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MARKET SIZE, THE INFORMATIONAL CONTENT OF STOCK PRICES AND RISK: A MULTIASSET MODEL AND SOME EVIDENCE

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ABSTRACT

Market thinness can be an important determinant of the riskiness of stock returns, because it reduces the reliability of stock prices as predictors of future dividends. This paper analyses the relationship between market size and risk as the outcome of rational expectations equilibrium in a multiasset model with transaction costs, and shows that: (i) the conditional and measured variance of stock returns should be higher for thin issues ceteris paribus, and (ii) this thinness-variability relationship should arise only from the unsystematic component of the variance. These predictions are tested on data from the Milan Stock Exchange and appear to be supported by the evidence.

JEL classification: 026, 313

Keywords: market thinness, informational content of stock prices, variability of stock returns, unsystematic risk, transaction costs

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NON-TECHNICAL SUMMARY

One of the main functions of the stock market is to "pool" the private information available to market participants: the information thus contained in asset prices allows each trader to sharpen his forecast of future dividends. The model presented here shows that thinness may limit the market's ability to perform this task: in a thin market the individual errors contained in the private information of traders will not cancel out in the aggregate, due to the paucity of investors. As a result, the informational content of the price will be clouded by more noise than would be the case in a deeper market, and the conditional variance of stock returns will be correspondingly higher. This highlights the fact that, in addition to the risk originating in the intrinsic variability of dividends stockholders may also be exposed to another form of risk arising from the unreliability of the current price as a predictor of future dividends. This second source of risk is closely related to the degree of thinness of the market.

In other words, investing on a thin market can be riskier, irrespective of the variability of its "fundamentals". In this respect, the results of this paper complement those contained in CEPR Discussion Paper No. 146, where a similar link between thinness and risk is generated by differences in individual asset demand functions, rather than by informational imperfections. Beyond this general point, however, the focus of the two models is quite different: in Discussion Paper No. 146 the supply of equities, the number of transactors and risk are jointly endogenously determined within the model, and the analysis shows how their interaction may generate multiple equilibria within the market for a single stock. In this paper I take asset supplies as exogenously given (as is customary in finance theory) and analyse the effects of thin trading on the stochastic behaviour of stock prices within a multiasset pricing model of the Sharpe-Lintner variety.

This paper focuses on the testable implications of thin trading, based on explicit and consistent modelling of market equilibrium. This concern is not commonly found in the literature in this area, where more attention has been devoted so far to the statistical modelling of market thinness, its implications for the stock returns and to the problems of estimating asset pricing models when markets are thin. In general, thin trading has been imposed as an assumption; it has not been modelled consistently as an outcome of market equilibrium.

The logical structure of the model can be explained as follows. Each trader is assumed to possess some "noisy" private information about the future prospects of each firm. The price of a firm's stock summarizes this private information: the larger the number of traders, the larger the amount of information pooled in the stock price, and the more reliable the price becomes as an indicator of the firm's prospects. To put it differently, as the number of investors increases, the <u>conditional</u> variance of the stock return - i.e. the subjective uncertainty of each trade - decreases.

The number of traders in each market, on the other hand, is determined endogenously in the model. This is done by assuming that trade involves fixed costs, which may differ across investors. This assumption is crucial: otherwise everyone would trade in all markets, and the number of transactors would not differ across markets. It is shown that the equilibrium number of traders for each stock is positively related to the supply of that stock. As a result, the informational content of stock prices is higher for stocks that are available in larger quantities.

To see why the equilibrium number of traders is positively related to the supply of the relevant asset, one must consider how investors decide whether to enter the market for a given stock: this decision turns on whether, in equilibrium, the expected utility associated with purchasing the extra asset - inclusive of the disutility associated with the fixed cost - is larger than the expected utility of not doing so. The entry of additional investors into the market produces two effects. On the one hand, they reduce the variance of the stock's return, because the larger volume of trade increases the precision of the market forecast of future dividends, thus leading the price to reflect more closely the future dividend movements; on the other hand, since the supply of the stock is assumed to be fixed, new entrants tend to put upward pressure on the price, and thus to lower the expected return on the stock. A unique equilibrium will exist at the point where the increase in price makes the stock too expensive for additional entrants: the higher the given supply of the stock, the larger the number of traders at which this equilibrium is achieved, and thus the lower the variance of its return.

Interestingly, the model also yields testable implications about the observed variances of asset returns. This measured variance is also a decreasing function of the number of investors and the size of the outstanding asset supply in the model. This inverse relationship between market thinness and variance of returns arises in the unsystematic component of the estimated return variance, i.e. in the variance of the residual in the regression of asset returns on the common market factor.

In addition, the model suggests that investors with comparatively low transaction costs will invest in a larger set of assets: in particular, they will be more likely to include in their portfolios the stocks of smaller firms; conversely, investors with comparatively high costs will confine their portfolio selection to the stocks issued by larger corporations. Thus investors with low transaction costs will hold portfolios that are more diversified. Their portfolios may nevertheless yield a more variable rate of return because they will also include stocks of the smaller corporations that are characterized by more volatile returns.

These predictions of the theory are tested using monthly data on prices and turnover from the Milan Stock Exchange between 1976 and 1984, and appear to be supported by the evidence. The theory suggests that there is a negative relationship between the unsystematic risk of a stock and the number of market participants (or the size of asset supplies). The number of market participants is not observable but can be proxied by average turnover, while the total capitalization of the corresponding firms can serve as a proxy for the size of asset supplies. The model also suggests the functional form of the relationship between unsystematic risk and number of traders: unsystematic risk not only decreases with the number of traders but does so at a decreasing rate. The data indicate a significant negative relationship between unsystematic risk and the volume of trade. In addition, the functional form of the relationship conforms to that predicted by the model. A similar relationship is apparent between unsystematic risk and firms' capitalization at market prices. The relationship between total risk and average turnover is also of interest, although the model presented in this paper focuses on unsystematic risk. The data indicate that there is a negative though statistically insignificant relationship between the variance of observed asset returns and turnover. Inspection of the data suggested a far larger dispersion of variance estimates at low than at high turnover levels. The same regressions were then re-estimated with the White procedure to obtain heteroskedasticity-consistent standard errors. These indicate that the negative relationship between total risk and the square of turnover is significant at conventional confidence levels. This result conforms to those reported in other empirical works on financial markets.

Introduction

One of the main functions of the stock market is to aggregate private information of market participants via the price, thus allowing each trader to sharpen his conditional forecast of future dividends. The model presented here shows that thinness may limit the market's ability to perform this task: in a thin market the individual biases contained in the private information of traders will not cancel out in the aggregate, due to the paucity of investors, and, as a result, the informational content of the price will be clouded by more noise than in a deeper market, and the conditional variance of stock returns will be correspondingly higher. This highlights the fact that, beside the risk stemming from the intrinsic variability of dividends (the "technological" component of risk), stockholders may also be exposed to another form of risk, that derives from the unreliability of the current price as predictor of future dividends and is closely related to the degree of thinness of the market (the "thinness-related" component of risk).

In other words, investing on a thin market can be riskier, irrespective of the variability of its "fundamentals". In this respect, the results of this paper complement those contained in Pagano (1986), where a similar link between thinness and risk is generated by idiosyncratic shocks in individual asset demand functions, rather than by informational imperfections. Beyond this general point, however, the focus of the two models is quite different: the analysis of Pagano (1986) is concerned with the issue that market size and risk may be both *endogenously* determined, and shows how their interaction may generate multiple equilibria within the market for a *single* stock; here instead, I take asset supplies as *exogenously* given (as traditionally done in finance) and try to show that the effects of thin trading on the stochastic behaviour of stock prices can be consistently analyzed within a *multiasset* pricing model of the Sharpe-Lintner variety. As a result, this model produces testable predictions about cross-sectional differences in the observable return variances: (i) in thin markets stock returns are expected to be characterized not only by ceteris paribus higher *conditional* variance but also by higher *measured* variance; (ii) the thinness-variance relationship only involves the *unsystematic* (or *diversifiable*) *component* of the variance of stock returns, i.e. the variance of the regression residuals of market model equations.

Efforts in this direction had been quite limited so far: most of the studies on the effects of thin trading on stock prices have devoted more attention to statistical than to economic modelling, being primarily motivated by the estimation problems implied by thinness in market model regressions (see Cohen et al. (1976,1978,1980), Dimson (1979), Fowler, Rorke and Jog (1979), Fowler and Rorke (1983), Marsh and Rosenfeld (1985)); in general, thin trading has been imposed as an assumption, rather than consistently modelled within an equilibrium framework.

The plan of the paper is as follows. Section 1 presents a one period version of the model, and Section 2 extends it to a multiperiod setup, to explore the robustness of the results obtained in the previous section. In Section 3, I first discuss the issues involved in drawing a testable hypothesis from the propositions of the model, and some statistical problems that arise in implementing the test; I then report results obtained using monthly data from the Milan Stock Exchange, and relate them to existing evidence on other financial markets. Section 4 concludes the paper by summarizing its main points.

1. The model

The model can be regarded as the result of cross-fertilization of two quite different traditions in the finance literature: the models with fixed transaction costs (such as Mayshar (1983)) and those on information aggregation in stock market economies (such as Grossman (1976)). Fixed transaction costs are required to pin down the number of investors in each market endogenously (otherwise everyone would trade on all markets, and the number of transactors would not differ across markets). Information aggregation issues are introduced by the fact that I postulate each investor to bring in additional (noisy) information about the future prospects of each firm: the ability of prices to aggregate all this private information is crucial to the result that as the number of investors increases, the price becomes a more reliable predictor of the corresponding firm's prospects.

The setup can be described as follows:

(i) There are M+1 stocks, indexed by $j = 0, 1, 2, \ldots, M$. The dividend of stock 0 is:

$$d_0 = a_0 + u_0 \quad ; \tag{1.1}$$

whereas that of any other stock j can be written as:

$$d_j = a_j + b_j u_0 + u_j \quad ; \tag{1.2}$$

I assume the u_j's to be normally distributed with unconditional mean $E(u_j) = 0$ and variance $E(u_j^2) = \sigma^2_{uj}$, for $j = 0, 1, \ldots, M$; I also impose $E(u_j u_k) = 0$, for all $j \neq k$. In words, the dividends of stocks indexed from 1 to M are generated by a common factor u_0 and by an asset-specific factor u_j, that is supposed to be uncorrelated across stocks and to have zero unconditional mean. The dividend of asset 0, instead, is perfectly correlated with the market factor u_0. The fact that it has no idiosyncratic factor can be rationalized by regarding asset 0 as a portfolio of many assets like those from 1 to M: by the law of large numbers the idiosyncratic components of the various assets in such portfolio would tend to "cancel out" and only the market factor would be left.

(ii) All agents can buy and sell the riskless asset and stock 0 (or, under the interpretation just sketched, the assets that compose portfolio 0) without paying any fixed cost on their transactions; for the remaining assets, instead, agents incur a fixed cost for every stock they decide to add to their portfolio. This fixed cost varies across investors: let the set S of all the T investors "born" at each date be partitioned into G subsets S_g of cardinality N_g ($g=1,\dots,G$), such that all the agents belonging to subset S_g face the same fixed cost f_g, and that f_g is a strictly increasing function of g, i.e. $f_g > f_{g-1}$, $\forall$ g. Such different costs can be rationalized as stemming from diverse efficiency in operating transactions, such as different brokerage fees.

(iii) Before trade occurs, each agent i costlessly obtains some noisy information about the idiosyncratic shock u_j of stock j's dividend d_j. In particular I assume him to observe $u_j + e_{ij}$, i.e. the idiosyncratic shock itself plus an agent-specific noise term. It should be noticed, however, that this is only part of the information set Ω_i available to agent i: I let in fact each agent also condition his net orders for each asset on the market price, which — as will be shown below — in this model efficiently aggregates all relevant agent-specific information (is a "sufficient statistic"). Thus in equilibrium each investor will effectively be able to condition the expectation of the dividend d_j relevant for his demand of asset j, k_{ij}, on the entire information set available to all market j participants, so that each of these investors will end up holding the same conditional expectation of d_j. I assume that the agent-specific noise term e_{ij} is normally distributed over the set of agents S, with mean $E_S(e_{ij}) = 0$ and variance $E_S(e_{ij}^2) = \sigma_{ej}^2$. I further impose $E_S(e_{ij}e_{hj}) = E(e_{ij}e_{ik}) = 0$, for all $i \neq h$ and $j \neq k$, i.e. the noise contained in the agent-specific information is uncorrelated across agents and across stocks. Finally, I require these noise terms to be uncorrelated with the market factor u_0 and the firm-specific dividend disturbance u_j, i.e. $E(e_{ij} u_j) = 0, \quad \forall\, i, j$.

(iv) The supply of each of the M+1 assets is fixed: the number of shares of asset j will be denoted by K_j $(j = 0, 1, \ldots, M)$.

(v) All agents maximize a mean-variance utility function in terminal wealth. Thus agent i's problem is:

$$(1.3) \quad \underset{\{k_{ij}\}}{\text{Max}} \sum_{j=0}^{M} E\left[d_j - Rp_j \,|\, \Omega_i\right] k_{ij} - \frac{b}{2} \text{Var}\left[\sum_{j=0}^{M} d_j k_{ij} \,|\, \Omega_i\right] + Rw_0 - \sum_{j=1}^{M} f(k_{ij}, i)$$

where Ω_i = information set of agent i,

w_0 = initial wealth

and $f(k_{ij}, i) = \begin{cases} f_h & \text{if } i \in S_h,\ 1 \le h \le G, \text{ and } k_{ij} \neq 0 \\ 0 & \text{otherwise} . \end{cases}$

A couple of remarks are in order at this point before proceeding with the solution of the model. The fact that agents can trade costlessly in asset 0 but not in the other M risky assets is unpalatable but necessary, if the individual decision about entering the market for each asset is to be characterized in any generality, as already noted by Mayshar (1983) (who also ends up adopting this assumption in a multi-asset model with fixed transaction costs). What essentially the assumption buys is the separation of the decision of entering each market from that of entering all the others. The reason is that with costless trade in stock 0 an investor that buys shares of one of the remaining M stocks can offset the implied systematic variance by an appropriate sale or purchase of the market factor at no additional cost, so that total portfolio risk only goes up because of the additional unsystematic variance. In essence, the investor can costlessly transform each of the assets from 1 to M into stocks with uncorrelated returns -- and this explains why, when deciding on entry into one of the corresponding M markets, he can neglect his investment decisions about the other M-1. Lacking this assumption, one should consider all possible combinations of assets for all agents, and all the equilibria associated with each, before concluding that an agent will participate or not in a given market.

A second point that I want to address is why I require agent-specific information to be costless and thus disregard different costs in the collection of information as the possible rationale for the assumed diversity in the fixed cost f_g across the agents: the reason is that here information is efficiently summarized by market prices, and these are freely observable by market participants, so that diversity in informational costs could hardly induce any difference in the economic behaviour of our investors. In fact, even worse, with costly information one would incur into the well-known equilibrium existence problem raised by Grossman and Stiglitz (1976).

Let us now turn to the portfolio choices of an arbitrarily selected investor i, $i \in S_h$, $1 \leq h \leq G$. Since trade in the riskless asset and in asset 0 is costless, it is clear that he will participate in both these two markets. Whether he will also trade in any of the other M risky asset is a more complicated question, that needs several steps to be answered. The answer clearly turns on whether the expected utility associated with purchasing the extra asset (inclusive of the disutility of incurring the fixed cost f_h) is greater than the expected utility of not doing so.

This comparison must however be performed by evaluating the expected utility of investor i at the equilibrium level of his holdings of the asset under consideration, i.e. at the level that he would in fact purchase had he paid the fixed cost f_h and were he faced with the equilibrium price of the asset in that time period. The computation of equilibrium in each of the M risky assets' markets, in turn, requires various steps. In these markets prices are used not only as clearing device but also as a source of information on next period dividends, and this feeds back on their joint distribution with dividends: the appropriate equilibrium concept then is that of a rational expectation equilibrium (REE), i.e. one in which the conjectures that market participants make about the joint distribution of prices and dividends are self-fulfilling, as in Lucas (1972), Green (1973), Grossman (1976), Kreps (1977) and Diamond and Verrecchia (1981). The steps to compute such an equilibrium are thus: (a) formulate a conjecture that market participants might hold about the process generating the price, and derive the implied conditional moments of dividends; (b) compute investors' demands conditional on such moments; (c) impose market clearing; (d) equate the market clearing price with that generated by the process conjectured by traders. This fixed point of the process is the REE price.

Assume that each agent i forms the following conjectures about the equilibrium asset prices:

$$p_0 = A_0 , \tag{1.4}$$

$$p_j = A_j + B_j \left[u_j + \sum_{i \in T_j} \frac{e_{ij}}{T_j} \right] , \qquad j = 1,\ldots,M, \tag{1.5}$$

where A_0, A_j and B_j are non-stochastic and T_j denotes the set of investors participating to the jth market as well as its cardinality.

Conjecture (1.4) about the price of asset 0 can be immediately used in computing the REE for that asset. The first order condition (FOC) for investor i with respect to his holdings of asset 0, k_{i0}, is:

$$E(d_0) - Rp_0 = b \left[k_{i0} + \sum_{j=1}^{M} b_j k_{ij} \right] \sigma_0^2 , \tag{1.6}$$

where the expectation on the LHS need not be conditioned on Ω_i because all private information in the economy is by assumption orthogonal to d_0. Substituting $E(d_0) = a_0$ and imposing market equilibrium by setting $\sum_{i \in T_j} k_{i0} = K_0$ and $\sum_{i \in T_j} k_{ij} = K_j$, where K_0 and K_j are the net supplies of the corresponding assets, we obtain:

$$(1.7) \qquad a_0 - Rp_0 = b \left(\frac{K_0}{T} + \sum_{j=1}^{M} \frac{b_j K_j}{T} \right) \sigma_0^2 \quad ;$$

so that, equating (1.4) and (1.7), we have:

$$A_0 = \frac{1}{R} \left[a_0 - b \left(\frac{K_0}{T} + \sum_{j=1}^{M} \frac{b_j K_j}{T} \right) \sigma_0^2 \right] \quad .$$

For any other asset j, $j = 1,\ldots,M$, we have to compute the conditional moments of its dividend first, and then plug them back into the FOCs of each agent in T_j, use these to obtain the total demand for that asset, impose market equilibrium and equate the resulting expression to the original conjecture to solve for the undetermined coefficients A_j and B_j. The dividend of asset j (d_j, as given by (1.2)), the private information of agent i about that dividend ($u_j + e_{ij}$) and the market price (p_j; from conjecture (1.5)), form a vector of jointly normal random variables:

$$y = (d_j,\ u_j + e_{ij};\ p_j),$$

with unconditional mean:

$$m = (a_j;\ 0,\ A_j)$$

and covariance matrix:

$$V = \begin{bmatrix} b_j\sigma_0^2 + \sigma_{uj}^2 & \sigma_{uj}^2 & B_j\sigma_{uj}^2 \\ \sigma_{uj}^2 & \sigma_{uj}^2 + \sigma_{ej}^2 & B_j\left(\sigma_{uj}^2 + \frac{\sigma_{ej}^2}{T_j}\right) \\ B_j\sigma_{uj}^2 & B_j\left(\sigma_{uj}^2 + \frac{\sigma_{ej}^2}{T_j}\right) & B_j^2\left(\sigma_{uj}^2 + \frac{\sigma_{ej}^2}{T_j}\right) \end{bmatrix}$$

Let us denote by y_2 the subvector $(u_j + e_{ij},\ p_j)$ that contains all the information about d_j included in Ω_i (the information set of our investor), by Y_2 its realization and by m_2 its unconditional mean $(0,\ A_j)$. Let us also partition the covariance matrix accordingly:

$$V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & V_{22} \end{bmatrix}$$

where V_{22} is the covariance matrix of y_2.

The conditional distribution of d_j, given Y_2, is normal with mean :

$$(1.8) \qquad E(d_j|Y_2) = E(d_j|\Omega_i) = a_j + v_{12}V_{22}^{-1}(Y_2 - m_2) = a_j + q_j\left[u_j + \sum_{i \in T_j} \frac{e_{ij}}{T_j}\right] ,$$

where $q_j \equiv \dfrac{\sigma_j^2}{\sigma_j^2 + \dfrac{\sigma_{ej}^2}{T_j}}$,

and variance :

(1.9) $$\mathrm{Var}(d_j|Y_2) = \mathrm{Var}(d_j|\Omega_i) = v_{11} - v_{12}v_{22}^{-1}v_{21} = b_j^2\,\sigma_0^2 + q_j\,\frac{\sigma_{ej}^2}{T_j} \; .$$

Expressions (1.8) and (1.9) show that individual private information is redundant in equilibrium, since all private information is efficiently aggregated and revealed by the market price. They also show that the precision with which agents can forecast the next period dividend is an increasing function of the number of participants in market j.

Now turn to the FOC with respect to k_{ij}:

(1.10) $$E(d_j|\Omega_i) - Rp_j = b\left(\mathrm{Var}(d_j|\Omega_i)k_{ij} + \sum_{\substack{h=0\\h\neq j}}^{M} \mathrm{Cov}(d_j, d_h|\Omega_i)k_{ij}\right)$$

Substituting for the conditional expectation and variance of d_j from (1.8) and (1.9),equation (1.10) becomes:

(1.10') $$a_j + q_j\left[u_j + \sum_{i\in T_j}\frac{e_{ij}}{T_j}\right] - Rp_j = b\left[q_j\,\frac{\sigma_{ej}^2}{T_j}\,k_{ij} + b_j\sigma_0^2\left[\sum_{h=1}^{M} b_h k_{ih} + k_{i0}\right]\right]$$

Using (1.6), this can in turn be rewritten as:

$$(1.10'') \qquad a_j + q_j \left[u_j + \sum_{i \in T_j} \frac{e_{ij}}{T_j}\right] - Rp_j = bq_j \left[\frac{\sigma^2_{ej}}{T_j}\right] k_{ij} + b_j(a_0 - Rp_0) \quad .$$

Aggregating over all market j traders and imposing market equilibrium, we get:

$$(1.11) \qquad a_j + q_j \left[u_j + \sum_{i \in T_j} \frac{e_{ij}}{T_j}\right] - Rp_j = bq_j \left[\frac{\sigma^2_{ej}}{T_j}\right] \frac{K_j}{T_j} + b_j(a_0 - Rp_0) \quad ,$$

and, equating (1.11) to the original conjecture (1.5), we find the value of the undetermined coefficients:

$$A_j = \frac{1}{R}\left[a_j - bq_j \left[\frac{\sigma^2_{ej}}{T_j}\right] \frac{K_j}{T_j} - b_j(a_0 - Rp_0)\right] \quad , \qquad B_j = \frac{q_j}{R} \quad .$$

Equations (1.7) and (1.11) completely characterize the REE of the model, except for the fact that the T_j's -- the equilibrium number of traders in each market j -- are still to be determined. To perform this final step, one has to evaluate the indirect utility function at equilibrium, to determine for which markets entry will be utility-increasing. Using the FOCs (1.6) and (1.10) in the utility function of agent i (the maximand in (1.3)), one obtains the indirect utility function:

$$(1.12) \qquad \frac{b}{2}\sum_{j=0}^{M}\left[\frac{q_j\ \sigma_{ej}^2}{T_j}\right]k_{ij}^2 + \frac{(a_0 - Rp_0)^2}{2b\sigma_0^2} + Rw_0 - \sum_{j=1}^{M} f(k_{ij}, i) \quad ,$$

Since the second term is independent of T_j (see (1.7)) and so is the third, the only terms that would be affected by entry in the jth market are the first and the last. It is then apparent that, for investor $i \in S_h$; entry would be utility-increasing if

$$(1.13) \qquad \frac{b}{2}\left[\frac{q_j\ \sigma_{ej}^2}{T_j}\right]k_{ij}^2 \ > \ f_h \quad .$$

Equating the RHSs of (1.10') and (1.10"), one finds that $k_{ij} = \frac{K_j}{T_j}$; i.e. in equilibrium every market participant holds an equal share of the outstanding supply (which is not surprising in view of the fact that the only assumed difference between investors, viz. their private information, is removed by the workings of the market). This implies that (1.13) can be rewritten as:

$$(1.13') \qquad \frac{b}{2}\left[\frac{q_j\ \sigma_{ej}^2}{T_j}\right]\left[\frac{K_j}{T_j}\right]^2 \ > \ f_h \quad .$$

This lead us to the determination of the equilibrium number of traders in market j, T_j^*. Two situations can occur in this case: (i) that for no agent the (1.13') holds with equality, so that there is no marginal investor in market j, and (ii) that there is instead an agent for which (1.13') holds with equality, i.e. who happens to be indifferent between entry and no entry. In the first case, equilibrium is described by the inequality:

$$(1.14a) \qquad f_{h-1} < \frac{b}{2} q_j^* \left(\frac{\sigma_{ej}^2}{T_j^*}\right) \left(\frac{K_j}{T_j^*}\right)^2 < f_h ,$$

where $T_j^* \equiv \sum_{g=1}^{h-1} N_g$, $q_j^* \equiv \dfrac{\sigma_j^2}{\sigma_j^2 + \dfrac{\sigma_{ej}^2}{T_j^*}}$,

implying that each agent for whom the fixed cost is equal to, or less than, f_{h-1} will want to participate in the market for asset j, whereas the reverse will be true for investors faced with larger transaction costs. Alternatively, it can happen that there is a marginal agent within a cost class, say within class S_h, so that the equilibrium number of agents in market j, T_j^*, would be defined by the equality:

$$(1.14b) \qquad \frac{b}{2} \left(\frac{q_j^* \sigma_{ej}^2}{T_j^*}\right) \left(\frac{K_j}{T_j^*}\right)^2 = f_h ,$$

The two cases are graphically represented in Figures 1 and 2: the downward-sloping curve traces out the set of values that the middle term of

(1.14a) takes as T_j is allowed to vary continuously, whereas the increasing stepwise function depicts the levels of the fixed cost f_g -- the height of each segment being the fixed cost f_g faced by agents belonging to class S_g, and its length being their number N_g. If the downward-sloping function intersects the f_g locus at a point of discontinuity, i.e. passes in between the segments corresponding to two successive cost classes, as in Figure 1, then we have the case described in (1.14a). If instead it happens to intersect one of the segments of the f_g locus, as in Figure 2, then we are in a situation like that described by equation (1.14b).

The model thus determines the equilibrium number of traders T_j^* for each stock endogenously and uniquely. Conditions (1.14a) and (1.14b) tell us that the equilibrium number of traders will be ceteris paribus higher in markets for thick issues (since T_j^* is increasing in K_j). Since the conditional variance of the return on stock j is a decreasing function of the number of traders (as shown by (1.9)), the following proposition has been established:

Proposition 1 *In the one-period model described by assumptions (i) to (v), the conditional variance of stock returns is ceteris paribus inversely related to the number of market participants and to the size of the outstanding asset supply (i.e. it is larger for thin issues than for thick ones).*

Heuristically, the mechanism behind this result can be understood as follows: as investors enter the market for a stock, they produce two effects: (i) on one hand, they reduce the variance of its return, because the larger volume of trade increases the precision of the market forecast of future dividends, thus leading the price to reflect more closely the future movements of the dividend and to covary more closely with the latter; (ii) on the other hand, they tend to put upward pressure on the price, and thus to lower the expected return. A unique equilibrium will exist at the point where the increase in price makes the stock too expensive for additional entrants: the higher the given supply of the stock, the larger the number of traders at which this equilibrium is achieved, and thus the lower the variance of its return.

Interestingly, the model also yields testable implications about the *measurable* variances of asset returns, i.e. those that would be estimated by an uninformed econometrician (as opposed to their *conditional* variances, i.e. those perceived by an informed investor). Denoting the measured return on asset j by $r_j (\equiv d_j - p_j)$, the variance of r_j turns out to be:

$$(1.15) \qquad \mathrm{Var}(r_j) = b_j^2\sigma_0^2 + \sigma_{uj}^2\left[1 + \frac{\sigma_{uj}^2\left(\frac{1}{R^2} - \frac{2}{R}\right)}{\sigma_{uj}^2 + \frac{\sigma_{ej}^2}{T_j^*}}\right].$$

This expression is also decreasing in T_j^*, provided $R > 1/2$ (a condition that is very likely to be met): in particular, this inverse relationship arises from the second term on the RHS of the expression. In the present model, this term is the unsystematic (or diversifiable) risk of asset j, __i.e.__ the portion of risk not generated by correlation with the market factor (that here is represented by the common factor $r_0 = d_0 - p_0$). Empirically, this component of total return variance is measured by the variance of the residual in a regression of the jth asset's return r_j on the common factor r_0 : the slope regression coefficient is b_j and the residual variance is equal to $Var(r_j) - b_j^2\sigma_0^2$; __i.e.__ to the second term on the RHS of expression (1.15). This leads to:

__Proposition 2__. *Under assumptions (i) to (v), the estimated variance of stock returns is* __*ceteris paribus*__ *inversely related to the number of market participants and to the size of the outstanding asset supplies if the rate of return on the safe asset is larger than - 1/2. This relationship derives only from the unsystematic component of the estimated return variance,* __*i.e.*__ *the variance of the residual obtained by regressing asset returns on their common factor.*

Another prediction of the model concerns the allocation of stocks across investors:

__Proposition 3__. *Investors with comparatively low transaction costs will invest in a larger set of assets; in particular, they will be more likely to include in their portfolios also the stocks of smaller firms; conversely, investors with* *comparatively high costs will confine their portfolio selection to the stocks issued by larger corporations.*

This proposition can easily be verified by examining the entry condition (1.13). Thus investors of the first type will hold portfolios that are more diversified than those of the second type, but may nevertheless yield a more variable rate of return, because they will also include stocks of the smaller corporations, that (from Proposition 1) are <u>ceteris paribus</u> characterized by more volatile returns -- a factor that may compensate the risk reduction arising from the greater diversification.

2. A multiperiod extension

In the model of section 1, an increase in the number of traders reduces the conditional variance of stock returns by making the purchase price of stocks covary more closely with their dividend. It is not obvious that this result would extend to a setup where firms last for several periods, rather than liquidating a final dividend at the end of the second period. The reason is that, while in a two-period model the conditional variance of the return on a stock coincides with the conditional variance of its dividend, in a multiperiod setup it also involves the conditional variance of the resale price and the conditional covariance between dividend and resale price: since in thick markets resale prices will reflect future (and currently unknown) information about dividend shocks more closely than in thin markets, they should be harder to forecast on the basis on current information, i.e. their conditional variance should be ceteris paribus higher. Does this imply that in a multiperiod model the conditional variance of the overall return will be higher for thick issues than for thin ones, reversing the result obtained in the previous section ? Not really: this section in fact establishes the following result:

Proposition 4. *The results of Propositions 1 and 2 also apply in a multiperiod setup, if and only if the rate of return on the safe asset is strictly positive. Proposition 3, instead, extends without this qualification.*

Although market depth actually increases the variability of future prices, it turns out that (under this condition on the rate of interest) it reduces even more the conditional variance of the dividend and the conditional covariance between dividend and resale price, with the result that the riskiness of the total return on equity is again decreased by an increase in the number of traders.

To extend the model to many periods with minimum additional complexity, I assume that assets are held for one period only -- an assumption that can be motivated, for instance, by modelling the inflow and outflow of investors within an overlapping generations framework (as in Pagano, 1986). The return on stock j over a one period horizon (r_{jt+1}) is obviously the sum of its dividend (d_{jt+1}) and of its appreciation over the period ($p_{jt+1} - p_{jt}$):

(2.1) $$r_{jt+1} = d_{jt+1} + p_{jt+1} - p_{jt} \quad , \quad j = 0,1,\dots,M \; .$$

Dividends are, as in section 1, generated by a common factor (u_{0t+1}) and an idiosyncratic factor (u_{jt+1}), except for the dividend of asset 0, whose stochastic behaviour is determined by the common factor alone:

(2.2a) $$d_{jt+1} = a_j + b_j u_{0t+1} + u_{jt+1} \quad ; \; b_0 = 0; \quad j = 0,1,\dots,M \quad ;$$

for the sake of realism, however, I also assume that the disturbances u_{jt} have an autoregressive component:

(2.2b) $$u_{jt+1} = \rho\, u_{jt} + \epsilon_{jt+1} \quad , \quad j = 0,1,\dots,M \quad ,$$

so that the dividend on asset j at time t+1 can be written as:

(2.3) $$d_{jt+1} = (1-\rho)a_j + \rho\, d_{jt} + b_j \epsilon_{0t+1} + \epsilon_{jt+1} \; , \quad b_0 = 0; \quad j = 0,1,\dots,M.$$

The disturbances ϵ_{jt} are zero-mean, normally distributed random variables, with no serial and cross-sectional correlation ($E(\epsilon_{jt}\epsilon_{jt-h}) = E(\epsilon_{jt}\epsilon_{kt}) = E(\epsilon_{jt}\epsilon_{kt-h}) = 0, \; \forall \; j \neq k, \; i > 0$) and with variance $\sigma^2_{\epsilon j}$ (implying that the variance of u_{jt} is

$\sigma^2_{u_j} = \sigma^2_{\epsilon_j}/(1-\rho^2))$. Obviously the parameter ρ could well be assumed to differ across assets, but this would render the notation more cumbersome without adding any insight to the analysis.

At date t, each agent i costlessly obtains some noisy information concerning next period dividends on assets j to M: let us denote this noisy signal by $\epsilon_{jt+1} + e_{ijt+1}$ $(j = 1,2,\ldots,M)$, where $E(e_{ijt}e_{ijt-h-1}) = E(e_{ijt}e_{ikt-h}) = E(e_{jt}\epsilon_{jt-h}) = E(e_{jt}\epsilon_{kt-h}) = 0$, $\forall$ i, j, $k \neq j$, $h \geq 0$, and $e_{ijt} \sim N(0, \sigma^2_{e_j})$, $\forall$ j, t. The information set of agent i at time t is $\Omega_{it} = \{d_{0t}, p_{0t}; \epsilon_{jt+1} + e_{ijt+1}; d_{jt}, p_{jt}$, for $j = 1,\ldots,M\}$. Conditioning on this information, agent i solves the following problem:

$$(2.4)\quad \underset{\{k_{ijt}\}}{\text{Max}} \sum_{j=0}^{M} E(p_{jt+1} + d_{jt+1} - Rp_{jt} | \Omega_{it})k_{ijt} - \frac{b}{2}\,\text{Var}\left[\sum_{j=0}^{M}(p_{jt+1} + d_{jt+1})k_{ijt} | \Omega_{it}\right] + Rw_{0t} - \sum_{j=1}^{M} f(k_{ijt}; i) \quad ,$$

which is the same problem as that under (1.3) in the last section, except for the fact that all variables are now subscripted with a time index and that also the resale price (p_{jt+1}) of each asset appears in the maximand together with the corresponding dividend (d_{jt+1}).

Apart from these changes, the assumptions of section 1 remain valid. And so does, of course, the solution method illustrated there. In the multiperiod model, the REE prices turn out to be (see Appendix for derivations):

$$(2.5)\quad p_{0t} = \frac{\rho}{R-\rho} d_{0t} + \frac{1}{r}\left[\frac{R(1-\rho)}{R-\rho} a_0 - b\,\frac{R^2}{(R-\rho)^2}\left(\frac{K_0}{T} + \sum_{j=1}^{M} \frac{b_j K_j}{T_j}\right)\sigma^2_{\epsilon 0}\right] ,$$

$$(2.6)\quad p_{jt} = \frac{\rho}{R-\rho} d_{jt} + \frac{1}{r}\left[\frac{R(1-\rho)}{R-\rho} a_j - b\,\frac{q_j}{(R-\rho)^2}\left(R^2\,\frac{\sigma^2_{ej}}{T_j} + \sigma^2_{\epsilon j}\right)\frac{K_j}{T_j}\right.$$

$$\left. - b_j\left(E(p_{0t+1} + d_{0t+1}) - Rp_{0t}\right)\right] + \frac{q_j}{R-\rho}\left(\epsilon_{jt+1} + \sum_{i \in T_j} \frac{e_{ijt}}{T_j}\right) , \quad j = 1,\ldots,M, \ \forall\, t,$$

where $q_j \equiv \dfrac{\sigma^2_{\epsilon j}}{\sigma^2_{\epsilon j} + \dfrac{\sigma^2_{ej}}{T_j}}$,

and the conditional variances of the dividend, of the resale price and the overall return are respectively:

$$(2.7)\qquad \mathrm{Var}(d_{jt+1} | \Omega_{it}) = b_j^2 \sigma^2_{\epsilon 0} + q_j\,\frac{\sigma^2_{ej}}{T_j} ;$$

$$(2.8)\qquad \mathrm{Var}(p_{jt+1} | \Omega_{it}) = \left(\frac{1}{R-\rho}\right)^2\left[\rho^2\left(b_j^2\sigma^2_{\epsilon 0} + q_j\,\frac{\sigma^2_{ej}}{T_j}\right) + q_j \sigma^2_{\epsilon j}\right] ,$$

$$(2.9)\qquad \mathrm{Var}(r_{jt+1} | \Omega_{it}) = \left(\frac{1}{R-\rho}\right)^2\left[R^2 b_j^2 \sigma^2_{\epsilon 0} + q_j\left(R^2\,\frac{\sigma^2_{ej}}{T_j} + \sigma^2_{\epsilon j}\right)\right] .$$

Using the definition of q_j reported above in expressions (2.7),(2.8) and (2.9), it can easily be seen that: (i) the conditional variance of the dividend is decreasing and that of the resale price is increasing in the number of traders in market j; (ii) the conditional variance of the total return is decreasing in the number of traders in that market if and only if R > 1, i.e. for strictly positive rates of return on the safe asset.

The next step is to determine T_j^*, i.e. the equilibrium number of traders in market j: as in section 1, we evaluate the utility function at the optimum to find the set of agents that will derive a utility increase from investing in asset j. The value of the utility of agent i at time t, at the optimum, is (see Appendix for derivations):

$$(2.10) \qquad \frac{b}{2}\sum_{i=1}^{M}\left(\frac{1}{R-\rho}\right)^2 q_j\left(R^2\,\frac{\sigma_{ej}^2}{T_j} + \sigma_{\epsilon j}^2\right)k_{ijt}^2 + \frac{1}{2b\sigma_{\epsilon 0}^2}\,[E(p_{0t+1} + d_{0t+1}) - Rp_{0t})]$$

$$+ Rw_{0t} + \sum_{i=1}^{M} f(k_{ijt};\ i) \quad .$$

Since the second and third term of (2.10) are independent of T_j, entry in market j will be utility increasing provided:

$$(2.11) \qquad \left(\frac{1}{R-\rho}\right)^2 q_j\left(R^2\,\frac{\sigma_{ej}^2}{T_j} + \sigma_{\epsilon j}^2\right)k_{ijt}^2 \;>\; f_h \quad ,$$

for investor $i \in S_h$. Since in equilibrium it can be shown that $k_{ijt} = K_j/T_j$: the equilibrium number of investors in market j, T_j^*: is pinned down by the following inequality:

$$\text{(2.12a)} \qquad f_{h-1} \;>\; \left(\frac{1}{R-\rho}\right)^2 q_j^* \left(R^2 \frac{\sigma_{ej}^2}{T_j^*} + \sigma_{\epsilon j}^2\right)\left(\frac{K_j}{T_j^*}\right)^2 \;>\; f_h \;;$$

where $T_j^* = \sum_{g=1}^{h-1} S_g$ and $q_j^* \equiv \dfrac{\sigma_{\epsilon j}^2}{\dfrac{\sigma_{ej}^2}{T_j^*} + \sigma_{\epsilon j}^2}$,

if there is no marginal investor; alternatively, it is implicitly determined by the equality:

$$\text{(2.12b)} \qquad \left(\frac{1}{R-\rho}\right)^2 q_j^* \left(R^2 \frac{\sigma_{ej}^2}{T_j^*} + \sigma_{\epsilon j}^2\right)\left(\frac{K_j}{T_j^*}\right)^2 = f_h \;;$$

if there is a marginal investor within cost class S_h. The two cases are exactly analogous to those of (1.14a) and (1.14b) in section 1, and can be equally well illustrated by figures 1 and 2, provided the downward-sloping locus is renamed to be the expression in the middle of (2.12a) (and on the LHS of (2.12b)): this expression can in fact be shown to be monotonically decreasing in T_j. This proves that, also in a multiperiod setup, the model with fixed asset supplies produces a unique equilibrium, with a number of traders T_j^* that is increasing in the size of the outstanding asset supply K_j. Since it has been shown above (point (ii)) that the conditional variance of the total return on asset j, $Var(r_{jt+1}|\Omega_{it})$, is a decreasing function of the number of traders T_j *if (and only if) $R > 1$*, we have established that, *with this qualification*, Proposition 1 applies also in this multiperiod framework: assets in larger supply will be characterized by a ceteris paribus lower conditional return variance.

The next step is to prove that the same can be said of the *measured* return variance (i.e. that also Proposition 2 generalizes if $R > 1$). Simple computations show that the measured variance of the total return of asset j is:

$$(2.13) \qquad Var(r_{jt+1}) = \left(\frac{1}{R-\rho}\right)^2 \left[\left(\frac{r^2\rho^2}{1-\rho^2} + R^2\right)\left(b_j^2\sigma_{\epsilon 0}^2 + \sigma_{\epsilon j}^2\right) + 2q_j^*\sigma_{\epsilon j}^2(1-R^2)\right] \quad ;$$

and its unsystematic (or diversifiable) component is:

$$(2.14) \qquad Var(r_{jt+1} - b_j r_{0t+1}) = \left(\frac{1}{R-\rho}\right)^2 \sigma_{\epsilon j}^2 \left[\left(\frac{r^2\rho^2}{1-\rho^2} + R^2\right) + 2q_j^*(1-R^2)\right] \quad .$$

It is easy to see that also in this case the condition $R > 1$ is necessary and sufficient to make both variances decreasing functions of T_j^*: q_j^* is in fact increasing in T_j^*, and it is multiplied by a negative sign if and only if that condition is met. It is also clear that the negative relationship between the measured return variance and the number of transactors T_j^* arises only from the nonsystematic component of the variance -- exactly as in the one-period framework of section 1. And, just as in that instance, also in this case the slope regression coefficient of r_{jt} on the common factor r_{0t} is equal to b_j, so that the magnitude in expression (2.14) can be obtained simply by computing the variance of the estimated residuals from such regression.

Heuristically, the reason why the multiperiod extension of the results of section 1 hinges so crucially on a positive rate of return on the safe asset can be explained as follows. As explained at the start of this section, in a multiperiod setup an increase in the number of traders has two effects: (i) it improves the informational content of the current price, thus increasing its covariance with future dividends and prices and reducing the variance of the overall return; (ii) it also increases the variance of future prices themselves, that will covary more closely with dividends even further into the future. However, since prices discount future dividend shocks at the rate of interest, the increase in the variance of future prices will be dampened if the rate of interest is positive, and the effect under (ii) will be more than compensated by that under (i).

To complete the proof of Proposition 4, it remains to be shown that the allocation of assets across investors predicted in Proposition 3 extends to the multiperiod case, irrespective of the value of the rate of interest: that this is the case, can immediately be seen by inspecting the entry condition (2.11).

3. Testing the theory

The propositions derived in the preceeding sections can be tested on a cross-section of stocks listed on the same Exchange. There are, however, a few preliminary issues to be tackled. First, due to the stylized character of the analysis, the propositions of the model do not translate immediately into testable predictions: it is then important to check if the model can be regarded as an approximation to a more realistic (and complex) one, and thus its predictions should be treated as empirically relevant. Second, having selected empirical counterparts for the theoretical variables, one has to extract from the model all the testable restrictions on the relationships to be estimated. Third, since implementing the test will require estimates of (total and unsystematic) risk, it will be necessary to face the statistical problems that market thinness poses in this respect. After dealing in turn with these three issues, this section will present results obtained on data from the Milan Stock Exchange.

(i) Robustness of the theory to more realistic assumptions.

Even in its multiperiod version, the model retains two features that are visibly at odds with the functioning of actual equity markets: (1) the *entire* outstanding supply of each asset changes hands in every period, so that there is no distinction between supply of the asset and volume of trade (and between holders and traders); (2) dividends are distributed at *every* date in which trade occurs.

In reality only a portion of the total outstanding supply of an asset is traded at each date, so that the trading volume and the supply of an asset are distinct empirical magnitudes. The model above could accommodate this distinction by postulating a holding period longer than unity -- for instance, by assuming that each generation of investors lives for several periods and that the transaction cost f_h makes portfolio rearrangements too expensive for all classes of investors. If this were the case, the variance of stock returns would still be a decreasing function of the *number of market participants*, since this is the variable that affects the informational content of stock prices; moreover, the number of traders would be positively related -- though not identical -- to that of asset holders (and trading volume to the supply of the asset). Thus the predictions of the model of the preceeding sections would still stand up.

As for the fact that the distribution of dividends occurs more infrequently than trade on the stock market, the multiperiod version of the model can be easily reinterpreted to encompass this case. Suppose that dividends are distributed at fixed intervals of length τ ($\tau > 1$), and let us denote the dividend distributed by firm j at time τ by $D_{j\tau}$. Also assume that the d_{jt}'s (rather than being actual dividends) are pieces of *certain* information revealed at time t about (the discounted value of) a portion of the dividend to be distributed at time τ by firm j, so that $\sum_{h=1}^{\tau} R^{\tau-h} d_{jh} = D_{\tau}$ (one could imagine the firm as setting aside d_{jh} in each period h out of its earnings for future distribution to its shareholders, and reinvesting it at the safe rate of return until time τ). If so reinterpreted, the model's results would apply exactly also to the case of intermittent distribution of dividends. One can conjecture that a modified version of those results may still apply if (more realistically) the d_{jt}'s were assumed to be *uncertain* information about future dividends.

(ii) Test specification

Proposition 2 states the hypothesis to be tested: that of a negative relationship between unsystematic risk and the number of market participants (or the size of asset supplies). The number of market participants, that is not observable per se, can be proxied by average turnover (and the size of asset supplies by the total capitalization of the corresponding firms). In the multiperiod model the functional form of the predicted relationship between unsystematic risk and number of traders (T_j^*) is shown in equation (2.14): if R > 1, it can be easily shown that unsystematic risk (hereafter USR_j) not only decreases in T_j^*, but does so at a decreasing rate. Thus, if indeed average turnover is a valid proxy for T_j^*; our hypothesis constrains the functional form linking unsystematic risk to turnover to be decreasing and strictly convex.

In fact one can go even further than this: one can try to fit the very functional form in (2.14) to the data. Turning (2.14) into an identified relationship requires however two bold assumptions: (i) that $\sigma^2_{\epsilon j}$ and $\sigma^2_{e j}$ have the same values for all assets (to be denoted σ^2_{ϵ} and σ^2_{e} respectively), so that they can be treated as parameters in a cross-sectional regression; (ii) that T_j^* is linked by a non-stochastic relationship to average turnover VOL_j (e.g. $VOL_j = \lambda T_j$, λ being a fixed proportionality factor). Unfortunately these assumptions leave us with the problem that the relationship to be estimated would be exact, rather than stochastic. But the model can easily be modified so as to produce an additive disturbance term in equation (2.14). For instance, suppose that we add another asset-specific disturbance z_{jt+1} in expression (2.2a) for the dividend on asset j:

$$(3.1) \qquad d_{jt+1} = a_j + b_j u_{0t+1} + u_{jt+1} + z_{jt+1} \quad ,$$

where z_{jt+1} is a zero-mean noise term orthogonal to all information known at time t by all agents (a pure surprise term). For simplicity, assume that it has no persistence, and denote its variance by σ^2_{zj}, and the average value of σ^2_{zj} by σ^2_z. Then the expression for unsystematic risk (the analogue of (2.14)) can be shown to be:

$$(3.2) \quad Var(r_{jt+1} - b_j r_{0t+1}) = \sigma^2_z + \left(\frac{1}{R-\rho}\right)^2 \sigma^2_\epsilon \left[\left(\frac{r^2\rho^2}{1-\rho^2} + R^2 \right) + 2q^*(1-R^2) \right] + (\sigma^2_{zj} - \sigma^2_z),$$

where $q^* \equiv \dfrac{\sigma^2_\epsilon}{\dfrac{\sigma^2_e}{T^*_j} + \sigma^2_\epsilon}$.

The deviations from the mean $(\sigma^2_{zj} - \sigma^2_z)$ can thus be regarded as an additive disturbance in a cross-sectional regression. If these assumptions are accepted, the equation to be estimated is then:

$$(3.3) \qquad USR_j = \alpha_0 + \frac{\alpha_1}{1 + \frac{\alpha_2}{VOL_j}} + \eta_j$$

where $\alpha_0 = \sigma^2_z + \left(\frac{1}{R-\rho}\right)^2 \sigma^2_\epsilon \left[\left(\frac{r^2\rho^2}{1-\rho^2} + R^2\right]\right.$, $\alpha_1 = \left(\frac{1}{R-\rho}\right)^2 \sigma^2_{\epsilon j} 2(1-R^2)$, $\alpha_2 = \frac{\sigma^2_{ej}}{\lambda\sigma^2_\epsilon}$

and $\eta_j = \sigma^2_{zj} - \sigma^2_z$.

The mantained hypothesis predicts then that, if $R > 1$, the parameter α_1 will be negative and that (irrespective of the value of R) α_0 and α_2 will be positive.

(iii) Thinness-related problems in the measurement of risk.

Trading volume -- the same variable that according to the mantained hypothesis should be negatively related to risk -- is also at the root of potentially severe biases in the measurement of risk. The reason for this is that low trading volume generally implies less frequent transactions, and this can in turn cause transaction prices for thin issues to be desynchronized from those of most other assets. This desynchronization can take two forms: that of different timing of transaction prices within the measurement interval, and that -- more extreme -- of complete absence of recorded transactions at certain dates. In the latter case, since no transaction prices are available for those dates, in most Exchanges it is customary to substitute the last recorded transaction price for the missing observation, so that for thinly traded issues one will often encounter strings of "stale" prices with the same value, generally followed by sudden jumps when an actual transaction takes place and the price has an opportunity to adjust to the intervened change in market fundamentals.

Both facts -- desynchronization within the measurement interval and intermittent trading -- cause conventional estimates of variances and covariances of asset returns to be biased. In addition, intermittent trading can also induce substantial losses in efficiency.

Let us consider first the *biases arising from desyncronization within the measurement interval*, a problem effectively tackled by Scholes and Williams (1977) [1]. Desynchronization implies that "under plausible restrictions on the trading processes, measured variances for single securities overstate true variances, while measured contemporaneous covariances understate in absolute magnitude true covariances" (ibidem, p. 310). In particular, consider an asset n with measured logarithmic return r^s_{nt}, and let the mean and variance of its true (unobserved) return r_{nt} be respectively μ_n and σ^2_n. If the nontrading intervals for that asset -- denoted by s_n -- are independently and identically distributed over time with variance $Var(s_n)$, it can be shown that the conventional estimator of the variance of logarithmic returns, $Var(r^s_{nt})$, equals:

(3.4) $$Var(r^s_{nt}) = \sigma^2_n + 2\,Var(s_n)\,\mu^2_n \;;$$

i.e. the measured variance is upward biased. The bias is greater the more unevenly traded is the security (the larger $Var(s_n)$) and the larger the mean rate of return (μ_n). As the latter is generally farther away from zero as the measuring interval lengthens, the lower the frequency of observations, the more severe the potential bias (Marsh and Rosenfeld (1985) in fact contend that for daily returns the bias is virtually absent). Since the first autocovariance of the measured rate of return is:

$$(3.5) \qquad Cov(r^s_{nt}, r^s_{nt-1}) = - Var(s_n)\ \mu^2_n \quad ,$$

the bias in (3.4) can be removed by making use of the variance estimator:

$$(3.6) \qquad \sigma^2_n = (1 + 2\ \hat{\rho}^s_n)\ Var(r^s_{nt}) \quad ,$$

where $\hat{\rho}^s_n$ is the sample first (simple) autocorrelation coefficient of the measured rate of return r^s_{nt}.

Similarly, it can be shown that the measured covariance between the return on asset n and asset m is given by:

$$(3.7) \quad Cov(r^s_{nt}, r^s_{mt}) = (1 - E[max(s_n, s_m) - min(s_n, s_m)])\ \sigma_{nm} + 2Cov(s_n, s_m)\ \mu_n\mu_m \ ,$$

where $\sigma_{nm} \equiv Cov(r_{nt}, r_{mt})$, the covariance between the true returns of the two assets. Assuming that the measurement interval is small, so that the magnitude of the means product $\mu_n \mu_m$ can be regarded as second order, the bias will arise from the term in curly brackets: the measured covariance will in general be biased towards zero relative to the true covariance σ_{nm}, and will be largest when one of the two securities is traded much more infrequently than the other.

These biases in the conventional moment and comoment estimators obviously extend to the OLS estimates of the constant and slope coefficients in the market model equation:

$$r^s_{nt} = \alpha^s_n + \beta^s_n r^s_{Mt} + \epsilon^s_{nt} \quad , \tag{3.8}$$

where r^s_{Mt} is the measured logarithmic rate of return on the market portfolio. Moreover, since the problem arises from errors in variables, the bias of the OLS estimators will not disappear as the sample size increases. This also applies to the estimates of the error term $\hat{\epsilon}^s_{nt}$, and thus to the conventional estimate of unsystematic risk -- the variance of OLS residuals from market model regressions. Scholes and Williams (1977) have shown how to correct for these biases, deriving the following consistent estimators of the constant and slope coefficients of market model equations [2]:

$$\alpha_n = \sum_{t=2}^{T-1} r^s_{nt}/(T-2) - \beta_n \sum_{t=2}^{T-1} r^s_{Mt}/(T-2) \quad ; \tag{3.9a}$$

and

$$(3.9b) \qquad \beta_n = (b_n^{-1} + b_n + b_n^{+1})/(1 + 2\,\hat{\rho}_{1,M}) \quad ,$$

where

b_n^{-1} = OLS estimate of the slope coefficient from the regression of r_{nt}^s on r_{Mt-1}^s,
b_n = OLS estimate of the slope coefficient from the regression of r_{nt}^s on r_{Mt}^s,
b_n^{+1} = OLS estimate of the slope coefficient from the regression of r_{nt}^s on r_{Mt+1}^s,
and $\hat{\rho}_{1,M}$ = first order sample autocorrelation for r_{Mt}^s.

The variance of the estimated residuals obtained from such estimators, i.e.:

$$(3.9c) \qquad \sum_{t=2}^{T-1} \epsilon_{nt}^2/(T-4) = \sum_{t=2}^{T-1} (r_{nt}^s - \alpha_n - \beta_n\, r_{Mt}^s)^2/(T-4)$$

will correspondingly be an unbiased and consistent estimator of the unsystematic risk of asset n [3].

It has been seen above that conventional estimates of contemporaneous covariances are biased towards zero for assets with widely different trading frequencies. The market portfolio, being a value-weighted average of all assets, can be regarded as an asset with an "average" trading frequency: it follows that "securities trading very infrequently, plus possibly some trading very frequently, have estimators asymptotically biased ... downward for b_n"; whereas "most remaining securities have OLSE asymptotically biased in the opposite directions" (ibidem, p. 316). A corollary of this proposition is that the estimated residuals will be asymptotically biased at the two extreme ends of the trading frequency scale; furthermore, in the likely event that the market index is dominated by large firms with relatively high trading frequency, this bias in the OLS residuals should be concentrated at the low end of the scale. This has an important implication for the test proposed in the last section, since it suggests that the OLS estimate of unsystematic risk is biased at low turnover levels.

The *biases arising from intermittent trading*, being just a more extreme form of the desynchronization problem, do not require a separate analysis. In addition, nontrading also induces potentially large *losses of efficiency* in conventional variance estimators, as shown by the experimental evidence in Marsh and Rosenfeld (1985) [4]. A simple way out in this case is to calculate stock returns on a trade-to-trade basis in computing simple variances, and regress these returns on changes in the market index calculated over the same trade-to-trade intervals when estimating market model equations (as in Marsh (1979) and Schwert (1977)). Alternatively, the biases induced by intermittent trading can be handled by a straightforward extension of the method proposed by Scholes and Williams (1977) (see Fowler and Rorke (1983)).

Some caution is needed, however, against excessive confidence in these bias-correcting techniques: experimental evidence by Fowler, Rorke and Jog (1980) has shown that these tecniques tend "to introduce large amounts of noise that swamp [their] bias-reducing properties". Due to such efficiency cost, all the estimates below are performed employing both the conventional and the Scholes-Wilson method, hoping that the test results will prove robust to the implied bias-inefficiency trade-off.

(iv) Evidence from the Milan Stock Exchange .

The data set used for the estimation consists of monthly observations on prices and turnover for all the stocks continuously listed on the Milan Stock Exchange from August 1976 to September 1984 (121 stocks) [5]. Returns have been calculated as differences of the logarithms of the corresponding prices [6]. Returns on the market portfolio have been computed as changes in the logarithms of the value-weighted index published by the Banca Commerciale Italiana, that includes all listed shares (including those that have been cancelled or have been admitted to listing during the above-mentioned interval).

The first step has been to obtain estimates of unsystematic risk from the variance of the estimated residuals of market model equations. Two alternative sets of estimates of market model equations have been computed: with OLS and with the Scholes and Williams procedure. With both procedures, I have also omitted all non-trading observations, to correct for the bias deriving from intermittent trading. The main difference is then that the OLS estimates do not correct for desynchronization *within* the monthly measurement interval, whereas those obtained with the Scholes-Wilson procedure also correct for this source of bias. The variances of the estimated residuals obtained with the two alternative procedures will be respectively denoted by USR1 and USR2.

Figures 3 and 4 display scatter diagrams of the relationship between unsystematic risk and average turnover, employing respectively USR1 and USR2 to measure the former. Tables 1.a and 1.b report the results of regressions between the two variables, using various functional forms (USR_1 and USR_2 are in both cases multiplied by 10^4 -- so that they measure the unsystematic variance of *percentage* returns -- to make their scale comparable with that of the regressor). The results appear almost completely insensitive to which of the two measures of unsystematic risk is used. The equations in the top line of either table are specified so as to nest linear and non-linear relationships between the unsystematic risk and volume: the two variables are inversely correlated, and their relationship appears to be strongly non-linear. Unsystematic risk appears to correlate well with the inverse of turnover, and its logarithm is strongly inversely related to the level and the logarithm of turnover. This is encouraging, as it suggests that the relationship displays the general characteristics required by the maintained hypothesis: not only it is negative, but it appears to be stricly convex.

Similar results are obtained in Table 2, where market capitalization replaces turnover as a proxy for the number of traders (capitalization is measured as of December 30, 1983). Only the results for USR2 are reported, as those for USR1 are almost identical. The only appreciable difference with Table 1.b is that unsystematic risk correlates well with the logarithm of capitalization, and not with its inverse, whereas the reverse is true for turnover.

The next step has been to impose on the data the precise functional relationship derived above in (3.3), estimating it by non-linear least squares. The results, reported in Table 3, show that all parameter estimates have the sign predicted by the maintained hypothesis (although the parameter α_2 is estimated rather imprecisely).

Finally, I have investigated the relationship between total risk and average turnover: although the maintained hypothesis strictly concerns unsystematic risk, completeness and comparability with existing evidence from other studies attach a certain interest to the empirical relationship between total risk and volume of trade. Again, I have employed two different risk measures: (i) the simple variance of observed returns (TR1), and (ii) the variance of returns corrected for desynchronized trading (according to the estimator in (3.6)) (TR2). In both cases, observations for months when no trade occurred have been skipped, so as to avoid the bias deriving from intermittent trading. The results are displayed in Table 4: the relationship of either measure with turnover and its square is negative, but very imprecisely estimated, according to OLS t-statistics. Since however inspection of the data suggested a far larger dispersion of variance estimates at low turnover levels than at high levels, I have performed Goldfield-Quandt tests for heteroskedasticity on the four equations reported in Table 4, and have found that indeed the null hypothesis of homoskedasticity can be rejected both at the 5% and at the 1% confidence level. The same regressions were then re-estimated with the White procedure to obtain heteroskedasticity-consistent standard errors. The corresponding t-statistics, reported in square brackets in Table 4, show that the relationship between total risk and the square of turnover is significant at conventional confidence levels.

The results reported in this section conform to those reported in empirical work on other financial markets. Fowler, Rorke and Jog (1979) (and the studies quoted there) point to the fact that the R^2 of market model equations in general is substantially lower in thin markets than in thick ones, implying that, if two stocks have the same total return variance, the unsystematic component will be higher for the stock with the thinner market. Cohen et al. (1976) analyze a stratified random sample of stocks from four different exchanges (the New York Stock Exchange, the American Stock Exchange, the Tokyo Stock Exchange and the Rio de Janeiro Stock Exchange) and find that the logarithm of the total variance of daily returns have a strong negative relationship with the logarithm of the turnover and the floating supply of the corresponding stocks [6]. Similarly, Teiser and Higimbotham (1977) report an inverse correlation between price variability and trading activity on futures markets.

4 Conclusions

This paper has shown that market thinness can be an important determinant of the riskiness of equilibrium stock returns, because a low number of traders per unit time implies that stock prices have low informational content. Assuming that trade involves a fixed cost, the number of traders and its effect on the variance of stock returns have been analyzed as the outcomes of rational expectation equilibrium in a multiasset framework: the key results of the theory are that one should expect thin issues to be ceteris paribus characterized by a higher conditional *and* measured variance, and that this thinness-variability relationship arises only from the unsystematic component of the variance of stock returns.

These predictions of the theory have been tested on data from the Milan Stock Exchange, and shown to be supported by the evidence: a significant negative relationship has been found between unsystematic risk, as measured by the variance of market model equations, and the volume of trade; also the functional form of the relationship conforms to that predicted by the model. A similar cross-sectional relationship is found between unsystematic risk and firm capitalization at market prices. The results are not sensitive to the choice between conventional and unbiased estimates of unsystematic risk. Finally, also total risk has been found to be inversely related to the volume of trade as well, when the relationship is estimated with heteroskedastic-consistent methods. These results conform to the evidence available for other financial markets.

Appendix

1. Derivation of the REE solution for the multiperiod model of section 2.

As in Section 1, we start by postulating price conjectures and then solve for their indetermined coefficients. Assume then that all investors form the following conjectures:

$$p_{0t} = A_0 d_{ot} + C_0 \quad , \forall\, t\, , \tag{A1}$$

$$p_{jt} = A_j d_{jt} + B_j \left(\epsilon_{jt+1} + \sum \frac{e_{ijt+1}}{T_j} \right) \quad , \forall\, t\; ;\; j = 1,2,\ldots,M, \tag{A2}$$

where A_0, C_0; A_j, B_j and C_j are indetermined coefficients and T_j is the number of traders in market j.

Using these conjectures, investor i's first order condition (FOC) with respect to his holdings of asset 0 (k_{i0t}) can be written as:

$$E(p_{0t+1} + d_{0t+1}) - Rp_{0t} = b\,(1+A_0)\left[(1+A_0)k_{i0t} + \sum_{j=1}^{M} b_j (1+A_j) k_{ijt}\right] \sigma^2_{\epsilon 0} \quad , \tag{A3}$$

where expectations are not conditioned on the information set Ω_{it} because all private information is irrelevant to the forecast of the return on asset 0.

Summing then in (A3) over all i, and imposing market equilibrium, we find the equilibrium condition:

$$\text{(A4)} \qquad E(p_{0t+1} + d_{0t+1}) - Rp_{0t} = b\,(1+A_0)\left[(1+A_0)\frac{K_0}{T} + \sum_{j=1}^{M} b_j(1+A_j)\frac{K_j}{T}\right]\sigma^2_{\epsilon 0} \quad .$$

Using then the fact that $E(d_{0t+1}) = (1-\rho)a_0 + \rho d_{0t}$ and solving forward for p_{0t}, we obtain:

$$\text{(A4')} \quad p_{0t} = \frac{\rho}{R-\rho}\,d_{0t} + \frac{(1-\rho)R}{(R-\rho)r}\,a_0 - \frac{b}{r}\,(1+A_0)\left[(1+A_0)\frac{K_0}{T} + \sum_{j=1}^{M} b_j(1+A_j)\frac{K_j}{T}\right]\sigma^2_{\epsilon 0} \; ;$$

so that the parameter $A_0 = \frac{\rho}{R-\rho}$, whereas the parameter C_0 cannot yet be determined, since it depends on the A_j's -- which in turn have to be obtained from the REE solution for the remaining M markets.

To derive the REE in the remaining M markets, the first step is to compute the conditional moments of their dividends and prices, conditional on conjectures (A2) and on the information set Ω_{it}. The next period dividend and price of asset j (d_{jt+1}, p_{jt+1}), the private information of agent i about the next period dividend ($\epsilon_{jt+1} + e_{ijt+1}$) and the current dividend and price of the stock (d_{jt}, p_{jt}) form a 5-dimensional vector y of jointly normal random variables:

$$y = (d_{jt+1};\ p_{jt+1},\ \epsilon_{jt+1} + e_{ijt+1};\ d_{jt},\ p_{jt}) \quad ;$$

with unconditional mean:

$$m = (a_j;\ A_j a_j + C_j;\ 0,\ a_j,\ A_j a_j + C_j) \quad ;$$

and covariance matrix:

$$V = \begin{bmatrix} \sigma^2_{dj} & A_j\sigma^2_{dj} & \sigma^2_{\epsilon j} & \rho\sigma^2_{dj} & A_j\rho\sigma^2_{dj} + B_j\sigma^2_{\epsilon j} \\ A_j\sigma^2_{dj} & A^2_j\sigma^2_{dj} + B^2_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right) & A_j\sigma^2_{\epsilon j} & A_j\rho\sigma^2_{\epsilon j} & A^2_j\rho\sigma^2_{dj} + A_jB_j\sigma^2_{\epsilon j} \\ \sigma^2_{\epsilon j} & A_j\sigma^2_{\epsilon j} & \sigma^2_{\epsilon j} + \sigma^2_{ej} & 0 & B_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right) \\ \rho\sigma^2_{dj} & A_j\rho\sigma^2_{dj} & 0 & \sigma^2_{dj} & A_j\sigma^2_{dj} \\ A_j\rho\sigma^2_{dj} + B_j\sigma^2_{\epsilon j} & A^2_j\rho\sigma^2_{dj} + A_jB_j\sigma^2_{ej} & B_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right) & A_j\sigma^2_{dj} & A^2_j\sigma^2_{dj} + B^2_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right) \end{bmatrix}$$

where $\sigma^2_{dj} \equiv \dfrac{b^2_j\sigma^2_{\epsilon 0} + \sigma^2_{\epsilon j}}{1-\rho^2}$.

We want the mean and the covariance matrix of the subvector $y_1 = (d_{jt+1};\ p_{jt+1})$ conditional on the values of the subvector $y_2 = (\epsilon_{jt+1} + e_{ijt+1},\ d_{jt};\ p_{jt}) \subset \Omega_{it}$, <u>i.e.</u> conditional on the information on asset j that is available to agent i at time t.

Partition the matrix V as follows:

$$V = \begin{bmatrix} \underset{(2 \times 2)}{V_{11}} & \underset{(2 \times 3)}{V_{12}} \\ \underset{(3 \times 2)}{V_{21}} & \underset{(3 \times 3)}{V_{22}} \end{bmatrix} ,$$

where the numbers below each submatrix indicate its dimension, and recall that the vector $y_1|y_2$ is normal with mean:

$$E(y_2|y_1) = m_1 + V_{12}V_{22}^{-1}(Y_2 - m_2) ,$$

and covariance matrix:

$$V(y_2|y_1) = V_{11} - V_{12}V_{22}^{-1}V_{21} .$$

Applying these formulae, one finds that the conditional expectations of the dividend and the price are:

$$\text{(A5)} \qquad E(d_{jt+1}|\Omega_{it}) = a_j + \frac{\rho B_j\left(\sigma_{\epsilon j}^2 + \frac{\sigma_{ej}^2}{T_j}\right) - A_j\sigma_{\epsilon j}^2}{B_j\left(\sigma_{\epsilon j}^2 + \frac{\sigma_{ej}^2}{T_j}\right)}(d_{jt} - a_j)$$

$$+ \frac{\sigma_{\epsilon j}^2}{B_j\left(\sigma_{\epsilon j}^2 + \frac{\sigma_{ej}^2}{T_j}\right)}(p_{jt} - A_j a_j - C_j)$$

$$= (1-\rho)a_j + \rho d_{jt} + \frac{\sigma^2_{\epsilon j}}{\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}} \left[\epsilon_{jt+1} + \sum_{i \in T_j} \frac{e_{ijt+1}}{T_j}\right]$$

(A6) $$E(p_{jt+1}|\Omega_{it}) = A_j a_j + C_j + \frac{\rho A_j B_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right) - A^2_j \sigma^2_{\epsilon j}}{B_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right)} (d_{jt} - a_j)$$

$$+ \frac{A_j \sigma^2_{\epsilon j}}{B_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right)} (p_{jt} - A_j a_j - C_j)$$

$$= A_j(1-\rho)a_j + C_j + A_j\left[\rho d_{jt} + \frac{\sigma^2_{\epsilon j}}{\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}} \left[\epsilon_{jt+1} + \sum_{i \in T_j} \frac{e_{ijt+1}}{T_j}\right]\right];$$

where conjecture (A2) has been substituted for p_{jt} in the second step of (A5) and (A6). Similarly, one finds that the conditional covariance matrix is:

$$V((d_{jt+1}, p_{jt+1})|\Omega_{it}) = \begin{bmatrix} b^2_j \sigma^2_{\epsilon 0} + \frac{\sigma^2_{\epsilon j}\frac{\sigma^2_{ej}}{T_j}}{\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}} & A_j\left[b^2_j \sigma^2_{\epsilon 0} + \frac{\sigma^2_{\epsilon j}\frac{\sigma^2_{ej}}{T_j}}{\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}}\right] \\ A_j\left[b^2_j \sigma^2_{\epsilon 0} + \frac{\sigma^2_{\epsilon j}\frac{\sigma^2_{ej}}{T_j}}{\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}}\right] & A^2_j\left[b^2_j \sigma^2_{\epsilon 0} + \frac{\sigma^2_{\epsilon j}\frac{\sigma^2_{ej}}{T_j}}{\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}}\right] + B^2_j\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right) \end{bmatrix}$$

(A7)

The next step is to compute market equilibrium using these conditional moments. To do this, let us write down investor i's FOC with respect to asset j:

$$\text{(A8)}\qquad E(p_{jt+1}+d_{jt+1}|\Omega_{it})-Rp_{jt}=b\left[Var(p_{jt+1}+d_{jt+1}|\Omega_{it})k_{ijt}+\sum_{\substack{h=0\\h\neq j}}^{M}Cov(p_{jt+1}+d_{jt+1},p_{ht+1}+d_{ht+1}|\Omega_{it})\right],\quad j=1,2,\ldots,M,$$

and, using (2.3) and (A2), notice that $Cov(p_{jt+1}+d_{jt+1},p_{ht+1}+d_{ht+1}|\Omega_{it})=b_jb_h(1+A_j)(1+A_h)\sigma^2_{\epsilon0}$ and $Cov(p_{jt+1}+d_{jt+1},p_{0t+1}+d_{0t+1}|\Omega_{it})=b_j(1+A_j)(1+A_0)\sigma^2_{\epsilon0}$. Substituting these expressions and those in (A5), (A6) and (A7) into (A8), we find:

$$\text{(A8')}\qquad (1+A_j)\left[(1-\rho)a_j+\rho d_{jt}+\frac{\sigma^2_{\epsilon j}}{\sigma^2_{\epsilon j}+\frac{\sigma^2_{ej}}{T_j}}\left(\epsilon_{jt+1}+\sum_{i\in T_j}\frac{e_{ijt+1}}{T_j}\right)\right]+C_j-Rp_{jt}$$

$$=b\left[(1+A_j)^2\left(b_j^2\sigma^2_{\epsilon0}+\frac{\sigma^2_{\epsilon j}\frac{\sigma^2_{ej}}{T_j}}{\sigma^2_{\epsilon j}+\frac{\sigma^2_{ej}}{T_j}}\right)+B_j^2\left(\sigma^2_{\epsilon j}+\frac{\sigma^2_{ej}}{T_j}\right)\right]k_{ijt}$$

$$+\,b\,b_j(1+A_j)\left[(1+A_0)k_{i0t}+\sum_{\substack{h=1\\h\neq j}}^{M}b_h(1+A_h)k_{iht}\right]\sigma^2_{\epsilon0}\quad.$$

Substituting in (A8') from the FOC (A3) and imposing market equilibrium, we finally have:

$$\text{(A9)}\quad (1+A_j)\left[(1-\rho)a_j + \rho d_{jt} + q_j\left(\epsilon_{jt+1} + \sum_{i\in T_j}\frac{e_{ijt+1}}{T_j}\right)\right] + C_j - Rp_{jt}$$

$$= b\left[(1+A_j)^2\, q_j\, \frac{\sigma^2_{\epsilon j}}{T_j} + B_j^2\left(\sigma^2_{\epsilon j} + \frac{\sigma^2_{ej}}{T_j}\right)\right]\frac{K_j}{T_j} + b_j\,\frac{1+A_j}{1+A_0}\left[E(p_{0t+1} + d_{0t+1}\,|\,\Omega_{it}) - Rp_{0t}\right],$$

where, as in the text, I use the short-hand $q_j \equiv \dfrac{\sigma^2_{\epsilon j}}{\sigma^2_{\epsilon j} + \dfrac{\sigma^2_{ej}}{T_j}}$.

The last step is to equate expression (A9) with the initial conjecture (A2) to solve for the indetermined coefficients:

$$A_j = \frac{\rho}{R-\rho}\ ,\quad B_j = \frac{q_j}{R-\rho}\ ;\quad C_0 = \frac{1}{r}\left[\frac{R(1-\rho)}{R-\rho}\,a_0 - b\,\frac{R^2}{(R-\rho)^2}\left[\frac{K_0}{T} + \sum_{j=1}^{M} b_j(1+A_j)\frac{K_j}{T}\right]\sigma^2_{\epsilon 0}\right],$$

$$C_j = \frac{1}{r}\left[\frac{R(1-\rho)}{R-\rho}\,a_j - b\,\frac{q_j}{(R-\rho)^2}\left(\sigma^2_{\epsilon j} + R^2\,\frac{\sigma^2_{ej}}{T_j}\right)\frac{K_j}{T_j} - b_j\left[E(p_{0t+1} + d_{0t+1}\,|\,\Omega_{it}) - Rp_{0t}\right]\right].$$

At this point, we can write down the complete REE solution of the model:

$$\text{(A10)}\quad p_{0t} = \frac{\rho}{R-\rho} d_{0t} + \frac{1}{r}\left[\frac{R(1-\rho)}{R-\rho} a_0 - b\,\frac{R^2}{(R-\rho)^2}\left(\frac{K_0}{T} + \sum_{j=1}^{M} \frac{b_j K_j}{T_j}\right)\sigma^2_{\epsilon 0}\right] ;$$

$$\text{(A11)}\quad p_{jt} = \frac{\rho}{R-\rho} d_{jt} + \frac{1}{r}\left[\frac{R(1-\rho)}{R-\rho} a_j - b\,\frac{q_j}{(R-\rho)^2}\left(R^2\,\frac{\sigma^2_{ej}}{T_j} + \sigma^2_{\epsilon j}\right)\frac{K_j}{T_j}\right.$$

$$\left. - b_j\left[E(p_{0t+1} + d_{0t+1}) - Rp_{0t}\right]\right] + \frac{q_j}{R-\rho}\left(\epsilon_{jt+1} + \sum_{i\in} \frac{e_{ijt}}{T_j}\right) ,\ j = 1,\ldots,M,\ \forall\, t,$$

that correspond to equations (2.5) and (2.6) in the text.

2. Derivation of expression (2.10) in section 2 .

To find the value of the utility function of agent i at the optimum, reported in the text as expression (2.10), let us substitute for the indetermined coefficients A_0, C_0, A_j; B_j and C_j in the FOC's (A3) and (A8'), to find respectively:

$$\text{(A12)}\qquad E(p_{0t+1} + d_{0t+1}) - Rp_{0t} = b\,\frac{R^2}{(R-\rho)^2}\left(k_{i0t} + \sum_{j=1}^{M} b_j k_{ijt}\right)\sigma^2_{\epsilon 0} ,$$

$$\text{(A13)}\qquad E(p_{0t+1} + d_{0t+1}) - Rp_{jt} = b\,\frac{q_j}{(R-\rho)^2}\left(\sigma^2_{\epsilon j} + R^2\,\frac{\sigma^2_{ej}}{T_j}\right)k_{ijt}$$

$$+ b_j\left[E(p_{0t+1} + d_{0t+1}|\Omega_{it}) - Rp_{0t}\right] .$$

Using (A12) and (A13) in the utility function (the maximand in (2.3)), one immediately obtains expression (2.10).

FOOTNOTES

[1] For detailed derivations of most of the following expressions, see the appendix contained there. I conform to their notation almost everywhere.

[2] A similar estimator, proposed by Dimson (1979), has been shown to be inconsistent by Fowler and Rorke (1983) and to perform generally worse than the Scholes and Williams estimator by Fowler, Rorke and Jog (1980).

[3] The (T-4) in the denominator derives from having T-2 sample observations and two lost degree of freedom.

[4] These authors also insist on the point that nontrading causes no bias in conventional volatility estimates. As it has been seen above, however, this statement is approximately true only when the measurement interval is very small. It does not apply, for instance, the data used in the present study, which have monthly frequency.

[5] The price data have been provided by the Banca Commerciale Italiana, whereas those on turnover have been drawn from The Performance of Listed Shares, a yearly publication of the Exchange.

[6] Lessard (1976), in commenting on their results, has remarked that the

authors of this study could more effectively isolate the trading-induced component of total variance by analysing the unsystematic variance alone, which is the approach suggested by the model of sections 1 and 2 and adopted here.

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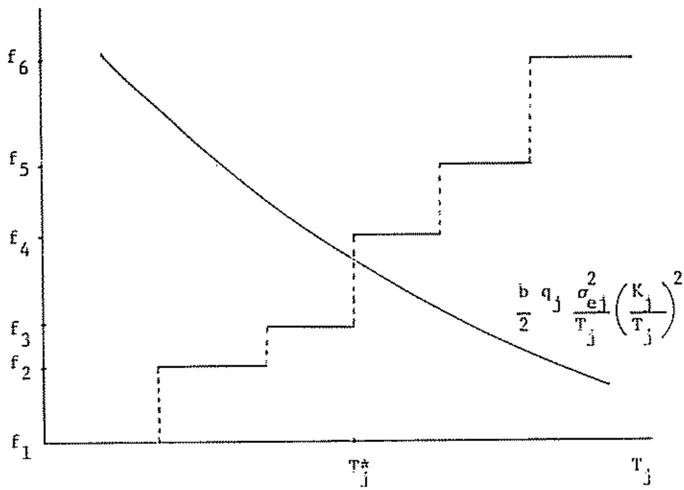

Figure 1

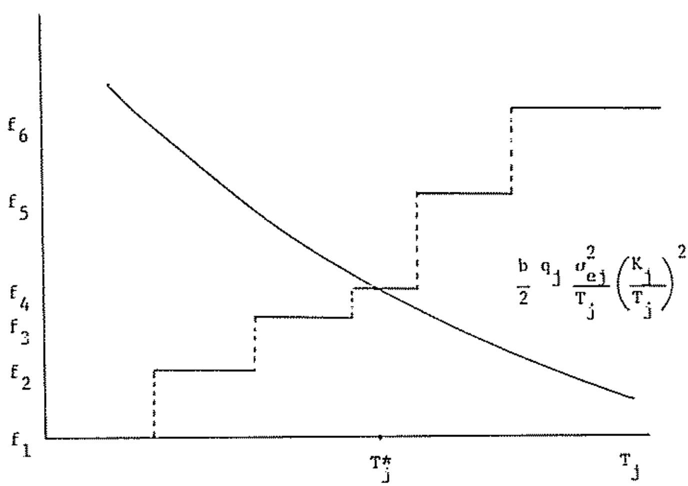

Figure 2

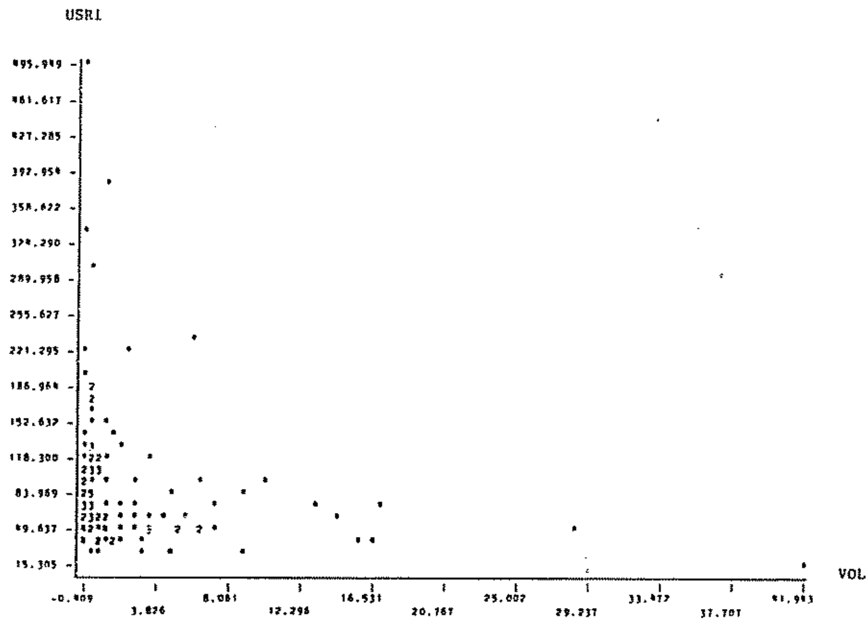

Figure 3

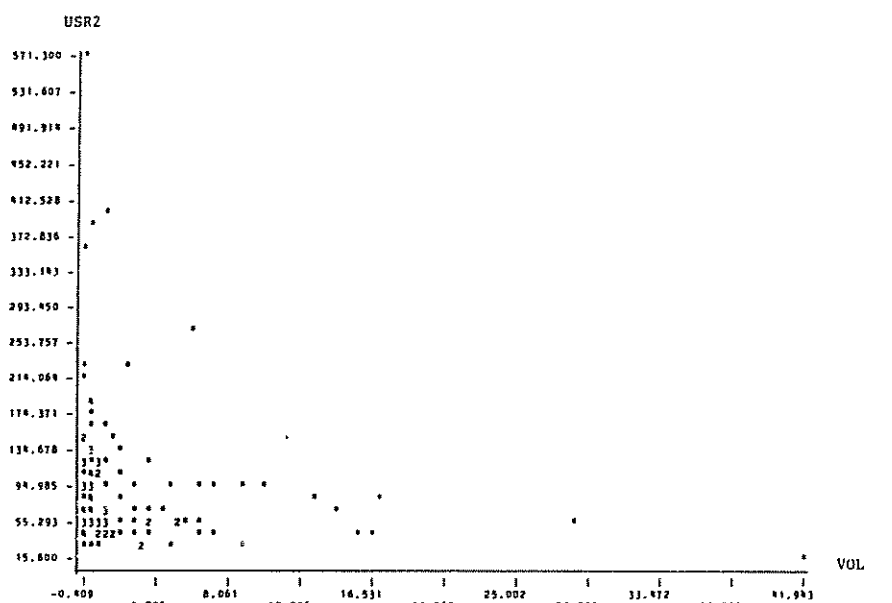

Figure 4

TABLE 1.a

Relationship between unsystematic risk and average turnover*

Dependent variable	Independent variables				Summary statistics		
USR1 x 10^4	Constant	VOL	log(VOL)	1/VOL	R^2	Standard error of the regression	Mean of the dependent variable
	88.97 (10.26)	- 1.91 (-1.04)	0.09 (0.01)	2.14 (2.43)	0.132	73.44	93.75
	88.94 (10.71)	-1.89 (-1.57)		2.13 (3.54)	0.132	73.13	93.75
	82.54 (11.35)			2.32 (3.92)	0.114	73.57	93.75

Dependent variable	Constant	VOL	log(VOL)	R^2	Standard error of the regression	Mean of the dependent variable
log(USR1 x 10^4)						
	4.31 (82.30)		-0.12 (-3.89)	0.113	0.57	4.33
	4.43 (75.46)	-0.03 (-3.75)		0.105	0.57	4.33

*Estimation method: OLS. The numbers in brackets below coefficient estimates are t-statistics.

USR1 = unsystematic risk, estimated as variance of market model regression residuals with *no* correction for thinness-related bias;

VOL = average monthly turnover at market prices (bn. Lit)

TABLE 1.b

Relationship between unsystematic risk and average turnover*

Dependent variable	Independent variables				Summary statistics		
USR2 x 10^4	Constant	VOL	log(VOL)	1/VOL	R^2	Standard error of the regression	Mean of the dependent variable
	94.11 (10.65)	- 1.88 (-1.01)	-1.04 (-0.13)	1.84 (2.06)	0.116	74.85	97.74
	94.43 (11.16)	-2.06 (-1.68)		1.92 (3.14)	0.116	74.54	97.74
	87.45 (11.78)			2.13 (3.53)	0.095	75.11	97.74

log(USR2 x 10^4)	Constant	VOL	log(VOL)	R^2	Standard error of the regression	Mean of the dependent variable
	4.36 (82.91)		-0.12 (-3.85)	0.111	0.57	4.38
	4.48 (76.23)	-0.03 (-3.81)		0.105	0.57	4.38

*Estimation method: OLS. The numbers in brackets below coefficient estimates are t-statistics.

USR2 = unsystematic risk, estimated as variance of market model regression residuals with correction for thinness-related bias;
VOL = average monthly turnover at market prices (bn. Lit)

TABLE 2

Relationship between unsystematic risk and market capitalization*

Dependent variable	Independent variables				Summary statistics		
USR2 x 10^4	Constant	CAP	log(CAP)	1/CAP	R^2	Standard error of the regression	Mean of the dependent variable
	58.94 (2.31)	3.96 (0.18)	-15.20 (-1.66)	-0.05 (-0.39)	0.065	77.32	97.63
	64.74 (4.77)		-12.54 (-2.83)		0.063	76.71	97.63

Dependent variable	Constant	CAP	log(CAP)	R^2	Standard error of the regression	Mean of the dependent variable
log(USR2 x 10^4)						
	4.03 (39.67)		-0.13 (-3.90)	0.114	0.57	4.37
	4.47 (82.91)	-0.40 (-3.71)		0.105	0.58	4.37

*Estimation method: OLS. The numbers in brackets below coefficient estimates are t-statistics. The means of the dependent variables differ slightly from those of Table 1.a because an observation in the CAP variable was missing, and the corresponding observation for the dependent variable had to be dropped.

USR2 = unsystematic risk, estimated as variance of market model regression residuals *with* correction for thinness-related bias;
CAP = stock capitalization at market prices (thousand bn. Lit, as of December 30, 1983)

TABLE 3

Estimation of equation (3.3)*

Dependent variable	Parameter estimates			Summary statistics		
	α_0	α_1	α_2	R^2	Standard error of the regression	Mean of the dependent variable
USR1	0.012 (7.18)	-0.006 (-2.94)	1.258 (0.71)	0.072	0.007	0.009375
USR2	0.013 (6.22)	-0.007 (-3.03)	0.836 (0.72)	0.072	0.008	0.009774

*Estimation method: NLLS. The numbers in brackets below coefficient estimates are t-statistics. For variable definitions, see footnotes to Tables 1.a and 1.b.

TABLE 4

Relationship between total risk and average turnover

Dependent variable	Independent variables			Summary statistics		
	Constant	VOL	VOL^2	R^2	Standard error of the regression	Mean of the dependent variable
TR1	147.81 (17.48) [16.68]	-1.17 (-0.88) [-1.41]		0.007	87.72	144.41
	146.21 (19.04) [18.90]		-0.05 (-1.13) [-4.42]	0.011	85.55	144.41
TR2	160.85 (9.09) [16.98]	-0.34 (-0.24) [-0.31]		0.001	88.97	159.86
	161.29 (19.53) [19.45]		-0.04 (-0.81) [-2.27]	0.005	88.75	159.86

*Estimation method: OLS. The numbers in round brackets below coefficient estimates are OLS t-statistics; those in square brackets are White-consistent t-statistics.

TR1 = total risk, estimated as the variance of *percentage* returns with *no* correction for thinness-related bias; TR2 = same, *with* correction for thinness-related bias.