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This paper develops a random-matching model of a frictional labor market with firm and worker dynamics. Multi-worker firms choose whether to shrink or expand their employment in response to shocks to their decreasing returns to scale technology. Growing entails posting costly vacancies, which are filled either by the unemployed or by employees poached from other firms. Firms also choose when to enter and exit the market. Tractability is obtained by proving that, under a parsimonious set of assumptions, all workers' and firm decisions are characterized by their joint marginal surplus, which in turn only depends on the firm's productivity and size. As frictions vanish, the model converges to a standard competitive model of firm dynamics which allows a quantification of the misallocation cost of labor market frictions. An estimated version of the model yields cross-sectional patterns of net poaching by firm characteristics (e.g., age and size) that are in line with the micro data. The model also generates a drop in job-to-job transitions as firm entry declines, offering an interpretation to U.S. labor market dynamics around the Great Recession. All these outcomes are a reflection of the job ladder in marginal surplus that emerges in equilibrium.

JEL Classification: D22, E23, E24, E32, J23, J63, J64, J69

Keywords: Decreasing Returns to Scale, Firm Dynamics, frictions, Job turnover, Marginal Surplus, Net Poaching, On the job search, unemployment, Vacancies, worker flows

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Firm and Worker Dynamics in a Frictional Labor Market*

Adrien Bilal[†] *Niklas Engbom*[‡] *Simon Mongey*[§] *and* *Giovanni L. Violante*[¶]

December 20, 2019

Abstract

This paper develops a random-matching model of a frictional labor market with firm and worker dynamics. Multi-worker firms choose whether to shrink or expand their employment in response to shocks to their decreasing returns to scale technology. Growing entails posting costly vacancies, which are filled either by the unemployed or by employees poached from other firms. Firms also choose when to enter and exit the market. Tractability is obtained by proving that, under a parsimonious set of assumptions, all workers' and firm decisions are characterized by their joint marginal surplus, which in turn only depends on the firm's productivity and size. As frictions vanish, the model converges to a standard competitive model of firm dynamics which allows a quantification of the misallocation cost of labor market frictions. An estimated version of the model yields cross-sectional patterns of net poaching by firm characteristics (e.g., age and size) that are in line with the micro data. The model also generates a drop in job-to-job transitions as firm entry declines, offering an interpretation to U.S. labor market dynamics around the Great Recession. All these outcomes are a reflection of the job ladder in marginal surplus that emerges in equilibrium.

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1 Introduction

Aggregate production in the economy is divided into millions of firms, each facing idiosyncratic fluctuations in its productivity and demand. Understanding the process of labor reallocation across these production units is important for several reasons. In the long run, reallocating labor away from unproductive firms toward more productive firms enhances aggregate productivity and growth. In the short run, the propagation of sectoral and aggregate shocks depends on how quickly labor flows across firms and between unemployment and employment. From a normative perspective, understanding the potential welfare losses or gains due to reallocation is necessary for assessing the efficacy of policies that subsidize jobless workers, protect employment, or advantage particular sectors/firms.

The labor reallocation process has three key properties. First, it has distinct layers: the entry and exit of firms, the creation and destruction of positions (i.e., jobs) at existing firms, and the turnover of workers across jobs at existing firms. Second, it is intermediated by labor markets that are frictional, as revealed by the coexistence of vacancies and job seekers. Third, around half of worker turnover occurs through direct job-to-job transitions: most new hires come from another firm rather than from unemployment.

Conceptually, therefore, addressing labor reallocation requires a framework with (i) a theory of the firm (i.e., its boundaries) and of firm dynamics (entry, growth, separations, exit); and (ii) a theory of worker flows intermediated by frictional labor markets that allows for on-the-job search and job-to-job mobility (i.e., poaching). Quantitatively, such a framework should account for a new body of time series and cross-sectional evidence—emerging from matched employer-employee data—that describes the relationship between firm characteristics and the direction and composition of worker flows.[1]

In this paper, we present a new model with these traits. A firm is a profit maximizing owner of a technology with decreasing returns to scale and stochastic productivity, that chooses optimally whether to enter and when to exit the market.[2] Firms grow by posting costly vacancies that are randomly matched to either unemployed or employed workers. Worker flows occur when matched workers determine that the value of working at the matched firm exceeds their value of unemployment or employment in their current firm. In general, with decreasing returns to scale in production, these values are a complicated function of a high dimensional state vector that includes distributions of wages or worker values inside the firm. This makes the problem intractable.

The first contribution of our paper is to set out a parsimonious set of assumptions that are *sufficient*

[1]If we consider hires for a particular firm type (e.g., young, small and fast-growing), by *composition* we mean the split between hires from unemployment and those from employment. Within hires from employment, *direction* refers to the characteristics of the employers between which workers are reallocated.

[2]Or, equivalently, a monopolistic producer facing a downward sloping demand curve with a stochastic shifter. These two interpretation are isomorphic in our model.

for tractability. Our assumptions place a minimal structure on bargaining and surplus sharing such that, as a result, the state vector becomes manageable. Three assumptions are common to many single-worker firm environments: (i) lack of commitment in firing and quit decisions; (ii) wage contract renegotiation by mutual consent; (iii) Bertrand competition among employers for employed jobseekers. Two further assumptions are required in our new multi-worker firm environment: (iv) internal wage renegotiations between the firm and its incumbent workers are a zero-sum game, i.e. no surplus gets lost; and (v) privately efficient vacancy posting—for which we offer an explicit microfoundation. Under these assumptions firm and workers' decisions are privately efficient, as if the firm and incumbent workers maximize their *joint value*. The state variables of the joint value function are only firm size (n) and productivity (z), allowing us to cleanly study firm and worker dynamics in a frictional labor market.

Two other ingredients are vital to achieve tractability. First, we work in continuous time. In a small interval of time only one random event may occur. A firm, for example, only needs to deal with one of its employee meeting another firm, not all combinations of its employees meeting other firms. Second, we take the continuous limit of a discrete workforce. Worker flows are determined by comparing the change in joint surplus that would arise if a worker either joins or leaves a firm. With a continuous measure of workers, this *marginal surplus* can be conveniently expressed as a partial derivative of total surplus.

We show that total and marginal surplus are sufficient for characterizing firm and worker dynamics. Marginal surplus pins down hiring: facing a convex vacancy cost, firms post vacancies until the marginal cost of a vacancy is equal to the expected marginal surplus of hiring. Marginal surplus also pins down separations: facing a decreasing marginal product of labor, firms fire workers until the marginal surplus of a worker equals the value of unemployment. When total surplus is less (more) than the firm's private outside option the firm exits (enters). Finally, in equilibrium, marginal surpluses determine the direction of worker flows. Workers climb a *job ladder* in marginal surplus, quitting when on-the-job search delivers a match with a higher marginal surplus firm. An intuitive Bellman equation accounts for the evolution of surplus, while a law of motion reflecting frictional labor reallocation accounts for the evolution of the firm size and productivity distribution.[3]

Our second contribution is to exploit the tractability of this simple representation to analytically characterize equilibrium firm and worker reallocation. First, we characterize firm dynamics and job turnover graphically in (n, z)-space by describing the regions in which a firm exits, fires and hires. Firms that exit and fire always destroy jobs. Hiring firms may either grow on net (creating jobs) or shrink on net

[3]This representation uniquely pins down firm and worker dynamics, the subject of this paper, but is consistent with multiple wage determination mechanisms that determine how this joint value is split. Wages, therefore, are not allocative in that the distribution of firms and flows of workers across firms is independent of wage dynamics. In order to study the model's implication for wage dynamics, one has to make additional assumptions. We return on this point in Section 2.

(destroying jobs) because some of their workers quit to firms with a higher rank in the marginal surplus ladder. Second, we decompose net growth of hiring firms into the different types of gross flows: hires and separation from/to unemployment and from/to employment via poaching. This decomposition varies systematically with the firm states (n, z) since they determine the firm marginal surplus. Third, we study the limiting behaviors of our economy. As *decreasing returns to scale* vanish, the economy converges to one in which single-worker firms operate in a frictional labor market (à la Postel-Vinay and Robin, 2002). As *frictions* vanish, the economy converges to one in which multi-worker firms operate in a competitive labor market (à la Hopenhayn, 1992). We show that this limit obtains only in the presence of on-the-job search, which provides the key force toward equating marginal products across firms. As opposed to an economy with constant returns to scale, our economy features a non-degenerate size distribution in the limit.

Our third contribution is to exploit the tractability of our framework to implement the model quantitatively. We estimate the model by Simulated Method of Moments, targeting cross-sectional moments of the size distribution of firms, firm dynamics, job flows and worker flows for the U.S. economy. We argue that parameters are well-identified. We then validate the model, conducting a data-model comparison of how firm-level hiring is split into inputs, i.e. between vacancy rates and vacancy yields (as in Davis, Faberman, and Haltiwanger, 2013) and outputs, i.e. between hires from employment and from unemployment (as in Bagger, Fontaine, Galenianos, and Trapeznikova, 2019). A firm experiencing a positive productivity shock increases hiring by posting more vacancies, but more so by filling those vacancies faster as a higher marginal surplus makes it more attractive to jobseekers. For the same reason, hires from employment increase disproportionately more than hires from unemployment. These patterns are quantitatively consistent with the data.

Finally, we use the parameterized model to address three quantitative questions that require an environment with decreasing returns to scale and on-the-job search. No other existing structural equilibrium model can address them.

We begin by quantifying the misallocation effects of labor market frictions. A doubling of match efficiency, which approximately corresponds to a doubling of contact rates, raises output by 15 percent relative to our estimated benchmark. Our key finding is that four fifths of the increase in output is caused by lower labor misallocation due to faster job-to-job transitions rather than by higher scale of production due to more employment.

Next, we direct our attention to a new set of facts about job-to-job flows and net poaching. Job-to-job flows and net poaching vary systematically by firm characteristics in the data (Haltiwanger, Hyatt, Kahn, and McEntarfer, 2018). Young firms poach workers from older firms, but firm size is only weakly correlated with net poaching. Our theory offers an interpretation of these facts. A young firm is far from its optimal size, and since decreasing returns generates a high marginal surplus, this firm will be near the top of the job ladder. However, a firm may be small because unproductive, or because young even if productive. These two types of small firms will be on opposite ends of the job ladder. To guide future measurement we show that labor productivity and firm growth are observables that are strongly positively correlated with marginal surplus, so predictive of net poaching and job ladder rank.

We conclude the quantitative analysis with an application to the U.S. Great Recession. Two distinguishing features were the sharp drop in firm entry and a decline in job-to-job reallocation of workers that led to a 'failure of the job ladder', i.e. a slow down of the process through which workers climb toward better firms (Siemer, 2014; Moscarini and Postel-Vinay, 2016). Our model suggests that the former accounts for the latter. A transitory shock to the discount rate (a commonly used shortcut for worsening financial frictions) lowers the value of entry and shrinks the population of young, high marginal surplus firms with high equilibrium net poaching rates. With fewer firms at the top of the job ladder and less vacancy posting among these firms, labor reallocation up the ladder breaks down. The resulting misallocation causes a persistent slump in output.

Overall, these applications demonstrate that our new theoretical framework offers a useful platform to jointly analyze the microeconomic dynamics of firms and workers in a frictional labor market and how these relate to macroeconomic fluctuations.

Literature

Our paper connects two strands of literature. The common core between the two is the idea that diminishing returns in production and heterogeneity in productivity are the dominant forces that delivers a non-degenerate firm-size distribution. This idea goes back at least to Lucas (1978) span of control model.

The first strand is the large literature on equilibrium models of single-product firm dynamics with competitive labor markets. Classic examples are Hopenhayn (1992), Hopenhayn and Rogerson (1993), and Luttmer (2011).[4] Recent examples, with applications to the Great Recession, are Arellano, Bai, and Kehoe (2016), Clementi and Palazzo (2010) and Sedláček (2014).

Like these models, our framework features entry, exit, and non degenerate distributions of firm size and age. Unlike these models, the employment adjustment costs that firms face are endogenous. They depend on the firm's probability of poaching and the expected transfers required to hire a worker away from a competing firm. Both are a function of the firm rank on the marginal surplus ladder, which itself

[4] For a review of the literature see also Luttmer (2010).

is an equilibrium object. The frictionless limit of our model, where these endogenous costs vanish, is a version of Hopenhayn (1992).

The second literature comprises a number of papers that model multi-worker firms in frictional labor markets. Here, two parallel approaches have been taken: directed search and random search.

The directed search models of Kaas and Kircher (2015) and Schaal (2017) generate firm employment dynamics resembling those in the micro data.[5] Building on Menzio and Shi (2011), the model in Schaal (2017) also allows for on the job search, and thus is the closest counterpart of our framework within the directed search approach. A drawback of directed search is that the probability that a firm hires from another firm or from unemployment is not determined.[6] As a result, this class of models cannot speak to systematic variation in net poaching rates or the composition of hires across firm types that have been documented in micro data. A model consistent with these facts is one of the objectives of our analysis.

In the random search strand, Elsby and Michaels (2013) and Acemoglu and Hawkins (2014) solve models where firms face decreasing returns in production, stochastic productivity, linear vacancy costs, and wages determined by Nash bargaining.[7] Both generate employment relationships with a large average surplus and small marginal surplus. Elsby and Michaels (2013) demonstrate that the latter property yields a volatile job-finding rate over the cycle, while the former avoids an excessively high separation rate, thus resolving the tension identified by Shimer (2005) in the Diamond-Mortensen-Pissarides framework. Gavazza, Mongey, and Violante (2018) generalize this model by introducing a hiring effort decision and financial constraints and show that it accounts for the sharp drop in aggregate recruiting intensity around the Great Recession. All of these models abstract from search on-the-job.[8]

Random search models with wage posting feature both on-the-job search and a firm-size distribution despite constant returns to scale. These follow Burdett and Mortensen (1998) and its generalizations to out of steady-state dynamics in Moscarini and Postel-Vinay (2013, 2016), Coles and Mortensen (2016), Engbom (2017a), Gouin-Bonenfant (2018) and Audoly (2019). In these models the size distribution is non degenerate only because of the existence of search frictions. As frictions disappear, all workers become

[5]It is worth remarking that these two papers had very different objectives to ours. Kaas and Kircher (2015) illustrate that a key advantage of directed search, the efficiency and block-recursivity properties of equilibrium, extends to models with 'large' firms. Schaal (2017) proves this property is also robust to the addition of on-the-job-search and studies aggregate uncertainty shocks in the context of the Great Recession.

[6]A key feature of the equilibrium is that net hiring costs are equated across firms through free entry, which implies that firms are indifferent across the markets in which they search for workers. The probability that a separation from a firm is to employment or unemployment, however, is determined.

[7]Bertola and Caballero (1994) derive closed form results under a linear approximation to both marginal product and convex vacancy costs, and a two state Markov process for productivity.

[8]Fujita and Nakajima (2016) introduce on-the-job search and study the dynamics of job-job flows over the business cycle. However, in their model all workers are always indifferent between searching/working and staying/moving because, to solve for the equilibrium, they must assume that the worker outside option is always the value unemployment.

employed at the most productive firm.[9] In our framework, instead, we can decompose how much of size dispersion is due to technology and how much is due to frictions.

Within the random search literature, we build on the set-up developed by Postel-Vinay and Robin (2002), which pairs Bertrand competition for workers among employers with wage renegotiation under mutual consent. This environment has become another workhorse of the literature due to its tractability and empirically plausible wage dynamics.[10] Kiyotaki and Lagos (2007) develop a version of this protocol which is a step closer to us. Their firms have fixed capacity of exactly one position and thus feature an extreme version of decreasing returns to scale. In their model, when a matched firm meets another worker, it also engages in a negotiation with its current incumbent. Internal renegotiation is a prominent feature of our model. Our contribution is to generalize this sequential auction protocol to multi-worker firms, show how one can still solve the model's equilibrium through the notion of joint surplus, and in doing so maintain a great deal of tractability. As opposed to the original Postel-Vinay and Robin (2002) framework the probability of hiring is not a function of the exogenous distribution of firm productivity, but is determined by the *endogenous* distribution of marginal surpluses, which itself depends on how the equilibrium of the frictional labor market has allocated workers across heterogeneous firms.

The final expression for joint surplus that features among our equilibrium conditions is reminiscent of that in Lentz and Mortensen (2012), a version of Klette and Kortum (2004) with on-the-job search in which a firm's demand for labor is limited by demand for its portfolio of products. While they *assume* that all decisions are based on joint firm-workers values, we derive this result from primitives, provide a characterization of the equilibrium and illustrate how to use the model for a quantitative analysis of newly documented empirical patterns. Our central finding that a job ladder in marginal surplus arises in equilibrium is closely related to contemporaneous work by Elsby and Gottfries (2019) who elegantly characterize a special case of our environment with linear vacancy costs and no firm entry/exit. In this setting a firm's values and its decisions are only a function of a single state variable reflecting the marginal product of labor. In our model marginal surplus is related to the marginal product of labor, but includes continuation values that depend on average surplus due to exit. In our calibrated model, we find that the rank-correlation between marginal surplus and marginal product is high. To the extent that marginal and average products of labor are strongly correlated, our result implies that the average product of labor—which is easily measurable—is an informative proxy for the rank of a firm in the job ladder, even in environments richer than Elsby and Gottfries (2019).

[9] Another implication of such environments is that large firms, which pay higher wages in the model, should poach from small firms, while the data suggest otherwise.

[10] Recent examples are Postel-Vinay and Turon (2010); Jarosch (2015); Lindenlaub and Postel-Vinay (2016); Borovicková (2016); Lise and Robin (2017).

Outline. In Section 2 we establish the environment and our key assumptions on how firm and workers share value following various stochastic events. In Section 3, we state our joint value representation and, to provide intuition on how we achieve tractability, we apply our assumptions in a simplified, static framework. Section 4 returns to the fully dynamic model and, after defining an equilibrium, characterizes firm dynamics and worker flows. In Section 5 we estimate the model with US data, discuss identification and validate our parameterization. Section 6 present our three quantitative exercises. Section 7 concludes. The Appendix contains all proofs and details on the computation of the model's equilibrium.

2 Model

In this section we describe the characteristics of the agents in the economy, how meetings take place in the labor market, and the timing of events. We then lay out our key assumptions regarding the contractual environment, i.e. on how the value generated by production is shared. Finally, we state our main result: the joint value representation of the economy. Under this representation, all allocations are *privately efficient*, meaning that they maximize the joint value of all the agents involved in the decision.

2.1 Physical environment

Time is continuous and there is no aggregate uncertainty. There are two types of agents. An exogenous mass $\bar{n}$ of ex-ante identical, infinitely-lived *workers* that are risk neutral, discount the future at rate ρ and are endowed with one unit of time each period which is inelastically supplied to production. An infinite mass of homogeneous *potential firms*, of which an endogenous mass become *operating firms*.

Production technology. There is a single homogeneous good. Workers may either be employed or unemployed. Unemployed workers produce b units of the final good. Employed workers are organized into firms which are heterogeneous in their productivity $z \in Z$. A firm employing n workers produces $y(z, n)$ units of the final good, where $y(z, n)$ is strictly increasing in z and n and concave in n, i.e. $y_{nn}(z, n) \leq 0$.[11, 12]

[11] In addition, we assume that for any z the Inada conditions hold with respect to n: (i) $y(z, 0) = 0$, (ii) $\lim_{n \to 0} y_n(z, n) = +\infty$, and (iii) $\lim_{n \to +\infty} y_n(z, n) = 0$.

[12] Key for firms' decisions is that this specification yields a decreasing returns to scale *revenue function* $r(z, n) = Py(z, n)$, with the normalization $P = 1$. An alternative foundation for a decreasing returns to scale *revenue function* is *constant returns to scale* in production of a differentiated final good which would yield a *decreasing marginal revenue* as under monopolistic competition. If firms' goods are imperfectly substitutable and household preferences are CES with elasticity $\eta > 1$, the firm faces a demand curve $p(z, n) \propto y(z, n)^{-1/\eta}$, such that the *revenue function* is $r(z, n) = p(z, n) \times y(z, n) \propto y(z, n)^{(\eta-1)/\eta}$. Our framework therefore accommodates imperfect substitutability in the goods market which is a key ingredient of trade models and macroeconomic models with nominal rigidities.

Firm demographics. A potential firm becomes an operating firm by paying a fixed cost c_0. Paying the fixed cost c_0 produces a draw of productivity z from the distribution $\Pi_0(z)$ and n_0 workers, taken from unemployment. If a potential firm enters and becomes an operating firm then its productivity z evolves stochastically. At any point in time a firm may exit, at which point all of its workers become unemployed and the firm produces $\vartheta > 0$ units of the final good which we refer to as its *scrap value*.[13] Denote the mass of entrants $\mathtt{m}_0$ and the mass of operating firms $\mathtt{m}$.

Matching technology. Workers join firms from both employment and unemployment through a frictional matching process. The total number of meetings between firms and workers is given by a CRS aggregate matching technology $m(\mathtt{s},\mathtt{v})$. Inputs to the matching technology are total vacancies $\mathtt{v}$ and total units of search efficiency $\mathtt{s} = \mathtt{u} + \xi(\bar{\mathtt{n}} - \mathtt{u})$, where the parameter ξ determines the relative search efficiency of employed workers. Search is random in the following sense. A firm pays a cost $c(v; z, n)$ to post v vacancies, where c is increasing and convex in v and $c(0; \cdot, \cdot) = 0$. Each vacancy of the firm is matched with a worker at rate $q(\mathtt{s},\mathtt{v}) = m(\mathtt{s},\mathtt{v})/\mathtt{v}$. With probability $\phi = (\mathtt{u}/\mathtt{s})$ the worker is unemployed, with probability $(1-\phi)$ the worker is employed. A worker faces no cost of search. An unemployed worker meets a firm at rate $\lambda^U(\mathtt{s},\mathtt{v}) = m(\mathtt{s},\mathtt{v})/\mathtt{s}$. An employed worker meets a firm at rate $\lambda^E(\mathtt{s},\mathtt{v}) = \xi\lambda^U(\mathtt{s},\mathtt{v})$. The rates q and λ^U can be expressed in terms of *market tightness* $\theta = (\mathtt{v}/\mathtt{s})$ If constituted, the match of a worker to a firm exogenously expires at rate δ, upon which the worker becomes unemployed.

States. Let x be the vector of state-variables for the firm. This vector includes all individual state variables of all workers at the firm. For now, we do not specify exactly what is in x and, along the way, define a number of functions that map x at instant t into a new state vector at $t + dt$. The vector x is common knowledge among all workers and the firm. Let i be an indicator function (possibly also a vector) that selects the particular entries of x that identify the worker within a firm (i.e., i is the unique identity of a worker in the firm x).[14] Let $H(x)$ be the measure of x across firms in the economy, $v(x)$ the number of vacancies created by a firm with state x, and $n(x)$ employment at firm x. The total mass of vacancies and employed workers in the economy are

$$\mathtt{v} = \int v(x)\, dH(x) \quad , \quad \mathtt{n} = \bar{\mathtt{n}} - \mathtt{u} = \int n(x)\, dH(x).$$

[13] A positive scrap value plays the same role as a fixed operation cost in generating endogenous exit.

[14] For example, x is a complete description of IBM and of all its workers. It might contain IBM productivity z, its size n, and all those features of the contracts of the current employees that are needed to forecast IBM's value and the value of each of its workers. The state (x, i) should be read: here is IBM, characterized by x, and we are assessing the characteristics of the worker named i within IBM.

Probability densities that will show up in firm and worker problems describe the vacancy-weighted and employment weighted distributions of firms:

$$h_v(x) = \frac{v(x)h(x)}{\mathtt{v}} \quad , \quad h_n(x) = \frac{n(x)h(x)}{\mathtt{n}}.$$

Information. Information in the economy is complete. Workers and firms know the relevant aggregate variables, i.e. $\mathtt{u}$, $\mathtt{m}$, the measure $H(x)$ and distributions $H_v(x)$, $H_n(x)$. The states x and x' of firms in competition for a worker are observable to both firms and to all incumbent workers of the two firms. Similarly, unemployed workers know the vector x of the firm they meet when searching, and incumbents of firm x know whether the firm has met with an unemployed worker.

Timing. We separate the within-dt timing of events in the model into two parts.

First, events up to the opening of the labor market are described in Figure 1. A firm's productivity z is first realized. Next, incumbent workers are fired, choose whether to quit the firm, or their employment contracts are renegotiated. Next, the firm decides whether to stay in operation or exit. An operating firm produces $y(z,n)$, pays wages according to contracts with its workers, and posts vacancies in the labor market.

Second, the mutually exclusive events that may occur to a worker or firm are described in Figure 2.[15] The first branch in Figure 2 describes events that may occur to an unemployed worker. The second and third branch distinguish between direct and indirect events that may affect the value of incumbent worker i. *Direct* events involve worker i meeting with another firm, or the destruction of the worker's job. *Indirect* events involve worker i's co-worker j meeting with another firm, or the destruction of a co-worker's job. The final branch describes events that directly impact the firm. The firm may meet an employed or unemployed worker, emerge either with a new hire or not and new allocation of values to its workers, reflected in updates to the state x. Following any of these events, the state vector x changes, potentially affecting the value of the match to worker i. Through the following assumptions, we put structure on the states in which these events occur and how values evolve in each case.

2.2 Contractual Environment

In this section we state a set of assumptions on the contractual environment sufficient to derive our main theoretical results. It is useful to begin from the definition of a wage contract. A contract between the firm and one of its workers is a binding agreement which specifies a constant wage, i.e. a fixed payment

[15] The mutual exclusivity property is a consequence of continuous time.

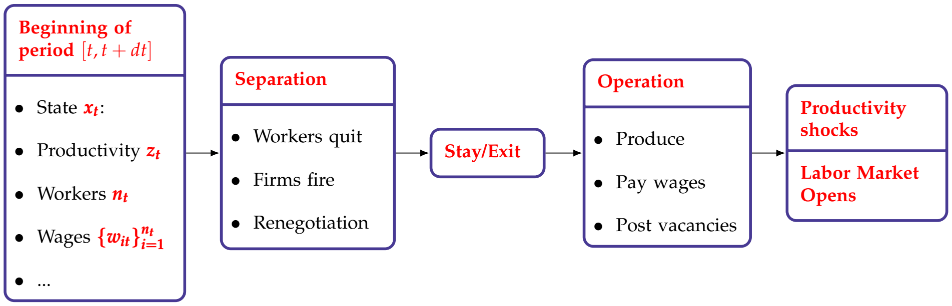


Figure 1: Timing of events prior to the opening of the labor market

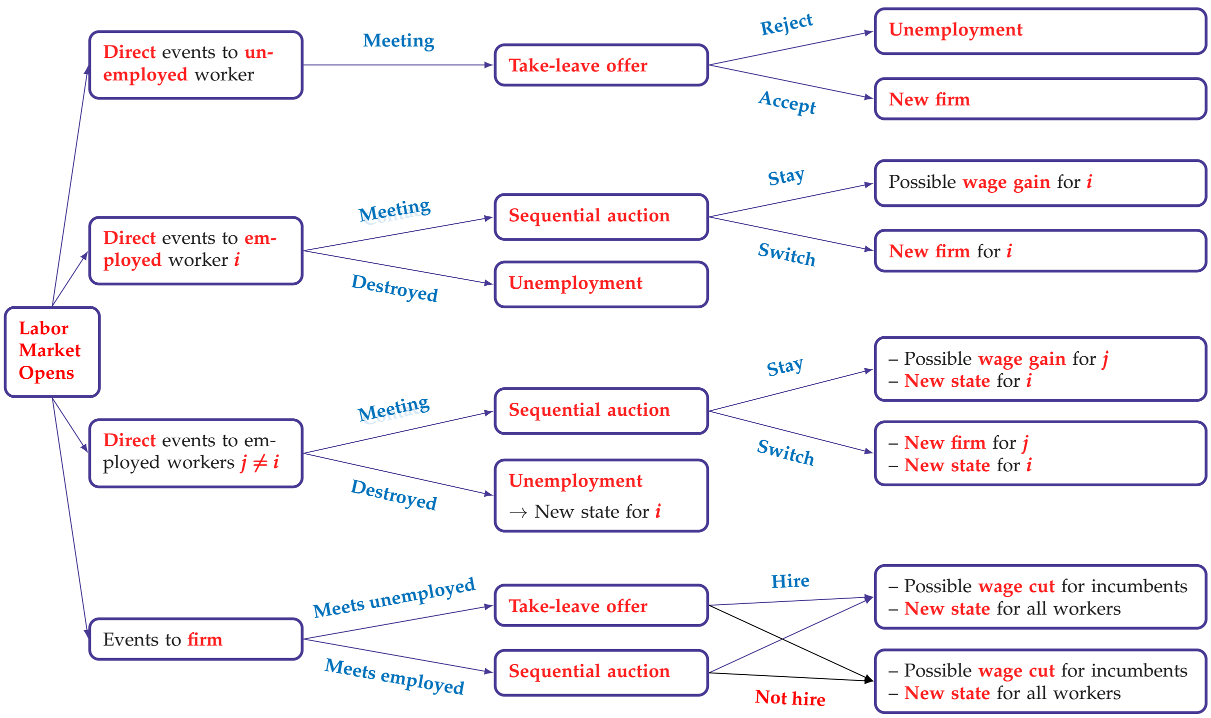


Figure 2: Labor market: Set of mutually exclusive possible labor market events

from the firm to the worker, in exchange for labor services. The contract satisfies five assumptions:

(A-LC) **Limited commitment.** All parties are subject to limited commitment. In particular,

(a) **Layoffs -** Firms can fire workers at will.

(b) **Quits -** Workers can always quit into unemployment or to another firm when they meet one.

(c) **Collective agreements -** Workers cannot commit to any other worker inside the firm. *De facto* this assumption rules out transfers among workers.

(A-MC) **Mutual consent.** The wage (contract) can be renegotiated only by mutual consent, i.e. only if one party can credibly threaten to dissolve the match (the firm by firing, the worker by quitting). A threat is credible when one of the two parties has an outside option that provides her with a value that is higher than the value under the current contract.

(A-EN) **External negotiation.** An *external negotiation* is a situation where, through search, the firm comes into contact with an external job seeker or an incumbent worker comes into contact with another firm. In external negotiations, all offers are *take-it-or-leave-it*.

- In a meeting an unemployed worker, the firm makes a take-leave offer to the worker.
- In a meeting with an employed worker, the two firms Bertrand compete through a *sequential auction*. First, the poaching firm makes the take-leave wage offer. Second, the target firm makes a take-leave counteroffer to the worker. Third, the worker decides.

(A-IN) **Internal negotiation.** An *internal negotiation* is any other situation where contracts between firm and any incumbent workers are modified (following **(A-MC)**, an internal negotiation takes place when any party has a credible threat). The only parties involved in an internal negotiation are those that have a threat and those that are under that threat. We assume that—with respect to worker and firm values—the internal negotiation is a *zero-sum game* and that participation is individually rational for all parties.[16] Apart from these assumptions we leave internal negotiation unrestricted.

(A-VP) **Vacancy posting.** The firm posts the privately efficient amount of vacancies, which is the one that maximizes the sum of the values of the firm and its workers. Below we propose one possible micro-foundation for **(A-VP)**.

Discussion. First, our simple wage contracts are rooted in incomplete contract theory, in which a key tenet is that contracting is only allowed on features that are verifiable to a third party e.g. a court. In our context, the only verifiable and hence contractible features are the wage, whether the firm made the wage payment, and whether the worker provided labor services. As a result, more complex state contingent contracts are ruled out. In the context of such incomplete contracts, renegotiation under mutual consent is a natural assumption consistent with many existing legal frameworks (as argued by Malcomson, 1999), and in the terminology of MacLeod and Malcomson (1989) yields *self-enforcing contracts*.

[16] We adopt the standard definition of a zero-sum game: each individual's gain or loss is exactly offset by losses and gains of other participants. We also adopt the standard definition of individual rationality: after internal negotiation each player who remains employed at the firm receives at least the outside option that was present before internal negotiation.

Second, **(A-LC, a,b)**, **(A-EN)** and **(A-MC)** amount to the contractual environment in Postel-Vinay and Robin (2002). The authors show that they lead to a convenient joint value representation in the one-worker-one-firm model. We now discuss how **(A-IN)** and **(A-VP)** are sufficient to extend this convenient representation to an environment with a diminishing marginal product of labor.

Our zero-sum game assumption on internal negotiation **(A-IN)** allows for a large class of possible micro-foundations for the internal renegotiation game. Each would imply different wage dynamics. The central takeaway is that, no matter the details of such a game, if **(A-IN)** is satisfied then our following representation of *allocations* as determined by joint value dynamics holds. Since this paper focuses on the *allocations* that result from firm and worker dynamics in a frictional labor market, we leave for future research a detailed theoretical and empirical investigation of the implications of different internal renegotiation games.[17]

Absent **(A-VP)**, the firm would have strong incentives to over-post vacancies relative to the privately efficient amount. The incentives to over post come in two forms: over-hiring and generating what we call *swapping threats*. The firm may post vacancies in order to over-hire, lowering marginal product and so credibly threaten some of its incumbents with wage cuts, as extensively discussed by Stole and Zwiebel (1996) and Brügemann, Gautier, and Menzio (2018). The firm may also post vacancies with no intention of hiring—which only occurs when marginal products are decreasing—hoping to use a match to threaten to *swap* an incumbent worker with a job seeker, extracting a wage cut from the incumbent. Proceeding under either would require the full distribution of wages as a state variable, ruling out tractability. Assumption **(A-VP)** resolves these issues.[18]

The presence of these inefficiencies and the need for an assumption like **(A-VP)** is unique to an environment with DRS, on-the-job search and endogenous vacancy posting. In a model with constant returns and on-the-job search, over-hiring does not arise due to a constant marginal product of labor (Postel-Vinay and Robin, 2002). Constant returns also implies that filling a vacancy with a matched job seeker is always profitable, removing the swapping threat. In a model with extreme decreasing returns—a capacity constraint of one worker—and on-the-job search but without endogenous vacancy posting there is no inefficient vacancy posting to resolve (Kiyotaki and Lagos, 2007). In a multi-worker firm model with decreasing returns and endogenous vacancies but without on-the-job search, incumbents are all hired

[17] As a start, a companion note available on our websites, (Bilal, Engbom, Mongey, and Violante, 2019) shows how wages would be pinned down under a particular internal renegotiation mechanism. We assume that workers make take-leave offers to the firm in internal renegotiation. Since this satisfies **(A-IN)**, then *allocations* are consistent with our general representation (1). In this setting we can compute wages without having to keep track of the entire within-firm wage distribution under the assumption that exit is exogenous. We show that if, instead, the firm made take-leave offers to workers in internal renegotiation—which also satisfies **(A-IN)**, complex transfers would be needed to implement **(A-VP)** and compute wages.

[18] In a different environment that allows full commitment to paying a fixed wage, as in Hawkins (2015), wage cuts are assumed away which eliminates privately inefficient vacancy posting.

from unemployment and with the same outside option are paid the same wage (Elsby and Michaels, 2013; Acemoglu and Hawkins, 2014). Swapping is not a threat because the job seeker and incumbent are paid the same wage. An over-hiring inefficiency is present, but with a degenerate distribution of wages within the firm, accommodating this inefficiency does not impede tractability. On-the-job-search generates a distribution of wages inside the firm due to the origin of hire and accumulated outside offers. If not addressed, the over-hiring inefficiency would render the model intractable.

We propose one possible micro-foundation that implements assumption **(A-VP)**. The idea is to remove any gains to the firm from expected future wage cuts that would otherwise encourage excess vacancy posting. We assume that workers anticipate that firm's behavior and offer a preemptive wage cut that leaves the firm indifferent between the efficient vacancy policy and the firm's privately optimal policy.[19] We formalize this assumption below.

(A-VPI) After the firm announces its proposed vacancies for dt, a randomly selected incumbent worker has the opportunity to make a take-leave counter-offer to the firm. The counter-offer specifies acceptable wages for all (or some) incumbents in exchange for an alternative spot vacancy policy.[20]

Having described the economy's environment and the contract space, we now state our main result.

3 Joint value representation

In this section we describe the main theoretical result of the paper. For presentation purposes, the environment is specialized in two ways. First, each firm employs a continuum of workers n. Second, productivity follows a diffusion $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$.[21]

Result. All *allocative decisions* in the economy—entry, exit, vacancy posting and mobility of workers between firms—are determined by the *joint value*. The *joint value* $\Omega(z, n)$ is the sum of the present discounted value of an operating firm's profits plus the present discounted value of lifetime utility of its workers, and satisfies the following, where U is lifetime utility of an unemployed worker:

[19] Alternative implementations could be based on the introduction of 'social norms' that prevent firms from cutting the wage of a worker and swapping an incumbent worker with a new worker. Because they would involve deviations from lack of commitment **(A-LC)**, we do not emphasize these alternative implementations in this paper.

[20] This assumption does not require commitment because it is a 'spot contract' between the parties involved: a transfer in exchange for an immediate action.

[21] As shown in Appendix II, our results also hold with an integer-valued workforce and when the productivity process is a jump-diffusion.

$$\begin{aligned}
\rho\Omega\left(z,n\right) = \max_{v\geq 0} \quad & y\left(z,n\right) - c\left(v;z,n\right) \\
\textit{EU destruction:} \quad & -\ \delta n\Big[\Omega_n\left(z,n\right) - U\Big] \\
\textit{UE hire:} \quad & +\ \phi q(\theta) v\Big[\Omega_n\left(z,n\right) - U\Big] \\
\textit{EE hire:} \quad & +\ (1-\phi) q(\theta) v \int \max\Big\{\Omega_n\left(z,n\right) - \Omega_n\left(z',n'\right)\ ,\ 0\Big\} dH_n\left(z',n'\right) \\
\textit{Shock:} \quad & +\ \mu(z)\Omega_z\left(z,n\right) + \frac{\sigma(z)^2}{2}\Omega_{zz}\left(z,n\right).
\end{aligned} \tag{1}$$

Firms' operation requires (z,n) to be interior to an exit boundary. An additional boundary condition determines when separations occur:[22]

$$\textit{Exit boundary:} \quad \Omega(z,n) \geq \vartheta + nU, \quad , \quad \textit{Layoff boundary:} \quad \Omega_n(z,n) \geq U. \tag{2}$$

The first term in (1) is simply output net of vacancy costs. Next, the firm exogenously loses a worker at rate δn with a net loss of $\Omega_n - U$ to the initial coalition of firm and workers. The change in value has two pieces: the change in value of the firm and its non-separating workers which is simply the marginal value of the lost worker $(-\Omega_n)$ and the value of unemployment attained by the separated worker $(+U)$. The firm hires by posting vacancies which are matched to a worker at rate $q(\theta)$, the probability that this worker is unemployed is ϕ. The firm always hires unemployed workers, which increases the value of the firm and incumbents by Ω_n but requires a pledge of U to the new worker.

The firm also hires from and loses workers to other firms by poaching. Workers at other firms are met according to the employment-weighted distribution of productivity and size, H_n. Upon meeting, the net coalition value increases by $\Omega_n(z,n) - \Omega_n(z',n')$, so poaching is successful if the firm's marginal value is largest. Note that $\Omega_n(z',n')$ is the highest value the other firm will offer to its incumbent, and hence it is the take-leave offer the poaching firm will make, as long as it is lower than $\Omega_n(z,n)$.[23]

The firm's current workers may also quit to higher marginal value firms. The firm and non-poached workers will lose $\Omega_n(z,n)$ and so are prepared to increase the poached worker's value by this amount to retain them. Knowing this, the external firm offers the poached worker exactly $\Omega_n(z,n)$ in order to hire them. The joint value—that of firm, non-poached workers and poached worker—is therefore unchanged and reminiscent of Postel-Vinay and Robin (2002), no '*EE Quit*' term appears in (1).

[22]More formally, Appendix II states the full Hamilton-Jacobi-Bellman-Variational-Inequality formulation of the joint value problem.

[23]This term of the Bellman equation reads as if the poaching coalition, which induces a breach of contract between the worker and the losing coalition, compensates the latter exactly for its loss of value associated with the quit. This scheme is reminiscent of the result in Diamond and Maskin (1979) (also present in Kiyotaki and Lagos, 2007) that compensatory damages in breach of contracts restore efficiency.

Boundary conditions (2) describe firm exit and layoffs. Firms keep operating if the value of doing so exceeds the total value of exit: the private scrap value ϑ plus unemployment for all its workers. If productivity falls, the marginal value of a worker will fall, but must remain above the opportunity cost of employment. To ensure this, firms layoff workers to sustain $\Omega_n(z, n) \geq U$.

This joint value representation has three appealing properties.

3.1 Properties

(1) Parsimony. Firm and worker policies are characterized by a low-dimensional state vector: productivity and size. Given decreasing returns to scale in production and on-the-job search, this simplification is a contribution. With decreasing returns spillovers exist as bargaining moves from one worker to the next. This problem has been addressed in the literature by following the approach of Stole and Zwiebel (1996), recently revisited by Brügemann, Gautier, and Menzio (2018), which delivers tractability. However this approach fails when workers have heterogeneous outside options such as due to on-the-job search. Previous frameworks have therefore restricted their analysis to the case of homogeneous outside options which requires confining attention only to labor market transitions between employment and unemployment, ignoring job-to-job flows and poaching altogether. In models with on-the-job search and heterogeneous outside options these bargaining spillovers are assumed away either (i) by constant returns to scale, which reduces decision making units to one-worker-one-firm pairs and impedes a proper study of firm dynamics; or (ii) by the combination of full commitment to complex state-contingent contracts and directed search. Our contribution is to prove that a plausible set of minimal assumptions on the contractual environment (featuring limited commitment) is sufficient to micro-found a parsimonious representation of allocations.

(2) Private efficiency. All agents' decisions (entry, exit, separations, vacancies, and hires) maximize their joint value. Put differently, in external and internal negotiations all privately attainable gains from trade are exploited. For the parties involved, no transfer could yield a Pareto improvement. We have therefore shown how the Coase theorem arises in our context without the need to assume full commitment and complex state contingency in contracting.[24]

[24] We do not solve for the socially efficient allocations in this paper, but note that the decentralized and planner's allocations will not coincide. Besides the standard congestion externality à la Hosios, an additional composition externality arises. As in Acemoglu (2001), low-productivity firms do not internalize that when posting vacancies on-the-job search will result in them diverting workers away from high-productivity firms. These distorted vacancy decisions affect the equilibrium distributions of workers across firms H_n which, in turn, influences the hiring opportunities of all other firms.

(3) Job ladder. In one-worker-one-firm models, it is the firm's *exogenous* productivity that fully determines its position on the job ladder. Here the ladder is in *endogenous marginal values* of labor $\Omega_n(z, n)$. These equilibrium objects are determined by the current marginal product of labor together with expectations of future productivity, worker mobility and exit. Hence life-cycle firm dynamics and worker dynamics across firms determine their equilibrium distribution. In particular, the equilibrium allocation is affected by a job ladder in marginal values that endogenously lowers the cost of hiring for firms at the top of the ladder.

Proof. The proof of the joint value representation for the full dynamic model, which needs much additional notation, is in Appendix II. To convey the economics of how our assumptions lead to this result, we use a static model. The approach and the arguments are the same as in the dynamic model, but the proof is much shorter. This static example covers the construction of each term in (1), one by one. We describe the *UE hire* term here, and detail the construction of the other terms in Appendix A.

3.2 Static example

Set up. Consider a firm with decreasing returns to scale technology $y(z, n)$ such that $y(z, 0) = 0$. Suppose the firm starts with productivity z and $n = 1$ worker. The current contract between the firm and the incumbent specifies a wage $w_1 \in (b, y(z, 1))$, where $b = U$ is the value of unemployment. At this point, the incumbent worker does not have a credible threat to quit into unemployment nor the firm has a credible threat to fire the worker. Then, the labor market opens. For now we also assume that the firm has sunk the cost of a vacancy c, in the Appendix we explicitly consider the decision to post a vacancy.

UE ***Hire.*** We describe how to obtain the 'UE hire' term in (1). Assume the firm's vacancy meets an unemployed worker. Four different cases can arise from the combination of hiring/not hiring and renegotiating/not renegotiating the wage with the incumbent. Our assumption on external negotiation **(A-EN)** requires that in all cases the take-leave wage offer of the firm to the outside worker is $w_2 = b$. Our internal negotiation assumption **(A-IN)** requires that the joint value with and without renegotiation is the same and simply equals output $y(z, n)$. Let w_1^* be the incumbent wage after the internal negotiation.

If the firm hires the new worker, its profits are as follows:

$$\underbrace{y(z,2) - w_1 - b}_{\text{Without renegotiation}} \quad , \quad \underbrace{y(z,2) - w_1^* - b}_{\text{With renegotiation}},$$

If the firm does not hire the new worker, its profits are

$$\underbrace{y(z,1) - w_1}_{\text{Without renegotiation}} \quad , \quad \underbrace{y(z,1) - w_1^*}_{\text{With renegotiation}}$$

We now describe which case occurs. This requires understanding when our mutual consent assumption **(A-MC)** coupled with limited commitment on layoffs **(A-LC)** bind. In particular, the firm may obtain a credible threat to trigger renegotiation of w_1. Since we are interested in allocations only, we focus first on when a hire occurs.

Hire. A hire *without* renegotiation occurs when the following two conditions hold:

$$\underbrace{y(z,2) - w_1 - b \geq y(z,1) - b}_{\text{No credible threat}} \quad , \quad \underbrace{y(z,2) - w_1 - b \geq y(z,1) - w_1}_{\text{Optimal to hire w/o renegotiation}} \tag{3}$$

The first condition illustrates that the threat to fire the incumbent worker is not credible, which under **(A-MC)** implies no renegotiation. Keeping the incumbent worker at w_1 and employing the outside worker at b delivers a higher value to the firm than the threat of "swapping": firing worker one and hiring the unemployed worker in his place. Given no renegotiation, the second condition ensures hiring is privately optimal for the firm.

A hire *with* renegotiation occurs when the following two conditions hold:

$$\underbrace{y(z,2) - w_1 - b < y(z,1) - b}_{\text{Credible threat}} \quad , \quad \underbrace{y(z,2) - w_1^* - b > y(z,1) - w_1^*}_{\text{Optimal to hire w/ renegotiation}}. \tag{4}$$

The firm has now a credible threat to fire the incumbent worker according to **(A-LC)**. This is possible only under decreasing returns to scale: even though $w_1 < y(z,1)$, the first inequality in (4) implies $w_1 > y(z,2) - y(z,1)$, i.e. the incumbent wage is above its own marginal product. Employing the outside worker at b and keeping the incumbent worker at w_1 delivers a lower value than 'firing and swapping'. The second condition is necessary for hiring to be optimal under the renegotiated wage w_1^* to the incumbent worker.

Under the zero sum game assumption **(A-IN)**, the renegotiated wage w_1^* only redistributes value between the incumbent worker and the firm and does not affect total value.[25] In addition, it must be individually rational, and so $w_1^* \in [b, y(z,2) - y(z,1)]$. Without further assumptions we cannot say exactly what this wage is, but we can nonetheless pin down allocations.

Rearranging the optimal hiring conditions, we observe that both are satisfied as long as

$$y(z,2) - y(z,1) > b. \tag{5}$$

[25] Two relevant cases that would violate this condition are (i) if worker's effort depends on the wage and enters the production function, and (ii) concave utility.

Note that without internal renegotiation **(A-IN)**, the hiring condition would differ in the two cases. If wages could not be cut and the firm had a credible threat, the incumbent worker would be fired and the firm would always hire the unemployed worker ($y(z,1) > b$). As a result, to determine when a hire occurs, one would need to know the incumbent's wage to distinguish between the two cases (thus, in the general model with n workers, the whole wage distribution). Similarly, if a fraction of output were to be lost because of the internal negotiation, a violation of **(A-IN)**, the hiring conditions in (3) and (4) would differ and, again one would need to know wages to determine whether a hire occurs.

We can write inequality (5) in terms of joint value. Workers' values are simply equal to their wage w_i for $i \in \{1,2\}$. The firm's value is simply equal to its profits. The fact that wages are valued linearly by both worker and firm implies that the joint value $\Omega(z,n)$ is independent of wages:

$$\Omega\left(z,n\right) = \underbrace{y(z,n) - \sum_{i=1}^{n} w_i}_{\text{Firm value}} + \underbrace{\sum_{i=1}^{n} w_i}_{\text{Sum of workers' values}}, \quad \text{for any } \left(w_i\right)_{i=1}^{n}.$$

Using the definition of joint value, equation (5) characterizes when the UE hire occurs:

$$\Omega\left(z,2\right) - \Omega\left(z,1\right) > U. \tag{6}$$

Thus, the decision of hiring from unemployment does not depend on wages, but only on productivity, size, and the value of unemployment $U = b$.

No hire. For completeness, consider the cases where no hiring occurs. No hire with renegotiation occurs when the following two conditions hold:

$$\underbrace{y(z,1) - b > y(z,1) - w_1}_{\text{Credible threat}}, \quad \underbrace{y(z,1) - w_1^* \geq y(z,2) - w_1^* - b}_{\text{Optimal to not hire}} \tag{7}$$

When incentive compatible for the firm to not expand its workforce, the firm always has a credible threat to swap out its incumbent worker since $w_1 > b$. In this case we can pin down w_1^* from the worker's individual rationality constraint. If $w_1^* > b$, then the firm would still have a credible threat to swap the worker, hence $w_1^* = b$. Since this outcome represents a redistribution of value between firm and worker then, consistent with **(A-IN)**, the joint value remains $\Omega(z,1)$.[26] Finally, the no-hiring condition in (7) can be re-written as in (5) with the opposite inequality, $\Omega(z,2) - \Omega(z,1) \leq U$.

[26]The value before renegotiation was $\Omega\left(z,1\right) = z - w_1 + w_1 = z$. The joint value after renegotiation is $\Omega\left(z,1\right) = z - w_1^* + w_1^* = z$.

Combined. The firm hires from unemployment when its vacancy meets an unemployed worker and the *marginal value* of the job seeker exceeds the value of unemployment:

$$\frac{\Omega(z,2)-\Omega(z,1)}{2-1} > U. \tag{8}$$

In addition, the joint value of the firm and its workers rises by $\frac{\Omega(z,2)-\Omega(z,1)}{2-1} - U$ when the hire occurs. This is exactly the *UE hire* term in the HJB equation (1). In the case of a hire, incumbent wages may or may not be renegotiated but this has no impact on whether hiring occurs, or how the joint value changes. When this condition fails, the firm does not hire, wages are renegotiated, but the joint value remains constant. All decisions require knowledge of (z,n) only, but not of incumbents' wages.

Appendix A shows how our assumptions deliver all the other terms of (1): *EE* hires, vacancy posting, layoff, quits, entry and exit. It also contains an extension of the *UE* hire case to a scenario in which the hiring firm has $n=2$ incumbents and either one or both may be under threat of layoff.

4 Equilibrium and characterization

Returning to the joint value representation (1) and (2), we define an equilibrium and characterize firm behavior and allocations.

4.1 Surplus formulation

A convenient formulation of (1) is in terms of *joint surplus*, defined as $S(z,n)=\Omega(z,n)-nU$, such that

$$S_n(z,n)=\Omega_n(z,n)-U \quad , \quad S_z(z,n)=\Omega_z(z,n) \quad , \quad S_{zz}(z,n)=\Omega_{zz}(z,n).$$

The marginal (joint) surplus $S_n(z',n')$ at a competitor is sufficient to characterize how surplus changes over an *EE* hire. We therefore directly compute the value of a vacancy using the employment-weighted distribution of marginal surplus in the economy: $H_n(S_n')$. Recall that $\rho U = b$. With these definitions (1) becomes

$$\begin{aligned} \rho S(z,n) \;=\; \max_{v\geq 0} \;\; & y(z,n) - nb - \delta n S_n(z,n) \\ & + q(\theta)v \underbrace{\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} \left(S_n(z,n)-S_n'\right)\, dH_n\left(S_n'\right)\right]}_{\text{Return on a vacancy: } R(z,n)=\widetilde{R}(S_n(z,n))} - c\big(v;z,n\big) \\ & + \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n) \end{aligned} \tag{9}$$

subject to the two boundary conditions expressed in terms of surplus:

$$\textit{Exit boundary:} \quad S(z,n) \geq \vartheta \quad , \quad \textit{Layoff boundary:} \quad S_n(z,n) \geq 0. \tag{10}$$

The entry decision can be written as: $\int S(z,n_0)\, d\Pi_0(z) \geq c_0$.

One-worker-one-firm models. The Bellman equation (9) is a natural extension of expressions of firm value found in earlier single worker job ladder models. In these settings, constant returns to scale production imply that firms can be treated as arbitrary groups of one-worker-one-firm pairs, each with match output $y(z)$. The surplus from such a firm-worker match in our model follows closely Postel-Vinay and Robin (2002) and Lise and Robin (2017). It can be obtained as a special case of (9) when the functions y and c are linear in n:[27]

$$\begin{aligned}\rho S(z) \quad = \quad \max_{v\geq 0}\; & y(z) - b - \delta S(z) + q(\theta)v\left[\phi S(z) + (1-\phi)\int_0^{S(z)}\left(S(z)-S'\right)dH_n(S')\right] - c(v;z) \\ & + \mu(z)S_z(z) + \frac{\sigma^2(z)}{2}S_{zz}(z)\end{aligned} \tag{11}$$

Surplus depends only on exogenous productivity z, and with one worker firms the unweighted and employment weighted measures of firms are identical. The expected return to a vacancy is therefore computed by integrating over $H_n(S') = H(z)$. In our framework $H_n(S')$ is an equilibrium outcome, while here it coincides with the *exogenous* productivity distribution. Thus, the rank of a firm on the job ladder is determined only by its productivity z.

4.2 Equilibrium

A stationary equilibrium with positive entry consists of: (i) a joint surplus function $S(z,n)$; (ii) a vacancy policy $v(z,n)$; (iii) a law of motion for firm level employment $\frac{dn}{dt}(z,n)$; (iv) a stationary distribution of firms $H(z,n)$; (v) vacancy and employment weighted distributions of marginal surplus $H_v(S_n)$ and $H_n(S_n)$; (vi) a positive mass of entrants $\mathtt{m}_0$, (vii) a vacancy meeting rate $q(\theta)$ and conditional probability of meeting an unemployed worker ϕ, such that:

(i) Total surplus $S(z,n)$ satisfies the HJB equation (9) and boundary conditions (10).

[27] The comparison is clearest with Lise and Robin (2017) which features convex vacancy costs. To be precise in the comparison split (11) into three pieces. The value of the coalition is $(\delta+\rho)S(z) = y(z) - b + \mu(z)S_z(z) + \left(\sigma^2(z)/2\right)S_{zz}(z)$, corresponding to their equation (3). The vacancy decision can be split off and given by $v(z)$ that satisfies $c_v(v(z),z) = q(\theta)\widetilde{S}(z)$, where the *"expected value of a contact"* is $\widetilde{S}(z) = \phi S(z) + (1-\phi)\int\left[S(z)-S(z')\right]^+ h_n(z')dz'$. These are their equations (6) and (7), respectively.

(ii) The vacancy policy $v(z,n)$ satisfies the first order condition:

$$c_v(v(z,n);z,n) = q(\theta)\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} \left(S_n(z,n) - S_n'\right)\, dH_n(S_n')\right].$$

(iii) The law of motion for firm level employment is

$$\frac{dn}{dt}(z,n) = \begin{cases} -\frac{n}{dt} & n < n_E^*(z) \\ q(\theta)v(z,n)\Big[\phi + (1-\phi)H_n(S_n(z,n))\Big] - n\Big[\delta + \lambda^E(\theta)(1-H_v(S_n(z,n)))\Big] & n \in \Big[n_E^*(z), n_L^*(z)\Big) \\ \frac{n_L^*(z)-n}{dt} & n \geq n_L^*(z), \end{cases}$$

where the notation $\frac{n}{dt}$ denotes a jump of size n, and where the layoff and exit thresholds satisfy

$$\underbrace{S_n\big(z,n_L^*(z)\big) = 0 \quad , \quad S\big(z,n_E^*(z)\big) = \vartheta}_{\text{From (10)}} \quad , \quad \underbrace{S_z\big(z,n_E^*(z)\big) = 0 \quad , \quad S_n\big(z,n_E^*(z)\big) = 0 \;\; \text{if} \;\; \frac{dn}{dt}\big(z,n_E^*(z)\big) < 0}_{\text{Smooth pasting}}$$

(iv) The vacancy and employment weighted distributions of marginal surplus $H_v(S_n)$ and $H_n(S_n)$ are consistent with $H(z,n)$:

$$H_v(S_n) = \int \mathbb{1}_{[S_n(z,n)\leq S_n]}\frac{v(z,n)}{\mathtt{v}}dH(z,n) \quad , \quad \mathtt{v} = \int v(z,n)dH(z,n)$$
$$H_n(S_n) = \int \mathbb{1}_{[S_n(z,n)\leq S_n]}\frac{n(z,n)}{\mathtt{n}}dH(z,n) \quad , \quad \mathtt{n} = \int n(z,n)dH(z,n)$$

(v) The measure of firms $H(z,n)$ is stationary, and admits a density function $h(z,n)$ that satisfies:

$$0 = -\frac{\partial}{\partial n}\left(\frac{dn}{dt}(z,n)\, h(z,n)\right) - \frac{\partial}{\partial z}\Big(\mu(z)\, h(z,n)\Big) + \frac{\partial^2}{\partial z^2}\left(\frac{\sigma(z)^2}{2}h(z,n)\right) + \mathtt{m}_0\, \pi_0(z)\, \mathbf{\Delta}(n)$$

where $\mathbf{\Delta}$ is the Dirac delta "function" which is zero everywhere except $n = n_0$ where it is infinite.[28]

(vi) Free-entry implies that the entry condition holds with equality, which determines entry $\mathtt{m}_0$:

$$c_0 = \int S(z,n_0)d\Pi_0(z),$$

(vii) The vacancy meeting rate $q(\theta)$ and conditional probability of meeting an unemployed worker ϕ are consistent with the aggregate matching function given unemployment ($\mathtt{u} = \bar{\mathtt{n}} - \int n\, dH(n,z)$) and aggregate vacancies ($\mathtt{v} = \int v(z,n)dH(z,n)$).

The numerical procedure to compute the equilibrium of the model is described in Appendix D.

[28]For notational brevity we have slightly abused notation by writing the Kolmogorov-Forward equation in the space of Schwarz distributions.

4.3 Vacancy policy

From (9), the first order condition for the firm's vacancy decision gives

$$q(\theta)R(S_n(z,n)) = c_v(v;z,n) \quad , \quad \text{where} \quad R(S_n) = \phi S_n + (1-\phi)\int_0^{S_n} \left(S_n - S_n'\right) dH_n\left(S_n'\right) \tag{12}$$

The return on a vacancy is independent of v, and is a strictly increasing and strictly convex function of only marginal surplus:

$$R'(S_n) = \underbrace{[\phi + (1-\phi)H_n(S_n)]\cdot 1}_{\uparrow \text{ Surplus on each hire}} + \underbrace{(1-\phi)h_n(S_n)\cdot 0}_{\text{Surplus on additional hires}=0} \quad , \quad R''(S_n) = (1-\phi)h_n(S_n)$$

Since the function c is convex in v and $c(0;z,n) = 0$, optimal vacancies are uniquely determined. On the *intensive margin*, a rise in S_n increases the return to hiring an unemployed or employed worker one-for-one. On the *extensive margin*, increasing S_n widens the set of firms from which the firm will poach, increasing the probability of a hire by $(1-\phi)h_n(S_n)$, but hiring from these additional firms yields zero additional value as the target firm's marginal surplus associated with the worker is close to that of the poaching firm.

4.4 Endogenous hiring cost

The literature on firm dynamics models employment adjustment costs parametrically. Search frictions and the job ladder induce, instead, an endogenous *hiring cost function* which depends on both equilibrium market tightness and on the rank of the firm on the job ladder.

The hiring rate per vacancy for a firm with marginal surplus $S_n(z,n)$ is $p = q(\theta)[\phi + (1-\phi)H_n(S_n)]$. Attaining $\widetilde{h}$ hires therefore requires $\widetilde{h}/p$ vacancies and costs $\mathcal{C}(h,n,z,S_n)$, given by

$$\mathcal{C}\left(h,z,n,S_n\right) = c\left(v\left(\widetilde{h},S_n\right);\, z,\, n\right) \quad = \quad c\left(\frac{\widetilde{h}}{q(\theta)\left[\phi + (1-\phi)H_n(S_n)\right]};z,n\right). \tag{13}$$

The reduced form hiring cost function implied by the model is convex in $\widetilde{h}$ and decreasing in marginal surplus. It is also determined by two equilibrium objects: overall market tightness via $q(\theta)$ and the macroeconomic distribution of marginal surplus $H_n(S_n)$. The cost function (13) therefore makes clear the role of frictions and on-the-job search as endogenous sources of adjustment cost.[29]

[29] Compare this cost function, for example, to the standard convex adjustment cost in firm dynamics models, which depends only on the net growth rate but not equilibrium objects, or to the effective firm-level employment adjustment cost functions in the directed search model of Kaas and Kircher (2015) or the random search model of Gavazza, Mongey, and Violante (2018) which do not feature on the job search and so depend on the distribution of firms in the economy only through the 'price' $q(\theta)$.

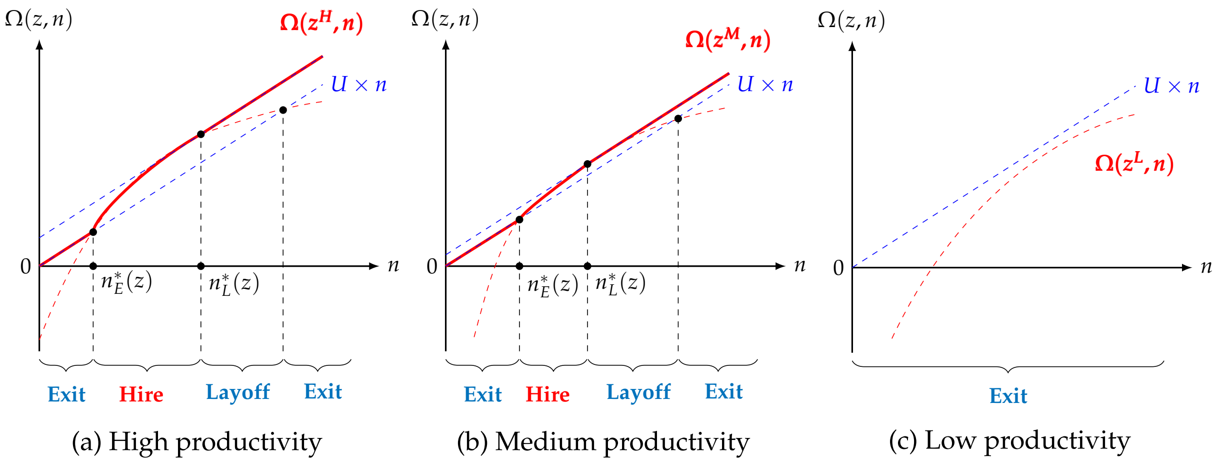


Figure 3: Values of exit, hiring and layoff for fixed levels of productivity z

4.5 Hire and separation policies

Properties of $S(z,n)$. It is useful to establish some properties of the joint surplus function under standard assumptions on technologies. Suppose (i) productivity follows a geometric Brownian motion with $\mu(z) = \mu \cdot z$, $\sigma(z) = \sigma \cdot z$, (ii) the vacancy cost function is isoelastic in vacancies only $c(v) = c_0 v^{1+\gamma}$, and (iii) the production function satisfies $y_z > 0, y_n < 0, y_{nn} < 0, y_{zn} > 0$.[30] In Appendix B we show that under these assumptions $P1 - P3$ hold:

(P1) S is increasing and concave in employment: $S_n > 0$, $S_{nn} < 0$

(P2) S is increasing in productivity: $S_z > 0$

(P3) S is supermodular in productivity and labor: $S_{zn} > 0$

Optimal policies. Figure 3 exploits these properties to characterize the firm's policies for alternative levels of productivity. The red dashed line describes the value of hiring minus the scrap value: $\Omega(z,n) - \vartheta$. The lower blue dashed line extending from the origin gives the total value of unemployment to the firms' employees: $U \times n$. The exit threshold $n^*_E(z)$ is determined by their intersection. At this point the per worker value net of ϑ is equal to the value unemployment: $(\Omega(z, n^*_E(z)) - \vartheta)/n = U$. As opposed to this condition on *average values*, the layoff threshold $n^*_L(z)$ equates the *marginal value* to U.

The solid red line is the upper envelope describing the pre-separation/exit value $\mathbf{\Omega}(z,n) = \mathbb{1}_{\{n<n^*_L(z)\}} \max\{nU, \Omega(z,n) - \vartheta\} + \mathbb{1}_{\{n \geq n^*_L(z)\}} \left[\Omega(z, n^*_L(z)) + (n - n^*_L(z))U\right]$. For example, if $n > n^*_L(z)$, the firm fires $(n - n^*_L(z))$ incumbents who each receive U. The joint value, given by the solid red line, is therefore given by

$$\mathbf{\Omega}(z,n) = \Omega(z, n^*_L(z)) + (n - n^*_L(z))\, U.$$

[30] All of which are satisfied by $y(z,n) = zn^\alpha$ with $\alpha \in (0,1)$, the functional form assumed in our quantitative analysis.

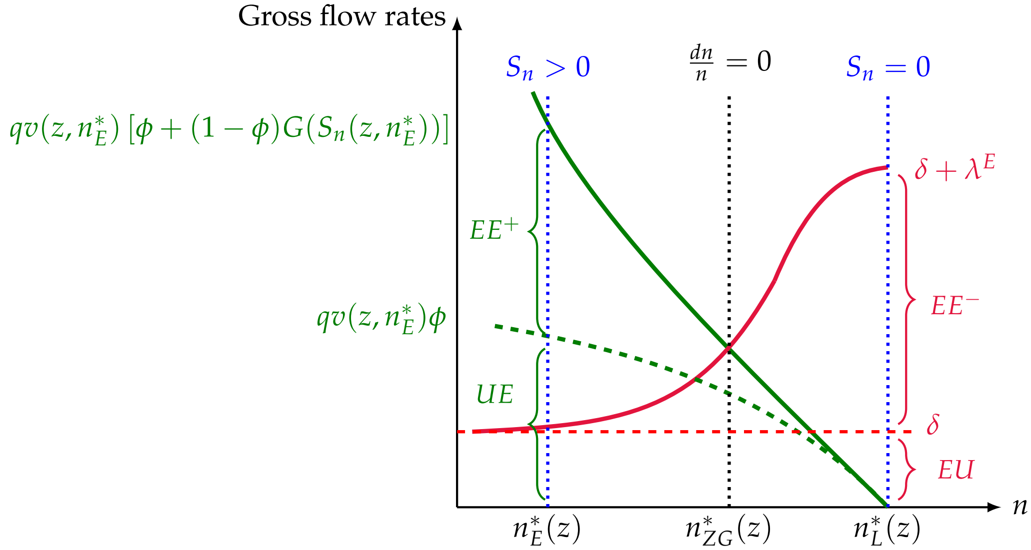


Figure 4: Gross worker flows by employment level, for given productivity

Notes: The solid red curve represents total separations $(EU + EE^-)$ and the dashed red horizontal line exogenous quits EU. The green curve represents total hires $(UE + EE^+)$ and the dashed green curve hires from unemployment (UE).

Panel (b) shows that under a lower productivity, the exit and layoff regions extend, while the hiring region shrinks. At an even lower $z^L < z^M$ it is optimal for the firm to exit for all n (panel (c)).

4.6 Worker reallocation

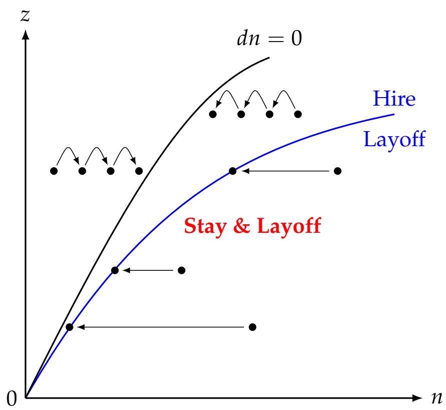
The model enables us to decompose firms' job flows (i.e. growth) into the four worker flows discussed in the introduction. Firm job growth in the hiring region is given by

$$\frac{dn}{n} = \underbrace{q(\theta)\frac{v(z,n)}{n}\Big[\phi + (1-\phi)H_n(S_n(z,n))\Big]}_{UE+EE^+} - \underbrace{\Big[\delta + \lambda^E(\theta)\overline{H}_v(S_n(z,n))\Big]}_{EU+EE^-}.$$

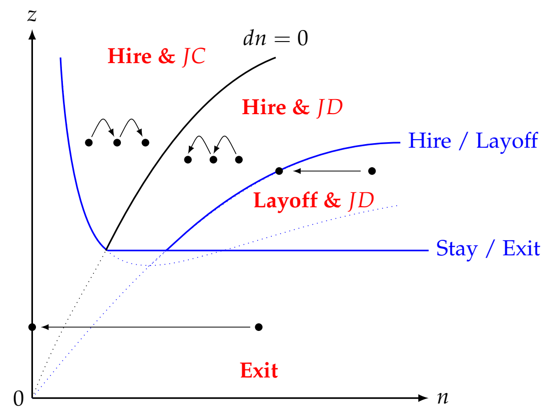
Under assumptions (i)-(iii) stated above, we can also prove (see Appendix B):

(P4) Net employment growth dn/n is increasing with productivity z and decreasing with size n.

Figure 4 illustrates how the four worker flows which determine net firm growth vary with n for a given level of z. Consider a firm that is at the layoff frontier, $n = n_L^*(z)$. Marginal surplus is zero so the firm posts zero vacancies and shrinks due to exogenous separations and poaching. Conditional on a meeting, any worker employed in that firm leaves ($\overline{H}_v(0) = 1$), and so separations occur at rate $\delta + \lambda^E(\theta)$. As the firm shrinks, decreasing returns in production cause the firm's marginal surplus to increase (P1). In terms of outflows, the firm loses fewer workers to competitors. In terms of inflows, the firm posts vacancies which always generate hires from unemployment and, as marginal surplus increases further,



(a) No endogenous exit ($\vartheta = 0$)



(b) Endogenous exit ($\vartheta > 0$)

Figure 5: Exit, layoff and no-growth frontiers in the (n, z)-space

Notes: This figure plots exit, layoff and no-growth frontiers for two cases: without and with positive scrap value. It also includes examples of hypothetical firm paths, in each case keeping productivity fixed. A firm (black dot) that begins in the layoff region jumps to the layoff frontier, firing $n - n^*_S(z)$ workers. Subsequent declines in productivity smoothly move the firm along the layoff frontier until, possibly, exit. A firm that is located in the hiring region smoothly converges toward the $dn = 0$ line by growing or shrinking.

hires from employment too. Firms shrink towards $n^*_{ZG}(z)$ where gross flows are positive but there is zero growth. For any given productivity z, the firm with the highest marginal surplus has the smallest size compatible with operating, i.e. size $n^*_E(z)$, and grows quickly away from $n^*_E(z)$ with high vacancy posting and net poaching.

Moreover, if $c(v; z, n) = c(v, S_n)$, then:

(1) Rates of EE^- and EE^+ are, respectively, decreasing and increasing in the firm's growth rate. Faster growing firms have higher rates of net-poaching: $(EE^+ - EE^-)$.

(2) Share of hires from unemployment decreases (from employment increases) in firm growth rate.

(3) Share of separations to unemployment increases (to employment decreases) in firm growth rate.

The intuition is simply that fast growing firms have high marginal surplus. For example, the pattern in (2) can be observed from Figure 4. As one moves leftward along the x-axis, S_n and firm's growth rate increases and EE^+ as a the share of total hires increases goes up as well.

We conclude by noting that this type of analysis on the composition of hires by firm size and productivity cannot be performed in directed search models. As explained in the Introduction, in that class of models, the composition of hires at the firm level is indeterminate.

4.7 Firm dynamics

Combining the characterization above with properties $P1 - P4$, we can fully theoretically represent firm reallocation (exit), job reallocation (net growth), and worker reallocation (hires and separations) in (n, z)-space. Figure 5 attains this by describing the functions that determine the stay/exit frontier $n_E^*(z)$, hire/layoff frontier $n_L^*(z)$, and the zero growth locus $n_{ZG}^*(z)$.

Panel (a) considers the model without a scrap value such that there is no endogenous exit. We can tightly characterize the layoff frontier. From (10) the layoff frontier has slope $dz/dn = -S_{nn}/S_{zn}$. Given properties P1 and P3, the frontier is therefore positively sloped. To understand firm dynamics to the left of this frontier, note that fractionally to the left $S_n \approx 0$, vacancy posting is low and the firm shrinks due to EE^- and EU flows. The zero growth locus along which $dn = 0$ must therefore be located strictly to the left of the layoff frontier. To the right of the zero-growth locus, firms hire but lose even more workers, and so experience net job destruction (JD). To the left of the zero-growth locus, marginal surplus is sufficiently large that firms are successful in hiring and retaining workers to experience net job creation (JC), but some endogenous separations through quits also occur. Thus, the model generates both hires for shrinking firms and endogenous separations for growing firms. To the right of the frontier, firms layoff workers, destroying jobs en masse, jumping back to the frontier.

Panel (b) introduces a positive scrap value and endogenous exit. First, consider ignoring smooth-pasting conditions. In this case the exit frontier would have gradient $dz/dn = -S_n/S_z$. Since $S_{nn} < 0$ (P1) and $S_z > 0$, the frontier would have a minimum when $S_n = 0$. The exit frontier therefore crosses the layoff frontier at its lowest point, increasing on either side.

Now let's consider how incorporating smooth pasting conditions affects exit. A necessary condition for optimal exit is that $S_n = 0$ on the boundary: if marginal surplus was positive, the firm would not want to exit. Since $S_n = 0$ on the layoff boundary and by (P1) S is strictly concave in n then it cannot be the case that S_n is zero again in the hiring region. This has two implications. First, firms cannot be exiting along the downward sloping section of the exit boundary in the *Hire & JC* region. This is consistent with employment dynamics as in this region firms drift to the right: $dn/n > 0$. Second, firms cannot be located in the *Hire & JD* region below the z at which $n_{ZG}^*(z)$ crosses the exit frontier. A firm located here would be drifting toward exit with $S_n > 0$, and exit with $S_n > 0$ is sub-optimal. As a result, to the right of the intersection of the zero-growth locus and the $S(z, n) = \vartheta$ locus, the exit frontier is flat.[31]

The stationary distribution of firms in the economy has support along the layoff frontier, and to its

[31]Note that the sufficient condition for this flat exit boundary to exist is that the $S_n = 0$ locus lies strictly below the $dn = 0$ locus. This is always the case, since if $S_n = 0$ the firm must always be shrinking and $S_{nz} > 0$ (P3). Therefore, a higher S_n such that the firm is not shrinking must be associated with a strictly higher z.

left. The distribution has zero mass along the left exit frontier. Growing firms do not exit, but shrinking firms may experience productivity shocks that force them to crossing the horizontal section of the $S(z, n) = 0$ exit frontier. All firms—except those on the layoff frontier—post vacancies, and hire workers both from employment and unemployment,and lose workers both to employment and unemployment.

The results derived in Section 4.6 regarding gross flows fully describe employment dynamics of the firm when interior to these boundaries.[32]

4.8 Frictionless limits

The frictionless limit of our economy is identical to that of a competitive, Hopenhayn-style model of firm dynamics with no dispersion in the marginal product of labor. Absent job-to-job mobility, this limit cannot be obtained. This theoretical limiting behavior benchmarks our exercise in Section 6.2 where we quantify the output effects of labor market frictions. In the rest of this section we offer an intuition for these results. The formal proof is in Appendix C.

Frictionless limit without on-the-job-search. Let A be matching efficiency, the scalar in front of the matching function, and take $A \to \infty$. Unemployment decreases and, without on-the-job search, firms can only hire from the shrinking pool of unemployed workers. From the perspective of the firm, the increase in A raises meeting rates, while the decrease in unemployment reduces meeting rates. In equilibrium these two forces exactly offset because the free entry condition uniquely pins down $q(A)$, independently of A. With q unaffected by the increase in A, the firm problem is unaltered, so firm employment dynamics are unchanged and, conditional on age, the dispersion in the marginal product of labor is unchanged.[33] Positive dispersion in marginal products is not a property of a frictionless competitive economy, but that of a competitive economy with adjustment costs. We now show that

Frictionless limit with on-the-job search. As $A \to \infty$ unemployment decreases but, with on-the-job search, firms can still hire from the non-shrinking pool of employed workers. Worker search efficiency, therefore, becomes constant, while aggregate feasibility ensures finite vacancies, so $q(A) = A(\mathtt{s}/\mathtt{v})^{-(1-\alpha)}$ increases in A. The increase in $q(A)$ accelerates labor reallocation from low to high marginal surplus firms. With decreasing returns to scale, marginal surplus increases at firms that lose workers, and decreases at the firms that poach them. The limit features the hallmark of a competitive model: zero

[32]One can think of Figure 4 as describing gross firm hiring along a straight horizontal line drawn in the (n, z) space of Figure 5 and running from the exit to the layoff frontier.

[33]As A increases the only change in the distribution of marginal products in the economy comes from the shifting age composition of firms as entry increases.

dispersion in marginal surplus. Job-to-job mobility is the key equalizing force.[34]

In the limit firm behavior is described by the following Bellman equation

$$\rho\overline{S}(z) = \max_n \; y(z,n) - bn + \mu(z)\frac{\partial \overline{S}}{\partial z}(z) + \frac{\sigma^2(z)}{2}\frac{\partial^2 \overline{S}}{\partial z^2}(z) \quad , \quad \overline{S}(z) \geq \vartheta \tag{14}$$

which makes clear the key properties of the limit. Without dispersion in marginal surpluses the on-the-job search terms drop out. The allocation is as if firms choose their optimal size each instant, where these hires are realized through immediate job-to-job reallocation. The only state variable is therefore z, and the productivity-size distribution is degenerate along $(z, n^*(z))$, where $y_n(z, n^*(z)) = b$, and marginal products are equalized. Firm exit is determined by a cut-off rule on productivity z. For new firms, the value of jumping from n_0 to $n^*(z)$ upon entry is finite, positive, and still depends on market tightness θ.

Thus in the limit the model is isomorphic to Hopenhayn (1992) with respect to job reallocation and firm exit. The only conceptual difference between the frictionless limit and Hopenhayn (1992) is that free-entry determines θ in the former, while it pins down the wage in the latter.

5 Estimation

We estimate the model on U.S. data. Because the model is set and solved in continuous time, we can construct correctly time aggregated measures at any desired frequency.

We make the following functional form assumptions. The vacancy cost function is $c(v,n) = \bar{c}\left(\frac{v}{n}\right)^{\gamma+1} v$ as in Kaas and Kircher (2015), such that the per vacancy cost is increasing in the vacancy rate. The production function is $y(z,n) = zn^{\alpha}$. The matching function is Cobb-Douglas with vacancy elasticity β: a worker meets a vacancy at rate $p(\theta) = A\theta^{\beta}$ and a vacancy meets a worker at rate $q(\theta) = A\theta^{-(1-\beta)}$. The distribution of entrant productivity draws is Pareto with a minimum of one and shape parameter ζ. We add exogenous firm exit at rate d.

We set two parameters exogenously and normalize three, as summarized by Table 1. The discount rate ρ implies an annual real interest rate of five percent. The elasticity of the matching function $\beta = 0.5$ is based on standard values in the literature. Without loss of generality we normalize the scrap value, ϑ and the scalar in the vacancy posting cost, $\bar{c}$.[35] The entry cost c_e is always pinned down by an average

[34]The proof requires characterization of the limiting behavior of the entire general equilibrium. This involves (i) the Bellman equation of the coalition, and (ii) the Kolmogorov-Forward equation of the distribution of coalitions. These two partial differential equations are coupled through the equilibrium distribution of marginal surplus and the firm vacancy policy. Given this complexity, we keep the proof manageable by assuming some additional structure. In particular, we assume that the entry productivity distribution has a sufficiently fat tail.

[35]Increasing the scrap value shifts up the exit frontier. Since productivity follows a geometric Brownian motion increasing the exit frontier simply increases mean productivity, so normalizing the scrap value is isomorphic to normalizing productivity which we are obviously free to do. The first order condition for vacancies obtained from (9) will have $\bar{c}$ multiply the marginal

	Parameter	Value	Target
ρ	Discount rate	0.004	5% annual real interest rate
β	Elasticity of matches w.r.t. vacancies	0.5	Petrongolo and Pissarides (2001)
ϑ	Scrap value	250	Normalization $(= 1/\rho)$
$\bar{c}$	Scalar in the cost of vacancies	100	Normalization
$\mathtt{m}$	Number of active firms	(1-0.06)/22	Average firm size (BDS)

Table 1: Externally chosen parameters

firm size of 22 in 2016 (U.S. Census Business Dynamics Statistics; BDS). We first identify a number of active firms $\mathtt{m}$ that delivers an average firm size of 22 when there is a unit measure of workers and an unemployment rate of six percent: $\widetilde{\mathtt{m}} = (1 - 0.06)/22 = 0.043$. While $\mathtt{m}$ is an equilibrium outcome, the fact that a higher $\mathtt{m}$ decreases the value of entry through a tighter labor market implies that there is always a unique c_e that satisfies the free-entry condition under $\mathtt{m} = \widetilde{\mathtt{m}}$.

5.1 Internally estimated - Minimum Distance

We estimate the 11 remaining parameters to minimize the objective function

$$\mathcal{G}(\boldsymbol{\psi}) = \Big(\widehat{\boldsymbol{m}} - \boldsymbol{m}(\boldsymbol{\psi})\Big)' \mathbf{W}^{-1} \Big(\widehat{\boldsymbol{m}} - \boldsymbol{m}(\boldsymbol{\psi})\Big) \quad , \quad \boldsymbol{\psi} = \Big\{\mu, d, \sigma, \zeta, n_0, \alpha, \gamma, A, \xi, \delta, b\Big\} ,$$

where $\widehat{\boldsymbol{m}}$ is a vector of empirical moments and $\boldsymbol{m}(\boldsymbol{\psi})$ are their model counterpart. The matrix $\mathbf{W}$ contains squares of the data moments on the main diagonal and zeros elsewhere.[36] We target 11 moments that are relatively standard to firm dynamics and frictional labor market literatures. While $\boldsymbol{\psi}$ is jointly estimated, some moments are particularly informative about some parameters. We briefly outline our logic then study identification more formally. Table 2 summarizes the estimated parameter values and the model fit with respect to the targeted moments.

Firm dynamics. The negative drift of productivity, μ, is informed by the exit rate of firms. The larger the drift, the faster firms hit the exit threshold. However, most of these firms that exit are small. The exogenous exit rate d induces large firms to exit and is informed by the employment-weighted exit rate. The standard deviation of productivity shocks, σ, is informed by cross-sectional dispersion in TFP, while the shape of the productivity distribution for entrants, ζ, affects productivity dispersion among young firms (Decker, Haltiwanger, Jarmin, and Miranda, 2018).[37] The size of initial firms n_0 is informed by the

cost of a vacancy and A multiply the marginal benefit of a vacancy. We therefore cannot identify $\bar{c}$ from A.

[36]Our moments are taken from various data sources and in most instances we cannot compute variances of the moments, let alone covariances with other moments.

[37]A natural alternative would have been to target the productivity gap between entrants (younger than 1 year old) and incumbents. The model does well in this respect. At the estimated parameter vector, this gap is 27 (35) percent in the model (data) (Gavazza, Mongey, and Violante, 2018).

Parameter		Value	Moment	Model	Data
μ	Mean of productivity shocks	-0.004	Exit rate (unweighted)	0.084	0.076
d	Exogenous exit rate	0.001	Exit Rate (employment weighted)	0.019	0.020
σ	Std. of productivity shocks	0.054	Std. deviation of log TFP	0.484	0.500
ζ	Shape of entry distribution	3.735	Std. deviation of log TFP (age 1-5)	0.362	0.400
n_0	Size of entrants	2.058	Job creation rate at age 1	0.237	0.244
α	Curvature in production	0.587	Employment share 500+	0.551	0.518
γ	Curvature of vacancy cost	6.023	Std. deviation of employment growth	0.337	0.420
A	Matching efficiency	0.157	UE rate	0.243	0.242
ξ	Search efficiency of employed	0.142	EE rate / UE rate	0.068	0.076
δ	Exogenous separation rate	0.017	EU rate	0.018	0.016
b	Flow value of leisure	0.295	Value of leisure to average output	0.367	0.400

Table 2: Estimated parameters and targeted moments

Notes: Exit rates and size distribution of firms are from HP-filtered Census BDS data between 2013–2016, which refer to firms (not establishments) and are annual. Moments specific to productivity dynamics and dispersion in productivity are taken from Decker, Haltiwanger, Jarmin, and Miranda (2018) (cross-sectional dispersion). The standard deviation of annual growth rates is taken from Elsby and Michaels (2013). The UE, EE and EU monthly mobility rates are averages of HP-filtered CPS data between 2013–2016, estimated on matched micro data for workers 16 years and older. The flow value of leisure to average output of 40 percent based on the average replacement rate reported in Shimer (2005).

job creation rate of age one firms. Given a distribution of entrant productivity, a smaller n_0 implies firms start further to the left of the zero growth locus, implying a faster rate of job creation. A narrower span of control parameter allows for fewer large firms, so α is informed by the employment share of firms with more than 500 employees. Finally, curvature in vacancy costs, γ, is informed by cross-sectional dispersion in annual employment growth (Elsby and Michaels, 2013). Conditional on σ, more convexity lowers responsiveness to productivity shocks, reducing dispersion in growth rates. We note that the data ask for both decreasing returns to scale and convexity in vacancy costs.[38]

Frictional labor market. Matching efficiency, A, is set to match the monthly UE rate. If matching is more efficient, workers find jobs faster. The relative search efficiency of employed workers, ξ, is then set to match the ratio of monthly EE rate to UE rate. The exogenous idiosyncratic separation rate δ maps into the monthly EU hazard. Finally, we set b to target a standard value for the flow value of leisure relative to average output per worker (Shimer, 2005).

5.2 Identification

To illustrate the identification of the model's parameters, we conduct two exercises. First, we demonstrate that, in a sizable hypercube around the estimated value of our 11-dimension parameter vector, the model is globally identified. Our argument proceeds parameter by parameter. We move each parameter ψ_i in steps in a wide range around ψ_i^*. Fixing ψ_i' we re-optimize all other parameters $\boldsymbol{\psi}_{-i}$ to minimize $\mathcal{G}\left(\boldsymbol{\psi}_{-i}, \psi_i'\right)$. We argue that the model is identified if $G\left(\boldsymbol{\psi}^*_{-i}(\psi_i'), \psi_i'\right)$ plotted as a function of ψ_i', traces a

[38] Similar degrees of convexity in vacancy costs have been estimated by Kaas and Kircher (2015) and Lise and Robin (2017).

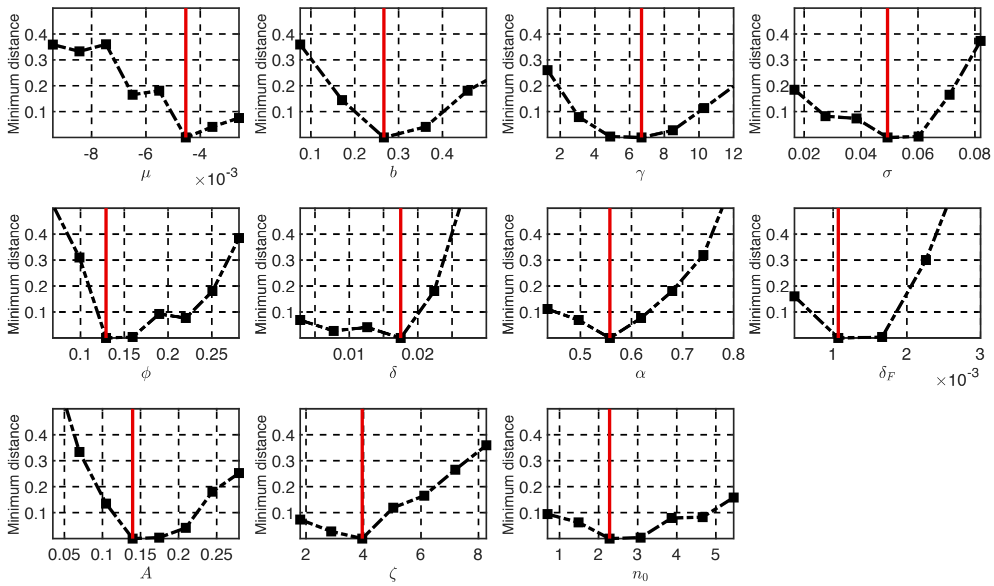


Figure 6: Global identification

Notes For each parameter $\psi_i \in \{\mu, \ldots, b\}$, the black line plots the minima of the minimum distance function $\mathcal{G}\left(\overline{\psi}_i\right) = \mathcal{G}(\boldsymbol{\psi}_{-i}, \overline{\psi}_i)$. when ψ_i is fixed at $\overline{\psi}_i$ (plotted on the x-axis), and minimization is with respect to all other parameters $\boldsymbol{\psi}_{-i}$. The red vertical line marks the estimated value ψ_i^* listed in Table 2

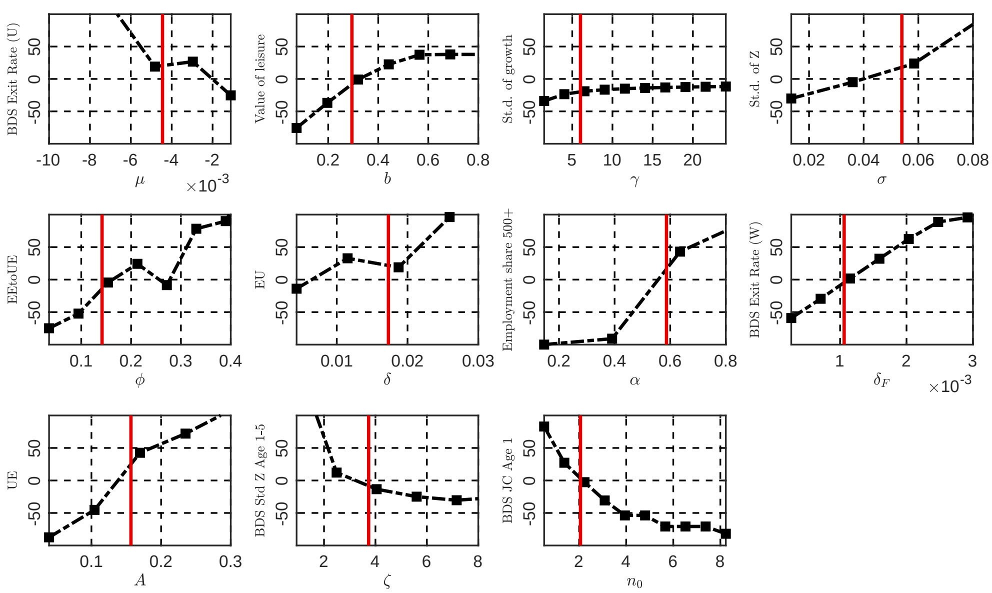


Figure 7: How informative specific moments are for individual parameters

Notes This figure plots the relationship between each parameter $\psi_i \in \{\mu, \ldots, b\}$ and the moment aligned with the parameter in Table 2. For each panel, the x-axis plots alternative values of the parameter. The y-axis plots the change in the corresponding moment in the steady state of the model obtained when all other parameters are as in Table 2.

A. Distributions of firms and employment

Group	A. Firms		B. Employment	
	Model	Data	Model	Data
A. By firm size				
0-19	0.909	0.881	0.182	0.180
20-49	0.053	0.075	0.069	0.097
50-249	0.032	0.036	0.133	0.154
250-499	0.004	0.004	0.065	0.057
500+	0.003	0.004	0.551	0.512
A. By firm age				
0-1	0.158	0.142	0.020	0.038
2-3	0.134	0.100	0.024	0.036
4-5	0.110	0.084	0.027	0.034
6-10	0.198	0.187	0.072	0.083
11+	0.400	0.487	0.856	0.808

B. Model dynamics of employment-productivity

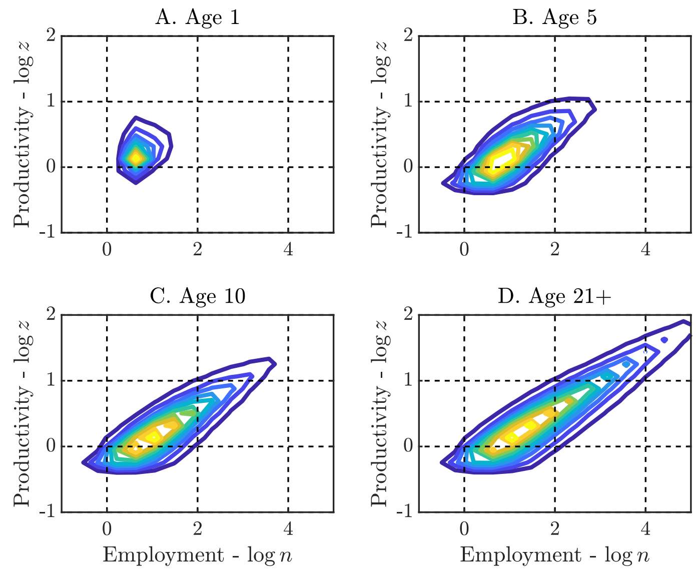


Figure 8: Distribution of firms in data and model

steep "U" with a minimum at ψ_i^*. Figure 6 plots this exercise and gives us confidence that our parameter vector is a global minimum.This is a straight-forward exercise for showing global identification.

Second, we show that our argument for identification is valid locally around $\boldsymbol{\psi}^*$. We discussed how each parameter is especially informed by a particular moment, despite the model being jointly identified. To support this argument, Figure 7 plots each one of the 11 moments as a function of the corresponding parameter in Table 2, keeping all other parameters at their estimated values. All panels show significant variation in the moment of interest as a function of its respective parameter.

5.3 Non-targeted moments

The aim of our theory is to describe the mechanics behind the reallocation of workers, in particular poaching flows, across the distribution of firms. Before exploring the importance of poaching flows directly we show that the model is consistent with data on (i) the distribution of firms, (ii) job and worker flows across the distribution, (iii) vacancy rates and vacancy yields, and (iv) the composition of hires and separations in response to firm-level productivity shocks.

1. Distribution of firms. Figure 8A shows that the model reproduces the skewed firm distribution. By size, in both data and model, around 90 percent of firms are small (less than 20 employees), but these account for only around 18 percent of employment. Symmetrically, firms with more than 500 employees

	A. Job reallocation				B. Worker reallocation				C. Firm reallocation	
	Job creation		Job destruction		Hires		Separations		Exit	
	Model	Data	Model	Data	Model	Data	Model	Data	Model	Data
A. By firm size										
0-19	0.045	0.049	0.038	0.038	0.116	0.122	0.109	0.110	0.023	0.020
20-49	0.029	0.033	0.029	0.029	0.104	0.106	0.105	0.102	0.003	0.009
50-249	0.031	0.031	0.031	0.027	0.105	0.102	0.105	0.097	0.003	0.008
250-499	0.032	0.031	0.033	0.025	0.105	0.101	0.107	0.096	0.003	0.006
500+	0.029	0.029	0.031	0.024	0.103	0.087	0.106	0.084	0.003	0.002
A. By firm age										
0-1	0.161	0.161	0.024	0.038	0.233	0.212	0.096	0.140	0.016	0.024
2-3	0.049	0.049	0.030	0.051	0.121	0.150	0.102	0.144	0.023	0.033
4-5	0.042	0.040	0.032	0.044	0.115	0.134	0.104	0.130	0.025	0.025
6-10	0.035	0.035	0.032	0.037	0.109	0.125	0.106	0.121	0.025	0.019
11+	0.028	0.026	0.033	0.024	0.102	0.086	0.107	0.084	0.020	0.013

Table 3: Job, worker and firm reallocation by size and age

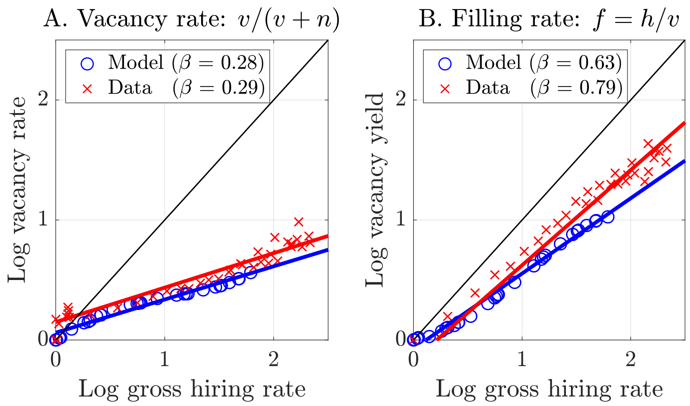
Notes: **Data**: Census BDS firm data for annual job creation, job destruction and exit. Quarterly rates constructed by dividing by four. Census J2J firm data for quarterly hiring and separation rates. Authors aggregate data into bins given in table which reflect the granularity of J2J data. Census J2J separations (hires) include separations to (hires from) non-employment. **Model**: Time aggregated to a quarterly frequency.

represent around 0.4 percent of firms, but more than 50 percent of employment. By age, less than half of firms are older than 10 years, but these account for more than 80 percent of employment.

Figure 8B plots the distribution of firms over size and productivity at ages 1, 5, 10 and 21+ firms. Output is higher when the correlation between employment and productivity is higher. In our model search frictions impede a perfect correlation and, thus, reduce output relative to the frictionless benchmark. Firms all start at n_0, but with a sizable dispersion in productivity. After a year the correlation between productivity and size is still low. Firms continue to grow rapidly during the first five years, while productivity gradually drifts down. Despite this rapid growth, there is still a significant dispersion in employment conditional on productivity. As firms get older, the productivity-size correlation increases, while the exit and layoff frontiers compress the dispersion in size conditional on productivity from the left and right, respectively. These frontiers are clearly delineated in (n, z) space, verifying our theoretical characterization of (n, z)-space (Figure 5).

2. Firm, job and worker reallocation. Table 3 shows that the model matches the key fact that worker reallocation rates are around three times as large as job reallocation rates. This difference is generated entirely by the presence of job-to-job mobility. That the model appears to generate a steady-state level of replacement hiring consistent with the data positions it well to understand the importance of job-to-job mobility in the Great Recession.

A. Vacancy rates and yields by gross hires



B. Decomposing growth

Moment	BFGT (2019)	Model
A. Net flows		
From/to employment	0.72	0.90
From/to unemployment	0.28	0.10
B. Gross flows		
Increasing hires		
From employment	0.44	0.51
From unemployment	0.14	0.10
Decreasing separations		
To employment	0.28	0.39
To unemployment	0.14	—

Figure 9: Vacancy filling and decomposing response to shocks

Notes **Panel A. Data**: Establishment-month observations in JOLTS microdata 2002-2018 are pooled in bins, where bins are determined by net monthly growth rate, and have a width of 1 percent. Growth rates computed as in DFH. Within bin b, total hires h_b, total vacancies v_b, total employment n_b are computed. From these, the gross hiring rate h_b/n_b, implied daily filling rate f_b and implied daily vacancy posting rate $vr_b = v_b/(v_b + n_b)$ are computed using the daily recruiting model of DFH. **Model**: The filling rate in the model is $f(n,z) = q(\theta)[\phi + (1-\phi)F(n,z)]$. The daily vacancy rate is $vr(n,z) = v(n,z)/(v(n,z)+n)$. Points plotted are logs of these variables, differenced about the bin representing a one percent net growth rate. **Panel B. Data**: Authors calculations from Table 8 of Bagger, Fontaine, Galenianos, and Trapeznikova (2019). **Model**: This column describes the response of firm employment to a 20 percent increase in productivity in the model, split into each margin. Aggregating across firms, the table gives the percentage of the increase in net job creation due to changes in different hiring margins. For example, 89.6 percent of the *increase* in net job creation is due to *increased* net poaching: $(EE^+ - EE^-)$.

In the cross-section, job flows display a degree of the empirical pattern of *'up-or-out'* dynamics. Job creation and exit rates peak for the young firms, albeit for exit the pattern is not as pronounced as in the data. The model also accounts for job creation by age. In the data (model) 18 percent (16 percent) of all jobs are created by new firm births and 26 percent (25 percent) by firms aged 1-10. We conclude that our framework accounts for the distribution of firms across age and size, job flows across these types of firms, as well as entry and exit dynamics.

Turning to worker flows, the data hiring rates display a negative relationship with both firm age and firm size, but the slope with respect to age is much more pronounced. In line with these facts, hiring rates are strongly decreasing in age whereas the rate at which gross flows decline with size is much more moderate.

3. Vacancy rates and yields. As pointed out by Davis, Faberman, and Haltiwanger (2013), the rate at which firms fill their vacancies varies systematically in the cross-section: firms with larger hiring rates also have higher filling rates. Our model shares this implication. We compare the model to BLS micro-data in Figure 9A, taking the same approach to data and model as Davis, Faberman, and Haltiwanger

(2013).[39] First we bin firms by their net growth rate, then within bins compute the average hiring rate $\widetilde{h}$, vacancy rate $\widetilde{v}$ and filling rate f of vacancies:

$$\widetilde{h}(z,n) = \frac{h(z,n)}{n} \quad , \quad \widetilde{v}(z,n) = \frac{v(z,n)}{n} \quad , \quad f(z,n) = q(\theta)\Big[\phi + (1-\phi)H_n(S_n(z,n))\Big].$$

The model replicates the key empirical observations of DFH: (i) both the vacancy rate and the filling rate are increasing in the hiring rate, (ii) the relationships are essentially log-linear, and (iii) with respect to the hiring rate, the filling rate is much more elastic than the vacancy rate.

In the model firms with high marginal surplus have higher hiring rates, post more vacancies and fill them more quickly as they poach from more firms. The slope of the vacancy rate is mediated by the degree of convexity of the vacancy cost γ. The fact that the model matches the data along this dimensions gives us more confidence on the estimated value for this parameter. In our model, on-the-job search is entirely responsible for the positive slope in the filling rate. Note, however, that in the data, nearly 80 percent of increases in the hiring rate are driven by changes in the filling rate, whereas in the model this effect accounts for just above 60 percent. We conclude that there is residual scope for other sources of firm's search effort margins, collectively interpreted as *recruiting intensity* by DFH and Gavazza, Mongey, and Violante (2018).

4. Firm-level response to productivity shocks The *cross-sectional* relationships documented above do not address the extent to which the employment growth of a firm is attained through more hires from unemployment, more poaching hires, fewer poaching separations or fewer separations to unemployment. Our theory has sharp qualitative predictions for this decomposition of growth into changes in constituent gross flows (Figure 4). A recent paper using Danish registry data by Bagger, Fontaine, Galenianos, and Trapeznikova (2019) decomposes the response of firm-level net employment *following a permanent value-added shock* into these margins. Their key finding is that job creation is achieved predominantly through increasing poaching inflows and decreasing poaching outflows.[40]

We replicate the exercise underlying the empirics of Bagger, Fontaine, Galenianos, and Trapeznikova (2019) and find that the model shares this central result. Table 9B shows that in Danish data 72 percent of the net increase in employment following a value added shock is due to *increasing net poaching*, and only 28 percent is due to increasing net hiring from unemployment. The model shares this feature, with

[39] Data is our own computations from BLS JOLTS microdata, in which we extend the sample period of Davis, Faberman, and Haltiwanger (2013) from 2002-2006 to 2002-2018. The filling rate and vacancy rate are the *daily filling rate* and *daily vacancy flow rate* implied by the model of daily hiring used in their paper.

[40] The data burden on producing these results is beyond what is available in the U.S. Two ingredients are necessary: high frequency data on (i) on job-to-job flows, (ii) firm value-added and employment. In the U.S., the LEHD contains the former, but revenue data—which can be used to proxy for value added under certain assumptions—is only available annually in the LBD. Data of this quality is, however, available in France and Sweden, for example.

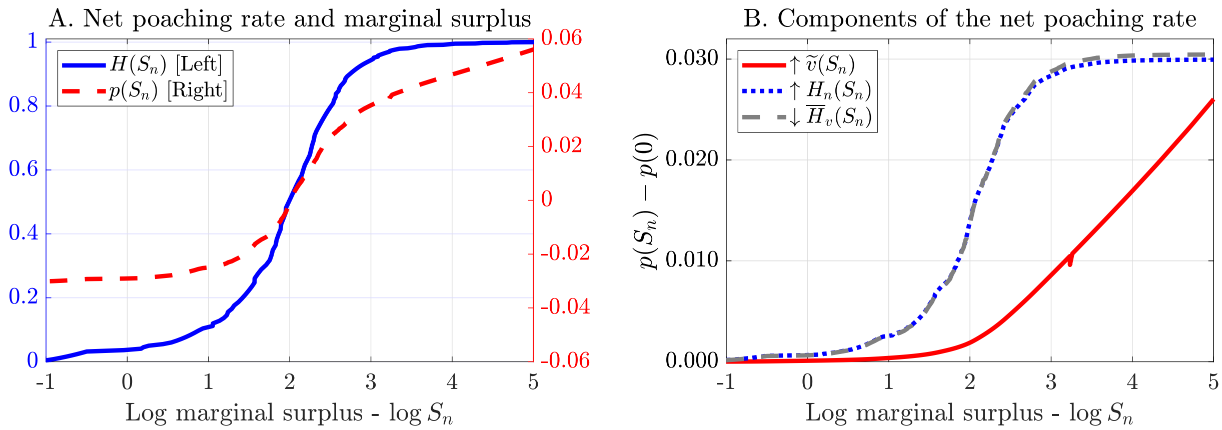


Figure 10: Net poaching and marginal surplus distribution

Notes: **Panel A.** Net poaching rate $p(S_n)$ by log marginal surplus S_n and the CDF of log marginal surplus. **Panel B.** Decomposition of the change in net poaching rate as S_n rises into three components: (i) higher vacancies (red line), (ii) more poaching hires due to higher rank on the job ladder (green line), and (iii) lower poaching separations due to higher rank on the job ladder (blue line).

the bulk of the growth owing to increased net poaching. In terms of gross flows, around 60 percent of growth comes from increasing hires and 40 percent from decreasing separations. Of these, poaching hires from employment plays the dominant role in increased hiring.[41]

6 Three applications

We now exploit our model parameterized to the US labor market to address three questions that require a model featuring proper notions of both firm dynamics and worker dynamics.

6.1 Net poaching by firm characteristics

The first question we ask is: who poaches from whom? What does the model tell us with respect to firms' key characteristics that determine their rank on the job ladder? The direct answer is: marginal surplus. Figure 10A plots the distribution of S_n together with the net poaching rate as a function of marginal surplus. The CDF reveals that the equilibrium density $h(S_n)$ displays quite a lot of mass in the middle and relatively long tails. As expected, net poaching is strictly increasing in S_n with a marked 'S shape'. What explains this particular shape? Figure 10B helps answering this questions.

Under our assumptions on vacancy costs, the vacancy rate of the firm ($\tilde{v} = v/n$) depends only on

[41] In the model the constant EU rate away from the layoff frontier means that separations can only fall on the EE margin.

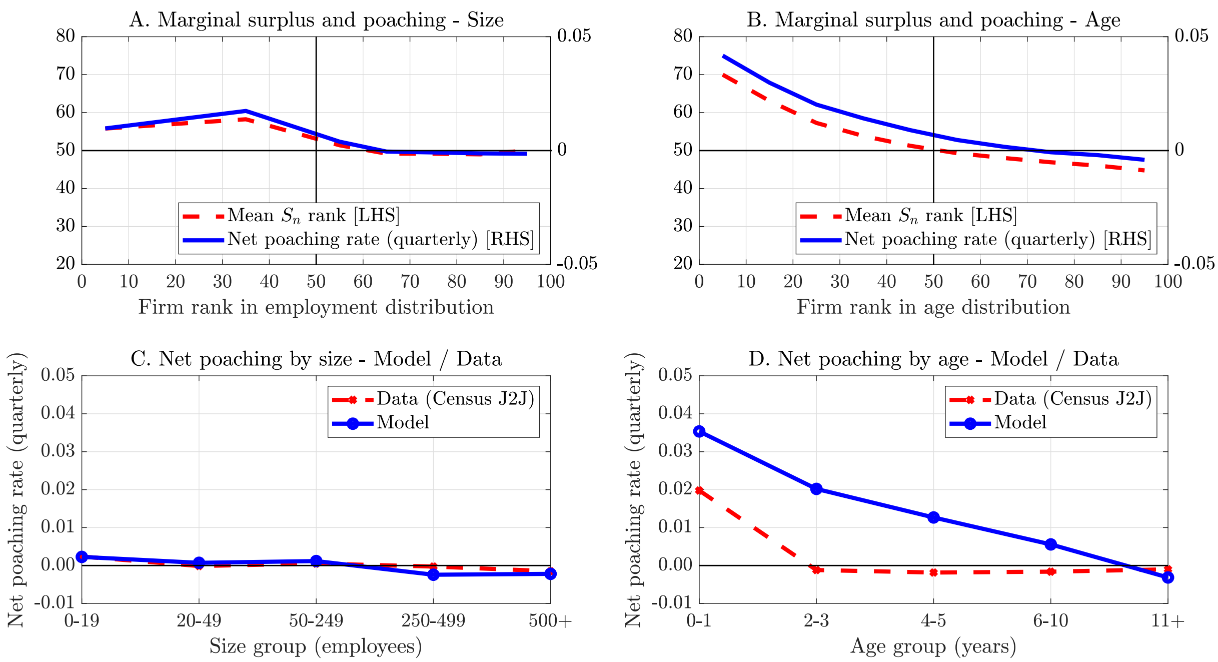


Figure 11: Net poaching rates by size and age

marginal surplus.[42] The net poaching rate is:

$$p(S_n) = \tilde{v}(S_n)q(\theta)(1-\phi)H_n(S_n) - \lambda^E(\theta)\overline{H}_v(S_n) \quad , \quad \tilde{v}(S_n) = q(\theta)^{1/\gamma}\left[\phi S_n + (1-\phi)\int_0^{S_n} S_n - u \; dH_n(u)\right]^{1/\gamma}.$$

As S_n rises net poaching increases through three channels: (i) a higher return to vacancies leads to higher vacancy posting, increasing EE hires ($\uparrow \tilde{v}(S_n)$); (ii) conditional on any vacancy policy a greater fraction of meetings result in a hire ($\uparrow H_n(S_n)$); (iii) firm incumbents bump into fewer vacancies that result in an EE quit ($\downarrow \overline{H}_v(S_n)$). Figure 10B plots these three forces using the following decomposition

$$\underbrace{p(S_n) - \lambda^E}_{p(0)} = \int_0^{S_n} \left[\underbrace{\frac{\partial \tilde{v}(u)}{\partial u} q(\theta)(1-\phi)H_n(u)}_{\text{Increasing} \uparrow \tilde{v}(S_n)} + \underbrace{q(\theta)(1-\phi)\frac{\partial H_n(u)}{\partial u}\tilde{v}(u)}_{\text{Increasing} \uparrow H_n(S_n)} - \underbrace{\lambda^E(\theta)\frac{\partial \overline{H}_v(u)}{\partial u}}_{\text{Decreasing} \downarrow \overline{H}_v(S_n)} \right] du.$$

Firms with very low marginal surplus basically do not hire and lose all their employees who are successful in their search on the job, so net poaching for them approaches $-\lambda^E$. In the range of $\log S_n$ between zero and two—any rise in marginal surplus increases net poaching almost entirely through changes in marginal surplus rank. For a given percentage change in S_n, firms climb the ranks of the job ladder especially fast in the middle of the distribution and this is reflected in the high slope of net

[42] To see this note that with $c(v,n) \propto (v/n)^\gamma v$, marginal cost is $c_v(v,n) \propto (v/n)^\gamma$ and, as characterized in Section 4, the marginal benefit of a vacancy depends only on S_n.

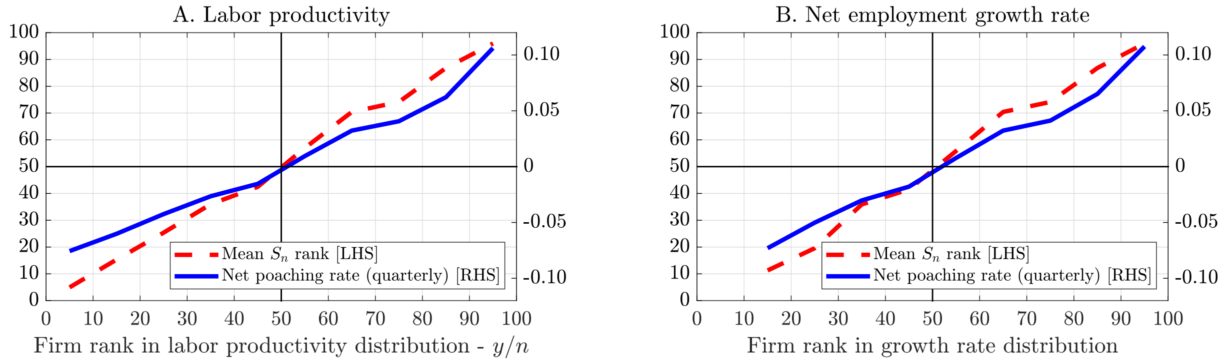


Figure 12: Additional determinants of marginal surplus and net poaching

poaching in that range (left panel).[43] The vacancy rate initially rises slowly but high marginal surplus firms want to grow fast and in order to achieve high growth post lots of vacancies.

Size and age. We now project this relationship between net poaching and marginal surplus onto observables in order to compare the model to empirical patterns documented by Haltiwanger, Hyatt, Kahn, and McEntarfer (2018). A key result is a negligible gradient of net poaching by size, but a steep gradient by age as young poach from old. Figure 11 shows that the model matches these patterns quite well. Size is not a particularly good predictor of where a firm sits on the marginal surplus job ladder. Consider a vertical slice of Figure 5. At a given size some firms are highly productive, have a high S_n, have positive net poaching and create jobs on net. Meanwhile some firms are less productive, have a low S_n, negative net poaching and destroy jobs on net. In contrast, young firms are on average small and productive, sitting to the left of $dn = 0$ and having not yet had time to grow and are high in the marginal surplus job ladder. They therefore display large, positive net poaching rates.

Figure 12 plots marginal surplus and net poaching as a function of two other observable firm characteristics, labor productivity and net employment growth rate. The model predicts a much higher gradient between these two variables and net poaching rates compared to size and age. Marginal surplus is highly correlated with the static marginal product of labor, and the latter is proportional to the average product under our functional form for $y(z, n)$.[44] Finally, as documented in Table 8B, firms grow mostly through hires from employment. Thus, the model implies a tight positive relation between net growth rate and net poaching rate.

[43] H_n and H_v are different distributions, but have strikingly similar properties given that vacancies are increasing in marginal surplus and, under our cost function, scale with n conditional on S_n.

[44] One generalization of the model that would weaken this relation is the addition of heterogeneity in the scale of production parameter α, as in (Gavazza, Mongey, and Violante, 2018). This would create an additional source of cross-sectional variation in marginal surplus that is orthogonal to z.

A. Changes in A to half, double UE rate

	Baseline	Half UE/U	Double UE/U
$\Delta \log A$	—	-0.47	0.56
UE/U rate	0.22	0.11	0.44
EE/E rate	0.016	0.008	0.032
u rate	0.069	0.153	0.028
$corr(n,z)$	0.822	0.755	0.888
$\Delta \log Y$	—	-0.231	0.152
$\Delta \log TFP$	—	-0.175	0.126
$\Delta \log TFP / \Delta \log Y$	—	75.8%	82.9%

B. Large changes in match efficiency

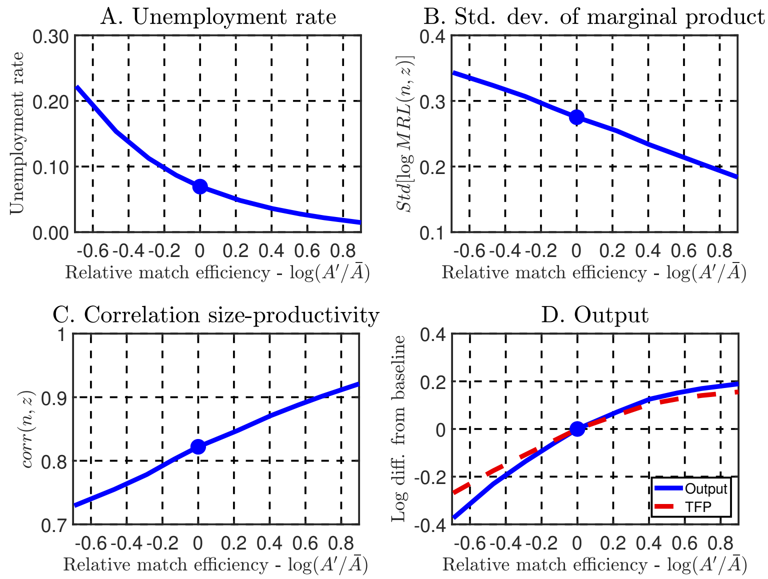


Figure 13: Effect of increasing match efficiency

6.2 Misallocation cost of labor market frictions

Next, we ask the model to quantify the misallocation costs of search frictions. Recall that frictionless limits do not make sense in models (i) with OJS but without DRS because they predict that the most productive firm would hire the entire labor force, or (ii) with DRS but without OJS because, as explained in Section 4.8, in these environments one does not recover a competitive economy in the limit.

In our counterfactual we shift the value of matching efficiency A holding all other parameters fixed at our baseline calibration. In each case, we resolve the model and recompute key moments of interest. In Figure 13A we focus our quantitative analysis on the range for A between -50% and +50% of our point estimate, which roughly corresponds to observed differences in job finding rates across developed countries (Engbom, 2017b), but Figure 13B plots model outcomes for a wider range as well.

As frictions vanish, unemployment falls, the dispersion of marginal products across firms shrinks and the correlation between size and productivity rises. Doubling match efficiency (and roughly doubling the UE rate) would increase aggregate output by 15 percent (7 percent net of the value of leisure).

To isolate the role of misallocation, we decompose the change in output into the component due to the allocation of workers across firms, and the component due to more employment in the economy as a

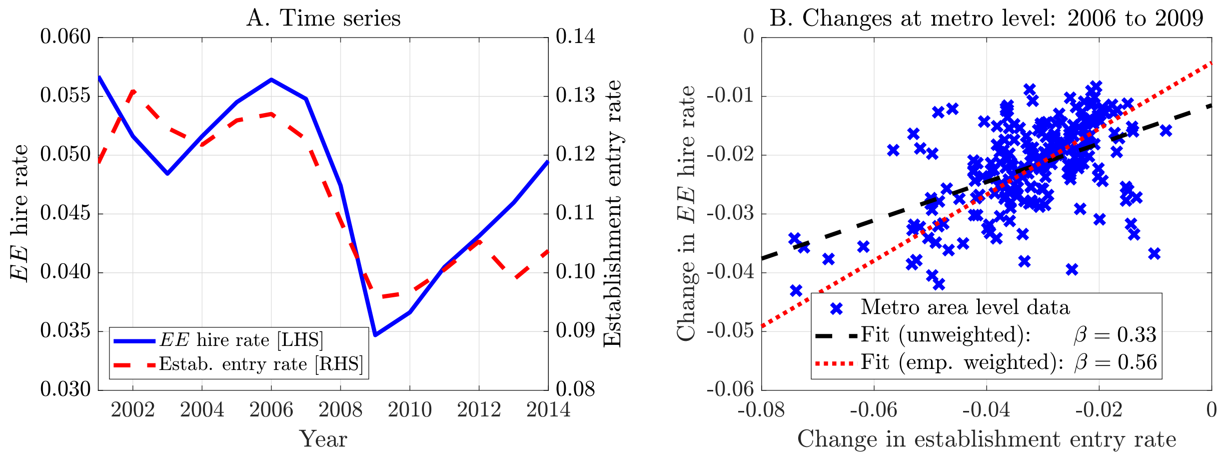


Figure 14: Entry and job-to-job hiring rates over the Great Recession: Aggregate and Cross-section

Notes Both panels are constructed from the same data at the metro level. Establishment entry and number of establishments are from the Census BDS data, and used to construct *establishment entry rate*. Job-to-job hires and employment are from the Census J2J data, and used to construct *EE hire rate*. The data cover the subset of states that participate in these Census data release programs. These cover more than half of the US population.

whole (the scale effect). Imposing an aggregate production function $Y = Z\bar{n}^{\alpha}$, then across steady states

$$\Delta \log Y = \Delta \log Z + \alpha \Delta \log \bar{n} \quad , \quad Z := \int_{\mathcal{N} \times \mathcal{Z}} z \left(\frac{n}{\bar{n}}\right)^{\alpha} dH(n,z).$$

The *TFP* term Z captures misallocation and is constant if the distribution of employment across productive units is constant. Lower misallocation accounts for over 80 percent of the increase in output. Interestingly, the relationship between frictions and output is concave because the unemployment rate is convex in match efficiency. Hence doubling frictions leads to a larger output loss, relative to the symmetric case, because of a stronger scale effect. Nonetheless, for both higher and lower labor market frictions the dominant channel determining output is higher or lower TFP due to labor misallocation.

6.3 Firm and worker dynamics in the Great Recession

Two of the defining features of the U.S. Great Recession were: (i) a strong decline in firm entry, and (ii) a sharp reduction in job-to-job worker reallocation associated with a failure of the job ladder. Firm entry (measured as the number of firms less than 1 year old in the BDS) dropped by 25-30 percent between 2007 and 2009 and since then it remained below trend (Siemer, 2014). The *EE* rate also fell around 25-30 percent over the same period (Shigeru, Moscarini, and Postel-Vinay, 2019; Haltiwanger, Hyatt, Kahn, and McEntarfer, 2018).

The decline in job to job transitions implied a marked slowdown of worker movements up the ladder.

Haltiwanger, Hyatt, Kahn, and McEntarfer (2018) document a decline in net poaching of high wage firms, those who are presumably at the top of the job ladder. Similarly, Moscarini and Postel-Vinay (2016) use a structural model to rank firms on the job ladder and estimate that high-rank firms curtailed their demand for new labor in the recession. As a result, the process of upgrading to better jobs, through job-to-job quits from low-rank to high-rank firms slowed down considerably. In short, as they put it: *the job ladder failed, starting from the upper rungs.*

These two facts have not been connected in the literature. Our environment suggests a natural mechanism to establish a causal link between the two. New entrants and young firms account for a substantial share of vacancies and have higher marginal surplus than other firms in the economy. Thus they account for a large fraction of the poaching of employed workers. Following a drop in the number of entrants, poaching would fall at the top of the ladder which reduces worker reallocation through the middle of the ladder, and so on down to unemployment.

Figure 14 shows that drawing this link is also consistent with the cross-region patterns. We combine newly released Census $J2J$ data with Census BDS data at the metro level. The time-series decline in entry and job-to-job mobility is mirrored in the cross-section of metropolitan labor markets: metro areas with larger decline in establishment entry were associated with larger decline in job-to-job mobility.

The aggregate shock that best describes the Great Recession is one that worsens financial frictions. To proxy for a financial shock in our framework, we solve the model under an unexpected temporary increase in the discount rate ρ (as in Hall, 2017). We calibrate the initial jump and the rate of convergence of ρ to match the 5 ppt increase in the unemployment rate and the six years it took to return to pre-recession levels. Appendix D provides more details on the computation.

Figure 15 describes the response of aggregates to the shock. The direct effect of the shock is to lower the valuation of future revenues at all firms. As a result, both average and marginal surplus fall, leading to an increase in EU separations from incumbents as well as from a spike in firm exit. Symmetrically, declining marginal surplus reduces the return on vacancies (13): vacancies collapse, job creation contracts, and so UE hires decrease. The combination of higher layoff rates and lower job finding rates induces the observed dynamics of unemployment. Young firms, which have a disproportionate fraction of their present value of revenues in the future, are especially hard hit, causing entry to collapses by almost 20 percent.

We illustrate the link between firm dynamics and the frictional labor market by plotting the dynamics of the job ladder in Figure 16. The top left panel shows that the job-to-job mobility rate drops upon impact and slowly recovers. The size of the drop is in line with the data. As young firms are disproportionately affected and have high-marginal surpluses, the share of vacancies originating from

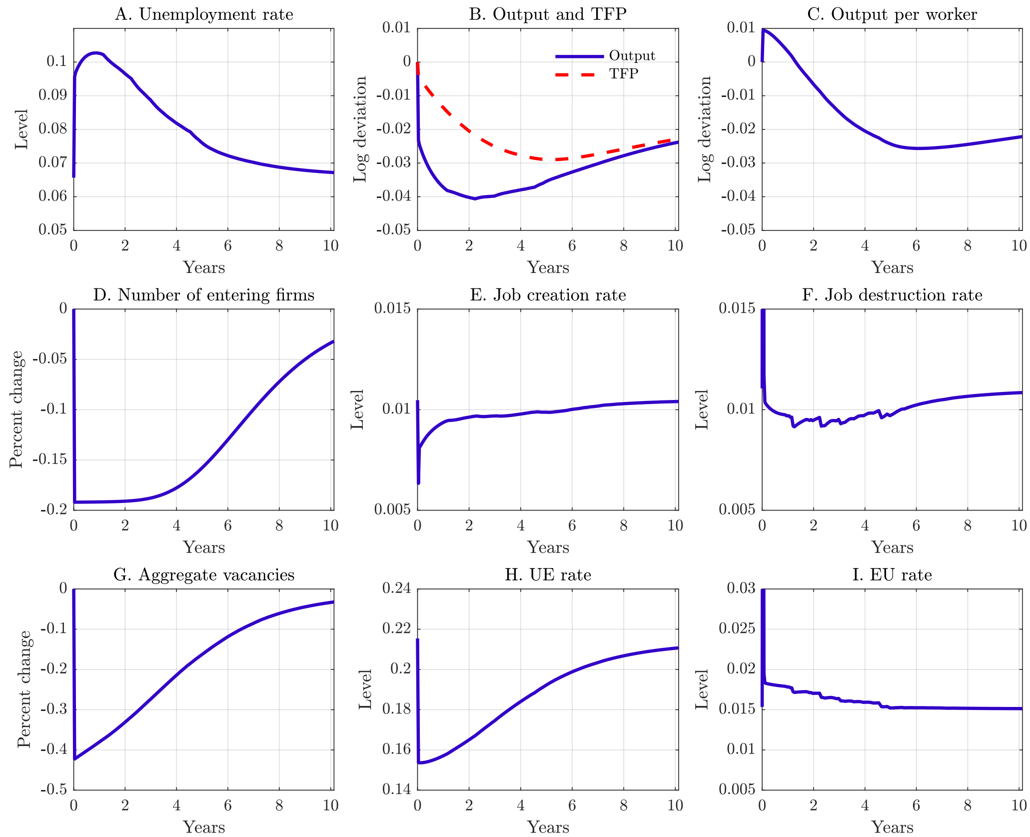


Figure 15: Response of aggregates following a discount rate shock

high-marginal surplus firms drops substantially. The decline of vacancies is much less pronounced in other regions of the ladder because of a general equilibrium rise in the rate at which vacancies get filled. With less vacancies and more unemployed workers, the aggregate vacancy yield rises. As in the data, the vacancy yield grows much more for small firms (Moscarini and Postel-Vinay, 2016).

The shifting vacancy distribution causes net poaching to collapse at high-marginal surplus firms and grow at low-marginal surplus firms. This compositional effect reduces the probability that a worker moves to a high-marginal surplus firm relative to a low-marginal surplus firm causing the observed 'failure' of the job ladder.

The collapse in the job ladder grinds down aggregate productivity persistently. Consistent with the data, aggregate output per worker increases for a short period. Initially, as low-productive firms shed more workers than high-productive firms, and as average firm size falls, output per worker increases, reflecting a short-lived cleansing effect of the recession. However, throughout the recession and its slow recovery, job-to-job worker flows towards high-marginal surplus firms slow down, exacerbating the misallocation that arises from labor market frictions. The result is a persistent decline in productivity. Note that, even after unemployment is back to trend, a decade after the onset of the recession, TFP is still

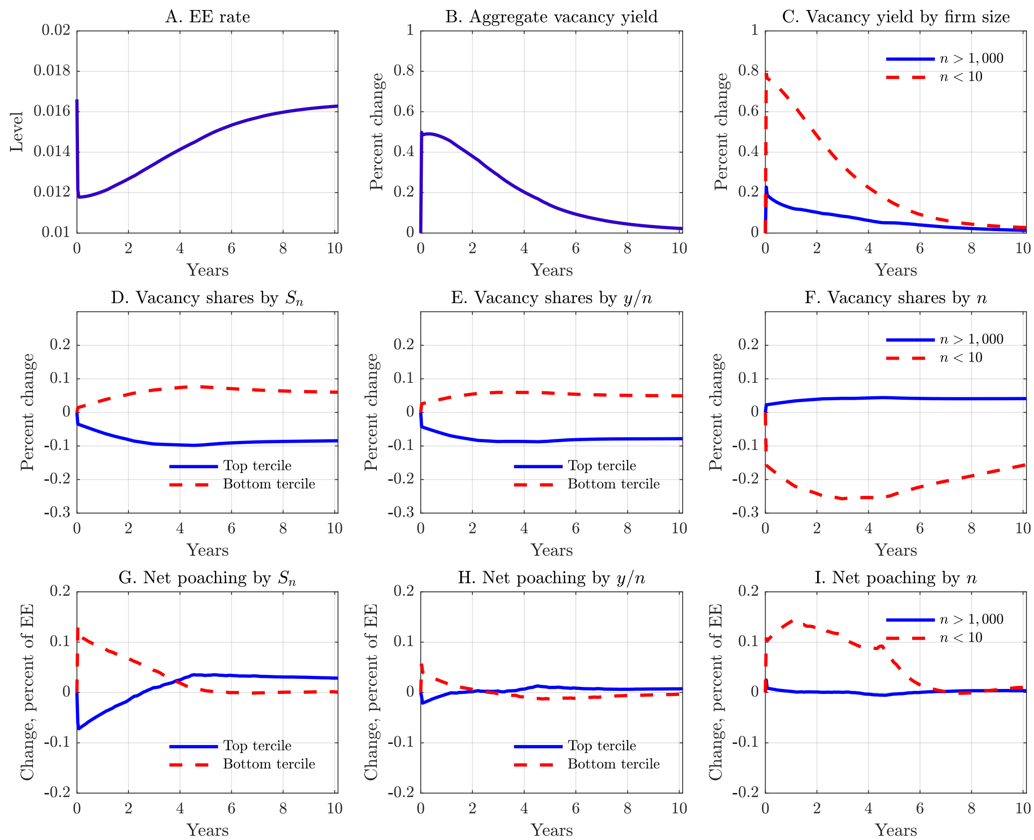


Figure 16: Response of job ladder following a discount rate shock

2.5 percent below steady state. The recovery of aggregate productivity is sluggish, with the scars of the recession encoded in the slow moving dynamics of the employment-weighted firm distribution.

7 Conclusion

We have set out a new and tractable framework to jointly study firm dynamics and worker reallocation. Consistent with the data—and novel with respect to existing multi-worker firm dynamics models with random search—firms hire from both employment and unemployment. In the limit as frictions vanish, our economy converges to a standard competitive firm dynamics model, in contrast to existing models with constant returns to scale in production where in the limit the most productive firm employs the whole workforce. This limiting behavior allowed us to estimate the productivity loss from labor misallocation due to labor market frictions.

The model features a *marginal surplus ladder*. Firms with higher marginal surplus attract workers more easily, and thus post more vacancies and grow faster. According to the model, value added per worker, firm growth, and age are tightly connected to marginal surplus, whereas firm size is only weakly correlated. The model, estimated on US micro data on firm dynamics, job reallocation and worker flows, produces the observed patterns of net poaching by size and age. Finally, the model offers a natural interpretation for the collapse of the job ladder during the Great Recession: the sharp drop in firm entry reduced the poaching from young and productive units and this weaker pull from the top trickled-down and muted poaching across the entire job ladder.

There are three natural directions to expand the research agenda. First, incorporating wage determination into the model. In Bilal, Engbom, Mongey, and Violante (2019) we make a first step in this direction. We plan to analyze data on the evolution of the wage distribution at the firm level (frequency and size of wage cuts, correlation of wage changes across workers, etc.) in order to discriminate between alternative protocols.

Second, adding aggregate uncertainty in order to analyze the cyclicality of labor market flows and of net poaching. As shown, we can compute transitional dynamics very efficiently. Thus, exploiting the impulse response as a numerical derivative (as in Boppart, Krusell, and Mitman, 2018) seems the most direct way to study aggregate fluctuations.

Finally, we applied our model to the US economy, but publicly available data from other countries (e.g., France, Germany, Italy, Sweden) would allow to compute poaching statistics by many additional firm characteristics. An interesting question is whether differences in stylized facts arise across countries and whether such differences can be interpreted as the result of heterogeneity in the degree of labor market frictions, institutions, technology, or other factors.

Because of the contemporaneous presence of a well defined notion of firm boundaries (through decreasing returns in technology or downward sloping demand) and a comprehensive model of workers' frictional reallocation across firms, our framework can be potentially useful to study a number of questions in macroeconomics, labor and trade.

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APPENDIX I

Firm and Worker Dynamics in a Frictional Labor Market

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This Appendix is organized as follows. Section A provides intuition for how our assumptions **(A)** yield a tractable Bellman equation for joint value by completing the analysis of static example started in the main text. Section B provides a characterization of the surplus function. Section C derives the limiting behavior of our economy when frictions vanish. Section D details the algorithms used in the paper to compute and estimate the model.

A Static Example

In this section, we continue the derivation of each term in the joint value equation (1). We start by generalizing the *UE* hire case analyzed in the main text to a firm with multiple incumbents.

A.1 *UE hire* when the internal renegotiation involves with multiple workers

It is sufficient to consider the case of two incumbent workers, $n = 2$. Without loss of generality, assume that the second worker is paid more than the first, $w_2 > w_1$. As in the approach taken earlier, suppose the firm has posted a vacancy that has met an unemployed worker. We have three cases to consider which illustrate how the firm may use a worker outside the firm to sequentially reduce wages of workers inside the firm.

First, the firm hires *without* renegotiation if:

$$\underbrace{y(z,3) - w_1 - w_2 - b > y(z,2) - w_1 - b}_{\text{No credible threat to } w_2} \quad , \quad \underbrace{y(z,3) - w_1 - w_2 - b > y(z,2) - w_1 - w_2}_{\text{Optimal to hire under } (w_1, w_2)}.$$

Hiring with current wages is preferred to replacing the most expensive incumbent—there is no credible threat—, and given no renegotiation, hiring is optimal. Since $w_2 > w_1$, no credible threat to worker 2 implies no credible threat to worker 1.

Second, the firm hires *with* renegotiation with worker 2 if:

$$\underbrace{y(z,2) - w_1 - b > y(z,3) - w_1 - w_2 - b > y(z,2) - w_2 - b}_{\text{Credible threat for worker 2 only}} \quad , \quad \underbrace{y(z,3) - w_1 - w_2^* - b > y(z,2) - w_1 - w_2^*}_{\text{Optimal to hire under } (w_1, w_2^*)}.$$

The threat is credible for worker 2, but is not for worker 1, and, conditional on renegotiating to (w_1, w_2^*), hiring is optimal.

Third, the firm hires *with* renegotiation with *both* workers if:

$$\underbrace{y(z,2) - w_1 - b > y(z,2) - w_2 - b > y(z,3) - w_1 - w_2 - b}_{\text{Credible threat for both workers}} \quad , \quad \underbrace{y(z,3) - w_1^* - w_2^* - b > y(z,2) - w_1^* - w_2^*}_{\text{Optimal to hire under } (w_1^*, w_2^*)}.$$

In all three cases, the optimal hiring condition can be written as:

$$\Omega(z,3) - \Omega(z,2) > U. \tag{15}$$

This last inequality does not depend on the order of the internal negotiation between firm and workers. In conclusion, the distribution of wages among incumbents again determines the patterns of wage renegotiation, but is immaterial for the sufficient condition for hiring.

Assumption **(A-LC-c)** that was not present in the one worker example plays a role here. Suppose that the renegotiated wage for worker 2 is pushed all the way down to b, making her indifferent between staying and quitting. Worker 1 could transfer a negligible amount to worker 2 in exchange of her quitting, which would raise the firm's marginal product and, possibly, remove its own threat. This is problematic for the representation because in this latter case the hiring condition becomes $y(z,2) - y(z,1) - w_1 - b > y(z,1) - w_1$, distinct from (15). Thus, to know whether a firm hires or not, one would need to know the wage distribution inside the firm. **(A-LC-c)** is sufficient to rule out transfers among workers and to prevent this scenario from happening.

Note that, this transfer scheme between workers occurring during the internal negotiation changes the joint value, and hence one can think of **(A-LC-c)** as being subsumed into **(A-IN)** already.

A.2 *EE* ***hire***

Now suppose that the worker matched with the firm's vacancy is currently employed at another firm with productivity z' and a single worker $n' = 1$. The situation is not that different from *UE hire*, except that the potential hire may have a better outside option in the form of the retention offer made to her by her current employer under **(A-EN)**. To see the similarity for now we fix this wage offer at $\overline{w}$. The same four cases can arise, except with $\overline{w}$ playing the role of b.[45] We can therefore reason as before and jump to the result that hiring will occur if and only if the following counterpart to (6) holds:

$$\Omega(z,2) - \Omega(z,1) > \overline{w}.$$

We now determine the poached worker's outside option $\overline{w}$. The poached firm's willingness to pay is a wage $\widetilde{w}$ that makes it indifferent between retaining and releasing the worker: $y(z',1) - \widetilde{w} = 0$. Hence, the contacted worker switches to the new employer as long as the poaching firm offers $\overline{w} \geq \widetilde{w} = y(z',1)$.

[45] Renegotiation will happen for different values of w_1 in the no hire case. Indeed, to establish the presence of a credible threat w_1 must be compared to $\overline{w}$ instead of b, but this has no allocative implications for the hiring decisions.

Bertrand competition between the two firms implies that the poaching firm offers $\overline{w} = y(z', 1)$, which is exactly the marginal value of the worker at the poached firm. As in the case of *UE hire*, whether *EE hire* occurs can be summarized by joint values:

$$\frac{\Omega\left(z,2\right)-\Omega\left(z,1\right)}{2-1}>\frac{\Omega\left(z',1\right)-\Omega\left(z',0\right)}{1-0}. \tag{16}$$

The *EE hire* decision is entirely characterized by knowledge of the pair (z, n) for the two firms.[46] The value gain to the firm and its workers is the difference between the left-hand side and right-hand side of equation (16). This comparison of marginal values is precisely the *EE hire* term in the HJB equation (1).

Finally, this exercise explains the absence of a *EE quit* term in (1). The payment received by its poached worker is equal to the poached coalition's willingness to pay, which is in turn exactly equal to the worker's marginal value to the coalition. The joint value of the poached coalition therefore does not change as it loses its worker. *EE quit* events play an important role in the dynamics of employment at the firm—which we describe below—but no role in the dynamics of $\Omega(z, n)$.

A.3 Vacancy posting

We now explain the private inefficiency in vacancy posting and why **(A-VP)** is crucial for tractability.

Recall that in the hiring scenarios just analyzed, two cases arise when the firm can credibly force a wage cut: (i) when it hires and the incumbent wage is above the post-hire new marginal product; (ii) when hiring is not profitable, but the firm can credibly 'fire and swap', i.e. as long as the reservation wage of the external worker met through search is below the incumbent wage. The firm has therefore incentives to spend resources on vacancy posting only to transfer value between agents, a privately inefficient outcome. The amount spent would depend on the incumbent's wage, breaking the tractability of our representation. Private efficiency reinstates tractability.

We start with the firm's preferred vacancy policy. Without loss of generality, suppose firms only meet unemployed workers (hence, upon a meeting, the 'fire and swap' threat is always credible). Let v be the number of vacancies posted, $c(v)$ the associated cost, and qv the probability a single vacancy meets a single worker. If no meeting occurs, then as per **(A-MC)**, w_1 does not change so the value of the firm does not change. The firm maximizing the expected return from vacancy posting net of costs is:

$$\max_{v} \quad -c(v)+qv\Big[\max\Big\{\underbrace{y(z,2)-w_1'-b}_{\text{Hire (cases 1\&2)}},\ \underbrace{y(z,1)-b}_{\text{No hire (case 3)}}\Big\}-\Big(y(z,1)-w_1\Big)\Big] \quad ,$$

[46]The case when the firm meets a worker at a firm with $(z', 2, w_1, w_2)$ is similar. Suppose the firm meets worker 1. The poached firm has the additional option of cutting w_2, but this is inconsequential for the argument because it only redistributes value within the poached-from firm.

Following a meeting, three cases may occur. In **Case 1**, the firm hires and there is no renegotiation, $w_1' = w_1$. This case arises when the wage of the incumbent worker is low enough. Then, adding a second worker does not reduce the marginal product of labor down to the point where the firm has a credible layoff threat. In **Case 2**, the firm hires but the wage of the incumbent is renegotiated down to $w_1' = w_1^*$. In this case, diminishing marginal returns drive the marginal product of labor with two workers below the incumbent's initial wage. In **Case 3**, the firm is better off not hiring, but under the threat of swapping out the incumbent, renegotiates w_1 down to b. The firm's preferred vacancy policy v^f then equates marginal cost to marginal expected return:

$$c_v\left(v^f\right) = q\Big[\max\Big\{y(z,2) - w_1' - b\,,\, y(z,1) - b\Big\} - \Big(y(z,1) - w_1\Big)\Big]. \tag{17}$$

The first-order condition (17) highlights that the firm's preferred vacancy policy depends on the incumbent's wage w_1 because this wage determines the gains from forcing a renegotiation through vacancy posting. This dependence is a source of intractability because, in the general model with n workers, (17) would depend on the entire wage distribution inside the firm.

Our assumption **(A-VP)** ensures that firms do not post v^f, but instead post the privately efficient amount of vacancies which does not depend on worker wages. We now show how our micro-foundation **(A-VPI)** implements **(A-VP)**.

Case 1 – Hire without renegotiation. In this case the outcome is already *privately efficient*. The worker's value does not decrease ($w_1' = w_1$), and by the fact that a hire occurs, the firm's value must increase. We can also write the expected return as $qv\left[\Omega(z,2) - \Omega(z,1) - U\right]$. Since the return is independent of w_1, then the efficient vacancy policy is independent of w_1. The firm is choosing vacancies as if it were maximizing the joint surplus without having to appeal to additional assumptions.

In cases 2 and 3, the outcome is *privately inefficient* because the firm may profit from vacancies that, if met by a job seeker, deliver a credible threat to cut the incumbent's wage to $w_1' < w_1$.

Our assumption **(A-VPI)** allows the worker to correct for this over-posting. The worker can then concede a pay cut in all states in exchange for an alternative level of vacancies. Such a pay cut is not covered under **(A-MC)** since when the vacancy policy is announced there is not yet a credible threat to layoff workers.[47] The firm will accept this wage cut and choose the worker's preferred vacancies if it delivers at least the value obtained under the firm's preferred vacancies v^f. We show that the worker's preferred package satisfying incentive compatibility restores efficiency in vacancy posting.

[47] A pay cut regardless of the outcome of the search for a new worker maps exactly into a transfer from worker to firm, which is how we approach the proof in the Appendix. We could have allowed for *state-contingent* wage-cuts that depend on who the firm meets or whether a meeting occurs. Even if these states were verifiable the result would only be for the worker to offer a menu of wage-cuts across states. This would increase worker value but not change allocations, hence for consistency with the rest of our assumptions, we assume a single wage cut.

Case 2 – Hire with renegotiation. In this case, the incumbent's wage w_1 is high enough that the firm finds it profitable to raise the contact probability with an unemployed worker beyond what would be efficient. Although the hiring outcome is efficient ex-post, too much resources are spent on vacancies ex-ante. Let w_1^* be the renegotiated wage after a meeting. The worker chooses a package of vacancies and a wage cut in all states (v^w, x) that solves:

$$\max_{v^w, x} \; qv^w\left(w_1^* - w_1\right) - x \tag{18}$$

subject to

$$\begin{aligned} & qv^w\left[\left(y(z,2) - (w_1 - x) - b\right) - \left(y(z,1) - w_1\right)\right] - c(v^w) \\ \geq \; & qv^f\left[\left(y(z,2) - w_1^* - b\right) \qquad - \left(y(z,1) - w_1\right)\right] - c(v^f) \end{aligned} \tag{IC}$$

The worker anticipates that after a meeting their wage will be renegotiated to $w_1^* < w_1$. Given this wage cut, the worker seeks to limit the probability of this event by cutting back on vacancies. Incentive compatibility (IC) requires that as the worker cuts vacancies it also cuts its wage so that the firm accepts the proposed policy v^w over v^f.

The Pareto problem (18) yields the result that vacancy posting is independent of w_1. First, given the linear objective function, (IC) holds with equality. Thus, we can substitute out x. Second, the zero-sum game assumption **(A-IN)** implies that w_1^* is a renegotiated wage that only redistributes value and hence drops out. Third, all terms that do not depend on (x, v^w) are irrelevant to the worker's decision. This leaves the following objective function:

$$\max_{v^w} qv^w\left[\left(\Omega(z,2) - U\right) - \Omega(z,1)\right] \; - \; c\left(v^w\right).$$

The decision can therefore be characterized by the *privately efficient return*, which is the change in joint value net of the cost of the new hire, $\Omega(z,2) - \Omega(z,1) - U$.

Case 3 – No hire with renegotiation. In this case the 'fire and swap' threat is credible. The incumbent's wage w_1 is high enough and the marginal product of an additional worker is below b. Replacing the return to hiring by the wage cut for the incumbent worker, the previous logic delivers

$$\max_{v^w} \; qv^w\left[\Omega(z,1) - \Omega(z,1)\right] \; - \; c\left(v^w\right) \qquad \Longrightarrow \qquad v^w = 0$$

Absent the transfer from worker to firm, the firm would post positive vacancies v^f even if the return from hiring is negative, i.e. $\Omega(z,2) - \Omega(z,1) < U$ to induce a wage cut, and v^f would depend on w_1. Under **(A-VPI)**, the worker takes a preemptive wage cut, and vacancies are zero, the efficient amount in this case.

Combined. Combining all three cases, privately efficient vacancies solve

$$\max_v qv \left[\max \left\{ \frac{\Omega(z,2) - \Omega(z,1)}{2-1} - U, 0 \right\} \right] - c(v).$$

Note three properties of this solution. First, the firm always hires when it meets an unemployed worker. Second, optimal vacancy posting equates the marginal gain in joint value to the marginal cost of a vacancy, and it only depends on (z, n). Third, this condition is the flip-side of the separation frontier. In (1) we said that if $\Omega_n(z, n) > U$, then the firm will not separate with workers. The terms inside the max expression say that if this is true, then the firm will post vacancies.[48]

We conclude that under **(A-VPI)**, the joint value is sufficient to characterize the vacancy decision. The distribution of wages in the firm is immaterial.

Multiple incumbents. When the firm employs more than one worker, the efficient transfer scheme can be implemented by randomly selecting a worker under threat to offer a package of wage-cuts and vacancies. In exchange, the firm posts the efficient number of vacancies. Under such a scheme, the initiating worker is strictly better off while the firm and the other workers are indifferent. We establish this case in detail in Appendix II.

A.4 Layoffs, quits, exit, entry

Having described most of the terms in the HJB (1), we conclude with the boundary conditions for exit, layoffs and the free entry condition.

Layoffs. Consider now a firm with $n = 2$ workers paid (w_1, w_2), and assume that $w_1 < y(z, 1)$ such that worker 1 is never under threat of layoff. The firm has a credible threat to fire worker 2 if

$$y(z,1) - w_1 > y(z,2) - w_1 - w_2.$$

Such a situation may occur if, for example, productivity has just declined. The firm has a credible threat to negotiate down to a wage level w_2^* such that $y(z,1) - w_1 = y(z,2) - w_1 - w_2^*$ and keep worker 2 employed. From the worker's perspective, it is individually rational to accept any wage w_2^* above b. Worker 2 is laid off if $y(z,1) - w_1 > y(z,2) - w_1 - b$. In terms of joint value, this can be written in exactly the form of the layoff frontier (2):

$$\frac{\Omega(z,2) - \Omega(z,1)}{2-1} < U.$$

[48]It is possible to determine the optimal wage cut x that delivers the efficient policy, but throughout the paper we focus on allocations only. See Bilal, Engbom, Mongey, and Violante (2019) for more details on wage determination.

The firm lays off workers until the marginal joint value of the worker is equal to the value of unemployment.[49] Note that this is the complement to the condition for posting vacancies. The special case with $n = 1$ of this scenario also arises in the one worker-one firm model with productivity shocks of Postel-Vinay and Turon (2010).

Quits to unemployment. Since in this static model workers will accept a renegotiated wage down to $w_i^* = b$, they will only quit at the point where the firm has a credible threat to lower wages below b. This is exactly the point at which the marginal value is equal to the value of unemployment. In this sense *layoffs* as described above are indistinguishable from quits to unemployment, as in any model with privately efficient separations. For ease of language all *endogenous* UE transitions are referred to as *layoffs*, and we use *quits* to refer only to EE transitions.

Finally, recall that in the dynamic model unemployed job seekers are promised a wage that implements a value U to them. If events occur in the firm that reduce the continuation value to that worker below U (e.g., a negative productivity shock), the incumbent may have a credible threat to quit and renegotiate her wage to restore its value at U, or above it, depending on the details of the internal negotiation. However, such renegotiation is, again, only a transfer of value within the firm. Separations remain privately efficient even in the dynamic model.

Exit. Now consider the exit decision of a firm with one worker. The private value of exit to the firm is the scrap value $\vartheta > 0$. The firm therefore exits if and only if $y(z,1) - w_1^* < \vartheta$, where w_1^* is a possibly renegotiated wage contingent on the firm remaining in operation. If the profit from operating at the lowest possible renegotiated wage $w_1^* = b$ is greater than ϑ, then the firm will continue to operate. Hence, the firm exits if $y(z,1) - b < \vartheta$, and the renegotiated wage only affects the distribution of value.[50] The exit condition can be written as $\Omega(z,1) - U < \vartheta$, and in the general case of n workers is exactly the boundary condition in (1): $\Omega(z,n) - nU < \vartheta$.

Entry. Upon entry the firm has n_0 workers hired from unemployment. The private entry cost of the firm is c_0, so entry requires $\int y(z,n_0) d\Pi_0(z) - n_0 b > c_0$. Using $\Omega(n,z) = y(z,n)$ and $U = b$, this requires $\int \Omega(z,n_0)\, d\Pi_0(z) > c_0 + n_0 U$.

[49] Note that, when both workers are under threat, the particular order in which values of workers are reduced is immaterial to the condition $\Omega(z,2) - \Omega(z,1) < U$. One could for example lower the wages of both workers proportionally, increasing the value of the firm, but a worker *must* be fired if $J(z,2,w_1^*,b) < J(z,1,w_1^*,\cdot)$ for any $w_1^* \geq b$.

[50] The firm has no credible threat to reduce w_1 if $y(z,1) - w_1 > \vartheta$. The firm can credibly threaten exit if $\vartheta \in (y(z,1) - w_1), y(z,1) - b)$, but in this case w_1 can be reduced to a point where this threat is no longer credible.

A.5 From static to dynamic

This static example showcases how to obtain every component of (1) from our set of assumptions. Appendix B generalizes these insights to the dynamic case. Two insights assist us. First, the proof begins with a discrete workforce. Here we are helped by continuous time, which removes complicated binomial probabilities of one, two, three, etc. incumbent workers meeting a competitor's vacancy. Second, we take the continuous workforce limit of the discrete workforce HJB equation. This limit delivers the joint value representation (1) in terms of the derivative of the joint value function rather than differences of values which, when moving up or down by one worker, are symmetric due to continuous differentiability.

B Characterization of surplus function

First define the surplus as $S(z,n) = \Omega(z,n) - nU$. Given that $\rho U = b$, this implies that $\rho S(z,n) = \rho\Omega(z,n) - nb$. We also have that $S_n(z,n) = \Omega_n(z,n) - U$. Combining these with the Bellman equation for Ω:

$$\begin{aligned}\rho S(z,n) &= \max_{v\geq 0} y(z,n) - c(v,z,n) - nb \\ &+ [q\phi v - \delta n]S_n(z,n) \\ &+ q(1-\phi)v\int_0^{S_n(z,n)}[S_n(z,n)-s]dH_n(s) \\ &+ \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n)\end{aligned}$$

where we sligtly abuse notation and use $H_n(s)$ to also denote here the employment-weighted cumulative distribution function of marginal surpluses. The value-pasting conditions become

$$\begin{aligned}S(z,n) &\geq \vartheta \\ S_n(z,n) &\geq 0\end{aligned}$$

We now make a number of assumptions to characterize the surplus. They are not all strictly necessary for each individual comparative static, but for convenience of exposition we present them all at the same time.

- The production function $y(z,n)$ satisfies $y_{\log z}, y_n, y_{\log z,n} > 0 > y_{nn}$.
- Productivity follows a geometric Brownian motion $\mu(z) = \mu z$ and $\sigma(z) = \sigma z$.
- Vacancy costs depend only on v and are isoelastic: $c(v) = c_0 v^{1+\gamma}$.

- The surplus function is twice continuously differentiable up to the boundary of the continuation region.

We now proceed to show the comparative statics discussed in the main text.

B.1 S is increasing in n

The no-endogenous-separations condition $S_n \geq 0$ immediately implies that the surplus is increasing in n.

B.2 S is increasing in z

Re-write the problem in terms of $x = \log z$. Denote with a slight abuse of notation $y(x, n) = y(e^x, n)$. Then, as a function of (x, n), the joint surplus solves

$$\begin{aligned}
\rho S(x,n) &= \max_{v \geq 0} y(x,n) - c(v) - nb \\
&+ [q\phi v - \delta n] S_n(x,n) \\
&+ q(1-\phi) v \mathcal{H}(S_n(x,n)) \\
&+ \left(\mu - \frac{\sigma}{2}\right) S_x(x,n) + \frac{\sigma^2}{2} S_{xx}(x,n)
\end{aligned}$$

where we integrated by parts, and denoted $\mathcal{H}(s) = \int_0^s H_n(r) dr$. Denote $\zeta(x, n) = S_x(x, n)$. Differentiate the Bellman equation w.r.t. x and use the envelope theorem to obtain

$$\begin{aligned}
\rho \zeta(x,n) &= y_x(x,n) \\
&+ \Big\{ \left[q(1-\phi) H_n(S_n(x,n)) + q\phi \right] v^*(x,n) - \delta n \Big\} \zeta_n(x,n) \\
&+ \mu \zeta_x(x,n) + \frac{\sigma^2}{2} \zeta_{xx}(x,n)
\end{aligned}$$

Now consider the stochastic process defined by

$$\begin{aligned}
dx_t &= \mu dt + \sigma dW_t \\
dn_t &= \Big\{ \left[q(1-\phi) H_n(S_n(x_t,n_t)) + q\phi \right] v^*(x_t,n_t) - \delta n_t \Big\} dt
\end{aligned} \tag{19}$$

This correponds to the true stochastic process for productivity, but a hypothetical process for employment, that in general differes from the realized one. We can now use the Feynman-Kac formula (Pham

2009) to go back to the sequential formulation:

$$\zeta(x,n) = \mathbb{E}\left[\int_0^T e^{-\rho t} y_x(x_t,n_t) + e^{-\rho T}\zeta(x_T,n_T) \,\middle|\, x_0 = x, n_0 = n, \{x_t,n_t\} \text{ follows (19)}\right]$$

and where T is the hitting time of either the separation of exit region. By assumption, $y_x > 0$, so the contribution of the first part is always positive. On the exit region, smooth-pasting requires that $\zeta = 0$. In the interior of the separation region, $\zeta = 0$. Under our regularity assumption, we thus get $\zeta = 0$ on the layoff boundary. Thus,

$$\zeta(x,n) = \mathbb{E}\left[\int_0^T e^{-\rho t} y_x(x_t,n_t)dt \,\middle|\, x_0 = x, n_0 = n, \{x_t,n_t\} \text{ follows (19)}\right] > 0$$

which concludes the proof.

B.3 S is concave in n

Denote $s(z,n) = S_n(z,n)$. Differentiate the Bellman equation w.r.t. n on the interior of the domain, use the envelope theorem and integrate by parts to obtain:

$$\begin{aligned}
(\rho+\delta)s(z,n) &= y_n(z,n) - b \\
&+ \Big\{\big[q\phi + q(1-\phi)H_n(s(z,n))\big]v^*(z,n) - \delta n\Big\} s_n(z,n) \\
&+ \mu(z)s_z(z,n) + \frac{\sigma^2(z)}{2} s_{zz}(z,n)
\end{aligned}$$

Recall that

$$(1+\gamma)c_0[v^*(z,n)]^\gamma = q\phi s(z,n) + q(1-\phi)\mathcal{H}(s(z,n))$$

In particular, differentiating w.r.t. n,

$$\gamma(1+\gamma)c_0[v^*(z,n)]^{\gamma-1}v_n^*(z,n) = \big[q\phi + q(1-\phi)H_n(s(z,n))\big]s_n(z,n)$$

and so

$$\gamma\frac{v_n^*(z,n)}{v^*(z,n)} = \frac{\phi + (1-\phi)H_n(s(z,n))}{\phi + (1-\phi)\overline{H}(s(z,n))}\frac{s_n(z,n)}{s(z,n)}$$

where $\overline{H}(s) = \frac{\mathcal{H}(s)}{s} \leq 1$. Now denote $\zeta(z,n) = s_n(z,n) = S_{nn}(z,n)$. Differentiate the recursion for s w.r.t. n to obtain

$$\begin{aligned}
&\left(\rho + 2\delta - q(1-\phi)H_n'(s(z,n)v^*(z,n)s_n(z,n) - q[\phi + (1-\phi)H_n(s(z,n))v_n^*(z,n)\right)\zeta(z,n) \\
= \;& y_{nn}(z,n) \\
+ \;& \Big\{\big[\lambda\phi + \lambda(1-\phi)H_n(s(z,n)\big]v^*(z,n) - \delta n\Big\}\zeta_n(z,n) \\
+ \;& \mu(z)\zeta_z(z,n) + \frac{\sigma^2(z)}{2}\zeta_{zz}(z,n)
\end{aligned}$$

Now define the "effective discount rate"

$$\begin{aligned}
R(z,n,s_n(z,n)) &= \rho + 2\delta - q(1-\phi)H_n'(s(z,n)v^*(z,n)s_n(z,n) - q[\phi + (1-\phi)H_n(s(z,n))]v_n^*(z,n) \\
&= \rho + 2\delta - qv^*(z,n)s_n(z,n)\underbrace{\left\{(1-\phi)H_n'(s(z,n)) + \frac{\phi + (1-\phi)H_n(s(z,n))}{\gamma s(z,n)}\frac{\phi + (1-\phi)H_n(s(z,n))}{\phi + (1-\phi)\overline{H}(s(z,n))}\right\}}_{\equiv P(z,n)>0}
\end{aligned}$$

where the second equality uses the expression for v_n^* derived above. Define the stochastic process

$$\begin{aligned}
dz_t &= \mu(z_t)dt + \sigma(z_t)dW_t \\
dn_t &= \Big\{\big[q(1-\phi)H_n(S_n(z_t,n_t)) + q\phi\big]v^*(z_t,n_t) - \delta n_t\Big\}dt
\end{aligned} \tag{20}$$

As before, we can use the Feynman-Kac formula to obtain

$$\zeta(z,n) = \mathbb{E}\left[\int_0^T e^{-\int_0^t R(z_\tau,n_\tau,\zeta(z_\tau,n_\tau))d\tau} y_{nn}(z_t,n_t)dt + e^{-\int_0^T R(z_\tau,n_\tau,\zeta(z_\tau,n_\tau))d\tau T}\zeta(z_T,n_T) \;\middle|\; z_0 = z, n_0 = n, \{z_t,n_t\} \text{ follows } (20)\right]$$

for T the first hitting time of the exit/separation region. The contribution of the first term is always negative. Note that ζ enters in the effective discount rate. Inside the separation region and in the exit regions, $\zeta = 0$. We restrict attention to twice continuously differentiable functions, so $\zeta = 0$ on the exit and separation frontiers. Then

$$\zeta(z,n) = \mathbb{E}\left[\int_0^T e^{-\int_0^t R(z_\tau,n_\tau,\zeta(z_\tau,n_\tau))d\tau} y_{nn}(z_t,n_t)dt \;\middle|\; z_0 = z, n_0 = n, \{z_t,n_t\} \text{ follows } (20)\right] < 0$$

which concludes the proof.

B.4 S is supermodular in $(\log z, n)$

Denote again $s(x,n) = S_n(x,n)$, where $x = \log z$. Recall

$$\begin{aligned}(\rho+\delta)s(x,n) &= y_n(x,n) - b \\ &+ \Big\{\big[q\phi + q(1-\phi)H_n(s(x,n)\big]v^*(x,n) - \delta n\Big\}s_n(x,n) \\ &+ \mu s_x(x,n) + \frac{\sigma^2}{2}s_{xx}(x,n)\end{aligned}$$

and that

$$(1+\gamma)c_0[v^*(x,n)]^\gamma = q\phi s(x,n) + q(1-\phi)\mathcal{H}(s(x,n))$$

In particular, differentiating w.r.t. x,

$$\gamma\frac{v_x^*(x,n)}{v^*(x,n)} = \frac{\phi + (1-\phi)H_n(s(x,n))}{\phi + (1-\phi)\overline{H}(s(x,n))}\frac{s_x(x,n)}{s(x,n)}$$

Now denote $\zeta(x,n) = s_x(x,n) = S_{xn}(x,n)$. Differentiate the recursion for $s(x,n)$ w.r.t. x to obtain

$$\begin{aligned}&\left(\rho+\delta - q(1-\phi)H_n'(s(x,n)v^*(x,n)s_x(x,n) - q[\phi + (1-\phi)H_n(s(x,n))v_x^*(x,n)\right)\zeta(x,n) \\ = \;& y_{nx}(x,n) \\ + \;& \Big\{\big[\lambda\phi + \lambda(1-\phi)H_n(s(x,n)\big]v^*(x,n) - \delta n\Big\}\zeta_n(x,n) + \mu\zeta_x(x,n) + \frac{\sigma^2}{2}\zeta_{xx}(x,n)\end{aligned}$$

As before, define the "effective discount rate"

$$\begin{aligned}R(x,n,s_x(x,n)) &= \rho+\delta - q(1-\phi)H_n'(s(x,n)v^*(x,n)s_x(x,n) - q[\phi + (1-\phi)H_n(s(x,n))]v_x^*(x,n) \\ &= \rho+\delta - qv^*(x,n)s_x(x,n)\underbrace{\left\{(1-\phi)H_n'(s(x,n)) + \frac{\phi+(1-\phi)H_n(s(x,n))}{\gamma s(x,n)}\frac{\phi+(1-\phi)H_n(s(x,n))}{\phi+(1-\phi)\overline{H}(s(x,n))}\right\}}_{\equiv P(x,n)>0}\end{aligned}$$

where the second equality uses the expression for v_n^* derived above. As before, define the stochastic process

$$\begin{aligned}dx_t &= \mu dt + \sigma dW_t \\ dn_t &= \Big\{\big[q(1-\phi)H_n(S_n(e^{x_t},n_t)) + q\phi\big]v^*(x_t,n_t) - \delta n_t\Big\}dt \end{aligned} \tag{21}$$

As before, we can use the Feynman-Kac formula to obtain

$$\zeta(x,n) = \mathbb{E}\left[\int_0^T e^{-\int_0^t R(x_\tau,n_\tau,\zeta(x_\tau,n_\tau))d\tau} y_{nx}(x_t,n_t)dt + e^{-\int_0^T R(x_\tau,n_\tau,\zeta(x_\tau,n_\tau))d\tau T}\zeta(x_T,n_T) \;\middle|\; x_0 = z, n_0 = n, \{x_t,n_t\} \text{ follows } (21)\right]$$

for T the first hitting time of the exit/separation region. The contribution of the first term is always positive. Inside the separation region and in the exit regions, $\zeta = 0$. We restrict attention to twice continuously differentiable functions, so $\zeta = 0$ on the exit and separation frontiers. Then

$$\zeta(x,n) = \mathbb{E}\left[\int_0^T e^{-\int_0^t R(x_\tau,n_\tau,\zeta(x_\tau,n_\tau))d\tau} y_{nx}(x_t,n_t)dt \;\middle|\; x_0 = z, n_0 = n, \{x_t,n_t\} \text{ follows } (21)\right]$$

which concludes the proof.

B.5 Net employment growth

Net employment growth in the continuation region is

$$\frac{dn_t}{dt} = q\Big[\phi + (1-\phi)H_n(S_n(z,n))\Big]v^*(z,n) - \lambda^E(1 - H_v(S_n(z,n)))n - \delta n \equiv g(z,n)$$

Using the expression above for $v^*(z,n)$:

$$\begin{aligned} g(z,n) = {} & \frac{q^{1+1/\gamma}}{[(1+\gamma)c_0]^{1/\gamma}}\Big(\phi + (1-\phi)H_n(S_n(z,n))\Big)\Big(\phi S_n(z,n) + (1-\phi)\mathcal{H}(S_n(z,n))\Big)^{1/\gamma} \\ & -\lambda^E(1 - H_v(S_n(z,n)))n - \delta n \end{aligned}$$

From the previous comparative statics on $S_n(z,n)$, it is straightforward to see that $g(z,n)$ is increasing in $\log z$ and decreasing in n.

C Frictionless limits

C.1 Setup

Frictional problem. Start by recalling the Bellman equation for the joint surplus in the frictional case:

$$
\begin{aligned}
\rho S(z,n) \quad &= \quad \max_v y(z,n) - nb - c(v) - \delta n S_n(z,n) \\
&+ \quad q(\theta)v\left\{\phi S_n + (1-\phi)\int_0^{S_n} H_n(s)ds\right\} \\
&+ \quad (\mathbb{L}S)(z,n) \\
\text{s.t.} \qquad & S(z,n) \geq 0,\ S_n(z,n) \geq 0
\end{aligned}
\tag{22}
$$

where H_n is the employment-weighted cumulative distribution function of marginal surpluses. $\mathbb{L}$ is the differential operator that encodes the continuation value from productivity shocks. For instance, for a diffusion, $(\mathbb{L}S)(z,n) = \mu(z)S_z(z,n) + \frac{\sigma(z)^2}{2}S_{zz}(z,n)$. Recall that $\phi = \frac{u}{u+\xi(1-u)}$ is the probability that a vacancy meets an unemployed worker, and q is the vacancy meeting rate.

Note that we abstracted from exogenous separations for simplicity, but endogenous separations when $S(n,z) < 0$ still occur. Denote by Δ the aggregate endogenous separation rate.

Inside the continuation region, the density function $h(z,n)$ of the distribution of firms by productivity and size is determined by the stationary KFE

$$
0 \quad = \quad -\frac{\partial}{\partial n}\Big(h(z,n)g(z,n)\Big) + (\mathbb{L}^* h)(z,n)
$$

where $\mathbb{L}^*$ is the formal adjoint of the operator $\mathbb{L}$, and $g(z,n)$ is the growth rate of employment

$$
g(z,n) = q(\theta)v^*(z,n)\Big[\phi + (1-\phi)H_n(S_n(z,n))\Big] - \xi\lambda^U n\Big[1 - H_v(S_n(z,n)),\Big] \tag{23}
$$

where λ^U is the meeting rate from unemployment, and ξ the relative search efficiency of the employed.

Finally, the mass of entrant firms $\mathtt{m}_0$ is determined by the free-entry condition

$$
c_e = \mathbb{E}^{\text{Entry}}[\max\{S(z,n_0),0\}] \tag{24}
$$

where n_0 is initial employment which is a parameter, and $\mathbb{E}^{\text{Entry}}$ is the expectation operator under the productivity distribution for entrants $\Pi_0(z)$. The surplus is a function of $\mathtt{m}_0$ through the vacancy meeting rate $q(\theta)$, since θ is increasing in $\mathtt{m}_0$.

Functional forms. For ease of exposition, we consider isoelastic vacancy cost functions

$$c(v) = \frac{c_0}{1+\gamma} v^{1+\gamma},$$

and normalize $c_0 = 1$, but the result does not depend on the particular functional form nor on the normalization. Also, we specialize to a Cobb-Douglas matching function $\mathtt{m}(s,v) = As^{\beta}v^{1-\beta}$, where A is match efficiency, a proxy for labor market frictions. Finally, for ease of exposition, we set to zero exogenous separations to unemployment $\delta = 0$.

Comparative statics. We describe behavior of the economy in the limit when match efficiency $A \to \infty$. We do so for two different configurations of the economy:

1. No on-the-job-search: $\xi = 0$
2. On-the-job search: $\xi > 0$

Notation. We write $B \approx C$ for a first-order Taylor expansion. Denote $||S_n|| = \mathbb{E}^{Steady-state}\left[S_n^{1/\gamma}\right]^{\gamma}$, where $\mathbb{E}^{Steady-state}$ denotes the expectation under the steady-state distribution of marginal surpluses. This is also the Lebesgue $(1/\gamma)$-norm of S_n under the steady-state probability measure.

C.2 No on-the-job search

Since $\xi = 0$, $\phi = 1$. From (22), the FOC for vacancies gives

$$v^*(z,n) \quad = \quad \Big(qS_n\Big)^{1/\gamma}. \tag{25}$$

using this optimality condition in the value function of hiring firms:

$$\begin{aligned} \rho S(z,n) \quad &= \quad y(z,n) - nb + \frac{\gamma}{1+\gamma} \cdot q(\theta)^{\frac{1}{1+\gamma}} S_n^{\frac{1}{1+\gamma}} + (\mathbb{L}S)(z,n) \\ \text{s.t.} \qquad & S(z,n) \geq 0, \; S_n(z,n) \geq 0 \end{aligned}$$

which now only depends on $q(\theta)$ as the sole aggregate. Hence, free-entry (24) uniquely pins down $q(\theta)$ to the same value no matter what value A takes. Therefore, the value function always satisfies the same Bellman equation, irrespective of A. Hence, throughout the state space, at any given (n,z), marginal surpluses $S_n(z,n)$ remain the same as A varies. Moreover, since the value $S(z,n)$ is independent from A, so are all the decisions by firms. As a result, the endogenous separation rate Δ always remains the same – and in particular, finite.

We now study how aggregates $\mathtt{v}, \mathtt{u}, \theta$ evolve along this limiting path. Given the matching function these determine all other equilibrium objects: λ^U, λ^E, q. In characterizing the limit we make use of the simple fact that both $\mathtt{m}_0$ and $\mathtt{v}$ must remain finite. If this were not the case, then infinite entry and vacancy costs would violate the economy's resource constraint.

C.2.1 Aggregates in the limit

Integrating both sides of the FOC for vacancies under the firm distribution, and using the matching function which implies that $q = A\theta^{-\beta}$, aggregate vacancies are

$$\mathtt{v} = \mathtt{m}_0 q^{\frac{1}{\gamma}} ||S_n||^{\frac{1}{\gamma}} = \mathtt{m}_0 A^{\frac{1}{\gamma}} \theta^{-\frac{\beta}{\gamma}} ||S_n||^{\frac{1}{\gamma}}$$

Since q remains constant, and $\mathtt{v}$ and $\mathtt{m}_0$ are finite in the limit, then the first equality implies that $||S_n||$ remains finite in the limit.

In the limit, the unemployment rate is $\mathtt{u} \approx \frac{\Delta}{\lambda^U}$. The matching function implies $\lambda^U = A\theta^{1-\beta}$. Combined, the unemployment rate is $\mathtt{u} \approx \Delta A^{-1}\theta^{-(1-\beta)}$. Combining these expressions with the expression for aggregate vacancies $\mathtt{v}$, tightness satisifies

$$\theta = \frac{\mathtt{v}}{\mathtt{u}} \approx \frac{\mathtt{m}_0 A^{\frac{1}{\gamma}} \theta^{-\frac{\beta}{\gamma}} ||S_n||^{\frac{1}{\gamma}}}{\Delta A \theta^{1-\beta}}$$

so that

$$\theta^{\beta\frac{1+\gamma}{\gamma}} \approx \left(\frac{\mathtt{m}_0}{\Delta}\right) ||S_n||^{\frac{1}{\gamma}} A^{\frac{1+\gamma}{\gamma}}.$$

Since $\mathtt{m}_0$, Δ, and $||S_n||$ are finite, θ diverges with A. Therefore, λ_U diverges as well. On the worker side, since λ_U, diverges to infinity, $\mathtt{u}$ goes to zero. On the firm side, $\mathtt{m}_0$ remains finite, but changes such that q remains constant and vacancies remain finite.

C.2.2 Invariant distribution of marginal surpluses

We now turn to the invariant distribution $h(z, n)$. After substituting optimal vacancies into (23) evaluated at $\xi = 1 - \phi = 0$, one obtains that the growth of employment in the hiring region is:

$$g(z, n) = q\Big(qS_n(z, n)\Big)^{\frac{1}{\gamma}}.$$

Since $S_n(z, n)$ remains constant throughout the state space, then employment growth in the hiring region remains constant throughout the state space. The firm loses no workers to employment because there is no on-the-job search. Since $S_n(z, n)$ and $U = b/\rho$ both stay unchanged, then the employment losses to unemployment are still unchanged. Since $S(z, n)$ is unchanged, then the exit decision is also unchanged.

Hence, the law of motion of employment is independent of A. Thus, the steady-state distribution $h(z,n)$ is also independent from A. Therefore the values of firms $S(z,n)$ are the same across the state space and the relative mass of firms at each (z,n) is unchanged, despite higher but finite $\mathtt{m}_0$.

C.2.3 Summary

Summarizing this case: as $A \to \infty$, even though unemployment vanishes, the allocations in the search model without on-the-job search do not converge to those of a competitive firm-dynamics model. The free entry condition requires the vacancy meeting rate q to remain finite and thus a non degenerate dispersion of marginal products of labor survives even in the limit as firms face the same adjustment frictions regardless of A. In contrast, in the competitive benchmark, marginal products of labor are equalized across firms.

C.3 On-the-job search with a fat-tailed entry distribution

We now turn to the case in which on-the-job search remains positive at some fixed value $\xi > 0$, and thus $\phi < 1$. We follow the same logic as before, with some additional steps due to on-the-job search. To simplify algebra we abstract from exogenous job destruction, setting $\delta = 0$.

To keep the arguments manageable, we also introduce an additional assumption. We require the entry productivity distribution to have a "fat enough" tail. With decreasing returns to scale, the optimal frictionless size of the firm grows without bound as productivity becomes large. We assume that the productivity distribution of entrants is unbounded, and assume that it is fat-tailed enough that the rate at which the optimal size of a firm grows with productivity is faster rate than the decay of the productivity distribution. More precisely, the frictionless optimal size is $n^*(z) = \arg\max_n y(z,n) - bn$. We assume that the entry productivity distribution $\Pi_0(z)$ is such that

$$\lim_{z \uparrow +\infty} n^*(z)\Pi_0(z) = +\infty$$

This is satisfied for the production function $y(z,n) = zn^\alpha$ and the entry distribution $\Pi_0(z) \propto z^{-\zeta}$, when $\frac{1}{1-\alpha} - \zeta \geq -1$. Our empirical implementation uses these functional forms and satisfies these restrictions.[51]

[51] An alternative approach is to assume a constant arrival rate of "The Godfather" shocks that leave workers *unable to refuse* any job offer. Additional details available on request.

Consider (22) written in terms of the return on a vacancy

$$\begin{aligned} \rho S(z,n) &= \max_v \; y(z,n) - nb - c(v) + q(\theta)vR(S_n) + (\mathbb{L}S)(z,n) \\ \text{s.t.} & \quad S(z,n) \geq 0,\; S_n(z,n) \geq 0 \end{aligned}$$

where

$$R(S_n) = \phi S_n + (1-\phi)\int_0^{S_n} H_n(s)ds \tag{26}$$

is the return to a vacancy. The growth of employment is

$$g(z,n) = qv^*(z,n)\Big[\phi + (1-\phi)\,H_n\left(S_n(z,n)\right)\Big] - \xi\lambda^U n\Big[1 - H_v\left(S_n(z,n)\right)\Big] \tag{27}$$

C.3.1 Aggregates in the limit

We restrict attention to the economically meaningful case in which (1) output remains finite and strictly positive in the limit, and (2) the rate at which workers separate into unemployment remains finite in the limit. These restrictions are equivalent to a guess and verify strategy, in which we guess that (1-2) hold and then verify those conditions. The logic of our approach is then to exhibit a solution in which (1-2) hold – but in principle other cases may arise.

Consider the set of meeting rates. Because some measure $\mathtt{n}$ employed jobseekers are always present regardless of A, the amount of effective search effort $\mathtt{s} = \mathtt{u} + \xi\mathtt{n}$ remains finite and positive even if $\mathtt{u}$ goes to zero. By (1), vacancies also remain finite. Combined, these imply that market tightness $\theta = \mathtt{v}/\mathtt{s}$ remains finite. Since $q = A\theta^{-\beta}$ and $\lambda^U = A\theta^{-(1-\beta)}$, then both meeting rates diverge to infinity at the same rate as A.[52]

Consider unemployment and aggregate vacancies. (2) requires that the rate at which workers separate into unemployment is a positive constant Δ in the limit. Since $\mathtt{u} \approx \frac{\Delta}{\lambda^U}$, and λ_U diverges, then the unemployment rate converges to zero. Since the unemployment rate converges to zero, then ϕ also converges to zero. Firm level and aggregate vacancies are given by

$$v = q^{\frac{1}{\gamma}} R\left(S_n\right)^{\frac{1}{\gamma}} \qquad , \qquad \mathtt{v} = \mathtt{m}_0 q^{\frac{1}{\gamma}} \left|\left|R\left(S_n\right)\right|\right|^{\frac{1}{\gamma}}. \tag{28}$$

(1) implies that both aggregate vacancies $\mathtt{v}$ the mass of entering firms $\mathtt{m}_0$ remain finite. Since $\mathtt{v}$ is finite and $\mathtt{m}_0$ is finite, while q diverges at the same rate as A, then $\gamma > 0$ requires $\left|\left|R\left(S_n\right)\right|\right|$ must go to zero at the same rate as A goes to infinity.

[52]Strictly speaking, free-entry then ensures that theta is pinned down to a strictly positive value. This proof is more lengthy but does not require any additional assumptions and is available upon request.

C.3.2 Invariant distribution of marginal surpluses

We now show that the distribution of marginal surpluses degenerates to a single value on the support of the invariant distribution.

First, we use (28) to express firm level vacancies as a share of aggregate vacancies, where that share is determined by the firms' return on a vacancy relative to the average return:

$$v = \frac{1}{m_0}\left(\frac{R\left(S_n\right)}{||R\left(S_n\right)||}\right)^{\frac{1}{\gamma}} \mathtt{v} = \frac{1}{m_0}\left(\frac{R\left(S_n\right)}{||R\left(S_n\right)||}\right)^{\frac{1}{\gamma}}\left(\frac{\lambda^U \xi}{q}\right)$$

where the second equality uses $q = A(\mathtt{v}/\xi)^{-\beta}$, and $\lambda^U = A(\mathtt{v}/\xi)^{-(1-\beta)}$, which jointly imply that $\mathtt{v} = \lambda^U \xi / q$. Now consider the expression for growth of employment inside the continuation region (27), under the limiting case of $\phi = 0$:

$$g(z,n) \approx qvH_n\left(S_n(z,n)\right) - \xi\lambda^U n\Big[1 - H_v\left(S_n(z,n)\right)\Big]$$

Substituting in the expression for firm vacancies and collecting $\lambda^U\xi$ terms:

$$g(z,n) \approx \lambda^U \xi \left\{\frac{1}{\mathtt{m}_0}\left(\frac{R(S_n)}{||R(S_n)||}\right)^{\frac{1}{\gamma}} H_n(S_n) - n\Big[1 - H_v(S_n)\Big]\right\}.$$

Consider some (n,z) that has positive mass in steady state. Since λ^U diverges but growth must remain finite, the term in braces must be equal to zero in the limit:

$$\frac{1}{\mathtt{m}_0}\left(\frac{R(S_n)}{||R(S_n)||}\right)^{\frac{1}{\gamma}} H_n(S_n) = n\Big[1 - H_v(S_n)\Big]$$

Using this we can show that the distribution of marginal surplus converges point-wise to a degenerate limiting distribution H_n^∞.

We proceed by contradiction. Suppose that H_n converges to a limiting distribution H_n^∞ that is non-degenerate.[53] Consider a firm at the top of the distribution, such that $1 - H_v(S_n) = 0$. The probability that the firm loses a worker is zero, so the right-hand side is zero. However, by the supposition that H_n is non-degenerate, then $R(S_n)$ converges to a non-zero value, since the firm can increase its value by poaching from workers below it on the ladder. Since there is some $R(S_n)$ that is non-zero, then $||R(S_n)||$ also converges to a non-zero value. Therefore flows out of the firm are zero, but flows into the firm are positive. This violates the above equality, which would imply infinite growth as λ^U diverges. This is a contradiction. Hence, in the limit H_n^∞ must be degenerate, and marginal surpluses of firms converge to

[53] So the probability measure of S_n in the cross-section would converge in distribution to a non-degenerate limit.

a common limit which we denote S_n^*.

We have shown that the limiting distribution H_n^∞ is degenerate. This implies that the invariant distribution of employment and productivity lines up along a strip $\{z, n^*(z)\}$ where $n^*(z)$ is implicitly defined by $S_n(n^*(z), z) = S_n^*$, so is strictly increasing.[54]

C.3.3 Unique value for S_n^* on the limiting strip

We have shown that $||R(S_n)||$ and $R(S_n)$ converge to zero in the limit, yet this does not necessarily imply a particular value for S_n^*. Here we show that $S_n^* = 0$. We guess the following, which we verify below:

$(\star)$ $n^*(z) = \arg\max_n \; y(z,n) - bn$.

From the concavity of marginal surplus and $n^*(z) > n_0$, we have

$$S(z, n_0) \leq S(z, n^*(z)) - S_n(z, n^*(z)) \times (n^*(z) - n_0)$$

In the limit $S_n(z, n^*(z)) \equiv S_n^*$ is equalized, which delivers the following upper bound to the value of entry:

$$\int S(z, n_0)\Pi_0(z)dz \leq \int S(z, n^*(z))\Pi_0(z)dz - S_n^* \int \Big(n^*(z) - n_0\Big)\Pi_0(z)dz$$

We show that $S_n^* = 0$ by contradiction. Suppose that $S_n^* > 0$. From our assumption on the entry distribution then in the limit $\int n^*(z)\Pi_0(z)dz$ is infinity. Since all other terms on the right-hand side of the above inequality are finite,[55] then a necessary condition for the above inequality to be satisfied is that $\int S(z, n_0)\Pi_0 dz < 0$, which violates the free-entry condition. Therefore it must be that $S_n^* = 0$.

Intuitively, a strictly positive marginal surplus S_n^* reflects that there is an excess supply of firms in the economy relative to the supply of workers. The fat tail assumption implies that this excess supply translates into a very negative value of entry, which cannot be an equilibrium in which firms enter freely.

C.3.4 Limiting value function

We now return to the limiting Bellman equation for marginal surplus. Given that $q(\theta)R(S_n^*) = 0$,[56] making the generator $\mathbb{L}$ explicit, applying the result that $n = n^*(z)$, and noting that $S_n(n, z) \geq 0$ is

[54] To see that this is a strip, recall that $S(z, n)$ is such that $S_{nn} < 0$ and $S_{zn} > 0$. Therefore $S_n(n^*(z), z) = S_n^*$ implicitly defines an strictly increasing function $n^*(z)$.

[55] $\int S(z, n^*(z))\Pi_0(z)dz$ remains finite because $S(z, n^*(z))$ satisfies (22) evaluated at $(z, n^*(z))$. It can then be shown that, in the limit, $q(\theta)R(S_n^*)$ depends only on S_n^* and θ, but not on A directly. The details of the derivation are available upon request.

[56] Details are available upon request.

satisfied with equality, we have

$$\begin{aligned}\rho S(z,n^*(z)) &= y(z,n^*(z)) - n^*(z)b + \mu(z)S_z(z,n^*(z)) + \frac{\sigma(z)^2}{2}S_{zz}(z,n^*(z)) \\ \text{s.t.} \quad & S(z,n^*(z)) \geq 0\end{aligned}$$

Our key result in the text was that the limiting economy featured a value function that depended only on z. However, the continuation value terms in the above value function contain *partial* derivatives with respect to z, not *total* derivatives. To argue that it is enough to focus on the value function evaluated on the strip, we must show that the partial derivatives approximate the total derivatives in the limit. The following shows that this is the case in the limit

$$\lim_{A\to\infty}\frac{dS(z,n^*(z))}{dz} = \lim_{A\to\infty}\left\{\left.\frac{\partial S(z,n)}{\partial z}\right|_{n=n^*(z)} + \underbrace{\left.\frac{\partial S(z,n)}{\partial n}\right|_{n=n^*(z)}}_{\to S_n^*=0} \cdot \underbrace{\frac{dn^*(z)}{dz}}_{\text{Finite constant}}\right\} = \lim_{A\to\infty}\left.\frac{\partial S(z,n^*(z))}{\partial z}\right|_{n=n^*(z)}$$

Therefore, in the limit, exit can be described by the value function evaluated on the strip, $\overline{S}(z) := S(z,n^*(z))$ which evolves according to

$$\rho\overline{S}(z) = y(z,n^*(z)) - n^*(z)b + \mu(z)\overline{S}_z(z) + \frac{\sigma(z)^2}{2}\overline{S}_{zz}(z)$$

and an exit cut-off determined by $\overline{S}(z) = 0$.

C.3.5 Optimal size

We now characterize the optimal size of incumbents in the limit and verify ($\star$). We note that, if for a small period of time dt, the firm was away from the exit cutoff but close to its optimal size, then

$$S(z,n) \approx \Big[y(z,n) - bn\Big]dt + e^{-\rho dt}\mathbb{E}\Big[S\Big(z_{dt}, n^*(z_{dt})\Big)\Big|z_0 = z\Big]$$

because the other contributions in $n - n^*(z)$ scale with $||S_n|| = 0$. Therefore,

$$S_n(z,n^*(z)) \approx \Big[y_n(z,n^*(z)) - b\Big]dt$$

and so it must be that $y_n(z,n^*(z)) = b$. This confirms guess ($\star$).[57]

[57] To make this argument strictly formal, differentiate the Bellman equation and represent marginal surplus as an integral with the Feynan-Kac formula as in Appendix B. The derivation details are available upon request.

C.3.6 Summary

With on-the-job search, as $A \to +\infty$, the value function converges to the one of the Hopenhayn model. The mass of active firms converges to some finite value. Free-entry pins down the mass of firms, and converges to a condition that differs from the Hopenhayn model's. There is an additional term that stems from the value gains that entrant realize along their (very fast) growth towards their optimal size. The equilibrating variable is the limiting market tightness, that governs the size of these gains.

D Algorithm

We want to solve the Bellman equation:

$$\begin{aligned} \rho S(z,n) &= \max_{v\geq 0} \quad y(z,n) - \delta n S_n(z,n) \\ &+ \mathcal{H}(S_n(z,n))\, v - c(v,n) \\ &+ \mu(z) S_z(z,n) + \frac{\sigma^2(z)}{2} S_{zz}(z,n) - \rho n U \end{aligned}$$

with

$$\begin{aligned} \mathcal{H}(S_n(z,n)) &= q\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} \left[S_n(z,n) - S_n'\right] dH_n\left(S_n'\right)\right] \\ &= q\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} H_n(s)\, ds\right] \end{aligned}$$

where the last line uses integration by parts. We assume that the vacancy cost satisfies $c(v,n) = \bar{c}\left(\frac{v}{n}\right) v$, where $\bar{c}$ is iso-elastic with elasticity γ. Therefore, $c_v(v,n) = (\gamma+1)\,\bar{c}\left(\frac{v}{n}\right)$. Along with the first order condition $c_v(v,n) = \mathcal{H}(S_n(z,n))$, this implies

$$c(v,n) = \bar{c}\left(\frac{v}{n}\right) v = \frac{1}{\gamma+1} c_v(v,n)\, v = \frac{1}{\gamma+1}\mathcal{H}(S_n(z,n))\, v$$

Therefore the total value of vacancy posting is

$$\begin{aligned} \mathcal{H}(S_n(z,n))\, v - c(v,n) &= \frac{\gamma}{\gamma+1}\mathcal{H}(S_n(z,n))\, v \\ \mathcal{H}(S_n(z,n))\, v - c(v,n) &= \frac{\gamma}{\gamma+1}\mathcal{H}(S_n(z,n))\left(\frac{v}{n}\right) n \end{aligned}$$

Letting $\bar{c}\left(\frac{v}{n}\right) = \frac{\kappa}{1+\gamma}\left(\frac{v}{n}\right)^{\gamma}$ and using $\bar{c}\left(\frac{v}{n}\right) = \frac{1}{\gamma+1}\mathcal{H}(S_n(z,n))$ then

$$\frac{v}{n} = \kappa^{-1/\gamma}\mathcal{H}(S_n(z,n))^{\frac{1}{\gamma}}$$

and

$$\mathcal{H}\left(S_n\left(z,n\right)\right)v-c\left(v,n\right)=\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}n$$

Substituting this into the Bellman equation, acknowledging that $\rho U = b$, we arrive at the formulation:

$$\begin{aligned}\rho S\left(z,n\right) &= y\left(z,n\right)-bn \\ &+ \left[\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\frac{\mathcal{H}\left(S_n\left(z,n\right)\right)^{\frac{\gamma+1}{\gamma}}}{S_n\left(z,n\right)}-\delta\right]S_n\left(z,n\right)n \\ &+ \mu(z)S_z(z,n)+\frac{\sigma^2(z)}{2}S_{zz}(z,n)\end{aligned} \tag{29}$$

subject to $S(z,n)\geq 0$ and $S_n(z,n)\geq 0$.

D.1 Algorithm

The algorithm consists of three steps, implemented in MATLAB called from master file `MAIN.m`.

Step 0: Construct an initial guess. Start by constructing a $n_z \times n_n$ grid for log productivity and log size. Let $\boldsymbol{\pi} = y(z,n) - bn$ denote the stacked $(n_z * n_n) \times 1$ vector of flow payoffs on this grid. Guess an initial surplus $\boldsymbol{S}^0$ on this grid (a $(n_z * n_n) \times 1$ column vector); a distribution of firms over productivity and size $\boldsymbol{h}^0$ (a $(n_z * n_n) \times 1$ column vector); aggregate finding rates q^0 and λ^0; and an efficiency-weighted share of unemployed searchers, θ^0. Construct marginal surplus. Construct exit regions, separation regions and the vacancy policy. File `InitialGuess.m` does this.

Step I: Iterate to convergence the coalition's problem for given aggregate states. For $t \geq 1$, given q^{t-1}, θ^{t-1}, h^{t-1} and $\boldsymbol{S}^{t-1}$, solve the coalition's problem to update the coalition value to $\boldsymbol{S}^t$. The solution to the coalition's surplus function is obtained in an inner iteration τ. Denote by $\boldsymbol{S}^{t,\tau}$ the surplus in outer iteration t during inner iteration τ, initiated with $\boldsymbol{S}^{t,0} = \boldsymbol{S}^t$; $\boldsymbol{T}_n(z,n)$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\boldsymbol{S}_n^{t,\tau} = \boldsymbol{T}_n(z,n)\boldsymbol{S}^{t,\tau}$, where $\boldsymbol{S}_n^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of derivatives of S w.r.t. n during outer iteration t and inner iteration τ; $\boldsymbol{T}_z$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\boldsymbol{S}_z^{t,\tau} = \boldsymbol{T}_z\boldsymbol{S}^{t,\tau}$, where $\boldsymbol{S}_z^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of derivatives of S w.r.t. z during outer iteration t and inner iteration τ; and $\boldsymbol{T}_{zz}$ a $(n_z * n_n) \times (n_z * n_n)$ matrix such that $\boldsymbol{S}_{zz}^{t,\tau} = \boldsymbol{T}_{zz}\boldsymbol{S}^{t,\tau}$, where $\boldsymbol{S}_{zz}^{t,\tau}$ is the stacked $(n_z * n_n) \times 1$ vector of second derivatives of S w.r.t. z during outer iteration t and inner iteration τ. Note that the matrix $\boldsymbol{T}_n(z,n)$ depends on (z,n) in the sense that the approximation is done either forward or backward depending on the endogenous drift for n at (z,n) (note that the drift of and innovations to z are independent of (z,n)). Within each outer iteration t, we iteratively update $\boldsymbol{S}^{t-1,\tau}$ for $\tau \geq 1$ following equation (29) based on

$$\left[\left(\rho+\frac{1}{\Delta}\right)\mathbb{1}-\left[\frac{\gamma\kappa^{-\frac{1}{\gamma}}}{\gamma+1}\frac{\mathcal{H}\left(\boldsymbol{S}_n^{t-1,\tau-1}\right)^{\frac{\gamma+1}{\gamma}}}{\boldsymbol{S}_n^{t-1,\tau-1}}-\delta\mathbb{1}\right].*\ T_n\left(z,n\right)-\mu T_z-\frac{\sigma^2}{2}T_{zz}\right]\boldsymbol{S}^{t-1,\tau} \quad = \quad \boldsymbol{\pi}+\frac{1}{\Delta}\boldsymbol{S}^{t-1,\tau-1}$$

where Δ is the step size, $.*$ denotes the element-by-element product, and $\mathcal{H}\left(\boldsymbol{S}_n^{t-1,\tau-1}\right)^{\frac{\gamma+1}{\gamma}}/\boldsymbol{S}_n^{t-1,\tau-1}$ is a $(n_z * n_n)\times(n_z * n_n)$ matrix constructed using the previous iteration's derivative of S stacked $(n_z * n_n)$ times in the column dimension. The step size cannot be too large for the problem to converge. These iterations are performed by iterating on τ until convergence by file `IndividualBehavior.m`, and the solution is assigned as the updated $\boldsymbol{S}^t$. We also obtain from the converged solution the updated separation, exit and a vacancy policies.

Step II: Iterate to convergence the aggregate states for given individual behavior. Given updated individual behavior in outer iteration t, obtain through iteration in an inner loop τ the distribution of firms $\boldsymbol{h}^t$, the aggregate meeting rates q^t and λ^t, the share of unemployed searchers θ^t, the distribution of workers over marginal surplus $\boldsymbol{H}_{\boldsymbol{n}}^t$, and the distribution of vacancies over marginal surplus $\boldsymbol{H}_{\boldsymbol{v}}^t$. File `AggregateBehavior.m` proceeds to do this in four steps.

Initiate each aggregate object with the previous outer iteration solution, $\boldsymbol{x}^{t-1,0}=\boldsymbol{x}^{t-1}$. Then:

Step II-a. Update the distribution of workers over marginal surplus to $\boldsymbol{H}_{\boldsymbol{n}}^{t-1,\tau}$ given a distribution of firms $\boldsymbol{h}^{t-1,\tau-1}$ and marginal surplus $\boldsymbol{S}_n^t$, where the latter was obtained in **Step I** above. This is done by file `CdfG.m`.

Step II-b. Update the distribution of vacancies over marginal surplus $\boldsymbol{H}_{\boldsymbol{v}}^{t-1,\tau}$ given a distribution of firms $\boldsymbol{h}^{t-1,\tau-1}$, the vacancy policy $\boldsymbol{v}^t$ and the ranking of firms in marginal surplus space. This is done by file `CdfF.m`.

Step II-c. Update the finding rates $q^{t-1,\tau}$, $\lambda^{t-1,\tau}$ and $\theta^{t-1,\tau}$ that is consistent with the vacancy policy $\boldsymbol{v}^t$ and the distribution of firms $\boldsymbol{h}^{t-1,\tau-1}$. This is done by file `HazardRates.m`.

Step II-d. Given $\boldsymbol{H}_{\boldsymbol{n}}^{t-1,\tau}$, $\boldsymbol{H}_{\boldsymbol{v}}^{t-1,\tau}$, $q^{t-1,\tau}$, $\lambda^{t-1,\tau}$ and $\theta^{t-1,\tau}$, update the distribution of firms $\boldsymbol{h}^{t-1,\tau}$ following the Kolmogorov forward equation in steady-state. This is executed by file `Distribution.m`.

Iterate over the four sub-steps *Step II-a*–*Step II-d* until convergence and assign the updated aggregate states q^t, λ^t, θ^t and $\boldsymbol{h}^t$. We subsequently return to step **Step I** and iterate on step **Step I**–**Step II** until both the surplus function and the aggregate states have converged.

D.2 Estimation

The criterion function that we minimize is highly-dimensional and potentially has many local minima. Furthermore, the equilibrium does not exist for some regions of the parameter space. For example, if the drift in productivity is not sufficiently negative, there is no ergodic distribution for productivity. For these reasons, using a sequential hill-climbing optimizer that updates its initial guess sequentially through a gradient-based method is prohibitive. Our solution is to use an algorithm that we can easily parallelize, that efficiently explores the parameter space, and for which we can ignore cases with no equilibrium.

We set up a hyper-cube in the parameter space and then initialize a Sobol sequence to explore it. A Sobol sequence is a quasi-random low-discrepancy sequence that maintains a maximum dispersion in each dimension and far outperforms standard random number generators. We then partition the sequence and submit each partition to a separate CPU on a high performance computer (HPC). From each evaluation of the parameter hyper-cube, we save the vector of model moments. We then collect them, splice them all together, and choose the one that minimizes the criterion function. Starting with wide bounds on the parameters, we run this procedure a number of times, shrinking the hyper-cube step by step until we achieve the global minimum.

Compared to standard optimizers, this procedure has the advantage that, as a byproduct of the estimation, we can learn a lot about model identification. From an optimizer one may retrieve the moments of the model only along the path of the parameter vector chosen by the algorithm. In our case, we retrieve tens of thousands of evaluations, knowing that the low-discrepancy property of the Sobol sequence implies that for an interval of any one parameter, the remaining parameters are drawn uniformly. Plotting each single moment against parameters therefore shows the effect of a parameter on a certain moment, conditional on local draws of all other parameters.

APPENDIX II

Firm and Worker Dynamics in a Frictional Labor Market

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This Appendix contains the proof of the joint surplus representation in the fully dynamic model, the main result of Section 3 in the main text. Section A lays out the notation for the fully dynamic model. Section B develops the proof.

A Notation for dynamic model

We first specify the value function of an individual worker i in a firm with arbitrary state x: $V(x,i)$. We then specify the value function of the firm: $J(x)$. Combining all workers' value functions with that of the firm we define the joint value: $\Omega(x)$. We then apply the assumptions from Section 2.2 which allow us to reduce (x) to only the number of workers and productivity of the firm, (z,n). Finally we take the continuous work force limit to derive a Hamilton-Jacobi-Bellman (HJB) equation for $\Omega(z,n)$ Applying the definition of total surplus used above, we obtain a HJB equation in $S(z,n)$ which we use to construct the equilibrium.

A.1 Worker value function: V

As in the static example, let U be the value of unemployment. It is convenient to define separately worker i's value when employed at firm x *before* the quit, layoff and exit decisions, $\boldsymbol{V}(x,i)$, and their value *after* these decisions, $V(x,i)$.[58]

Value of unemployment. Let $h_U(x)$ denote how the state of firm x is updated when it hires an unemployed worker.[59] Let $\mathcal{A}$ denote the set of firms making job offers that an unemployed worker would accept. The value of unemployment U therefore satisfies

$$\rho U = b + \lambda^U(\theta) \int_{x\in\mathcal{A}} \left[\boldsymbol{V}(h_U(x),i) - U\right] dH_v(x)$$

where H_v is the vacancy-weighted distribution of firms. If $x \notin \mathcal{A}$, then the worker remains unemployed.

[58] In terms of Figure 1, the value $\boldsymbol{V}$ is computed after the first stage of the flow chart, and the value V after the second stage, in the case that the firm stays in operation.

[59] For example, size would be update from n to $n+1$ and possibly some of the incumbent wages would be bargained down.

Stage I. To relate the value of the worker pre separation, $\boldsymbol{V}(x,i)$, to that post separation, $V(x,i)$, we require the following notation regarding firm and co-worker actions. Since workers do not form 'unions' within the firm, all of these actions are taken as given by worker i.

- Let $\epsilon(x) \in \{0,1\}$ denote the exit decision of firm, and $\mathcal{E} = \{x : \epsilon(x) = 1\}$ the set of x's for which the firm exits.

- Let $\ell(x) \in \{0,1\}^{n(x)}$ be a vector of zeros and ones of length $n(x)$, with generic entry $\ell_i(x)$, that characterizes the firm's decision to lay off incumbent worker $i \in \{1,\ldots,n(x)\}$, and $\mathcal{L} = \{(x,i) : \ell_i(x) = 1\}$ the set of (x,i) such that worker (x,i) is laid off.

- Let $q^U(x) \in \{0,1\}^{n(x)}$ be a vector of length $n(x)$, with generic entry $q_i^U(x)$ that characterizes an incumbent workers' decisions to quit, and $\mathcal{Q}^U = \left\{(x,i) : q_i^U(x) = 1\right\}$ the set of (x,i) such that worker (x,i) quits into unemployment.

- Let $\kappa(x) = (1 - \boldsymbol{\ell}(x)) \circ (1 - q_U(x))$ be an element-wise product vector that identifies workers that are kept in the firm, and $\mathcal{S} = \mathcal{L} \cup \mathcal{Q}^U = \{(x,i) : \kappa_i(x) = 0\}$, the set of (x,i) such that worker (x,i) separates into unemployment.

- Let $s(x,\kappa(x))$ denote how the state of firm x is updated when workers identified by $\kappa(x)$ are kept. This includes any renegotiation.

Given these sets and functions, the pre separation value $\boldsymbol{V}(x,i)$ satisfies:

$$\boldsymbol{V}(x,i) = \underbrace{\epsilon(x)U}_{\text{Exit}} + (1-\epsilon(x))\Big[\underbrace{\mathbb{I}_{\{(x,i)\notin\mathcal{S}\}}V(s(x,\kappa(x)),i)}_{\text{Continuing employment}} + \underbrace{\mathbb{I}_{\{(x,i)\in\mathcal{S}\}}U}_{\text{Separations and Quits}}\Big]$$

Stage II. It is helpful to characterize the value of employment post separation decisions, $V(x,i)$, in terms of the three distinct types of events described in Figure 2. First, the value changes due to *'Direct'* labor markets shocks to worker i, $V_D(x,i)$. These include her match being destroyed exogenously or meeting a new potential employer. Second, the value changes due to labor market shocks hitting other workers in the firm, $V_I(x,i)$, including their matches being exogenously destroyed or them meeting new potential employers. These events have an *'Indirect'* impact on worker i. Third, the value changes due to events on the *'Firm'* side, $V_F(x,i)$, including the firm contacting new workers and receiving productivity shocks. Combining events and exploiting the fact that in continuous time they are mutually exclusive, we obtain the following, where $w(x,i)$ is the wage paid to worker i:

$$\rho V(x,i) = w(x,i) + \rho V_D(x,i) + \rho V_I(x,i) + \rho V_F(x,i).$$

We note that the wage function $w(x,i)$ includes the transfers between worker i and the firm that may occur at the stage of vacancy posting (after separations and before the labor market opens), as discussed in Section A.3 in the context of the static example. These transfers can depend on the entire wage distribution inside the firm which is subsumed in the state vector x.

Direct events. We first characterize changes in value due to labor market shocks directly to worker i in firm x, $V_D(x,i)$. Exogenous separation shocks arrive at rate δ and draws of outside offers arrive at rate $\lambda^E(\theta)$ from the vacancy-weighted distribution of firms H_v. If worker i receives a sufficiently good outside offer from x', she quits to the new firm. We denote by $\mathcal{Q}^E(x,i)$ the set of such quit-firms x' for i. Otherwise, the worker remains with the current firm but with an updated contract. Therefore $V_D(x,i)$ satisfies

$$\rho V_D(x,i) = \underbrace{\delta\left[U - V(x,i)\right]}_{\text{Exogenous separation}} + \underbrace{\lambda^E(\theta)\int_{x'\in\mathcal{Q}^E(x,i)}\left[\boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) - V(x,i)\right]dH_v\left(x'\right)}_{EE\ \text{Quit}}$$
$$+ \underbrace{\lambda^E(\theta)\int_{x'\notin\mathcal{Q}^E(x,i)}\left[\boldsymbol{V}\left(r\left(x,i,x'\right),i\right) - V(x,i)\right]dH_v\left(x'\right)}_{\text{Retention}},$$

where $h_E(x,i,x')$ describes how the state of a poaching firm x' gets updated when it hires worker i from firm x. Similarly, $r(x,i,x')$ updates x when—after meeting firm x'—worker i in firm x is retained and renegotiates its value. In all functions with three arguments (x,i,x'), the first argument denotes the origin firm, the second identifies the worker, and the third the potential destination firm.

Indirect events. We next characterize changes in value due to the same labor market shocks hitting other workers in firm x, $V_I(x,i)$. The value $V_I(x,i)$ satisfies

$$\rho V_I(x,i) = \sum_{j\neq i}^{n(x)}\Bigg\{\underbrace{\delta\left[\boldsymbol{V}(d(x,j),i) - V(x,i)\right]}_{\text{Exogenous separation}} + \underbrace{\lambda^E(\theta)\int_{x'\in\mathcal{Q}^E(x,j)}\left[\boldsymbol{V}\left(q_E\left(x,j,x'\right),i\right) - V(x,i)\right]dH_v\left(x'\right)}_{EE\ \text{Quit}}$$
$$+ \underbrace{\lambda^E(\theta)\int_{x'\notin\mathcal{Q}^E(x,j)}\left[\boldsymbol{V}\left(r\left(x,j,x'\right),i\right) - V(x,i)\right]dH_v\left(x'\right)}_{\text{Retention}}\Bigg\},$$

where $d(x,j)$ updates x when worker j exogenously separates, and $q_E(x,j,x')$ when worker j quits to firm x'.

Firm events. Finally, we characterize changes in value due to events that directly impact the firm and hence indirectly its workers, $V_F(x,i)$. Taking as given the firm's vacancy posting policy $v(x)$ and other actions, $V_F(x,i)$ satisfies

$$
\begin{aligned}
\rho V_F(x,i) &= \\
UE\text{ Hire} \quad & \phi q(\theta) v(x) \left[\boldsymbol{V}(h_U(x),i) - V(x,i)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE\text{ Threat} \quad & +\phi q(\theta) v(x) \left[\boldsymbol{V}(t_U(x),i) - V(x,i)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE\text{ Hire} \quad & +(1-\phi) q(\theta) v(x) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E(x',i',x),i\right) - V(x,i)\right] dH_n(x',i') \\
EE\text{ Threat} \quad & +(1-\phi) q(\theta) v(x) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E(x',i',x),i\right) - V(x,i)\right] dH_n(x',i') \\
\text{Shock} \quad & +\Gamma_z[\boldsymbol{V},V](x,i)
\end{aligned}
$$

where $t_U(x)$ updates x when an unemployed worker is met and not hired, but could be possibly used as a threat in firm x. Similarly, $t_E(x',i',x)$ updates x when worker i' employed at firm x' is met, not hired, but could be used as a threat. And, with a slight abuse of notation, $H_n(x',i')$ gives the joint distribution of firms x' and worker types within firms i'.

Finally, $\Gamma_z[\boldsymbol{V},V](x,i)$ identifies the contribution of productivity shocks z to the Bellman equation. At this stage we only require that the productivity process is Markovian with an infinitesimal generator. Later we will specialize this to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ such that

$$
\begin{aligned}
\Gamma_z[\boldsymbol{V},V](x,i) &= \mu(z) \lim_{dz \to 0} \frac{\boldsymbol{V}((x,z+dz),i) - V(x,z,i)}{dz} \\
&+ \frac{\sigma^2(z)}{2} \lim_{dz \to 0} \frac{\boldsymbol{V}((x,z+dz),i) + \boldsymbol{V}((x,z-dz),i) - 2V(x,z,i)}{dz^2}
\end{aligned} \tag{30}
$$

In the case that $\boldsymbol{V} = V$, this becomes the standard expression for a diffusion featuring the first and second derivatives of V with respect to z: $\Gamma_z[V](x,i) = \mu(z)V_z(x,z,i) + \frac{1}{2}\sigma(z)^2 V_{zz}(x,z,i)$.[60]

In the event productivity changes or $n(x)$ changes because of exogenous labor market events, the worker will want to reassess whether to stay with the firm or not. Additionally, the firm may want to reassess whether to exit or fire some workers. Bold values $\boldsymbol{V}$ capture any case where the state changes.

[60] Note that in (30) we abuse notation and write the state as (x,z) with some redundancy since z is clearly a member of x. We also note that we are not constrained to a diffusion process. We could also consider a Poisson process where, at exogenous rate η, z jumps according to the transition density $\Pi(z,z')$: $\Gamma_z[\boldsymbol{V},V](x,i) = \eta[\sum_{z' \in Z} \boldsymbol{V}((x,z'),i)\,\Pi(z',z) - V(x,z,i)]$.

A.2 Firm value function: J

Consistent with the notation we used for workers' values, let $\boldsymbol{J}(x)$ and $J(x)$ be the values of the firm at the corresponding points of an interval dt. For now, we take the vacancy creation decision $v(x)$ as given. At the end of the section we describe the expected value of an entrant firm.

Stage I. Consistent with the first stage worker value function, we define the firm value before the exit/layoff/quit decision, where we recall that ϑ is the firm's value of exit, or scrap value:

$$\boldsymbol{J}(x) = \epsilon(x)\vartheta + [1-\epsilon(x)]\, J(s(x,\kappa(x))).$$

Stage II. Given a vacancy policy $v(x)$, let $J(x)$ be the value of a firm with state x *after* the layoff/quit, exit. It is convenient to split the value of the firm, as we did for the worker, into three components

$$\rho J(x) = \underbrace{y(x) - \sum_{i=1}^{n(x)} w_i(x,i)}_{\text{Flow profits}} \quad + \quad \underbrace{\rho J_W(x)}_{\text{Workforce events}} \quad + \quad \underbrace{\rho J_F(x) - c(v(x),x)}_{\text{Firm events net of vacancy costs}} \quad .$$

For a given policy $v(x)$ there is a set of associated transfers between workers and the firm which, as for the worker value function, are implicit in the wage function $w(x,i)$.

The component $J_W(x)$ is given by

$$\begin{aligned}
&\rho J_W(x) \quad = \\
&\text{Destruction} \qquad \delta \sum_{i=1}^{n(x)} \left[\boldsymbol{J}(d(x,i)) - J(x)\right] \\
&EE\ \text{Quit} \qquad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(q_E(x,i,x')\right) - J(x)\right] dH_v(x') \\
&\text{Retention} \qquad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(r(x,i,x')\right) - J(x)\right] dH_v(x').
\end{aligned}$$

The component $J_F(x)$ is given by

$$
\begin{aligned}
\rho J_F(x) &= \\
UE\text{ Hire} \quad & \phi q(\theta)v(x)\left[\boldsymbol{J}(h_U(x)) - J(x)\right]\cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threat} \quad & +\ \phi q(\theta)v(x)\left[\boldsymbol{J}(t_U(x)) - J(x)\right]\cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hire} \quad & +\ (1-\phi)\,q(\theta)v(x)\int_{x\in\mathcal{Q}^E(x',i')}\left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) - J(x)\right]dH_n\left(x',i'\right) \\
EE\text{ Threat} \quad & +\ (1-\phi)\,q(\theta)v(x)\int_{x\notin\mathcal{Q}^E(x',i')}\left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) - J(x)\right]dH_n\left(x',i'\right) \\
\text{Shock} \quad & +\Gamma_z\left[\boldsymbol{J},J\right](x)
\end{aligned}
$$

It is useful to recall that, in continuous time at most one contact is made per instant. That is, either one worker is exogenously separated, or one worker is contacted by another firm, or one worker is met by posting vacancies (at rate $q(\theta)v(x)$), or a shock hits the firm. Note also that we have bold $\boldsymbol{J}$'s in each line since after any of these events, the firm may want to layoff some workers or exit, and workers may want to quit.

Entry. The expected value of an entrant firm is

$$J_0 = -c_0 + \int \boldsymbol{J}(x_0)\,d\Pi_0(z_0) \tag{31}$$

where x_0 is the state of the entrant firm which includes only the random productivity value z_0 drawn from Π_0 since we assumed the initial number of workers is 0. The argument of the integral is $\boldsymbol{J}$, which incorporates the firm's decision to exit or operate after observing z_0. Entry occurs when $J_0 > 0$.

B Derivation of the joint value function Ω

We define the **joint value** of the firm and its employed workers $\Omega(x) := J(x) + \sum_{i=1}^{n(x)} V(x,i)$. We also define the joint value before exit/quit/layoff decisions: $\boldsymbol{\Omega}(x) := \boldsymbol{J}(x) + \sum_{i=1}^{n(x)} \boldsymbol{V}(x,i)$.

B.1 Combinining worker and firm values

In this section, we show that summing firm and worker values, then applying these definitions delivers the following Bellman equation for the joint value:

$$
\begin{aligned}
\rho\Omega(x) \quad &= \quad y(x) - c(v(x), x) \\
\text{Destruction} \quad &+ \sum_{i=1}^{n(x)} \delta\left[\boldsymbol{\Omega}(d(x,i)) + U - \Omega(x)\right] \\
\text{Retention} \quad &+ \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega(x)\right] dH_v\left(x'\right) \\
EE\text{ Quit} \quad &+ \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right), i\right) - \Omega(x)\right] dH_v\left(x'\right) \\
UE\text{ Hire} \quad &+ \phi q(\theta) v(x)\left[\boldsymbol{\Omega}\left(h_U(x)\right) - U - \Omega(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE\text{ Threat} \quad &+ \phi q(\theta) v(x)\left[\boldsymbol{\Omega}\left(t_U(x)\right) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE\text{ Hire} \quad &+ (1-\phi) q(\theta) v(x) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right), i'\right) - \Omega(x)\right] dH_n\left(x',i'\right) \\
EE\text{ Threat} \quad &+ (1-\phi) q(\theta) v(x) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x)\right] dH_n\left(x',i'\right) \\
\text{Shock} \quad &+ \Gamma_z\left[\boldsymbol{\Omega}, \Omega\right](x).
\end{aligned}
\tag{32}
$$

Note that this joint value is only written in terms of other joint values and worker values. However, it involves both firm and worker decisions through the sets $\mathcal{A}$, $\mathcal{Q}^E$ and the vacancy policy, $v(x)$.

Derivation. We start by computing the sum of the workers' values at a particular firm. Summing values of all the employed workers

$$\begin{aligned}
\rho \sum_{i=1}^{n(x)} V(x,i) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[U - V(x,i) \right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(r\left(x,i,x' \right), i \right) - V(x,i) \right] dH_v\left(x' \right) \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x' \right) \right) - V(x,i) \right] dH_v\left(x' \right) \\
\text{Incumbents} \quad & + \sum_{i=1}^{n(x)} \rho V_I(x,i) \\
\text{Firm} \quad & + \sum_{i=1}^{n(x)} \rho V_D(x,i)
\end{aligned}$$

where the indirect term due to incumbents can be written as:

$$\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_I(x,i) &= \\
\text{Destructions} \quad & \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \delta \left[\boldsymbol{V}\left(d(x,j), i \right) - V(x,i) \right] \\
\text{Retentions} \quad & + \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \lambda^E \int_{x' \notin \mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(r\left(x,j,x' \right), i \right) - V(x,i) \right] dH_v\left(x' \right) \\
EE \text{ Quits} \quad & + \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \lambda^E \int_{x' \in \mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(q_E\left(x,j,x' \right), i \right) - V(x,i) \right] dH_v\left(x' \right)
\end{aligned}$$

and the indirect term due to the firm can be written as:

$$
\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_F(x,i) &= \\
UE \text{ Hires} \quad & qv(x)\,\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(h_U(x),i\right) - V(x,i)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} \quad & +qv(x)\,\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(t_U(x),i\right) - V(x,i)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv(x)\,(1-\phi) \sum_{i=1}^{n(x)} \int_{x\in\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - V(x,i)\right] dH_n\left(x',i'\right) \\
EE \text{ Threats} \quad & +qv(x)\,(1-\phi) \sum_{i=1}^{n(x)} \int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - V(x,i)\right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & +\sum_{i=1}^{n(x)} \Gamma_z[\boldsymbol{V},V](x,i)
\end{aligned}
$$

We now collect terms.

Destructions. When worker i separates from firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$
\begin{aligned}
\text{Destructions} &= \delta\left[U - V(x,i)\right] + \delta \sum_{j\neq i}^{n(x)} \left[\boldsymbol{V}\left(d(x,i),j\right) - V(x,j)\right] \\
&= \delta\left[U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V(x,j)\right]
\end{aligned}
$$

Retentions. When i renegotiates at firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$
\begin{aligned}
\text{Retentions} &= \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) - V(x,i)\right] dH_v\left(x'\right) \\
&\quad +\lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \sum_{j\neq i}^{n(x)} \left[\mathbf{V}\left(r\left(x,i,x'\right),j\right) - V(x,j)\right] dH_v\left(x'\right) \\
&= \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V(x,j)\right] dH_v\left(x'\right) \\
&= \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\sum_{j=1}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V(x,j)\right] dH_v\left(x'\right)
\end{aligned}
$$

Quits. Similarly, when i quits firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$EE\text{ Quits} = \lambda^E \int_{x' \in Q(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right) \right] dH_v\left(x'\right)$$

Combining terms. Before summing up all these terms, define for convenience the total worker value:

$$\begin{aligned}
\rho \overline{V}\left(x\right) \quad &= \quad \sum_{i=1}^{n(x)} w\left(x,i\right) \\
\text{Destructions} \quad &+ \sum_{i=1}^{n(x)} \delta \left[U + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right) \right] \\
\text{Retentions} \quad &+ \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right) \right] dH_v\left(x'\right) \\
EE\text{ Quits} \quad &+ \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right) \right] dH_v\left(x'\right) \\
UE\text{ Hires} \quad &+ qv\left(x\right)\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(h_U\left(x\right),i\right) - V\left(x,i\right) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE\text{ Threats} \quad &+ qv\left(x\right)\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(t_U\left(x\right),i\right) - V\left(x,i\right) \right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE\text{ Hires} \quad &+ qv\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - V\left(x,i\right) \right] dH_n\left(x',i'\right) \\
EE\text{ Threats} \quad &+ qv\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - V\left(x,i\right) \right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad &+ \sum_{i=1}^{n(x)} \Gamma_z[\boldsymbol{V}, V](x,i)
\end{aligned}$$

Now sum, up all the previous terms, collect terms and use the definition of $\overline{V}(x)$:

$$
\begin{aligned}
\rho\overline{V}(x) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}(d(x,i),j) - \overline{V}(x) \right] \\
\text{Retentions} \quad & + \lambda^{E} \sum_{i=1}^{n(x)} \int_{x'\notin \mathcal{Q}^{E}(x,i)} \left[\sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \overline{V}(x) \right] dH_{v}\left(x'\right) \\
EE\ \text{Quits} \quad & + \lambda^{E} \sum_{i=1}^{n(x)} \int_{x'\in \mathcal{Q}^{E}(x,i)} \left[\boldsymbol{V}\left(h_{E}\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_{E}\left(x,i,x'\right),j\right) - \overline{V}(x) \right] dH_{v}\left(x'\right) \\
UE\ \text{Hires} \quad & + qv(x)\,\phi \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_{U}(x),i\right) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\ \text{Threats} \quad & + qv(x)\,\phi \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_{U}(x),i\right) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\ \text{Hires} \quad & + qv(x)(1-\phi) \int_{x\in \mathcal{Q}^{E}(x',i')} \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_{E}\left(x',i',x\right),i\right) - \overline{V}(x) \right] dH_{n}\left(x',i'\right) \\
EE\ \text{Threats} \quad & + qv(x)(1-\phi) \int_{x\notin \mathcal{Q}^{E}(x',i')} \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_{E}\left(x',i',x\right),i\right) - \overline{V}(x) \right] dH_{n}\left(x',i'\right) \\
\text{Shocks} \quad & + \Gamma_{z}[\overline{\boldsymbol{V}},\overline{V}](x)
\end{aligned}
$$

Adding this last equation to the Bellman equation for $J(x)$ yields

$$\begin{array}{rcl}
\rho\Omega(x) & = & y(x) - c(v(x), x) \\
\text{Destructions} & & + \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{J}(d(x,i)) + U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}(d(x,i),j) - J(x) - \overline{V}(x) \right] \\
\text{Retentions} & & + \lambda^E \sum_{i=1}^{n(x)} \int_{x'\notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - J(x) - \overline{V}(x) \right] dH_v\left(x'\right) \\
EE\text{ Quits} & & + \lambda^E \sum_{i=1}^{n(x)} \int_{x'\in \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - J(x) - \overline{V}(x) \right] dH_v\left(x'\right) \\
UE\text{ Hires} & & + qv(x)\phi \left[\boldsymbol{J}(h_U(x)) + \sum_{i=1}^{n(x)} \boldsymbol{V}(h_U(x), i) - J(x) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threats} & & + qv(x)\phi \left[\boldsymbol{J}(t_U(x)) + \sum_{i=1}^{n(x)} \boldsymbol{V}(t_U(x), i) - J(x) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hires} & & + qv(x)(1-\phi) \int_{x\in \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - J(x) - \overline{V}(x) \right] dH_n\left(x',i'\right) \\
EE\text{ Threats} & & + qv(x)(1-\phi) \int_{x\notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - J(x) - \overline{V}(x) \right] dH_n\left(x',i'\right) \\
\text{Shocks} & & + \Gamma_z[\boldsymbol{J} + \overline{\boldsymbol{V}}, J + \overline{V}](x) - J(x) - \overline{V}(x)
\end{array}$$

Collecting terms and using the definition of Ω :

$$\begin{array}{rcl}
\rho\Omega(x) & = & y(x) - c(v(x), x) \\
\text{Destructions} & & + \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{\Omega}(d(x,i)) + U - \Omega(x) \right] \\
\text{Retentions} & & + \lambda^E \sum_{i=1}^{n(x)} \int_{x'\notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega(x) \right] dH_v\left(x'\right) \\
EE\text{ Quits} & & + \lambda^E \sum_{i=1}^{n(x)} \int_{x'\in \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) - \Omega(x) \right] dH_v\left(x'\right) \\
UE\text{ Hires} & & + qv(x)\phi \left[\boldsymbol{\Omega}(h_U(x)) - U - \Omega(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threats} & & + qv(x)\phi \left[\boldsymbol{\Omega}(t_U(x)) - \Omega(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hires} & & + qv(x)(1-\phi) \int_{x\in \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right),i'\right) - \Omega(x) \right] dH_n\left(x',i'\right) \\
EE\text{ Threats} & & + qv(x)(1-\phi) \int_{x\notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x) \right] dH_n\left(x',i'\right) \\
\text{Shocks} & & + \Gamma_z[\overline{\boldsymbol{\Omega}}, \overline{\Omega}](x)
\end{array}$$

B.2 Value sharing

To make progress on (32), we begin by stating seven intermediate results, conditions **(C-RT)-(C-E)** which we prove from the assumptions listed in Section 2.2. These results establish how worker values $\boldsymbol{V}$ in (32) evolve in the six cases of hiring, retention, layoff, quits, exit and vacancy creation. Next, we apply conditions **(C-RT)-(C-E)** to (32).

To highlight the structure of the argument, we note a key implication our zero-sum game assumption **(A-IN)**: during internal negotiation, any value lost to one party must accrue to the other. This feature is obvious in the static model, and extends readily to our dynamic environment. In other words, the joint value of the firm plus its incumbent workers is invariant during the negotiation. We use this property extensively in the proof. This generalizes pairwise efficient bargaining—commonly used in one-worker firm models with linear production—to an environment with multi-worker firms and decreasing returns in production.

We now state the seven conditions that we apply to (32). In section B.3 below, we prove how each of them is implied by the assumptions of Section 2.2.

(C-RT) **Retentions and Threats.** First, if firm x meets an unemployed worker and the worker is not hired but only used as a threat, then the joint value of coalition x does not change since threats only redistribute value within the coalition. Second, when firm x uses employed worker i' from firm x' as a threat, the joint value of coalition x does not change. Third, when firm x meets worker i' at x' and the worker is retained by firm x', the joint value of coalition x' does not change. Formally,

$$\boldsymbol{\Omega}\left(r\left(x',i',x\right)\right)=\Omega(x') \quad , \quad \boldsymbol{\Omega}\left(t_U\left(x\right)\right)=\Omega(x) \quad , \quad \boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right)=\Omega(x).$$

Respectively, these imply that the *Retention*, *UE Threat* and *EE Threat* components of (32) are equal to zero.

(C-UE) *UE* **Hires.** An unemployed worker that meets firm x is hired when $x \in \mathcal{A}$. This set consists of firms that have a joint value after hiring that is higher than the pre-hire joint value plus the outside value of the hired worker. Due to the take-leave offer, the new hire receives her outside value, which is the value of unemployment:

$$\mathcal{A}=\{x|\boldsymbol{\Omega}(h_U(x))-\Omega(x)\geq U\} \quad , \quad \boldsymbol{V}\left(h_U\left(x\right),i\right)=U.$$

(C-EE) *EE* **Hires.** An employed worker i' at firm x' that meets firm x is hired when $x \in \mathcal{Q}^E\left(x',i'\right)$. This set consists of firms that have a higher marginal joint value than that of the current firm:

$$\mathcal{Q}^E(x', i') = \left\{ x \middle| \boldsymbol{\Omega}\left(h_E(x', i', x)\right) - \Omega(x) \geq \Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right) \right\}.$$

Due to the take-leave offer, the new hire receives her outside value, which is the marginal joint value at her current firm:

$$\boldsymbol{V}\left(h_E(x', i', x)\right) = \Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right).$$

(C-EU) *EU* **Quits and Layoffs.** An employed worker i at firm x quits to unemployment when $(x, i) \in \mathcal{Q}^U$. This set consist of states x such that the marginal joint value is less than the value of unemployment:

$$\begin{aligned} \mathcal{Q}^U &= \left\{ (x, i) \middle| \Omega\left(\widehat{s}_{q1}(x, i)\right) + U > \Omega\left(\widehat{s}_{q0}(x, i)\right) \right\}, \\ \text{where} \quad \widehat{s}_{q1}(x, i) &= s\left(x, (1 - [q_{U,-i}(x); q_{U,i}(x) = 1]) \circ (1 - \ell(x))\right), \\ \widehat{s}_{q0}(x, i) &= s\left(x, (1 - [q_{U,-i}(x); q_{U,i}(x) = 0]) \circ (1 - \ell(x))\right). \end{aligned}$$

The first expression captures when worker i quits, and the second where worker i does not. Similarly, an *EU* layoff will be chosen by the firm when $(x, i) \in \mathcal{L}$:

$$\begin{aligned} \mathcal{L} &= \left\{ (x, i) \middle| \Omega\left(\widehat{s}_{\ell 1}(x, i)\right) + U > \Omega\left(\widehat{s}_{\ell 0}(x, i)\right) \right\}, \\ \text{where} \quad \widehat{s}_{\ell 1}(x, i) &= s\left(x, (1 - [\ell(x); \ell_i(x) = 1]) \circ (1 - q_U(x))\right), \\ \widehat{s}_{\ell 0}(x, i) &= s\left(x, (1 - [\ell(x); \ell_i(x) = 0]) \circ (1 - q_U(x))\right). \end{aligned}$$

The first expression captures when worker i is laid off, and the second when worker i is not.

(C-X) **Exit.** A firm x exits when $x \in \mathcal{E}$. This set consists of the states in which the total outside value of the firm and its workers is larger than the joint value of operation:

$$\mathcal{E} = \left\{ x \middle| \vartheta + n\left(s\left(x, \kappa(x)\right)\right) \cdot U > \Omega\left(s\left(x, \kappa(x)\right)\right) \right\}.$$

(C-V) **Vacancies.** The expected return to a matched vacancy $R(x)$ depends only on the joint value, and so the firm's optimal vacancy policy $v(x)$ depends only on the joint value. The policy $v(x)$ solves

$$\max_v \; q(\theta) v R(x) - c(v, x),$$

where the expected return to a matched vacancy is

$$R(x) = \phi \underbrace{\left[\boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}}_{\text{Return from unemployed worker match}}$$

$$+ \quad (1-\phi) \underbrace{\int_{x \in \mathcal{Q}^E(x',i')} \left\{ \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right] \right\} dH_n\left(x',i'\right)}_{\text{Expected return from employed worker match}}.$$

(C-E) Entry. A firm enters if and only if

$$\int \boldsymbol{\Omega}\left(x_0\right) d\Pi_0(z) \geq c_0 + n_0 U.$$

Summarizing (C). The substantive result is that all firm and worker decisions and employed workers' values can be expressed in terms of joint value Ω and exogenous worker outside option U.

B.3 Proof of Conditions (C)

B.3.1 Proof of C-UE and C-RT (UE Hires and UE Threats)

In this subsection, we consider a meeting between a firm x and an unemployed worker. Following **A-IN** and **A-EN**, the firm internally renegotiates according to a zero-sum game with its incumbent workers and makes a take-leave offer to the new worker. Intuitively, having the worker "at the door" is identical to having her hired at value U for the firm and for all incumbent workers: the firm can always make new take-leave offers to its incumbents after hiring the new worker. Hence, we expect the firm to make one take-leave offer to the new worker and its incumbents at the time of the meeting, and not make a new, different offer to is incumbents afer hiring has taken place.

We start by showing this equivalence formally. To do so, when meeting an unemployed worker, we let the firm conduct internal renegotiation with its incumbent workers and make an offer to the new worker. Then, we let a second round of internal offers take place after the hiring. We introduce some notation to keep track of values throughout the internal and external negotiations. To fix ideas, we denote by (IR1) the first round of internal negotiation, pre-external negotiation. We denote by (IR2) the second round of internal negotiation, post-hire.

Post-hire and post-internal negotiation (IR2) values are denoted with double stars. Post-internal-negotiation (IR1) but pre-external-negotiation values are denoted with stars.

$$\Omega^{**} := J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**}$$

$$\Omega^{*} := J^{*} + \sum_{j=1}^{n(x)} V_j^{*}$$

$$\Omega := J + \sum_{j=1}^{n(x)} V_j$$

Proceeding by backward induction, under **A-EN** the firm makes a take-it-or-leave-it offer to the unemployed worker, therefore

$$V_i^{**} = U$$

We now divide the proof in several steps. We start by proving that for all incumbent workers $j = 1...n(x)$, $V_j^{**} = V_j^{*}$. We then use **A-IN** to argue that $\Omega^* = \Omega$. Once these claims have been proven, we move on to proving **C-UE** (UE Hires) and the part of for threats from unemployment **C-RT** (UE Threats). Finally, we show that our microfoundations for the renegotiation game deliver **A-IN**.

Claim 1: For all incumbents workers $j = 1...n(x)$, we have $V_j^{**} = V_j^{*}$.

We proceed by backwards induction using our assumptions **A-EN** and **A-IN**. Immediately after (IR1) has taken place, only the following events can happen:

1. Hire/not-hire
   - Either the worker is hired from unemployment (H),
   - Or the worker is not hired from unemployment (NH)
2. Possible new round of internal negotiation (IR2). This possible second round of internal negotiation (now including the newly hired worker) leads to values V_j^{**}.

We focus on subgame perfect equilibria in this multi-stage game. Therefore, after (IR1), workers perfectly anticipate what the outcome of the hire/not-hire stage will be. That is, after (IR1), they know perfectly what hiring decision (H or NH) the firm will make. Now suppose that internal renegotiation (IR2) actually happens after the hire/not-hire decision, that is, that for some incumbent worker $j \in \{1, ..., n(x)\}$, $V_j^{**} \neq V_j^{*}$. Note that the firm has no incentives to accept a change in the new worker's value to anything above U, so by **A-MC** her value does not change in the second round (IR2).

We construct the rest of the proof by contradiction. Consider for a contradiction an incumbent worker j whose value changed in (IR2). Because of **A-MC**, her value can change only in the following cases:

- The firm has a credible threat to fire worker j, in which case $V_j^{**} < V_j^*$
- Worker j has a credible threat to quit, in which case $V_j^{**} > V_j^*$

In addition, those credible threats can lead to a different outcome than in (IR1), and thus $V_j^{**} \neq V_j^*$, only if the threat on either side was not available in (IR1). If that same threat was available in the first round (IR1), then the outcome of the bargaining (IR1) would have been V_j^{**}.

Recall that both incumbent worker j and the firm understand and anticipate which hire/not-hire decision the firm will make after the first round (IR1). They also understand and anticipate that, in case of hire, the value of the new worker will remain U in the second round (IR2).

Therefore, the firm can *credibly threaten* to hire the new worker in the first round *if and only if* it actually hires her after the first round (IR1) is over. This implies that the firm can credibly threaten worker to fire j in the second round (IR2), by **A-LC**, *if and only if* it could credibly threaten her with hiring the new worker *in the first round of internal renegotiation (IR1).* This in turn entails that any credible threat the firm can make in the second round (IR2) was already available in the first round.

On the worker side, quitting into unemployment is a credible threat when her value is below the value of unemployment. So this threat does not change between the first round (IR1) and the second round (IR2), because the equilibrium value to that worker will always be above the value of unemployment.

In sum, the set of credible threats both to the firm and to worker j does not change between the initial round of internal renegotiation (IR1) and the post-hiring-decision round (IR2). This finally implies that the outcome of the initial round of internal renegotiation (IR1) for any incumbent j remains unchanged in the second round (IR2), that is:

$$V_j^{**} = V_j^*$$

which proves **Claim 1**.

We can now move on to proving **C-UE**.

Proof of C-UE. Using the definitions of Ω^{**} and Ω, we can write

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**} \right] - \left[J + \sum_{j=1}^{n(x)} V_j \right]$$

Now using $V_i^{**} = U$, we obtain

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(x)} V_j^{**} \right] - \left[J + \sum_{j=1}^{n(x)} V_j \right] + U$$

Using **Claim 1:** $V_j^{**} = V_j^{*}$, and adding and subtracting J^* we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(x)} V_j^* \right] - \left[J + \sum_{j=1}^{n(x)} V_j \right] + U$$

Subsituting in the definition of Ω and of Ω^*,

$$\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] + U$$

Finally recall that internal renegotiation is (1) individually rational, and (2) is a zero-sum game, according to **A-IN**. Thus, all incumbent workers remain in the coalition after internal renegotiation, and the joint value is unchanged: $\Omega^* = \Omega$. Using $\Omega^* = \Omega$

$$\Omega^{**} - \Omega = [J^{**} - J^*] + U$$

which can be re-written

$$J^{**} - J^* = [\Omega^{**} - \Omega] - U$$

Now under **A-LC**, the firm will only hire if its value after hiring is higher than its value after internal renegotiation: $J^{**} - J^* \geq 0$. This inequality requires

$$\Omega^{**} - \Omega \geq U$$

$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U$$

The firm does not hire when its value of hiring is below its value of renegotiation $J^{**} < J^*$. This inequality implies

$$\Omega^{**} - \Omega < U$$

When the firm does not hire, we obtain using again **A-IN** and $\Omega^* = \Omega$:

$$\Omega^{**} - \Omega^* < U$$

which finally implies

$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \mathbf{\Omega}\left(t_U(x)\right) < U$$

Now, we argue that conditional on not hiring, $\Omega^{**} = \Omega^{*} = \Omega$, where in this case Ω^{**} denotes the value of the coalition without hiring, and thus does not include the value of the unemployed worker. Just as before, this is a direct consequence from **A-IN** and that the internal renegotiation game is zero-sum.

Therefore:

$$\boldsymbol{\Omega}\left(t_U(x)\right) = \Omega(x)$$

We have therefore shown **C-UE** and part of **C-RT (*UE* Hires and *UE* Threats)**: An unemployed worker that meets x is hired when$x \in \mathcal{Q}^U$, where

$$\mathcal{A} = \left\{x \middle| \boldsymbol{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U\right\}$$

and upon joining the firm, has value

$$\boldsymbol{V}\left(h_U\left(x,i\right)\right) = U.$$

and

$$\boldsymbol{\Omega}(t_U(x)) = \Omega(x).$$

B.3.2 Proof of C-EE and C-RT (*EE* Hires, *EE* Threats and Retentions)

Consider firm x that has met worker i' at firm x'. We first seek to determine $\mathcal{Q}^E\left(x', i'\right)$. Under **A-IN** and **A-EN**, upon meeting an employed worker, internal negotiation may take place at the poaching firm x, and x makes a take-it-or-leave-it offer. Internal negotiation may take place at x' with all workers including i'.

Proceeding by backward induction, we again define intermediate values but here at x', noting that $q_E\left(x', i', x\right)$ gives the number of employees in x' if the worker leaves:

$$\Omega = J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}$$

$$\Omega^{*} = J^{*} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{*} + V_{i'}^{*}$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}$$

Note, in the second line we are describing the values of the firm in renegotiation where i' stays with the firm, so $V_{i'}^{*}$ is the outcome of internal negotiation. In the third line we consider the firm having lost the worker. Under **A-EN** the firm will respond to an offer $\overline{V}$ from x with

$$V_{i'}^{*} = \overline{V}$$

The same result as in **Claim 1** from section B.3.1 obtains: under **A-EN** and **A-IN**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)_j$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)_j$, that is

$$V_j^{**} = V_j^*$$

The argument are exactly the same.

Using these two results and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] \\
&= \left[J^{**} + J^* - J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**} + V_{i'}^* - V_{i'}^*\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] \\
&= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^*\right] - \left[J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'}\right] - V_{i'}^* \\
&= [J^{**} - J^*] + [\Omega^* - \Omega] - V_{i'}^* \\
&= [J^{**} - J^*] + [\Omega^* - \Omega] - \overline{V}
\end{aligned}$$

In this setup, **A-IN** again implies that any value lost to the firm must accrue to its workers, while any value lost to a worker must accrue either to the firm, or to another worker, which we earlier formulated as *"the joint value stays constant before and after an internal negotiation"*. Mathematically, this statement translates into

$$\Omega^* = \Omega$$

Subsituting into the equation that we obtained above $\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] - \overline{V}$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \overline{V}$$

Now under **A-LC**, the firm x' will only try to keep the worker if $J^* > J^{**}$, which requires

$$\begin{aligned}
\Omega - \Omega^{**} &\leq \overline{V} \\
\mathbf{\Omega}\left(r(x', i', x) - \mathbf{\Omega}\left(q_E\left(x', i', x\right)\right) \right. &\leq \overline{V}
\end{aligned}$$

This determined the maximum value that x' can offer to the worker to retain them. Knowing that firm x' can counter at most with $\overline{V} = \mathbf{\Omega}\left(r(x', i', x) - \mathbf{\Omega}\left(q_E\left(x', i', x\right)\right)\right.$, then will firm x successfully poach the worker?

First, note that the bargaining protocol implies that x firm will offer $\overline{V}$ if it is making an offer, since it need not offer more. For firm x the argument may proceed identically to the case of unemployment, simply replacing U with $\overline{V}$. The result is that the firm will hire only if

$$\boldsymbol{\Omega}\left(h_E\left(x', i', x\right)\right) - \Omega\left(x\right) \geq \overline{V}$$

or

$$\boldsymbol{\Omega}\left(h_E\left(x', i', x\right)\right) - \Omega\left(x\right) \geq \boldsymbol{\Omega}\left(r(x', i', x)\right) - \boldsymbol{\Omega}\left(q_E\left(x', i', x\right)\right)$$

Finally, when firm x does not hire, the same argument as in **Claim 32** in Section B.3.1 applies: $\Omega^{**} = \Omega^{*} = \Omega$. This observation implies

$$\boldsymbol{\Omega}(t_E(x', i', x)) = \Omega(x)$$

Similarly, the same argument as in **Claim 2** implies that when firm x' does not lose its worker, $\Omega^{**} = \Omega^{*} = \Omega$, thereby implying

$$\boldsymbol{\Omega}(r(x', i', x)) = \Omega(x')$$

The combination of these conditions deliver **C-UE** and part of **C-RT** (*EE* Hires, *EE* Threats and Retention):

1. The quit set of an employed worker is determined by

$$\mathcal{Q}^E\left(x', i'\right) = \left\{ x \middle| \boldsymbol{\Omega}\left(h_E\left(x', i', x\right)\right) - \Omega\left(x\right) \geq \Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x', i', x\right)\right) \right\}$$

2. The worker's value of being hired from employment from firm x' is

$$\boldsymbol{V}(h_E(x, x', i')) = \Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x', i', x\right)\right)$$

3. Worker i''s value of being retained at x' after meeting x is[61]

$$\boldsymbol{V}(r(x', i', x), i') = \boldsymbol{\Omega}\left(h_E\left(x', i', x\right)\right) - \Omega\left(x\right)$$

4. The joint value of the potential poaching firm x when the worker is not hired does not change:

$$\boldsymbol{\Omega}(t_E(x', i', x)) = \Omega(x)$$

[61] Because offers are made at no cost, both firms always make an offer, even when they know that they cannot retain/hire the worker in equilibrium. This is exactly the same as in Postel-Vinay Robin (2002).

5. The joint value of the potential poached firm x' does not change when the worker stays:

$$\Omega(r(x', i', x)) = \Omega(x')$$

B.3.3 Proof of C-EU (*EU* Quits and layoffs)

We first show that

$$\mathcal{L} = \Big\{(x,i)\Big|\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)+U \\ > \Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)\Big\}$$

from the firm side, then that

$$\mathcal{Q}^U = \Big\{(x,i)\Big|\Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right),i\right)+U \\ > \Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=0\right]\right)\right),i\right)\Big\}$$

on the worker side.

Part 1: Firm side Consider a firm x who is considering laying off worker i for whom $q_{U,i}(x) = 0$. As above, we start with definitions, noting that $n\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right)\right)$ is the number of workers if i is laid off.

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

Note that in the first line the coalition has still worker i in it. In the second line, the firm and the worker i have negotiated (and internal negotiation has determined V_i^* which is what i will get if they stay in the firm). In the third line, the worker has been fired and another round of negotiation has occurred among incumbents.

The same result as in **Claim 1** from section B.3.1 obtains: under **A-BP**, the values accepted by the incumbent workers *after the internal renegotiation* $\left(V_j^*\right)$ will be equal to the values they receive *after the external negotiation* $\left(V_j^{**}\right)$, that is $V_j^{**} = V_j^*$.

Using this result and the above definitions

$$
\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} - J^* + J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* - V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] - V_i^* \\
&= [J^{**} - J^*] + [\Omega^* - \Omega] - V_i^*
\end{aligned}
$$

Using again **A-IN** to conclude that $\Omega^* = \Omega$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

Now under **A-LC**, the firm x will only layoff the worker if $J^{**} > J^*$, which requires

$$\Omega - \Omega^{**} < V_i^*$$

As long as $V_i^* > U$ the worker would be willing to transfer value to the firm to avoid being laid off, implying

$$\Omega - \Omega^{**} < U.$$

which we can re-write

$$\Omega\left(s\left(x, (1 - [\ell(x); \ell_i(x) = 1]) \circ (1 - q_U(x))\right), i\right) + U > \Omega\left(s\left(x, (1 - [\ell(x); \ell_i(x) = 0]) \circ (1 - q_U(x))\right), i\right)$$

where the LHS is $\Omega^{**} + U$ (under the layoff) and the RHS is Ω. This concludes the proof for the firm side.

Part 2: Worker side Consider worker i in firm x who is considering quitting to unemployment for whom $\ell_i(x) = 0$. As above, we start with definitions, noting that

$n\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right)\right)$ is the number of workers if i quits. As before,

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*$$

$$\Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

The same result as in **Claim 1** from section B.3.1 obtains $V_j^{**} = V_j^*$.

Using this result and the above definitions

$$\begin{aligned}
\Omega^{**} - \Omega &= \left[J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} + J^* - J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* - V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] \\
&= \left[J^{**} - J^*\right] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i\right] - V_i^* \\
&= \left[J^{**} - J^*\right] + \left[\Omega^* - \Omega\right] - V_i^*
\end{aligned}$$

Again, $\Omega^* = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = \left[J^{**} - J^*\right] - V_i^*$$

Now under **A-LC**, worker i will quit into unemployment iff $V_i^* < U$, which requires

$$J^{**} - J^* + \left[\Omega - \Omega^{**}\right] < U$$

As long as $J^{**} < J^*$, the firm is willing to transfer value to worker i to retain her. Therefore, worker i quits into unemployment iff the previous inequality holds at $J^{**} = J^*$, i.e.

$$\Omega - \Omega^{**} < U$$

Therefore, the worker quits iff

$$\begin{aligned}
&\Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=1\right]\right)\right),i\right) + U \\
&> \Omega\left(s\left(x,\left(1-\ell\left(x\right)\right)\circ\left(1-\left[q_{U,-i}\left(x\right);q_{U,i}\left(x\right)=0\right]\right)\right),i\right)
\end{aligned}$$

which concludes the proof of the worker side. This delivers **C-EU**.

B.3.4 Proof of C-X (Exit)

Consider a firm x who contemplates exit after all endogenous quits and layoffs, thus when its employment is $n\left(s\left(x,\kappa\left(x\right)\right)\right)$. As before we define values conditional on exiting:

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j$$

$$\Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^*$$

$$\Omega^{**} = J^{**} + 0$$

Notice that the joint value after exit is simply the value of the firm, since all other workers have left because of exit. We can compute:

$$\begin{aligned}
\Omega^{**} - \Omega &= J^{**} - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j\right] \\
\text{(add and subtract } J^*) &= [J^{**} - J^*] + J^* - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j\right] \\
\text{(add and subtract } \sum_{j=1}^{n(s(\cdot))} V_j^*) &= [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(s(\cdot))} V_j^*\right] - \left[J + \sum_{j=1}^{n(s(\cdot))} V_j\right] - \sum_{j=1}^{n(s(\cdot))} V_j^* \\
\text{(definition of } \Omega, \Omega^*) &= [J^{**} - J^*] + [\Omega^* - \Omega] - \sum_{j=1}^{n(s(\cdot))} V_j^*
\end{aligned}$$

Again, $\Omega^* = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

The firm exits iff $J^{**} \geq J^*$, that is, $\vartheta \geq J^*$. This is equivalent to

$$\Omega^{**} - \Omega \geq - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

Using again that $\Omega^{**} = J^{**} = \vartheta$, the firm exits iff

$$\vartheta + \sum_{j=1}^{n(s(\cdot))} V_j^* \geq \Omega$$

Since any worker is better off under $V_i^* \geq U$ than unemployed, all workers are willing to take a value cut down to U if $\vartheta \geq \Omega - \sum_{j=1}^{n(s(\cdot))} V_j^*$ because then the firm can credibly exit.

This implies that the firm exits if and only if

$$\vartheta - \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) + n\left(s\left(x, \kappa\left(x\right)\right)\right) U \geq 0$$

This proves **C-X (Exit)**: the set of x such that the firm exits is given by

$$\mathcal{E} = \left\{ x \middle| \vartheta + n\left(s\left(x, \kappa\left(x\right)\right)\right) \cdot U \geq \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) \right\}$$

B.3.5 Proof of C-V (Vacancies)

We split the proof in two steps. First, we show that workers are collectively willing to transfer value to the firm in exchange for the joint value-maximizing vacancy policy function. Second, we show that a single worker can create a system of transfers that achieves the same outcome. These transfers are equivalent to wage renegotiation, which explains why we have subsumed them in the wage function $w(x, i)$ in the equations above. Similarly to wages, these transfers drop out from the expression for the joint value.

Part 1: Collective transfers In this step, we show that workers are collectively better off transferring value to the firm in exchange of the firm posting the joint value-maximizing amount of vacancies.

The vacancy posting decision v^J that maximizes firm value is:

$$\frac{c_v\left(v^J\left(x\right), n\left(x\right)\right)}{q} = \phi\left[\boldsymbol{J}\left(h_U\left(x\right)\right) - J\left(x\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} + (1-\phi) \int_{x \in \mathcal{Q}^E(x', i')} \left[\boldsymbol{J}\left(h_E\left(x', i', x\right)\right) - J\left(x\right)\right] dH_n\left(x', i'\right).$$

Similarly, define v^Ω be the policy that maximizes the value of the coalition, and $v^{\overline{V}}$ be the policy that maximizes the value of all the employees. Let Ω^γ,J^γ, $\overline{V}^\gamma$ be the value of the coalition, firm and all workers under the v^γ, for $\gamma \in \left\{\Omega, J, \overline{V}\right\}$. We now prove our claim in several steps.

Part 1-(a) Collective value gains. The policy v^Ω will lead to $\overline{V}^\Omega \geq \overline{V}^J + [J^J - J^\Omega]$ where $J^J - J^\Omega \geq 0$.

Proof: By construction Ω^Ω is greater than Ω^J: $\Omega^\Omega \geq \Omega^J$. By definition: $\Omega^\Omega = J^\Omega + \overline{V}^\Omega$, and $\Omega^J = J^J + \overline{V}^J$. Use those definitions to obtain inequality $J^\Omega + \overline{V}^\Omega \geq J^J + \overline{V}^J$, which can be re-arranged into $\overline{V}^\Omega - \overline{V}^J \geq J^J - J^\Omega$. Since J^J is the value under the optimal policy for J, then $J^J \geq J^\Omega$. The above then implies that

$$\overline{V}^\Omega - \overline{V}^J \geq J^J - J^\Omega \geq 0$$

This implies that workers would be prepared to transfer $T = J^J - J^\Omega \geq 0$ to the firm in order for the firm to pursue policy v^Ω instead of v^J. This concludes the proof of **Part 1-(a).**

Part 1-(b) Infeasibility of $\overline{V}^{\overline{V}}$. There does not exist an incentive-compatible transfer from workers to firm that will lead to $\overline{V}^{\overline{V}}$.

Proof: Suppose workers consider transferring even more to induce the firm to follow policy $v^{\overline{V}}$ that maximizes their value. By construction $\Omega^\Omega \geq \Omega^{\overline{V}}$. Using definitions for each of these, then $J^\Omega + \overline{V}^\Omega \geq J^{\overline{V}} + \overline{V}^{\overline{V}}$. Rearranging this: $J^\Omega - J^{\overline{V}} \geq \overline{V}^{\overline{V}} - \overline{V}^\Omega$. Since $\overline{V}^{\overline{V}}$ is the value under the optimal policy for $\overline{V}$, then $\overline{V}^{\overline{V}} \geq \overline{V}^\Omega$. The above then implies that

$$J^\Omega - J^{\overline{V}} \geq \overline{V}^{\overline{V}} - \overline{V}^\Omega \geq 0$$

Taking v^Ω as a baseline, the above implies that a change to $v^{\overline{V}}$ causes a loss of $J^\Omega - J^{\overline{V}}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}} - \overline{V}^\Omega$ to the workers. This implies that workers could transfer all of their gains under $v^{\overline{V}}$ to the firm, but the firm would still not choose $v^{\overline{V}}$ over v^Ω. This concludes the proof of **Part 1-(b).**

Part 1-(c) Optimality of $\overline{V}^\Omega$. There does not exist an incentive-compatible transfer from workers to firm that will lead to $\overline{V}^* \in \left(\overline{V}^\Omega, \overline{V}^{\overline{V}}\right)$.

Proof: Call such a policy $v^{\overline{V}*}$. Then: $\Omega^\Omega \geq \Omega^{\overline{V}*}$, and by definitions

$$\begin{aligned} J^\Omega + \overline{V}^\Omega &\geq J^{\overline{V}^*} + \overline{V}^{\overline{V}^*} \\ J^\Omega - J^{\overline{V}^*} &\geq \overline{V}^{\overline{V}^*} - \overline{V}^\Omega \end{aligned}$$

Since by definition $\overline{V}^* \in \left(\overline{V}^\Omega, \overline{V}^{\overline{V}}\right)$, then $\overline{V}^{\overline{V}^*} - \overline{V}^\Omega \geq 0$. Therefore

$$J^\Omega - J^{\overline{V}^*} \geq \overline{V}^{\overline{V}^*} - \overline{V}^\Omega \geq 0$$

Taking v^Ω as a baseline, the above implies that a change to $v^{\overline{V}^*}$ causes a loss of $J^\Omega - J^{\overline{V}^*}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}^*} - \overline{V}^\Omega$ to the workers. This concludes the proof of **Part 1-(c).**

Part 1-(d) Conclusion. In summary, it is optimal for workers to transfer exactly $T = J^J - J^\Omega$ to the firm, in order for the firm to pursue v^Ω instead of v^J. Further transfers to the firm would be required to have the firm pursue a better policy for workers, but this is exceedingly costly to the firm and the workers are unwilling to make a transfer to cover these costs. This concludes the proof of **Step 1: Collective transfers.**

Part 2: Individual transfers In this step, we show that a single, randomly drawn worker can construct a system of transfers that induces the firm to post v^{Ω} instead of v^{J}, while leaving all agents better off.

Within dt, consider the single, randomly drawn worker j_0. Consider the following system of transfers. Worker j_0 makes a transfer $J^J - J^{\Omega}$ to the firm, in exchange of what (i) the firm posts v^{Ω} instead of v^J, and (ii) the worker gets a wage increase that gives her all the differential surplus $\overline{V}^{\Omega} - \overline{V}^{J}$.

Following the same steps as in **Part 1: Collective transfers**, the firm gets $J^{\Omega} + [J^J - J^{\Omega}] = J^J$ and is hence indifferent. Similarly, workers $j \neq j_0$ do not get any value change, and are thus indifferent Finally, worker j_0 gets a value increase of

$$[\overline{V}^{\Omega} - \overline{V}^{J}] - [J^J - J^{\Omega}] \geq 0$$

where the inequality similarly follows from **Part 1: Collective transfers**. This concludes the proof of **Part 2: Individual transfers**.

Conclusion. The previous arguments show that a single worker has an incentive to and can induce the firm to post v^{Ω}. Notice also that the same argument holds starting from any vacancy policy function $\widetilde{v} \neq v^J$ together with a value of the firm $\widetilde{J}$. Thus, even if some worker induces the firm to post a different vacancy policy function which is not v^{Ω} any other worker has an incentive to induce the firm to post v^{Ω}. Therefore, in equilibrium, the firm posts v^{Ω}, which concludes the proof of **C-V**.

B.4 Applying Conditions (C)

Having established that **Assumption (A)** can be used to prove **Conditions (C)**, we now apply conditions **(C)** to the Bellman equation for the joint value. The goal of this section is to show that for $x \in \mathcal{E}^c$ the complement of the exit set, we can considerably simplify the recursion for the joint value:

$$\begin{aligned}
\rho\Omega(x) &= y(z(x), n(x)) - c(v(x), n(x), z(x)) \\
\text{Destructions} \quad & -\delta \sum_{i=1}^{n(x)} [\Omega(x) - \mathbf{\Omega}(d(x,i)) - U] \\
UE \text{ Hires} \quad & +qv(x)\phi[\mathbf{\Omega}(h_U(x)) - \Omega(x) - U] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv(x)(1-\phi)\int_{x\in\mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E(x',i',x)\right) - \Omega(x)\right] - \left[\Omega(x') - \mathbf{\Omega}\left(q_E(x',i',x)\right)\right]\right] dH_n(x',i') \\
\text{Shocks} \quad & +\Gamma[\mathbf{\Omega}, \Omega]
\end{aligned}$$

with the sets

$$\mathcal{Q}^U = \Big\{ (x,i) \Big| \Omega\left(s\left(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x) ; q_{U,i}(x) = 1])\right), i\right) + U$$
$$> \Omega\left(s\left(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x) ; q_{U,i}(x) = 0])\right), i\right) \Big\}$$
$$\mathcal{L} = \Big\{ (x,i) \Big| \Omega\left(s\left(x, (1-[\ell(x) ; \ell_i(x) = 1]) \circ (1 - q_U(x))\right), i\right) + U$$
$$> \Omega\left(s\left(x, (1-[\ell(x) ; \ell_i(x) = 0]) \circ (1 - q_U(x))\right), i\right) \Big\}$$
$$\mathcal{E} = \Big\{ x \Big| \vartheta + n\left(s(x, \kappa(x))\right) \cdot U \geq \Omega(s(x, \kappa(x))) \Big\}$$
$$\mathcal{A} = \Big\{ x \Big| \boldsymbol{\Omega}\left(h_U(x)\right) - \Omega(x) \geq U \Big\}$$
$$\mathcal{Q}^E(x', i') = \Big\{ x \Big| \boldsymbol{\Omega}\left(h_E(x', i', x)\right) - \Omega(x) \geq \Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right) \Big\}$$

and—as per **(C-V)**—the vacancy policy $v(x)$ is given by the solution to the following:

$$\begin{aligned} \frac{c_v(v(x), n(x))}{q} &= \phi\left[\boldsymbol{\Omega}\left(h_U(x)\right) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\ &+ (1-\phi) \int_{x \in \mathcal{Q}^E(x', i')} \left[\left[\boldsymbol{\Omega}\left(h_E(x', i', x)\right) - \Omega(x)\right] - \left[\Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right)\right]\right] dH_n(x', i') \end{aligned}$$

In continuous time, the exit decision is captured by $x \in \mathcal{E}$. The Bellman equation above holds exactly for $x \in \mathcal{E}^c$. Exit is accounted for in the "bold" continuation values, which all include the possible exit decision should the firm's state fall into $\mathcal{E}$ after an event.

We first proceed one term at the time, working through (B.4.1) exogenous destructions, (B.4.2) retentions, (B.4.3) *EE* (poached) quits, (B.4.4) *UE* hires, (B.4.5) *UE* threats, (B.4.6) *EE* (poached) hires, and (B.4.7) *EE* threats.

B.4.1 Exogenous destructions

$$\begin{aligned} \text{Destructions} &= \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \boldsymbol{V}\left(d(x,i), j\right) + U - \Omega(x) \right] \\ &= \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{\Omega}\left(d(x,i)\right) + U - \Omega(x) \right] \end{aligned}$$

where we simply have used the definition $\boldsymbol{\Omega}\left(d(x,i)\right) := \boldsymbol{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \boldsymbol{V}\left(d(x,i), j\right)$.

B.4.2 Retentions

$$\begin{aligned}\text{Retentions} &= \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \Omega\left(x\right) \right] dH_v\left(x'\right) \\ &= \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega\left(x\right) \right] dH_v\left(x'\right)\end{aligned}$$

where we simply have used the definition $\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) = \boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right)$. Now using the result in **C-RT** that

$$\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) = \Omega(x')$$

we obtain that

$$\text{Retentions} = 0$$

B.4.3 *EE* Quits

$$EE\text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(q_E\left(x,i,x'\right),i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \Omega\left(x\right) \right] dH_v\left(x'\right)$$

Now by definition

$$\begin{aligned}\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) &= \boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j=1}^{n(q_E(x,i,x'))} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) \\ &= \boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right)\end{aligned}$$

Using this last equality in the term in square brackets

$$EE\text{ Quits} = \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\Omega\left(q_E\left(x,i,x'\right)\right) - \Omega\left(x\right) + \boldsymbol{V}\left(q_E\left(x,i,x'\right),i\right) \right] dH_v\left(x'\right)$$

Using **C-EE**, the value going to the poached worker is $\boldsymbol{V}\left(q_E\left(x,i,x'\right)\right) = \Omega\left(x\right) - \Omega\left(q_E\left(x,i,x'\right)\right)$. Substituting this into the last equation, we observe that the term in the square brackets is zero, and so

$$EE\text{ Quits} = 0$$

B.4.4 *UE* Hires

$$UE\text{ Hires} = qv\left(x\right)\phi \left[\boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U\left(x\right),i\right) - \Omega\left(x\right) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

Now by definition

$$\begin{aligned}\mathbf{\Omega}\left(h_U\left(x\right)\right) &= \boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(h_U(x))} \boldsymbol{V}\left(h_U\left(x\right), i\right) \\ &= \boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U\left(x\right), i\right) + \boldsymbol{V}\left(h_U\left(x\right), i\right)\end{aligned}$$

and so, re-arranging,

$$\boldsymbol{J}\left(h_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U\left(x\right), i\right) = \mathbf{\Omega}\left(h_U\left(x\right)\right) - \boldsymbol{V}\left(h_U\left(x\right), i\right)$$

Substituting this last equation into the term in the square brackets of the first equation,

$$UE \text{ Hires} = qv\left(x\right)\phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - \boldsymbol{V}\left(h_U\left(x\right), i\right)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}}$$

Following **C-UE**, the value going to the hired worker is $\boldsymbol{V}\left(h_U\left(x\right), i\right) = U$. Substituting in:

$$UE \text{ Hires} = qv\left(x\right)\phi\left[\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) - U\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}}$$

B.4.5 *UE* Threats

$$UE \text{ Threats} \quad = \quad qv\left(x\right)\phi\left[\boldsymbol{J}\left(t_U\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_U\left(x\right), i\right) - \Omega\left(x\right)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}}$$

Using the definition of $\mathbf{\Omega}(t_U(x))$, we can re-write this term as

$$UE \text{ Threats} \quad = \quad qv\left(x\right)\phi\left[\mathbf{\Omega}\left(t_U\left(x\right)\right) - \Omega(x)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}}$$

Now using our result in condition **C-UE** that $\mathbf{\Omega}\left(t_U\left(x\right)\right) = \Omega(x)$, we can conclude that

$$UE \text{ Threats} = 0$$

B.4.6 *EE* Hires

$$EE \text{ Hires} = qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^E(x',i')}\left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right), i\right) - \Omega\left(x\right)\right] dH_n\left(x',i'\right)$$

Now by definition

$$
\begin{aligned}
\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) &= \boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(h_E(x',i',x))} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) \\
&= \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\right] + \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)
\end{aligned}
$$

which can be re-arranged into

$$
\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) = \boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)
$$

Using this in the term in the square brackets

$$
EE \text{ Hires} = qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\right] dH_n\left(x',i'\right)
$$

Under **C-EE**, the value going to the hired worker is $\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) = \Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$. Substituting this in:

$$
EE \text{ Hires} = qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^E(x',i')} \left[\left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)\right] - \left[\Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)\right]\right] dH_n\left(x',i'\right)
$$

B.4.7 EE Threats

$$
EE \text{ Threats} \quad = \quad qv\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - J\left(x\right) - \overline{V}\left(x\right)\right] dH_n\left(x',i'\right)
$$

Using the definition of $\boldsymbol{\Omega}(t_E(x',i',x))$, we obtain

$$
EE \text{ Threats} \quad = \quad qv\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x)\right] dH_n\left(x',i'\right)
$$

Now using the result in condition C-RT that $\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) = \Omega(x)$, we obtain that

$$
EE \text{ Threats} = 0
$$

B.5 Reducing the state space

Now that we obtained the simplified recursion, we are in a position to argue that the only payoff-relevant states are (z,n), and that the details of the within-firm contractual structure do not affect allocations. The goal of this section is to show that we can express the joint values pre- and post- separation and exit decisions as follows. First, the exit and separation decisions are characterized by

$$\begin{aligned} \mathbf{\Omega}(z,n) &= \mathbb{I}_{\{(z,n)\in\mathcal{E}\}}\Big\{\vartheta+nU\Big\}+\mathbb{I}_{\{(z,n)\in\mathcal{Q}^U\}}\Big\{\mathbf{\Omega}(z,n-1)+U\Big\}+\mathbb{I}_{\{(z,n)\notin\mathcal{Q}^U\cup\mathcal{E}\}}\Omega(z,n), \quad (33)\\ \text{where} \quad \mathcal{E} &= \{n,z|\vartheta+nU>\Omega(z,n)\},\\ \mathcal{Q}^U &= \{z,n|\mathbf{\Omega}(z,n-1)+U>\Omega(z,n)\}. \end{aligned}$$

The first expression is the value of exit. A firm that does not exit, chooses whether to separate with a worker or not. If separating with a worker, the firm re-enters (33) with $\mathbf{\Omega}(z,n-1)$, having dispatched with a worker with value U, and again choosing whether to exit, fire another worker, or continue. Iterating on this procedure delivers

$$\mathbf{\Omega}(z,n)=\max\left\{\vartheta+nU\,,\max_{s\in[0,\ldots,n]}\Omega(z,n-s)+sU\right\}. \quad (34)$$

Second, the post-exit/separation decision joint value is given by the Bellman equation

$$\begin{aligned} \rho\Omega(z,n) &= \max_{v\geq 0} y(z,n)-c(v,n,z)\\ \text{Destruction} &+ \delta n\Big\{\big(\mathbf{\Omega}(z,n-1)+U\big)-\Omega(z,n)\Big\}\\ UE\text{ Hire} &+ \phi q(\theta)v\cdot\mathbb{I}_{\{(z,n)\in\mathcal{A}\}}\cdot\Big\{\mathbf{\Omega}(z,n+1)-\big(\Omega(z,n)+U\big)\Big\}\\ EE\text{ Hire} &+ (1-\phi)\,q(\theta)v\int_{(z,n)\in\mathcal{Q}^E(z',n')}\Big\{[\mathbf{\Omega}(z,n+1)-\Omega(z,n)]-\big[\Omega(z',n')-\mathbf{\Omega}(z',n'-1)\big]\Big\}dH_n(z',n')\\ \text{Shock} &+ \Gamma_z[\mathbf{\Omega},\Omega](z,n),\\ \text{where} \quad \mathcal{A} &= \{z,n|\mathbf{\Omega}(z,n+1)\geq\Omega(z,n)+U\},\\ \mathcal{Q}^E(z',n') &= \{z,n|\mathbf{\Omega}(z,n+1)-\Omega(z,n)\geq\Omega(z',n')-\mathbf{\Omega}(z',n'-1)\}. \end{aligned}$$

Finally, firms enter if and only if

$$\int\mathbf{\Omega}(z,0)\,d\Pi_0(z)\geq c_e. \quad (35)$$

This condition pins down the entry rate per unit of time.[62]

We proceed in three steps. First, we isolate (z,n) in the state vector x by writing $x=(z,n,\chi)$ where χ collects all other terms in x. Second, we introduce functions that update χ following events to the firm and worker. Third, we argue that χ is a redundant state. This delivers the final Bellman equation for the joint value function for the discrete workforce model, equation (35).

[62] Recall that $J_0=-c_e+\int\boldsymbol{J}(x_0)d\Pi(z_0)$. Given $\Omega(z_0,0)=J(z_0,0)$, we have $J_0=-c_e+\int\mathbf{\Omega}(z_0,0)d\Pi(z_0)$. Free-entry implies $J_0=0$, which delivers (35).

B.5.1 Isolate (z,n) in the state vector

It is immediate that x should contain at least the pair (z,n). Call everything else χ. Then we express $x=(z,n,\chi)$. Making this substitution into the above conditions:

$$\begin{aligned}
\rho\Omega\left(z,n,\chi\right) &= y\left(z,n\right)-c\left(v\left(z,n,\chi\right),n\right)\\
\text{Destructions} \quad & -\delta\sum_{i=1}^{n(x)}\left[\Omega\left(z,n,\chi\right)-\mathbf{\Omega}\left(d\left(z,n,\chi,i\right)\right)-U\right]\\
UE\ \text{Hires} \quad & +qv\left(z,n,\chi\right)\phi\left[\mathbf{\Omega}\left(h_U\left(z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)-U\right]\cdot\mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}}\\
EE\ \text{Hires} \quad & +qv\left(z,n,\chi\right)\left(1-\phi\right)\int_{(z,n,\chi)\in\mathcal{Q}^E(n',z',\chi',i')}\Big[\left[\mathbf{\Omega}\left(h_E\left(z',n',\chi',i',z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right]\\
& \qquad -\left[\Omega\left(z',n',\chi',i'\right)-\mathbf{\Omega}\left(q_E\left(n',z',\chi',i',z,n,\chi\right)\right)\right]\Big]\\
& \qquad \cdot\, dH_n\left(z',n',\chi',i'\right)\\
\text{Shocks} \quad & +\Gamma_z[\mathbf{\Omega},\Omega](z,n,\chi)
\end{aligned}$$

with sets

$$\begin{aligned}
\mathcal{Q}^U &= \Big\{(z,n,\chi,i)\Big|\Omega\left(s\left(z,n,\chi,\left(1-\ell\left(z,n,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(z,n,\chi\right);q_{U,i}\left(z,n,\chi\right)=1\right]\right)\right),i\right)+U\\
&\qquad >\Omega\left(s\left(z,n,\chi,\left(1-\ell\left(z,n,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(z,n,\chi\right);q_{U,i}\left(z,n,\chi\right)=0\right]\right)\right),i\right)\Big\}\\
\mathcal{L} &= \Big\{(z,n,\chi,i)\Big|\Omega\left(s\left(z,n,\chi,\left(1-\left[\ell\left(z,n,\chi\right);\ell_i\left(z,n,\chi\right)=1\right]\right)\circ\left(1-q_U\left(z,n,\chi\right)\right)\right),i\right)+U\\
&\qquad >\Omega\left(s\left(z,n,\chi,\left(1-\left[\ell\left(z,n,\chi\right);\ell_i\left(z,n,\chi\right)=0\right]\right)\circ\left(1-q_U\left(z,n,\chi\right)\right)\right),i\right)\Big\}\\
\mathcal{E} &= \Big\{z,n,\chi\Big|\vartheta+n\left(s(z,n,\chi,\kappa(z,n,\chi))\right)\cdot U\geq\Omega(s(z,n,\chi,\kappa(z,n,\chi)))\Big\}\\
\mathcal{A} &= \Big\{z,n,\chi\Big|\Omega\left(h_U\left(z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\geq U\Big\}\\
\mathcal{Q}^E\left(z',n',\chi',i'\right) &= \Big\{z,n,\chi\Big|\mathbf{\Omega}\left(h_E\left(z',n',\chi',i',z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\geq\Omega\left(n',z',\chi',i'\right)-\mathbf{\Omega}\left(q_E\left(z',n',\chi',i',z,n,\chi\right)\right)\Big\}
\end{aligned}$$

and vacancy posting

$$\begin{aligned}\frac{c_v\left(v\left(z,n,\chi\right),z,n\right)}{q} &= \phi\left[\boldsymbol{\Omega}\left(h_U\left(z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right]\cdot\mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}}\\ &+(1-\phi)\int_{(z,n,\chi)\in\mathcal{Q}^E(z',n',\chi',i')}\Big[\left[\boldsymbol{\Omega}\left(h_E\left(z',n',\chi',i',z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right]\\ &\qquad -\left[\Omega\left(z',n',\chi',i'\right)-\boldsymbol{\Omega}\left(q_E\left(n',z',\chi',i',z,n,\chi\right)\right)\right]\Big]\\ &\qquad \cdot dH_n\left(z',n',\chi',i'\right)\end{aligned}$$

Finally, note that the contribution of shocks writes explicitly

$$\Gamma_z[\boldsymbol{\Omega},\Omega]=\lim_{dt\to 0}E_t\left[\frac{\boldsymbol{\Omega}(z_{t+dt},n_{t+dt},\chi_{t+dt})}{dt}\right]$$

To avoid introducing too much stochastic calculus notation, we will show that χ is a redundant state under the special case that shocks z follow a multi-point Poisson jump process. The logic of the proof with other stochastic processes would be exactly the same, at the expense of more notation. In the Poisson case, we have

$$\Gamma_z[\boldsymbol{\Omega},\Omega]=\tau(z)E_z\Big[\boldsymbol{\Omega}(\eta,n,\chi'(z,n,\chi,\eta))-\Omega(z,n,\chi)\Big]$$

where $\tau(z)$ is the intensity at which the Poisson shocks hit, and η is a random variable following the distribution of those shocks conditional on arrival and conditional on the initial productivity z.

B.5.2 Introduce functions that update the residual χ

We define the following functions given that we know how n changes in each of the cases

$$
\begin{aligned}
s(z,n,\chi,\kappa(z,n,\chi)) &= (\mathcal{N}(z,n,\chi),z,s^{\chi}(z,n,\chi)) \\
d\left(z,n,\chi,i\right) &= \left(n-1,z,d^{\chi}\left(z,n,\chi,i\right)\right) \\
s\left(z,n,\chi,\left(1-\ell\left(z,n,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(z,n,\chi\right);q_{U,i}\left(z,n,\chi\right)=1\right]\right)\right) &= \left(\mathcal{N}(z,n,\chi)-\tau_1(z,n,\chi),z,\tau_1^{\chi}(z,n,\chi,i)\right) \\
s\left(z,n,\chi,\left(1-\ell\left(z,n,\chi\right)\right)\circ\left(1-\left[q_{U,-i}\left(z,n,\chi\right);q_{U,i}\left(z,n,\chi\right)=0\right]\right)\right) &= \left(\mathcal{N}(z,n,\chi),z,\tau_0^{\chi}(z,n,\chi,i)\right) \\
s\left(z,n,\chi,\left(1-\left[\ell\left(z,n,\chi\right);\ell_i\left(z,n,\chi\right)=1\right]\right)\circ\left(1-q_U\left(z,n,\chi\right)\right)\right) &= \left(\mathcal{N}(z,n,\chi)-\eta_1(z,n,\chi),z,\eta_1^{\chi}(z,n,\chi,i)\right) \\
s\left(z,n,\chi,\left(1-\left[\ell\left(z,n,\chi\right);\ell_i\left(z,n,\chi\right)=0\right]\right)\circ\left(1-q_U\left(z,n,\chi\right)\right)\right) &= \left(\mathcal{N}(z,n,\chi),z,\eta_0^{\chi}(z,n,\chi,i)\right) \\
h_U\left(z,n,\chi\right) &= \left(n+1,z,h_U^{\chi}\left(z,n,\chi\right)\right) \\
h_E\left(z',\chi',i',z,n,\chi,n'\right) &= \left(n+1,z,h_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right) \\
q_E\left(z',n',\chi',i',z,n,\chi\right) &= \left(n'-1,z',q_E^{\chi}\left(n',z',\chi',i',z,n,\chi\right)\right) \\
H_n\left(z',n',\chi',i'\right) &= \frac{1}{n'}H_n\left(z',n',\chi'\right) \\
g_z\left(z,n,\chi,\eta\right) &= \Big(\eta,n,g_z^{\chi}\left(z,n,\chi,\eta\right)\Big)
\end{aligned}
$$

The above uses the function $\mathcal{N}(z,n,\chi)$, which gives the number of workers the firm retains after endogenous quits and layoffs. It solves

$$
\mathcal{N}(z,n,\chi) = \arg\max_{k\in\{0,\ldots,n\}} \Omega(k,z,\chi)+(n-k)U
$$

In addition, $\tau_1(z,n,\chi),\eta_1(z,n,\chi)\in\{0,1\}$. $\tau_1(z,n,\chi)=0$ if $\ell_i(z,n,\chi)=1$. Similarly, $\eta_1(z,n,\chi)=0$ if $q_{U,i}(z,n,\chi)=1$. Using these definitions in the Bellman equation above:

$$
\begin{aligned}
\rho\Omega\left(z,n,\chi\right) \;=\;& y\left(z,n\right)-c\left(v\left(z,n,\chi\right),z,n\right) \\
\text{Destructions}\quad & -\delta\sum_{i=1}^{n(x)}\left[\Omega\left(z,n,\chi\right)-\boldsymbol{\Omega}\left(n-1,z,s^{\chi}\left(z,n,\chi,i\right)\right)-U\right] \\
UE\text{ Hires}\quad & +qv\left(z,n,\chi\right)\phi\left[\Omega\left(n+1,z,h_U^{\chi}\left(z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)-U\right]\cdot\mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}} \\
EE\text{ Hires}\quad & +qv\left(z,n,\chi\right)\left(1-\phi\right)\int_{(z,n,\chi)\in\mathcal{Q}^E(n',z',\chi',i')}\Big[\left[\boldsymbol{\Omega}\left(n+1,z,h_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right)-\Omega\left(z,n,\chi\right)\right] \\
& \qquad -\left[\Omega\left(z',n',\chi',i'\right)-\boldsymbol{\Omega}\left(n'-1,z',q_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right)\right]\Big] \\
& \qquad \cdot dH_n\left(z',n',\chi',i'\right) \\
\text{Shocks}\quad & +\tau(z)\mathbb{E}_z\Big[\boldsymbol{\Omega}\Big(\eta,n,g_z^{\chi}\left(z,n,\chi,\eta\right)\Big)-\Omega\left(z,n,\chi\right)\Big]
\end{aligned}
$$

and sets

$$\mathcal{E} = \left\{ z, n, \chi \middle| \vartheta + \mathcal{N}(z,n,\chi) \cdot U \geq \Omega(\mathcal{N}(z,n,\chi), z, s^{\chi}(z,n,\chi)) \right\}$$

$$\mathcal{Q}^{U} = \left\{ (z,n,\chi,i) \middle| \Omega(\mathcal{N}(z,n,\chi) - \tau_1(z,n,\chi), z, \tau_1^{\chi}(z,n,\chi,i)) + U \right. \\ \left. > \Omega(\mathcal{N}(z,n,\chi), z, \tau_0^{\chi}(z,n,\chi,i)) \right\}$$

$$\mathcal{L} = \left\{ (z,n,\chi,i) \middle| \Omega(\mathcal{N}(z,n,\chi) - \eta_1(z,n,\chi), z, \eta_1^{\chi}(z,n,\chi,i)) + U \right. \\ \left. > \Omega(\mathcal{N}(z,n,\chi), z, \eta_0^{\chi}(z,n,\chi,i)) \right\}$$

$$\mathcal{A} = \left\{ z, n, \chi \middle| \Omega\left(n+1, z, h_U^{\chi}(z,n,\chi)\right) - \Omega(z,n,\chi) \geq U \right\}$$

$$\mathcal{Q}^{E}\left(z', n', \chi', i'\right) = \left\{ z, n, \chi \middle| \mathbf{\Omega}\left(n+1, z, h_E^{\chi}\left(z,n,\chi,z',n',\chi',i'\right)\right) - \Omega(z,n,\chi) \right. \\ \left. \geq \Omega\left(z',n',\chi',i'\right) - \Omega\left(n'-1, z', p^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right) \right\}$$

and the definition

$$\mathcal{N}(z,n,\chi) = \arg\max_{k \in \{0,\ldots,n\}} \Omega(k,z,\chi) + (n-k)U$$

and vacancy posting

$$\frac{c_v\left(v(z,n,\chi), z, n\right)}{q} = \phi\left[\mathbf{\Omega}\left(n+1, z, h_U^{\chi}(z,n,\chi)\right) - \Omega(z,n,\chi)\right] \cdot \mathbb{I}_{\{(z,n,\chi)\in\mathcal{A}\}} \\ + (1-\phi)\int_{(z,n,\chi)\in\mathcal{Q}^E(z',n',\chi',i')} \Big[\left[\mathbf{\Omega}\left(n+1, z, h_E^{\chi}\left(z,n,\chi,n',z',\chi',i'\right)\right) - \Omega(z,n,\chi)\right] \\ - \left[\Omega\left(z',n',\chi',i'\right) - \mathbf{\Omega}\left(n'-1, z', q_E^{\chi}\left(z',n',\chi',i',z,n,\chi\right)\right)\right] \Big] \\ \cdot \, dH_n\left(z',n',\chi',i'\right)$$

B.5.3 Argue that (χ, i) are a redundant state

The system above defines a functional fixed point equation. Inspection of the Bellman equation reveals that χ has no *direct* impact on the flow payoff, continuation values, or mobility sets. Its only impact is through the dependence of Ω on χ. This observation implies that χ is a redundant state, and can be removed from the fixed point equation. The same argument ensures that the worker index i is redundant as well.

B.5.4 Bellman equation without (χ, i)

We can re-write our Bellman equation for $(z,n) \in \mathcal{E}^c$ as:

$$
\begin{array}{rcl}
\rho\Omega\left(z,n\right) & = & y\left(z,n\right) - c\left(v\left(z,n\right),n\right) \\
\text{Destructions} & & -\delta \sum_{i=1}^{n} \left[\Omega\left(z,n\right) - \mathbf{\Omega}\left(n-1,z\right) - U\right] \\
\text{Retentions} & & +\lambda^{E} \sum_{i=1}^{n} \int_{(n',z')\in\mathcal{R}(z,n)} \left[\mathbf{\Omega}\left(z,n\right) - \Omega\left(z,n\right)\right] dH_{v}\left(x'\right) \\
UE\text{ Hires} & & +qv\left(z,n\right)\phi\left[\mathbf{\Omega}\left(n+1,z\right) - \Omega\left(z,n\right) - U\right] \cdot \mathbb{I}_{\{(z,n)\in\mathcal{A}\}} \\
EE\text{ Hires} & & +qv\left(z,n\right)\left(1-\phi\right) \int_{(z,n)\in\mathcal{Q}^{E}(z',n')} \left[\left[\mathbf{\Omega}\left(n+1,z\right) - \Omega\left(z,n\right)\right] - \left[\Omega\left(z',n'\right) - \mathbf{\Omega}\left(n'-1,z'\right)\right]\right] d\widetilde{H_{n}}\left(z',n'\right) \\
\text{Shocks} & & +\Gamma_{z}[\mathbf{\Omega},\Omega](z,n)
\end{array}
$$

with the sets

$$
\mathcal{E}^{c} = \left\{ z,n \middle| \Omega(\mathcal{N}(z,n)) \geq \vartheta + \mathcal{N}(z,n)U \right\}
$$

$$
\mathcal{L} = \mathcal{Q}^{U} = \left\{ z,n \middle| \Omega(\mathcal{N}(z,n),z) - \mathbf{\Omega}(\mathcal{N}(z,n)-1,z) \leq U \right\}
$$

$$
\mathcal{A} = \left\{ z,n \middle| \mathbf{\Omega}\left(n+1,z\right) - \Omega\left(z,n\right) \geq U \right\}
$$

$$
\mathcal{Q}^{E}\left(z',n'\right) = \left\{ z,n \middle| \mathbf{\Omega}\left(n+1,z\right) - \Omega\left(z,n\right) \geq \Omega\left(z',n'\right) - \mathbf{\Omega}\left(n'-1,z'\right) \right\}
$$

and the definition

$$
\mathcal{N}(z,n) = \arg\max_{k\in\{0,\ldots,n\}} \Omega(k,z) + (n-k)U
$$

and the vacancy policy function:

$$
\begin{aligned}
\frac{c_{v}\left(v\left(z,n\right),z,n\right)}{q} = {} & \phi\left[\mathbf{\Omega}\left(n+1,z\right) - \Omega\left(z,n\right)\right] \cdot \mathbb{I}_{\{(z,n)\in\mathcal{A}\}} \\
& + \left(1-\phi\right) \int_{(z,n)\in\mathcal{Q}^{E}(z',n')} \left[\left[\mathbf{\Omega}\left(n+1,z\right) - \Omega\left(z,n\right)\right] - \left[\Omega\left(z',n'\right) - \mathbf{\Omega}\left(n'-1,z'\right)\right]\right] d\widetilde{H_{n}}\left(n',z'\right)
\end{aligned}
$$

B.5.5 Expressing "bold" values

In this step we express "bold" values – that encode the optimal quit, layoff and exit decisions – as simple functions of non-bold values.

From the definition of the exit and quit sets $\mathcal{E}, \mathcal{Q}^U$, we can express:

$$\boldsymbol{\Omega}(z,n) \quad = \quad \max\left\{\underbrace{\Omega(z,n)}_{\text{Operate}}, \quad \underbrace{\boldsymbol{\Omega}(n-1,z)+U}_{\text{Separate one worker and re-evaluate}}, \underbrace{\vartheta+nU}_{\text{Exit}}\right\}$$

We can iterate on this equation. To see the logic, consider the first few steps.

$$\begin{aligned}
\boldsymbol{\Omega}(z,n) &= \max\left\{\Omega(z,n)\,,\,\boldsymbol{\Omega}(n-1,z)+U\,,\,\vartheta+nU\right\} \\
&= \max\left\{\Omega(z,n)\,,\,\max\left\{\Omega(n-1,z)\,,\,\boldsymbol{\Omega}(n-2,z)+U\,,\,\vartheta+(n-1)U\right\}+U\,,\,\vartheta+nU\right\} \\
&= \max\left\{\Omega(z,n)\,,\Omega(n-1,z)+U\,,\,\boldsymbol{\Omega}(n-2,z)+2U\,,\,\vartheta+(n-1)U+U\,,\,\vartheta+nU\right\} \\
&= \max\left\{\Omega(z,n)\,,\Omega(n-1,z)+U\,,\,\boldsymbol{\Omega}(n-2,z)+2U\,,\,\vartheta+nU\right\}
\end{aligned}$$

By recursion, it is easy to see that

$$\begin{aligned}
\boldsymbol{\Omega}(z,n) &= \max\left\{\Omega(\mathcal{N}(z,n),z)+(n-\mathcal{N}(z,n))\cdot U\,,\,\vartheta+nU\right\} \\
&= \max\left\{\max_{k\in\{0,\dots,n\}}\Omega(k,z)+(n-k)U\,,\,\vartheta+nU\right\}
\end{aligned}$$

where recall that

$$\mathcal{N}(z,n)=\arg\max_{k\in\{0,\dots,n\}}\Omega(k,z)+(n-k)U$$

B.6 Continuous workforce limit

Up to this point the economy has featured a continuum of firms, but an integer-valued workforce. We now take the continuous workforce limit by defining the 'size' of a worker—which is 1 in the integer case—and taking the limit as this approaches zero. Specifically, denote the "size" of a worker by Δ, such that $n = N\Delta$ where N is the old integer number of workers. Now define $\Omega^{\Delta}(z,n) := \Omega(z,n/\Delta)$, and likewise define $y^{\Delta}(z,n) := y(z,n/\Delta)$ and $c^{\Delta}(v,n,z) := c(v/\Delta,n/\Delta,z)$. We also define $b^{\Delta} := b/\Delta$ and $\vartheta^{\Delta} := \vartheta/\Delta$. These imply, for example, that $\Omega(z,N) = \Omega^{\Delta}(z,N\Delta)$. Substituting these terms into (34) and (35), and taking the limit $\Delta \to 0$, while holding $n = N\Delta$ fixed, we would obtain a version of (36) in which all functions have the Δ super-script notation. We also specialize the productivity to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$.

The result is the joint value representation of section 3: a Hamilton-Jacobi-Bellman (HJB) equation for the joint value *conditional on the firm and its workers operating*:

$$\begin{aligned}
\rho\Omega(z,n) = \max_{v\geq 0} \quad & y(z,n) - c(v,n,z) \\
\text{Destruction} \quad & -\delta n[\Omega_n(z,n) - U] \\
UE \text{ Hire} \quad & +\phi q(\theta) v\,[\Omega_n(z,n) - U] \\
EE \text{ Hire} \quad & +(1-\phi) q(\theta) v \int \max\left\{\Omega_n(z,n) - \Omega_n(n',z')\,,\,0\right\} dH_n(z',n') \\
\text{Shock} \quad & +\mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n).
\end{aligned} \tag{36}$$

Boundary conditions for the firm and its workers operating require the state to be interior to the exit and separation boundaries:

$$\begin{aligned}
\text{Exit boundary:} \quad & \Omega(z,n) \geq \vartheta + nU, \\
\text{Layoff boundary:} \quad & \Omega_n(z,n) \geq U
\end{aligned}$$

Note the absence of $\boldsymbol{\Omega}$ terms. Since the value we track is that of a hiring firm subject to boundary conditions, then $\boldsymbol{\Omega} = \Omega$. This admits the simplification of 'Shock' terms we noted when discussing (30).

We proceed in three steps:

(A.5.1) Define worker size and the renormalization

(A.5.2) Take the limit as worker size goes to zero

(A.5.3) Introduce a continuous productivity process.

B.6.1 Define worker size and the renormalization

We denote the "size" of a worker by Δ. That is, we currently have an integer work-force $n \in \{1, 2, 3, \ldots\}$. We now consider $m \in \{\Delta, 2\Delta, 3\Delta, \ldots\}$. So then $n = m/\Delta$. We use this to make the following normalizations:

$$\begin{aligned}
\omega(z,m) &= \Omega\left(\frac{m}{\Delta}, z\right) \\
\mathcal{Y}(z,m) &= y\left(\frac{m}{\Delta}, z\right) \\
\mathcal{C}(z,m) &= c\left(\frac{v}{\Delta}, \frac{m}{\Delta}, z\right)
\end{aligned}$$

These definition imply

$$\begin{aligned}\Omega(z,n) &= \omega(n\Delta,z)\\ y(z,n) &= \mathcal{Y}(n\Delta,z)\\ c(v,z,n) &= \mathcal{C}(v\Delta,n\Delta,z)\end{aligned}$$

In addition, the value of unemployment solves

$$\rho U = b$$

Define

$$\mathcal{U} = \frac{b}{\rho\Delta} = \frac{U}{\Delta}$$

and

$$\theta = \frac{\vartheta}{\Delta}$$

Substituting these definitions into the Bellman equation, we obtain

$$\begin{aligned}
\rho\omega\left(n\Delta,z\right) = \max_{v\Delta\geq 0} \quad & \mathcal{Y}\left(n\Delta,z\right) - \mathcal{C}\left(v\Delta,n\Delta,z\right)\\
\text{Destructions} \quad & -\delta n\Delta\left[\frac{\omega\left(n\Delta,z\right)-\boldsymbol{\omega}\left(n\Delta-\Delta,z\right)}{\Delta}-\mathcal{U}\right]\\
UE\text{ Hires} \quad & +qv\Delta\phi\left[\frac{\boldsymbol{\omega}\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta}-\mathcal{U}\right]\cdot\mathbb{I}_{\{(n\Delta,z)\in\mathcal{A}\}}\\
EE\text{ Hires} \quad & +qv\Delta\left(1-\phi\right)\int_{(n\Delta,z)\in\mathcal{Q}^E(n'\Delta,z')}\left[\frac{\boldsymbol{\omega}\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta}-\frac{\omega\left(n'\Delta,z'\right)-\boldsymbol{\omega}\left(n'\Delta-\Delta,z'\right)}{\Delta}\right]d\widetilde{H}_n\left(n'\Delta,z'\right)\\
\text{Shocks} \quad & +\Gamma_z\left[\boldsymbol{\omega},\omega\right]\left(n\Delta,z\right)
\end{aligned}$$

with the set definitions

$$\begin{aligned}
\mathcal{E} &= \left\{n\Delta,z\left|\max_{k\Delta\in\{0,\ldots,n\Delta\}}\omega(k\Delta,z)+(n\Delta-k\Delta)\mathcal{U}<\theta+n\Delta\mathcal{U}\right.\right\}\\
\mathcal{A} &= \left\{n\Delta,z\left|\frac{\boldsymbol{\omega}\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta}\geq\mathcal{U}\right.\right\}\\
\mathcal{Q}^U &= \left\{n\Delta,z\left|\frac{\omega\left(n\Delta,z\right)-\boldsymbol{\omega}\left(n\Delta-\Delta,z\right)}{\Delta}\leq\mathcal{U}\right.\right\}\\
\mathcal{Q}^E\left(n'\Delta,z'\right) &= \left\{n\Delta,z\left|\frac{\boldsymbol{\omega}\left(n\Delta+\Delta,z\right)-\omega\left(n\Delta,z\right)}{\Delta}\geq\frac{\omega\left(n'\Delta,z'\right)-\boldsymbol{\omega}\left(n'\Delta-\Delta,z'\right)}{\Delta}\right.\right\}
\end{aligned}$$

and the definition:

$$\boldsymbol{\omega}(n\Delta, z) \quad = \quad \max\left\{ \max_{k\Delta \in \{0,\dots,n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U} \ , \ \theta + n\Delta\mathcal{U} \right\}$$

B.6.2 Continuous limit as worker size goes to zero

Now we take the limit $\Delta \to 0$, holding $m = n\Delta$ fixed. We note $\widehat{v} = \lim_{\Delta\to 0} v\Delta$. We see derivatives appear. We denote $\omega_m(z,m) = \frac{\partial\omega}{\partial m}(z,m)$.

First, we note that the following limit obtains:

$$\boldsymbol{\omega}(z,m) \quad = \quad \max\left\{ \max_{k\in[0,m]} \omega(k,z) + (m-k)\mathcal{U} \ , \ \theta + m\Delta\mathcal{U} \right\}$$

In particular, the exit set limits to

$$\mathcal{E} = \left\{ z, m \,\middle|\, \max_{k\in[0,m]} \omega(k,z) + (m-k)\mathcal{U} < \theta + m\mathcal{U} \right\}$$

In equilibrium, the $\boldsymbol{\omega}(z,m)$ terns on the right-hand-side of the Bellman equation are the result of endogenous quits, layoffs and hires. Because our continuous time assumption has been made *before* we take the limit to a continuous workforce limit, we need only consider those changes in the workforce one at a time. Hence, for any $(z,m) \in \text{Interior}(\mathcal{E}^c \cap \mathcal{A})$, the *interior* of the continuation set, there is always $\overline{\Delta} > 0$: such that for any $\Delta \leq \overline{\Delta}$:

$$\boldsymbol{\omega}(m \pm \Delta, z) = \omega(m \pm \Delta, z)$$

Using this observation in the Bellman equation, we can obtain derivatives on the right-hand-side. We obtain, for pairs (z,n) in the interior of the continuation set $(z,n) \in \text{Interior}(\mathcal{E}^c \cap \mathcal{A})$:

$$\begin{aligned}
\rho\omega\,(z,m) = \max_{\widehat{v}\geq 0} \quad & \mathcal{Y}\,(z,m) - \mathcal{C}\,(\widehat{v}, z, m) \\
\text{Destructions} \quad & -\delta m[\omega_m(z,m) - \mathcal{U}] \\
UE \text{ Hires} \quad & +q\widehat{v}\phi\,[\omega_m(z,m) - \mathcal{U}] \cdot \mathbb{I}_{\{(z,m)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +q\widehat{v}\,(1-\phi) \int_{(z,m)\in\mathcal{Q}^E(m',z')} \left[\omega_m(z,m) - \omega_m(m',z')\right] d\widetilde{H_n}\,(m',z') \\
\text{Shocks} \quad & +\Gamma_z\,[\boldsymbol{\omega}, \omega]\,(z,n)
\end{aligned}$$

with the set definitions

$$\mathcal{E} = \left\{ z, m \middle| \max_{k \in [0,m]} \omega(k,z) + (n-k)\mathcal{U} < \theta + m\mathcal{U} \right\}$$

$$\mathcal{A} = \left\{ z, m \middle| \omega_m(z,m) \geq \mathcal{U} \right\}$$

$$\mathcal{Q}^U = \left\{ z, m \middle| \omega_m(z,m) \leq \mathcal{U} \right\} = \overline{A} \text{ , the complement of } \mathcal{A}$$

$$\mathcal{Q}^E\left(z', m'\right) = \left\{ z, m \middle| \omega_m(z,m) - \omega_m(m', z') \geq 0 \right\}$$

and the definition

$$\boldsymbol{\omega}(z,m) = \max \left\{ \max_{k \in [0,m]} \omega(k,z) + (m-k)\mathcal{U} \text{ , } \theta + m\mathcal{U} \right\}$$

Note that now, the only place where $\boldsymbol{\omega}$ enters in the Bellman equation is the contribution of shocks. To replace it with ω, we need to apply the same argument to z as the one we applied to n. We thus need to specialize to a continuous productivity process.

B.6.3 Continuous productivity process

We now specialize to a continuous productivity process, as this makes the formulation of the problem very economical. It allows to simplify the contribution of productivity shocks and get rid of the remaining "bold" notation. We suppose that productivity follows a diffusion process:

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$$

In this case, for any (z, m) in the interior of the continuation set, productivity shocks in the interval $[t, t+dt]$ cannot move the firm towards a region in which it would endogenously separate or exit, when dt is small enough. In this case, we can write the following, where we have also replaced the $\mathcal{Q}^E$ set with the max operator:

$$
\begin{aligned}
\rho\omega\left(z,m\right) = \max_{v\geq 0} \quad & \mathcal{Y}\left(z,m\right) - \mathcal{C}\left(v,z,m\right) \\
\text{Destructions} \quad & -\delta m[\omega_m(z,m) - \mathcal{U}] \\
UE \text{ Hires} \quad & +qv\phi\left[\omega_m(z,m) - \mathcal{U}\right] \\
EE \text{ Hires} \quad & +qv\left(1-\phi\right)\int \max\left\{\omega_m(z,m) - \omega_m(z',m')\ ,\ 0\right\} d\widetilde{H_n}\left(m',z'\right) \\
\text{Shocks} \quad & +\mu(z)\omega_z(z,m) + \frac{\sigma(z)^2}{2}\omega_{zz}(z,m) \\
\text{s.t.} \quad & \\
\text{No Exit} \quad & \omega(z,m) \geq \theta + m\mathcal{U} \\
\text{No Separations} \quad & \omega_m(z,m) \geq \mathcal{U}
\end{aligned}
$$

To make the notation more comparable, we slightly abuse notation and use the same letters as before, but now for the continuous workforce case. We obtain finally:

$$
\begin{aligned}
\rho\Omega\left(z,n\right) = \max_{v\geq 0} \quad & y\left(z,n\right) - c\left(v,z,n\right) \\
\text{Destructions} \quad & -\delta n[\Omega_n(z,n) - U] \\
UE \text{ Hires} \quad & +qv\phi\left[\Omega_n(z,n) - U\right] \\
EE \text{ Hires} \quad & +qv\left(1-\phi\right)\int \max\left[\Omega_n(z,n) - \Omega_n(z',n')\ ,\ 0\right] d\widetilde{H_n}\left(z',n'\right) \\
\text{Shocks} \quad & +\mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n) \\
\text{s.t.} \quad & \\
\text{No Exit} \quad & \Omega(z,n) \geq \vartheta + nU \\
\text{No Separations} \quad & \Omega_n(z,n) \geq U
\end{aligned}
$$

When the coalition hits $\Omega_n(z,n) = U$, it endogenous separates worker to stay on that frontier. It exits when it hits the frontier $\Omega(z,n) = \vartheta + nU$.

In addition to these "value-pasting" boundary conditions, optimality implies necessary "smooth-pasting" boundary conditions (see Stokey 2009): $\Omega_z(z,n) = 0$ if the firm actually exits at (z,n) following productivity shocks, and $\Omega_n(z,n) = 0$ if the firm actually exits at (z,n) following changes in size. These are necessary and sufficient for the definition of our problem (Brekke and Øksendal 1991). Its general formulation terms of optimal switching between three regimes (operation, layoffs, exit) on the entire positive quadrant, can be made as a system of Hamilton-Jacobi-Bellman-Variational-Inequality (see Pham 2009), which we present here for completeness :

$$\max \Bigg\{ -\rho\Omega\,(z,n) + \max_{v\geq 0} -\delta n[\Omega_n(z,n) - U] + qv\phi\,[\Omega_n(z,n) - U] \\ +qv\,(1-\phi) \int \max \left[\Omega_n(z,n) - \Omega_n(z',n')\,,\,0 \right] d\widetilde{H_n}\,(z',n') + \mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n)\,; \\ \vartheta + nU - \Omega(z,n)\,;\ \max_{k\in[0,n]} \Omega(z,k) + (n-k)U - \Omega(z,n) \Bigg\} = 0 \quad , \quad \forall (z,n) \in \mathbb{R}^2_+$$

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