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Abstract

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1 Introduction

Consider a group of agents who must come together to make a choice from some set of potential outcomes that will affect each of them. This can be modeled as having the agents play a "game", taking turns choosing from sets of actions (possibly simultaneously), with the final outcome determined by the decisions made by all of the agents each time they were called to play. To ensure that the ultimate decision taken satisfies desirable normative properties (e.g., efficiency), the standard approach in mechanism design is to provide agents with incentives to play in a predictable optimal way. For instance, the designer may use a Bayesian or dominant-strategy incentive-compatible direct mechanism where it is optimal for agents to simply report all of their private information truthfully. This approach succeeds if the participants understand that being truthful is in their interest, but there is evidence many real-world agents do not report the truth, even in strategy-proof mechanisms.[1] In other words, Bayesian or dominant-strategy mechanisms, may not be sufficiently simple for participants to play optimally in practice. Simpler mechanisms are also appealing because they lower participation costs, attract participants, and equalize opportunities across participants with different levels of access to information and strategic sophistication. Additionally, designing simpler mechanisms requires less information about participants' beliefs.[2]

What mechanisms, then, are actually "simple to play"? We address this question by introducing a general class of simplicity standards that vary the foresight abilities required of agents in extensive-form imperfect-information games. We then use these standards to assess the restrictions simplicity imposes on the mechanism designer, as well as to characterize simple mechanisms for a broad range of social choice environments both with and without transfers.[3] Similarly to how the revelation principle allows a designer to limit the search for a Bayesian mechanism to the space of incentive-compatible direct mechanisms, our results construct classes of mechanisms that allows one to do the same when searching for simple mechanisms.[4] As applications, we provide microfoundations for popular simple mechanisms such as posted prices, ascending auctions, and Random Priority.

[1]See, e.g., Kagel, Harstad, and Levin (1987), Li (2017b), Hassidim, Romm, and Shorrer (2016), Rees-Jones (2017), Rees-Jones (2018), Shorrer and Sóvágó (2018), and Artemov, Che, and He (2017).

[2]See Vickrey (1961) for participation costs, Spenner and Freeman (2012) for attracting participants, Pathak and Sönmez (2008) for leveling the playing field, and Wilson (1987) and Bergemann and Morris (2005) for a designer's informational requirements.

[3]Examples include auctions (Vickrey, 1961; Riley and Samuelson, 1981; Myerson, 1981), voting (Arrow, 1963), school choice (Abdulkadiroğlu and Sönmez, 2003), organ exchange (Roth, Sönmez, and Ünver, 2004), course allocation (Sönmez and Ünver, 2010; Budish and Cantillon, 2012), and refugee resettlement (Jones and Teytelboym, 2016; Delacrétaz et al., 2016).

[4]Direct mechanisms are not necessarily simple, and hence the revelation principle does not extend to simple extensive form games, cf. Li (2017b).

The main innovation in our approach is a departure from the standard assumption that agents plan a complete strategy for every possible future contingency; rather, we consider agents that, at each information set, make plans for only those information sets that they perceive as simple from the current perspective. We refer to these plans as *partial strategic plans*.[5] A (partial) strategic plan is *simply dominant* if the called for action is weakly better than any alternative, irrespective of what happens at information sets that are not simple. As the game progresses, the agent's perception of which information sets are simple may change, and agents may update their strategic plans along the path of the game, which is what differentiates strategic plans from the standard game-theoretic concept of a strategy.[6]

Variations in the sets of information sets perceived as simple gives rise to a family of simple dominance standards that vary in strength. The stronger the simplicity standard—i.e., the fewer information sets in the future that are perceived as simple from today's perspective—the more robust the corresponding mechanisms is to agents who can plan for only limited future horizons (e.g., because of non-exponential discounting) or whose decision capabilities are otherwise constrained.[7] We focus on special cases of simple dominance in which agents are able to plan some exogenously given number $k \in \{0, 1, ..., \infty\}$ of future moves; that is, they perceive as simple their current information set and only the first k information sets at which they may be called to play in the continuation game. We show that the longer the foresight horizon of the agents, the more social choice rules a designer can implement in a simply-dominant way; furthermore, without loss of flexibility, the designer can restrict attention to perfect-information extensive-form games.

We analyze three special cases of simple dominance in detail.

- $k = \infty$: agents perceive all of their own information sets as simple, and all information sets of other agents as not simple—in other words, at each information set, an agent can plan the actions they will take at any future information set at which they may be called to play. This is equivalent to Li's (2017b) notion of obvious dominance; for this reason, we refer to the resulting simply dominant strategic plans as *obviously dominant*,

[5]Savage (1954) wrestles with whether decision-makers should be modeled as "look before you leap" (create a complete contingent plan for all possible future decisions one may face) or "you can cross that bridge when you come to it" (make choices as they arise). While standard strategic concepts of game theory formalize the former modeling option, our approach formalizes the latter.

[6]We are agnostic as to whether the agents are sophisticated and understand that their plans might be updated, or whether the agents are naive about this possibility. Simple dominance only requires that the initial action of the strategic plan is unambiguously better than other actions the agent could have chosen at the information set at which the plan is made; the subsequent actions of the strategic plan merely ensure the optimality of the initial action.

[7]We show that a strategic plan is simply dominant if and only if in every game an agent may confuse with the actual game being played, the strategic plan is weakly dominant in the standard sense (Theorem 3). Li (2017b) provides a related behavioral microfoundation for his obvious dominance, on which we build.

and the corresponding mechanisms as *obviously strategy-proof (OSP).*

- $k = 1$: agents perceive as simple their current information set and only the *first* information sets at which they may be called to play in the continuation game—in other words, agents are able to plan at most one move ahead at a time. We refer to the resulting simply dominant strategic plans as *one-step dominant*, and the corresponding mechanisms as *one-step simple (OSS).*

- $k = 0$: agents perceive as simple *only* their own current information set—in other words, agents cannot plan for any moves in the future. We refer to the resulting resulting simply dominant strategic plans as *strongly obviously dominant*, and the corresponding mechanisms as *strongly obviously strategy-proof (SOSP).*

The above concepts are nested: strongly obviously dominant strategic plans are one-step dominant, which in turn are obviously dominant. While obvious dominance is the most permissive standard, it relies on the assumption that agents can create a complete plan for all possible contingencies going forward, and further are able to perform backwards induction over at least their own future actions (though not over the actions of their opponents). As an example, consider the game of chess: assuming that White can always force a win, any winning strategy of White is obviously dominant; yet, the strategic choices in chess are far from obvious. On the other hand, winning strategies in chess are not one-step dominant dominant, nor strongly obviously dominant, as they require looking many steps into the future. Games that admit one-step and/or strongly obviously dominant strategies do not require agents to have such lengthy foresight.

For the above three simplicity standards we ask: which mechanisms are simple? For obvious dominance, we focus on social choice environments without transfers, hence complementing Li (2017b), who focuses on the case with transfers. We show that OSP games can be represented as *millipede games.* In a millipede game, each time an agent is called to move, she is presented with some subset of payoff-equivalent outcomes, or more simply *payoffs*, that she can 'clinch', after which she leaves the game; she also may be given the opportunity to 'pass' and remain in the game, with the potential of being offered better clinching options in the future. If this agent passes, another agent is presented with an analogous choice, etc., until one of them eventually clinches and leaves the game, and the process continues with the next agent. While some millipede games, such as serial dictatorships, are frequently encountered and are indeed simple to play, others are rarely observed in market-design practice, and their strategy-proofness is not necessarily immediately clear. In particular, similar to chess, some millipede games require agents to look far into the future and to perform potentially complicated backward induction reasoning (see Figure 2 in Section 4.2 for an example).

We next study one-step dominance in environments both with and without transfers. We first show that in the binary allocation environments with transfers studied by Li (2017b)—which encompass canonical special cases such as single-unit auctions and binary public good choice—any one-step simple mechanism is equivalent to a personal clock auction. This strengthens Li's result that personal clock auctions are OSP by showing that there is no loss in imposing OSS relative to OSP: any social choice rule that is implementable in obviously dominant strategies is also implementable in one-step dominant strategic plans. In no-transfer environments, one-step simplicity eliminates the complex, yet still formally, OSP millipede games discussed above (and also eliminates games such as chess). Indeed, we can characterize OSS millipede games as those that satisfy the following monotonicity property: each time an agent is called to move, at any next move in the continuation game at which the agent is called again (or terminal history), the agent is able to clinch a payoff that is either at least as good as anything she could have clinched previously, or at least as good as anything that was possible but not clinchable. Monotonic games seem particularly simple, both for a designer to implement, since the agent only needs to recognize that she can do no worse at her very next move if she remains in the game.[8]

For strong obvious dominance, we show that SOSP games do not require agents to look far into the future and perform lengthy backwards induction: in all such games, each agent has essentially at most one payoff-relevant move. Thus, strongly obviously dominant strategic plans are robust to agents who may be concerned about trembles, or have time-inconsistent preferences. Building on this insight, we show that all SOSP games can be implemented as sequential choice games in which each agent moves at most once, and, at this move, is offered a choice from a menu of options. If the menu has three or more options for the agent in question, then the agent's final payoff is what they choose from the menu. If the menu has only two options, then the agent's final payoff might depend on other agents' choices, but truthfully indicating the preferred option is the dominant choice. The offered menu may include prices, in which case we call the mechanism a *(sequential) posted price mechanism.* In this way, strong obvious dominance gives us a microfoundation for posted prices, a ubiquitous sales mechanism.[9]

As an application of our analysis, we provide an axiomatic characterization of the well-known Random Priority (RP; also known as Random Serial Dictatorship) mechanism using simplicity, efficiency, and fairness axioms. In the context of no-transfer allocation problems,

[8]Note also that our monotonicity property is a generalization of a similar feature of ascending auctions (and also Li's personal clock auctions). In an ascending auction, if an agent passes (continues in the auction), at any next move, she will be offered the opportunity to drop out (clinch the zero payoff), except if she wins.

[9]For earlier microfoundations of posted prices, see Hagerty and Rogerson (1987) and Copic and Ponsati (2016).

Random Priority works as follows: first Nature selects an ordering of agents, and then each agent moves in turn and chooses her favorite object among those that remain available given previous agents' choices. This mechanism has a long history, and is used in a wide variety of practical allocation problems, including school choice, worker assignment, course allocation, and the allocation of public housing. Random Priority is well-known to have good efficiency, fairness, and simplicity properties: it is Pareto efficient, it treats agents in a symmetric way, and it is obviously strategy-proof (as well as one-step simple and strongly obviously strategy-proof). However, it has until now remained unknown whether there are other such mechanisms, and if so, what explains the relative popularity of Random Priority over these alternatives.[10] We show that there are none, thus resolving positively the quest to establish Random Priority as the unique mechanism with good incentive, efficiency, and fairness properties and thereby explaining its popularity in practical market design settings.

Our construction of the simplicity criteria is inspired by Li (2017b), who formalized obvious strategy-proofness and established its desirability as an incentive property; we go beyond his work by allowing for the gradation of simplicity criteria—which allows us to assess the trade-off between simplicity and implementation flexibility—and by providing simplicity-based microfoundations for popular mechanisms such as posted prices and Random Priority. Following up on Li's work, but preceding ours, Ashlagi and Gonczarowski (2018) show that stable mechanisms such as Deferred Acceptance (DA) are not obviously strategy-proof, except in very restrictive environments where DA simplifies to an obviously strategy-proof game with a 'clinch or pass' structure similar to simple millipede games (though they do not describe it in these terms). Other related papers include Troyan (2019), who studies obviously strategy-proof allocation via the popular Top Trading Cycles (TTC) mechanisms, and provides a characterization of the priority structures under which TTC are OSP-implementable.[11] Following our work, Arribillaga et al. (2020) and Arribillaga et al. (2019) characterize the voting rules that are obviously strategy-proof on domains of single-peaked

[10] The efficiency and fairness of Random Priority were recognized already by Abdulkadiroğlu and Sönmez (1998), while its obvious strategy-proofness was established by Li (2017b). In single-unit demand allocation with at most three agents and three objects, Bogomolnaia and Moulin (2001) proved that Random Priority is the unique mechanism that is strategy-proof, efficient, and symmetric. In markets in which each object is represented by many copies, Liu and Pycia (2011) and Pycia (2011) proved that Random Priority is the asymptotically unique mechanism that is symmetric, asymptotically strategy-proof, and asymptotically ordinally efficient. While these earlier results looked at either very small or very large markets, ours is the first characterization that holds for any number of agents and objects.

[11] Li showed that the classic TTC mechanism of Shapley and Scarf (1974), in which each agent starts by owning exactly one object, is not obviously strategy-proof. Following our and Troyan's work, Mandal and Roy (2020) characterize the priority structures under which Hierarchical Exchange of Pápai (2000) and Trading Cycles (group strategy-proof and efficient mechanisms) of Pycia and Ünver (2017) are OSP-implementable; cf. also Mandal and Roy (2021).

preferences. Bade and Gonczarowski (2017) study obviously strategy-proof and efficient social choice rules in several environments. Mackenzie (2020) introduces the notion of a "round table mechanism" for OSP implementation and draws parallels with the standard Myerson-Riley revelation principle for direct mechanisms. There has been less work that goes beyond Li's obvious dominance. Li (2017a) extends his ideas to an ex post equilibrium context, while Zhang and Levin (2017a; 2017b) provide decision-theoretic foundations for obvious dominance and explore weaker incentive concepts.[12]

Our work contributes to the understanding of limited foresight and limits on backward induction. Other work in this area—with different approaches from ours—includes Jehiel (1995; 2001) on limited foresight equilibrium in which players' forecasts are correct, Gabaix et al. (2006) on directed cognition, Ke's (2019) axiomatization of bounded-horizon backward induction, as well as the rich literature on time-inconsistent preferences (e.g., Laibson (1997) and Gul and Pesendorfer (2001; 2004)). A major difficulty for models of imperfect foresight is the question of how an agent takes into account the future they are unable to foresee; we resolve this difficulty by designing games in which all resolutions of the unforeseen lead the agent to the same current decision.[13]

The paper also adds to our understanding of dominant incentives, efficiency, and fairness in settings with and without transfers. In settings with transfers, these questions were studied by e.g. Vickrey (1961), Clarke (1971), Groves (1973), Green and Laffont (1977), Holmstrom (1979), Dasgupta et al. (1979), and Hagerty and Rogerson (1987). In settings without transfers, in addition to Gibbard (1973, 1977) and Satterthwaite (1975) and the allocation papers mentioned above, the literature on mechanisms satisfying these key objectives includes Ehlers (2002) and Pycia and Unver (2020; 2017) who characterized efficient and group strategy-proof mechanisms in settings with single-unit demand, and Pápai (2001) and Hatfield (2009) who provided such characterizations for settings with multi-unit demand.[14] Liu and Pycia (2011), Pycia (2011), Morrill (2015), Hakimov and Kesten (2014), Ehlers and Morrill (2017), and Troyan et al. (2020) characterize mechanisms that satisfy incentive, efficiency, and fairness objectives.

[12] Also of note is Glazer and Rubinstein (1996), who argued that extensive-form games may simplify the solution of normal-form games, and Loertscher and Marx (2020), who study environments with transfers and construct a prior-free obviously strategy-proof mechanism that becomes asymptotically optimal as the number of buyers and sellers grows. A different strategic perspective on simplicity in mechanism design was explored by Börgers and Li (2019).

[13] The issue of accounting for the unforeseen is also crucial for the analyses of incomplete contracts (e.g., Maskin and Tirole, 1999) and unawareness (e.g., Karni and Viero, 2013). Agents who rely on incomplete models have been also studied in the context of persuasion (e.g., Schwartzstein and Sunderam, 2021).

[14] Pycia and Ünver (2020) characterized individually strategy-proof and Arrovian efficient mechanisms. For an analysis of these issues under additional feasibility constraints, see also Dur and Ünver (2015) and Root and Ahn (2020).

2 Model

2.1 Preferences

Let $\mathcal{N} = \{i_1, \ldots, i_N\}$ be a set of agents, and $\mathcal{X}$ a finite set of outcomes.[15] An outcome might involve a monetary transfer; we allow both environments with and without transfers. Each agent has a preference ranking over outcomes, where, for any two $x, y \in \mathcal{X}$, we write $x \succsim_i y$ to denote that x is weakly preferred to y. We allow for indifferences, and write $x \sim_i y$ if $x \succsim_i y$ and $y \succsim_i x$. For any $\succsim_i$, we let $\succ_i$ denote the corresponding strict preference relation, i.e., $x \succ_i y$ if $x \succsim_i y$ but not $y \succsim_i x$. We use $\mathcal{P}_i$ to denote the domain of agent i's preferences, and refer to $\succsim_i$ as agent i's **type**.

We allow incomplete information through the standard imperfect-information construction of a meta-game in which Nature moves first and determines agents' types, and only then the designed game/mechanism is played. Due to the nature of the dominance properties we study, we do not need to make any assumptions on agents' beliefs about others' types nor on how agents' evaluate lotteries.[16]

2.2 Extensive Form Games

To determine the outcome the planner designs a game Γ for the agents to play. We consider imperfect-information, extensive-form games with perfect recall, which are defined in the standard way: there is a finite collection of partially ordered **histories**, $\mathcal{H}$. We write $h' \subseteq h$ to denote that $h' \in \mathcal{H}$ is a subhistory of $h \in \mathcal{H}$, and $h' \subset h$ when $h' \subseteq h$ but $h \neq h'$. Terminal histories are denoted with bars, i.e., $\bar{h}$. Each $\bar{h} \in \mathcal{H}$ is associated with an outcome in $\mathcal{X}$. At every non-terminal history $h \in \mathcal{H}$, one agent, denoted i_h, is called to play and has a finite set of **actions** $A(h)$ from which to choose. We write $h' = (h, a)$ to denote the history h' that is reached by starting at history h and following the action $a \in A(h)$. To avoid trivialities, we assume that no agent moves twice in a row and that $|A(h)| > 1$ for all non-terminal $h \in \mathcal{H}$. To capture random mechanisms, we also allow for histories h at which a non-strategic agent, Nature, is called to move, and selects an action in $A(h)$ according to some probability distribution.

[15] Assuming $\mathcal{X}$ is finite simplifies the exposition and is satisfied in the examples listed in the introduction. This assumption can be relaxed. For instance, our analysis goes through with no substantive changes if we allow infinite $\mathcal{X}$ endowed with a topology such that agents' preferences are continuous in this topology and the relevant sets of outcomes are compact.

[16] It is natural to assume that an agent weakly prefers lottery μ over ν whenever for all outcomes $x \in \operatorname{supp}(\mu)$ and $y \in \operatorname{supp}(\nu)$ this agent weakly prefers x over y. This mild assumption is satisfied for expected utility agents, as well as for agents who prefer μ to ν as soon as μ first-order stochastically dominates ν. While our results do not rely on this assumption, it ensures that dominant actions always lead to weakly preferred lotteries over outcomes.

The set of histories at which agent i moves is denoted $\mathcal{H}_i = \{h \in \mathcal{H} : i_h = i\}$. The set $\mathcal{I}_i$ is a partition of $\mathcal{H}_i$ into **information sets**, where, for any information set $I \in \mathcal{I}_i$ and $h, h' \in I$ and any subhistories $\tilde{h} \subseteq h$ and $\tilde{h}' \subseteq h'$ at which i moves, at least one of the following two symmetric conditions obtains: either (i) there is a history $\tilde{h}^* \subseteq \tilde{h}$ such that $\tilde{h}^*$ and $\tilde{h}'$ are in the same information set, $A(\tilde{h}^*) = A(\tilde{h}')$, and i makes the same move at $\tilde{h}^*$ and $\tilde{h}'$, or (ii) there is a history $\tilde{h}^* \subseteq \tilde{h}'$ such that $\tilde{h}^*$ and $\tilde{h}$ are in the same information set, $A(\tilde{h}^*) = A(\tilde{h})$, and i makes the same move at $\tilde{h}^*$ and $\tilde{h}$. We denote by $I(h) \in \mathcal{I}_i$ the information set containing history h. We say that an information set I_1 **precedes** information set I_2 if there are $h_1 \in I_1$ and $h_2 \in I_2$ such that $h_1 \subseteq h_2$; we then write $I_1 \leq I_2$ (and $I_1 < I_2$ if $I_1 \neq I_2$) and we also say that I_2 **follows** I_1 and that I_2 is a **continuation** of I_1. We say that an outcome x is **possible** at information set I if there is $h \in I$ and a terminal history $\bar{h} \supseteq h$ such that x obtains at $\bar{h}$.

3 Simple Dominance

We propose a class of simplicity standards that relaxes the standard assumption of economic analysis that players can analyze and plan their actions arbitrarily far into the future of the game. Such foresight assumptions are embedded in standard game theoretic concepts of backward induction, dynamic programming, perfect Bayesian equilibrium, iterated dominance, weak dominance, and Li's obvious dominance. In relaxing the foresight assumption, we build on the pioneering approach of Li (2017b) whose obvious dominance allows for players who cannot fully analyze the actions of others, but maintains the assumption that players understand the set of possible outcomes following their own actions and the structure of precedence among their own information sets. While obvious dominance guarantees that when taking an action, agents do not have to reason carefully about what their *opponents* will do, it still may require that they search deep into the game with regard to their future self. It assumes that they know all of *their own* actions they will take in the future, and understand precisely the sets of outcomes that could possibly obtain, conditional on any sequence of their own actions that they plan to take. This is the reason that if White has a winning strategy in chess then—knowing at the start of the game what she needs to do at any possible future configuration of the board in order to ensure a victory—White has a strategy that is not only winning, but also obviously dominant. We relax Li's foresight assumptions, only maintaining that players know possible outcomes of actions at information sets they perceive as simple and the precedence relations among these information sets.

The key innovation in our framework is that the information sets an agent perceives as simple may update as the game is played. In other words, we allow the agent's perception of the strategic situation, and hence, the planned actions—referred to as a "strategic plan" below, to distinguish from the standard game-theoretic notion of a "strategy" as a *complete* contingent plan of action—to vary as the game progresses.

Formally, for each player i and information set $I^* \in \mathcal{I}_i$ at which i moves, there is a set of information sets $\mathcal{I}_{i,I^*} \subseteq \{I \in \mathcal{I}_i | I \supseteq I^*\}$ that are perceived as **simple information sets** from the perspective of I^*. We assume that $I^* \in \mathcal{I}_{i,I^*}$, but otherwise, the only restriction is that $\mathcal{I}_{i,I^*} \subseteq \mathcal{I}_i$.[17] A **(partial) strategic plan** $S_{i,I^*}(\succ_i)$ for agent i of type $\succ_i$ at information set I^* maps each simple information set $I \in \mathcal{I}_{i,I^*}$ to an action at this information set.[18] Note that a strategic plan does not specify the play at all continuation information sets at which i may be called to move, but rather only at the information sets that are simple from the perspective of I^*. Sets of strategic plans $(S_{i,I^*}(\succ_i))_{I^* \in \mathcal{I}_i}$ and $(S_{i,I^*}(\succ_i))_{I^* \in \mathcal{I}_i, \succ_i \in \mathcal{P}_i}$ of agent i are called **strategic collections**.

An **extensive-form mechanism** $(\Gamma, S_{\mathcal{N},\mathcal{I}})$, or simply a **mechanism**, is an extensive-form game Γ together with a profile of strategic collections, $S_{\mathcal{N},\mathcal{I}} = ((S_{i,I^*}(\succ_i))_{I^* \in \mathcal{I}_i, \succ_i \in \mathcal{P}_i})_{i \in \mathcal{N}}$. For any strategic collection $(S_{i,I^*}(\succ_i))_{I^* \in \mathcal{I}_i}$, we define the **induced strategy** $\hat{S}_i(\succ_i) : \mathcal{I}_i \to \cup_{I \in \mathcal{I}_i} A(I)$ as the mapping from information sets to actions defined by $\hat{S}_i(\succ_i)(I) = S_{i,I}(\succ_i)(I)$ for each $I \in \mathcal{I}_i$; that is, $\hat{S}_i(\succ_i)$ is a standard game-theoretic strategy (complete contingent plan of action) defined by agent i selecting the action that is called for by the strategic plan $S_{i,I}$ at information set I itself. For any $S_{\mathcal{N},\mathcal{I}}$ and type realization $\succ_{\mathcal{N}}$, we can find the terminal history/outcome that is reached when the game is played according to the profile of strategic collections $S_{\mathcal{N},\mathcal{I}}(\succ_{\mathcal{N}})$ by following the profile of induced strategies $\hat{S}_{\mathcal{N}}(\succ_{\mathcal{N}})$. For each player i and type $\succ_i$, the induced strategy $\hat{S}_i(\succ_i)$ allows us to define the set of **on-path information sets** for a strategic collection as the information sets $I \in \mathcal{I}_i$ such that there exists other players' and Nature's strategies such that I is on the path of play of $\hat{S}_i(\succ_i)$.

Induced strategies also allow us to define equivalence of mechanisms: two mechanisms $(\Gamma, S_{\mathcal{N},\mathcal{I}})$ and $(\Gamma', S'_{\mathcal{N},\mathcal{I}})$ are **equivalent** if, for every profile of types $\succ_{\mathcal{N}}$, the distribution over outcomes from the induced strategies $\hat{S}_{\mathcal{N}}(\succ_{\mathcal{N}})$ in Γ is the same as from the induced strategies $\hat{S}'_{\mathcal{N}}(\succ_{\mathcal{N}})$ in $\hat{\Gamma}$. This equivalence definition is purely outcome-based, and allows that $(\Gamma, S_{\mathcal{N},\mathcal{I}})$ and $(\Gamma', S'_{\mathcal{N},\mathcal{I}})$ have different classes of simple information sets. Given a mechanism, we can construct the corresponding **social choice rule**—that is, mapping from preference profiles to outcomes—that is implemented. All mechanisms in the same equivalence class implement

[17]The assumption that $\mathcal{I}_{i,I^*} \subseteq \mathcal{I}_i$ is made for simplicity; in its absence we need to endow players with beliefs of what other players do. A natural requirement on the collection of simple node sets is that if an agent classifies an information set $I > I_1$ as simple from the perspective of information set I_1 then the agent continues to classify I as simple from the perspective of all information sets $I_2 > I_1$ such that $I \geq I_2$; while we do not impose this requirement, it is satisfied in all of the examples of simple dominance that we study.

[18]We focus on pure strategies; the extension to mixed strategies is straightforward.

the same social choice rule.

Strategic plan $S_{i,I^*}(\succ_i)$ is **simply dominant at information set** I^* for type $\succ_i$ of player i if the worst possible outcome for i in the continuation game assuming i follows $S_{i,I^*}(\succ_i)(I)$ at all $I \in \mathcal{I}_{i,I^*}$ is weakly preferred by i to the best possible outcome for i in the continuation game if i plays some other action $a' \neq S_{i,I^*}(\succ_i)(I^*)$ at I^*. We say that a strategic collection $(S_{i,I^*}(\succ_i))_{I^* \in \mathcal{I}_i, \succ_i \in \mathcal{P}_i}$ is **simply dominant** if, for each type $\succ_i \in \mathcal{P}_i$, the strategic plan $S_{i,I^*}(\succ_i)$ is simply dominant at I^* for each on-path information set I^*.[19] We say that a game is simple dominant if it admits simply dominant strategies.

Note that the collections of simple information sets, $(\mathcal{I}_{i,I^*})_{I^* \in \mathcal{I}_i}$, is a parameter of the model. In the sequel, we focus on collections of simple information sets that vary in the foresight of the agents, though this is not necessary, and there are other ways to conceptualize what information sets are viewed as simple from a given perspective.[20] Given a fixed $k \in \{0, 1, 2, ..., \infty\}$, we say that agent i can plan k moves ahead but not more if $\mathcal{I}_{i,I^*} = \{I \in \mathcal{I}_i | I^* \leq I \text{ and } I^* < I_1 ... < I_k < I \Rightarrow \exists \ell \in \{1, ..., k\} \text{ s.t. } I_\ell \notin \mathcal{I}_i\}$. We refer to the resulting simply dominant strategic collections as **k-step dominant** and we say that a strategy is k-simple if it is the induced strategy for some k-step dominant strategic collection. Varying k allows us to embed in our model the following special cases:

- $k = \infty$ that is $\mathcal{I}_{i,I^*} = \{I \in \mathcal{I}_i | I^* \leq I\}$; i can plan all of her future moves. We refer to the resulting simply dominant strategic collections as **obviously dominant**, because the induced strategy is obviously dominant in the sense of Li (2017b), and any obviously dominant strategy S_i in the sense of Li (2017b) determines an obviously dominant strategic collection $(S_{i,I^*})_{I^* \in \mathcal{I}_i}$ by defining $S_{i,I^*}(I) = S_i(I)$ for any $I^* \leq I$. If a mechanism admits obviously dominant strategic collections, then we say it is **obviously strategy-proof (OSP)**.

- $k = 1$ that is $\mathcal{I}_{i,I^*} = \{I \in \mathcal{I}_i | I^* \leq I \text{ and } I^* < I' < I \Rightarrow I' \notin \mathcal{I}_i\}$; i can plan one move ahead but not more. We refer to the resulting simply dominant strategic collections as **one-step dominant**. The information sets in $\mathcal{I}_{i,I^*} - \{I^*\}$ are called i's **next** information

[19] When assessing $S_{i,I^*}(\succ_i)(I)$, we take the worst case over all game paths consistent with i following $S_{i,I^*}(\succ_i)(I)$ at all $I \in \mathcal{I}_{i,I^*}$, and compare to the best case over *all* game paths following any alternative action $a' \neq S_{i,I^*}(\succ_i)(I^*)$. While formulated slightly differently than Li (2017b), who invokes the notion of an earliest point of departure between two strategies, our definition is formally equivalent to his when $\mathcal{I}_{i,I^*}$ is the set of all continuation information sets at which i moves. Both we and Li (2017b) require simple dominance (respectively, obvious dominance) only on-path; this choice is in line with e.g. Pearce's (1984) extensive form rationalizability and Shimoji and Watson's (1998) conditional dominance. An alternative approach is to require simple dominance at all nodes (information sets) in the game, including off-path ones.

[20] For instance, the collection of simple information sets could be those at which a measure of computational complexity of a decision problem is below some threshold; cf. e.g. Arora and Barak (2009) for a survey of computational complexity criteria.

sets (from the perspective of I^*). If a mechanism admits one-step dominant strategic collections, then we say it is **one-step simple (OSS)**.

- $k = 0$ that is $\mathcal{I}_{i,I^*} = \{I^*\}$; i cannot plan any future moves. We refer to the resulting simply dominant strategic collections as **strongly obviously dominant**. In this case, we can also talk about strongly obviously dominant *strategies* because, as for obvious dominance, there is a one-to-one correspondence between strategic collections $(S_{i,I^*})_{I^* \in \mathcal{I}_i}$ and the induced strategies $\hat{S}_i(I^*) = S_{i,I^*}(I^*)$. If a mechanism admits strongly obviously dominant strategic collections, then we say it is **strongly obviously strategy-proof (SOSP)**.

Remark 1. Obviously dominant strategic collections and strongly obviously dominant strategic collections are consistent in the following sense: $S_{i,I^*}(I) = S_{i,I}(I)$ for all $I \in \mathcal{I}_{i,I^*}$ and all $I^* \in \mathcal{I}_i$. One-step dominant strategic collections, on the other hand, do not satisfy this property; ascending auctions, discussed in Section 4.2, are an example of such a case. The failure of this property does not mean that agents who plan only one step ahead are inconsistent or irrational. Indeed, such agents might understand that they may adjust their plans later, and think of the partial strategic plan S_{i,I^*} as an argument establishing that playing $S_{i,I^*}(I^*)$ is better than any other action they could take at I^*. The tentativeness of such partial plans is an important possibility in the under-explored game-theoretic paradigm of making choices as they arise, a paradigm that Savage (1954) describes as "you can cross that bridge when you come to it" (cf. Introduction).

A direct verification shows that the smaller the set of simple nodes, the stronger is the resulting simplicity requirement. To formulate this result, for any simple information sets $\mathcal{I}_{i,I^*}$ and $\mathcal{I}'_{i,I^*}$ such that $\mathcal{I}_{i,I^*} \subseteq \mathcal{I}'_{i,I^*}$, we say that a strategic collection $\left(S'_{i,I^*}(\succ_i)\right)_{\succ_i \in \mathcal{P}_i}$ on $\mathcal{I}'_{i,I^*}$ is an $\mathcal{I}'_{i,I^*}$**-extension** of a strategic collection $(S_{i,I^*}(\succ_i))_{\succ_i \in \mathcal{P}_i}$ on $\mathcal{I}_{i,I^*}$ if $S'_{i,I^*}(\succ_i)(I) = S_{i,I^*}(\succ_i)(I)$ for all $I \in \mathcal{I}_{i,I^*}$.

Theorem 1. ***(Nesting of Simplicity Concepts).*** *If simple information sets $\mathcal{I}_{i,I^*}$ and $\mathcal{I}'_{i,I^*}$ are such that $\mathcal{I}_{i,I^*} \subseteq \mathcal{I}'_{i,I^*}$ and strategic collection S_{i,I^*} is simply dominant at I^* for $\mathcal{I}_{i,I^*}$, then any $\mathcal{I}'_{i,I^*}$-extension of S_{i,I^*} is simply dominant at I^* for $\mathcal{I}'_{i,I^*}$.*

As a corollary, we conclude that the lower the parameter k, the more restrictive k-step simplicity becomes. Further, our class of simple dominance concepts has a natural lattice structure, with obvious dominance as its least demanding concept and strong obvious dominance as the most demanding one.

Corollary 1. *(i) Take $k, k' \in \{0, 1, 2, ..., \infty\}$ and assume $k < k'$. Then, any strategic collection that is k-step dominant is also k'-step dominant.*

(ii) If a strategic collection $(S_{i,I^*})_{I^*\in\mathcal{I}_i}$ *is simply dominant for some collection of simple information sets, then the induced strategy* $\hat{S}_i(I^*) = S_{i,I^*}(I^*)$ *is obviously dominant.*

(iii) If the induced strategy $\hat{S}_i(I^*) = S_{i,I^*}(I^*)$ *is strongly obviously dominant, then the strategic collection is simply dominant for any* $(\mathcal{I}_{i,I^*})_{I^*\in\mathcal{I}_i}$.

From an implementation perspective, an immediate consequence of Corollary 1 is that the set of k-step simple implementable social choice rules weakly expands as k is increased. The following result shows that in general, this inclusion is strict: that is, stronger simplicity constraints (lower k) reduce the flexibility of the designer.[21]

Theorem 2. *Let* $k, k' \in \{0, 1, 2, ..., \infty\}$ *and assume* $k' > k$. *There exist social choice rules that are implementable in* k'*-step simple strategic collections, but are not implementable in* k*-step simple strategic collections.*

The presence of the simplicity-implementability trade-off depends on the preference environment: for instance, Theorem 6 shows that in some environments there is no loss in imposing one-step simplicity ($k = 1$) relative to obvious strategy-proofness ($k = \infty$): in these environments, any social choice rule that is OSP-implementable is also OSS-implementable.

To get a sense of why the inclusion can be strict consider an environment with transfers in which there are at least two agents and each agents' values come from the same support with at least three distinct values. Suppose we want to allocate an object to the highest-value agent. This social choice rule can be implemented via an ascending auction and ascending auctions are OSS (we establish the one-step simplicity of ascending auctions in Theorem 6). At the same time, this social choice rule, and the price discovery it entails, cannot be implemented via SOSP mechanisms, which resemble posted prices (the posted price characterization of SOSP is given by our Theorem 8). For k, k' strictly larger than 0, our proof in the appendix constructs social rules that are k'-step simple implementable but not k-step simple implementable in no-transfer single-unit demand allocation environments.

3.1 Behavioral Microfoundations

We may think of simple strategic plans as providing guidance to a player that is unaffected even when they may be confused about the game they are playing, in the sense that they may mistake the game for a different game that has different players, actions, and precedence relations at non-simple information sets. An alternative interpretation is that the player is only given a partial description of the game: each time they are called to move, they are

[21] In particular, the theorem shows that for any $k < \infty$, there are social choice rules that are OSP-implementable but not k-step implementable.

told what happens at their own simple information sets, but not at any other non-simple information set. If players have simply dominant strategic plans, the prediction of play is unaffected by concerns that they might be so-confused.

To formalize this idea, say that game Γ' is **indistinguishable from Γ from the perspective of agent i at information set I^* of game** Γ if there is an injection λ from the set of agent i's simple information sets $\mathcal{I}_{i,I^*}$ in Γ into the set of agent i's information sets $\mathcal{I}'_i$ in Γ' such that:

1. If $I_1, I_2 \in \mathcal{I}_{i,I^*}$ and I_1 precedes I_2 in Γ, then $\lambda(I_1)$ precedes $\lambda(I_2)$ in Γ'.

2. For each $I \in \mathcal{I}_{i,I^*}$, there is a bijection η_I that maps actions at agent's i information set I in Γ onto actions at agent's i information set $\lambda(I)$ in Γ'.

3. An outcome is possible following action a at $I \in \mathcal{I}_{i,I^*}$ in Γ if and only if this outcome is possible following $\eta_I(a)$ at $\lambda(I)$ in Γ'.

We say that $\lambda(I)$ is the game Γ' **counterpart** of information set I and $\eta_I(a)$ is the game Γ' **counterpart** of action a at information set I in game Γ. The concept of indistinguishability captures the idea that agent i understands the precedence relation among simple information sets, as well as the available actions and possible outcomes at these information sets.

Simple dominance is equivalent the standard weak dominance on all games that are indistinguishable from the game played. We say that a strategy S_i of player i **weakly dominates** strategy S'_i in the continuation game beginning at I^* if following strategy S_i leads to weakly better outcomes for i than following strategy S'_i, irrespective of the strategies followed by other players. Note that here, S_i and S'_i denote full strategies in the standard game-theoretic sense of a complete contingent plan of action.

Theorem 3. ***(Behavioral Microfoundation).*** *For each game Γ, agent i, type $\succ_i$, and collection of simple information sets $(I_{i,I^*})_{I^* \in \mathcal{I}_i}$, the strategic plan S_{i,I^*} is simply dominant from the perspective of $I^* \in \mathcal{I}_i$ in Γ if and only if, in every game Γ' that is indistinguishable from Γ from the perspective of i at information set I^*, in the continuation game of Γ' starting at the counterpart of I^*, any strategy that at the counterpart of each $I \in \mathcal{I}_{i,I^*}$ selects the counterpart of $S_{i,I^*}(I)$ weakly dominates any strategy that does not select the counterpart of $S_{i,I^*}(I^*)$ at the counterpart of I^*.*

This theorem tells us the strategic collection $(S_{i,I^*})_{I^* \in \mathcal{I}_i}$ is simply dominant in Γ if and only if for every $I^* \in \mathcal{I}_i$ in every game Γ' that is indistinguishable from the perspective of information set I^* every strategy S'_i that agrees with the counterpart of S_i at all $I \in \mathcal{I}_{i,I^*}$ is weakly dominant in the continuation game starting at the counterpart of I^*. When the strategic collection is consistent, we can express this result equivalently in terms of simplicity of the induced global strategies $S_i(I) = S_{i,I}(I)$. When expressed in this way, this result corresponds to Li's (2017b) microfoundation for obvious strategy-proofness.[22]

3.2 Design Sufficiency of Perfect Information Games

Under perfect information, each information set I contains a single history (or node) h and, to keep the notation at the minimum, we identify history h and information set $\{h\}$. The key parameter of the simplicity definition then becomes the collection $(\mathcal{H}_{i,h^*})_{h^* \in \mathcal{H}_i}$ of **simple histories**, and we denote the corresponding strategic collections by $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$.

Perfect information games plays a special role in designing simply dominant mechanisms because for any imperfect-information simply dominant mechanism, we can find an equivalent perfect-information one.[23] To make this point precise, for any imperfect-information game Γ, define the corresponding perfect information game Γ' with the same set of histories as Γ. Given a collection of simple information sets $(\mathcal{I}_{i,I^*})_{I^* \in \mathcal{I}_i}$ in Γ, we define the induced collection of simple histories $(\mathcal{H}_{i,h^*})_{h^* \in \mathcal{H}_i}$ in Γ' such that $\mathcal{H}_{i,h^*}$ consists of all histories in $\mathcal{I}_{i,I^*}$. For a strategic collection $(S_{i,I^*})_{I^* \in \mathcal{I}_i}$, we define the induced strategic collection $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$ such that $S_{i,h^*}(h) = S_{i,I^*}(I)$, where I is a continuation information set of I^*, $h^* \in I^*$ and $h \in I$.

Theorem 4. ***(Perfect-Information Reduction).*** *If $(S_{i,I^*})_{I^* \in \mathcal{I}_i}$ is simply dominant in an imperfect-information game Γ with simple information sets $(\mathcal{I}_{i,I^*})_{I^* \in \mathcal{I}_i}$, then in the corresponding perfect information game Γ' with the induced simple histories $(\mathcal{H}_{i,h^*})_{h^* \in \mathcal{H}_i}$, the induced strategic collection $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$ is simply dominant.*

To prove the theorem, consider an agent i with type $\succ_i$. Notice that if some history h is on-path for the strategic collection $\left(\hat{S}_{i,h^*}(\succ_i)\right)_{h^* \in \mathcal{H}_i}$ in Γ', then the corresponding information set $I \ni h$ is on-path for the strategic collection $\left(\hat{S}_{i,I^*}(\succ_i)\right)_{I^* \in \mathcal{I}_i}$ in Γ. Furthermore, the worst outcome following $S_{i,h^*}(h) = S_{i,I^*}(I)$ in Γ' is weakly better than the worst outcome over the entire information set I when following this strategy. Similarly, the best outcome following an alternative action $a \neq S_{i,h^*}(h)$ at h is worse than the best outcome following an alternative action $a \neq S_{i,h^*}(h)$ over the entire information set h. Thus, if the strategic plan $S_{i,I^*}(I)$ is simply dominant in Γ, then the induced strategic plan $S_{i,h^*}(\succ_i)$ is simply dominant in Γ'.

[22] While the two results capture the same phenomenon, there is a slight difference between them even when restricted to OSP, as Li's (2017b) microfoundation assumes that λ is a bijection on all agent i's information sets. We could embed this assumption in our analysis by assuming that λ is a bijection from agent i's simple information sets in Γ to simple information sets in Γ'.

[23] An analogous property of obvious strategy-proofness was asserted in Ashlagi and Gonczarowski (2018). Following our work, Mackenzie (2020) extended this property of obvious strategy-proofness to extensive-form games without perfect recall.

In light of Theorem 4 we focus on perfect information games in the study of design of the next two sections.

4 Characterizing Simple Mechanisms

We now consider three special cases of the above simplicity standards—obvious dominance, one-step dominance, and strong obvious dominance—and characterize simple mechanisms and social rules in environments both with and without transfers. To make our analysis relevant for market design applications and to avoid general impossibility results such as the Gibbard-Satterthwaite Theorem, we must impose certain assumptions on the domains of agent preferences. We formalize this as follows: We take as a primitive a **structural dominance relation** over outcomes, denoted $\trianglerighteq$, where $\trianglerighteq$ is a reflexive and transitive binary relation on $\mathcal{X}$. The notation $x \trianglerighteq y$ is read as "x weakly dominates y" or "x trumps y". If $x \trianglerighteq y$ but not $y \trianglerighteq x$, then we write $x \triangleright y$, and say that x strictly dominates (or strictly trumps) y. For instance, in environments with transfers, outcome x trumps outcome y for an agent if the agent receives a higher transfer under outcome x, and all else is equal. We say that a preference ranking $\succsim_i$ is **consistent** with $\trianglerighteq$ if $x \trianglerighteq y$ implies that $x \succsim_i y$ and $x \triangleright y$ implies that $x \succ_i y$.

We allow the possibility that different agents have different dominance relations, $\trianglerighteq_i$, and therefore different preference domains. We assume that all rankings in $\mathcal{P}_i$ are consistent with $\trianglerighteq_i$. If $x \trianglerighteq_i y$ and $y \trianglerighteq_i x$ then x and y are $\trianglerighteq_i$-**equivalent**. Any $\trianglerighteq_i$ determines an **equivalence partition** of $\mathcal{X}$. We refer to each element $[x]_i = \{y \in \mathcal{X} : x \trianglerighteq_i y \text{ and } y \trianglerighteq_i x\}$ of the equivalence partition as a **payoff** of the agent in question. Consistency implies that each preference ranking in $\mathcal{P}_i$ induces a well-defined preference ranking over payoffs in the natural way: $[x]_i \succsim_i [y]_i$ if $x \succsim_i y$ and $[x]_i \succ_i [y]_i$ if $x \succ_i y$. To avoid unnecessary formalism, we use the same symbol for preferences over payoffs as for preferences over outcomes, and write "payoff x" for $[x]_i$ and "payoff x obtains" when the realized outcome belongs to $[x]_i$. Unless stated otherwise, we assume in this section that the preference domain $\mathcal{P}_i$ is **rich** in the following sense: the set of induced preferences over payoffs consists of all strict rankings over payoffs.[24]

[24] Our use of the term richness shares with other uses of the term in the literature the idea that the domain of preferences contains sufficiently many profiles: if certain preference profiles belong to the domain, then some other profiles belong to it as well (cf. Dasgupta, Hammond, and Maskin (1979) and Pycia (2012)). The more outcome pairs that are comparable by the structural dominance relation $\trianglerighteq_i$, the smaller the resulting preference domain, and hence—in Theorems 5, 7, and 8 below—simple dominance is required for fewer preference types. At one extreme, $\trianglerighteq_i$ is an identity relation for each $i \in \mathcal{N}$, agents' preference domains consist of all strict rankings, and the set of simple mechanisms resembles dictatorships as in Gibbard (1973) and Satterthwaite (1975) and our Corollary 2. At the other extreme, $\trianglerighteq_i$ compares all outcomes, each agent is

The framework of rich preference domains is flexible and encompasses many standard economic environments. Examples of rich domains without transfers include:

- **Voting:** Every agent has strict preferences over all alternatives in $\mathcal{X}$. This is captured by the trivial dominance relation $\trianglerighteq_i$ in which $x \trianglerighteq_i y$ implies $x = y$ for all i. Each agent's preference domain $\mathcal{P}_i$ partitions $\mathcal{X}$ into $|\mathcal{X}|$ individual subsets. Richness implies that each $\mathcal{P}_i$ consists of all strict preference rankings over $\mathcal{X}$.

- **Allocating indivisible goods without transfers:** Each $x \in \mathcal{X}$ describes the entire allocation of goods to each of the agents. Each agent has strict preferences over each bundle of goods she may receive, but is indifferent over how goods she does not receive are assigned to others. This is captured by a dominance relation $\trianglerighteq_i$ for agent i defined as follows: $x \trianglerighteq_i y$ if and only if agent i receives the same set of goods in outcomes x and y. Each element of agent i's equivalence partition can be identified with the set of objects she receives. Richness implies that every strict ranking of these sets belongs to $\mathcal{P}_i$ for each i.

With such examples in mind, we say that an environment is **without transfers** if the dominance relation $\trianglerighteq_i$ is symmetric for all i.[25] Non-symmetric dominance relations $\trianglerighteq_i$ allows us to model transfers: all else equal, having more money dominates having less. Examples of rich domains with transfers include:

- **Social choice with transfers:** Let $\mathcal{X} = \mathcal{Y} \times \mathcal{W}^{\mathcal{N}}$, where $\mathcal{Y}$ is a set of substantive outcomes and $\mathcal{W} \subsetneq \mathbb{R}$ a (finite) set of possible transfers. Each agent i prefers to pay less rather than more (for a fixed $y \in \mathcal{Y}$) and is indifferent between any two outcomes that vary only in other agents' transfers. The structural dominance relation is then $(y, w) \trianglerighteq_i (y', w')$ if and only if $y = y'$ and $w_i \geq w'_i$ (where $w \equiv (w_i)_{i \in \mathcal{N}}$ is the profile of transfers).

- **Auctions**: Let $\mathcal{X} \subseteq \mathcal{N}^{\mathcal{O}} \times \mathcal{W}^{\mathcal{N}}$ where $\mathcal{O}$ is a finite set of goods and $\mathcal{W} \subsetneq \mathbb{R}$ is a finite set of transfers. Each agent i prefers to win more goods and to pay less rather than more. Denoting by O_i the set of goods allocated to i and writing $O = (O_i)_{i \in \mathcal{N}}$, the structural dominance relation is given by $(O; w) \trianglerighteq_i (O'; w')$ if and only if $O_i \supseteq O'_i$ and $w_i \geq w'_i$.

indifferent among all outcomes, and all extensive-form games, with any strategies, are simple. In between these extremes, we have other classes of simple mechanisms, as we explore in this section. We would like to thank referees for these clarifications.

[25] A binary relation $\trianglerighteq_i$ is *symmetric* if $x \trianglerighteq_i y$ implies $y \trianglerighteq_i x$. It is easy to see that this holds in the examples without transfers above, but not in those with transfers below.

These are just a few examples of settings that fit into our general model. While richness is a flexible assumption, not all preference domains are rich. For instance, domains of single-peaked preferences are typically not rich and Arribillaga, Massó, and Neme (2020) show that our millipede construction does not extend to single-peaked preference domains.

4.1 Obvious Dominance

Obvious dominance (defined above) is the simplicity standard introduced by Li (2017b). Recall that in analyzing obvious dominance we do not need to distinguish between strategies and strategic plans; thus, for simplicity of exposition, we focus on strategies. If a game Γ admits a profile of obviously dominant strategies, then the game and the resulting mechanism (game and strategy profile) are said to be **obviously strategy-proof (OSP)**.

Li's (2017b) shows that in binary allocation environments with transfers, every OSP mechanism is equivalent to a personal clock auction. In this section, we focus on environments without transfers and show that any OSP game is equivalent to what we call a *millipede game*. Roughly speaking, a millipede game is a clinch-or-pass game similar to a centipede game (Rosenthal, 1981), but in general with more players and more actions (i.e., "legs") at each node. A simple example of a millipede game in an object allocation environment is a deterministic **serial dictatorship** in which there are no passing moves and all payoffs that are not precluded by the earlier choices of other agents are clinchable (cf. Sections 4.3 and 5).

As a preliminary step to define millipede games, we introduce the following definitions, which apply to any game Γ. Given some history h, we say that payoff x is **possible** for agent i at h if there is a terminal history $\bar{h} \supseteq h$ such that at the outcome associated with $\bar{h}$, agent i obtains payoff x. We use $P_i(h)$ to denote the set of possible payoffs for i at h. We say that agent i has **clinched** payoff x at history h if at all terminal histories $\bar{h} \supseteq h$, agent i receives payoff x. If i moves at h, takes action $a \in A(h)$, and has clinched x at the history (h, a), then we call action a a **clinching action**; any action at h that is not a clinching action is called a **passing action**. We denote by $C_i(h)$ the set of all payoffs x that are **clinchable** for i at h; that is, $C_i(h)$ is the set of payoffs for which there is an action $a \in A(h)$ such that i has clinched x at the history (h, a). At a terminal history $\bar{h}$, no agent is called to move and there are no actions; however, it is notationally useful to define $C_i(\bar{h}) = \{x\}$, where x is the payoff that i obtains at terminal history $\bar{h}$.

We further define $C_i^{\subseteq}(h) = \{x : x \in C_i(h') \text{ for some } h' \subseteq h \text{ s.t. } i_{h'} = i\}$ to be the set of payoffs that i can clinch at some subhistory of h, and $C_i^{\subset}(h) = \{x : x \in C_i(h') \text{ for some } h' \subsetneq h \text{ s.t. } i_{h'} = i\}$ to be the set of payoffs that i can clinch at some strict subhistory of h. Note that while the definition of $C_i(h)$ presumes that i moves at h or h is terminal, the payoff sets $P_i(h)$, $C_i^{\subseteq}(h)$ and $C_i^{\subset}(h)$ are well-defined for any h, whether i moves at h or not, and whether h is terminal or not. Finally, consider a history h such that $i_{h'} = i$ for some $h' \subsetneq h$ and either $i_h = i$ or h is a terminal history. We say that payoff x **becomes impossible** for i at h if $x \in P_i(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$, but $x \notin P_i(h)$. We say payoff x is **previously unclinchable** at h if $x \notin C_i^{\subset}(h)$.

Given a mechanism $(\Gamma, S_{\mathcal{N}})$ and a type $\succ_i$, we call strategy $S_i(\succ_i)$ a **greedy strategy** if at any history $h \in \mathcal{H}_i$ it satisfies the following: if the $\succ_i$-best still-possible payoff in $P_i(h)$ is clinchable at h, then $S_i(\succ_i)(h)$ clinches this payoff; otherwise, $S_i(\succ_i)(h)$ is a passing action. A greedy strategic plan is defined in the same way.[26]

Given these definitions, we define a **millipede game** as a finite extensive-form game of perfect information that satisfies the following properties:

1. Nature either moves once, at the empty history $h_{\varnothing}$, or Nature has no moves.

2. At any history at which an agent moves, all but at most one action are clinching actions, and following any clinching action, the agent does not move again.

3. At all h, if there exists a previously unclinchable payoff x that becomes impossible for agent i_h at h, then $C_{i_h}^{\subset}(h) \subseteq C_{i_h}(h)$.

We refer to millipede games with greedy strategies as **millipede mechanisms**. In a millipede game, it is obviously dominant for an agent to clinch the best possible payoff at h whenever it is clinchable. The last condition of the millipede definition ensures that passing at h is obviously dominant when an agent's best possible payoff at h is not clinchable.

Theorem 5. ***(Millipedes).*** *Consider an environment without transfers. Every OSP mechanism is equivalent to a millipede mechanism. Every millipede mechanism is OSP.*

This theorem is applicable in many environments. This includes allocation problems in which agents care only about the object(s) they receive, in which case, clinching actions correspond to taking a specified (set of) object(s) and leaving the remaining objects to be distributed amongst the remaining agents. Theorem 5 also applies to standard social choice problems in which no agent is indifferent between any two outcomes (e.g., voting), in which case clinching corresponds to determining the final outcome for all agents. In such environments, we have the following:

[26] A stronger concept of a greedy strategy would additionally require that when passing, the agent takes an action a such that they are indifferent between the best possible payoffs at h and (h, a). (Such an action a exists because $P_i(h) = \cup_{a \in A(h)} P_i((h, a))$.) This distinction is immaterial for millipede games, since they have at most one passing action at each history, and all of our results are valid for both concepts of greediness.

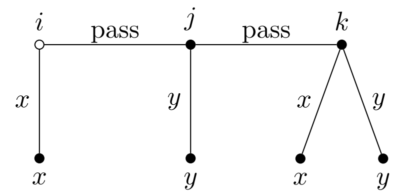

Figure 1: An example of a non-dictatorial millipede game in a voting environment with two outcomes, $\mathcal{X} = \{x, y\}$. The obviously dominant (greedy) strategy profile is for any agent to clinch if she is offered to clinch her preferred option among $\{x, y\}$, and otherwise pass.

Corollary 2. *Let each agent's preference domain $\mathcal{P}_i$ be the space of all strict rankings over outcomes $\mathcal{X}$. Then, every OSP game is equivalent to a game in which either:*

(i) the first agent to move can clinch any possible outcome and has no passing action; or

(ii) there are only two outcomes that are possible when the first agent moves, and the first mover can either clinch any of them, or can clinch one of them or pass to a second agent, who is presented with an analogous choice, etc.

The former case of Corollary 2 is the standard dictatorship, with a possibly restricted set of outcomes. The latter case is a generalization that allows an agent to enforce one of the two outcomes, but not the other, at her turn;see Figure 1 for an example. In particular, this corollary gives an analogue of the Gibbard-Satterthwaite dictatorship result, with no efficiency assumption.

The full proof of Theorem 5 is in the appendix; here, we provide a brief sketch of the more interesting direction that for any OSP game Γ, there is an equivalent millipede game. We construct this millipede game via the following transformations. Starting with any arbitrary game, we begin by breaking information sets; this only shrinks the set of possible outcomes any time an agent is called to play, which preserves the min/max obvious dominance inequality. For similar reasons, we can shift all of Nature's moves to the beginning of the game, and so now have a perfect information game Γ' in which Nature moves once, as the first mover.[27] Second, if there are two passing actions a and a' at some on-path history h, then there are (by definition) at least two payoffs that are possible for i following each. We show that obvious dominance then implies that i must have some continuation strategy that can guarantee his top possible payoff in the continuation game following at least one of a or a'.

[27] The first part of this transformation is a special case of Theorem 4. That every OSP game is equivalent to an OSP game with perfect information was first pointed out in a footnote by Ashlagi and Gonczarowski (2018), which also notes that de-randomizing an OSP game leads to an OSP game. For completeness, Lemma A.4 provides the details.

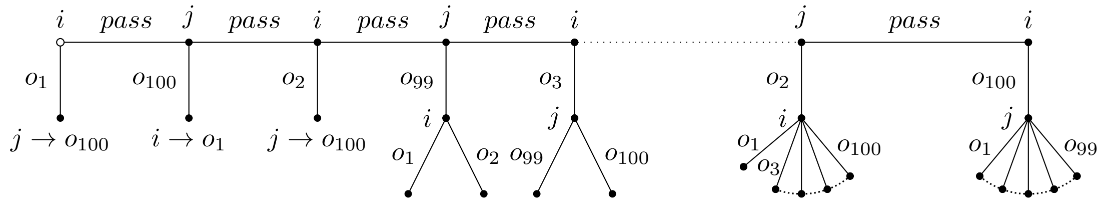

Figure 2: An example of a millipede game with two agents $\{i, j\}$ and 100 objects $\{o_1, o_2, \ldots, o_{100}\}$. If the first clinching is in an agent's first 50 moves, then the other agent is given the choice of clinching any object he or she could have clinched previously; if the first clinching is after the clinching agent's first 50 moves, then the other agent is given the choice of clinching any still-available object.

Then, we can construct an equivalent game via a transformation in which we add an action that allows i to clinch this payoff already at h by making all such "future choices" today. We also rely on Li's pruning, in which the actions no type chooses are removed from the game tree, cf. Appendix A.1. We repeat these transformations until there is at most one passing action remaining. The final step of the proof is to show that these transformations give us a millipede game. This last step relies on richness and shows that if there remains some h such that agent i cannot clinch her favorite possible payoff at h, the game must promise i that she will never be strictly worse off by passing, which is condition 3.

4.2 One-Step Dominance

In this section, we analyze the stronger simplicity concept of one-step simple dominance. To see why strengthening of obvious dominance might be useful, recall that obviously dominant strategies may not be intuitively simple; an already discussed stark example is White's winning strategy in chess. As another example, consider a no-transfer object allocation environment and the two-player millipede game in Figure 2. At the first move, type $o_{100} \succ_i o_1 \succ_i o_2 \succ_i \cdots \succ_i o_{99}$ is offered her second-favorite object, o_1, while her top choice, o_{100}, is possible. The obviously dominant greedy strategy of this type is to pass; however, if she does so, she may not be offered the opportunity to clinch her top object, o_{100}, or even go back to her second-best object, o_1, until far into the future. Thus, while passing is obviously dominant, comprehending this requires the ability to reason far into the future of the game and to perform lengthy backwards induction.[28]

[28]The first 100 moves of this millipede cannot be substantially shortened because, given the players' greedy strategies, for $k = 1, \ldots, 50$, i can obtain o_{k+1} if and only if j's top choice is $o_{100-k+1}$ or a lower-indexed object, and j can obtain $o_{100-k+1}$ if and only if i's top choice is o_k or a higher-indexed object.

The more demanding concept of one-step simplicity eliminates the intuitively complex, yet still formally obviously dominant, strategies such as White winning strategy in chess and the greedy strategy in the millipede of Figure 2, while still classifying greedy strategies in serial dictatorships and ascending auctions as simple.

Binary allocation with transfers

Li (2017b) illustrates the usefulness of obvious dominance in the setting of binary allocation with transfers, defined as follows. The set of outcomes is $\mathcal{X} = Y \times \mathbb{R}^{\mathcal{N}}$, where $Y \subseteq \{0,1\}^{\mathcal{N}}$ is a set of feasible allocations and $\mathbb{R}^{\mathcal{N}}$ is the set of profiles of transfers, one for each agent; a generic allocation is denoted y and a generic profile of transfers $w = (w_i)_{i\in\mathcal{N}}$. In this section, we denote types by $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$, where $0 \leq \underline{\theta}_i < \overline{\theta}_i < \infty$, and assume each agent has preferences represented by a quasilinear utility function: $u_i(\theta_i, y, w) = \theta_i y_i + w_i$.[29] This framework captures many important environments of economic interest, including single-unit auctions, procurement auctions, and binary public goods games.

For these environments, Li introduces the class of *personal clock auctions*, which generalize the ascending auction in several ways: agents may face different individualized prices ("clocks"); at any point, there may be multiple quitting actions that allow agents to drop out of the auction, or multiple continuing actions that allow them to stay in the auction; and when an agent quits, her transfer need not be zero. The key restrictions are that each agent's clock must be monotonic, and that whenever the personal price an agent faces strictly changes, she must be offered an opportunity to quit. The formal definition of a personal clock auction can be found in Appendix B.3, where we also prove Theorem 6.

Li (2017b) shows that in binary allocation settings, OSP games are equivalent to personal clock auctions. Using our new conception of simplicity, we can strengthen this result to show that personal clock auctions are also OSS, and so, perhaps surprisingly, there is no loss in imposing one-step dominance: any OSP-implementable social choice rule is also implementable in one-step dominant strategic collections.

Theorem 6. ***(OSS and Personal Clock Auctions).*** *In binary allocation settings with transfers, every one-step simple mechanism is equivalent to a personal clock auction with one-step dominant strategic collections. Furthermore, every personal clock auction is one-step simple.*

Because our Corollary 1 shows that any OSS mechanism is also OSP, the first part of the theorem follows from Li's (2017b) result that any OSP mechanism is equivalent to

[29] We allow for a continuum of types and transfers here in order to reproduce the binary allocation environment of Li (2017b). Our simplicity concepts extend to this environment when we substitute inf for min and sup for max in our definitions. Richness plays no role in the binary allocation results.

a personal clock auction with greedy strategies, provided we can find a profile of one-step dominant strategic collections that replicates the play of Li's greedy strategies. We construct these collections in the proof of the second part of the theorem. For the special case of a standard ascending auction for a single good, this can be easily done as follows: at any information set I_i^* such that the current price p is weakly lower than the bidder i's value v_i: i stays In, with a plan to drop Out at any next-information set $I_i \supset I_i^*$. For any information set I_i^* such that the current price is $p > v_i$: i drops Out immediately. This strategic collection leads to the same outcome as the greedy strategy of staying in at prices weakly below v_i and dropping out at prices strictly above. The collection is one-step dominant because at any p at which the agent's stay In, she can plan to quit at the next step and get a payoff of 0, which is no worse than quitting now. For personal clock auctions more generally, a key feature is whenever an agent's price changes, she must be offered an opportunity to quit. This feature allows us to construct one-step dominant strategic plans analogous to those just defined for ascending auctions. The complete argument can be found in the Supplementary Appendix.

Environments without transfers

In environments without transfers, we have seen millipede games that, while OSP, may still be quite complex and require lengthy foresight on the part of the agents (e.g., Figure 2). Imposing the stronger standard of OSS eliminates these complex millipede games, and leaves only games that are monotonic in the following sense: a millipede game Γ is **monotonic** if, for any agent i and any histories h, h' such that: $(h, a^*) \subseteq h'$ where a^* is a passing action at h, $i_h = i$, $i_{h'} = i$ or h' is terminal, and $i_{h''} \neq i$ for any h'' such that $h \subsetneq h'' \subsetneq h'$, either (i) $C_i(h) \subseteq C_i(h')$ or (ii) $P_i(h) \setminus C_i(h) \subseteq C_i(h')$. In words, this says that if an agent passes at h, the next time she moves, she is offered to clinch either (i) everything she could have clinched at h or (ii) everything that was possible, but not clinchable at h. Some millipede games, such as serial dictatorships in which each agent only moves once and has no passing action, are trivially monotonic; for a less trivial example of a monotonic millipede game, see Figure 3 in Section 5. We say that a mechanism is monotonic when the underlying game is.

Theorem 7. ***(Monotonic Millipedes).*** *In environments without transfers, every one-step simple millipede mechanism is equivalent to a monotonic millipede mechanism with one-step dominant strategic collections. Furthermore, every monotonic millipede mechanism is one-step simple.*

At any history h in a monotonic millipede game, the one-step dominant strategic plan is as follows: if the agent can clinch her top still-possible outcome at h, then she does so; otherwise, the agent passes at h, and for any next-history h', the strategic plan is to clinch her top possible object in $C_i(h')$. If clause (i) of monotonicity holds, then this is at least as good as anything she could clinch at h (since the clinchable set weakly expands); if clause (ii) of monotonicity holds, then she obtains her best possible payoff in $P_i(h)$, which is again at least as good as anything that was clinchable at h.

From the perspective of an agent playing in a game, monotonic games seem particularly simple: each time an agent is called to move, she knows that if she chooses to pass, at her next move, she will either be able to clinch everything she is offered to clinch currently, or she will be able to clinch her top remaining choice. On the other hand, in a non-monotonic game such as that in Figure 2, an agent's possible clinching options may be strictly worse for many moves in the future, before eventually being re-offered what she was able to clinch in the past (or something better). If agents are unable to plan far ahead in the game tree, it may be difficult to recognize that passing is obviously dominant in such a game; in a monotonic game, however, agents only need to be able to plan at most one step at a time to recognize that passing is a dominant choice.

Further, from a practical implementation perspective, monotonic games are also particularly simple for a designer to run dynamically: at each step, the designer only need tell an agent her possible clinching options today, plus that if she passes, at her very next move, her clinchable set will either weakly expand, or she will be offered everything possible that she was not offered today. Such a partial, one-step-at-a-time description is simpler than trying to describe all of the possibilities many moves in the future that would be necessary to implement more complex, non-monotonic OSP games.

4.3 Strong Obvious Dominance, Choice Mechanisms, and Posted Prices

In light of Theorem 1, the strongest simplicity standard in our class is strong obvious dominance. If a game Γ admits a profile of strongly obviously dominant strategic collections, we say that it is **strongly obviously strategy-proof (SOSP)**. Random Priority is SOSP, but ascending auctions are not. Thus, SOSP mechanisms further delineate the class of games that are simple to play, by eliminating millipede games that require even one-step forward-looking behavior. As for obvious dominance, there is a one-to-one correspondence between strongly obviously dominant strategic collections and strongly obviously dominant strategies, and so for simplicity of exposition, in this section we focus on strategies. Additionally, in this section we make use of the concept of an undominated payoff, where we say that a payoff x is **undominated** (or untrumped) in a subset of payoffs for agent i if there is no payoff y in this subset such that $y \rhd_i x$. A mechanism $(\Gamma, (S_i(\succ_i))_{i\in\mathcal{N}})$ is **pruned** if every information set in Γ is on path for some type of some player; Li (2017b) observed that every OSP mechanism is equivalent to a pruned OSP mechanism and the same is true for SOSP, cf. Appendix A.1.

Strongly obvious strategy-proof games are particularly simple to play. Any strongly obviously dominant strategy is greedy. Further, SOSP games can be implemented so that each agent is called to move at most once and has at most one history at which her choice of action is payoff-relevant. Formally, we say a history h at which agent i moves is payoff-irrelevant for this agent if i receives the same payoff at all terminal histories $\bar{h} \supset h$; if i moves at h and this history is not payoff-irrelevant, then it is **payoff-relevant** for i. The definition of SOSP and richness of the preference domain give us the following.

Lemma 1. *Along each game path of a pruned SOSP mechanism, there is at most one payoff-relevant history for each agent.*

This result—proven in Supplementary Appendix B.5—allows us to further conclude that, for a given game path, the unique payoff-relevant history (if it exists) is the first history at which an agent is called to move. While an agent might be called to act later in the game, and her choice might influence the continuation game and the payoffs for other agents, it cannot affect her own payoff.

Building on Lemma 1, we show that SOSP effectively implies that agents—in a sequence—are faced with choices from personalized menus, e.g., in allocation with transfers this may be menus of object-price pairs. At the typical payoff-relevant history an agent is offered a menu of payoffs that she can clinch, she selects one of the alternatives from the menu, and she is never called to move again. More formally, we say that Γ is a **sequential choice game** if it is a perfect-information game in which Nature moves first (if at all). The agents then move sequentially, with each agent called to play at most once. The ordering of the agents and the sets of possible outcomes at each history are determined by Nature's action and the actions taken by earlier agents. As long as there are either at least three distinct undominated payoffs possible for the agent who is called to move or there is exactly one such payoff, the agent can clinch any of the possible payoffs. When exactly two undominated payoffs are possible for the agent who moves, the agent can be faced with either (i) a set of clinching actions that allow the agent to clinch either of the two payoffs, (ii) a passing action and a set of clinching actions that allow the agent to clinch exactly one of these payoffs. Note that we allow potentially many ways of clinching the same payoff; we can conceptualize the many was of clinching a fixed payoff as clinching it and sending a message from a predetermined set of messages. Note also that (ii) does not allow the agent to clinch the other payoff.

Theorem 8. ***(Sequential Choice).*** *Every strongly obviously strategy-proof mechanism is equivalent to a sequential choice mechanism with greedy strategies. Every sequential choice mechanism with greedy strategies is strongly obviously strategy-proof.*

Theorem 8 applies to any rich preference environment, including both those with and without transfers. In an object allocation model without transfers, every SOSP mechanism resembles a sequential dictatorship, in which agents are called sequentially and offered to clinch any object that still can be clinched given earlier clinching choices; they pick their most preferred object and leave the game. The key difference between a sequential choice game and a sequential dictatorship is that at an agent's turn, she need not be offered all still-available objects.

In environments with transfers, sequential choice games can be interpreted as sequential posted-price games. In a binary allocation setting with a single good and transfers, each agent is approached one at a time, and given a take-it-or-leave-it (TIOLI) offer of a price at which she can purchase the good; if an agent refuses, the next agent is approached, and given a (possibly different) TIOLI offer, etc. If there are multiple objects for sale, each agent is offered a menu consisting of several bundles of objects with associated transfers, and selects her most preferred option from the menu.

Price mechanisms are ubiquitous in practice. Even on eBay, which began as an auction website, Einav et al. (2018) document a dramatic shift in the 2000s from auctions to posted prices as the predominant selling mechanism. Posted prices have also garnered significant attention in the computer science community. For instance, computing the optimal allocation in a combinatorial Vickrey auction can be complex even from a computational perspective, and several papers have shown good performance using sequential posted price mechanisms (e.g., Chawla, Hartline, Malec, and Sivan (2010) and Feldman, Gravin, and Lucier (2014)). By formalizing a strategic simplicity-based explanation for the popularity of these mechanisms, our Theorem 8 complements this literature.[30]

5 Random Priority

As an application we show that OSP can be combined with natural fairness and efficiency axioms to provide a characterization of the popular Random Priority (RP) mechanism. In Random Priority, first Nature selects an ordering of agents, and then each agent moves in turn and chooses her favorite object among those that remain available given previous agents'

[30]Prior economic studies on the focal role of posted prices in mechanism design—e.g., Hagerty and Rogerson (1987) and Copic and Ponsati (2016)—focused on bilateral trade, while our analysis is applicable to any economic environment satisfying our richness assumption.

choices. Random Priority succeeds on three important design dimensions: it is simple to play, efficient, and fair.[31] However, this is only a partial explanation of its success, as to now, it has remained unknown whether there exist other such mechanisms, and, if so, what explains the relative popularity of RP over these alternatives (cf. footnote 10). Theorem 9 provides an answer to this question: not only does Random Priority have good efficiency, fairness, and incentive properties, it is the *only* mechanism that does so, thus explaining the widespread popularity of Random Priority in practice.

We consider a canonical object allocation model with single-unit demand, a special case of our general framework. There is a set $\mathcal{N}$ of agents, a set of objects, also of cardinality $|\mathcal{N}|$, and global outcomes are bijections between agents and objects. Each agent has a strict preference ranking $\succ_i$ over the objects. Our efficiency concept is Pareto efficiency: an outcome is **Pareto efficient** when no other outcome is weakly preferred by all participants and strictly preferred by at least one; a mechanism $(\Gamma, S_{\mathcal{N}})$ is **Pareto efficient** if it generates Pareto efficient outcomes for all Nature's choices and agents' types.[32] Our fairness concept is symmetry: a mechanism $(\Gamma, S_{\mathcal{N}})$ is **symmetric** if, for any two agents $i, j \in \mathcal{N}$, the outcome distribution of the mechanism does not change when we transpose the preference rankings of i and j and at the same time transpose the objects the two agents obtain. Informally, the outcome of the mechanism would not change if i played the role of j and vice versa.[33] The symmetry condition fails in a serial dictatorship in which player 1 chooses first among all outcomes and then player 2 chooses among all remaining outcomes: if they have the same most preferred object then 1 obtains this object in the original serial dictatorship but not in the transposed one. Random Priority orders the agents randomly, and in effect the probability agent 1 obtains the preferred object is the same before and after the transposition.

Theorem 9. ***(Random Priority).*** *An obviously strategy-proof mechanism is symmetric and Pareto efficient if and only if it is equivalent to Random Priority.*

As discussed above, it is well-known Random Priority satisfies OSP, symmetry, and Pareto efficiency. The converse implication is new. Theorem 9 remains true if we replace OSP with OSS, SOSP, or any other of our simplicity standards; this is implied by combining Theorem 9 with Theorems 1 and 8.

[31] Pareto efficiency and fairness of RP have been recognized at least since Abdulkadiroğlu and Sönmez (1998), while Li (2017b) established OSP of RP. It is easy to see that the standard extensive-form implementation of RP also satisfies all of our more demanding simplicity requirements.

[32] Because we focus on obviously strategy-proof mechanisms, $S_{\mathcal{N}}$ here denotes a profile of strategies in the standard game-theoretic sense, rather than strategic plans.

[33] We formalize the concept of a role in the appendix. Because any permutation can be decomposed into a composition of transpositions, we can equivalently state the symmetry property as $\sigma^{-1} \circ (\Gamma, S_{\mathcal{N}}) \circ \sigma = (\Gamma, S_{\mathcal{N}})$ for all permutations $\sigma : \mathcal{N} \to \mathcal{N}$.

The first step in proving Theorem 9 is to recognize that it is sufficient to prove it for any uniform randomization over Pareto efficient deterministic millipedes. The reduction to symmetric randomizations over Pareto efficient deterministic OSP mechanisms follows because every symmetric mechanism is a lottery over symmetric randomizations, and if each of these randomizations is equivalent to Random Priority than so is the the lottery over them (details are in the appendix). The further reduction to randomizations over Pareto efficient millipedes follows from our Theorem 5.

At the core of the reminder of the proof is the construction of a bijection between permutations of any deterministic Pareto-efficient millipede and permutations of serial dictatorships such that the outcomes of the permuted millipede and permuted serial dictatorship are exactly the same. The existence of such a bijection implies that uniform randomizations over permutations of a deterministic Pareto-efficient millipede give the same resulting outcome distribution as Random Priority.[34] The full construction is lengthy and involved, and its details can be found in the appendix. Here, we provide a simple three-agent example to showcase the general idea.[35]

Consider the millipede game presented in Figure 3. The game allocates three objects A, B, and C to three agents (or players) $1, 2$ and 3. Agent 1 moves first and can clinch one of the objects A and B or pass. The second move is made by agent 2, who either clinches an object (in which case the allocation is fully determined) or passes (the passing move is only possible following a pass by 1). Agent 3 only moves following two passes; this player can then clinch any object. If Agent 3 clinches A or B then the allocation is determined, and if Agent 3 clinches C then Agent 1 can choose between A and B. This game is a millipede (and OSP), and is Pareto efficient for any preference profile.

We can apply any permutation of agents, $\sigma : \mathcal{N} \to \mathcal{N}$, to permute an entire mechanism. In a serial dictatorship, this corresponds to changing the order in which the agents select. Similarly, in any arbitrary base game each permutation σ creates a **permuted game** in which agent i is given the moves and payoffs of agent $\sigma(i)$. For instance, if in game Γ in such Figure 3 the agents are permuted by σ such that $\sigma(1) = 1$, $\sigma(2) = 3$, and $\sigma(3) = 2$, then the first agent to move is still agent 1 but the second agent to move is agent 3, and agent 2

[34]The bijection idea was first employed by Abdulkadiroğlu and Sönmez (1998), and has since been used by others (e.g., Pathak and Sethuraman (2011) and Carroll (2014)). Our construction is different from the bijections in the earlier literature, and relies on the properties of millipede games established by us, and on the properties of Pareto efficient OSP mechanisms subsequently obtained by Bade and Gonczarowski (2017).

[35]For $|\mathcal{N}| = 1$, the equivalence follows from Pareto efficiency. For $|\mathcal{N}| = 2$, the equivalence is implied by Pareto efficiency when agents rank objects differently and it is implied by symmetry when they rank objects in the same way. Cf. Bogomolnaia and Moulin (2001) who also analyze the three-agent case; their approach is not applicable beyond three agents.

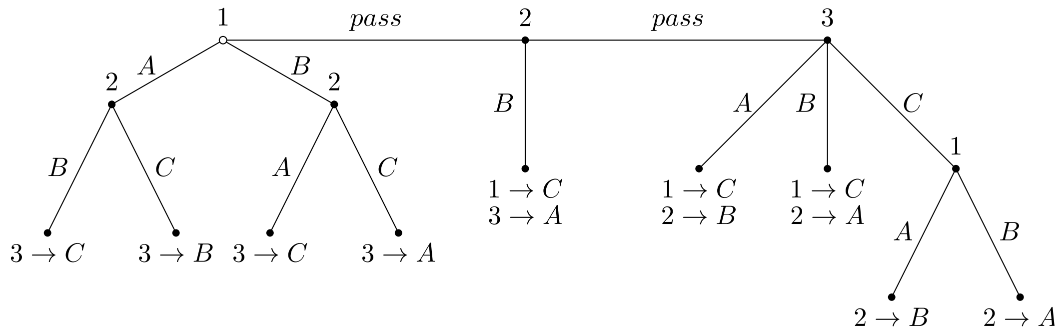

Figure 3: An OSP and Pareto efficient game Γ among three players 1, 2 and 3. Clinching move marked by the object being clinched (A, B,or C); passing moves marked "pass."

moves only after two passes of agents 1 and 3.

For the sake of illustration, suppose that the preferences are such that $A \succ_1 B \succ_1 C$ for agent 1, $A \succ_2 B \succ_2 C$ for agent 2, and $C \succ_3 B \succ_3 A$ for agent 3. We assume that all agents play greedy strategies. Then, under each permutation σ such that agents 1 or 2 are the first movers, game Γ is played as serial dictatorship would be played: the first mover picks their favorite object, A, and the second mover picks their favorite still-available object, thus also determining the allocation of the third mover. In constructing the bijection between permutations of Γ and permutations of serial dictatorships, we map each of the above permutations $\sigma(\Gamma)$ to the corresponding serial dictatorships. As we prove in the appendix, whenever the game starts with several agents choosing clinching moves, we can map it into a serial dictatorship that starts with the same agents moving in the same order.

The mapping of games that involve passing is more subtle. In our example, passing is on the game path if $\sigma(1) = 3$. There are two such permutations: if $\sigma(2) = 2$ then the resulting outcome is $\{(1, A), (2, B), (3, C)\}$, and if $\sigma(2) = 1$ then the resulting outcome is $\{(1, B), (2, A), (3, C)\}$. To what serial dictatorships should we map these two permutations? The unique mapping achieving the bijection—given how we mapped other permutations—maps the first of the two permutations into the serial dictatorship with agents ordered 3, 1, 2 and the second one into the serial dictatorship with agents ordered 3, 2, 1. There is no simple rule of thumb in mapping permutations that entails passing on the path of play—notice e.g. that in the present example the serial dictatorship order is not the order in which the agents move, nor is it the order in which the agents clinch—and the bulk of the proof is devoted to the general construction of such a mapping.

6 Conclusion

We study the question of what makes a game "simple to play", and introduce a general class of simplicity standards that vary the foresight abilities required of agents in extensive-form imperfect-information games. We consider agents that form a strategic plan only for a limited horizon in the continuation game, though they may update these plans as the game progresses and the future becomes the present. The least restrictive simplicity standard include in our class is Li's (2017b) obvious strategy-proofness, which presumes agents have unlimited foresight of their own actions, while the strongest, strong obvious strategy-proofness, presumes no foresight. For each of these standards, as well as an intermediate standard of one-step simplicity, we provide characterizations of simple mechanisms in various environments with and without transfers, and show that our simplicity standards delineate classes of mechanisms that are commonly observed in practice. Among these results, we show that Li's characterization of OSP mechanisms as personal clock auctions can be strengthened to OSS, and that SOSP mechanisms are equivalent to price mechanisms, which are ubiquitous in practice. Finally, in the context of object allocation without transfers, we provide an explanation for the popularity of Random Priority by showing that it is the essentially unique mechanism that is OSP, efficient, and symmetric. Along the way, we provide a logically consistent—though limited to simple games—approach to the analysis of agents with limited foresight.

Our results contribute to the understanding of the fundamental trade-off between simplicity of mechanisms and the ability to implement other social objectives, such as efficiency and revenues. In environments with transfers, Vickrey (1961), Riley and Samuelson (1981), Myerson (1981), Manelli and Vincent (2010), and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) show that the efficiency and revenues achieved with Bayesian implementation can be replicated in dominant strategies; thus the accompanying increase in simplicity may come without efficiency and revenue costs. Li (2017b) and our paper advance this insight further and establish that obviously strategy-proof and one-step simple mechanisms can also implement efficient outcomes (and revenue-maximizing outcomes). At the same time, strong obvious dominance is more restrictive, and more severely limits the class of implementable objectives. In environments with transfers, SOSP generally precludes efficiency and revenue maximization.[36] In environments without transfers, however, even SOSP mech-

[36] For instance, when we want to allocate an object to the highest value agent with transfers and with at least two agents and agents' values are drawn iid from among at least three values, an impossibility result obtains: no SOSP and efficient mechanism exists. This is implied by Theorem 8. This also shows that SOSP mechanisms raise less revenue than optimal auctions. On the other hand, Armstrong (1996) shows that posted prices achieve good revenues when bundling allows the seller to equalize the valuations of buyers, and Chawla, Hartline, Malec, and Sivan (2010) and Feldman, Gravin, and Lucier (2014) show that sequential

anisms—serial dictatorships—can achieve efficient outcomes. Combining our results with the mechanism equivalence analysis of Pycia (2017) allows us to conclude that, in single-unit demand allocation problems without transfers, the restriction to strongly obvious strategy-proof mechanisms allows the designer to achieve virtually the same efficiency and many other objectives as those achievable in merely strategy-proof mechanisms. Thus in many environments, simplicity entails no efficiency loss. In other environments, the trade-off between simplicity and efficiency is more subtle. Our Theorem 2 shows that in general, in environments both with and without transfers, imposing more restrictive simplicity standards on the mechanisms limits the set of implementable social choice functions.[37]

Our work is complementary to the experimental literature on how mechanism participants behave and what elements of design enable them to play equilibrium strategies, cf. e.g. Kagel et al. (1987) and Li (2017b). While this literature identifies implementation features that facilitate play and confirms that obviously strategy-proof mechanisms are indeed simpler to play than merely strategy-proof mechanisms, while strongly obviously strategy-proof mechanisms are easier still and nearly all participants play them as expected (see Bo and Hakimov, 2020),[38] our general theory of simplicity opens new avenues for experimental investigations. For instance, we may define the simplicity level of a game in terms of the smallest (in an inclusion sense) set of histories that an agent must see as simple in the sense of Section 4 in order to play the equilibrium strategy correctly; or as the highest k that still allows the agent to play k-simple strategies correctly. We may similarly define the measure of sophistication of experimental subjects as the highest k that allows the subjects to play k-simple strategies correctly.

In sum, the sophistication of agents may vary across applications, and so it is important to have a range of simplicity standards. For sophisticated agents, a weaker simplicity standard ensures they play the intended strategies, allowing the designer more flexibility on other objectives; however, for less sophisticated agents, a stronger standard of simplicity may need to be imposed to ensure the intended strategies are played, with potential limitations on flexibility. Understanding the simplicity of games and the simplicity-flexibility tradeoff requires an adaptable approach to thinking about simplicity. This paper puts forth one such proposal, though there is much work still to be done in fully exploring this trade-off and testing various simplicity standards empirically.

price mechanisms achieve decent revenues even without the bundling/equalization assumption.

[37]A different approach to the trade-off between simplicity and flexibility was proposed by Li and Dworczak (2020), who study strategy-proofness, obvious strategy-proofness, and strong obvious strategy-proofness. While we evaluate this tradeoff for designers who never confuse the mechanism participants, they evaluate it for designers who can confuse participants. See also work in progress by Catonini and Xue (2021), who study a weakening of one-step simplicity.

[38]For a test of the first claim see also Breitmoser and Schweighofer-Kodritsch (2019).

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A Appendix: Proofs

This appendix contains the central elements of the proofs of our main theorems. All lemmas used in these proofs, as well as Theorem 6 and Lemma 1 from the main text, are proven in the Supplementary Appendix.

A.1 Pruning Principle

Given a game Γ and strategy profile $(S_i(\succ_i))_{i\in\mathcal{N}}$, the **pruning** of Γ with respect to $(S_i(\succ_i))_{i\in\mathcal{N}}$ is a game Γ' that is defined by starting with Γ and deleting all histories of Γ that are never reached for any type profile. Li (2017b) introduced the following **pruning principle**: if $(S_i(\succ_i))_{i\in\mathcal{N}}$ is obviously dominant for Γ, then the restriction of $(S_i(\succ_i))_{i\in\mathcal{N}}$ to Γ' is obviously dominant for Γ', and both games result in the same outcome. Thus, for any OSP mechanism, we can find an equivalent OSP pruned mechanism. For strong obvious dominance the pruning principle remains valid: if $(S_i(\succ_i))_{i\in\mathcal{N}}$ is strongly obviously dominant for Γ, then the restriction of $(S_i(\succ_i))_{i\in\mathcal{N}}$ to its pruning Γ' is strongly obviously dominant for Γ', and both games result in the same outcome.

A.2 Proof of Theorem 2

In light of Corollary 1, it is sufficient to prove the result for $k < \infty$ and $k' = k + 1$. For $k = 0$, the result follows from Theorems 6 and 8, applied to a single-unit auction with transfers. Theorem 6 shows that in such a setting, personal clock auctions are efficient and OSS, while Theorem 8 implies that an efficient, SOSP ($k = 0$) mechanism does not exist when there are at least two agents whose valuations are drawn iid from at least 3 values (see also footnote 36). For $k = 1$ we construct below a 2-step simple social choice rule that cannot be one-step implemented; we conclude the proof by extending this example to any larger k.

Consider an object allocation environment without transfers in which agents demand exactly one object each. There are at least three agents i, j, ℓ and the objects included in the game Γ are shown in Figure 4. Each branch of the game tree represents a clinching action where the agent clinches the labeled object ($x, \tilde{x}$, etc.). The notation such as "$\ell \to \gamma$" below terminal nodes denotes that agent ℓ is assigned to object γ at this node, without needing to take any action. The root of the game is agent i's choice between clinching x and passing. If i clinches x at the first move, then the game immediately ends with j assigned α_j and ℓ assigned α_ℓ, and further, this is the only terminal history at which j receives α_j and ℓ receives α_ℓ. Similarly, there are objects β_ℓ, γ_ℓ, and δ_ℓ that agent ℓ receives only at the denoted terminal histories, and nowhere else in the game.

It is straightforward to check that $(\Gamma, S_{\mathcal{N},\mathcal{H}})$, where $S_{\mathcal{N},\mathcal{H}}$ is a profile of greedy strategic collections, is k-step implementable for any $k \geq 2$; in particular, this implies that Γ is OSP. It is also easy to check that Γ itself is not OSS: the type of i that ranks $w \succ_i x \succ_i z$ has no one-step simple strategic plan when choosing between x and passing at the first move of the game. Showing that the social choice rule implemented by $(\Gamma, S_{\mathcal{N},\mathcal{H}})$ cannot be OSS-

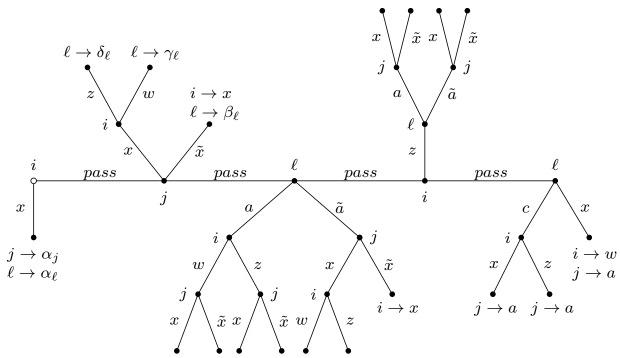

Figure 4: A game in which greedy strategies are two-step simple and for which no equivalent one-step simple mechanism exists.

implemented by any other mechanism is subtler, and we relegate the proof of the following lemma establishing this statement to Supplementary Appendix B.1.

Lemma A.1. *No one-step simple mechanism is equivalent to* $(\Gamma, S_{\mathcal{N},\mathcal{H}})$.

For $k = 2$, game $\Gamma^{(2)}$ in Figure 5 is an example that is k'-step simple for any $k' > k$, but for which no equivalent k-step simple mechanism exists. This game is similar in structure to that of Figure 4, but has the following additions:

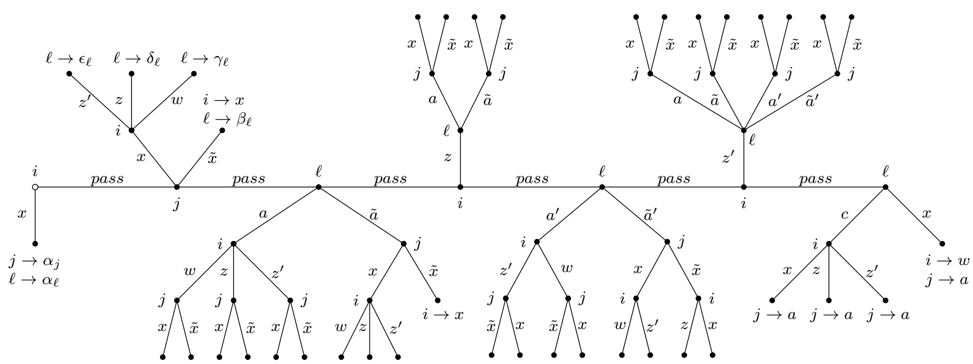

Figure 5: A game in which greedy strategies are three-step simple and for which no equivalent two-step simple mechanism exists.

(i) In the subgame following i passing and j clinching x at its first move, we add the possibility of i clinching z'. In this way we assure that i can then clinch any possible and not previously clinchable object.[39]

(ii) In the subgame following i and j passing and ℓ clinching a at its first move, we add the possibility of i clinching z' (following which j can clinch x and $\tilde{x}$). In this way we assure

[39] This property and the property in (ii) were also true in game Γ in Figure 4 and these two modifications simply reestablish these properties for the game $\Gamma^{(2)}$ in Figure 5, in which z' becomes possible for i.

that i can clinch any possible and not previously clinchable object.

(iii) Following i's pass at its second move on the focal path, we add a node at which ℓ can clinch two new objects a' and $\tilde{a}'$ (following the clinching of a', agent i can clinch any possible not previously clinchable object, and then j can clinch any previously clinchable object; following the clinching of $\tilde{a}'$, agent j can clinch any previously clinchable object, and then following the clinching of x agent i can clinch any possible but not previously clinchable objects while following the clinching of $\tilde{x}$ agent i can clinch any previously clinchable object).

(iv) Following the pass at the added node for ℓ, we add a node at which i can clinch an additional object z'. Following i clinchhing z', ℓ and then j can clinch any object they could clinch previously).

To prove the theorem for arbitrary $k \geq 2$, we recursively create game $\Gamma^{(k)}$ by adding to game $\Gamma^{(k-1)}$ further objects $z^{(k)}$, $a^{(k)}$, and $\tilde{a}^{(k)}$, and then adding the analogues of subgames (i)-(iv). In the analogues of subgames (i)-(ii), we now allow i to additionally clinch $z^{(k)}$; in the analogue of (iii), $a^{(k)}$ and $\tilde{a}^{(k)}$ play the roles of a and $\tilde{a}$, and in the analogue of (iv) $z^{(k)}$ plays the role of z.

It is straightforward to check that $\left(\Gamma^{(k)}, S^{(k)}_{\mathcal{N},\mathcal{H}}\right)$ is $(k+1)$-step simple but not k-step simple, where $S^{(k)}_{\mathcal{N},\mathcal{H}}$ is a profile of greedy strategic collections. Showing that no equivalent mechanism is k-step simple is done similarly to the $k = 1$ case. The details can be found in Supplementary Appendix B.1.

Lemma A.2. *For any $k \geq 2$, no k-step-simple mechanism is equivalent to* $\left(\Gamma^{(k)}, S^{(k)}_{\mathcal{N},\mathcal{H}}\right)$.

Lemmas A.1 and A.2 establish the result for $k \geq 1$. ■

A.3 Proof of Theorem 3

The proof develops the proof of the similar result for OSP in Li (2017b). For one direction of implication, suppose the strategic plan S_{i,I^*} is simply dominant from the perspective of $I^* \in \mathcal{I}_i$ in Γ. Then any outcome that is possible after playing S_{i,I^*} at all information nodes $I \in \mathcal{I}_{i,I^*}$ is weakly better than any outcome that is possible after playing $S'_i(I^*) \neq S_{i,I^*}(I^*)$ in Γ, and hence the analogue of this "weakly better" comparison applies to the counterparts of these actions in any game Γ' that is indistinguishable from Γ from the perspective of i at I^* (by condition (3) of indistinguishability). Hence, in any such Γ', every strategy S'_i that calls for playing the counterparts of actions $S_{i,I^*}(I)$ for counterparts of all $I \in \mathcal{I}_{i,I^*}$ weakly dominates any strategy S''_i that does not call for playing the counterpart of $S_{i,I^*}(I^*)$ at the counterpart of I^*.

For the other direction of implication, fix information set I^* at which i moves, preference ranking $\succsim_i$ of agent i, and a partial strategic plan S_{i,I^*} such that in every game Γ' that is indistinguishable from Γ from the perspective of agent i at I^*, any strategy S_i' that plays counterparts of $S_{i,I^*}(I)$ for all counterparts of $I \in \mathcal{I}_{i,I^*}$ weakly dominates any strategy S_i'' that plays at the counterpart of I^* another action than the counterpart of $S_{i,I^*}(I^*)$. Our goal is to show that any outcome that is possible when i follows S_{i,I^*} at information sets $\mathcal{I}_{i,I^*}$ is $\succsim_i$ -weakly preferred to any outcome that is possible after i plays any $a \neq S_{i,I^*}(I^*)$ at I^* in game Γ. To prove it consider Γ' that differs from Γ only in that all moves of agent i and other agents that follow history h^* but are not not in $\mathcal{I}_{i,h^*}$ are made by Nature instead of the party making them in Γ and that Nature puts positive probability on all its possible moves. Notice that such Γ' is indistinguishable from Γ from the perspective of i at I^*. As in Γ' any strategy that selects counterparts of S_{i,I^*} at any counterpart of $I \in \mathcal{I}_{i,I^*}$ weakly dominates any strategy S_i'' that selects a at the counterpart of I^*, we conclude from condition (3) of indistinguishability that, in Γ, any outcome that is possible after i follows S_{i,I^*} at information sets in $\mathcal{I}_{i,I^*}$ is weakly better than any outcome that is possible following a. ∎

A.4 Proof of Theorem 5

Section 4 introduces the notions of possible and clinchable payoffs at a history h, and the sets of such payoffs, denoted $P_i(h)$ and $C_i(h)$, respectively. For the proof, we also need the notion of a guaranteeable payoff: a payoff x is **guaranteeable** for i at h if there is some continuation strategy S_i such that i receives payoff x at all terminal histories $\bar{h} \supseteq h$ that are consistent with i following S_i. We use $G_i(h)$ to denote the set of payoffs that are guaranteeable for i at history h.

The proof is broken down into five steps, stated as Lemmas A.3-A.7 below. The proofs of these lemmas can be found in Supplementary Appendix B.2. First, we check there that all millipede games with greedy strategies are OSP, establishing one direction of the theorem.

Lemma A.3. *Millipede games with greedy strategies are obviously strategy-proof.*

Given Li's pruning principle (see Subsection A.1), the converse implication of Theorem 5—that all OSP mechanisms are equivalent to millipedes—follows from the remaining four lemmas.[40] Lemma A.4 develops Theorem 4 (see this theorem for a discussion):

Lemma A.4. *Every OSP game is equivalent to an OSP game with perfect information in which Nature moves at most once, as the first mover.*

[40] We actually prove a slightly stronger statement, which is that every OSP game is equivalent to a millipede game that satisfies the following additional property: for all i, all h at which i moves, and all $x \in G_i(h)$, there exists an action $a_x \in A(h)$ that clinches x (see Lemma A.6 below).

Lemma A.5 shows that if a game is OSP, then at every history, for all actions a with the exception of possibly one special action a^*, all payoffs that are possible following a are also guaranteeable at h.[41]

Lemma A.5. *Let Γ be an obviously strategy-proof game of perfect information that is pruned with respect to the obviously dominant strategy profile $(S_i(\succ_i))_{i\in\mathcal{N}}$. Consider a history h where agent $i_h = i$ is called to move. There is at most one action $a^* \in A(h)$ such that $P_i((h, a^*)) \notin G_i(h)$.*

The above lemma leaves open the possibility that there are several actions that can ultimately lead to multiple final payoffs for i, which can happen when different payoffs are guaranteeable for i by following different strategies in the future of the game. The next lemma shows that if this is the case, we can always construct an equivalent OSP game such that all actions except for possibly one are clinching actions.

Lemma A.6. *For any OSP game Γ, there exists an equivalent OSP game Γ' such that the following hold at each $h \in \mathcal{H}$ (where i is the agent called to move at h):*

(i) At least $|A(h)| - 1$ actions at h are clinching actions.

(ii) For every payoff $x \in G_i(h)$, there exists an action $a_x \in A(h)$ that clinches x for i.

(iii) If $P_i(h) = G_i(h)$, then all $a \in A(h)$ are clinching actions and $i_{h'} \neq i$ for any $h' \supsetneq h$.

The final lemma of the proof establishes the payoff guarantees in the game constructed in the previous lemmas.

Lemma A.7. *Let $(\Gamma, S_{\mathcal{N}})$ be an obviously strategy-proof mechanism that satisfies the conclusions of Lemmas A.4 and A.6. At all h, if there exists a previously unclinchable payoff z that becomes impossible for agent i_h at h, then $C^{c}_{i_h}(h) \subseteq C_i(h)$.*

This lemma concludes the proof of Theorem 5. ∎

A.5 Proof of Theorem 7

We first prove the second statement. Let Γ be a monotonic millipede game. Fix an agent i, and, for any history h^* at which i moves, let $\bar{x}_{h^*} = Top(\succ_i, P_i(h^*))$ and $\bar{y}_{h^*} = Top(\succ_i, C_i(h^*))$. Let $\mathcal{H}_{i,h^*} = \{h \in \mathcal{H}_i | h^* \subsetneq h' \subsetneq h \implies h' \notin \mathcal{H}_i\}$ be the set of one-step simple nodes. Consider the following strategic plan for any h^*:

- If $\bar{x}_{h^*} \in C_i(h^*)$, then $S_{i,h^*}(h^*) = a_{\bar{x}_{h^*}}$, where $a_{\bar{x}_{h^*}} \in A(h^*)$ is a clinching action for $\bar{x}_{h^*}$.

[41] We emphasize the distinction between a payoff x being "guaranteeable" vs. "clinchable": the latter means the agent receives x at all terminal histories, while the former means there is a continuation strategy S_i such that she receives x at all terminal histories consistent with S_i.

- If $\bar{x}_{h^*} \notin C_i(h^*)$, then $S_{i,h^*}(h^*) = a^*$ (i passes at h^*), and, for any other $h \in \mathcal{H}_{i,h^*}$:
  - If $P_i(h^*) \setminus C_i(h^*) \subseteq C_i(h)$, then $S_{i,h^*}(h^*) = a_{\bar{x}_{h^*}}$.
  - Else, we have $C_i(h^*) \subseteq C_i(h)$ (by monotonicity) and we set $S_{i,h^*}(h^*) = a_{\bar{y}_{h^*}}$.

It is straightforward to verify that this strategic plan is one-step dominant at any h^*, and thus the corresponding strategic collection $(S_{i,h^*})_{h^* \in \mathcal{H}_i}$ is also one-step dominant.

In order to prove the first statement, let $(\Gamma, S_{\mathcal{N},\mathcal{H}})$ be a millipede mechanism with a profile of one-step dominant strategic collections $S_{\mathcal{N},\mathcal{H}}$. Begin by constructing an equivalent millipede mechanism that satisfies Lemma A.6. Note that the transformations used in the proof to construct the equivalent millipede mechanism are one-step dominance preserving—i.e., if $(\Gamma, S_{\mathcal{N},\mathcal{H}})$ was an OSS millipede mechanism before the transformation, then the transformed game $(\Gamma', S'_{\mathcal{N},\mathcal{H}})$ is another OSS millipede mechanism that satisfies Lemma A.6. It remains to show:

Lemma A.8. *Any OSS millipede mechanism that, at each $h \in \mathcal{H}$, satisfies conditions (i), (ii), and (iii) of Lemma A.6 is monotonic.*

We prove this lemma in Supplementary Appendix B.4. ∎

A.6 Proof of Theorem 8

That sequential choice mechanisms are SOSP is immediate from the definition, and so we focus on proving that every SOSP mechanism is equivalent to a sequential choice mechanism. Following the same reasoning as in the proof of Lemma A.4, given any SOSP mechanism, we can construct an equivalent SOSP mechanism of perfect information in which Nature moves at most once, as the first mover. It remains to analyze the subgame after a potential move by Nature and to show that every perfect-information SOSP mechanism in which there are no moves by Nature is equivalent to a sequential choice mechanism.

Let $(\Gamma, S_{\mathcal{N}})$ be such a mechanism. In line with the discussion in Section A.1, we can assume that Γ is pruned. By Lemma 1, each agent i can have at most one payoff-relevant history along any path of Γ, and this history (if it exists) is the first time i is called to play. Consider any such history h_0^i. If there is some other history $h' \supset h_0^i$ at which i is called to play, then history h' must be payoff-irrelevant for i; in other words, there is some payoff x such that $P_i((h', a')) = \{x\}$ for all $a' \in A(h')$. Using the same technique as in the proof of Lemma A.6, we construct an equivalent pruned game in which at history h_0^i, i is asked to also choose her actions for all successor histories $h' \supset h_0^i$ at which she might be called to play, and then is not called to play again after h_0^i. Since all of these future histories were payoff-irrelevant for i, the new game continues to be strongly obvious dominant for i. Strong obvious dominance is also preserved for all $j \neq i$, since having i make all of her choices earlier only shrinks the set of possible outcomes any time j is called to move, and thus, if some action was strongly obviously dominant in the old game, the analogous action(s) will be strongly obviously dominant in the new game. Repeating this for every agent and every history, we construct a pruned SOSP game Γ' that is equivalent to Γ and in which each agent is called to move at most once along any path of play. It remains to show

Lemma A.9. Γ' *with greedy strategies is a sequential choice mechanism.*

We prove this lemma in Supplementary Appendix B.6. ∎

A.7 Proof of Theorem 9

In the proof we use the concepts of roles and role assignment functions. Let $\mathcal{R}$ be a set of players such that $|\mathcal{R}| = |\mathcal{N}|$; we call each $r \in \mathcal{R}$ a **role**. Let Σ be the set of bijections $\sigma : \mathcal{R} \to \mathcal{N}$ between the set of roles and the set of agents $\mathcal{N}$; we call these bijections **role assignment** functions. Given a game Γ, let the function $\rho : \mathcal{H} \to \mathcal{R}$ map each history h in game Γ to the role $\rho(h)$ that moves at this history. In the environment of the theorem, each outcome in any game is an assignment of the objects to the agents playing the game. Given a mechanism (Γ, S),[42] each role assignment bijection σ determines a mechanism $(\Gamma_\sigma, S_\sigma)$ for the agents in $\mathcal{N}$ as follows: Γ_σ is the extensive-form game with the same game tree as Γ and such that at each non-terminal history h, the agent called to move is $\sigma(\rho(h))$; at each terminal history in Γ_σ the object assigned to agent i is the same as the object assigned to role $\sigma^{-1}(i)$ at the corresponding terminal history in Γ; the strategy S_i of agent i in Γ_σ is the same as the strategy of role $\sigma^{-1}(i)$ in Γ. There are $|\Sigma| = N!$ possible mechanisms $(\Gamma_\sigma, S_\sigma)$; we call them the permuted mechanisms.[43]

We further define the **symmetrization of mechanism** (Γ^*, S^*) to be the following random mechanism: first, Nature chooses a role assignment function σ uniformly at random from the set of all possible role assignment functions, and then, the agents play Γ_σ with strategies S_σ.[44] To formally ensure that the symmetrization of a millipede is a millipede, we

[42] For brevity, we denote a profile of strategies as S rather than $S_\mathcal{R}$ or $S_\mathcal{N}$.

[43] Our terminology of roles generalizes Carroll's (2014) priority roles from Pápai (2000)'s hierarchical exchanges to general extensive-form games.

[44] While this construction implies that different agents play the same strategies in the same role, our arguments only rely on the weaker assumption that an agent's strategy $S_{\sigma,i}(\succ_i)$ depends only on her own preferences and her role assignment, and not on the roles assigned to other agents. In other words, in any two subgames Γ_A and Γ_B following Nature's selection of role assignments σ_A and σ_B, if $\sigma_A^{-1}(i) = \sigma_B^{-1}(i) = r_n$, then $S_{A,i}(\succ_i)(h_A) = S_{B,i}(\succ_i)(h_B)$ for any equivalent histories h_A and h_B in these two games.

assume that Nature draws the role assignment σ and its initial decision in the subgame Γ_σ in the same move.

Lemma A.10. *Suppose that, for every deterministic OSP and Pareto-efficient perfect-information mechanism, its symmetrization is equivalent to Random Priority. Then, every symmetric, OSP and Pareto-efficient mechanism is equivalent to Random Priority.*

Proof. Take a symmetric, OSP, and Pareto-efficient mechanism (Γ, S). By Lemma A.4, we can assume that (Γ, S) has perfect information and that Nature moves only at the beginning of the game. Because (Γ, S) is symmetric, its symmetrization (Γ^*, S^*) is equivalent to (Γ, S). Furthermore, (Γ^*, S^*) is a lottery over symmetrizations of each deterministic perfect-information continuation game Γ' after Nature's move in (Γ, S). The mechanism given by game Γ', together with the strategy profile induced from Γ, is OSP and Pareto efficient, and hence by the assumption of the lemma it is equivalent to Random Priority. Because every lottery over Random Priority lotteries is still equivalent to Random Priority, the lemma obtains. ■

In light of the above lemma, it is sufficient to prove Theorem 9 for symmetrizations. To do so, we build on the bijective argument used by Abdulkadiroğlu and Sönmez (1998) to show the equivalence of Random Priority and the Core from Random Endowments (see also Pathak and Sethuraman, 2011 and Carroll, 2014). Let Ord denote the set of total linear orders over the set of agents $\mathcal{N}$. Given a function $f : \Sigma \to Ord$, for any $\sigma \in \Sigma$, let $f_\sigma(n)$ denote the n^{th} ranked agent under f_σ.

Lemma A.11. *Let (Γ, S) be a deterministic OSP and Pareto-efficient perfect-information mechanism, and fix a preference profile $\succ_\mathcal{N}$. There exists a bijection $f : \Sigma \to Ord$ such that, for each $\sigma \in \Sigma$, the permuted mechanism $(\Gamma_\sigma, S_\sigma(\succ_\mathcal{N}))$ results in the same final allocation as the serial dictatorship in which the agents choose their most preferred object in the order $f_\sigma(1), f_\sigma(2), \ldots, f_\sigma(N)$.*

We construct the bijection f and establish the lemma in Supplementary Appendix B.7.[45] Because f is a bijection, we can associate to each role assignment function σ a unique serial dictatorship that produces the same final allocation, thereby showing that the symmetrization of (Γ, S) is equivalent to Random Priority. As this holds for any mechanism, applying Lemma A.10 completes the proof of Theorem 9. ■

[45]The bijection f is constructed for a fixed mechanism and preference profile, i.e., different mechanisms and preference profiles might have different bijections. While the full argument constructing f, and showing it is bijective and results in the same final allocation as the serial dictatorship, is highly involved, Section 5 of the main text gives a simple example to showcase the idea.

B Supplementary Appendix: Omitted Proofs (For Online Publication)

This supplementary appendix contains the proofs of the lemmas used in the proofs of the main theorems in Appendix A, as well as the full proofs of Theorem 6 and Lemma 1 from the main text.

B.1 Proofs of Lemmas for Theorem 2

Proof of Lemma A.1. In order to show that there is no OSS mechanism that is equivalent to Γ, suppose, by way of contradiction, that there is such mechanism with game $\tilde{\Gamma}$ and a profile of OSS strategic plans. Let $\tilde{S}$ be the profile of strategies in $\tilde{\Gamma}$ induced by the strategic plans; by Theorem 1, this profile is obviously dominant.

The proof proceeds in a series of steps, which we label 1.1-1.6. (The labeling $k.1 - k.6$ is used because, after proving the result for $k = 1$, we use analogues of these steps to prove Lemma A.2 for arbitrary k.)

Step 1.1. In $\tilde{\Gamma}$, the first mover must be i, and x must be guaranteeable for i. Furthermore, at the empty history, w and z are not guaranteeable for i, but there is a unique action after which w and z are possible. This action is taken by all types of player i that rank either w or z first; we call this action i's focal action.

Proof of Step 1.1. First notice that i must be the first mover. Indeed, in mechanism Γ, agent j receives α_j if and only if agent i prefers x to w and z. Assume that, under $\tilde{\Gamma}$, agent j moves first. Something must be guaranteeable for agent j at this history, say λ.[46] If $\lambda = \alpha_j$, then we have non-equivalence when j prefers α_j the most and agent i does not prefer x to w and z. If $\lambda \neq \alpha_j$, then, we have non-equivalence when j prefers λ the most and i prefers x to w and z. Therefore, the first mover cannot be j. As the same argument works for agent ℓ, the first mover must be i.

Second, note that equivalence implies that i obtains x for any preference profile such that i prefers x the most, and therefore, x is guaranteeable at the first move in $\tilde{\Gamma}$. Analogously, w and z must be possible but not guaranteeable for i at the first move. To see that w cannot be guaranteeable, note that if it were, i would receive w for all preference profiles where she ranked it first, which is not the case in Γ, and so equivalence is violated; the same holds for z. By equivalence, both w and z are possible for i, i.e., $w, z \in P_i(h)$. Further, there must be a unique action a^* such that $w, z \in P_i((h, a^*))$. If there were two actions a_1, a_2 such that

[46] That something must be guaranteeable follows because each history has at least two actions, and in any OSP game, there can be at most one action such that there is some payoff that is possible, but not guaranteeable (see the proof of Theorem 5).

w were possible after both, then any type that prefers w the most would have no obviously dominant action, since w is not guaranteeable; the same holds for z. Therefore, each of w and z are possible after exactly one action, label them a_w and a_z. If $a_w \neq a_z$, then any type that ranks w first and z second would have no obviously dominant action.[47] Therefore, $a_w = a_z$; we call this action i's focal action. Since w and z are possible following only the focal action, all types that rank either w or z first must select it. This completes the proof of Step 1.1.

Step 1.2. In $\tilde{\Gamma}$, at the history following the first focal action by i, agent j moves. At this history, both $\tilde{x}$ and x are guaranteeable for j, while a is not guaranteeable. Further, there is a unique action after which a is possible, and this action is taken by all types of j who rank a first; we call this action j's focal action.

Proof of Step 1.2. Since, per Step 1.1, both w and z are possible for i following the focal action, the focal action cannot lead to a terminal history, and so there must be an agent who moves. We start by showing that the mover must be j. Note that in Γ, agent ℓ receives β_ℓ if and only if agent i prefers either w or z to x, and agent j prefers $\tilde{x}$ the most out of $\{x, \tilde{x}, a\}$. Suppose that i prefers either w or z to x, so that i follows the focal action at the initial history. By the same logic as in Step 1.1, if agent ℓ is the next mover, she must be able to guarantee some payoff, say γ. If $\gamma = \beta_\ell$, this would lead to a non-equivalence when ℓ ranks γ first and j ranks x first. If $\gamma \neq \beta_\ell$, then we have a non-equivalence when ℓ ranks γ first and j ranks $\tilde{x}$ first. Therefore, ℓ cannot be the next mover, and neither can be i (as i just moved) and so it must be j.

The equivalence of Γ and $\tilde{\Gamma}$ implies that for any profile such that i prefers w or z over x and j prefers x the most, j receives x. Because, per Step 1.1, all types of i take the focal action in $\tilde{\Gamma}$, we conclude that following i's focal action, j must be able to guarantee himself x. The same argument applies for $\tilde{x}$. Similarly, equivalence implies that there must be an action for j such that a is possible. Outcome a cannot be guaranteeable for j, because if it were, then j would receive a for all preference profiles where i ranks w or z first and j ranks a first, which is not the case in Γ. By an argument similar to Step 1.1, there cannot be any other actions after which a is possible, and all types of j that rank a first must select this action. We label this action j's focal action.

Step 1.3. In $\tilde{\Gamma}$, following i's focal action and j's focal action, there might be any finite number of consecutive histories at which i and j move. At these histories where i moves, i can clinch x, but neither w nor z are guaranteeable, and there is a unique action (the

[47] Since w is not guaranteeable and z is not possible after a_w, the worst case from any strategy that selects a_w is strictly worse than z, which is possible from a_z. Similarly, since w is not possible following a_z, the worst case is strictly worse than w, which is possible from a_w. Note that an analogous argument would apply to any type that ranks z first and w second.

focal action) after which w and z are possible and that is taken by all types of i that rank w or z first. At these histories where j moves, both $\tilde{x}$ and x are guaranteeable, but a is not guaranteeable, and there is a unique action (the focal action) after which a is possible and is taken by all types of j that rank a first. Following this sequence of focal actions, agent ℓ moves.

Proof of Step 1.3. Since, per Step 1.2, a is possible, but not guaranteeable following j's focal action, the focal action cannot lead to a terminal history, and so must lead to a history at which an agent moves. As j just moved, the next mover must be either i or ℓ. If the next mover is i, as the history is on-path for all types of i who prefer w or z over x, the OSS property of $\tilde{\Gamma}$ implies that either x or else both w and z are clinchable for i. Equivalence implies that neither w nor z can be clinchable for i: if w were clinchable, then i receives w for all profiles such that i prefers w the most and j prefers a the most, which is not the case in Γ; an analogous argument applies for z. Therefore, x must be clinchable. Furthermore, w and z are possible but not guaranteeable for i, and so, as in Step 1.1, OSP implies that there is a unique action after which both w and z are possible, and all types that rank either w or z first takes this action (note that these types must have taken the focal action at i's initial move, and so are on-path); we call this action the focal action.

Following the focal action by i, the next mover must be j or ℓ. If it is j, then an analogous argument as for i shows that this agent must have both $x, \tilde{x}$ clinchable, and that there must be a unique action after which a is possible but not guaranteeable; we call it the focal action.

Following j's focal action, the next move is by i or ℓ. If it is by i then the above argument applies again. We might then have a sequence of moves by i and j to which the above two arguments apply. As the game is finite and at the end of every focal action in the sequence more than one outcome is possible, the focal path of the game must lead to a history at which ℓ is called to play. This proves Step 1.3.

Step 1.4. In $\tilde{\Gamma}$, at ℓ's move following the sequence of focal actions described in Step 1.3, both $\tilde{a}$ and a are guaranteeable for ℓ, while neither c nor x are guaranteeable. There is also a unique action (the focal action) after which c and x are possible for ℓ. This action is taken by all types of ℓ that rank c first.

Proof of Step 1.4. Using arguments similar to Step 1.2, equivalence implies that at ℓ's move, both $\tilde{a}$ and a are guaranteeable for ℓ, while neither c nor x are guaranteeable, but both c and x are possible following a unique action that is taken by all types of agent ℓ that rank c first. Since c is not guaranteeable, this action cannot lead to a terminal history. Since c is possible following only the focal action, all types of ℓ that rank c first must select this action. This proves Step 1.4.

Step 1.5. In $\tilde{\Gamma}$, following the above sequence of focal actions that ends with the first focal *action by ℓ, there might be any finite number of consecutive histories at which j and ℓ move. Each of these histories has a unique action (the focal action) after which a is possible for j's moves, and c and x are possible for ℓ's moves. All types of j that rank a first and all types of ℓ that rank c first take their respective focal actions. Following this sequence of focal actions, the next mover is i.*

Proof of Step 1.5. Since there are multiple possible outcomes for k following her focal action, the focal action cannot lead to a terminal history. As k just moved, the next mover must be either i or j. First consider the case in which j moves next. The OSS property implies that either both x and $\tilde{x}$ are clinchable for j, or a is clinchable for j. Consider the latter case. If this were true, then under a preference profile where i prefers w most and z second, j prefers a most, and ℓ prefers c most, j would receive a, which is not the case in Γ. Therefore, j must be able to clinch x and $\tilde{x}$. By equivalence, a must be possible for j, but not guaranteeable, and so once again there must be a unique focal action after which a is possible and that is taken by all types of j that prefer a the most (note that all of these types have passed at j's prior moves, and so are on-path). Following the focal action, the next mover is i or ℓ. If it is ℓ, then an analogous argument implies that ℓ must be able to clinch a and $\tilde{a}$, with c possible but not guaranteeable following a unique focal action. There may again be a sequence of moves by j and ℓ for which this argument can be repeated. As the game is finite and at the end of every focal action more than one outcome is possible, the focal path must lead to a history at which i is called to play. This proves step 1.5.

Step 1.6. In $\tilde{\Gamma}$, at i's move following the sequence of focal actions described in Step 1.5, x is not clinchable for i.[48] *At this move, there is a unique action (the focal action) after which w is possible for i; the focal action is also the unique action after which x is possible for i. This focal action is taken by all types of i that rank w first.*

Proof of Step 1.6. By way of contradiction, suppose x is clinchable for i. Then OSP implies that in the continuation game following i's clinching of x, there must be a terminal history at which j receives a: if there were not, then the type of j that prefers a the most and x second would have no obviously dominant action at j's prior moves. At this terminal history, agent ℓ must be assigned something other than x (which was assigned to i) or a (which was assigned to j). But then, the type of ℓ that prefers x the most and a second has no obviously dominant action at ℓ's prior moves, which is a contradiction.[49]

An analogous argument to that which showed that there is a unique action after which w is possible for i in Step 1.1, tell us that there is a unique action (the focal action) after

[48] The argument shows that x not only is not clinchable for i but also not guaranteeable.

[49] Note that by equivalence, x must be possible for ℓ at these prior moves, since in Γ, k receives x for type profiles such that i ranks w first, j ranks a first, and ℓ ranks x first.

which w is possible for i. By OSP, types of i ranking w first take this action. An analogous argument shows that the focal action is the unique action after which x is possible.

Finishing the proof for $k = 1$.

As the previous step shows that x is not clinchable at the move of i considered there, OSS implies that both w and z must be clinchable for i. This implies that for preference profiles such that i ranks w first and x second, j ranks a first, and k ranks c first, agent i is assigned w. However, under such profiles in Γ, i receives x, which is a contradiction to equivalence. ■

Proof of Lemma A.2. Take any $k \geq 2$. By way of contradiction, suppose that $\tilde{\Gamma}^{(k)}$ with a profile of strategic plans is a k-step simple mechanism equivalent to $\Gamma^{(k)}$ with greedy strategic plans. The proof begins by repeating steps 1.1-1.6 from the proof of Lemma A.1 above, with the only change being that $\Gamma^{(k)}$ plays the role of Γ and $\tilde{\Gamma}^{(k)}$ plays the role of $\tilde{\Gamma}$. Then, we continue with the addition of steps k'.3-k'.6 for $k' = 2, 3, \ldots, k$. Each step k'.3-k'.6 is analogous to the corresponding step 1.3-1.6 from above, except that $a^{(k)}$ plays the role of a, $\tilde{a}^{(k)}$ plays the role of $\tilde{a}$, and $z^{(k)}$ plays the role of z. Finally, the proof for arbitrary k concludes with a final step that is the direct analogue of the finishing step for $k = 1$, except that we apply k-step simplicity instead of OSS. ■

B.2 Proofs of Lemmas for Theorem 5

Proof of Lemma A.3. Let Γ be a millipede game. For a set X of payoffs of agent i and a type $\succ_i$, let $Top(\succ_i, X)$ be the best payoff in X according to preferences $\succ_i$. Consider some profile of greedy strategies $(S_i(\cdot))_{i \in \mathcal{N}}$. If $Top(\succ_i, C_i(h)) = Top(\succ_i, P_i(h))$, then clinching a top payoff is obviously dominant at h. What remains to be shown is if $Top(\succ_i, C_i(h)) \neq Top(\succ_i, P_i(h))$, then passing is obviously dominant at h.

Assume that there exists a history h that is on the path of play for type $\succ_i$ when following $S_i(\succ_i)$ such that $Top(\succ_i, C_i(h)) \neq Top(\succ_i, P_i(h))$, yet passing is not obviously dominant at h; further, let h be any earliest such history for which this is true. To shorten notation, let $x_P(h) = Top(\succ_i, P_i(h))$, $x_C(h) = Top(\succ_i, C_i(h))$, and let $x_W(h)$ be the worst possible payoff from passing and continuing to follow $S_i(\succ_i)$ at all future nodes.

First, note that $x_W(h) \succsim_i x_W(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$. Since passing is obviously dominant at all $h' \subsetneq h$, we have $x_W(h') \succsim_i x_C(h')$, and together, these imply that $x_W(h) \succsim_i x_C(h')$ for all such h'. At h, since passing is not obviously dominant and all other actions are clinching actions, we have $x_C(h) \succ_i x_W(h)$; further, since $Top(\succ_i, C_i(h)) \neq Top(\succ_i, P_i(h))$, there must be some $x' \in P_i(h) \smallsetminus C_i(h)$ such that $x' \succ_i x_C(h) \succ_i x_W(h)$. The above implies that $x' \succ_i x_C(h) \succ_i x_C(h')$ for all $h' \subsetneq h$ such that $i_{h'} = i$.

Let $X_0 = \{x' : x' \in P_i(h) \text{ and } x' \succ_i x_C(h)\}$; in words, X_0 is a set of payoffs that are possible at all $h' \subseteq h$, and are strictly better than anything that was clinchable at any $h' \subseteq h$ (and therefore have never been clinchable themselves). Order the elements in X_0 according to $\succ_i$, and without loss of generality, let $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_M$.

Consider a path of play starting from h that is consistent with $S_i(\succ_i)$ and ends in a terminal history $\bar{h}$ at which i receives $x_W(h)$. For every $x_m \in X_0$, let h_m denote the earliest history on this path such that $x_m \notin P_i(h_m)$ and either (i) $i_h = i$ or (ii) h_m is terminal. Note that because i is ultimately receiving payoff $x_W(h)$, such a history h_m exists for all $x_m \in X_0$. Let $\hat{h}_{-m}$ be the earliest history at which i moves and at which all payoffs strictly preferred to x_m are no longer possible.

Claim. For all $x_m \in X_0$ and all $h' \subseteq \bar{h}$, we have $x_m \notin C_i(h')$.

Proof of claim. First, note that $x_m \notin C_i(h')$ for any $h' \subseteq h$ by construction. We show that $x_m \notin C_i(h')$ at any $\bar{h} \supseteq h' \supset h$ as well. Start by considering $m = 1$, and assume $x_1 \in C_i(h')$ for some $\bar{h} \supseteq h' \supset h$. By definition, $x_1 = Top(\succ_i, P_i(h))$; since $h' \supset h$ implies that $P_i(h') \subseteq P_i(h)$, we have that $x_1 = Top(\succ_i, P_i(h'))$ as well. Since $x_1 \in C_i(h')$ by supposition, greedy strategies direct i to clinch x_1, which contradicts that she receives $x_W(h)$.[50]

Now, consider an arbitrary m, and assume that for all $m' = 1, \ldots, m-1$, payoff $x_{m'}$ is not clinchable at any $h' \subseteq \bar{h}$, but x_m is clinchable at some $h' \subseteq \bar{h}$. Let $x_{m'} \succ_i x_m$ be a payoff that becomes impossible at $\hat{h}_{-m} \subseteq \bar{h}$; if such payoff does not exist then the argument of the paragraph above applies. There are two cases:

Case (i): $h' \subsetneq \hat{h}_{-m}$**.** This is the case in which x_m is clinchable while there is some strictly preferred payoff $x_{m'} \succ_i x_m$ that is still possible. By assumption, all $\{x_1, \ldots, x_{m-1}\}$ are previously unclinchable at $\hat{h}_{-m}$, and so $x_{m'}$ is previously unclinchable at $\hat{h}_{-m}$. By definition of a millipede game (part 3), we we have $x_m \in C_i(\hat{h}_{-m})$. Thus, x_m is the best remaining payoff at $\hat{h}_{-m}$, and is clinchable, and so greedy strategies direct i to clinch x_m at $\hat{h}_{-m}$, which contradicts that she receives $x_W(h)$ (as in footnote 50, the argument still applies if $\hat{h}_{-m}$ is a terminal history).

Case (ii): $h' \supseteq \hat{h}_{-m}$**.** In this case, x_m becomes clinchable after all strictly preferred payoffs are no longer possible. Thus, again, greedy strategies instruct i to clinch x_m, which contradicts that she is receiving $x_W(h)$. ∎

To finish the proof of Lemma A.3, let $\hat{h} = \max\{h_1, h_2, \ldots, h_M\}$ (ordered by $\subset$); in words, $\hat{h}$ is the earliest history on the path to $\bar{h}$ at which no payoffs in X_0 are possible any longer. Let $\hat{x}$ be a payoff in X_0 that becomes impossible at $\hat{h}$. The claim shows that no $x \in X_0$ is clinchable at any $h' \subseteq \hat{h}$, and so we can further conclude that $\hat{x}$ is previously unclinchable at

[50] If h' is terminal, then, even though i takes no action at h', according to our notational convention we define $C_i(h') = \{x_1\}$, which also contradicts that she receives payoff $x_W(h)$.

$\hat{h}$. Therefore, by part 3 in the definition of a millipede game, $x_C(h) \in C_i(\hat{h})$. Since $x_C(h)$ is the best possible remaining payoff at $\hat{h}$, greedy strategies direct i to clinch $x_C(h)$, which contradicts that she receives $x_W(h)$ (as in footnote 50, the argument still applies if $\hat{h}$ is a terminal history). ■

Proof of Lemma A.4. Ashlagi and Gonczarowski (2018) briefly mention this result in a footnote; here, we provide the straightforward proof for completeness. That every OSP game is equivalent to an OSP game with perfect information is implied by our more general Theorem 4. To show that we can furthermore assume that Nature moves at most once, as the first mover, consider a perfect-information game Γ. Let $\mathcal{H}_{\text{nature}}$ be the set of histories h at which Nature moves in Γ. Consider a modified game Γ' in which at the empty history Nature chooses actions from $\times_{h \in \mathcal{H}_{\text{nature}}} A(h)$. After each of Nature's initial moves, we replicate the original game, except at each history h at which Nature is called to play, we delete Nature's move and continue with the subgame corresponding to the action Nature chose from $A(h)$ at $\varnothing$. Again, note that for any agent i and history h at which i is called to act, the support of possible outcomes at h in Γ' is a subset of the support of possible outcomes at the corresponding history in Γ (where the corresponding histories are defined by mapping the $A(h)$ component of the action taken at $\varnothing$ by Nature in Γ' as an action made by Nature at h in game Γ). When the support of possible outcomes shrinks, the worst-case outcome from any fixed strategy can only improve, while the best-case can only diminish, and so if a strategy was obviously dominant in Γ, the corresponding strategy will continue to be obviously dominant in Γ', and the two games will be equivalent. ■

Proof of Lemma A.5. For any history h, let $PnG_i(h) = P_i(h) \setminus G_i(h)$ (where "**PnG**" is shorthand for "possible but not guaranteeable"). Now, consider any h at which i moves, and assume that at h, there are (at least) two such actions $a_1^*, a_2^* \in A(h)$ as in the statement. We first claim that $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \varnothing$, where $h_1^* = (h, a_1^*)$ and $h_2^* = (h, a_2^*)$. Indeed, if not, then let x be a payoff in this intersection. By pruning, some type $\succ_i$ is following some strategy such that $S_i(\succ_i)(h) = a_1^*$ that results in a payoff of x at some terminal history $\bar{h} \supseteq (h, a_1^*)$. Note that $Top(\succ_i, P_i(h)) \neq x$, because otherwise a_1^* would not be obviously dominant for this type (since $x \notin G_i(h)$ and $x \in P_i(h_2^*)$). Thus, let $Top(\succ_i, P_i(h)) = y$. Note that $y \notin G_i(h)$ (or else it would not be obviously dominant for type $\succ_i$ to play a strategy such that x is a possible payoff). Further, we must have $y \in P_i(h_1^*)$ and $y \notin P_i(h_2^*)$. To see the former, note that if $y \notin P_i(h_1^*)$, then a_1^* is not obviously dominant for type $\succ_i$, which contradicts that $S_i(\succ_i)(h) = a_1^*$; given the former, if $y \in P_i(h_2^*)$, then once again a_1^* would not be obviously dominant for type $\succ_i$. Now, again by pruning, there must be some type $\succ_i'$ such that $S_i(\succ_i')(h) = a_2^*$ that results in payoff x at some terminal history $\bar{h} \supseteq (h, a_2^*)$. By similar reasoning as previously, $Top(\succ_i', P_i(h)) \neq x$, and so $Top(\succ_i', P_i(h)) = z$ for some $z \in P_i(h_2^*)$. Since $y \notin P_i(h_2^*)$, we have $z \neq y$, and we can as above conclude that $z \notin G_i(h)$. It is without loss of generality to consider a type $\succ_i'$ such that $Top(\succ_i', P_i(h) \setminus \{z\}) = y$. Note that, for this type, no action $a \neq a_2^*$ can obviously dominate a_2^* (since $z \notin G_i(h)$). Further, a_2^* itself is not obviously dominant for this type, since the worst case from a_2^* is strictly worse than y (since $y \notin P_i(h_2^*)$ and $z \notin G_i(h)$), while $y \in P_i(h_1^*)$. Therefore, this type has no obviously dominant action at h, which is a contradiction.

Thus, $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \varnothing$, which means there must be distinct x, y such that (i) $x, y \in PnG_i(h)$ (ii) $x \in P_i(h_1^*)$ but $x \notin P_i(h_2^*)$ and (iii) $y \in P_i(h_2^*)$ but $y \notin P_i(h_1^*)$. Next, for all types of agent i that reach h, it must be that $Top(\succ_i, P_i(h)) \neq x, y$. To see why, assume there were a type that reaches h such that $Top(\succ_i, P_i(h)) = x$. Then, by richness, there is a type that reaches h such that $Top(\succ_i, P_i(h) \setminus \{x\}) = y$. But, note that this type has no obviously dominant action at h. An analogous argument applies switching x with y.

Now, by pruning, there is some type that reaches h that plays a strategy such that $S_i(\succ_i)(h) = a_1^*$ and x is a possible payoff. Let $Top(\succ_i, P_i(h)) = z$ for this type, where, as just noted, $z \neq x, y$. The fact that $S_i(\succ_i)(h) = a_1^*$ implies that $z \in P_i(h_1^*)$ and $z \notin G_i(h)$; if either of these were false, it would not be obviously dominant for this type to play a strategy such that $S_i(\succ_i)(h) = a_1^*$ and x is a possible payoff. In other words, $z \in PnG(h)$ and $z \in P_i(h_1^*)$. Since we just showed that $PnG_i(h) \cap P_i(h_1^*) \cap P_i(h_2^*) = \varnothing$, we have $z \notin P_i(h_2^*)$. Finally, consider a type $\succ_i$ such that $Top(\succ_i, P_i(h)) = z$ and $Top(\succ_i, P_i(h) \setminus \{z\}) = y$. Note that this type has no obviously dominant action at h, which is a contradiction. ■

Proof of Lemma A.6. Given an OSP mechanism $(\Gamma, S_{\mathcal{N}})$, begin by using Lemma A.4 to construct an equivalent OSP game of perfect information in which Nature moves only at the initial history (if at all). Further, prune this game according to the obviously dominant strategy profile $S_{\mathcal{N}}$. With slight abuse of notation, we denote this pruned, perfect information mechanism by $(\Gamma, S_{\mathcal{N}})$. Consider some history h of Γ at which the mover is $i_h = i$. By Lemma A.5, all but at most one action (denoted a^*) in $A(h)$ satisfy $P_i((h,a)) \subseteq G_i(h)$; this means that any obviously dominant strategy for type $\succ_i$ that does not choose a^* guarantees the best possible outcome in $P_i(h)$ for type $\succ_i$. Define the set

$$\mathcal{S}_i(h) = \{S_i : S_i(h) \neq a^* \text{ and at all terminal } \bar{h} \text{ consistent with } S_i, i \text{ receives the same payoff}\}.$$

In words, each $S_i \in \mathcal{S}_i(h)$ guarantees a unique payoff for i if she plays strategy S_i starting from history h, no matter what the other agents do.

We create a new game Γ' that is the same as Γ, except we replace the subgame starting from history h with a new subgame defined as follows. If there is an action a^* such that

$P_i((h, a^*)) \notin G_i(h)$ in the original game (of which there can be at most one), then there is an analogous action a^* in the new game, and the subgame following a^* is exactly the same as in the original game Γ. Additionally, there are $M = |\mathcal{S}_i(h)|$ other actions at h, denoted $a_1, \dots, a_M$. Each a_m corresponds to one strategy $S_i^m \in \mathcal{S}_i(h)$, and following each a_m, we replicate the original game, except that at any future history $h' \supseteq h$ at which i is called on to act, all actions (and their subgames) are deleted and replaced with the subgame starting from the history (h', a'), where $a' = S_i^m(h')$ is the action that i would have played at h' in the original game had she followed strategy $S_i^m(\cdot)$. In other words, if i's strategy was to choose some action $a \neq a^*$ at h in the original game, then, in the new game Γ', we ask agent i to "choose" not only her current action, but all future actions that she would have chosen according to $S_i^m(\cdot)$ as well. By doing so, we have created a new game in which every action (except for a^*, if it exists) at h clinches some payoff x, and further, agent i is never called upon to move again.[51]

We construct strategies in Γ' that are the counterparts of strategies from Γ, so that for all agents $j \neq i$, they continue to follow the same action at every history as they did in the original game, and for i, at history h in the new game, she takes the action a_m that is associated with the strategy S_i^m in the original game. By definition if all agents follow strategies in the new game analogous to the their strategies from the original game, the same outcome is reached, and so Γ and Γ' are equivalent under their respective strategy profiles.

We must also show that if a strategy profile is obviously dominant for Γ, this modified strategy profile is obviously dominant for Γ'. To see why the modified strategy profile is obviously dominant for i, note that if her obviously dominant action in the original game was part of a strategy that guarantees some payoff x, she now is able to clinch x immediately, which is clearly obviously dominant; if her obviously dominant strategy was to follow a strategy that did not guarantee some payoff x at h, this strategy must have directed i to follow a^* at h. However, in Γ', the subgame following a^* is unchanged relative to Γ, and so i is able to perfectly replicate this strategy, which obviously dominates following any of the clinching actions at h in Γ'. In addition, the game is also obviously strategy-proof for all $j \neq i$ because, prior to h, the set of possible payoffs for j is unchanged, while for any history succeeding h where j is to move, having i make all of her choices earlier in the game only shrinks the set of possible outcomes for j, in the set inclusion sense. When the set of possible outcomes shrinks, the best possible payoff from any given strategy only decreases (according to j's preferences) and the worst possible payoff only increases, and so,

[51] More precisely, all of i's future moves are trivial moves in which she has only one possible action; hence these histories may further be removed to create an equivalent game in which i is never called on to move again. Note that this only applies to the actions $a \neq a^*$; it is still possible for i to follow a^* at h and be called upon to make a non-trivial move again later in the game.

if a strategy was obviously dominant in the original game, it will continue to be so in the new game. Repeating this process for every history h, we are left with a new game where, at each history, there are only clinching actions plus (possibly) one passing action, and further, every payoff that is guaranteeable at h is also clinchable at h, and i never moves again following a clinching action. This shows parts (i) and (ii). Part (iii) follows immediately from part (ii), due to greedy strategies and the pruning principle. ■

Proof of Lemma A.7. Let h be any earliest history where some agent i moves such that there is a previously unclinchable payoff z that becomes impossible at h (the case for terminal histories is dealt with separately below). This means that i moves at some strict subhistory $h' \subsetneq h$ and the following are true: (a) $z \notin P_i(h)$; (b) $z \in P_i(h')$ for all $h' \subsetneq h$ such that $i_h = i$; and (c) $z \notin C_i^{\subset}(h)$. Points (b) and (c) imply that z is possible at every $h' \subsetneq h$ at which i is called to move, but it is not clinchable at any of them; thus, for any type of agent i that ranks z first, any obviously dominant strategy has the agent choosing the unique passing action at all $h' \subsetneq h$.

We want to show that $C_i^{\subset}(h) \subseteq C_i(h)$. Towards a contradiction, assume that $C_i^{\subset}(h) \nsubseteq C_i(h)$, and let $x \in C_i^{\subset}(h)$ but $x \notin C_i(h)$. Consider a type $\succ_i$ that ranks z first and x second. By the previous paragraph, this type must be playing some strategy that passes at any $h' \subsetneq h$, and so h is on the path of play for type $\succ_i$. Since $z \notin P_i(h)$ and $x \notin C_i(h)$, by Lemma A.6, part (ii), the worst case outcome from this strategy is some y that it is strictly worse than both z and x according to $\succ_i$. However, we also have $x \in C_i(h')$ for some $h' \subsetneq h$, and so the best case outcome from clinching x at h' is x. This implies that passing is not obviously dominant, and thus Γ is not OSP, a contradiction.

Last, consider a terminal history $\bar{h}$. As above, let z be a payoff such that (a), (b), and (c) hold (replacing h with $\bar{h}$). Recall that for terminal histories, we define $C_i(\bar{h}) = \{y\}$, where y is the payoff that obtains at h for i. Towards a contradiction, assume that there is some $x \in C_i(h')$ for some $h' \subsetneq \bar{h}$ but $x \notin C_i(\bar{h})$. Note that (i) $z \neq y$, by (a); (ii) $z \neq x$, by (c); and (iii) $x \neq y$, since $x \notin C_i(\bar{h})$. In other words, x, y, and z are all distinct payoffs for i. Thus, consider the type $\succ_i$ that ranks z first, x second, and y third, followed by all other payoffs. By (b) and (c), z is possible at every $h'' \subsetneq \bar{h}$ at which i moves, but is not clinchable at any such h''. Thus, any obviously dominant strategy for type $\succ_i$ must have agent i passing at all such histories. This implies that y is possible for this type. However, at h', i could have clinched x, and so the strategy is not obviously dominant, a contradiction. ■

B.3 Proof of Theorem 6

Before proving the theorem, we first formally define a personal clock auction. Given some perfect-information game Γ, define outcome functions g as follows: $g_y(\bar{h}) \subseteq \mathcal{N}$ is the set of agents who are in the allocation $\bar{y}$ that obtains at terminal history $\bar{h}$ (that is, $i \in g_y(\bar{h})$ if and only if $\bar{y}_i = 1$), and $g_{w,i}(\bar{h}) \in \mathbb{R}$ is the transfer to agent i at $\bar{h}$. The following definition of a personal clock auction is adapted from Li (2017b). Note that the game is deterministic, i.e., there are no moves by Nature.[52]

Γ is a **personal clock auction** if, for every $i \in \mathcal{N}$, at every earliest history h_i^* at which i moves, either **In-Transfer Falls**: there exists a fixed transfer $\bar{w}_i \in \mathbb{R}$, a going transfer $\tilde{w}_i : \{h_i : h_i^* \subseteq h_i\} \to \mathbb{R}$ and a set of "quitting actions" A^q such that

- For all terminal $\bar{h} \supset h_i^*$, either (i) $i \notin g_y(\bar{h})$ and $g_{w,i}(\bar{h}) = \bar{w}_i$ or (ii) $i \in g_y(\bar{h})$ and $g_{w,i}(\bar{h}) = \inf\{\tilde{w}_i(h_i) : h_i^* \subseteq h_i \subsetneq \bar{h}\}$.
- If $\bar{h} \supsetneq (h, a)$ for some $h \in \mathcal{H}_i$ and $a \in A^q$, then $i \notin g_y(\bar{h})$.
- $A^q \cap A(h_i^*) \neq \varnothing$
- For all $h_i', h_i'' \in \{h_i \in \mathcal{H}_i : h_i^* \subseteq h_i\}$:
  - If $h_i' \subsetneq h_i''$, then $\tilde{w}_i(h_i') \geq \tilde{w}_i(h_i'')$
  - If $h_i' \subsetneq h_i''$, $\tilde{w}_i(h_i') > \tilde{w}_i(h_i'')$ and there is no h_i''' such that $h_i' \subsetneq h_i''' \subsetneq h_i''$, then $A^q \cap A(h_i'') \neq \varnothing$
  - If $h_i' \subsetneq h_i''$ and $\tilde{w}_i(h_i') > \tilde{w}_i(h_i'')$, then $|A(h_i') \setminus A^q| = 1$
  - If $|A(h_i') \setminus A^q| > 1$, then there exists $a \in A(h_i')$ such that, for all $\bar{h} \supseteq (h_i', a)$, $i \in g_y(\bar{h})$;[53]

or, **Out-Transfer Falls**: as above replacing every instance of "$i \in g_y(\bar{h})$" with "$i \notin g_y(\bar{h})$" and vice-versa.

We now prove Theorem 6. As discussed in the main text, the first part of this theorem follows from our Corollary 1, Li (2017), and the construction of the one-step simple strategic

[52] We slightly simplify Definition 15 of Li (2017b) by restricting it to perfect information games: by Theorem 4, for any personal clock auction that satisfies Definition 15 of Li (2017b), there is an equivalent mechanism that satisfies the definition we work with. This also applies to the minor correction provided by Li in a corrigendum available on his website; cf. footnote 53 for further details.

[53] The corrigendum issued by Li replaces this statement with one that says if there is more than one non-quitting action at h_i', there is a *continuation strategy* (rather than an action) that guarantees that $i \in g_y(\bar{h})$. The corrigendum also notes, though, that this change does not expand the set of implementable choice rules, because for any newly admissible mechanism, there is always an equivalent mechanism satisfying the original definition in which the agent reports her type at h_i' and does not move again. Thus, our notion of equivalence allows us to work directly with this simpler definition of personal clock auctions.

collections for each agent that we now present. This construction also proves the second part of the theorem.

Let Γ be a personal clock auction. We present the construction and argument for in-transfer falls; the case of out-transfer falls is analogous. Consider any $h_i \in \mathcal{H}_i$ and simple-node set $\mathcal{H}_{i,h_i} = \{h' \in \mathcal{H}_i : h_i \subsetneq h'' \subsetneq h' \implies h'' \notin \mathcal{H}_i\}$, and define the strategic plan $S_{i,h_i}(h')$ at $h' \in \mathcal{H}_{i,h_i}$ as follows:

- If $\theta_i + \tilde{w}_i(h_i) > \bar{w}_i$ and $A(h_i) \setminus A^q \neq \varnothing$:
  - **[Action at h_i]** Choose $S_{i,h_i}(h_i) = a \in A(h_i) \setminus A^q$; if it further holds that $|A(h_i) \setminus A^q| > 1$, then choose $S_{i,h_i}(h_i) = a \in A(h_i) \setminus A^q$ such that $i \in g_y(\bar{h})$ for all $\bar{h} \supseteq (h_i, a)$.
  - **[Actions at next-histories]** For $h' \in \mathcal{H}_{i,h_i} \setminus \{h_i\}$, if there exists $a \in A(h') \cap A^q$, then $S_{i,h_i}(h') = a$ for some $a \in A(h') \cap A^q$. Else, $S_{i,h_i}(h') = a'$ for some $a' \in A(h')$ such that for all $\bar{h} \supseteq (h, a')$, $i \in g_y(\bar{h})$.
- Else, choose actions such that $S_{i,h_i}(h') \in A^q$ for all $h' \in \mathcal{H}_{i,h_i}$.

To show that this is a one-step simple strategic collection, first consider h_i such that $A(h_i) \setminus A^q = \varnothing$. Then the only actions available at h_i are quitting actions. Thus, the best- and worst-cases from any action are all $\bar{w}_i$, and one-step dominance holds. Second, consider $\theta_i + \tilde{w}_i(h_i) \leq \bar{w}_i$. Then, the worst case from quitting at h_i is a payoff of $\bar{w}_i$. Since the going transfer can only fall, the best case from playing a non-quitting action at h_i is at most $\theta_i + \tilde{w}_i(h_i) \leq \bar{w}_i$, and so again one-step dominance holds. Third, consider the remaining case in which $\theta_i + \tilde{w}_i(h_i) > \bar{w}_i$ and there exists some $a \in A(h_i) \setminus A^q$. There are two subcases:

First, if $|A(h_i) \setminus A^q| = 1$, then all other actions at h_i are quitting actions, and i's best case and worst case payoff from following any such action is $\bar{w}_i$. We must show that the worst case from the perspective of node h_i from following the specified strategic plan gives a weakly greater payoff than $\bar{w}_i$. For any next-history $h'_i \in \mathcal{H}_{i,h_i}$ at which there is a quitting action (i.e., $A(h'_i) \cap A^q \neq \varnothing$), the worst case from the perspective of h_i of following the strategic plan is $\bar{w}_i$. If there is no quitting action at h'_i (i.e., $A(h'_i) \cap A^q = \varnothing$), then, by construction of a personal clock auction, we have that (i) $\tilde{w}_i(h_i) = \tilde{w}_i(h'_i)$, and (ii) there exists an $a' \in A(h'_i)$ such that, for all $\bar{h} \supseteq (h'_i, a')$, we have $i \in g_y(\bar{h})$. Further, for any $h''_i \supsetneq h'_i$, $\tilde{w}_i(h''_i) = \tilde{w}_i(h'_i) = \tilde{w}_i(h_i)$, and so, for any $\bar{h} \supseteq (h'_i, a')$, $g_{w,i}(\bar{h}) = \tilde{w}_i(h_i)$. Therefore, the worst case from following the strategic plan from the perspective of h_i conditional on reaching any such h'_i is $\theta_i + \tilde{w}_i(h_i)$. In either case, the worst case from the strategic plan from the perspective of h_i is weakly better than taking any other action at h_i.

Second, if $|A(h_i) \setminus A^q| > 1$, then the strategic plan instructs i to follow the action $a \in A(h_i)$ such that $i \in g_y(\bar{h})$ for all $\bar{h} \supseteq (h_i, a)$; further, by construction of a personal clock auction, at any $\bar{h} \supseteq (h_i, a)$, we have $g_{w,i}(\bar{h}) = \tilde{w}_i(h_i)$. Since $\theta_i + \tilde{w}_i(h_i) > \bar{w}_i$, this is strictly preferred to the payoff from taking any quitting action at h_i, and since the going transfer cannot rise, it is also weakly preferable to taking any other non-quitting action at h_i. ■

B.4 Proof of Lemma for Theorem 7

Proof of Lemma A.8. By way of contradiction, let $(\Gamma, S_{\mathcal{N},\mathcal{H}})$ be a millipede mechanism that satisfies (i)-(iii) at each history but is not monotonic. The failure of monotonicity implies that there exists an agent i, history $h^* \in \mathcal{H}_i$, history h that follows i's passing move at h^* that is either terminal or in $\mathcal{H}_i$ and such that i does not move between h^* and h, and payoffs x and y such that $x \in (P_i(h^*) \smallsetminus C_i(h^*)) \smallsetminus C_i(h)$ and $y \in C_i(h^*) \smallsetminus C_i(h)$; in particular, $x \neq y$. Without loss of generality, assume that h^* is an earliest history at which monotonicity is violated in this way. This implies that $x \notin C_i(h')$ for any $h' \subseteq h^*$ such that $i_{h'} = i$.[54] In particular, any type $\succ_i$ of agent i that ranks payoff x first passes at any $h' \subseteq h^*$ at which this agent moves.

As $x, y \notin C_i(h)$ by the choice of these payoffs, there is some third payoff $z \neq x, y$ such that $z \in C_i(h)$. Let $\succ_i$ be such that $\succ_i$: $x, y, z \ldots$ and $\succ_i'$ be such that $\succ_i'$: $x, z, \ldots$; these types exist by richness, given that we are in a no-transfer environment. Ranking x first, these types are passing at all nodes $h' \subseteq h^*$ at which they move: $S_{i,h'}(\succ_i)(h') = S_{i,h'}(\succ_i')(h') = a^*(h')$ where $a^*(h')$ denotes the passing action at h'.

We conclude the indirect argument by showing that none of the following two cases is possible:

Case $y \notin P_i(h)$. If also $x \notin P_i(h)$, then $P_i(h)$ contains some $w \neq x, y$. If $x \in P_i(h)$, then $x \notin C_i(h)$ implies that $x \in P_i((h, a^*(h)))$ and by definition of a passing action, there is some $w \neq x$ such that $w \in P_i((h, a^*))$; furthermore $w \neq y$ because $y \notin P_i(h)$. In either case, passing at h^* might lead to w which is worse for $\succ_i$ than y, and i can clinch y at h^*; thus $S_{i,h^*}(\succ_i)$, which passes at h^*, is not one-step dominant; a contradiction.

Case $y \in P_i(h)$. If $z \in P_i((h, a^*))$ then $x, y \notin C_i(h)$ implies that the worst case for type $\succ_i$ from passing at h^* is at best z, which is worse than clinching y at h^*. Therefore, the passing action $S_{i,h^*}(\succ_i)$ is not one-step dominant at h^* for $\succ_i$, a contradiction. We may thus assume that $z \notin P_i((h, a^*))$. Because $x \notin C_i(h)$, the assumptions of Lemma A.6 imply that x is not guaranteeable at h, and in particular it is not guaranteeable at $(h, a^*(h))$. Thus,

[54] If $x \in C_i(h')$ for some h', then, by monotonicity, at any next history $h'' \supsetneq h'$ following a pass where i moves, either $x \in C_i(h'')$ or $P_i(h') \smallsetminus C_i(h') \subseteq C_i(h'')$. If the latter holds, then at h'', i has been offered to clinch everything that is possible for her, and so, by greediness, h is not on-path for any type of agent i, and we can construct an equivalent game in which monotonicity is not violated at h^*. Therefore, $x \in C_i(h'')$. Repeating this argument for every history between h' and h^* at which i moves delivers that $x \in C_i(h^*)$, which is a contradiction.

the worst case for type $\succ_i'$ from passing at h is strictly worse than z; since $z \in C_i(h)$, this implies that $S_{i,h}(\succ_i')$ clinches at h. Thus $x \notin C_i(h)$ allows us to conclude that $x \notin P_i(h)$, as otherwise $S_{i,h}(\succ_i')$ could not be clinching at h. Since $y \notin C_i(h)$ and $y \in P_i(h)$, we infer that $y \in P_i((h, a^*(h)))$. As at least two payoffs are possible following passing and $x \notin P_i(h)$, there is some $w \neq x, y$ that is possible at $(h, a^*(h))$ and hence also at h. As x is not possible and y is not clinchable at h, the worst case for type $\succ_i$ from the perspective of node h^* from following $S_{i,h^*}(\succ_i)$ is at best w, which is strictly worse than clinching y at h^*. Thus $S_{i,h^*}(\succ_i)$ is not one-step dominant. ■

B.5 Proof of Lemma 1

Recall that any strongly obviously dominant strategy is greedy. We first note the following lemmas. To state the lemmas, define $\hat{P}_i(h) = \{x \in P_i(h) : \nexists y \in P_i(h) \text{ s.t. } y \rhd_i x\}$ to be the set of possible payoffs for i at h that are undominated in $P_i(h)$.

Lemma B.1. *Let Γ be a pruned SOSP game. If a history h at which agent i moves is payoff-relevant, then $|\hat{P}_i(h)| \geq 2$.*

Proof of Lemma B.1. Assume not, and let $\hat{P}_i(h) = \{x\}$, where x is the unique undominated payoff at h.[55] In particular, $x \rhd_i x'$ for all $x' \in P_i(h)$, and $Top(\succ_i, P_i(h)) = x$ for all types of agent i. Because x is possible at h, there is an action $a \in A(h)$ such that $x \in P_i((h, a))$. Action a does not clinch x; indeed if $P_i((h, a)) = \{x\}$ then greediness would imply that only actions clinching x are taken, and in a pruned game h would not be payoff relevant. Thus, there is another $x' \in P_i((h, a))$ such that $x \succ_i x'$ for all types of agent i. Let $a' \neq a$ be an action at $A(h)$. If $x \in P_i((h, a'))$, then, analogously as for a, there is some other $x'' \in P_i((h, a'))$. It is then easy to check that neither a nor a' strongly obviously dominates the other. If $x \notin P_i((h, a'))$ then it would not be strongly obviously dominant (SOD, for shortness) for any type to take action a', which would contradict the game being pruned. ■

Lemma B.2. *Let (Γ, S) be a pruned SOSP mechanism. Let h_0^i be any earliest history at which agent i is called to play. Then, $|\hat{P}_i((h_0^i, a))| \leq 2$ for all $a \in A(h_0^i)$, with equality for at most one $a \in A(h_0^i)$.*

Proof of Lemma B.2. Since h_0^i is the first time i is called to move, it is on-path for all types of agent i. We first show that $|\hat{P}_i((h_0^i, a))| \leq 2$ for all $a \in A(h_0^i)$. By way of contradiction assume that there exists some h_0^i such that $|\hat{P}_i((h_0^i, a))| \geq 3$. Let $x, y, z \in \hat{P}_i((h_0^i, a))$ be three distinct undominated payoffs that are possible following a. As (Γ, S) is pruned, there must

[55] There must be at least one undominated payoff, since $\trianglerighteq_i$ is transitive and the number of payoffs is finite.

be some type, $\succ_i$, for which action a is SOD at h_0^i. Possibly by renaming the outcomes, richness allows us to assume that $Top(\succ_i, P_i(h_0^i)) = x$ and $x \succ_i y \succ_i z$. For a to be strongly obviously dominant, for all other actions $a' \neq a$ at h_0, the best case outcome for type $\succ_i$ following a' must be no better than z; in particular, this implies that for all $a' \neq a$ and all $w \in \hat{P}_i((h_0^i, a'))$, $w \not\trianglerighteq_i y$. Let $a'' \neq a$ be an action at h_0. If there is $w \in \hat{P}_i((h_0^i, a''))$ such that $x \not\trianglerighteq_i w$, then there is a type $\succ_i'$ such that $Top(\succ_i', P_i(h_0^i)) = y$ and $y \succ_i' w \succ_i' x$. For this type, the worst case from a is at best x, while w is possible following a'' , so a is not strongly obviously dominant; for any $a' \neq a$, the worst case is strictly worse than y as argued above, while the best case from a is y, and so no $a' \neq a$ is SOD either. Therefore, type $\succ_i'$ has no SOD action, a contradiction showing that no $w \in \hat{P}_i((h_0^i, a''))$ satisfies $x \not\trianglerighteq_i w$. An analogous argument—with z playing the role of x—shows that no $w \in \hat{P}_i((h_0^i, a''))$ satisfies $z \not\trianglerighteq_i w$. Thus, for all a'' and all $w \in \hat{P}_i((h_0^i, a''))$, $x \trianglerighteq w$ and $z \trianglerighteq w$. As x and z are distinct, for any type $\succ_i'$, either $x \succ_i' w$ or $z \succ_i' w$, and in either case a'' is not a dominant action for a type contrary to (Γ, S) being pruned. This contradiction shows that $|\hat{P}_i((h_0^i, a))| \leq 2$ for all $a \in A(h_0^i)$.

Finally, we show that $|\hat{P}_i((h_0^i, a))| = 2$ for at most one $a \in A(h_0^i)$. Towards a contradiction, let a and a' be two actions such that there are two possible undominated payoffs for i following each, and, for notational purposes, let $\hat{P}_i((h_0^i, a)) = \{x, y\}$ and $\hat{P}_i((h_0^i, a')) = \{w, z\}$, where, a priori, it is possible that $w, z \in \{x, y\}$. As the mechanism is pruned, there is some type $\succ_i$ that selects action a as an SOD action; without loss of generality, let $Top(\succ_i, P_i(h_0^i)) = x$. Since y is possible following a, in order for a to be SOD, the best case from any $a' \neq a$ must be no better than y, which implies that $w, z \not\trianglerighteq_i x$, and thus $x \neq w, z$. Pruning also implies that some type $\succ_i'$ is selecting action a'' as an SOD action; without loss of generality, let $Top(\succ_i', P_i(h_0^i)) = z$. Since w is possible following a'', in order for a'' to be SOD, the best case from a must be no better than w for type $\succ_i'$, thus $x, y \not\trianglerighteq_i z$, and so $z \neq x, y$. Thus, we have shown that x, y, z are all distinct, that no outcome in $P_i(h_0^i)$—including z and y—structurally dominates x, and that $y \not\trianglerighteq_i z$. Richness then implies that there is a type $\succ_i$ such that $Top(\succ_i, P_i(h_0^i)) = x$ and $x \succ_i z \succ_i y$. This type has no SOD action: only a can be SOD because only a makes x possible, but a is not SOD because the worst case from a is at best y, while the best case from a' is z. ■

Continuing with the proof of Lemma 1, assume that there was a path of the game with two payoff-relevant histories $h_1 \subsetneq h_2$ for some agent i. It is without loss of generality to assume that h_1 and h_2 are the first and second times i is called to play on the path. First, we claim that there are at least two structurally undominated payoffs at h_1, i.e., $|\hat{P}_i(h_1)| \geq 2$. To show it by way of contradiction, suppose that $\hat{P}_i(h_1) = \{x\}$, which implies that $x \triangleright_i x'$ for all other $x' \in P_i(h_1)$. Then $P_i((h_1, a)) = \{x\}$ for all $a \in A(h_1)$. Indeed, suppose that

$x' \neq x$ is possible after some action a at h_1. Then $x, x' \in P_i((h_1, a))$ because otherwise no type of i finds a to be SOD, which is impossible as the game is pruned. If $x \in P_i((h_1, a'))$ for some action $a' \neq a$ at h_1, then a is not SOD for any type of i, which again is impossible as the game is pruned. Thus $x \notin P_i((h_1, a'))$ and no type of i finds a' to be SOD, which yet again is impossible in a pruned game. Thus, no $x' \neq x$ is possible after any $a \in A(h_1)$, which contradicts that h_1 is payoff-relevant. This contradiction shows that $\hat{P}_i(h_1)$, being non-empty, has at least two elements.

Let a_1^* be the action such that $h_2 \supseteq (h_1, a_1^*)$. By Lemma B.2, one of the below two cases would need to obtain, and to conclude the indirect argument we now show that neither of them obtains.

Case $|\hat{P}_i((h_1, a_1^*))| = 1$. Let z be the unique undominated payoff that is possible after a_1^*; $z \in \hat{P}_i(h_1)$ as otherwise no type of i would find a_1^* to be SOD, which is impossible in a pruned mechanism. Because h_2 is payoff-relevant, Lemma B.1 tells us that $|\hat{P}_i(h_2)| \geq 2$, and thus $z \notin \hat{P}_i(h_2)$ as z weakly structurally dominates all outcomes in $P_i(h_2) \subseteq P_i((h_1, a_1^*))$. Let $x \neq z$ be an outcome in $\hat{P}_i(h_1)$ and let $z', z'' \in \hat{P}_i(h_2)$ be distinct undominated payoffs that are possible at h_2, and consider a type $\succ_i$ that ranks the outcomes $z \succ_i x \succ_i z'$. For this type, a_1^* is not SOD at h_1 because z' is possible following a_1^* while $x \notin \{z\} = \hat{P}_i((h_1, a_1^*))$ is possible following some other action at h_1. No action $a \neq a_1^*$ is SOD for $\succ_i$ if $z \notin P_i((h_1, a))$. Hence $z \in P_i((h_1, a))$ but then a_1^* would not be SOD for any type; impossible as the mechanism is pruned. This contradiction shows that the present case is impossible.

Case $|\hat{P}_i((h_1, a_1^*))| = 2$. Then a_1^* is the unique action with two undominated payoffs from Lemma B.2; let us label these payoffs x and y. As the game is pruned, there is some type $\succ_i$ for which a_1^* is strongly obviously dominant; in particular the payoff $Top(\succ_i, P_i(h_1))$ is possible following a_1^* and by renaming payoffs we can set $x = Top(\succ_i, P_i(h_1))$. For each action $a \neq a_1^*$ at h_1, Lemma B.2 implies that $\hat{P}_i((h_1, a)) = \{w_a\}$, for some payoff w_a; action a_1^* being SOD for type $\succ_i$ implies that $w_a \not\succeq_i x$ (and in particular $w_a \neq x$); and a being SOD for some other type implies that $y \not\succ_i w_a$. If $w_a \neq y$ then $y \not\succeq_i w_a$, and, given that x and y are mutually undominated, richness would give us a type $\succ_i^a$ such that $x \succ_i^a w_a \succ_i^a y$, but for this type neither a_1^* nor a nor any other action a' at h_1 is SOD because as shown above $w_{a'} \neq x$. We conclude that $w_a = y$ for all actions $a \neq a_1^*$ at h_1.

To continue the indirect argument we now show that $\hat{P}_i(h_2) = \{x, y\}$. The set $\hat{P}_i(h_2)$ has two elements, by Lemma B.1, because h_2 is payoff relevant. Thus, if $\hat{P}_i(h_2) \neq \{x, y\}$ then there would be some $z \neq x, y$ such that $z \in \hat{P}_i(h_2) \subseteq P_i(h_1)$. As x and y are undominated at $(h_1, a_1^*) \subsetneq h_2$, richness would give us type $\succ_i^2$ such that $x \succ_i^2 y \succ_i^2 z$ and for this type a_1^* would not be SOD at h_1 because z would be possible following a_1^* while, as shown above, y would be possible following another action; further, no $a \neq a_1$ would be SOD at h_1 because

y would be possible following a while x would be possible following a_1^*. The lack of an SOD action is a contradiction showing that $\hat{P}_i(h_2) = \{x, y\}$. Thus any $z \in P_i(h_2)$ is structurally dominated by either x or y and for, each type, x or y is the top payoff in $P_i(h_2)$. Since $\hat{P}_i((h_1, a)) = y$ for all $a \neq a_1^*$, strong obvious dominance implies that all and only types $\succ_i^1$ with $x = Top(\succ_i^1, P_i(h_1))$ select action a_1^* at h_1 and hence these are the types for whom h_2 is on path. As y is possible at h_2, there is at least one action $a_2^* \in A(h_2)$ after which y is possible. As at each history agents have at least two actions, there is another action $a_2 \in A(h_2)$, and, as the mechanism is pruned, there are two types $\succ_i^{a_2^*}$ and $\succ_i^{a_2}$ for which h_2 is on path such that $\succ_i^{a_2^*}$ selects a_2^* and $\succ_i^{a_2}$ selects a_2 at h_2. Because we established that x is possible at h_2 and that it is the top possible payoff for both these types, SOSP implies that $x \in P_i((h_2, a_2^*))$ and $x \in P_i((h_2, a_2))$. By construction, $y \in P_i((h_2, a_2^*))$, and hence a_2^* is not SOD for type $\succ_i^{a_2^*}$; a contradiction that concludes the proof of the lemma. ■

B.6 Proof of Lemma for Theorem 8

Proof of Lemma A.9. By way of contradiction suppose that game Γ', together with greedy strategies, is not a sequential choice mechanism. Let h be an earliest history where the definition of a sequential choice mechanism is violated. As such h is payoff relevant and Γ' is pruned, Lemma 1 implies that h is a first history at which i moves. Since Γ' is not a sequential choice mechanism, there must be some payoff $x \in P_i(h)$ that i cannot clinch at h. We may assume that x is not dominated, i.e., $x \in \hat{P}_i(h)$; indeed, if all $x' \in \hat{P}_i(h)$ were clinchable at h, then greediness would imply that all dominated actions were pruned in Γ'. Since x is not clinchable, for any action $a \in A(h)$ such that $x \in P_i((h, a))$, there is some payoff in $P_i((h, a))$ that is different from x. We fix one such action a.

Case $|P_i(h)| = 2$. Let y be the other payoff in $P_i(h)$. If y were clinchable then the mechanism would satisfy the definition of sequential choice at h. Since we assumed that the definition is not satisfied at h, neither x nor y is clinchable. Thus, for all $a \in A(h)$, $P_i((h, a)) = \{x, y\}$. As x and y are different payoffs, at least one of $x \succ_i y$ or $y \succ_i x$ holds for some type at h. Because there are at least two actions in $A(h)$, this type does not have a strongly obviously dominant (SOD) action at h, which is a contradiction.

Case $|P_i(h)| \geq 3$ **and** $x \rhd_i y$ **for all** $y \neq x$ **in** $P_i((h, a))$**.** There is an action $a' \neq a$ at h and, because x is not clinchable at h, there is some $w \neq x$ that belongs to $P_i((h, a'))$. We have $y \unrhd_i w$; indeed, if not, then x being undominated implies that there would exist type $\succ_i$ such that $x \succ_i w \succ_i y$, and, taking into account that x is not clinchable at h, this type would have no SOD action at h. Thus, $x \rhd_i y \unrhd_i w$; but this implies that a' is not SOD for any type, which contradicts the mechanism being pruned.

Case $|P_i(h)| \geq 3$ and there exists $y \in P_i((h,a))$ such that x and y do not dominate each other. By Lemma B.2, for any $a' \neq a$, the set $\hat{P}_i((h,a'))$ is a singleton. We first claim that for any $a' \neq a$, $\hat{P}_i((h,a')) = \{y\}$. Assume not, i.e., there exists some $a' \neq a$ and $w' \neq y$ such that $\hat{P}_i((h,a')) = \{w'\}$. Then also $w' \neq x$; indeed, if $w' = x$ then, as x is both structurally undominated and unclinchable at h, there would be $w \in P_i((h,a'))$ such that $x \rhd_i w$, and—with w possible after a' and x possible after a—no type would find a' to be SOD, contrary to pruning. If $w' \rhd_i x$ then no type would find a to be SOD, which contradicts pruning; we conclude that $w' \ntriangleright_i x$. In particular, $x \notin P_i((h,a'))$. If $y \rhd_i w'$, then $y \notin P_i((h,a'))$ because $w' \in \hat{P}_i((h,a'))$ is undominated at (h,a'). Thus, a' would not be SOD for any type, a contradiction to pruning. We conclude that $y \ntriangleright_i w'$. If $w' \rhd_i y$, then this and the previously established $w' \ntriangleright_i x$, gives us the existence of type $\succ_i$ such that $x \succ_i w' \succ_i y$. This type has no SOD action at h, a contradiction to pruning. We conclude that $w' \ntriangleright_i y$. If $x \rhd_i w'$, then type $x \succ_i w' \succ_i y$ exists and has no SOD action at h; we conclude that $x \ntriangleright_i w'$. The above four conclusions imply that x, y, w' are mutually undominated at h. Thus, there is a type such that $x \succ_i w' \succ_i y$ and this type has no SOD action at h. This final contradiction shows that $\hat{P}_i((h,a')) = \{y\}$ for all $a' \neq a$.

We further claim that $P_i((h,a')) = \{y\}$ for all $a' \neq a$; indeed, if this were not the case, then there is some a' and some $w' \in P_i((h,a'))$ such that $y \rhd_i w'$. As the mechanism is pruned, some type $\succ'_i$ takes action a'; but, the worst case from a', for all types, is at best w', while y is possible following a; thus a' is not SOD for type $\succ'_i$. This contradiction shows that $P_i((h,a')) = \{y\}$ for all $a' \neq a$.

Finally, let $z \neq x, y$ be some third payoff that is possible at h. In light of the previous paragraph, $z \in P_i((h,a))$, and $z \notin P_i((h,a'))$ for all other $a' \neq a$. As $\hat{P}_i(h) = \{x,y\}$, z dominates neither x nor y, and richness gives us a type such that $x \succ_i y \succ_i z$. This type has no SOD action at h; this contradicts the mechanism being SOSP and established the theorem.) ■

B.7 Proof of Lemma A.11 for Theorem 9

To prove Lemma A.11, we first construct a mapping from role assignment functions σ to partial orderings $\succ_\sigma$ over agents in Subsection B.7.2. In Subsection B.7.3, we then verify that any mapping f_σ to from role assignment functions to total orderings that is consistent with $\succ_\sigma$ satisfies the Same-Allocations claim of Lemma A.11:

- **Same Allocations**: For each $\sigma \in \Sigma$, the permuted mechanism $(\Gamma_\sigma, S_\sigma(\succ_\mathcal{N}))$ results in the same final allocation as a serial dictatorship in which the agents choose their most preferred object in the order $f_\sigma(1), f_\sigma(2), \ldots, f_\sigma(N)$.

We complete the proof of Lemma A.11 in B.7.4, by constructing a mapping f_σ consistent with $\succ_\sigma$ that satisfies the Bijectivity claim of the lemma:

- **Bijectivity**: f_σ is a bijection between Σ and Ord.

With a slight abuse of notation, in this section, we use $\mathcal{X}$ to denote the set of objects to be allocated (rather than global outcomes), and use x, y, z, etc. to refer to objects from $\mathcal{X}$. As objects determine agents' payoffs, we identify the two concepts and talk about possible, guaranteeable, clinchable, and clinched objects referring to the corresponding payoffs. We refer to objects that are clinched as being assigned.

B.7.1 Efficient Millipedes

We study randomization over a deterministic efficient millipede $(\Gamma_\sigma, S_\sigma(\succ_\mathcal{N}))$. Lemma A.6 allows us to assume that Γ has the following properties:[56]

1. At each history h, there is at most one passing action in $A(h)$; this action, if it exists, is denoted $a^* \in A(h)$. With slight abuse of notation, when the context is clear, we use the symbol a^* to represent the unique passing action at any history h (if such an action exists), and write $h' = (h, \overbrace{a^*, \ldots, a^*}^{n \text{ times}})$ to denote that history h' is the superhistory of h that is reached by starting at h and following n passing actions in a row; since there is at most one passing action at any given history, h' is uniquely defined.

2. If i moves at h and $x \in G_i(h)$, then there exists a clinching action $a_x \in A(h)$ that clinches x for i.

3. If i is the unique active agent for whom $P_i(h) = G_i(h)$, then i moves at h.

4. If i moves at h and $P_i(h) = G_i(h)$, then $C_i(h) = P_i(h)$, there there is no passing action at h, and i is not called to move at any $h' \supsetneq h$.

We can impose on the millipede several further assumptions. In order to formulate them we say that agent i is **active** at h if she has been previously called to play at some $h' \subseteq h$, and has not yet clinched an object at h. Let $\mathcal{A}(h)$ denote the set of active agents at h. Among active agents we distinguish the class of lurkers that is a more restrictive version of

[56]Property 1 is the basic structure of the millipede established in the lemma; we restate it in order to introduce the a^* notation. That, without loss of generality, we can assume property 2 is established in the proof of the lemma, and the same argument allows us to assume property 3. We can assume property 4 because by property 2 and greedy strategies, any passing move at h can be pruned.

a similar concept introduced by Bade and Gonczarowski (2017, hereafter BG).[57] Informally, a lurker is an active agent who has been offered to clinch all objects that are possible for him except for exactly one, which he is said to "lurk". If an agent lurks some object x, then we can infer that x is the agent's favorite possible object, and so we might exclude x from other agents without violating Pareto efficiency.

To formally define a lurker, recall that $C_i^{\subseteq}(h) = \{x : x \in C_i(h') \text{ for some } h' \subseteq h\}$ is the objects agent i has been offered to clinch at some subhistory of h and $C_i^{\subsetneq}(h) = \{x : x \in C_i(h') \text{ for some } h' \subsetneq h\}$ is the objects agent i has been offered to clinch at some strict subhistory of h. We consider a history h and an active agent i who has moved at a strict subhistory of h. Let $h' \subsetneq h$ be the maximal strict subhistory such that $i_{h'} = i$. Agent i is said to be a **lurker** for object x at h if (i) $P_i(h) \neq G_i(h)$, (ii) $x \in P_i(h')$, (iii) $C_i^{\subseteq}(h') = P_i(h') \setminus \{x\}$, and (iv) $x \notin C_j^{\subseteq}(h')$ for any other active $j \neq i$ that is not a lurker at h'. If some agent i is a lurker for an object x at a history h, then we say x is a **lurked object** at h. We use the term **BG lurker** to refer to any agent that satisfies (i), (ii), and (iii).[58] We will see below that at any history at most two active agents are not lurkers (cf. Lemma B.11).

At any h, we partition the set of active agents as $\mathcal{A}(h) = \mathcal{L}(h) \cup \bar{\mathcal{L}}(h)$, where $\mathcal{L}(h) = \{\ell_1^h, \ldots, \ell_m^h\}$ is the set of lurkers and $\bar{\mathcal{L}}(h)$ is the set of active non-lurkers. Let $\mathcal{X}(h)$ denote the set of still-available (unclinched) objects at h, and partition this set as $\mathcal{X}(h) = \mathcal{X}^{\mathcal{L}}(h) \cup \bar{\mathcal{X}}^{\mathcal{L}}(h)$, where $\mathcal{X}^{\mathcal{L}}(h) = \{x_1^h, \ldots, x_{\lambda(h)}^h\}$ is the set of lurked objects and $\bar{\mathcal{X}}^{\mathcal{L}}(h) = \mathcal{X}(h) \setminus \mathcal{X}^{\mathcal{L}}(h)$ is the set of unlurked objects at h. Each ℓ_m^h has a unique object that she lurks, x_m^h, and each x_m^h has a unique lurker. We order the sets so that if $m' < m$, then lurker $\ell_{m'}^h$ is **older** than lurker ℓ_m^h, in the sense that $\ell_{m'}^h$ first became a lurker for $x_{m'}^h$ at a strict subhistory of the history at which ℓ_m^h became a lurker for x_m^h; we then say that lurker ℓ_m^h is **younger** than lurker $\ell_{m'}^h$. We use the same older and younger comparisons for BG lurkers. Lemmas B.9-B.12 show that this construction is well-defined.

Any efficient millipede game satisfies the following additional conditions. If there are lurkers at a history then the last agent to moved along this history passed and, as long as no one has taken a clinching action, the set of lurkers and the set of lurked objects continue to grow, until eventually, we reach a history h where some agent i clinches some object x.[59]

[57] They focus on understanding which OSP mechanisms are Pareto efficient. In this proof we build on their insights, and in turn their analysis follows our 2016 characterization of OSP mechanisms through millipede games.

[58] BG lurkers were studied in BG. Because we impose condition (iv), our definition of a lurker is more restrictive than BG (Definition E.9): all lurkers in our sense are BG lurkers, but the converse need not hold. On the other hand, our definition of a non-lurker is more permissive: a non-lurker in our usage may not be a BG non-lurker. We include (iv) in the definition of a lurker because it simplifies the definition of our coding algorithm that maps role assignment functions to agent orderings; our coding algorithm treats BG lurkers who do not satisfy (iv) the same as other non-lurkers and differently from how it treats lurkers.

[59] It is immediate that lurker conditions (i)-(iii) continue to hold at each history reached by passing from

When i clinches at h, this allows us to determine the assignments of all lurkers as follows:

- If $x \in \bar{\mathcal{X}}^{\mathcal{L}}(h)$, each lurker $\ell^h_m \in \mathcal{L}(h)$ receives her lurked object, x^h_m.
- If $x = x^h_{m_1}$ for some lurked $x^h_{m_1} \in \mathcal{X}^{\mathcal{L}}(h)$, then all older lurkers $\ell^h_{m'}$ for $m' < m_1$ receive their lurked objects $x^h_{m'}$; lurker $\ell^h_{m_1}$, whose lurked object is assigned to i, receives her favorite object from the remaining set of unclinched objects, $\mathcal{X}(h) \smallsetminus \{x^h_1, \ldots, x^h_{m_1}\}$.
  - If $\ell^h_{m_1}$ is assigned an unlurked object, then all remaining lurkers get their lurked objects; if $\ell^h_{m_1}$ is assigned a lurked object $x^h_{m_2}$ for some $m_2 > m_1$, then all older unmatched lurkers ($\ell^h_{m'}$ for $m_1 < m' < m_2$) receive their lurked objects. Lurker $\ell^h_{m_2}$ gets his favorite object from $\mathcal{X}(h) \smallsetminus \{x^h_1, \ldots, x^h_{m_2}\}$.
  - This process is repeated until some lurker $\ell^h_{\bar{m}}$ receives an unlurked object, at which point all remaining unassigned lurkers are assigned their lurked objects.

These assignments are implied by Lemma E.17 in BG (who show that it is valid under the definition of BG lurkers) and by our Lemma B.8, which shows that, at any history, there is at most one BG lurker who is not a lurker and it is the youngest BG lurker.

Via the argument from Lemma A.6, this structure of assignments allows us, without loss of generality, to further restrict attention to millipedes satisfying the following:

5. Following the clinching action, the next moves are taken by lurkers who are not assigned to their lurked objects (if there are any), from oldest to youngest. At each of these moves, the lurker is offered for clinching all objects that have not been assigned prior to the move. The remaining lurkers (if there are any) never move after h.

In the above structure of assignments, there is a unique agent j who clinches an unlurked object y; this agent might be the agent i above or one of the lurkers. Lemmas B.11 and B.13 show that there might be at most one additional active agent, j', who is neither i nor one of the lurkers. If such a j' exists and $y \in C^{\subseteq}_{j'}(h)$ then j' receives her favorite object that was neither assigned prior to h nor to other active agents at h.[60] This allows us to further restrict the attention to millipedes satisfying the following:

the history at which agent i became a lurker. To see that (iv) also continues to hold, suppose it fails at a history $(h, a^*, ..., a^*)$ because some agent j was offered object x lurked by i till this history, then Lemma B.7 implies that at the history at which j passed, she was offered all objects possible for her; a contradiction with greedy strategies of the millipede.

[60]Let y' be the top choice for j' among objects that were neither assigned prior to h nor to other active agents at h. Then j' can at best receive y'. As there is a preference profile of other agents at which they rank y' lowest, making y' impossible for j' would violate Pareto efficiency. Thus y' is possible for j'. At the same time, the payoff guarantee properties of a millipede imply that j' is offered for clinching all objects that were possible but not clinchable for her when j' passed on y. Thus, the footnoted claim follows.

6. If the agent j' just described exists and $y \in C^{\subsetneq}_{j'}(h)$, then j' moves right after agent j and is offered for clinching all objects that have not been assigned prior to her move.

7. If the agent j' just described exists and $y \notin C^{\subsetneq}_{j'}(h)$, then j' moves right after j and is offered for clinching all objects in $C^{\subsetneq}_{j'}(h)$; j' might also have other clinching moves or a passing move.

Remark 2 (*Recursive structure*). Properties 5 and 6 guarantee that the games we study have a recursive structure. The continuation game following the last move described in properties 5 and 6 (by j or j') or by agent $i = j$ (if neither condition 5 nor 6 are applicable) is just a smaller Pareto efficient millipede game on the remaining unmatched agents and objects. This continuation game has the structure described above. Property 7 guarantees that, if j' moves in the continuation game, then, after the property 7 move, the set of objects j' could have clinched till a history in the continuation game is the same as the set of objects this agent could have clinched till this history in the entire game.

B.7.2 Constructing the mapping from role assignment functions σ to partial orderings $\succ_\sigma$

We provide an algorithm that, for a given mechanism (Γ, S) and fixed preference profile, follows the path of the game from the root node $h_\varnothing$ to the terminal node $\bar{h}$ and outputs a partial ordering—also called a *coding*—of the agents, denoted $\succ$. This ordering is only partial because agents may tie. Each role assignment function $\sigma \in \Sigma$ induces a game Γ_σ and an associated coding, $\succ_\sigma$, via our coding algorithm. Running the algorithm on all $N!$ role assignment functions gives $N!$ codings. We then argue that it is possible to break the ties in a way that gives us a bijection $f : \Sigma \to Ord$ such that for each σ, a serial dictatorship (**SD** for shortness) run under serial dictatorship ordering (**SD orderings** for shortness) f_σ results in the same allocation as game Γ_σ.

The intuitive idea behind constructing the coding $\succ$ is as follows. We start by finding the first agent to clinch some object x (after a possibly empty series of passes) at some history h. This induces a chain of assignments of the active agents $\mathcal{A}(h)$ as described above in Section B.7.1 above. We create $\succ$ by ordering agents who receive lurked objects in order of when the lurked object became lurked, i.e., the first agent in the ordering is the agent who receives the object that became lurked first, etc. After this is done, there are at most 2 active agents who have yet to be coded, one of whom has clinched an unlurked object, say y; if y was previously offered to the remaining active agent, then we add both remaining agents to the order without distinguishing between them, i.e., these two agents tie; if y was not previously offered to the other remaining active agent, then we only add to the ordering the agent who clinched y, The other active agent (if such an agent exists) will be added in a later segment triggered by clinching; at the beginning of the next segment this agent is still active with the carried over "endowment" $C_j^{\subseteq}(h)$ (cf. Remark 2). After clearing this first segment of agents, we continue along the game path and find the first unordered agent to clinch an object, and repeat.

Coding Algorithm. Consider any game path from the root node $h_\varnothing$ to a terminal node $\bar{h}$, which is associated with a unique allocation of objects to agents. Each step k of the algorithm below produces a partial ordering $\tilde{\succ}^k$ on the set of agents who are processed in step k. At the end of the final step K, we concatenate the K components to produce $\succ$, the final coding on the set of all agents $\mathcal{N}$.

Step 1 Find the first object to be clinched along the game path, say x^1 at history h^1 by agent i^1.[61] Let $\mathcal{L}(h^1) = \{\ell_1, \ldots, \ell_{\lambda(h^1)}\}$ be the set of lurkers, and $\mathcal{X}^{\mathcal{L}}(h^1) = \{x_1, \ldots, x_{\lambda(h^1)}\}$ be the set of lurked objects at h^1, where x_k is the k-th object to become lurked and ℓ_k the lurker of this object; if these sets are empty, skip directly to step 1.2 below.

1. For $x_k \in \mathcal{X}^{\mathcal{L}}(h^1)$, let i_{x_k} be the agent who receives x_k at $\bar{h}$.[62]
2. Let $j \in \mathcal{L}(h^1) \cup \{i^1\}$ be the unique agent that is not one of the agents $i_{x_1}, ..., i_{x_{\lambda(h^1)}}$ from step 1.1. By the properties introduced in Subsection B.7.1, j receives an unlurked object $y \in \bar{\mathcal{X}}^{\mathcal{L}}(h^1)$ and there may be at most one active agent $j' \in \mathcal{A}(h^1) \setminus (\mathcal{L}(h^1) \cup \{i^1\})$.
   (a) If such a j' exists and $y \in C_{j'}^{\subseteq}(h^1)$, then define $\tilde{\succ}^1$ as:
   $$i_{x_1} \tilde{\succ}^1 i_{x_2} \tilde{\succ}^1 \cdots \tilde{\succ}^1 i_{x_{\lambda(h^1)}} \tilde{\succ}^1 \{j, j'\}$$
   (b) Otherwise, define $\tilde{\succ}^1$ as
   $$i_{x_1} \tilde{\succ}^1 i_{x_2} \tilde{\succ}^1 \cdots \tilde{\succ}^1 i_{x_{\lambda(h^1)}} \tilde{\succ}^1 j$$
   In particular, if j' exists and $y \notin C_{j'}^{\subseteq}(h^1)$ then we do not yet order agent j'.

Step k Find the first object to be clinched along the game path by an agent that has not yet been ordered, say x^k at history h^k by agent i^k. Let $\mathcal{L}(h^k) = \{\ell_1, \ldots, \ell_{\lambda(h^k)}\}$ be the set of lurkers, and $\mathcal{X}^{\mathcal{L}}(h^k) = \{x_1, \ldots, x_{\lambda(h^k)}\}$ be the set of lurked objects, and carry out a procedure analogous to that from step 1 to produce the step k order $\tilde{\succ}^k$.

[61] That is, $i_{h^1} = i^1$, and i^1 selects a clinching action $a_{x^1} \in A(h^1)$ that clinches x^1. By Lemma B.13, $i^1 \notin \mathcal{L}(h^1)$. Notice the difference between superscript in x^1, which refers to the step of the algorithm, and the subscripts in lurked objects, which refer to the order in which they were lurked. In the notation for lurkers $\ell_k^{h^1}$ and lurked objects $x_k^{h^1}$ we suppress the history superscript.

[62] Note that i_{x_k} is not necessarily the agent who lurks x_k at h^1.

This produces a collection of codings $(\tilde{\succ}^1, \ldots, \tilde{\succ}^K)$, where each $\tilde{\succ}^k$ is a partial order on the agents processed in step k. We then create the final $\succ$ in the natural way: for any two agents i, j who were processed in the same step k, $i \succ j$ if and only if $i \tilde{\succ}^k j$. For any two agents i, j processed in different steps $k < k'$, respectively, we order $i \succ j$ and say that i **precedes** j.

The output of the coding algorithm is a partial order, $\succ$, on $\mathcal{N}$, the set of agents. If there are two agents j and j' such that $j \not\succ j'$ and $j' \not\succ j$, then we say j and j' **tie** under $\succ$. Note that by construction, all ties are of size at most 2, and agents can only tie if they are processed in the same step of the algorithm.

Remark 3. The coding algorithm divides the game path from the root to the terminal node into a series of K steps. At the end of each coding step, there may be one agent, say j', who was active during the step, and was not coded in the step. When this occurs, at the the initial history of the continuation game that begins after all agents from the previous step have been assigned their objects, agent j' is called to move, and is offered the to clinch everything that she has been offered to clinch previously in the game (and might have other moves). The next step of the coding algorithm is initiated the first time an agent clinches an object in this continuation game, and the process is repeated. This recursive structure is further discussed in Remark 2.

B.7.3 Proof of the Same-Allocations claim

Take a role assignment function σ, corresponding game Γ_σ, and the partial ordering $\succ_\sigma$ that results from applying the coding algorithm to Γ_σ. Let f_σ be a total (strict) ordering of the agents, where $f_\sigma(1) = i$ is the first agent, $f_\sigma(2) = j$ is the second agent, etc. We say that f_σ is **consistent** with $\succ_\sigma$ if, for all j, j': $j \succ_\sigma j'$ implies $f_\sigma^{-1}(j) < f_\sigma^{-1}(j')$. In other words, given some coding $\succ_\sigma$, total order f_σ is consistent if there is some possible way to break the ties in $\succ_\sigma$ that delivers f. We further say that f_σ is **consistent with $\succ_\sigma$ on an initial segment till an agent** i if, for all j, j' that either precede i or tie with i, if $j \succ_\sigma j'$ then $f_\sigma^{-1}(j) < f_\sigma^{-1}(j')$.

Lemma B.3. *For any agent i and any total order f_σ consistent with $\succ_\sigma$ on an initial segment till i, the allocation of agents who precede or tie with i under the serial dictatorship with agent ordering f_σ is the same as their allocation in Γ_σ. In particular, given two games Γ_A and Γ_B played under role assignment functions σ_A and σ_B, respectively, if $\succ_A = \succ_B$, then Γ_A and Γ_B end with the same final allocations to all agents.*

We prove this lemma in subsection B.7.6 below. Given $\succ_\sigma$, any way of breaking the ties (if any tie exist) between agents produces a total order f_σ that is consistent with $\succ_\sigma$. Thus, by Lemma B.3, the mechanism $(\Gamma_\sigma, S_\sigma(\succ_\mathcal{N}))$ ends with the same allocation as the serial dictatorship with agent ordering f_σ, which proves the Same Allocations claim.

B.7.4 Proof of the Bijectivity claim

We prove the Bijectivity claim using two lemmas—Lemmas B.4 and B.5—on the properties of the partial orders produced by the coding algorithm applied to games with different role assignments. To streamline the presentation, the proofs of these lemmas can be found in subsection B.7.6.

Let h_A^k be the history that initiates step k of the coding algorithm when it is applied to game Γ_A. For instance, $h_A^1 = (h_\varnothing, a^*, \ldots, a^*)$ is a history following a sequence of passes such that agent $i_{h_A^1}$ moves at h_A^1 and is the first agent to clinch in the game. This induces a chain of assignments of the agents in $\mathcal{L}(h_A^1) \cup \{i_{h_A^1}\}$, plus possibly one other active non-lurker at h_A^1, as described above. History $h_A^2 \supsetneq h_A^1$ is then the next time along the game path that an agent who was not ordered in step 1 of the coding algorithm clinches an object, etc. Define h_B^k analogously, and let K_A and K_B be the total number of steps in the coding algorithm when applied to games Γ_A and Γ_B, respectively.

Lemma B.4. *Let σ_A and σ_B be two role assignment functions, and Γ_A and Γ_B their associated games. Let $\succ_A^k$ be the initial segment of $\succ_A$ consisting of agents ordered till step k of the coding algorithm in game Γ_A. If ordering $\succ_A^k$ equals to an initial segment of $\succ_B$, then $h_A^{k'} = h_B^{k'}$ for all $k' = 1, \ldots, k$ and $\sigma_A^{-1}(i) = \sigma_B^{-1}(i)$ for all agents i who are coded up to step k. In particular, if $\succ_A = \succ_B$, then $h_A^k = h_B^k$ for all k, $K_A = K_B$, and $\sigma_A^{-1}(i) = \sigma_B^{-1}(i)$ for all $i \in \mathcal{N}$.*

The previous lemma shows that the mapping from role assignments to codings (partial orderings) is injective. As there may be ties in some codings, what remains to show is that it is possible to break the ties in all codings in such a way that preserves the injectivity. The next lemma provides the key tool needed to do this.

We write $j_1 \cdots j_P \succ i \succ j \cdots$ when $\succ$ ranks $j_1, \ldots, j_P$ first, possibly with ties; ranks i immediately (and strictly) after, and then either (a) ranks singleton j immediately (and strictly) after i, or (b) there is some k such that the tie $\{j, k\}$ is ranked immediately after i. We also write $j_1 \cdots j_P \succ i \succ j \succ \cdots$ for case (a) and $j_1 \cdots j_P \succ i \succ \{j, k\} \cdots$ for case (b).

Lemma B.5. *Assume that there exist positive integers $n, m \geq 1$ and two sequences of role assignment functions, $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_{n+1}\}$ and $\Sigma' = \{\sigma'_1, \sigma'_2, \sigma'_3, \ldots, \sigma'_m, \sigma'_{m+1}\}$ such that*

$\sigma_1 = \sigma_1'$ and the resulting codings are:

$$\begin{aligned}
\textit{Sequence } \Sigma\textit{:}\quad & j_1 \cdots j_P \succ_1 \{i, k_1\} \succ_1 \cdots \\
& j_1 \cdots j_P \succ_2 k_1 \succ_2 \{i, k_2\} \succ_2 \cdots \\
& j_1 \cdots j_P \succ_3 k_1 \succ_3 k_2 \succ_3 \{i, k_3\} \succ_3 \cdots \\
& \vdots \\
& j_1 \cdots j_P \succ_n k_1 \succ_n k_2 \succ_n k_3 \succ_n \cdots \succ_n k_{n-1} \succ_n \{i, k_n\} \succ_n \cdots \\
& j_1 \cdots j_P \succ_{n+1} k_1 \succ_{n+1} k_2 \succ_{n+1} k_3 \succ_{n+1} \cdots \succ_{n+1} k_{n-1} \succ_{n+1} k_n \succ_{n+1} i \cdots
\end{aligned}$$

$$\begin{aligned}
\textit{Sequence } \Sigma'\textit{:}\quad & j_1 \cdots j_P \succ_1' \{i, k_1\} \succ_1' \cdots \\
& j_1 \cdots j_P \succ_2' i \succ_2' \{k_1, k_2'\} \succ_2' \cdots \\
& j_1 \cdots j_P \succ_3' i \succ_3' k_2' \succ_3' \{k_1, k_3'\} \succ_3' \cdots \\
& \vdots \\
& j_1 \cdots j_P \succ_m' i \succ_m' k_2' \succ_m' k_3' \succ_n' \cdots \succ_m' k_{m-1}' \succ_m' \{k_1, k_m'\} \succ_m' \cdots \\
& j_1 \cdots j_P \succ_{m+1}' i \succ_{m+1}' k_2' \succ_{m+1}' k_3' \succ_{m+1}' \cdots \succ_{m+1}' k_{m-1}' \succ_{m+1}' k_m' \succ_{m+1}' k_1 \cdots
\end{aligned}$$

where the partial order on $j_1 \cdots j_P$ is the same in all above codings. Then, one of the following must hold:

(I) In $\succ_{n+1}$, agent i ties with some agent k_{n+1}; or

(II) In $\succ_{m+1}'$, agent k_1 ties with some agent k_{m+1}'.

Notice the symmetry between sequences Σ and Σ', to which we also refer as **arms**. They have the following properties:

- Each arm starts with the same role assignment and codings, i.e., $\sigma_1 = \sigma_1'$ and $\succ_1 = \succ_1'$.
- In arm Σ, every subsequent coding ranks k_1 strictly ahead of all other agents (besides the j_p's), while in Σ', every subsequent coding ranks i ahead of all other agents (besides the j_p's).
- Within arm Σ, the only difference from ℓ to $\ell + 1$ is that the agent k_ℓ who tied with i in $\succ_\ell$ is now ranked strictly above i, with i now tied with a different agent, $k_{\ell+1}$ (except for $\succ_{n+1}$, in which case i is ranked next, but may or may not tie with another agent). A similar remark applies to Σ'.
- Across the two arms, it is possible that some or all of the agents $k_2, \ldots, k_n$ are the same as the agents $k_2', \ldots, k_m'$, though it is not necessarily assumed. We also do not require $m = n$.

By Lemma B.4, the mapping from role assignments σ to codings $\succ_\sigma$ generated by the coding algorithm is injective. Using Lemma B.5, we break the ties to create from each $\succ_\sigma$ a consistent total order f_σ in a way that preserves the injectivity. We proceed with the following two tie-breaking steps:

Tie-Breaking Step 1. For all permutations σ, in coding $\succ_\sigma$ we break any tie $\{i, k_1\}$ so that $i \succ_\sigma k_1$ if and only if, in the original set of codings, there is an arm of the form Σ from Lemma B.5 in which the second coding starts with $j_1 \cdots j_P \succ k_1 \succ$ for some $j_1, ..., j_P \neq i$ and in the last coding agent i does not tie; analogously, we break any tie $\{i, k_1\}$ so that $k_1 \succ_\sigma i$ if and only if there is an arm of the form Σ' from Lemma B.5 in which the second coding starts with $j_1 \cdots j_P \succ i \succ$ for some $j_1, ..., j_P \neq k_1$ and in the last coding agent k_1 does not tie.

Lemma B.5 guarantees that the tie-breaking procedure just described is well-defined, in the sense that it will produce no conflicts in how to break a given tie. In particular, if there is an arm that forces a tie-break such that, say, $i \succ_\sigma k_1$, then Lemma B.5 implies that there cannot be an arm that forces a tie-break such that $k_1 \succ_\sigma i$.

Lemma B.5 further implies that, if $\succ_\sigma$ starts with $j_1 \cdots j_P \succ_1 \{i, k_1\}$ and we broke the tie $i \succ_{\sigma'} k_1$ (the other fully case is symmetric) then (i) no other coding starts with $j_1 \cdots j_P \succ_{\sigma'} i \succ_{\sigma'} k_1 \succ_{\sigma'}$ and (ii) no other coding starts with $j_1 \cdots j_P \succ_{\sigma'} i \succ_{\sigma'} \{k_1, k_2\}$ for some k_2 and the above tie-breaking procedure breaks the tie so that $k_1 \succ_{\sigma'} k_2$. By applying observations (i) and (ii) to tie breaks, starting at the end of each coding, we infer that the resulting mapping from permutations to partially tie-broken codings remain injective.

Importantly, the above tie-breaking procedure did not create any new ties that could be broken as in Tie-Breaking Step 1. Indeed, if, say, a broken tie $\{i, k_\ell\}$ creates a new arm that would allow a tie break at $\{i, k_1\}$ then, the structure of the arms in the statement of Lemma B.5 implies that before the former tie-break, the latter tie is broken by the union of the arm from $\{i, k_1\}$ till $\{i, k_\ell\}$ and the arm that allowed us to break the tie $\{i, k_\ell\}$.

Tie-Breaking Step 2. After the end of Tie-Breaking Step 1, there may still be ties remaining. If there are no ties remaining, then Step 1 has already produced an injective mapping from codings to consistent total orderings, and we skip to the last paragraph of the proof. If there are ties remaining, then it must be that all arms that begin with these ties end with the last agent being in a tie. We then proceed recursively. We look over all ties in the partial orders created in Tie-Breaking Step 1 across all permutations σ and find a tie—say $\{i, k_1\}$—that has the largest number of agents ranked above it. If such a tie $\{i, k_1\}$ exists then we break this tie arbitrarily. Because we broke only one such tie, the "at least one tie" structure of arms stated in Lemma B.5 holds for the resulting set of partial orderings. We can thus perform the same tie breaking as was done in Tie-Breaking Step 1 and, as above, the resulting mapping from permutations to partially tie-broken codings remain injective and, in all remaining ties, all arms end with the last agent being in a tie.

We repeat the above tie-breaking procedure iteratively: we look over all ties in partial orders created so far in Tie-Breaking Step 2, across all permutations σ, and again find a tie that has the largest number of agents ranked above it and repeat the Step-2 tie break procedure above. We proceed in this way till all ties are broken and we have constructed an injective mapping from permutations to total orderings.

As the resulting total orderings are created by breaking ties in the original codings, the complete orderings are consistent with the original codings. Hence we created an injective mapping from permutations to total orderings that are consistent with codings. In this way we obtain an injection from role assignments σ to serial dictatorships with orders f_σ. Because in this injection the domain of role assignments σ and the range of serial dictatorship orderings f_σ are finite and have equal size, this injection is a bijection. ∎

B.7.5 Preliminary Results for the Proofs of Lemmas B.3, B.4, and B.5

In this section, we present several auxiliary lemmas elucidating the properties of lurkers and other active agent in Pareto efficient millipede games satisfying properties 1-4; these properties allow us to then restrict attention in the rest of the proof to millipede games satisfying also Assumptions 6-8. We first present two new results—Lemmas B.6 and B.8—on the connection between lurkers and BG lurkers. Then, we give five lemmas that are analogues of corresponding lemmas first given by BG for BG lurkers in Pareto efficient OSP games, and show that these lemmas continue to hold for our definition of lurkers. We finish with two additional lemmas.

Given some history h, let h' be the maximal superhistory of the form $h' = (h, a^*, \ldots, a^*)$. Following BG, we call h' a **terminating history**, and the agent who moves at h' a **terminator**. The terminating history provides an upper bound on the number of passes that can be taken in a row, i.e., at the terminating history, the agent that moves has only clinching actions, and his action triggers a chain of assignments as described in Section B.7.1. Note that there may be many terminating histories along the full game-path, and that the definition of the terminating history is only a function of the game form Γ, and is independent of the lurker definition that is considered.

Lemma B.6. *Let h be a history such that there is an active BG non-lurker j such that $x \in C_j^{\subseteq}(h)$ for some object x that is BG-lurked at h. Then, h is a terminating history, and j is the terminator.*

Proof. Let $\bar{h}$ be the largest proper subhistory of h, $\bar{h} \subsetneq h$, such that the set of BG-lurked objects at $\bar{h}$ is empty. It is sufficient to show that for the smallest superhistory $h \supseteq \bar{h}$

that satisfies the statement of the lemma, h is a terminating history. Define h' such that $h = (h', a^*)$, i.e., h' is the immediate predecessor of h; such a predecessor exists because there are BG-lurked objects at h. By the supposition that h is the smallest superhistory of $\bar{h}$ that satisfies the statement of the lemma, we have that either (i) x is not BG-lurked at h' or (ii) x is BG-lurked at h', but $x \notin C_j^{\subseteq}(h')$.

For case (i), x first becomes BG-lurked at h. Let ℓ be the agent that BG-lurks x at h, and notice that it must be ℓ that moves at h'.[63] This implies that both j and ℓ are active at h', and neither are BG lurkers. Because there can be at most two active BG non-lurkers at any history, all other active agents at h' are BG lurkers. Now, consider h. At h, $x \in C_j^{\subseteq}(h)$, and so Lemma E.14 of BG implies $P_j(h) = G_j(h)$. Further, j is the unique active agent such that $P_j(h) = G_j(h)$.[64] Thus, by properties 4 and 5 of Section B.7.1, j moves at h and $P_i(h) = G_i(h) = C_i(h)$, and there is no passing action at h. Thus, h is the terminating history.

For case (ii), $x \notin C_j^{\subsetneq}(h')$ but $x \in C_j^{\subseteq}(h)$ implies that j must move at h, and $x \in C_j(h)$. By BG Lemma E.14, $P_j(h) = G_j(h)$. By property 4 in Section B.7.1, $P_i(h) = G_i(h) = C_i(h)$, and there is no passing action at h. Thus, h is the terminating history. ∎

Lemma B.7. *If $i \in \bar{\mathcal{L}}(h)$ and $x_\ell \in C_i^{\subseteq}(h)$ for some $x_\ell \in \mathcal{X}^{\mathcal{L}}(h)$, then $i_h = i$, $P_i(h) = G_i(h) = C_i(h)$, and there is no passing action at h (that is, h is a terminating history).*

Proof. If x_ℓ is lurked at h then x_ℓ is BG-lurked at h; thus if i is a BG non-lurker at h, then the result follows from Lemma B.6. So, assume that i is a non-lurker that is a BG lurker at h. We claim that for any lurked object $x_\ell \in \mathcal{X}^{\mathcal{L}}(h)$, we have $x_\ell \notin C_i^{\subseteq}(h)$, and so the result holds vacuously. To show it, let h' be such that $h = (h', a^*)$, i.e., h' is the immediate predecessor of h. By Lemma B.8, h must be a terminating history, agent i moves at h' and passes, and becomes a BG lurker at h. Note that x_ℓ is BG-lurked at h. If $x_\ell \in C_i^{\subseteq}(h)$, then, since i does not move at h, we have $x_\ell \in C_i^{\subseteq}(h')$ as well. Because x_ℓ cannot be the object i BG lurks at h, object x_ℓ must be BG-lurked at h' by some other agent. But then, at h', i

[63]Assume not, i.e., assume some $k \neq \ell$ moved at h'. Then, the maximal strict subhistory of h where ℓ moves is some $h'' \subsetneq h'$, and by definition of a BG lurker (i) $P_\ell(h) \neq G_\ell(h)$, (ii) $x \in P_\ell(h'')$, and (iii) $C_\ell^{\subseteq}(h'') = P_\ell(h'') \setminus \{x\}$ hold.

This implies that ℓ is already a lurker for x at h': since $h'' \subsetneq h'$, (i) and (ii) continue to hold at h', while for for (iii), if $P_\ell(h') = G_\ell(h')$, then, since the game is a millipede game that satisfies properties 1-4, there is no passing action at h'. This contradicts that x is not lurked at h'.

[64]For any active lurker ℓ at h, $P_\ell(h) \neq G_\ell(h)$ by definition. The only other possibility is that some k becomes active at h, and is such that $P_k(h) = G_k(h)$. If this is the case, by BG Lemma E.11, all BG-unlurked objects are possible for k at h. If $P_k(h) = G_k(h)$, then she can clinch any BG-unlurked object at h, by property 4. Consider k clinching some BG-unlurked object y. By BG Lemma E.17, all BG lurkers at h are assigned their BG lurked objects, and so no BG-lurked object is in $G_j(h)$. But, y was arbitrary, and so no BG-unlurked object is in $G_j(h)$ either, and so $G_j(h)$ is empty, which contradicts that $P_j(h) = G_j(h)$.

is not a BG lurker, and has previously been offered to clinch a BG-lurked object. Thus, by Lemma B.6, h' is a terminating history, which is a contradiction. ■

Lemma B.8. *At any h, there is at most one BG lurker that is not a lurker. If such an agent i exists, then i is the youngest BG lurker at h, and h is a terminating history. Further, i does not move at h.*

Proof. Consider a history $\bar{h}$ at which there are no BG lurkers (and thus, also no lurkers). Because at each history, only one new BG lurker can be added, it is sufficient to show that if $h \supsetneq \bar{h}$ is the smallest superhistory of $\bar{h}$ such that there is a BG lurker that is not a lurker, then h is a terminating history. Thus, let $h = (h', a^*)$, where at h', all BG lurkers are lurkers, but at h, there is a BG lurker that is not a lurker; label this agent i. Then, it must be that i first becomes a BG lurker at h, and at h, point (iv) fails, i.e., there is some active BG non-lurker $j \neq i$ that has been previously offered to clinch the object that i BG lurks. Lemma B.6 implies that h is the terminating history, and agent j moves at h. Since no new agent has entered the game at h, and all agents other than j are BG lurkers at h, there is only one BG lurker that is not a lurker. The rest of the statements follow easily from the fact that h is a terminating history. ■

The next four lemmas are analogues of statements derived for BG lurkers in BG; we give the analogous BG lemmas in parentheses. Recall that $\mathcal{L}(h)$ and $\mathcal{X}^{\mathcal{L}}(h)$ are the sets of lurkers and lurked objects, respectively, at history h. Let $\mathcal{L}^{BG}(h)$ and $\mathcal{X}^{\mathcal{L},BG}(h)$ denote the sets of BG lurkers and BG-lurked objects. Notice that $\mathcal{L}(h) \subseteq \mathcal{L}^{BG}(h)$ and $\mathcal{X}^{\mathcal{L}}(h) \subseteq \mathcal{X}^{\mathcal{L},BG}(h)$, by definition. Further, by Lemmas B.6 and B.8, if $\mathcal{L}(h) \subsetneq \mathcal{L}^{BG}(h) = \{\ell_1, \ldots, \ell_{\lambda^{BG}(h)}\}$, then $\mathcal{L}(h) = \mathcal{L}^{BG}(h) \smallsetminus \{\ell_{\lambda^{BG}(h')}\}$, where $\ell_{\lambda^{BG}(h')}$ is the youngest BG lurker. Similarly, if $\mathcal{X}^{\mathcal{L}}(h) \subsetneq \mathcal{X}^{\mathcal{L},BG}(h) = \{x_1, \ldots, x_{\lambda(h)}\}$ then $\mathcal{X}^{\mathcal{L}}(h) = \mathcal{X}^{\mathcal{L},BG}(h) \smallsetminus \{x_{\lambda^{BG}(h)}\}$, where $x_{\lambda^{BG}(h)}$ is the youngest BG-lurked object.

Lemma B.9. *(BG Lemma E.11) If agent i is active at h, then $\bar{\mathcal{X}}^{\mathcal{L}}(h) \subseteq P_i(h) \cup C_i^{\subsetneq}(h)$. If $i \in \mathcal{L}(h)$, then $\bar{\mathcal{X}}^{\mathcal{L}}(h) \subseteq C_i^{\subsetneq}(h)$.*

Proof. For the first part, for any $x \in \bar{\mathcal{X}}^{\mathcal{L}}(h)$ that is also BG-unlurked, the statement follows from BG Lemma E.11. So, consider some $x \in \bar{\mathcal{X}}^{\mathcal{L}}(h)$ but $x \in \mathcal{X}^{\mathcal{L},BG}(h)$. As shown above, there is only one such object, and it is $x = x_{\lambda(h)}$, the youngest lurked object at h. Further, by Lemma B.8, this condition only obtains when h is a terminating history, and the active agents at h are $\ell_1, \ldots, \ell_{\lambda(h)}, j$ where: $\ell_1, \ldots, \ell_{\lambda(h)-1}$ are both lurkers and BG lurkers, $\ell_{\lambda(h)}$ is a BG lurker but not a lurker, and j is the terminator (and neither a lurker nor a BG lurker). By BG Lemma E.16, $x_{\lambda(h)} \in P_{\ell'}(h)$ for all $\ell' \in \{\ell_1, \ldots, \ell_{\lambda(h)}\}$, while by BG Lemma E.18, $x_{\lambda(h)} \in C_j^{\subseteq}(h)$.

The second part follows from the first part and the definition of a lurker. ■

Corollary B.1. *If, at history h, agent i clinches $x \in \bar{\mathcal{X}}^{\mathcal{L}}(h)$ that is unlurked at h, then $x = Top(\succ_i, \bar{\mathcal{X}}^{\mathcal{L}}(h))$.*

Lemma B.10. *(BG Lemma E.16) Let $\mathcal{L}(h) = \{\ell_1^h, \ldots, \ell_{\lambda(h)}^h\}$ be the set of lurkers at h and $\mathcal{X}^{\mathcal{L}}(h) = \{x_1^h, \ldots, x_{\lambda(h)}^h\}$, with ℓ_1^h lurking x_1^h, ℓ_2^h lurking x_2^h, etc., where $m < m'$ if and only if ℓ_m^h became a lurker at a strict subhistory of the history at which $\ell_{m'}^h$ became a lurker. Then,*

1. *$x_1^h, \ldots, x_{\lambda(h)}^h$ are all distinct objects.*
2. *For all $m = 1, \ldots \lambda(h)$, $P_{\ell_m^h}(h) = \mathcal{X}(h) \setminus \{x_1^h, \ldots, x_{m-1}^h\}$.*

Proof. Because any lurker is a BG lurker, and the same applies to lurked objects, this is immediate from BG Lemma E.16. ∎

Lemma B.11. *(BG Lemma E.19) For all h, $|\bar{\mathcal{L}}(h)| \le 2$.*

Proof. By BG Lemma E.19, there can be at most two BG non-lurkers at h. If there exists a non-lurker that is not a BG non-lurker, by Lemmas B.6 and B.8, all active agents except for one are BG lurkers, and at most one BG lurker is a non-lurker. Thus, there are at most two non-lurkers at h. ∎

Lemma B.12. *(BG Lemma E.18, E.20) Let h be a history with lurked objects and let $i_{h'} = t$ be the agent who moves at the maximal superhistory of the form $h' = (h, a^*, \ldots, a^*)$. Then:*
(i) Agent t is not a lurker at h.
(ii) $C_t^{\subseteq}(h') = \mathcal{X}(h)$.
(iii) If $i_h \neq t$, then $C_{i_h}(h) \cap C_t^{\subseteq}(h) = \varnothing$.
(iv) If $x_\ell \in P_j(h)$ for some non-lurker j and lurked object $x_\ell \in \mathcal{X}^{\mathcal{L}}(h)$, then $j = t$.
(v) $C_t^{\subseteq}(h') = \mathcal{X}(h)$.

Proof. Notice first that parts (ii), (iii), and (v) do not make any reference to lurkers or lurked objects, and thus these parts follow immediately from the corresponding statements in BG Lemma E.18. BG Lemma E.18 part (i) says that agent t is not a BG lurker, and thus, agent t is not a lurker either. What remains is to show part (iv). For all $h \subsetneq h'$, any non-lurker is also a BG non-lurker by Lemmas B.6 and B.8, and any lurked object is also a BG lurked object, and so the result follows from the corresponding lemma of BG. Thus, consider h'. By Lemma B.6 and Corollary B.8, at h', either $\mathcal{L}^{BG}(h') = \mathcal{L}(h')$ or $\mathcal{L}(h') = \mathcal{L}^{BG}(h') \setminus \{\ell_{\lambda^{BG}(h')}\}$. Similarly, either $\mathcal{X}^{\mathcal{L},BG}(h) = \mathcal{X}^{\mathcal{L}}(h)$ or $\mathcal{X}^{\mathcal{L}}(h') = \mathcal{X}^{\mathcal{L},BG}(h') \setminus \{x_{\lambda^{BG}(h)}\}$. If j is a BG non-lurker, then the result is immediate from the corresponding lemma of BG. It remains to consider j who is a non-lurker but a BG lurker. By Corollary B.8, j is a BG lurker for $x_{\lambda^{BG}(h')}$. Notice that $x_{\lambda^{BG}(h')}$ is not lurked at h' (though it is BG-lurked).

Thus, the lurked objects at h' are $\mathcal{X}^{\mathcal{L}}(h') = \{x_1, \dots, x_{\lambda^{BG}(h')-1}\}$. By Lemma E.16 from BG , $P_j(h') = \mathcal{X}(h') \setminus \{x_1, \dots, x_{\lambda^{BG}(h')-1}\}$; in other words, for any $x \in \mathcal{X}^{\mathcal{L}}(h')$, we have $x \notin P_j(h')$, and so the statement holds vacuously. ■

Lemma B.13. *For any history h and any superhistory $h' \supseteq h$ of the form $h' = (h, a^*, a^*, \dots, a^*)$, we have $i_{h'} \notin \mathcal{L}(h)$ and $i_{h'} \notin \mathcal{L}(h')$.*

Proof. The claim is immediate if $\mathcal{L}(h) = \varnothing$. Suppose $\mathcal{L}(h) \neq \varnothing$. We only show $i_{h'} \notin \mathcal{L}(h)$ as $i_{h'} \notin \mathcal{L}(h')$ then follows by setting $h' = h$. Let $\mathcal{L}(h) = \{\ell_1^h, \dots, \ell_{\lambda(h)}^h\}$ be the set of lurkers at h and $\mathcal{X}^{\mathcal{L}}(h) = \{x_1^h, \dots, x_{\lambda(h)}^h\}$ the set of lurked objects.

First, assume $h \neq h'$. Assume that the statement was false, and let $h' = (h, a^*, a^*, \dots, a^*)$ be the smallest superhistory of h such that $i_{h'} = \ell_m^h$ for a lurker ℓ_m^h (that is, $i_{h''} \notin \mathcal{L}(h)$ for all $h \subseteq h'' \subsetneq h'$). Note first that, for any h'' such that $h \subseteq h'' \subsetneq h'$, $i_{h'''} = j \in \bar{\mathcal{L}}(h)$, and if there exists some lurked $x_m^h \in C_j^{\subsetneq}(h'')$, by Lemma B.7, there is no passing action at h'', which is a contradiction. Therefore, any clinching action $a_y \in A(h'')$ clinches some $y \in \mathcal{X}(h) \setminus \mathcal{X}^{\mathcal{L}}(h)$, and for all terminal histories $\bar{h} \supset (h'', a_y)$, each lurker $\ell_m^h \in \mathcal{L}(h)$ receives his lurked object x_m^h. Finally, consider history h'. By Lemma B.10, for each $\ell_m^h \in \mathcal{L}(h)$, $P_{\ell_m^h}(h') = P_{\ell_m^h}(h) = \mathcal{X}(h) \setminus \{x_1^h, \dots, x_{m-1}^h\}$ (note that h' is reached from h via a series of passes, and so $\mathcal{X}(h) = \mathcal{X}(h')$), and $Top(\succ_{\ell_m^h}, P_{\ell_m^h}(h')) = x_m^h$ for all types $\succ_{\ell_m^h}$ such that h' is on the path of play. Therefore, by property 4 and greedy strategies, at h', there is no clinching action a_x for any $x \in P_{\ell_m^h}(h') \setminus \{x_m^h\}$. Thus, the only possibility is that every action $a \in A(h')$ clinches x_m^h.[65] This then implies that ℓ_m^h gets x_m^h at all terminal $\bar{h} \supset h'$. Combining this with the previous statement that ℓ_m^h gets x_m^h for all terminal $\bar{h} \supset (h'', a_y)$ for any $h \subseteq h'' \subsetneq h'$ and clinching action $a_y \in A(h'')$, we conclude that ℓ_m^h gets x_m^h for all terminal $\bar{h} \supset h$, i.e., ℓ_m^h has already clinched his object x_m^h at h. Thus, by definition of a millipede game, $i_{h'} \neq \ell_m^h$, which is a contradiction proving the first claim for $h' \neq h$.

Second, if $h = h'$ then let $h^* \subsetneq h$ be the immediate predecessor history of h. By the just proven part of the lemma, i_h is not a lurker at h^*, and because i_h moves at h, she cannot move at h^*, and hence she is not a lurker at h. ■

Lemma B.14. *Let i and j be active non-lurkers at a history h, and let $y \in \bar{\mathcal{X}}^{\mathcal{L}}(h)$ be an unlurked object at h. Further, assume that $i_h = i$ and $y \in C_i(h) \cap C_j^{\subsetneq}(h)$. Consider a type $\succ_j$ that reaches h, and define $\bar{x} = Top(\succ_j, \bar{\mathcal{X}}^{\mathcal{L}}(h))$. Then, $\bar{x} \succ_j y$.*

Proof. By Lemma B.12, part (iii), agent j cannot be the terminator. By Lemma B.12, part (iv), $P_j(h) \subseteq \bar{\mathcal{X}}^{\mathcal{L}}(h)$. Since i can clinch y at h, there must be some $x \in P_j(h)$ such that $x \succ_j y$, by OSP. Since $P_j(h) \subseteq \bar{\mathcal{X}}^{\mathcal{L}}(h)$, we have $x \in \bar{\mathcal{X}}^{\mathcal{L}}(h)$, i.e., $Top(\succ_j, \bar{\mathcal{X}}^{\mathcal{L}}(h)) \succ_j y$. ■

[65] Note that there cannot be a passing action either: if there were, then, since every history is non-trivial, there must be another action. But, as just argued, there can be no clinching actions for any other $x \neq x_m^h$, and thus there must be a clinching action for x_m^h, and the passing action would be pruned.

B.7.6 Proofs of Lemmas B.3, B.4, and B.5

In the proofs that follow, we refer to roles in a game form Γ to state properties of Γ that are independent of the specific agent that is assigned to that role; for a formal definition of roles see Appendix A.7. Below, we write $C_r(h)$ to refer to the set of outcomes that are clinchable for the role $r \in \mathcal{R}$ at h and $P_r(h)$ for the set of outcomes that are possible for role r; these sets do not depend on the role assignment function σ; if $\sigma(r) = i$, then $C_i(h) = C_r(h)$, $P_i(h) = P_r(h)$, etc. Analogously to the sets $\mathcal{A}(h)$ and $\mathcal{L}(h)$ for active agents and lurkers at a history h, we write $\mathcal{A}_R(h)$ for the set of active roles at a history h, and $\mathcal{L}_R(h)$ for the set of roles that are lurkers at h. When we want to refer to the game form with agents assigned to roles via a specific role assignment function σ_A, we write Γ_A. In the proofs, we often move fluidly between agents and roles; to avoid confusion, we use the notation i, j, k to refer to specific agents, and the notation r, s, t to refer to generic roles. Finally, note that while the set of agents who are lurkers at any h may differ depending on the role assignment function, the set of lurked objects, the order in which they become lurked, and the set of lurker roles depend only on h, and are independent of the specific agent assigned to the role that moves at h.

Unless otherwise specified, when we write the phrase "i clinches x at h" (or similar variants), what is meant is that i moves at h, takes some clinching action $a_x \in A(h)$, and receives object x at all terminal histories $\bar{h} \supseteq (h, a_x)$.

The following is a restatement of part (iv) of the definition of a lurker, but deserves an emphasis, as it arises frequently in the arguments below.

Remark 4. If, at a history h, object x is such that $x \in C_j(h)$ for an active non-lurker j at h, then, for any history $h' = (h, a^*, \dots, a^*)$ such that $\mathcal{L}(h) = \mathcal{L}(h')$, x is not lurked at h; in other words, if x has been offered to an active non-lurker, it cannot become the next lurked object along the passing path.

Proof of Lemma B.3

We show the first statement; the second statement is then an immediate corollary. Suppose agent i is ordered in step k of the ordering algorithm. First consider the case $k = 1$ and let agent i^* be the first agent to clinch in game Γ_σ and let h^* be the history at which i^* clinches; this clinching induces the ordering of the first segment of agents in step 1 of the ordering algorithm. Let $\mathcal{X}^{\mathcal{L}}(h^*) = \{x_1, \dots, x_n\}$ be the set of lurked objects at h^*; this set may be empty.

Case: $\mathcal{A}(h) = \mathcal{L}(h) \cup \{i^*\}$. If i^* clinches an unlurked object $y \in \bar{\mathcal{X}}^{\mathcal{L}}(h^*)$, then, in Γ_σ, all lurkers get their lurked objects (the oldest lurker ℓ_1 gets x_1, the second oldest lurker ℓ_2

gets x_2, etc.), and in the resulting SD f_σ, the agents are ordered $f_\sigma : \ell_1, \ell_2, \ldots, \ell_n, i^*$. By Lemma B.10, for each lurker ℓ_m, we have $x_m = Top(\succ_{\ell_m}, \mathcal{X} \smallsetminus \{x_1, \ldots, x_{m-1}\})$. When it is agent ℓ_m's turn in the SD, she is offered to choose from $\mathcal{X} \smallsetminus \{x_1, \ldots, x_{m-1}\}$, and thus selects x_m. Finally, consider agent i^*. In game Γ_σ, when she clinches y at h^*, it is unlurked. By Corollary B.1, $y = Top(\succ_{i^*}, \bar{\mathcal{X}}^{\mathcal{L}}(h^*))$. At her turn in the SD, the set of objects remaining is precisely $\bar{\mathcal{X}}^{\mathcal{L}}(h^*)$, and so i^* selects y.

In the remaining possibility, i^* clinches some lurked object x_m. Then all older lurkers $\ell_1, \ldots, \ell_{m-1}$ get their lurked objects in Γ_σ, and the resulting SD begins as $f_\sigma : \ell_1, \ldots, \ell_{m-1}, i^*$. By an argument equivalent to the previous paragraph, each of the lurkers once again gets the same object under the SD. For agent i^*, since she took a lurked object at h^* in Γ_σ, we have $x_m = Top(\succ_i, \mathcal{X})$, and thus, at her turn in the SD, she once again selects x_m, since it is still available. Then, in Γ_σ, agent ℓ_m is offered to clinch anything from $\mathcal{X} \smallsetminus \{x_1, \ldots, x_m\}$. If ℓ_m takes another lurked object $x_{m'}$ for some $m' > m$, then each lurker $\ell_{m+1}, \ldots, \ell_{m'-1}$ is assigned to their lurked object, and we add to the SD order as $f_\sigma : \ell_1, \ldots, \ell_{m-1}, i^*, \ell_{m+1}, \ldots, \ell_{m'-1}, \ell_m$. By the same argument as above, at their turn in the resulting SD, each agent $\ell_{m+1}, \ldots, \ell_{m'-1}, \ell_m$ gets the same object in the SD.[66] This process continues until someone eventually takes an unlurked object, all remaining lurkers are ordered, and step 1 is completed.

Case: $\mathcal{A}(h) = \mathcal{L}(h) \cup \{i^*, j\}$ for some $j \in \mathcal{A}(h) \smallsetminus (\mathcal{L}(h) \cup \{i^*\})$. First consider the case that i^* clinches an unlurked object $y \in \bar{\mathcal{X}}^{\mathcal{L}}(h^*)$. If $y \notin C_j^{\subsetneq}(h^*)$, then the argument is exactly the same as in Case (1) (note that j is not ordered in step 1 in this case). If $y \in C_j^{\subsetneq}(h^*)$, then the step 1 partial order is $\ell_1 \tilde{\triangleright}^1 \cdots \tilde{\triangleright}^1 \ell_n \tilde{\triangleright}^1 \{i^*, j\}$. We must show that any SD run under $f_\sigma : \ell_1, \ldots, \ell_n, i^*, j, \ldots$ and $f'_\sigma : \ell_1, \ldots, \ell_n, j, i^*, \ldots$ result in the same outcome as Γ_σ for these agents. For the lurkers, the argument is as above in either case. For i^* and j, in game Γ_σ, by construction, $y \in C_j(h')$ for some $h' \subsetneq h^*$. Let $z = Top(\succ_j, \bar{\mathcal{X}}^{\mathcal{L}}(h^*))$, and note that by Lemma B.14, $z \succ_j y$. Since i clinched y at h^*, we have $y \succ_i z$. In the SD, after all lurkers have picked, the set of remaining objects is precisely $\bar{\mathcal{X}}^{\mathcal{L}}(h^*)$. Thus, it does not matter whether i^* or j is ordered next in the SD, as there is no conflict between them: in both cases, i^* takes y, and j takes z, and both f_σ and f'_σ give the same allocation as Γ_σ. For the case where i^* begins by clinching some lurked object $x_m \in \mathcal{X}^{\mathcal{L}}(h^*)$, we consider agent j and the lurker who, in the chain of assignments, eventually takes an unlurked object y; otherwise, the argument is analogous.

The proof so far has shown that we get the same allocation for all agents ordered in step 1 of the ordering algorithm. If $k > 1$ then we proceed recursively through steps $2, ..., k$, as

[66] When it is agent ℓ_m's turn in the SD, the set of available objects is a subset of the set of objects that were offered to her when she clinched in $\Gamma_\sigma : \mathcal{X} \smallsetminus \{x_1, \ldots, x_{m'-1}\} \subseteq \mathcal{X} \smallsetminus \{x_1, \ldots, x_m\}$. However, $x_{m'}$ belongs to both sets, and so since ℓ_m takes $x_{m'}$ in Γ_σ, she also takes it at her turn in the SD, when her offer set is smaller.

follows: If all active agents at $\mathcal{A}(h^*)$ are processed in step 1 of the ordering algorithm, then we repeat the same argument for the continuation subgame following the clinching by i^* at h^*; the second step of the coding algorithm for the original game is the same as the first step of the coding algorithm for this continuation subgame. If not all active agents at $\mathcal{A}(h^*)$ are processed in step 1, then there is at most one active agent $j \in \mathcal{A}(h^*)$ who is not processed in this step. Agent j has been previously offered some objects in the set $C_j^{\subsetneq}(h^*)$ where $C_j^{\subsetneq}(h^*) \subseteq \bar{\mathcal{X}}^{\mathcal{L}}(h)$. The coding in the continuation subgame following the clinching at h^* is the same as coding in the Pareto efficient auxiliary millipede that begins with agent j being offered clinching from $C_j^{\subsetneq}(h^*)$ and passing, and that then moves into the above continuation subgame; the second step of the coding algorithm for the original game is the same as the first step of the coding algorithm for this auxiliary millipede. ■

Proof of Lemma B.4

First consider $k = 1$ and suppose $\tilde{\succ}_A^1$ is equal to the initial part of the ordering $\succ_B$. Define the function $g_A(i) = |j \in \mathcal{N} : j \succ_A i| + 1$, which is the number of agents ranked strictly ahead of i under $\succ_A$. This function will almost corresponds to i's picking order in the resulting serial dictatorship, except if i ties under $\succ_A$; if i and i' tie, then $g_A(i) = g_A(i')$. Define g_B similarly.

Claim B.1. If $\tilde{\succ}_A^1$ is equal to an initial segment of $\succ_B$, then $h_A^1 = h_B^1$.

Proof of Claim B.1. Note that both h_A^1 and h_B^1 consist of a, possibly empty, sequence of passing moves, and so one of these histories must be a subset of the other. Towards a contradiction, assume that $h_A^1 \neq h_B^1$.

First, consider the case $h_A^1 \subsetneq h_B^1$. Define i_A to be the agent that clinches at h_A^1, and x_A to be the object that is clinched. Since there is a passing action at h_A^1, object x_A is unlurked at h_A^1, by Lemma B.7. Since i_A clinches an unlurked object at h_A^1, we have $x_A = Top(\succ_{i_A}, \bar{\mathcal{X}}^{\mathcal{L}}(h_A^1))$ by Corollary B.1. By construction of the coding algorithm, $g_A(i_A) = \lambda(h_A^1) + 1$, where $\lambda(h_A^1) = |\mathcal{L}_R(h_A^1)|$ is the number of lurkers (and hence also the number of lurked objects) that are present at h_A^1. Since $\tilde{\succ}_A^1$ is equal to an initial segment of $\succ_B$ and i_A is ordered in step 1 of Γ_A, we have $g_B(i_A) = \lambda(h_A^1) + 1$ as well.[67]

We claim that $\mathcal{X}^{\mathcal{L}}(h_A^1) = \mathcal{X}^{\mathcal{L}}(h_B^1)$. First, notice that $h_A^1 \subsetneq h_B^1$ implies $\mathcal{L}_R(h_A^1) \subseteq \mathcal{L}_R(h_B^1)$ and $\mathcal{X}^{\mathcal{L}}(h_A^1) \subseteq \mathcal{X}^{\mathcal{L}}(h_B^1)$, which follows because at each history in the millipede at most one object becomes lurked, and once an object is lurked, it remains lurked until it is clinched. If

[67]This is a key point, and its analogue remains true in the alternate case $h_B^1 \subsetneq h_A^1$. There, $g_B(i_B) = \lambda(h_B^1) + 1$, and we infer that also $g_A(i_B) = \lambda(h_B^1) + 1$. This follows because $h_B^1 \subsetneq h_A^1$ implies $\lambda(h_A^1) \geq \lambda(h_B^1)$, and so at least $\lambda(h_B^1) + 1$ agents are coded in step 1 of $\tilde{\succ}_A^1$. Thus, at least the first $\lambda(h_B^1) + 1$ agents in $\succ_B$ are in the same position in $\succ_A$, which includes agent i_B.

$\mathcal{X}^{\mathcal{L}}(h_B^1) \supsetneq \mathcal{X}^{\mathcal{L}}(h_A^1)$, then the $(\lambda(h_A^1)+1)^{th}$ lurked object in Γ_B (denoted $x_{\lambda(h_A^1)+1}$) must be x_A because (i) the coding algorithm puts the agent who receives $x_{\lambda(h_A^1)+1}$ as the $(\lambda(h_A^1)+1)^{th}$ agent, and hence this agent is i_A, and (ii) by Lemma B.3, i_A receives the same object under both σ_A and σ_B. But, because $x_A \in C_r(h_A^1)$, where r is the role that moves at h_A^1 and is not a lurker, x_A cannot be the $(\lambda(h_A^1)+1)^{th}$ lurked object, by part (iv) of the definition of a lurker, which is a contradiction. Therefore, $\mathcal{X}^{\mathcal{L}}(h_A^1) = \mathcal{X}^{\mathcal{L}}(h_B^1)$. This also means that $\mathcal{L}_R(h_A^1) = \mathcal{L}_R(h_B^1)$ and $\lambda(h_A^1) = \lambda(h_B^1)$; for simplicity, define $\lambda^1 := \lambda(h_A^1) = \lambda(h_B^1)$. Since x_A is unlurked at h_A^1, it is also unlurked at h_B^1.

Next, notice that some $j \neq i_A$ moves at h_A^1 in Γ_B, because otherwise, i_A would take the same (clinching) action at h_A^1 in Γ_B, which contradicts $h_A^1 \subsetneq h_B^1$. Let $s = \rho(h_A^1)$ be the role that moves at h_A^1, and so by definition, $\sigma_A(s) = i_A$ and $\sigma_B(s) = j$. At h_B^1, there are two active non-lurker roles: role s and another role s'. This follows because role s moves at h_A^1, and there is a passing action, so the history $h' = (h_A^1, a^*)$ must be controlled by a different active non-lurker role. Since there are no new lurkers at h_B^1, and there can be at most two active non-lurkers at any history, both roles s and s' remain active non-lurkers at h_B^1.

We claim that i_A must tie with another agent in $\succ_B$. To see this, note that if role s' moves at h_B^1, then i_A will tie with agent j in $\succ_B$, since $x_A \in C_s^{\subsetneq}(h_B^1)$ and $\sigma_B(s) = j$. If role s moves at h_B^1, then it is j that clinches at h_B^1 in Γ_B. If j clinches an unlurked object at h_B^1, then $g_B(j) = \lambda^1 + 1$, and so i_A ties with j in $\succ_B$. If j clinches a lurked object, then role s is the terminator role. Therefore, agent i_A was in the terminator role in Γ_A, and, since she clinched x_A first, we have $x_A = Top(\succ_A, \mathcal{X})$, which follows because all available objects are possible for the agent in the terminator role, by Lemma B.12. This implies that i_A cannot be a lurker at h_B^1 in Γ_B, because if she were, she would have been offered to clinch x_A, and since it is her top object, would have clinched it prior to h_B^1, by greedy strategies. Thus, the only way for agent i_A to be such that $g_B(i_A) = \lambda^1 + 1$ is if she is an active non-lurker that does not move at h_B^1, which means that she must tie in $\succ_B$ with some agent.

Thus, we have shown that i_A must tie with some agent k in $\succ_B$, i.e., $g_B(i_A) = g_B(k) = \lambda^1+1$ for some k. Since i_A is coded in step 1 of Γ_A, and $\tilde{\succ}_A^1$ is equal to an initial segment of $\succ_B$, we further have $g_A(i_A) = g_A(k) = g_B(i_A) = g_B(k) = \lambda^1 + 1$; in other words, agent i_A ties with agent k in both $\succ_A$ and $\succ_B$.

Since i_A ties with k in Γ_A, at h_A^1, we have $x_A \in C_{s'}^{\subsetneq}(h_A^1)$ for the other active non-lurker role s' at h_A^1. We have seen that $\sigma_B^{-1}(i_A) \neq s$. If $\sigma_B(s') = i_A$, then in Γ_B, i_A passed at some history $h' \subsetneq h_A^1$ at which she was offered to clinch x_A in Γ_B. By Lemma B.14, $Top(\succ_{i_A}, \bar{\mathcal{X}}^{\mathcal{L}}(h_A^1)) \succ_{i_A} x_A$, which is a contradiction. Since we know that i_A is coded in step 1 of Γ_B, the only other possibility is that in Γ_B, i_A is a lurker for some object z at h_B^1, which implies that $z \succ_{i_A} x_A$. It also means that the agent that moves at h_B^1 in Γ_B is clinching a lurked object (because if an unlurked object were clinched, then i_A would be assigned to z, a contradiction). This implies that h_B^1 is the terminating history, by Lemma B.7, and $\rho(h_B^1)$ is the terminator role. We cannot have $\rho(h_B^1) = s$, because then role s is the terminator role, and i_A is in the terminator role in Γ_A and would not clinch x_A first in Γ_A, a contradiction. Thus, $\rho(h_B^1) = s'$, and s' is the terminator role. Finally, notice that at h_A^1, role s is offered x_A and $x_A \in C_{s'}^{\subsetneq}(h_A^1)$, which contradicts Lemma B.12, part (iii).

The case $h_B^1 \subsetneq h_A^1$ follows an analogous argument; cf. footnote 67 for the needed adjustments. ■

Thus far, we have shown that if $\tilde{\succ}_A^1$ is equal to the initial part of the ordering $\succ_B$, then $h_A^1 = h_B^1$. We next show that the same roles are coded in step 1 of Γ_A and Γ_B, and further that $\sigma_A(r) = \sigma_B(r)$ for all such roles r.

Define $h^1 := h_A^1 = h_B^1$. In both games, the first clinching is taken by the agent in role $\rho(h^1)$, and the set of lurked objects and active lurker-roles are equivalent at the first clinching in both Γ_A and Γ_B. Letting $r_0 = \rho(h^1)$, write

$$\sigma_A(r_0) \to x_{a_1} \to \sigma_A(r_{a_1}) \to x_{a_2} \to \cdots \to \sigma_A(r_{a_M}) \to x_{a_{M+1}} \tag{A}$$

to represent the chain of clinching that is initiated in Γ_A by agent $\sigma_A(r_0)$ at h^1: agent $\sigma_A(r_0)$ clinches some (possibly lurked) object x_{a_1}, the agent $\sigma_A(r_{a_1})$ who was lurking x_{a_1} clinches lurked object x_{a_2}, etc., until eventually agent $\sigma_A(r_{a_M})$ ends the chain by being the first agent to clinch an unlurked object $x_{a_{M+1}}$. Similarly, for Γ_B, write

$$\sigma_B(r_0) \to x_{b_1} \to \sigma_B(r_{b_1}) \to x_{b_2} \to \cdots \to \sigma_B(r_{b_{M'}}) \to x_{b_{M'+1}}. \tag{B}$$

Note that the agents who begin the chains, $\sigma_A(r_0)$ and $\sigma_B(r_0)$ are not lurkers in their respective games, while all of the remaining agents are lurkers.[68] Also, not all of the agents ordered in step 1 need to appear in the corresponding chain; in particular, any lurker who receives their lurked object does not appear, nor does the other active non-lurker, if such an agent exists. If $M = M'$ and $\sigma_A(r_{a_m}) = \sigma_B(r_{b_m})$ for all $m = 0, \ldots, M$, then we say (A) and (B) are **equivalent chains**.

Claim B.2. Suppose that (A) and (B) are equivalent chains. Then, the same roles are coded in step 1 in Γ_A and Γ_B, and further, for all such roles, $\sigma_A(r) = \sigma_B(r)$.

Proof of Claim B.2. By construction of the coding algorithm, the set of roles coded during the coding step initiated at h_A^1 consists of (i) all lurker-roles at h_A^1, (ii) the non-lurker-role that moves at h_A^1, and potentially (iii) the active non-lurker role that does not move at h_A^1;

[68]If there are no lurkers at h^1, this is obvious; if there are lurkers, it follows from Lemma B.13.

label this role s. Since $h^1_A = h^1_B$, (i) and (ii) are the same in Γ_A and Γ_B. For (iii), role s is coded in Γ_A if and only if the first unlurked object in the chain, $x_{a_{M+1}}$, has been offered to role s to clinch prior to h^1_A. Since the chains are equivalent, this holds in Γ_A if and only if it holds in Γ_B, which establishes the first statement.

To see that $\sigma_A(r) = \sigma_B(r)$ for all roles that are coded in step 1 of Γ_A (and hence also step 1 of Γ_B), note that because (A) and (B) are equivalent, the statement holds for any role that appears in the chain. For roles that do not appear in the chain, if r' is a lurker role that is active at h^1, the corresponding lurked object x' is assigned to its lurker in both Γ_A and Γ_B, and so $\tilde{\succ}^1_A$ equivalent to the initial part of the ordering $\succ_B$ implies that $\sigma_A(r') = \sigma_B(r')$ for all such roles, by Lemma B.3.

It remains to consider the active non-lurker role s that does not move at h^1. Note that $M = M'$ and $\sigma_A(r_M) = \sigma_B(r_M)$ implies, by Lemma B.3, that $x_{a_{M+1}} = x_{b_{M'+1}}$; let $x_{M+1} := x_{a_{M+1}} = x_{b_{M'+1}}$, and recall that x_{M+1} is unlurked. If there is no such active role s, or if $x_{M+1} \notin C^{\subsetneq}_s(h^1)$, then this role is not coded in step 1, and we are done. Thus, assume that s exists, and that $x_{M+1} \in C^{\subsetneq}_s(h^1)$. In this case, the agent assigned to role s is ordered in step 1 in both Γ_A and Γ_B, and by construction, ties with agent $\sigma(r_M) := \sigma_A(r_M) = \sigma_B(r_M)$ in both $\succ_A$ and $\succ_B$. Once again, $\tilde{\succ}^1_A$ equivalent to the initial part of the ordering $\succ_B$ implies that $\sigma_A(s) = \sigma_B(s)$. ■

Claim B.3. Chains (A) and (B) are equivalent.

Proof of Claim B.3. We begin by showing that $\sigma_A(r_0) = \sigma_B(r_0)$. Towards a contradiction, assume that $\sigma_A(r_0) \neq \sigma_B(r_0)$, which implies also that that $x_{a_1} \neq x_{b_1}$ Lemma B.3. If $M = M' = 0$, then both chains have only one agent, $\sigma_A(r_0)$ and $\sigma_B(r_0)$, who immediately clinch unlurked objects. Define $\sigma_A(r_0) = i$ and $\sigma_B(r_0) = j$, where $i \neq j$, since they are clinching different objects in their respective games. Since $\tilde{\succ}^1_A$ is equal to the initial part of $\succ_B$, and both i and j clinch unlurked objects, this implies that i and j must tie under $\succ_A$ and $\succ_B$. Thus, by construction of the coding algorithm, there must be another non-lurker role $s \neq r_0$ that is active at h^1, and $\sigma_A(s) = j$ and $\sigma_B(s) = i$, and $x_{a_1}, x_{b_1} \in C^{\subsetneq}_s(h^1)$. Since i clinches an unlurked object x_{a_1} at h^1 in Γ_A, we have $x_{a_1} = Top(\succ_i, \bar{\mathcal{X}}^{\mathcal{L}}(h^1))$, by Corollary B.1. Now, consider game Γ_B. Since $\sigma_B(s) = i$ and $x_{a_1} \in C^{\subsetneq}_s(h^1)$, in game Γ_B, there is some history $h' \subsetneq h^1$ such that $x_{a_1} \in C_i(h')$. By Lemma B.14, we have $Top(\succ_i, \bar{\mathcal{X}}^{\mathcal{L}}(h^1)) \succ_i x_{a_1}$, which is a contradiction.

Now, consider the case that $M > 0$. This implies that a lurked object, x_{a_1}, is clinched at h^1 in Γ_A, which means that role r_0 is the terminator role by Lemma B.7. It also implies that that x_{a_1} is agent $\sigma_A(r_0)$'s favorite object (among all objects $\mathcal{X}$). So, in game Γ_B, agent $\sigma_A(r_0)$ must be lurking object x_{a_1}, i.e., she is in role r_{a_1} in Γ_B.[69] Agent $\sigma_A(r_{a_1})$—the agent

[69]Since x_{a_1} is lurked, it is only possible for "older" lurkers and the terminator. Agent $\sigma_A(r_0)$ cannot be

who lurks x_{a_1} in Γ_A—receives x_{a_2}, and so in Γ_B, must be the lurker for x_{a_2}.[70] Similarly, agent $\sigma_A(r_2)$ must lurk x_{a_3} in Γ_B, etc., until we reach agent $\sigma_A(r_M)$. By similar reasoning as footnote 70, we conclude that agent $\sigma_A(r_M)$ must be in role s in Γ_B. For shorthand, define $k := \sigma_A(r_M)$, and so $\sigma_B^{-1}(k) = s$.[71]

Finally, since $\sigma_B^{-1}(k) = s$ and k is ordered in step 1 of Γ_B (see footnote 71), there must be some other agent j such that $g_B(j) = \lambda(h^1) + 1$, and so $g_A(j) = g_A(k) = g_B(j) = g_B(k) = \lambda(h^1) + 1$. Since $g_A(j) = \lambda(h^1) + 1$, j must be clinching an unlurked object in Γ_A. Since the first person to clinch an unlurked object in Γ_A is k who clinches $x_{a_{M+1}}$, it must be that $\sigma_A^{-1}(j) = s$ and $x_{a_{M+1}} \in C_s^{\subseteq}(h^1)$. Finally, since $\sigma_B^{-1}(k) = s$, we have $x_{a_{M+1}} \in C_k^{\subseteq}(h^1)$ in Γ_B, and by Lemma B.14, $Top(\succ_k, \bar{\mathcal{X}}^{\mathcal{L}}(h^1)) \succ_k x_{a_{M+1}}$. However, since k chose to clinch $x_{a_{M+1}}$ in Γ_A and $x_{a_{M+1}}$ was unlurked, we have $Top(\succ_k, \bar{\mathcal{X}}^{\mathcal{L}}(h^1)) = x_{a_{M+1}}$, which is a contradiction.

The case where x_{b_1} is lurked is analogous, and the argument is omitted. We have thus shown that $\sigma_A(r_0) = \sigma_B(r_0)$.

If agent $\sigma_A(r_0)$ clinches an unlurked object, then the proof is complete. If not, then the above arguments can be repeated to show that $\sigma_A(r_{a_1}) = \sigma_B(r_{b_1})$, etc., until an unlurked object is reached. This completes the proof of Claim B.3. ■

Claims B.2 and B.3 imply the following:

Claim B.4. The same roles r' are coded in step 1 of the coding algorithm applied to games Γ_A and Γ_B, and for all these roles $\sigma_A(r') = \sigma_B(r')$.

To complete the proof we establish the claim of the lemma for steps $k > 1$ of the coding algorithm by an inductive argument. Suppose that the lemma obtains for steps $1, ..., k$ of the coding algorithm. After the chain of clinchings initiated at h_A^k (which is the same as h_B^k), we enter a subgame among agents and objects that were unmatched till step k. By the inductive assumption, these subgames begin at some history $\hat{h}^{k+1}$ that is the same under both σ_A and σ_B. As argued in Remark 2, these subgames continue to have the structure described in Section B.7.1. Let $h_A^{k+1} \supseteq \hat{h}^{k+1}$ be the first history at which a clinching action is taken following a (possibly empty) sequence of passes in the subgame of Γ_A starting at $\hat{h}^{k+1}$;

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an older lurker in Γ_B, because then she would have been offered x_{a_1}, and, by greedy strategies, would have clinched it. Nor can she be the the terminator, because $\sigma_B(r_0) \neq \sigma_A(r_0)$. Therefore, she must be in role r_{a_1} in Γ_B.

[70] This is because by definition of a lurker, agent $\sigma_A(r_{a_1})$ strictly prefers x_{a_1} to all younger lurked objects and all unlurked objects; thus, in Γ_B, she cannot be an older lurker, because she would have been offered x_{a_1}, and thus could not end up with something she strictly disprefers (recall that by Lemma B.3, all agents receive the same objects in both games). She cannot be the terminator, because then, since $h_A^1 = h_B^1$, and all objects are possible for the terminator, she would clinch x_{a_1}, which is again a contradiction to Lemma B.3.

[71] Note that k is coded in step 1 of the coding algorithm applied to Γ_A, and receives an unlurked object, so $g_A(k) = \lambda^1 + 1$, and therefore, $g_B(k) = \lambda^1 + 1$. Since at least $\lambda^1 + 1$ agents are coded in step 1 of Γ_B, this is only possible if agent k is also coded in step 1 of Γ_B, and thus she must be active at h^1, and so the only possibility is that $\sigma_B^{-1}(k) = s$.

define $h_B^{k+1} \supseteq h^{k+1}$ analogously. If now $\succ_A^{k+1}$ equals to an initial segment of $\succ_B$, then we can repeat the arguments developed for $k = 1$ above to show that $h_A^{k+1} = h_B^{k+1}$, the same roles are coded in step $k+1$ under σ_A and σ_B, and $\sigma_A(r') = \sigma_B(r')$ for all roles coded in step $k+1$. The inductive argument completes the proof. ■

Proof of Lemma B.5

For a (fixed) game form Γ, we let Γ_τ denote the specific game under role assignment σ_τ. Note that the set of objects that are lurked at any given history depends only on the game form, and is independent of the specific role assignment. We use the notation h_τ^* for the first history at which an object is clinched in Γ_τ; that is, $h_\tau^* = (h_\varnothing, a^*, \ldots, a^*)$, where a^* is the number of passes taken by the agents until the agent who moves at h_τ^* chooses to clinch at this history. The number of passes will depend on τ. For any agent j, we write x_j to denote the object that is ultimately received by agent j.

Note that it is without loss of generality to assume that for all games Γ_τ that we consider, at h_τ^*, the objects $x_{j_1}, \ldots, x_{j_P}$ are all lurked, in this order. To see this, note that if not, then, there is some game Γ_τ and $p' < P$ such that the last lurked object is $x_{j_{p'}}$. Consider the smallest such p'. Since $p' < P$, this means that the agents coded in step 1 of the coding algorithm are $j_1, \ldots, j_{p'}, j_{p'+1}$, and possibly $j_{p'+2}$, which can only occur if there is a tie at the end of the step.[72] Now, since all codings under consideration are exactly the same on the agents $j_1, \ldots, j_{p'}, j_{p'+1}, j_{p'+2}$, by Lemma B.4 we have that in all of the games we consider, all of these agents are in the same roles, and, at the end of the first coding step, we reach the same history in each game to begin the next coding step. Thus, we can disregard these agents, and begin the analysis for each game at this history. Repeating this argument, we continually eliminate all higher ranked agents until we reach a coding step at which all of the remaining agents ranked strictly head of k_1 are coded in the first step in all games.

Thus, for the entirety of this proof (including all sublemmas stated therein), we assume that the objects $x_{j_1}, \ldots, x_{j_P}$ are all lurked at h_τ^* for all games we consider. Note that this also implies that all agents $j_1, \ldots, j_P$ are ranked strictly, without ties, in all codings, and that there are at least $P+1$ agents coded in the first step of every game Γ_τ. We allow the case $P = 0$, in which case there are no agents j_p.

Since agent i ties in $\succ_1$, she receives an object that is unlurked at h_1^*, which means that $x_i = Top(\succ_i, \bar{\mathcal{X}}^{\mathcal{L}}(h_1^*))$. By the structure of the sequence, this also implies that for $n' \geq 2$, if $x_i \in \bar{\mathcal{X}}^{\mathcal{L}}(h_{n'}^*)$, then $x_i = Top(\succ_i, \bar{\mathcal{X}}^{\mathcal{L}}(h_{n'}^*))$ because each of the agents $i, j_1, ..., j_P$ receives the

[72]By construction of the coding algorithm, if there are p' lurked objects at the initiation of a coding step, then the number of agents coded in that step is either $p'+1$ or $p'+2$. Since all of the agents j_p are ranked strictly above the remaining agents, and $p' < P$, none of the agents i nor $k_{n'}$ can be coded in step 1 of the game.

same object under both σ_1 and $\sigma_{n'}$ (by Lemma B.3), and from the game Γ_1 we infer that i prefer the object received (x_i) to all objects except the objects assigned to $j_1, ..., j_P$, and in game $\Gamma_{n'}$ no other object belongs to $\bar{\mathcal{X}}^{\mathcal{L}}(h^*_{n'})$.

We begin with the following Lemmas B.15, B.16, and B.17, which show that, under certain conditions, either condition (I) or (II) in the statement of the lemma will hold. Then, we apply these lemmas to show that all cases are covered, which will prove the result. The proofs of these lemmas can be found following the conclusion of this proof.

The first of these lemmas shows that if there is a sequence Σ such that $n \geq 2$ and such that the lurked objects on the initial passing path of the game form are (in order) $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n-1}}$, then i must tie in $\succ_{n+1}$.

Lemma B.15. *Assume that there exists a sequence of role assignment functions Σ as defined in the statement of Lemma B.5, and such that $n \geq 2$. Further, assume that along the initial passing path of the game form, the first lurked objects are (in order) $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n-1}}$.*[73] *Then, at h^*_{n+1} in Γ_{n+1}, there is an agent $\ell \neq j_1, \ldots, j_P, k_1, \ldots, k_n, i$ such that ℓ is an active non-lurker at h^*_{n+1} that does not move at h^*_{n+1} and $x_i \in C^{\subsetneq}_{\ell}(h^*_{n+1})$. Further, i must tie with some other agent in $\succ_{n+1}$, and we label this agent k_{n+1}.*

Remark 5. A supposition in Lemma B.15 (and in Lemma B.18, below) is that the first lurked objects of the game form are $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n-1}}$, in this order, where $n \geq 2$. A sufficient condition for this to hold is the following: there is a game Γ_A such that $j_1 \cdots j_P \succ_A k_1 \succ_A \cdots \succ_A k_{n-1} \succ_A \{i, k_n\} \succ_A \cdots$ and i is coded in the initial step of the coding algorithm.

To see this, assume not, and let n' be the smallest n such that $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n'-1}}$ become lurked, but $x_{k_{n'}}$ is not the next lurked object. This means that at h^*_A (the history of the first clinching in Γ_A), there are at most $\lambda^*_A = P + n' - 1$ lurked objects. Consider agent $k_{n'}$. By construction, $n' < n$, and so $k_{n'}$ does not tie in $\succ_n$. Thus, in the coding step in Γ_A that begins at h^*_A, agent $k_{n'}$ must be the first agent to clinch an unlurked object. This ends the coding step at $k_{n'}$, without a tie, which contradicts that i is coded in this step in game Γ_A. ■

The second of these lemmas shows that if there is a sequence Σ, plus an additional role assignment function σ_0 in which all $j_1, \ldots, j_P$ are ranked strictly above i, who is ranked strictly above k_1, who is ranked strictly above all other remaining agents, then i must tie in $\succ_{n+1}$.

Lemma B.16. *Assume that there exists a sequence of role assignment functions Σ as defined in the statement of Lemma B.5. If there exists another role assignment function σ_0 with a*

[73] We allow for the possibility that $P = 0$, but whether $P = 0$ or $P > 0$, the assumption that $n \geq 2$ implies that along the initial passing path of the game form, at least x_{k_1} becomes lurked.

corresponding coding,

$$j_1 \cdots j_P \succ_0 i \succ_0 k_1 \succ_0 \cdots,$$

then in $\succ_{n+1}$ of Σ, i must tie with some agent k_{n+1}.

Remark 6 (Symmetry). Lemmas B.15 and B.16 were stated for sequence Σ, and concluded that i must tie in $\succ_{n+1}$. There are also symmetric versions of these lemmas that apply to sequence Σ' and conclude that k_1 must tie in $\succ_{m+1}$ that have the exact same proof.

The last of these lemmas deals with the case that neither x_i nor x_{k_1} are the $(P+1)^{th}$ lurked object on the initial passing path, nor does there exist a σ_0 as in Lemma B.16.

Lemma B.17. *Assume that there exist two sequences of role assignment functions Σ and Σ' as defined in the statement of Lemma B.5 such that $n, m \geq 2$. Further, assume that along the initial passing path of the game form, the objects $x_{j_1}, \ldots, x_{j_P}$ all become lurked, in this order, but neither x_i nor x_{k_1} is the $(P+1)^{th}$ lurked object.Then, one of the following is true:*

1. *In $\succ_{n+1}$, agent i must tie with some agent k_{n+1}.*
2. *In $\succ'_{m+1}$, agent k_1 must tie with some agent k'_{m+1}.*

With these lemmas in hand, we can complete the proof of Lemma B.5 as follows:

- If there exists σ_0 such that $j_1 \cdots j_P \succ_0 i \succ_0 k_1 \succ_0 \cdots$, then we apply Lemma B.16 to Σ conclude that (I) holds.
- If there exists σ'_0 such that $j_1 \cdots j_P \succ'_0 k_1 \succ'_0 i \succ'_0 \cdots$, then we apply the symmetric version of Lemma B.16 with k_1 and i swapped to Σ' to conclude that (II) holds.
- If neither of the above two cases hold (i.e., there do not exist σ_0 nor σ'_0):[74]
  - If x_{k_1} is the $(P+1)^{th}$ lurked object along the initial passing path, then we apply Lemma B.15 to Σ to conclude that (I) holds.
  - If x_i is the $(P+1)^{th}$ lurked object along the initial passing path, then we apply the symmetric version of Lemma B.15 with k_1 and i swapped to Σ' to conclude that (II) holds.
  - If neither x_{k_1} nor x_i is the $(P+1)^{th}$ lurked object along the initial passing path, then we apply Lemma B.17 to conclude that either (I) or (II) must hold. ■

[74]Notice that the assumption that there is no σ_0 or σ'_0 imply that $n, m \geq 2$, which is needed to apply Lemma B.15 below.

Proofs of Lemmas B.15, B.16, and B.17

Proof of Lemma B.15. We start with the following lemma.

Lemma B.18. *Assume that there exists a sequence of role assignment functions Σ as defined in the statement of Lemma B.5, and such that $n \geq 2$. Further, assume that along the initial passing path of the game form, the first lurked objects are (in order) $x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{n-1}}$.*[75] *Then:*

(a) For all $n' = 1, \dots, n-1$, the agent that moves at $h^*_{n'}$ in $\Gamma_{n'}$ is agent i, and at $h^*_{n'}$, the number of lurked objects is $P + n' - 1$.

(b) $h^*_1 \subsetneq h^*_2 \subsetneq \cdots \subsetneq h^*_{n-1} \subsetneq h^*_n$.

(c) For all $n' = 1, \dots, n$, the number of lurked objects at $h^*_{n'}$ is $P + n' - 1$.

(d) For all $n' = 1, \dots, n-1$, $p = 1, \dots, P$, and $n'' = 1, \dots, n'$, in $\Gamma_{n'}$, agent j_p is in the role that lurks x_{j_p} and agent $k_{n''}$ is in the role that lurks $x_{k_{n''}}$.

(e) $h^*_{n-1} \subsetneq h^*_{n+1}$ and the number of lurked objects at h^*_{n+1} is at least $P + n - 1$.

Proof of Lemma B.18. **Part (a).** Let $\lambda^*_{n'}$ be the number of lurked objects at history $h^*_{n'}$. Notice that since $\succ_{n'}$ has a tie in the $(P+n')^{th}$ place, we have $\lambda^*_{n'} \leq P + n' - 1$ for all $n' = 1, \dots, n$. Towards a contradiction, assume there was a game $\Gamma_{n'}$ for which i does not move at $h^*_{n'}$.. Since $\lambda^*_{n'} \leq P + n' - 1$, the structure of $\succ_{n'}$ implies that the lurked objects are $\{x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{\lambda^*_{n'}-P}}\}$,[76] and the agents coded in step 1 of $\Gamma_{n'}$ are $\{j_1, \dots, j_P, k_1, \dots, k_{\lambda^*_{n'}-P+1}\}$ (if $\lambda^*_{n'} < P + n' - 1$) or $\{j_1, \dots, j_P, k_1, \dots, k_{\lambda^*_{n'}-P+1}, i\}$ (if $\lambda^*_{n'} = P + n' - 1$). and the set of lurked objects is $\{x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{\lambda^*_{n'}-P}}\}$. Now, notice that it cannot be a lurked object that is clinched at $h^*_{n'}$. Indeed, if this were true, then $h^*_{n'}$ is the terminating history, which implies that $x_{k_{\lambda^*_{n'}-P}}$ is the last lurked object on the initial passing path of the game (Lemma B.7). But, this contradicts the assumption that $x_{k_{\lambda^*_{n'}-P+1}}$ is the next lurked object on the initial passing path, where notice that such an object exists because $\lambda^*_{n'} - P + 1 \leq n' \leq n-1$. Thus, it must be an unlurked object that is clinched at $h^*_{n'}$. In particular, by the structure of $\succ_{n'}$, the only possibilities are that agent $k_{\lambda^*_n-P+1}$ clinches object $x_{k_{\lambda^*_n-P+1}}$, or agent i clinches x_i, where the latter case is only possible if $\lambda^*_{n'} = P + n' + 1$. However, if agent $k_{\lambda^*_n-P+1}$ clinches object $x_{k_{\lambda^*_n-P+1}}$, then object $x_{k_{\lambda^*_n-P+1}}$ has been offered to an active non-lurker at $h^*_{n'}$, and so $x_{k_{\lambda^*_n-P+1}}$ cannot be the next lurked object along the initial passing path (Remark 4), a contradiction. Therefore, it must be that $\lambda^*_{n'} = P + n' - 1$, and agent i is the agent that moves at $h^*_{n'}$.

[75] We allow for the possibility that $P = 0$, but whether $P = 0$ or $P > 0$, the assumption that $n \geq 2$ implies that along the initial passing path of the game form, x_{k_1} is the $(P+1)^{th}$ lurked object. .

[76] This is implicitly assuming that $\lambda^*_{n'} > P$. An analogous argument works for the case that $\lambda^*_{n'} \leq P$, but, for brevity, this argument is omitted.

Parts (b). As shown in part (a), for $n' = 1, \dots, n-1$, there are $\lambda^*_{n'} = P + n' - 1$ lurked objects at $h^*_{n'}$, which immediately implies that $h^*_1 \subsetneq h^*_2 \subsetneq \cdots \subsetneq h^*_{n-2} \subsetneq h^*_{n-1}$ (because the number of lurked objects only grows as we go down the initial passing path).

It remains to show that $h^*_{n-1} \subsetneq h^*_n$. By way of contradiction, assume that $h^*_n \subseteq h^*_{n-1}$. Then, $\lambda^*_n \leq \lambda^*_{n-1} = P+n-2$, and the lurked objects at h^*_n are $\{x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{\lambda^*_n}}\}$. If a lurked object is clinched at h^*_n, then h^*_n is the terminating history, and there is no passing action at h^*_n (Lemma B.7). However, this contradicts that $x_{k_{\lambda^*_n+1}}$ is the next lurked object on the initial passing path. So, it must be an unlurked object that is clinched. By the structure of $\succ_n$, it must be $k_{\lambda^*_n+1}$ that clinches $x_{k_{\lambda^*_n+1}}$. But then, $x_{k_{\lambda^*_n+1}}$ has been offered to active nonlurker at h^*_n, and so $x_{k_{\lambda^*_n+1}}$ cannot be the next lurked object along the initial passing path (Remark 4), which is a contradiction. Therefore, $h^*_{n-1} \subsetneq h^*_n$.

Part (c). Part (a) shows this for $n' \leq n-1$. So, we must show $\lambda^*_n = P+n-1$. Notice that $h^*_{n-1} \subsetneq h^*_n$ implies that $\lambda^*_n \geq \lambda^*_{n-1} = P+n-2$, while the structure of $\succ_n$ (in particular, the tie between agent i and k_n), implies that $\lambda^*_n \leq P+n-1$. Thus, we need to show $\lambda^*_n \neq P+n-2$. Assume that $\lambda^*_n = P+n-2$. Then, the lurked objects are $x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{n-2}}$, and the agents coded in step 1 are $j_1, \dots, j_P, k_1, \dots, k_{n-2}, k_{n-1}$. If a lurked object is clinched at h^*_n, then this is the terminating history, which contradicts that $x_{k_{n-1}}$ is the next lurked object along the initial passing path (Lemma B.7). If an unlurked object is clinched, then it must be k_{n-1} clinching $x_{k_{n-1}}$, but since this is offered to an active non-lurker, $x_{k_{n-1}}$ cannot be the next lurked object along the initial passing path (4), a contradiction. Therefore, $\lambda^*_n = P+n-1$.

Part (d). By part (a), agent i moves at $h^*_{n'}$ in $\Gamma_{n'}$, and, since i ties in $\succ'_n$, object x_i is unlurked. Therefore, all lurked objects are immediately assigned to their lurkers, which delivers the result.

Part (e). To show $h^*_{n-1} \subsetneq h^*_{n+1}$, assume not. Then, $h^*_{n+1} \subseteq h^*_{n-1}$, and $\lambda^*_{n+1} = P+\bar{n}-1$ for some $\bar{n} \leq n-1$. So, the lurked objects at h^*_{n+1} are $x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{\bar{n}-1}}$, and the agents coded in step 1 are $j_1, \dots, j_P, k_1, \dots, k_{\bar{n}}$. Since $\bar{n} \leq n-1$, we know that $x_{k_{\bar{n}}}$ must be the next lurked object on the initial passing path. An argument analogous to those given above delivers a contradiction.

To show $\lambda^*_{n+1} \geq P+n-1$, note that $h^*_{n-1} \subsetneq h^*_{n+1}$ implies $\lambda^*_{n+1} \geq \lambda^*_{n-1} = P+n-2$. Thus, we must just show that $\lambda^*_{n+1} \neq P+n-2$. So, assume this was the case. Then, the lurked objects are $x_{j_1}, \dots, x_{j_P}, x_{k_1}, \dots, x_{k_{n-2}}$, and the agents coded in step 1 are $j_1, \dots, j_P, k_1, \dots, k_{n-2}, k_{n-1}$. If a lurked object is clinched at h^*_{n+1}, then this is the terminating history, which contradicts that $x_{k_{n-1}}$ is the next lurked object along the initial passing path (Lemma B.7). If an unlurked object is clinched, then it must be k_{n-1} clinching $x_{k_{n-1}}$, but since this is offered to an active non-lurker, $x_{k_{n-1}}$ cannot be the next lurked object along the initial passing path (Remark 4), a contradiction. Therefore, $\lambda^*_{n+1} \geq P+n-1$.

This completes the proof of Lemma B.18. ■

Continuing with the proof of Lemma B.15, we first show the first statement, that at h^*_{n+1} there is an agent $\ell \neq j_1, \ldots, j_P, k_1, \ldots, k_n, i$ such that ℓ is an active non-lurker at h^*_{n+1} that does not move at h^*_{n+1}, and $x_i \in C^{\subsetneq}_\ell(h^*_{n+1})$. By Lemma B.18, we have (i) $h^*_{n-1} \subsetneq h^*_n, h^*_{n+1}$ (ii) $\lambda^*_n = P + n - 1$ and (iii) $\lambda^*_{n+1} \geq P + n - 1$. In particular, the lurked objects at h^*_n are $\{x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n-1}}\}$. Since there is a tie in $\succ_n$, there are two active non-lurker roles at h^*_n, and both of these roles have been offered to clinch x_i at h^*_n. Let s be the role that moves at h^*_n, and s' be the other active non-lurker that does not move at h^*_n.

Case 1: x_{k_n} is the next lurked object along the initial passing path of the game form. Since x_{k_n} is the next lurked object along the initial passing path, it must be i that moves at h^*_n and clinches x_i, i.e., $\sigma_n(s) = i$. [77] Further, we have $h^*_n \subsetneq h^*_{n+1}$. To see this, note that if not, then $h^*_{n+1} \subseteq h^*_n$, and x_{k_n} is not lurked at h^*_{n+1}. Thus, it cannot be a lurked object that is clinched at h^*_{n+1}, because this would imply that h^*_{n+1} is the terminating history (Lemma B.7), which contradicts that x_{k_n} becomes lurked along the initial passing path. So, the object clinched at h^*_{n+1} must be unlurked, and so the set of lurked objects is $\{x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{\bar{n}-1}}\}$, where $x_{k_{\bar{n}}}$ is the unlurked object that is clinched, and $\bar{n} \leq n$, which follows because $h^*_{n+1} \subseteq h^*_n$. But then, $x_{k_{\bar{n}}}$ is offered to an active non-lurker at h^*_{n+1}, which contradicts that it is the next lurked object along the initial passing path (Remark 4). Therefore, $h^*_n \subsetneq h^*_{n+1}$.

Since x_{k_n} is the next lurked object along the initial passing path, we must have x_{k_n} becoming lurked at some h' such that $h^*_n \subsetneq h' \subseteq h^*_{n+1}$. But, notice that there is still some role r such that, at h', r is an active non-lurker, and $x_i \in C^{\subsetneq}_r(h')$. Thus, x_i cannot be the next lurked object along the initial passing path. Therefore, for i to be ranked immediately after k_n in $\succ_{n+1}$, she must clinch x_i while it is unlurked, either at h^*_{n+1}, or in the resulting step 1 assignment chain of the coding algorithm.

We next claim that in Γ_{n+1}, $\sigma^{-1}_{n+1}(i) \neq s, s'$. To see this, first note that if $\sigma^{-1}_{n+1}(i) = s$, then i has the same role in Γ_n and Γ_{n+1}, and thus would once again clinch at h^*_n in Γ_{n+1}, which contradicts $h^*_n \subsetneq h^*_{n+1}$. Therefore, $\sigma^{-1}_{n+1}(i) \neq s$. Next, assume that $\sigma_{n+1}(s') = i$. Notice that role s' cannot be the terminator role, by Lemma B.12(iii) and the fact that $x_i \in C_s(h^*_n)$ and $x_i \in C^{\subsetneq}_{s'}(h^*_n)$. Thus, only objects that are unlurked at h^*_n are possible for role s', and so if $\sigma_{n+1}(s') = i$, since x_i is i's top unlurked object, she would clinch it at some history $h' \subsetneq h^*_n \subseteq h^*_{n+1}$, which is a contradiction. Therefore, $\sigma^{-1}_{n+1}(i) \neq s, s'$.

We showed above that s' is not the terminator role. If s is the terminator role, then,

[77] Agent k_n cannot move at h^*_n, because then x_{k_n} would have been offered to an active non-lurker at h^*_n, which contradicts that x_{k_n} is the next lurked object along the initial passing path. Nor can it be any $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, x_{k_{n-1}}$, because then they would be clinching a lurked object, and so h^*_n is the terminating history, which again contradicts that x_{k_n} is the next lurked object along the initial passing path.

when i clinches at h_n^*, we conclude that x_i is her top possible object among all of those that are available. This implies that i cannot be in a role that is a lurker at h_n^*. So, we have shown that in Γ_{n+1}, agent i is not a lurker at h_n^*, nor is she is role s or s'. Thus, i is not active at h_n^* in Γ_{n+1}, and so there must be some agent $\ell \neq j_1, \dots, j_P, k_1, \dots, k_n$ such that $\sigma_{n+1}^{-1}(\ell) = s$ or s'. But then, since i is unlurked at h_{n+1}^*, we have that $x_i \in C_\ell^{\subsetneq}(h_{n+1}^*)$, as desired.

If s is not the terminator role, we once again claim that i cannot be in a role that is a lurker at h_n^*. Indeed, if this were true, then some agent j who is receiving a lurked object is not a lurker at h_n^*. Therefore, this agent must be in the terminator role, and clinch at h_{n+1}^*. Since the terminator role is not s or s', it is not yet active at h_n^*, and so j is not active at h_n^* in Γ_{n+1}. Therefore, there must be some $\ell \neq j_1, \dots, j_P, k_1, \dots, k_n$ such that $\sigma^{-1}(\ell) = s$ or s', and that is still active when j clinches at h_{n+1}^*, which implies that $x_i \in C_\ell^{\subsetneq}(h_{n+1}^*)$, as desired.

Case 2: x_{k_n} is not the next lurked object along the initial passing path. By Lemma B.18, at h_n^*, there are $P + n - 1$ lurked objects. This implies that both i and k_n are coded in step 1 of the coding algorithm for Γ_n, and thus that the first unlurked object that is clinched is either x_i or x_{k_n}.[78] This gives rise to two subcases.

Case 2.1: x_{k_n} is the first unlurked object that is clinched in the coding algorithm in Γ_n. In this case, $\sigma_n(s') = i$, and there is some history $\tilde{h} \subsetneq h_n^*$ such that $x_{k_n} \in C_i(h_n^*)$.

Claim B.5. The following are true: (a) $h_{n-1}^* \subsetneq h_{n+1}^* \subsetneq h_n^*$ and (b) agent k_n clinches x_{k_n} at h_{n+1}^* in Γ_{n+1}, and x_{k_n} is unlurked at this history.

Proof of Claim B.5. **Part (a).** First notice that $h_{n-1}^* \subsetneq h_{n+1}^*$ follows from Lemma B.18. So, we must show that $h_{n+1}^* \subsetneq h_n^*$. Towards a contradiction assume that $h_n^* \subseteq h_{n+1}^*$. Since $h_{n-1}^* \subsetneq h_n^*$, we have $h_{n-1}^* \subsetneq h_n^* \subseteq h_{n+1}^*$. Lemma B.18 also implies that $\lambda_n^* = P + n - 1$. Since i does not move at h_n^* in Γ_n, it must be some $j_1, \dots, j_P, k_1, \dots, k_n$ that does. If a lurked object is clinched at h_n^*, then h_n^* is the terminating history. It also implies that agent k_n is a lurker for some lurked object, and therefore in step 1 of the coding algorithm, some agent takes the object k_n lurks, and he ends the step by clinching x_{k_n}, which is unlurked. This means that x_{k_n} is his favorite object that is unlurked at h_n^*. Now, consider Γ_{n+1}, and note that $h_n^* \subseteq h_{n+1}^*$ and h_n^* being the terminating history implies that $h_n^* = h_{n+1}^*$. In Γ_{n+1}, the set of lurked objects is the same as in Γ_n, so x_{k_n} is again the first unlurked object that is clinched in step 1 of the coding algorithm. But, since $h_n^* = h_{n+1}^*$, there is again an agent in role s' who is an active non-lurker at h_{n+1}^*, and so this agent would once again tie with k_n in $\succ_{n+1}$, a contradiction. Therefore, it must be that k_n is the agent that moves at h_n^* in Γ_n, which means that x_{k_n} has been offered to both active non-lurker roles at h_n^*. Since we assumed that $h_n^* \subseteq h_{n+1}^*$, it is impossible for k_n to be ranked n^{th} strictly, without ties, in $\succ_{n+1}$,[79] which

[78] Note that this does not necessarily mean that the object clinched at h_n^* is x_i or x_{k_n}.

[79] Note that x_{k_n} cannot be the next lurked object, so, there must be no newly lurked objects at h_{n+1}^*

is a contradiction. Thus, we have shown that $h^*_{n+1} \subsetneq h^*_n$, which is part (a).

Part (b). Part (a) plus Lemma B.18 implies that $\lambda^*_{n+1} = P + n - 1$. Additionally, $h^*_{n+1} \subsetneq h^*_n$ means that h^*_{n+1} is not the terminating history, so it must be an unlurked object that is clinched there. Thus, since k_n is ordered $(P+n)^{th}$ without ties, it must be that k_n clinches x_{k_n} at h^*_{n+1}, and x_{k_n} is unlurked. ∎

By Lemma B.18, the agent that moves at h^*_{n-1} must be agent i, and therefore, at h^*_{n-1}, there are two active non-lurker roles that both have been offered x_i. Let the role that moves at h^*_{n-1} be denoted r, and the other active non-lurker at h^*_{n-1} be denoted r'. Thus, by definition, $\sigma_{n-1}(r) = i$.

We claim that in Γ_{n+1}, i cannot be active at h^*_{n-1}. At h^*_{n-1}, there are $P + n - 2$ active lurker roles, and two active non-lurker roles, r and r'. First, it is clear that $\sigma_{n+1}(r) \neq i$, because otherwise i is in the same role in Γ_{n-1} and Γ_{n+1}, and so would clinch at h^*_{n-1} in Γ_{n+1}, which contradicts $h^*_{n-1} \subsetneq h^*_{n+1}$ from Claim B.5. Second, assume that in Γ_{n+1}, agent i is in a lurker role for a lurked object at h^*_{n-1}, say y. By part (b) of Claim B.5, agent k_n clinches an unlurked object at h^*_{n+1}, and so all lurkers are immediately assigned to their lurked objects, which means that i would receive y which is a contradiction.

It remains to rule out that $\sigma^{-1}_{n+1}(i) = r'$. By construction, $x_i \in C_r(h^*_{n-1})$, where $x_i \in C_{r'}(\tilde{h})$ for some $\tilde{h} \subsetneq h^*_{n-1}$. This implies that role r' cannot be the the terminator role, by Lemma B.12(iii), and the fact that $x_i \in C_r(h^*_{n-1})$. Since role r' is not the terminator role, only unlukred objects are possible for role r', by Lemma B.12(iv). As x_i is agent i's most preferred unlurked object, by greedy strategies, she would clinch at $\tilde{h}$, which is a contradiction. Therefore, i is not active at h^*_{n-1} in Γ_{n+1}.

We also claim that i is not active at h^*_{n-1} in Γ_n, either. The arguments are the same as above for Γ_{n+1}, except for the case in which i lurks some lurked object at h^*_{n-1}. This is ruled out by the fact that $\sigma_n(s') = i$, and s' is a non-lurker at h^*_{n-1}.

Next, we claim that $\sigma_{n+1}(s) \neq i$. To see this, recall that $\sigma_n(s') = i$, and, as we showed, i is not active at h^*_{n-1} in Γ_n or Γ_{n+1}. This means that $s' \neq r, r'$, or in other words, s' is a role that becomes active after h^*_{n-1}. Thus, we must have $s = r$ or r', and so role s is active at h^*_{n-1}, which implies that $\sigma_{n+1}(s) \neq i$.

Next, we claim that $\sigma_{n+1}(s') = k_n$. Indeed, since $h^*_{n-1} \subsetneq h^*_{n+1} \subsetneq h^*_n$ and k_n moves at h^*_{n+1}, k_n must be in role either s or s'. If $\sigma_{n+1}(s) = k_n$, then, since she does not tie in $\succ_{n+1}$, she must clinch x_{k_n} at some history h' such that $h^*_{n-1} \subsetneq h' \subsetneq \hat{h}$, where $\hat{h}$ is the history at which role s' is offered to clinch x_{k_n}. This implies that $\sigma_n(s) \neq k_n$, or else in Γ_n, she would also

(Remark 4). If k_n clinches at h^*_{n+1}, she would tie with the other active non-lurker. If some other agent clinches at h^*_{n+1}, then either this agent is ranked strictly ahead of k_n, or she ties with k_n, which again is a contradiction.

clinch at h'. So, in Γ_n, $\sigma_n(s) = k_{n'}$ for some $n' < n$, and k_n is in the lurker role for some object $x_{k_{\bar{n}}}$. The former implies that h_n^* is the terminating history, while the latter implies that k_n strictly prefers $x_{k_{\bar{n}}}$ to x_{k_n}. But then, since $\sigma_{n+1}(s) = k_n$, agent k_n is in the terminator role in Γ_{n+1}, and thus $x_{k_{\bar{n}}}$ is a possible outcome for her, she would not choose to clinch x_{k_n} first at h_{n+1}^*, a contradiction. Therefore, $\sigma_{n+1}(s') = k_n$.

Concluding the argument for Case 2.1, because k_n clinches an unlurked object at h_{n+1}^* in Γ_{n+1}, all agents $j_1, \ldots, j_P, k_1, \ldots, k_{n-1}$ must be in the lurker role for their respective objects. Therefore, none of them are in role s. As just shown, $\sigma_{n+1}(s) \neq k_n$ or i, either. All of this means that $\sigma_{n+1}(s) = \ell$ for some $\ell \neq j_1, \ldots, j_P, k_1, \ldots, k_n, i$, and in Γ_{n+1}, we have $x_i \in C_\ell^{\subsetneq}(h_{n+1}^*)$, as desired.

Case 2.2: x_i is the first unlurked object that is clinched in step 1 of the coding algorithm in Γ_n. In this case, we have that $\sigma_n(s') = k_n$, and $x_i \in C_{s'}(\tilde{h})$ for some $\tilde{h} \subsetneq h_n^*$. There are two further subcases:

Case 2.2.1: $\sigma_n(s) \neq i$. In this subcase, $\sigma_n(s)$ is one of $j_1, \ldots, j_P, k_1, \ldots, k_{n-1}$, and is clinching a lurked object at h_n^*. This implies that h_n^* is the terminating history, and s is the terminator role, which also means that we have $h_{n-1}^* \subsetneq h_{n+1}^* \subseteq h_n^*$. This combined with Lemma B.18 implies that there are $P + n - 1$ lurkers at h_n^*, and the structure of $\succ_{n+1}$ means that x_{k_n} is the first unlurked object clinched in step 1 of Γ_{n+1}, and, at h_{n+1}^*, x_{k_n} has not been offered to the active non-lurker who does not move at h_{n+1}^*.

We also claim that role s cannot be active at history h_{n-1}^*. Indeed, since i clinches at h_{n-1}^* in Γ_{n-1} and ties, we know that there are two active non-lurker roles, say r and r', and they both have been offered x_i. If role s were one of these roles, then, since s is the terminator role, Lemma B.12 implies that $x_i \notin C_{s'}(\tilde{h})$, which is a contradiction. This implies that role s is a role that becomes active after h_{n-1}^*. Since there is only one new lurker between h_{n-1}^* and h_{n+1}^*, this further implies that role s' must have been active at h_{n-1}^*, and $x_i \in C_{s'}^{\subsetneq}(h_{n-1}^*)$.

We next claim that $\sigma_{n+1}(s') \neq i$. To see why this is true, notice that s' is not the terminator role (because that is role s). Thus, only unlurked objects are possible for role s' (Lemma B.12(iv)), and, since we know that x_i is i's favorite unlurked object, if she were in role s', she would clinch at $\tilde{h} \subsetneq h_{n+1}^*$, a contradiction. Therefore, $\sigma_{n+1}(s') \neq i$.

Now, if it is one of the $j_1, \ldots, j_P, k_1, \ldots, k_{n-1}$ that moves at h_{n+1}^*, then h_{n+1}^* is the terminating history, and so $h_{n+1}^* = h_n^*$. This implies that x_i has been offered to the agent in role $\sigma_{n+1}(s')$ (who is not coded in step 1). As we just showed that $\sigma_{n+1}(s') \neq i$, we have $\sigma_{n+1}(s') = \ell$ for some $\ell \neq j_1, \ldots, j_P, k_1, \ldots, k_n$, and $x_i \in C_\ell(h_{n+1}^*)$ in Γ_{n+1}, as desired.

Concluding subcase 2.2.1, assume that it is k_n that moves at h_{n+1}^* in Γ_{n+1}. This means that k_n is in role s or s' in Γ_{n+1}. Note that we cannot have $\sigma_{n+1}(s') = k_n$, because if this were true, then k_n has the same role in Γ_n as in Γ_{n+1}, and would pass at all histories in Γ_{n+1}, just as she did in Γ_n. Therefore, $\sigma_{n+1}(s) = k_n$. Again, as we know that $\sigma_{n+1}(s') \neq i$, we have that $\sigma_{n+1}(s') = \ell$ for some $\ell \neq j_1, \dots, j_P, k_1, \dots, k_n$, and $x_i \in C_\ell(h^*_{n+1})$ in Γ_{n+1}, as desired.

Case 2.2.2: $\sigma_n(s) = i$. In this subcase, i clinches x_i at h^*_n. If $h^*_n \subseteq h^*_{n+1}$, then notice that at h^*_n in Γ_{n+1}, there are two active non-lurker roles, s and s', that have been offered x_i. We claim that $\sigma^{-1}_{n+1}(i) \neq s, s'$. First, it is clear that $\sigma_{n+1}(s) \neq i$, as otherwise, i would clinch at h^*_n in Γ_{n+1}, just as she did in Γ_n. To see that $\sigma_{n+1}(s') \neq i$, notice that role s' cannot be the terminator role, by Lemma B.12 and the fact that $x_i \in C_s(h^*_n)$ and $x_i \in C^{\subsetneq}_{s'}(h^*_n)$. Thus, only unlurked objects are possible for role s', and so if $\sigma_{n+1}(s') = i$, since x_i is i's top unlurked object, she would clinch it at some history $h' \subsetneq h^*_n \subseteq h^*_{n+1}$, which is a contradiction. Therefore, $\sigma^{-1}_{n+1}(i) \neq s, s'$, and so there must be some $\ell \neq j_1, \dots, j_P, k_1, \dots, k_n$ such that $x_i \in C_\ell(h^*_{n+1})$, as desired.

It remains to consider $h^*_{n+1} \subsetneq h^*_n$. Then, there are $P+n-1$ lurkers at h^*_{n+1}, and, since h^*_{n+1} is not the terminating history, it must be agent k_n that moves at h^*_{n+1}. This also implies that k_n is in role s or s'. If $\sigma_{n+1}(s') = k_n$, then k_n is in the same role in Γ_{n+1} as in Γ_n, and would pass at h^*_{n+1} in Γ_{n+1} as she did in Γ_n,which is a contradiction. Therefore, $\sigma_{n+1}(s) = k_n$.

We claim that role s is not an active at history h^*_{n-1}. Indeed, notice that because i clinches at h^*_{n-1} in Γ_{n-1}, we have that $x_i \in C^{\subsetneq}_s(h^*_{n-1})$. This implies that role s is not the terminator role, which follows by Lemma B.12 and the fact that $x_i \in C_{s'}(h')$ for some $h' \supsetneq h^*_{n-1}$. This implies that only unlurked objects are possible for role s when she is called to play. Thus, if role s were an active non-lurker at history h^*_{n-1}, then, in Γ_n, when $\sigma_n(s) = i$, agent i is offered to clinch x_i at some $h' \subseteq h^*_{n-1}$. Since we know that only unlurked objects are possible, and x_i is i's top unlurked object, she would clinch at $h' \subsetneq h^*_n$ in Γ_n, which is a contradiction. Since role s is not active at h^*_{n-1}, there are two roles that are not s that are active non-lurkers at h^*_{n-1} and such that both have been offered to clinch x_i. At h^*_{n+1} in Γ_{n+1}, at least one of these roles must still be active and not assigned to any agent $j_1, \dots, j_P, k_1, \dots, k_n, i$. Thus, there must be some $\ell \neq j_1, \dots, j_P, k_1, \dots, k_n, i$ such that ℓ is an active non-lurker that does not move at h^*_{n+1} and $x_i \in C^{\subsetneq}_\ell(h^*_{n+1})$, as desired. This concludes the analysis of subcase 2.2.2, and hence of case 2.2.

The above shows that in all cases, there is some $\ell \neq j_1, \dots, j_P, k_1, \dots, k_n, i$ such that ℓ is an active non-lurker that does not move at h^*_{n+1} and $x_i \in C^{\subsetneq}_\ell(h^*_{n+1})$ in game Γ_{n+1}. Recall that, by Lemma B.18, $\lambda^*_{n+1} \geq P+n-1$. If $\lambda^*_{n+1} > P+n-1$, then there are at least $P+n$ lurked objects at h^*_{n+1}, and the only way i can be ranked in the $(P+n+1)^{th}$ position in $\succ_{n+1}$ is if she is coded in the first step. Since there is some agent $\ell \neq i$ such that $x_i \in C^{\subsetneq}_\ell(h^*_{n+1})$, i can at best tie with this agent. If $\lambda^*_{n+1} = P+n-1$, then by the structure of $\succ_{n+1}$, it must be agent k_n that clinches at h^*_{n+1}, and there is no tie at the end of step 1. This means that ℓ is not coded in step 1, and so the continuation game that begins step 2 of the coding algorithm starts with agent ℓ being offered x_i. Now, for i to be ranked immediately after k_n, she must be ordered first in step 2 of the coding algorithm, and for i to be ordered first without ties, either she must lurk x_i and it is the first lurked object, or i must clinch x_i while there are no lurked objects and before x_i has not been offered to another active non-lurker. However, neither of these can occur because ℓ begins the step 2 continuation game being offered x_i. Therefore, in $\succ_{n+1}$, i must tie with some agent that we label k_{n+1}. This completes the proof of Lemma B.15. ■

Before proving Lemmas B.16 and B.17, we first state and prove Lemma B.19, on which both rely. To state the lemma, we introduce the following notation: define Q to be the step of the coding algorithm in which i is coded in game Γ_n. Also, define h_n^{q*} to be the history at which the first object is clinched in step q of the coding algorithm for game Γ_n.

Lemma B.19. *Assume that there exists a sequence of role assignment functions Σ as defined in the statement of Lemma B.5, and such that $n \geq 2$. If either (i) $Q = 1$, or (ii) $Q \geq 2$ and at h_n^{1*}, there is an agent ℓ that is an active nonlurker at h_n^{1*} that does not move at h_n^{1*}, and $x_i \in C_\ell^{\subsetneq}(h_n^{1*})$, then, in $\succ_{n+1}$, agent i must tie with some agent k_{n+1}.*

Proof of Lemma B.19. We start with the following lemma.

Lemma B.20. *Consider two games Γ_A and Γ_B, with corresponding role assignment functions σ_A and σ_B, and resulting agent orderings $\succ_A$ and $\succ_B$. Assume that $\succ_A$ begins as $\{i,j\} \succ_A \cdots$, and $\succ_B$ begins as: $j \succ_B i \cdots$. Further, assume that in game Γ_A, there is some history h where j moves such that: (i) $h \subseteq h_A^*$, (ii) $x_i \in C_j(h)$ (iii) $x_j \notin C_j^{\subseteq}(h)$ (iv) $x_i, x_j \notin C_i^{\subseteq}(h)$. Then:*

(a) If agent j clinches at h_A^* in Γ_A, then in Γ_B, agent j clinches at $h_B^* \subsetneq h_A^*$, and there is some agent $k \neq i$ that is an active non-lurker at h_B^* such that $x_i \in C_k(h_B^*)$.

(b) In $\succ_B$, agent i must tie with some other agent k.

Proof of Lemma B.20. Let h_A^* and h_B^* be the first time an agent clinches in Γ_A and Γ_B. Notice that by the structure of $\succ_A$, at history h_A^*, there are two active roles, and both are nonlurkers at h_A^*; label the roles s and s', and, wlog, let $\sigma_A(s) = i$ and $\sigma_A(s') = j$. Using these definitions, we can write the presumptions of the lemma as (ii) $x_i \in C_j(h)$ (iii) $x_j \notin C_j^{\subseteq}(h)$ (iv)$x_i, x_j \notin C_i^{\subseteq}(h)$. Also, notice that $h \subseteq h_A^*$ implies that there are no lurkers at h, and so the only roles that may possibly be active at h are s and s'. Finally, since x_i and x_j tie for the top ranking in $\succ_A$, it must be that x_i is i's favorite object among all objects and x_j is j's favorite object among all objects. Therefore, by greedy strategies, if at any history i is able to clinch x_i, she will do so, and the same for j and x_j.

Part (a). The structure of $\succ_A$ implies that $x_j \in C_s(h')$ for some $h' \subsetneq h_A^*$. Now, consider Γ_B. The only way for j to be ranked first without ties is that $\sigma_B(s) = j$, and j clinches at

$h^*_B \subsetneq h^*_A$.[80] Let $k := \sigma_B(s')$, and notice that, by the assumptions of the lemma, $x_j \notin C^{\subseteq}_s(h)$, and so $h \subsetneq h^*_B$, and therefore $x_i \in C^{\subsetneq}_{s'}(h^*_B)$. It is clear that $k \neq j$. Further, $k \neq i$ because if $k = i$, then $x_i \in C_i(h)$ in Γ_B, and thus, i would clinch x_i at $h \subsetneq h^*_B$ in Γ_B, which contradicts that the first clinching in Γ_B is j clinching at h^*_B. Therefore, $\sigma_B(s') = k$ for some $k \neq i, j$, and k is an active non-lurker that does not move at h^*_B such that $x_i \in C_k(h^*_B)$ in Γ_B.

Part (b). If j clinches at h^*_A, ,then by part (a), there is an agent k such that $x_i \in C^{\subsetneq}_k(h^*_B)$ and k is not coded in the coding step initiated at h^*_B in Γ_B. Let $h^{**}_B \supsetneq h^*_B$ be the history at which the next clinching occurs in Γ_B. Since k was offered x_i in the previous coding step, but is still active, at the initial history of the continuation game that begins step 2, k is offered to clinch x_i again (see Remark 3). Thus, x_i cannot be the first lurked object on the initial passing path of the continuation game form (Remark 4), and so there must be no lurked objects at h^{**}_B. For i to be coded next, she must be active at h^{**}_B, and since there are no lurked objects, there are two active agents, i and k. If k clinches at h^{**}_B, it is obvious that i can at best tie; if i clinches at h^{**}_B, i once again ties with k, because $x_i \in C_k(h^{**}_B)$.

The other possibility is that i clinches at h^*_A, which implies that $x_i \in C_{s'}(h')$ for some $h' \subsetneq h^*_A$. For j to be ranked first without ties in $\succ_B$, at h^*_B, either (a) there are lurkers, and x_j is the first lurked object or (b) there are no lurkers, j clinches x_j, and x_j has not been offered to another non-lurker that is active at h^*_B. There are 3 cases:

Case: $\sigma_B(s') = i$. In this case, i would clinch x_i at h and would be ranked first in $\succ_B$, which is a contradiction.[81]

Case: $\sigma_B(s') = j$. Here, j is in the same role in both games, and therefore $\sigma_B(s) = \ell \neq i$, which follows because if $\ell = i$, then both j and i are in the same roles, and we would get the same initial orderings for $\succ_A$ and $\succ_B$, a contradiction. This implies that $h^*_B \supsetneq h^*_A$, because if $h^*_B \subseteq h^*_A$, then, since j is in the same role, she would clinch at h^*_B in Γ_A, a contradiction.[82] Now, notice that because x_i has been offered to both j and ℓ (weakly) prior to h^*_A, x_i cannot be the first or second lurked object of the game. This means that, for i to be ranked second, there can be at most one lurked object at h^*_B, and if it exists it must be x_j that is lurked.

If x_j is lurked at h^*_B, it must be by either j or ℓ. If it is lurked by ℓ, then x_j must clinch at h^*_B, but, since there is only one lurker, this implies that ℓ must clinch an unlurked object, and will be ranked second (possibly tied with i). If x_j is lurked by j, then ℓ is still an active non-lurker at h^*_B such that $x_i \in C_\ell(h^*_B)$. If i clinches x_i at h^*_B, she will tie with ℓ; if i does not clinch, she can at best tie with ℓ (and may be ranked strictly lower). In either case, the

[80] The only other way for j to be ranked first without ties is that x_j is the first lurked object; however, this cannot obtain, because $x_j \in C_s(h')$ at some history h' where there are no lurkers.

[81] Note that x_j has not been offered to any agent at h, by the presumptions of the lemma.

[82] The case $h^*_B = h^*_A$ is ruled out because i moves at h^*_A in Γ_A, and this history is controlled by role s, not s'.

result holds.

The final case is that nothing is lurked at h_B^*. This implies that x_j clinches at h_B^*, but again, $x_i \in C_\ell(h_B^*)$. Therefore, at the initial history of the continuation game that begins step 2 of the coding algorithm, x_i is offered to agent ℓ. Let h_B^{**} be the first time an object is clinched in this continuation game. Since x_i is offered to ℓ at the initial history, x_i cannot be the first lurked object, and so, for i to be ranked first in this continuation game without ties, she must clinch x_i while it is unlurked and has not been offered to another active non-lurker. But, we have just seen that x_i is offered to ℓ at the initial history, and so this cannot hold.

Case: $\sigma_B(s') = \ell'$ for some $\ell' \neq i, j$. First, notice that $\sigma_B(s) = \ell$ for some $\ell \neq i$. To see this, assume that $\ell = i$. Then, i is in the same role in Γ_A and Γ_B. This implies that $h_B^* \subsetneq h_A^*$, because if h_A^* is reached in Γ_B, i would clinch there, and be ranked above j. But, $h_B^* \subsetneq h_A^*$ implies that j is not ranked first in $\succ_B$ (since she is not yet active at h_B^*), which is a contradiction.

If $\sigma_B(s) = j$, then for j to be ranked first in $\succ_B$, either (a) x_j is the first lurked object on the path to h_B^* or (b) there are no lurked objects at h_B^*, j clinches x_j at h_B^*, and x_j has not been offered to another active non-lurker. Notice that $h_B^* \supsetneq h$,[83] which implies that agent $x_i \in C_{\ell'}(h_B^*)$. But, then it is impossible for i to be ranked immediately after $j \succ_B$ without ties, which is a contradiction.

If $\sigma_B(s) \neq j$, then roles s and s' are assigned to agents ℓ and ℓ' in Γ_B, neither of which are j or i. So, for j to be ranked first without ties, x_j must be the first lurked object (and be lurked by either ℓ or ℓ'), and j must clinch it at some $h_B^* \supsetneq h_A^*$. For i to be ranked second without ties in this case, there must be two lurked objects at h_B^*,[84] and x_i must be the second lurked object (after x_j). But, at the history $h'' \supsetneq h_A^*$ where x_j becomes lurked, one of agents ℓ or ℓ' is an active non-lurker who has been previously offered to clinch x_i, and so x_i cannot be the next lurked object, a contradiction. ■

Continuing with the proof of Lemma B.19, first, consider $Q = 1$. Then, all agents $j_1, \ldots, j_P, k_1, \ldots, k_n, i$ are coded in step 1 of game Γ_n. By Remark 5, $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n-1}}$ all become lurked on the initial passing path of the game form, and further, since $n \geq 2$, we can apply Lemma B.15 to conclude that i ties in $\succ_{n+1}$.

It remains to consider $Q \geq 2$. Since we have assumed that $P + 1$ agents are coded in step 1, all agents j_p have been coded in the first step, and so the agent who is coded first in step Q of the coding algorithm of Γ_n is $k_{\bar{n}}$ for some $\bar{n} < n$. So, the subcoding of $\succ_n$ starting from

[83]In case (a), this follows because there are no lurkers at h; in case (b), it follows from the assumption of the lemma that $x_j \notin C_s^{\subseteq}(h)$.

[84]Since j clinches at h_B^*, if there is no other lurked object at h_B^*, the only active agents are ℓ, ℓ', and j, and so one of ℓ or ℓ' will be ranked above i in $\succ_B$, which is a contradiction.

step Q is:

$$k_{\bar{n}} \succ_n k_{\bar{n}+1} \succ_n \cdots \succ_n k_{n-1} \succ_n \{i, k_n\}.$$

Consider the sequence of games $\Gamma_{\bar{n}}, \Gamma_{\bar{n}+1} \ldots, \Gamma_n, \Gamma_{n+1}$. Notice that the codings for all of these games are exactly the same, up to agent $k_{\bar{n}-1}$. Therefore, by Lemma B.4, all agents $j_1, \ldots, j_P, k_1, \ldots, k_{\bar{n}-1}$ are in the same roles in all of these games. In particular, agent $k_{\bar{n}-1}$ is the last agent coded in step $Q-1$ in all of these games, and the initial history of the continuation game that begins step Q is the also the same in all of these games; label this history $h^Q_\varnothing$. Now, applying the coding algorithm to the sequence of continuation games of $\Gamma_{\bar{n}}, \ldots, \Gamma_n, \Gamma_{n+1}$ starting from history $h^Q_\varnothing$, we get the sub-codings:

$$\begin{aligned}
&\{i, k_{\bar{n}}\} \succ_{\bar{n}} \cdots \\
&k_{\bar{n}} \succ_{\bar{n}+1} \{i, k_{\bar{n}+1}\} \succ_{\bar{n}+1} \cdots \\
&\vdots \\
&k_{\bar{n}} \succ_n k_{\bar{n}+1} \succ_n \cdots \succ_n k_{n-1} \succ_n \{i, k_n\} \succ_n \cdots. \\
&k_{\bar{n}} \succ_{n+1} k_{\bar{n}+1} \succ_{n+1} \cdots \succ_{n+1} k_n \succ_{n+1} i \cdots
\end{aligned}$$

There are two cases.

Case 1: $\bar{n} < n$**.** In this case, we can apply Lemma B.15 to the game form starting from $h^Q_\varnothing$ to conclude that i must tie in $\succ_{n+1}$. To see this, simply note that upon reindexing to start from $h^Q_\varnothing$ rather than $h_\varnothing$, the condition "$n \geq 2$" becomes "$n \geq \bar{n}+1''$. Then, we have that $x_{k_{\bar{n}}}, \ldots, x_{k_{n-1}}$ all become lurked on the initial passing path of the game form starting from $h^Q_\varnothing$, which follows from Remark 5, $n \geq \bar{n}+1$, and the fact that i is coded in the initial step of the continuation game of Γ_n starting from $h^Q_\varnothing$. Thus, all of the conditions of Lemma B.15 are satisfied.

Case 2: $\bar{n} = n$**.** In this case, the games we are concerned with are Γ_n and Γ_{n+1}, with subcodings:

$$\begin{aligned}
&\{i, k_n\} \succ_n \cdots \\
&k_n \succ_{n+1} i \cdots.
\end{aligned} \tag{C}$$

Notice that here, we can no longer apply Lemma B.15, since we do not have at least two games in which i ties in the sequence. Our goal is to apply Lemma B.20 instead, but to do so, we must show that the conditions (i)-(iv) of Lemma B.20 are satisfied at $h^Q_\varnothing$.

For each each coding step $q = 1, \ldots, Q$ of game Γ_n, let h^{q*}_n denote history at which the first object is clinched in the q^{th} coding step, and let $h^{\varnothing_q}_n$ denote the initial history that begins the continuation game for the next step, after all of the agents in step $q-1$ are coded (in particular, $h_n^{\varnothing_1} = h_\varnothing$, and $h_n^{1*} = h_n^*$ in our earlier notation). In $\succ_n$, all agents who are coded in steps $q < Q$ are ranked strictly, without ties. Let k_{n^q} denote the agent who is coded ***last*** in the q^{th} step. With this notation, the subcoding from the q^{th} step is:

$$k_{n^{q-1}+1} \succ_n k_{n^{q-1}+1} \succ_n \cdots \succ_n k_{n^q},$$

where we define $n^0 = 0$. It is possible that $k_{n^{q-1}+1} = k_{n^q}$, in which case only one agent is coded in step q. Since there are no ties, agent k_{n^q} ends the coding step by clinching an unlurked object that has not been offered to another non-lurker who is active at h_n^{q*}.

Claim B.6. For all $q < Q$, there is an agent $\ell \neq k_1, \ldots, k_{n^q}, i$ such that ℓ is an active nonlurker at h_n^{q*} that does not move at h_n^{q*}, and $x_i \in C_\ell^{\subsetneq}(h_n^{q*})$.

Claim B.6 (whose proof can be found immediately after the proof of this lemma) implies that when we reach step Q in Γ_n, at the initial history of the continuation game $h_n^{\varnothing_Q}$ that begins this step, there is some agent $\ell \neq k_1, \ldots, k_{n-1}, i$ such that $x_i \in C_\ell(h_n^{\varnothing_Q})$. Since the subcodings for $\succ_n$ in this step begin with a tie between i and k_n (see Equation C), it must be that $\ell = k_n$. Finally, we apply Lemma B.20 by setting $A = n$, $B = n+1$, $h = h_n^{\varnothing_Q}$, $j = k_n$, and $i = i$ to conclude that i must tie in $\succ_{n+1}$.[85] ■

Proof of Claim B.6. By the supposition of the lemma, at h_n^{1*}, there is an agent ℓ that is an active nonlurker at h_n^{1*} that does not move at h_n^{1*}, and $x_i \in C_\ell^{\subsetneq}(h_n^{1*})$. It is clear that ℓ is not coded (since there is no tie in step 1), and so $\ell \neq k_1, \ldots, k_{n^1}$. To see that $\ell \neq i$, note that if $\ell = i$, then step 2 begins with agent i being offered to clinch x_i. If i is not coded in step 2, then step 3 begins with i being offered x_i, etc.. The same continues up to and including step Q, in which i is coded. Since i is coded first in step Q (tying with k_n) x_i is her top object among those that remain at the beginning of step Q. Since $x_i \in C_i(h_n^{(Q-1)*})$, agent i begins step Q by being offered to clinch x_i at the initial history of this step. Since x_i is her top remaining object, she would clinch it, and thus would not tie with k_n, which is a contradiction. Thus, the statement holds for $q = 1$.

Now, consider step $q = 2$ of game Γ_n, which begins at $h_n^{\varnothing_2}$ and produces the subcoding:

$$k_{n^1+1} \succ_n k_{n^1+2} \succ_n \cdots \succ_n k_{n^2} \succ_n .$$

Case 1: $n^2 = n^1 + 1$. Then only one agent, agent k_{n^1+1}, is coded in step 2 of game Γ_n, which begins with the continuation game that starts at history $h_n^{\varnothing_2}$. The result from step 1

[85] Condition (i) of Lemma B.20 is immediate. For condition (ii) was just shown. Condition (iii) holds because, if $x_{k_n} \in C_{k_n}(h_n^{\varnothing_Q})$, then k_n would immediately clinch it at $h_n^{\varnothing_n}$, and would not tie with i in $\succ_n$. Condition (iv) is also immediate, as i has not yet been called to move at $h_n^{\varnothing_Q}$.

implies that at $h^{\varnothing_2}_n$, some agent $\ell \neq k_1, \ldots, k_{n^1}, i$ moves and $x_i \in C_\ell(h^{\varnothing_2}_n)$.

Since k_{n^1+1} is the only agent coded in step 2 of Γ_n, and does not tie, she must clinch $x_{k_{n^1+1}}$ at h^{2*}_n in Γ_n while it is unlurked, and before it is offered to another active non-lurker. Now, since $\succ_n$ and $\succ_{n^1+1}$ are the same up til agent k_{n^1}, Lemma B.4 implies that $h^{\varnothing_2}_n = h^{\varnothing_2}_{n^1+1}$; for shorthand, define $h^{\varnothing_2} := h^{\varnothing_2}_n = h^{\varnothing_2}_{n^1+1}$. The second step continuation games of Γ_{n^1+1} and Γ_n both start from $h^{\varnothing_2}$, and lead to the initial subcodings:

$$\{i, k_{n^1+1}\} \succ_{n^1+1} \cdots$$
$$k_{n^1+1} \succ_n \cdots$$

Let role s be the role that moves at $h^{\varnothing_2}$, and role s' be the second role that becomes active on the initial passing path of the game form starting from $h^{\varnothing_2}$. These two roles exist because there is an initial tie in $\succ_{n^1+1}$, and in Γ_{n^1+1}, s and s' are assigned to k_{n^1+1} and i, in some manner. If $\sigma_{n^1+1}(s) = i$, then i would clinch at $h^{\varnothing_2}$ in Γ_{n^1+1}, and would not tie, a contradiction. Therefore, $\sigma_{n^1+1}(s) = k_{n^1+1}$, which implies that $x_{k_{n^1+1}} \notin C_{k_{n^1+1}}(h^{\varnothing_2}_n)$; indeed, if this were true, then k_{n^1+1} would clinch it at $h^{\varnothing_2}_{n^1+1}$ in Γ_{n^1+1}, which contradicts that k_{n^1+1} ties in $\succ_{n^1+1}$.

Now, if $\sigma_n(s) = k_{n^1+1}$, then k_{n^1+1} is in the same role in both games, and so it must be i that clinches at $h^{2*}_{n^1+1}$, which means that $x_i \in C_{s'}(h^{2*}_{n^1+1})$.[86] It also means that $h^{2*}_n \supsetneq h^{2*}_{n^1+1}$, and that $\sigma_n(s') \neq i$, and so, there exists some agent $\ell' \neq i$ such that in Γ_n, $x_i \in C_{\ell'}(h^{2*}_n)$, which is what we wanted to show.

Last, if $\sigma_n(s) \neq k_{n^1+1}$, then $\sigma_n(s') = k_{n^1+1}$. Thus, in this case, there is some agent other agent ℓ such that $\sigma_n(s) = \ell$. Again, $\ell \neq i$, because $x_i \in C_s(h^{\varnothing_2}_n)$. Thus, when k_{n^1+1} clinches at h^{2*}_n in Γ_n, we have $x_i \in C^{\subsetneq}_\ell(h^{2*}_n)$, as desired.

Case 2: $n^2 > n^1 + 1$. Consider games $\Gamma_{n^1+1}, \Gamma_{n^1+2}, \ldots, \Gamma_n$ and notice that the codings for all of these games are equivalent up to agent k_{n^1}. Therefore, by Lemma B.4, all agents $k_1, \ldots, k_{n^1}$ are in the same roles in all of these games, and so these agents will take the same actions, which implies that, for each of these games, step 2 of the coding algorithm begins at the same history of the game form, which we denote $h^{\varnothing_2}$.

Consider the continuation game form starting at $h^{\varnothing_2}$, and recall that $h^{2*}_{n'}$ is the first time an object is clinched in step 2 of game $\Gamma_{n'}$, which is also the first time an object is clinched in step 1 of the continuation game beginning at $h^{\varnothing_2}$. Notice that by the structure of $\succ_n$, the objects $x_{k_{n^1+1}}, \ldots, x_{k_{n^2-1}}$ are lurked at h^{2*}_n in Γ_n, while $x_{k_{n^2}}$ is not, i.e., objects $x_{k_{n^1+1}}, \ldots, x_{k_{n^2-1}}$ are the first lurked objects (in order) along the initial passing path of the

[86]If k_{n^1+1} clinched first in Γ_{n^1+1} and Γ_n, and is in the same role, then the subcodings $\succ_{n^1+1}$ and $\succ_n$ would be the same up to k_{n^1+1}, which is a contradiction.

game form, beginning at $h^{\varnothing_2}$.

The subcodings of games $\Gamma_{n^1+1}, \Gamma_{n^1+2}, \ldots, \Gamma_{n^2+1}$ beginning at history $h^{\varnothing_2}$ are:

$$\{i, k_{n^1+1}\} \succ_{n^1+1} \cdots$$
$$\vdots$$
$$k_{n^1+1} \succ_{n^2} k_{n^1+2} \succ_{n^2} \cdots \succ_{n^2} k_{n^2-1} \succ_{n^2} \{i, k_{n^2}\} \succ_{n^2} \cdots.$$
$$k_{n^1+1} \succ_{n^2+1} k_{n^1+2} \succ_{n^2+1} \cdots \succ_{n^2+1} k_{n^2} \succ_{n^2+1} \{i, k_{n^2+1}\} \cdots$$

By Lemma B.15 applied to the continuation game and subcodings beginning at $h^{\varnothing_2}$, in Γ_{n^2+1}, at $h^{2*}_{n^2+1}$, there is an agent ℓ such that ℓ is an active non-lurker at $h^{2*}_{n^2+1}$ that does not move at $h^{2*}_{n^2+1}$ and $x_i \in C^{\subsetneq}_{\ell}(h^{2*}_{n^2+1})$. Since $\succ_n$ is equivalent to $\succ_{n^2+1}$ up to agent k_{n_2}, and agent k_{n_2} is the last agent in a coding step of game Γ_n, we have that $h^{2*}_n = h^{2*}_{n^2+1}$, by Lemma B.4. This implies that at h^{2*}_n, there is an agent ℓ that is an active non-lurker at h^{2*}_n that does not move at h^{2*}_n and $x_i \in C^{\subsetneq}_{\ell}(h^{2*}_n)$ (which may or may not be the same such agent in Γ_{n^2+1}, depending on the role assignment functions).

It remains to show that $\ell \neq k_1, \ldots, k_{n^2}, i$. It is clear that $\ell \neq k_1, \ldots, k_{n^2}$, since all of these agents are coded by the end of step 2 in Γ_n, while agent ℓ is not. If $\ell = i$, step 3 begins with agent i being offered to clinch x_i. If i is not coded in step 3, then i continues to be active in step 4, which begins with i being offered x_i, etc.. The same continues up to and including step Q, in which i is coded. Since i is coded first in step Q (tying with k_n) x_i is her top object among those that remain at the beginning of step Q. Since $x_i \in C_i(h^{(Q-1)*}_n)$, agent i begins step Q by being offered to clinch x_i at the initial history of this step. Since x_i is her top remaining object, she would clinch it, and thus would not tie with k_n, which is a contradiction. Therefore, $\ell \neq i$. This completes the result for $q = 2$.

We then just repeat the arguments for the $q = 2$ case for all $q = 3, 4, \ldots, Q-1$, which completes the proof of Lemma B.19. ■

Proof of Lemma B.16. We begin by showing the result for $n = 1$, as part of the following claim.

Claim B.7. *Assume that there exist σ_0 and σ_1 such that:*

$$j_1 \cdots j_P \succ_0 i \succ_0 k_1 \succ_0 \cdots$$
$$j_1 \cdots j_P \succ_1 \{i, k_1\} \succ_1 \cdots.$$

Then:

(a) We have $h^*_0 \subsetneq h^*_1$, and the agent that moves at h^*_0 in Γ_0 is agent i.

(b) If there exists a σ_2 such that $j_1 \cdots j_P \succ_2 k_1 \succ_2 i \cdots$, then $h^*_0 \subsetneq h^*_2$. Further, in $\succ_2$, agent i

must tie with some other agent k_2.

(c) If x_{k_1} is not the $(P+1)^{th}$ lurked object on the initial passing path, then in Γ_2, agent k_1 clinches at $h_2^* \subsetneq h_1^*$. Further, at h_2^*, there is an active non-lurker $\ell \neq j_1, \ldots, j_P, i, k_1$ such that $x_i \in C_{k_2}^{\subsetneq}(h_2^*)$.

The proof of this claim can be found at the end of the proof of the lemma. Now, consider a sequence Σ such that $n \geq 2$. We will show that i must tie in $\succ_{n+1}$.

In game Γ_n, i is coded in some step of the coding algorithm with some subset of the agents $j_1 \ldots, j_P, k_1, \ldots, k_{n-1}$. Let Q be the step number in which i is coded in game Γ_n. The goal is to apply Lemma B.19, which the following claim allows us to do.

Claim B.8. If $Q \geq 2$, then at h_n^*, there is an agent ℓ that is an active non-lurker at h_n^* that does not move at h_n^* and $x_i \in C_\ell^{\subsetneq}(h_n^*)$.

The proof of this claim is found below, immediately after the proof of Claim B.7. Given Claim B.8, we can apply Lemma B.19 to conclude that i must tie in $\succ_{n+1}$, which completes the proof of Lemma B.16. ■

Proof of Claim B.7. Since we assume there are at least P lurkers at h_1^*, by the structure of $\succ_1$, there are exactly P lurkers at h_1^*. This implies that the first P lurked objects are $x_{j_1}, \ldots, x_{j_P}$. Additionally, objects x_i and x_{k_1} are unlurked at h_1^*, and so x_i and x_{k_1} are agent i and k_1's favorite objects among the set of those that are unlurked at h_1^*, respectively.

Part (a). Suppose not, then the passing structure of histories implies that $h_1^* \subseteq h_0^*$. Notice that at h_1^*, there must be two active non-lurker roles.

Case 1: $P = 0$**.** In this case, there are no agents j_p, so at h_1^*, there are exactly two active roles, label them s and s', and wlog, let $\sigma_1(s) = i$ and $\sigma_1(s') = k_1$. If i clinches at h_1^* in Γ_1, then $x_i \in C_{s'}^{\subsetneq}(h_1^*)$ and $x_i \in C_s(h_1^*)$. Now, for i to be ranked first without ties in $\succ_0$ is either (i) x_i is the first lurked object of the game or (ii) i clinches x_i first as an unlurked object, and it has not been offered to another active non-lurker. However, $h_1^* \subseteq h_0^*$ implies that neither (i) nor (ii) can obtain, as x_i has been offered to both active non-lurkers at h_1^*, which is a contradiction.

If k_1 clinches at h_1^* in Γ_1, then $x_{k_1} \in C_s^{\subsetneq}(h_1^*)$ and $x_{k_1} \in C_{s'}^{\subseteq}(h_1^*)$. Now, $h_1^* \subseteq h_0^*$ implies that in Γ_0, $\sigma_0^{-1}(k_1) \neq s, s'$.[87] Since k_1 is not in either of these roles, there is some $\ell \neq i, k_1$ that is active at h_1^* in Γ_0 and is such that $x_i \in C_\ell^{\subseteq}(h_1^*)$. Notice also that since x_{k_1} has been offered to both active agents at h_1^*, it cannot be the second lurked object along the initial passing path (Remark 4), and so for k_1 to be ranked second, there can be at most 3 active agents at h_0^*, in particular agents i, k_1, and ℓ. If k_1 moves at h_0^*, i must be lurking x_i, and k_1 will

[87]If $\sigma_0^{-1}(k_1) = s$, then k_1 would clinch at some $h' \subsetneq h_1^*$; if $\sigma_0^{-1}(k_1) = s'$, then k_1 is in the same role in Γ_0 and Γ_1, and thus would clinch at $h_0^* = h_1^*$, and would once again tie for first in $\succ_0$.

tie with agent ℓ. If ℓ moves at h_0^*, it is clear k_1 will not be ranked second without ties. If i moves at h_0^*, then there must be no lurked objects at h_0^*.[88] But, since $h_1^* \subseteq h_0^*$, we have $x_{k_1} \in C_\ell(h_0^*)$, and so, since ℓ was not coded in the first step, she begins the second step by being offered x_{k_1} at the initial history of the continuation game. Thus, it is impossible for k_1 to be ranked first without ties in this continuation game, a contradiction.

Case 2: $P \geq 1$**.** In this case, there is at least one lurker j_p at h_1^*. Further, at h_1^*, there are P active lurker roles for the objects $x_{j_1}, \ldots, x_{j_P}$, and 2 active non-lurkers roles; label the role that moves at h_1^* as s, and the other active nonlurker at h_1^* as s'. There are three subcases, depending on who is in role s.

Subcase 2.1. $\sigma_1(s) = i$**.** In this case, we have $\sigma_1(s') = k_1$ and $x_i \in C_{s'}^{\subsetneq}(h_1^*)$. We first claim that i cannot be active at h_1^* in Γ_0. First, notice that i cannot move at h_1^* in Γ_0, because if she did, she would choose the same action at h_1^* in both games, and would tie in $\succ_0$, just as she did in $\succ_1$. So, $\sigma_0(s) \neq i$. Next, assume i is a lurker at h_1^* in Γ_0, for some lurked object $x_{j_1}, \ldots, x_{j_P}$. Note that x_i cannot be the next object lurked along the initial passing path because it has been offered to (both) active non-lurkers at h_1^*, so at h_0^*, there must be no newly lurked objects, and roles s and s' are still active non-lurkers. The first coding of step Γ_0 thus ends when i clinches x_i, which is unlurked. But, because $h_1^* \subseteq h_0^*$, x_i has been offered to both role s and s' at h_0^*, and one of these is an active non-lurker who does not move at h_0^*, and so i would tie with this agent in $\succ_0$.

Second, assume that $\sigma_0(s') = i$. Then, notice that $x_i \in C_{s'}(h')$ for some $h' \subsetneq h_1^*$. We claim that i would clinch x_i at this history. Indeed, at h', role s' is an active non-lurker that is not the terminator.[89] This means that only unlurked objects are possible for the agent in this role, and since x_i is i's favorite unlurked object, she will clinch it at h', by greedy strategies. Therefore, i is not active at h_1^* in Γ_0.

Now, i is not active at h_1^* in Γ_0, but there are two active non-lurkers, those in roles s and s', and both of these have been offered x_i. Thus, x_i cannot be the next lurked object along the initial passing path of the game form, and so there can be no newly lurked objects at h_0^*. But then, i is not active at h_0^* (since no new agent can become active unless something else becomes lurked), and so i is not coded in this step, which contradicts that she is ranked $(P+1)^{th}$ in $\succ_0$.

Subcase 2.2: $\sigma_1(s) = k_1$**.** In this case, we have $x_{k_1} \in C_{s'}(h')$ for some $h' \subsetneq h_1^* \subseteq h_0^*$ and $x_{k_1} \in C_s(h_1^*)$. This implies that x_{k_1} cannot be either of the next two lurked objects on the

[88] If there were, it must be x_i. It cannot be lurked by k_1, since this would mean x_i is her top object, which is a contradiction. So, it must be lurked by some $\ell \neq i, k_1$, and so ℓ will be ranked ahead of or tie with k_1 in $\succ_1$.

[89] This follows from Lemma B.12. If this role were the terminator, then role s could not be offered x_i at $h_1^* \supsetneq h'$.

initial passing path of the game form (if they exist). Since k_1 is ordered immediately after i in $\succ_0$ and k_1 does not tie, there can be at most one newly lurked object at h_0^*, and it must be x_i.

We next claim that k_1 cannot be active at h_1^* in Γ_0. It is clear that $\sigma_0(s) \neq k_1$, because otherwise k_1 would clinch at h_1^* in Γ_0, and once again tie in $\succ_0$. We also have that $\sigma_0(s') \neq k_1$. To see why, notice that s' is not the terminator role (see footnote 89). So, only unlurked objects are possible for the agent in this role, and thus, if k_1 was in this role, she would clinch x_{k_1} at $h' \subsetneq h_0^*$, since it is her favorite unlurked object. Last, if k_1 lurks some object x_{j_p} at h_1^*, then she strictly prefers x_{j_p} to x_{k_1}. It then must be some agent $j_{p'}$ that moves at h_0^* and clinches a lurked object $x_{j_{p'}}$. This means that $j_{p'}$ is in the terminator role. We claim that $\sigma_0^{-1}(j_{p'}) \neq s, s'$. We know (see footnote 89) that s' is not the terminator role, so $\sigma_0^{-1}(j_{p'}) \neq s'$. If $\sigma_0(s) = j_{p'}$, then s is the terminator role. But, this contradicts that k_1 clinched x_{k_1} first at h_1^* in Γ_1, since in that game she was in the terminator role and so x_{j_p} is possible for her, and she strictly prefers it. Therefore, in Γ_0, $j_{p'}$ is in some role s'' that was not active at h_1^*. This implies that one of s or s' is still active at h_0^* in Γ_0, and whoever it is, this agent has been offered x_{k_1} prior to h_0^*. So, k_1 would tie with this agent in $\succ_0$, a contradiction.

So, k_1 is not active at h_1^* in Γ_0. So, there is some agent $\ell \neq j_1, \dots, j_P, i, k_1$ that is active at h_1^* in Γ_0. This agent cannot be a lurker at h_0^*, since if she were, she would necessarily be coded in step 1, and, as x_{k_1} is not lurked at h_0^*, k_1 could at best tie with her. Thus, $\sigma_0^{-1}(\ell) = s$ or s', and no matter which, we have $x_{k_1} \in C_\ell(h_1^*)$. If x_{k_1} is clinched in step 1, then k_1 can at best tie with ℓ. If k_1 is not coded in step 1, then in at the start of the continuation game for step 2, ℓ is offered x_{k_1}. But, if this is the case, then k_1 cannot be ordered first without ties in step 2, which contradicts the definition of $\succ_0$.

Subcase 2.3: $\sigma_1(s) = j_p$ for some $p = 1, \dots, P$. In this case, agent j_p is clinching a lurked object at h_1^*, and so h_1^* is the terminating history. Then, $h_1^* \subseteq h_0^*$ implies that $h_1^* = h_0^*$. Thus, in Γ_0, x_i is the first (and only) unlurked object clinched in step 1, and so $x_i \notin C_{s'}^{\subsetneq}(h_1^*)$. So, because there is a tie in Γ_1, it must be that $x_{k_1} \in C_{s'}^{\subsetneq}(h_1^*)$.

Next, we claim that in Γ_0, k_1 is not active at h_1^*. Indeed, k_1 is not in role s (as that is occupied by j_p). She also cannot be a lurker, because she is not coded in step 1 (which ends with i). Finally, consider role s'. Notice that s' is not the terminator role (because that is role s), and so, if k_1 were in role s', she would clinch x_{k_1} at some history $h' \subsetneq h_1^*$ at which it was offered to her, a contradiction.

Therefore, there is some $\ell \neq j_1, \dots, j_P, i, k_1$ that is such that $\sigma_0(s') = \ell$ and $x_{k_1} \in C_\ell(h_1^*)$. Since ℓ is not coded in step 1, she begins the continuation game for step 2 by being offered x_{k_1}. Thus, k_1 cannot be ordered first in step 2 without ties, which is a contradiction.

The above shows that $h_0^* \subsetneq h_1^*$. To finish the proof of part (a), we must show that agent

i moves at h_0^* in Γ_0. Notice that $h_0^* \subsetneq h_1^*$ and the structure of $\succ_1$ implies there can be at most P lurkers at h_0^*. First, if there are no lurkers ($P = 0$) at h_0^*, then, it is clear that i must move at h_0^*, as that is the only way she can be ranked first without ties. Now, presume that $P > 0$. If it is some j_p that moves at h_0^*, then j_p clinches a lurked object x_{j_p}, which implies that h_0^* is the terminating history, which contradicts $h_0^* \subsetneq h_1^*$. Therefore, no agent $j_1, \ldots, j_P$ can move at h_0^*. Since there can be at most P lurkers at h_0^*, given that i is ranked $(P+1)^{th}$ without tying, the only other possibility is that it is agent i that moves at h_0^* and clinches x_i.

Part (b). We first show that $h_0^* \subsetneq h_2^*$. By part (a), $h_0^* \subsetneq h_1^*$. This means that agent i cannot move at h_0^* in Γ_1. Nor can any potential agent j_p, because if they did, they would be clinching a lurked object, which means h_0^* is the terminating history, which contradicts $h_0^* \subsetneq h_1^*$. Therefore, it must be k_1 that moves at h_0^* in Γ_1.

By way of contradiction suppose that $h_0^* \subsetneq h_2^*$ fails; because of the passing structure of this histories, it means that $h_2^* \subseteq h_0^*$. The structure of $\succ_2$ implies that k_1 clinches at h_2^* in Γ_2, which also means that h_2^* and h_0^* are controlled by different roles, and further $h_2^* \subsetneq h_0^*$.[90] So, in Γ_0, it must be some agent $\ell \neq j_1, \ldots, j_P, i, k_1$ that moves at h_2^*. But then, we have $x_{k_1} \in C_\ell(h_0^*)$, so at the initial history of the continuation game that begins step 2, agent ℓ is offered x_{k_1}, and so k_1 cannot be ordered first in step 2, which is a contradiction to the definition of Γ_0. Therefore, $h_0^* \subsetneq h_2^*$.

Thus, we have $h_0^* \subsetneq h_1^*, h_2^*$, and so agent i does not move at h_0^* in Γ_1 or Γ_2.

Case 1: Agent k_1 moves at h_0^* in Γ_2. Here, k_1 is in the same role as in Γ_1, and so $h_1^* \subsetneq h_2^*$. This implies that i must clinch at h_1^* in Γ_1, and so i does not move at h_1^* in Γ_2. If some j_p moves at h_1^* in Γ_2, then this agent must also clinch at h_2^*, and she must clinch a lurked object. This means that i must be a lurker for some $x_{j_{p'}}$, and so she strictly prefers $x_{j_{p'}}$ to x_i. But then, the agent that moves at h_1^* is in the terminator role, and so in Γ_1, i is in the terminator role, and since she clinches x_i at h_1^* , this implies that x_i is her top object (lurked or unlurked) by Lemma B.12(v), which is a contradiction. So, it must be some $\ell \neq j_1, \ldots, j_P, i, k_1$ that moves at h_1^* in Γ_2, and so $x_i \in C_\ell^{\subsetneq}(h_2^*)$ in Γ_2. Since ℓ is not coded in step 1, she is offered x_i at the initial history of the continuation game that begins step 2. Therefore, i cannot be ranked first without ties in this continuation game.

Case 2: Some agent $j_1, \ldots, j_P$ moves at h_0^* in Γ_2. This agent, say j_p, must be the one clinching at h_2^* (since j_p is not a lurker at h_0^*, but ultimately receives a lurked object), and she must clinch a lurked object. This implies that the agent who moves at h_0^* is in the terminator role, and that h_2^* is the terminating history, so $h_1^* \subseteq h_2^*$. Let r be the other role

[90]If they were the same role, then k_1 is in this role in Γ_1, and would clinch at h_2^* in Γ_1, which is a contradiction.

that is active at h_1^*. Since there is a tie in $\succ_1$, this role must be such that either $x_i \in C_r^{\subseteq}(h_1^*)$ or $x_{k_1} \in C_r^{\subseteq}(h_1^*)$. In the latter subcase, x_{k_1} cannot be the next lurked object along the passing path (from h_1^*), and so there must be no newly lurked objects at h_2^*. Next, notice that $\sigma_2(r) \neq k_1$, because otherwise, k_1 would clinch x_{k_1} at the history $h' \subsetneq h_1^*$ where it was offered in Γ_2. Thus, k_1 can at best tie with the agent $\sigma_2(r)$, which is a contradiction.

For the subcase $x_i \in C_r^{\subseteq}(h_1^*)$, if $\sigma_2(r) = k_1$, then there is some agent $\ell \neq j_1, \ldots, j_P, k_1$ who is a lurker for some $x_{j_1}, \ldots, x_{j_P}$. We also have $\ell \neq i$. This is because the agent who moves at h_0^* is in the terminator role, and so in Γ_0, i is in this role, and since she clinches, x_i is her top available object (lurked or unlurked), and therefore i cannot lurk any of the x_{j_p}'s. Therefore, agent ℓ will be ranked ahead of i in $\succ_2$, a contradiction.[91] We also cannot have $\sigma_2(r) = i$, because i would clinch x_i at the history $h' \subsetneq h_1^*$ at which she was offered x_i. Thus, $\sigma_2(r) = \ell$ for some $\ell \neq j_1, \ldots, j_P, i, k_1$. Agent ℓ is not coded in step 1, and thus, she is offered x_i at the initial history of the continuation game that begins step 2, and so i cannot be ranked first without tying in step 2.

Part (c). If x_{k_1} is not the $(P+1)^{th}$ lurked object, then, because k_1 is ordered without tying in $\succ_2$, at h_2^*, k_1 must clinch x_{k_1}, and it has not been offered to another active non-lurker. Notice also that $h_0^* \subsetneq h_2^*$ implies that i does not move at h_0^* in Γ_1 or Γ_2, and that k_1 moves at h_0^* in Γ_1. If k_1 moves at h_0^* in Γ_2, then she is in the same role in both games, and so $h_1^* \subsetneq h_2^*$. This also means that i moves at h_1^* in Γ_1 (because if it was k_1, then x_{k_1} is offered to both active roles at h_1^*, and so in Γ_2, k_1 would clinch at some $h' \subsetneq h_2^*$). Thus, x_i has been offered to both active non-lurker roles at h_1^*. This implies that i cannot be active at h_1^* in Γ_2, and so there is some $\ell \neq j_1, \ldots, j_P, i, k_1$ such that $x_i \in C_\ell(h_2^*)$ in Γ_2. If k_1 does not move at h_0^* in Γ_2, then it is some $\ell \neq j_1, \ldots, j_P, i, k_1$ that moves at h_0^*. In either case, we have $x_i \in C_\ell^{\subsetneq}(h_2^*)$ in Γ_2. ∎

Proof of Claim B.8. (See above for the statement of the claim).Since it is without loss of generality to assume that there are at least P lurkers at h_n^*, there are two cases. Recall that k_1 is ranked strictly, without ties, in $\succ_n$.

Case 1: There are exactly P lurkers at h_n^*. In this case, k_1 is the last agent coded in step 1 of Γ_n. Consider game Γ_2, and notice that $\succ_n = \succ_2$ up to agent k_1. Since agent k_1 is the last agent in a coding step, by Lemma B.4, all agents $j_1, \ldots, j_P, k_1$ are in the same roles in Γ_2 and Γ_n, and $h_n^* = h_2^*$. Further, notice that x_{k_1} is not the $(P+1)^{th}$ lurked object along the initial passing path,[92] and so, by Claim B.7 part (c), there is an agent ℓ that is an active

[91] Note that x_i cannot be lurked at h_2^*, since it has been offered to agent j_p at h_0^*, who is the terminator.

[92] If k_1 clinches at h_n^*, then x_{k_1} is offered to an active non-lurker, and so cannot be the next lurked object along the initial passing path; if some j_p clinches at h_n^*, then they are clinching a lurked object, and so h_n^* is the terminating history, which again implies that x_{k_1} is not $(P+1)^{th}$ lurked object along the initial passing path (because no such object exists).

non-lurker at h_2^* that does not move at h_2^* and $x_i \in C_\ell^{\subsetneq}(h_2^*)$. Since $h_2^* = h_n^*$, the result holds.

Case 2: There are strictly greater than P lurkers at h_n^*. In this case, the objects $x_{j_1}, \ldots, x_{j_P}, x_{k_1}, \ldots, x_{k_{n'-1}}$ are lurked at h_n^*, while $x_{k_{n'}}$ is not, where $n > n' > 1$.[93] Consider game $\Gamma_{n'+1}$, and notice that $\succ_n$ is equivalent to $\succ_{n'+1}$ up to agent $k_{n'}$. Therefore, by Lemma B.4, all agents $k_1, \ldots, k_{n'}$ are in the same roles in all of these games, and $h_n^* = h_{n'+1}^*$. By Lemma B.15, in $\Gamma_{n'+1}$, at $h_{n'+1}^*$, there is an active agent ℓ such that ℓ is an active non-lurker at $h_{n'+1}^*$ that does not move at $h_{n'+1}^*$ and $x_i \in C_\ell^{\subsetneq}(h_{n'+1}^*)$. Since $h_n^* = h_{n'+1}^*$, the result holds. ■

Proof of Lemma B.17. By the assumption that $n, m \geq 2$ in Σ and Σ', we have that there exist (at least) the following codings:

$$j_1 \cdots j_P \succ_1 \{i, k_1\} \succ_1 \cdots$$
$$j_1 \cdots j_P \succ_2 k_1 \succ_2 \{i, k_2\} \cdots$$
$$j_1 \cdots j_P \succ_2' i \succ_2' \{k_1, k_2'\} \cdots.$$

We start by presenting the following two conditions, one of which, when combined with prior lemmas, will imply that Statement 1 of the lemma holds, and the other of which will imply Statement 2 of the lemma holds.

- Condition 2: In Γ_2, at h_2^* there is an active non-lurker ℓ such that ℓ does not move at h_2^* and $x_i \in C_\ell^{\subsetneq}(h_2^*)$.
- Condition 2′: In Γ_2', at $h_{2'}^*$, there is an active non-lurker ℓ such that ℓ does not move at $h_{2'}^*$ and $x_{k_1} \in C_\ell^{\subsetneq}(h_{2'}^*)$.[94]

We first show that these conditions imply the lemma. Then, we show that one of these conditions must hold.

We will show that Condition 2 implies that Statement 1 of Lemma B.17 holds. The two statements are symmetric, so this will also show that Condition 2′ implies Statement 2 of Lemma B.17.

To show Condition 2 implies Statement 1, we use Lemma B.19. So, consider the sequence

[93] Because $Q \geq 2$, the last agent coded in step 1 of Γ_n is at most k_{n-1}, which means that $x_{k_{n-1}}$ is not lurked, i.e., the last lurked object is at most $x_{k_{n-2}}$, which is why we have $n' < n$.

[94] We use $h_{2'}^*$ (instead of $h_2^{'*}$) to denote the first history at which an object is clinched in game Γ_2' (under role assignment σ_2').

of codings

$$j_1 \cdots j_P \succ_1 \{i, k_1\} \succ_1 \cdots$$
$$j_1 \cdots j_P \succ_2 k_1 \succ_2 \{i, k_2\} \cdots$$
$$j_1 \cdots j_P \succ_3 k_1 \succ_3 k_2 \succ_3 \{i, k_3\} \succ_3 \cdots$$
$$\vdots$$
$$j_1 \cdots j_P \succ_n k_1 \succ_n k_2 \succ_n k_3 \succ_n \cdots \succ_n k_{n-1} \succ_n \{i, k_n\} \succ_n \cdots$$
$$j_1 \cdots j_P \succ_{n+1} k_1 \succ_{n+1} k_2 \succ_{n+1} k_3 \succ_{n+1} \cdots \succ_{n+1} k_{n-1} \succ_{n+1} k_n \succ_{n+1} i \cdots$$

Recall that it is wlog to assume that there are at least P lurked objects at $h^*_{n'}$ for each n'. We claim further that in this case, there are exactly P lurked objects at $h^*_{n'}$ for each n'. For $n' = 1$, this follows from the fact that i and k_1 tie. For $n' > 1$, the next ordered agent is k_1. So, if there were $p > P$ lurked objects at $h^*_{n'}$, the $(p+1)^{th}$ lurked object would have to x_{k_1}, which contradicts the supposition of the lemma. Therefore, for all $n' = 1, \ldots, n+1$, at $h^*_{n'}$ in game $\Gamma_{n'}$, there are exactly P lurked objects, and by definition, these must be $x_{j_1}, \ldots, x_{j_P}$, in this order.

Next, notice that for all $n' \geq 2$, since there are exactly P lurked objects at $h^*_{n'}$, the set of agents coded in step 1 of $\Gamma_{n'}$ must be $j_1, \ldots, j_P, k_1$. In particular, this is true for Γ_2 and Γ_n, and since $\succ_2$ is equivalent to $\succ_n$ up to agent k_1, by Lemma B.4, all of these agents are in the same roles in both games, and $h^{1*}_n = h^*_2$. By Condition 2, there is some agent ℓ such that ℓ is an active non-lurker that does not move at h^*_2 and $x_i \in C^{\subsetneq}_\ell(h^*_2)$. Since $h^{1*}_n = h^*_2$, we have that in Γ_n, there is some agent ℓ' that is an active non-lurker at h^{1*}_n and that does not move at h^{1*}_n and $x_i \in C^{\subsetneq}_\ell(h^{1*}_n)$. Further, $Q \geq 2$. Thus, all of the conditions of Lemma B.19 are satisfied, and we conclude that i must tie with some agent k_{n+1} in $\succ_{n+1}$.

We complete the proof of Lemma B.17 by showing that at least one of Condition 2 or Condition 2′ must hold. This assertion is proven as Lemma B.21 below. ■

Lemma B.21. *Assume that there are three codings:*

$$j_1 \cdots j_P \succ_A \{i, k\} \succ_A \cdots$$
$$j_1 \cdots j_P \succ_B i \succ_B \cdots$$
$$j_1 \cdots j_P \succ_C k \succ_C \cdots$$

such that:

- *At each of h^*_A, h^*_B, h^*_C, the objects $x_{j_1}, \ldots, x_{j_P}$ are all lurked, in this order, and*
- *Neither x_i nor x_k are the $(P+1)^{th}$ lurked object on the initial passing path of the game.*

Then, one of the following conditions must hold:

Condition (B): In Γ_B, at h^*_B there is an active non-lurker ℓ such that ℓ does not move at h^*_B and $x_k \in C^{\subsetneq}_{\ell}(h^*_B)$.

Condition (C): In Γ_C, at h^*_C, there is an active non-lurker ℓ such that ℓ does not move at h^*_C and $x_i \in C^{\subsetneq}_{\ell}(h^*_C)$.

Proof of Lemma B.21. First, notice that in each of the games, there must be exactly P lurkers at h^*_γ for $\gamma = A, B, C$. It is a presumption of the lemma that there are at least P lurkers. To see that there are at most P lurkers, notice that, for Γ_A, this holds because i and k tie. In Γ_B, it holds because x_i is not the next lurked object along the initial passing path, and thus, x_i must be the first—and since there is no tie, only—unlurked object that is coded in step 1. The same applies to Γ_C. Therefore, in Γ_A, there are exactly $P+2$ agents coded in step 1, while in Γ_B and Γ_C, there are exactly $P+1$ agents coded in step 1.

In Γ_A, at h^*_A, there are P active lurker roles and two active non-lurker roles. The objects $x_{j_1}, \ldots, x_{j_P}$ are lurked, and x_i and x_k are unlurked. Let s be the active non-lurker role that moves at h^*_A, and s' the role of the other active non-lurker. One of x_i or x_k must be the first unlurked object that is clinched in step 1 of the coding algorithm, either at h^*_A itself, or in the chain of assignments that follows. Assume it is x_i (a symmetric argument works if it is x_k). This implies that $x_i \in C^{\subsetneq}_{s'}(h^*_A)$, and $\sigma_A(s') = k$. There are two cases, depending on who is in role s.

Case 1: $\sigma_A(s) = j_p$ for some p. Agent j_p must be clinching a lurked object at h^*_A, which implies that h^*_A is the terminating history, and s is the terminator role. This means that s' is *not* the terminator role, and so $x_k \notin C^{\subseteq}_{s'}(h^*_A)$; indeed, if this were true, then x_k would have clinched it in Γ_A, because it is her favorite unlurked object and only unlurked objects are possible for a non-lurker who is not the terminator (Lemma B.12(iv)). It also means that agent i must be a lurker for some object $x_{j_{\bar{p}}}$, and thus, agent i strictly prefers $x_{j_{\bar{p}}}$ to x_i.

Now, consider game Γ_C. The agents coded in step 1 of Γ_C are $j_1, \ldots, j_P, k$, and so it must be one of these agents that moves at h^*_C.

Subcase 1.1: The agent that clinches at h^*_C is some $j_{p'}$. Here, h^*_C must also be the terminating history, and so $\sigma_C(s) = x_{j_{p'}}$ and $h^*_A = h^*_C$. Since k is coded in step 1, she must then be a lurker, and so there is some other agent $\ell \neq j_1, \ldots, j_P, k$ such that $\sigma_C(s') = \ell$. We claim that $\ell \neq i$. Indeed, if $\ell = i$, then there is some history $h' \subsetneq h^*_C$ such that $x_i \in C_i(h')$. Since s' is not the terminator role, only unlurked objects are possible for i in Γ_C, and since x_i is her top unlurked object, she would clinch at h', a contradiction. Therefore, $\sigma_C(s') = \ell \neq i$, and Condition (C) holds.

Subcase 1.2: Agent k clinches at h^*_C in Γ_C. Here, we have $\sigma_C(s) = k$, because, as we saw above, $x_k \notin C^{\subseteq}_{s'}(h^*_A)$ and h^*_A is the terminating history, so $h^*_C \subseteq h^*_A$. Let $h' \subsetneq h^*_A$ be the history at which role s' is offered to clinch x_i.

If $h^*_C \supsetneq h'$, then, by similar logic to subcase 1.1, $\sigma_C(s') = \ell$ for some $\ell \neq j_1, \dots, j_P, k, i$, and Condition (C) holds.

Finally, consider $h^*_C \subsetneq h'$.[95] In Γ_B, since there are exactly $P+1$ agents coded in step 1, x_i is the first (and only) unlurked object that is clinched, and since there is no tie, it has not been offered to another active non-lurker. This implies that $h^*_B \subseteq \tilde{h} \subsetneq h^*_A$. Since h^*_B is not the terminating history, it must be an unlurked object that is clinched, and therefore, it must be i that clinches x_i. If $\sigma_B(s) = i$, then i is in the terminator role, and would not clinch x_i first at h^*_B (recall that she prefers $x_{j_{\bar{p}}}$ to x_i). Thus, it must be that $\sigma_B(s') = i$, and i clinches x_i at h^*_B. If $h^*_B \subsetneq h^*_C$, then by similar logic to the above, Condition (C) holds. If $h^*_C \subsetneq h^*_B$, then $x_k \in C^{\subsetneq}_s(h^*_B)$ for the agent in role s. Notice that $\sigma_B(s) \neq k$, because if so, then k has the same roles in Γ_B and Γ_C, and so would clinch at $h^*_C \subsetneq h^*_B$ in Γ_B, a contradiction. It is also immediate that $\sigma_B(s) \neq j_1, \dots, j_P$, since they must be in the lurker roles for their respective objects. Thus, $\sigma_B(s) = \ell$ for some $\ell \neq j_1, \dots, j_P, i, k$, and Condition (B) holds.

Case 2: $\sigma_A(s) = i$**.** We once again have that role s' is not the terminator role,[96] and so, as in Case 1, $x_k \notin C^{\subseteq}_{s'}(h^*_A)$. Once again, consider game Γ_C. As in Case 1, there are two subcases.

Subcase 2.1: The agent that clinches at h^*_C in Γ_C is some $j_{p'}$. Here, $j_{p'}$ clinches a lurked object at h^*_C, and so h^*_C is the terminating history. This implies that $h^*_A \subseteq h^*_C$, and $\sigma_C(s) = j_{p'}$. But then, notice that the agent in role s' is an active non-lurker at h^*_C that does not move at h^*_C, and $x_i \in C^{\subsetneq}_{s'}(h^*_C)$. Since this agent is not coded in step 1, we know that $\sigma_C(s') \neq j_1, \dots, j_P, k$. If $\sigma_C(s') = i$, then i is offered to clinch x_i at some $h' \subsetneq h^*_C$, and since s' is not the terminator role, only unlurked objects are possible for her, and therefore, since x_i is i's top object, she would clinch at h', a contradiction. Thus, $\sigma_C(s') = \ell$ for some $\ell \neq j_1, \dots, j_P, i, k$, and $x_i \in C^{\subsetneq}_{\ell}(h^*_C)$, i.e., Condition (C) holds.

Subcase 2.2: The agent that clinches at h^*_C in Γ_C is k. Since k clinches first, and x_k is unlurked, all lurked objects are immediately assigned to their lurkers, which implies that j_p is in the lurker role for x_{j_p} for all $p = 1, \dots, P$.

If $h^*_A \subseteq h^*_C$, then, at h^*_C, there are two active non-lurkers, $\sigma_C(s)$ and $\sigma_C(s')$, and both have been offered x_i. One of these must be k. If $\sigma_C(s') = k$, then notice that $\sigma_C(s) \neq i$, because if $\sigma_C(s) = i$, then i is in the same role in Γ_A and Γ_C, and would clinch at h^*_A in Γ_C, which contradicts that k clinches first in Γ_C. Thus, $\sigma_C(s) = \ell \neq i$. If $\sigma_C(s) = k$, then if $\sigma_C(s') = i$, then i is in the non-terminator role, and $x_i \in C_i(\tilde{h})$ for some $\tilde{h} \subsetneq h^*_A \subseteq h^*_C$, and since x_i is i's favorite unlurked object, she will clinch it at $\tilde{h}$, a contradiction. Therefore, in

[95]Note that $h^*_C = h'$ is ruled out because role s' moves at h', while role s moves at h^*_C.
[96]This follows from Lemma B.12.

either case, there is some agent $\ell \neq j_1, \ldots, j_P, i, k$ such that $x_i \in C_\ell^{\subsetneq}(h_C^*)$, and Condition (C) holds.

It remains to consider $h_C^* \subsetneq h_A^*$. Here, we must have $\sigma_C(s) = k$, because if $\sigma_C(s') = k$, then as we showed above, $x_k \notin C_{s'}^{\subsetneq}(h_A^*)$, which contradicts that k clinches at h_C^*. Now, consider Γ_B. In Γ_B, since there are exactly $P + 1$ agents coded in step 1, x_i is the first (and only) unlurked object that is clinched, and the agents coded in step 1 are $j_1, \ldots, j_P, i$.

If $h_B^* \subseteq h_C^*$, then, $h_B^* \subsetneq h_A^*$, and h_B^* is not the terminating history. Thus, in Γ_B, agent i must move at h_B^* and clinch x_i. This implies that $\sigma_B(s') = i$, because if $\sigma_B(s) = i$, then i has the same role in Γ_A and Γ_B and clinches at both h_B^* and h_A^*, which contradicts that $h_B^* \subsetneq h_A^*$. Further, this means that $h_B^* \neq h_C^*$, because role s moves at h_C^* and role s' moves at h_B^*. Thus, at h_C^* in Γ_C, we have $x_i \in C_{s'}^{\subsetneq}(h_C^*)$. We cannot have $\sigma_C(s') = i$, because i would clinch at h_B^* in Γ_C, a contradiction. Therefore, $\sigma_C(s') = \ell$ for some $\ell \neq j_1, \ldots, j_P, i, k$ and $x_i \in C_\ell^{\subsetneq}(h_C^*)$, and thus, Condition (C) holds.

If $h_C^* \subsetneq h_B^*$, then if some $j_{p'}$ clinches at h_B^* in Γ_B, then h_B^* is the terminating history, and $h_A^* \subseteq h_B^*$. But then, there is an active non-lurker—the agent $\sigma_B(s')$—that has been offered to clinch x_i prior to h_B^*, and so i would at best tie with this agent in $\succ_B$, a contradiction. Thus, it must be i that clinches at h_B^* in Γ_B, which implies that $\sigma_B^{-1}(i) = s$ or s'. If $\sigma_B^{-1}(i) = s$, then i has the same roles in Γ_A and Γ_B, and so $h_A^* = h_B^*$, and i would tie with the agent in role s' in $\succ_B$, a contradiction. Thus, $\sigma_B(s') = i$. This means that h_C^* and h_B^* are controlled by different roles, and $x_k \in C_s^{\subsetneq}(h_B^*)$. Finally, we cannot have $\sigma_B(s) = k$, because then k is in the same role as Γ_C, and would clinch at $h_C^* \subsetneq h_B^*$ in Γ_B. So, we must have $\sigma_B(s) = \ell$ for some $\ell \neq j_1, \ldots, j_P, i, k$, and in Γ_B, $x_k \in C_\ell^{\subsetneq}(h_B^*)$. Therefore, Condition (B) holds.

Finally, notice that all of this was done under the assumption that x_i was the first unlurked object that was clinched in step 1 of the coding algorithm in Γ_A. The other possibility is that this object is x_k. However, everything is symmetric, and so the exact same argument, swapping the i and k, shows that either Condition (B) or Condition (C) must hold in this case as well. ∎