2$ which are given by*

$$\begin{aligned} V(1 - F_2(x)) + \alpha_1 x &= \alpha_1 d \\ V(1 - F_1(x)) + \alpha_2 x &= \alpha_2 d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = 1 - \frac{\alpha_2(d - x)}{V} \tag{17}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = 1 - \frac{\alpha_1(d - x)}{V} \tag{18}$$

Proof. See Appendix B. ■

In that case, the agents' expected payoffs are given by

$$\pi_i = \alpha_i d, i = 1, 2$$

The condition $\alpha_1 d < V$ implies that both agents have an incentive to compete for the prize and this condition decreases their minimal effort to zero. Note that $\alpha_1 > \alpha_2$ implies that $\pi_1 > \pi_2$. The expected total effort is

$$\begin{aligned} TE_B &= \int_0^d xF_1'(x)dx + \int_0^d xF_2'(x)dx \\ &= \int_0^d \frac{a_2}{V} xdx + \int_0^d \frac{a_1}{V} xdx + \\ &= \frac{\alpha_1 + \alpha_2}{2V} d^2 \end{aligned} \tag{19}$$

3.3 Contests with asymmetric prizes

Here we assume that the agents are asymmetric in that they have different values for the prize where $V_i \geq V_2$. We also assume that agents have the same production function, $\alpha_i = \alpha$, and the same effort constraint $d_i = d$ for $i = 1, 2$. The next result shows that if the effort constraint is relatively low such that $d < \frac{V_2}{\alpha}$ both agents use mixed strategies and compete for the prize for which they have the same expected payoff.

Proposition 8 *In reverse contest B with two asymmetric agents when $V_1 \geq V_2$, if $\alpha d < V_2$, there is a mixed strategy equilibrium in which agents $1, 2$, randomize on the intervals $[0, d]$ according to their cumulative distribution functions $F_1(x), F_2(x)$ which are given by*

$$\begin{aligned} V_1(1 - F_2(x)) + \alpha x &= \alpha d \\ V_2(1 - F_1(x)) + \alpha x &= \alpha d \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = 1 - \frac{\alpha(d - x)}{V_2}; \tag{20}$$

agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = 1 - \frac{\alpha(d - x)}{V_1}; \tag{21}$$

Proof. See Appendix B. ■

The condition $\alpha d < V_2$ implies that both agents have an incentive to compete for the prize. In that case, although the agents have asymmetric values for the prize they have the same expected payoff

$$\pi_i = \alpha d \ , \ i = 1, 2$$

Their expected total effort is

$$\begin{aligned} TE_B &= \int_0^d xF_2'(x)dx + \int_0^d xF_1'(x)dx \\ &= \int_0^d \frac{\alpha}{V_1}xdx + \int_0^d \frac{\alpha}{V_2}xdx = \frac{\alpha d^2}{2V_1} + \frac{\alpha d^2}{2V_2} \end{aligned} \tag{22}$$

3.4 Contests with asymmetric effort constraints

We assume here that the two agents have asymmetric effort constraints where $d_1 \geq d_2$. We also assume that they have the same value of the prize $V_i = V$, and the same production function $\alpha_i = \alpha$ for $i = 1, 2$. The next result shows that the two agents use mixed strategies but they do not have the same expected payoff.

Proposition 9 *In reverse contest B with two asymmetric agents when $d_1 \geq d_2$, if $\alpha d_1 < V$, there is a mixed strategy equilibrium in which both agents randomizes on the interval $[0, d_2]$ according to his cumulative distribution function $F_i(x), i = 1, 2$, which is given by*

$$\begin{aligned} V(1 - F_1(x)) + \alpha x &= \alpha d_2 \\ V(1 - F_2(x)) + \alpha x &= \alpha d_1 \end{aligned}$$

Thus, agent 1's equilibrium effort is distributed according to the cumulative distribution function

$$F_1(x) = 1 - \frac{\alpha(d_2 - x)}{V}; \tag{23}$$

and agent 2's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \begin{cases} 1 - \frac{\alpha(d_1 - x)}{V}, x < d_2 \\ 1, x \geq d_2 \end{cases}; \tag{24}$$

Proof. See Appendix B. ■

The condition $\alpha d_2 < V$ implies that both agents have an incentive to compete for the prize. In that case, the agents' expected payoffs are given by

$$\pi_i = \alpha d_i \ , \ i = 1, 2$$

and their expected total effort is

$$TE_B = \int_0^{d_2} xF_2'(x)dx + \int_0^{d_2} xF_1'(x)dx + d_2\frac{\alpha(d_1-d_2)}{V} \quad (25)$$
$$= \frac{\alpha(d_2^2+d_1^2)}{2V} + d_2\frac{\alpha(d_1-d_2)}{V}$$

3.5 Contests with asymmetric reward functions, asymmetric prizes and asymmetric effort constraints

We assume here that agents have asymmetric effort constraints where $d_i \geq d_2$. The next result shows that if the effort constraints are relatively low such that $d_1 < \frac{V_1}{\alpha_1}$ and $d_2 < \frac{V_2}{\alpha_2}$ then the two agents use mixed strategies and have different expected payoffs.

Proposition 10 *In reverse contest B with two asymmetric agents where*

1) $d_1 \geq d_2$

2) $\alpha_1 d_1 < V_1$

3) $\alpha_2 d_2 < V_2$

there is a mixed strategy equilibrium in which both agents randomize on the intervals $[0, d_2]$ *according to their cumulative distribution functions* $F_1(x), F_2(x)$ *which are given by*

$$V_2(1-F_1(x)) + \alpha_2 x = \alpha_2 d_2$$
$$V_1(1-F_2(x)) + \alpha_1 x = \alpha_1 d_1$$

Thus, agent 1*'s equilibrium effort is distributed according to the cumulative distribution function*

$$F_1(x) = 1 - \frac{\alpha_2(d_2-x)}{V_2}; \quad (26)$$

agent n's equilibrium effort is distributed according to the cumulative distribution function

$$F_2(x) = \begin{cases} 1-\frac{\alpha_1(d_1-x)}{V_1}, x < d_2 \\ 1, x \geq d_2 \end{cases}; \quad (27)$$

Proof. See Appendix B. ■

The agents' expected payoffs are given by

$$\pi_i = \alpha_i d_i \text{ , } i = 1, 2$$

Note that if α_2 is sufficiently larger than α_1 then agent 2 has a higher expected payoff than agent 1. Also, the conditions $\alpha_i d_i < V_i, i = 1, 2$ imply that the distribution functions are well defined. In contrast to the case with only asymmetric effort constraints, here we require that both effort constraints be relatively small. Then, the agents' expected total effort is

$$\begin{aligned} TE_B &= \int_0^{d_2} xF_2'(x)dx + \int_0^{d_2} xF_1'(x)dx + d_2 \frac{\alpha_1(d_1 - d_2)}{V_{n-1}} \\ &= (\frac{\alpha_1}{V_1} + \frac{\alpha_2}{V_2})\frac{d_2^2}{2} + d_2 \frac{\alpha_1(d_1 - d_2)}{V_1} \end{aligned} \tag{28}$$

4 Reverse contests with more than two agents

4.1 Reverse contest A

Suppose that in reverse contest A there are n symmetric agents who have the same reward function, $\alpha_i = \alpha, i = 1, ..., n$, the same value for the punishment if they win, $P_i = P$, $i = 1, ..., n$, and the same effort constraint $d_i = d, i = 1, ..., n$. Then, agent i's expected payoff is

$$u_i(x_i) = \begin{cases} \alpha_i x_i - P_i \ \text{ if } \ x_i \geq \max_j x_j \\ \alpha_i x_i \ \text{ if } \ x_i < \max_j x_j \end{cases}$$

The next result shows that similar to the two-agent contest, all of the agents use the same mixed strategy equilibrium, and the expected payoff of all the agents is the same and positive.

Proposition 11 *In reverse contest A with n symmetric agents, if $\alpha d > P$, there is a mixed strategy equilibrium in which agents $1, ..., n$ randomize on the interval $[d - \frac{P}{\alpha}, d]$ according to their cumulative distribution function $F(x)$ which is given by*

$$-PF(x)^{n-1} + \alpha x = -P + \alpha d$$

Thus, each agent's equilibrium effort is distributed according to the cumulative distribution function

$$F(x) = \sqrt[n-1]{\frac{-P + \alpha(d - x)}{-P}} \ , \ d - \frac{P}{\alpha} \leq x \leq d$$

The expected payoff of each agent is

$$\pi_i = -P + \alpha d \ , \ i = 1, ..., n$$

If we assume that there are n asymmetric agents who have asymmetric reward functions, or asymmetric prizes, or asymmetric effort constraints there is no a simple generalization of the the equilibrium of the two-agent contest to larger contests with more than two agents.

4.2 Reverse contest B

Now assume that there are n symmetric agents who have the same reward function, $\alpha_i = \alpha, i = 1, ..., n$; the same value for the prize, $V_i = V$, $i = 1, ..., n$; and the same effort constraint $d_i = d, i = 1, ..., n$. Agent i's expected payoff is

$$u_i(x_i) = \begin{cases} \alpha_i x_i & \text{if } \ x_i \geq \min x_j \\ \alpha_i x_i + V_i & \text{if } \ x_i < \min x_j \end{cases}$$

Then, similar to the two-agent contest, the agents' mixed strategy equilibrium is given by

Proposition 12 *In reverse contest B with n symmetric agents, if $\alpha d < V$, there is a mixed strategy equilibrium in which agents $1, ..., n$ randomize on the interval $[0, d]$ according to their cumulative distribution function $F(x)$ which is given by*

$$V(1 - F(x))^{n-1} + \alpha x = \alpha d$$

Thus, each agent's equilibrium effort is distributed according to the cumulative distribution function

$$F(x) = 1 - \sqrt[n-1]{\frac{\alpha(d - x)}{V}}$$

The agents have the same expected payoff

$$\pi_i = \alpha d \ , \ i = 1, ..., n$$

If we assume that there are n asymmetric agents who have either asymmetric reward functions, asymmetric prizes, or asymmetric effort constraints then we obtain that two agents behave exactly as in reverse contest B with two agents and each of the other agents has no incentive to compete for the prize and he chooses an effort that is equal to his effort constraint.

5 Conclusions

We analyzed the equilibrium strategies of two agents in two types of contests referred to as reverse contest A in which the designer imposes a punishment on the agent with the highest effort and reverse contest B in which the designer awards a prize to the agent with the lowest effort. The main properties of these equilibria are as follows:

- The smallest point of the support of the agents' mixed strategies in contest A is zero while in contest B it is larger than zero.
- In the symmetric and asymmetric contests A and B, both agents use mixed strategies and have a positive expected payoff.
- If the agents have asymmetric reward functions in both contests A and B, they have different expected payoffs.
- If agents have asymmetric values for the prize/punishment in both contests A and B, all agents have the same expected payoff.
- In contest A, if agents have asymmetric effort constraints, the agents have the same expected payoff. In contest B, on the other hand, if agents have asymmetric effort constraints, they have different expected payoffs.

Accordingly, if the designer uses the right reverse contest, in the mixed-strategy equilibrium the agents have positive expected payoffs and therefore they both have an incentive to participate in the contests. In particular, the designer should use reverse contest A if the levels of the agents' effort constraints are relatively high and otherwise, if the levels of the agents' effort constraints are relatively small, he should use reverse contest B.

6 Appendix A

6.1 Proof of Proposition 1

The function $F(x)$ given by (1) is well-defined, strictly increasing on $[d-\frac{P}{\alpha}, d]$, continuous, satisfies $F(d-\frac{P}{\alpha}) = 0$ and $F(d) = 1$. Thus, $F(x)$ is a cumulative distribution function of a continuous probability distribution supported on $\left[d - \frac{P}{\alpha}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha d$ for any effort $x \in [d - \frac{P}{\alpha}, d]$. Since it can be easily shown that for agent 1, efforts below $d - \frac{P}{\alpha}$ would lead to a lower expected payoff than $-P + \alpha d$, and since efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha}, d]$ is a best response of agent 1 when agent 2 uses $F(x)$. By symmetry, any effort in $[d - \frac{P}{\alpha}, d]$ is a best response of agent 2 when agent 1 uses $F(x)$. Hence, $F(x)$ given by (1) is a symmetric mixed strategy equilibrium.

6.2 Proof of Proposition 2

The functions $F_i(x)$, $i = 1, 2$, given by (3) and (4) are well-defined, strictly increasing on $[d - \frac{P}{\alpha_1}, d]$, continuous, and satisfy both $F_1(d-\frac{P}{\alpha}) = F_2(d-\frac{P}{\alpha}) = 0$, and $F_2(d) = 1, F_1(d) = 1$, when agent 1 chooses the effort that is equal to d with a probability of $\frac{\alpha_1-\alpha_2}{\alpha_1} > 0$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $\left[d - \frac{P}{\alpha_1}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = -P + \alpha_1 d$ for any effort $x \in [d - \frac{P}{\alpha_1}, d]$. Since it can be easily shown that for agent 1, efforts below $d - \frac{P}{\alpha_1}$ would lead to a lower expected payoff than $-P + \alpha_1 d$ and that efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha_1}, d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = -\frac{\alpha_2}{\alpha_1}P + \alpha_2 d$ for any effort $x \in [d-\frac{P}{\alpha_1}, d)$. Since it can be easily shown that for agent 2, efforts below $d - \frac{P}{\alpha_1}$ and an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d - \frac{P}{\alpha_1}, d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$.Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (3) and (4) are a mixed strategy equilibrium.

6.3 Proof of Proposition 3

The functions $F_i(x)$, $i = 1, 2$, given by (6) and (7), are well defined, strictly increasing on $[d - \frac{P_1}{\alpha}, d]$, continuous, satisfy $F_1(d - \frac{P_1}{\alpha}) = F_2(d - \frac{P_1}{\alpha}) = 0$, and $F_2(d) = 1, F_1(d) = 1$, where agent 1 chooses the effort that is equal to d with a probability of $\frac{P_2 - P_1}{P_2} > 0$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $\left[d - \frac{P_1}{\alpha}, d\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = -P_1 + \alpha d$ for any effort $x \in [d - \frac{P_1}{\alpha}, d]$. Since it can be easily shown that for agent 1, efforts below $d - \frac{P_1}{\alpha}$ would lead to a lower expected payoff than $-P_1 + \alpha d$ and efforts above d are infeasible, any effort in $[d - \frac{P_1}{\alpha}, d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = -P_1 + \alpha d$ for any effort $x \in [d - \frac{P_1}{\alpha}, d)$. Since it can be easily shown that for agent 2, efforts below $d - \frac{P_1}{\alpha}$ as well as an effort that is equal to d would result in a lower expected payoff, and efforts above d are infeasible, any effort in $[d - \frac{P_1}{\alpha}, d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (6) and (7) are a mixed strategy equilibrium.

6.4 Proof of Proposition 4

The functions $F_i(x)$, $i = 1, 2$, given by (9) and (10), are well-defined, strictly increasing on $[d_1 - \frac{P}{\alpha}, d_2]$, continuous, and satisfy $F_1(d_1 - \frac{P}{\alpha}) = F_2(d_1 - \frac{P}{\alpha}) = 0$. Agent 2's mixed strategy satisfies $F_2(d_2) = 1$, when agent 2 chooses the effort that is equal to d_2 with a probability of $\frac{\alpha(d_1 - d_2)}{P} > 0$. Agent 1's mixed strategy satisfies $F_1(d_2) < F_1(d_1) = 1$, when agent 1 chooses the effort that is equal to d_1 with a probability of $\frac{\alpha(d_1 - d_2)}{P} > 0$.

Thus, $F_i(x), i = 1, 2$, is a cumulative distribution function of continuous probability distributions supported on $\left[d_1 - \frac{P}{\alpha}, d_i\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$. agent 1's expected payoff is $\pi_1 = -P + \alpha d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$. Since it can be easily shown that for agent 1, efforts below $d_1 - \frac{P}{\alpha}$ and between d_2 and $d_1$would lead to a lower expected payoff than $-P + \alpha d_1$ and that efforts above d_1 are infeasible, any effort in $[d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$

, agent 2's expected payoff is $\pi_2 = -P + \alpha d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2]$. Since it can be easily shown that for agent 2, efforts below $d_1 - \frac{P}{\alpha}$ would result in a lower expected payoff and efforts above d_2 are infeasible, any effort in $[d_1 - \frac{P}{\alpha}, d_2]$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (9) and (10) are a mixed strategy equilibrium.

6.5 Proof of Proposition 5

The functions $F_i(x)$, $i = 1, 2$, given by (12) and (13), are well-defined, strictly increasing on $[d_1 - \frac{P_1}{\alpha_1}, d_2]$, continuous, and satisfy $F_1(d_1 - \frac{P_1}{\alpha_1}) = F_2(d_1 - \frac{P_1}{\alpha_1}) = 0$. Agent 2's mixed strategy satisfies $F_2(d_2) = 1$, when agent 2 chooses the effort that is equal to d_2 with a probability of $\frac{\alpha_1(d_1-d_2)}{P_1} > 0$. Agent 1's mixed strategy satisfies $F_1(d_2) < F_1(d_1) = 1$, when agent 1 chooses the effort that is equal to d_1 with a probability of $1 - \frac{-\frac{\alpha_2}{\alpha_1}P_1+\alpha_2(d_1-d_2)}{-P_2} > 0$.

Thus, $F_i(x), i = 1, 2$, is a cumulative distribution function of continuous probability distributions supported on $\left[d_1 - \frac{P_1}{\alpha_1}, d_i\right]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = -P_1 + \alpha_1 d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2] \cup \{d_1\}$. Since it can be easily shown that for agent 1, efforts below $d_1 - \frac{P_1}{\alpha_1}$ and between d_2 and $d_1$would lead to a lower expected payoff than $-P_1 + \alpha_1 d_1$ and that efforts above d_1 are infeasible, any effort in $[d_1 - \frac{P_1}{\alpha_1}, d_2] \cup \{d_1\}$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = -\frac{\alpha_2}{\alpha_1}P_1 + \alpha_2 d_1$ for any effort $x \in [d_1 - \frac{P}{\alpha}, d_2]$. Since it can be easily shown that for agent 2, efforts below $d_1 - \frac{P_1}{\alpha_1}$ would result in a lower expected payoff and efforts above d_2 are infeasible, any effort in $[d_1 - \frac{P_1}{\alpha_1}, d_2]$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (12) and (13) are a hybrid equilibrium.

7 Appendix B

7.1 Proof of Proposition 6

The function $F(x)$ given by (15) is well-defined, strictly increasing on $[0, d]$, continuous, and satisfies both $F(0) = 1 - \frac{\alpha d}{V}$ and $F(d) = 1$. Thus, $F(x)$ is a cumulative distribution function of a continuous probability distribution supported on $[0, d]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F(x)$, agent 1's expected payoff is $\pi_1 = \alpha d$ for any effort $x \in [0, d]$. Since for agent 1 efforts above d are infeasible, any effort in $[0, d]$ is a best response of agent 1 when all the other agents use $F(x)$. By symmetry, any effort in $[0, d]$ is a best response of agent 2 when agent 1 uses $F(x)$. Hence, $F(x)$ given by (15) is a symmetric mixed strategy equilibrium.

7.2 Proof of Proposition 7

The functions $F_i(x)$, $i = 1, 2$, given by (17) and (18), are well-defined, strictly increasing on $[0, d]$, continuous, and satisfy $F_1(0) = \frac{\alpha_2 d}{V}$, $F_2(0) = \frac{\alpha_1 d}{V}$, and $F_1(d) = F_2(d) = 1$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $[0, d]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = \alpha_1 d$ for any effort $x \in [0, d]$. Since for agent 1, efforts above d are infeasible, any effort in $[0, d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$ agent 2's expected payoff is $\pi_2 = \alpha_2 d$ for any effort $x \in [0, d]$. Since for agent 2 efforts above d are infeasible, any effort in $[0, d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$.Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (17) and (18) are a mixed strategy equilibrium.

7.3 Proof of Proposition 8

The functions $F_i(x)$, $i = 1, 2$, given by (20) and (21), are well defined, strictly increasing on $[0, d]$, continuous, and satisfy $F_1(0) = \frac{\alpha d}{V_2}$, $F_2(0) = \frac{\alpha d}{V_1}$, and $F_2(d) = F_1(d) = 1$. Thus, $F_i(x)$, $i = 1, 2$ are cumulative distribution functions of continuous probability distributions supported on $[0, d]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = \alpha d$ for any effort $x \in [0, d]$. Since for agent 1 efforts above d are infeasible, any effort in $[0, d]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = \alpha d$ for any effort $x \in [0, d)$. Since for agent 2 efforts above d are infeasible, any effort in $[0, d)$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (20) and (21) are a mixed strategy equilibrium.

7.4 Proof of Proposition 9

The functions $F_i(x)$, $i = 1, 2$, given by (23) and (24) are well-defined, strictly increasing on $[0, d_2]$, continuous, and satisfy $F_1(0) = \frac{\alpha d_n}{V}$, and $F_2(0) = \frac{\alpha d_{n-1}}{V}$. Agent 2's mixed strategy satisfies $F_2(d_2) = 1$, when agent 2 chooses the effort that is equal to d_2 with the probability of $\frac{\alpha(d_1-d_2)}{V} > 0$. Agent 1's mixed strategy satisfies $F_1(d_2) = 1$. Thus, $F_i(x), i = 1, 2$, is a cumulative distribution function of continuous probability distributions supported on $[0, d_2]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent 1's expected payoff is $\pi_1 = \alpha d_1$ for any effort $x \in [0, d_2]$. Since it can be easily shown that for agent 1, efforts above d_2 would result in a lower expected payoff for agent 1, any effort in $[0, d_2]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = \alpha d_2$ for any effort $x \in [0, d_2]$. Since for agent 2 efforts above d_2 are infeasible, any effort in $[0, d_2]$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (23) and (24) are a mixed strategy equilibrium.

7.5 Proof of Proposition 10

The functions $F_i(x)$, $i = 1, 2$, given by (26) and (27) are well-defined, strictly increasing on $[0, d_2]$, continuous, and satisfy $F_1(0) = \frac{\alpha_2 d_2}{V_2}$, and $F_2(0) = \frac{\alpha_1 d_1}{V_1}$. Agent 2's mixed strategy satisfies $F_2(d_2) = 1$, when agent 2 chooses the effort that is equal to d_2 with the probability of $\frac{\alpha_1(d_1-d_2)}{V_1} > 0$. Agent 1's mixed strategy satisfies $F_1(d_2) = 1$. Thus, $F_i(x), i = 1, 2$, is a cumulative distribution function of continuous probability distributions supported on $[0, d_2]$. In order to see that the above strategies are an equilibrium, note that when agent 2 uses the mixed strategy $F_2(x)$, agent1's expected payoff is $\pi_1 = \alpha_1 d_1$ for any effort $x \in [0, d_2]$. Since it can be easily shown that for agent 1, efforts above d_2 would result in a lower expected payoff for agent 1, any effort in $[0, d_2]$ is a best response of agent 1 when agent 2 uses $F_2(x)$. Similarly, when agent 1 uses the mixed strategy $F_1(x)$, agent 2's expected payoff is $\pi_2 = \alpha_2 d_2$ for any effort $x \in [0, d_2]$. Since for agent 2 efforts above d_2 are infeasible, any effort in $[0, d_2]$ is a best response of agent 2 when agent 1 uses $F_1(x)$. Hence, the mixed strategies $(F_1(x), F_2(x))$ given by (26) and (27) are a mixed strategy equilibrium.

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