DISCUSSION PAPER SERIES

DP13946

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INTERNATIONAL TRADE AND REGIONAL ECONOMICS

ISSN 0265-8003

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Discussion Paper DP13946
Published 21 August 2019
Submitted 16 August 2019

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www.cepr.org

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JEL Classification: D11, D43, D62

Keywords: Intersectoral linkages, Sectoral Shocks, welfare changes, Complementarity, Substitutability

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Acknowledgements
We thank Yasusada Murata, Jacques-François Thisse and seminar and conference participants at HSE Moscow, HSE Saint Petersburg, Ifo, Aarhus, EEA-ESEM 2017 in Lisbon, SAET 2019 in Ischia, and the 2019 Australasian economic theory workshop in Sydney for very helpful comments and suggestions. This article was prepared within the framework of the HSE University Basic Research Program and funded by the Russian Academic Excellence Project '5-100'. Behrens gratefully acknowledge financial support from the CRC Program of the Social Sciences and Humanities Research Council (SSHRC) of Canada for the funding of the Canada Research Chair in Regional Impacts of Globalization.

Intersectoral linkages: Good shocks, bad outcomes?

Kristian Behrens* Sergey Kichko† Philip Ushchev‡

August 19, 2019

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We analyze multisector models with endogenous product variety and derive general results on the magnitude of welfare changes due to sector-specific price shocks. Intersectoral linkages magnify or dampen these shocks, depending on complementarity or substitutability in consumers' preferences. Under the widely used combination of Cobb-Douglas-CES utilities and monopolistic competition, intersectoral linkages disappear. This does not hold with more general preferences or market structures, where sector-specific price shocks that are a priori welfare improving can turn out to be welfare worsening economy-wide. We illustrate this result with several examples, in particular where one sector is 'granular' and the other is monopolistically competitive.

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1 Introduction

Assessing the welfare effects of economic shocks is of paramount importance in many applied fields of economics. While partial equilibrium approaches provide valuable insights into specific mechanisms, one important issue is to understand whether sector-specific shocks—i.e., shocks that directly affect only a single sector—are magnified or dampened in a general equilibrium context. How do, for example, productivity-improving shocks in one industry affect overall welfare when sectors are linked? Are there always welfare gains? And which properties of consumer preferences exacerbate or attenuate the magnitude of these shocks? Our key objective is to answer these questions and to better understand how intra-sectoral shocks translate into aggregate welfare changes in multisector models with imperfect competition.

Although multisector general equilibrium models are a staple in many applied fields of economics, little is known in general on their welfare properties. The general theory of the second-best (Lipsey and Lancaster, 1956) tells us that a potentially welfare improving shock to one sector need not be welfare improving when there are multiple distortions. We do not know much beyond that. In particular, we do not know to what extent the specific modeling choices in the literature—which draw heavily on particular preferences, like CES or Cobb-Douglas, and market structures, like monopolistic or oligopolistic competition—affect the normative properties of the models that we use. This is problematic since many of the welfare statements we make are likely to substantially hinge—quantitatively, as well as qualitatively—on these modeling choices.

The aim of this paper is twofold. First, we propose a fairly general multisector model that nests many of the approaches used in the applied literature. Using this model, we derive a 'welfare multiplier'—a statistic that tells us which share of the direct gains of the positive sector-specific shock materializes in general equilibrium—and establish precise conditions under which welfare-improving intra-sectoral shocks are magnified or dampenend in the aggregate. In a two-sector economy, we show that the magnitude and the sign of the welfare multiplier crucially hinge on complementarity or substitutability between goods in consumers' preferences. If a shock affects the sector that has a less elastic price index, when two goods are gross complements the welfare effects are magnified ceteris paribus; whereas they are dampened if goods are gross substitutes. Cobb-Douglas preferences are a limiting case for which a welfare-improving shock in one sector is always welfare improving in general and of the same magnitude as the sectoral shock itself.

Second, we show that a specific combination of assumptions that is widely used in the literature—namely CES preferences and a monopolistically competitive market structure—give rise to very specific results. In particular, the combination of these assumptions guarantees that a positive sector-specific shock always translates into aggregate welfare gains under 'reasonable' assumptions on the elasticities of substitution of different tiers of preferences. To fix ideas, in a two-sector model where both sectors are of the CES-monopolistic competition type, a welfare-improving shock to prices in one sector always has positive aggregate welfare effects when the cross-sectoral elasticity of substitution is lower than the elasticities of substitution within sectors. This result is, however, not robust as there exists a large class of homothetic preferences and different market structures under which it need not hold. We illustrate this finding with the help of a simple example and show that even small departures from the CES-monopolistic competition paradigm can substantially alter welfare statements with multiple sectors. The intuition is that a positive shock to one sector—followed by a reallocation of budget towards that sector—forces firms out of other sectors. If that effect is strong enough, welfare can decrease because the negative effects of higher prices (markups) and less variety in the sectors not exposed to the shock dominate the positive price and variety effects in the sector subject to the shock. This is, for example, likely to occur when the sector not exposed to the shock is oligopolistically competitive and sufficiently 'granular' (i.e., there are a small number of competing firms).[1] The bottom line is that both quantitative and qualitative welfare statements based on multisector CES models with monopolistic competition should be taken with a grain of salt and considered with some degree of caution.

Our paper is mainly linked to two strands of the literature. First, it is linked to the emerging literature on imperfect markets with multiple industries (e.g., Epifani and Gancia, 2011; d'Aspremont and Dos Santos Ferreira, 2016; Behrens, Mion, Murata, and Suedekum, 2016). It complements that literature by deriving sharper conditions on how intersectoral effects affect welfare changes due to sector-specific shocks. It also complements contributions that deal with the positive analysis of multisector monopolistic competition models (e.g., Matsuyama, 1995).[2]

Second, it is linked to the international trade literature with multiple industries and product differentiation. While some of that literature deals with the positive aspects only—e.g., trade elasticities in Ossa (2015) or changes in productivity cutoffs in Segerstrom and Sugita (2015)—another part of that literature sets out to explicitly quantify the welfare effects (e.g., Hsieh, Li,

[1]The mechanism is different from that in the immiserizing growth literature (Johnson, 1955; Bhagwati, 1958) and the 'Dutch desease' (Corden and Neary, 1982). Our effects are not driven by adverse changes in the terms of trade in international markets but go through consumer preferences only. Our findings are reminiscent of those in the normative trade literature where welfare losses can occur when product selection is not optimal (see, e.g., Dixit and Norman, 1980, Ch.9). They are also related to the literature on trade in 'cultural goods', where the point is made that specificities in consumer preferences (e.g., network externalities) can lead to losses from trade (e.g., Francois and van Ypersele, 2002; Janeba, 2007). We do not require any network externalities.

[2]We may view our paper as a 'normative companion' of Matsuyama's (1995) positive analysis of monopolistic competition models. While Matsuyama (1995) does not directly investigate welfare, he analyzes two-sector models and discusses some efficiency considerations.

Ossa, and Yang, 2016).

The remainder of the paper is organized as follows. Secction 2 develops two motivating examples which illustrate how good shocks can lead to different welfare consequences depending on arguably small variations in modeling choices. Section 3 presents our general multisector model, establishes the equilibrium, and derives general results on welfare changes due to sector-specific shocks. To obtain sharper results, Section 4 discusses the case of two sectors. Section 5 presents some applications of our general results. Finally, Section 6 concludes. Proofs and additional material are relegated to an extensive set of appendices.

2 Motivating examples

We illustrate the key ideas of our subsequent analysis using two simple examples. In both examples, there are two sectors—the first being perfectly competitive and the second being imperfectly competitive. In both cases, we consider a productivity-enhancing shock specific to the perfectly competitive sector. The difference between the two examples lies solely in how the market structure in the imperfectly competitive sector is specified.

As we will show, the 'good shock' has qualitatively different welfare consequences in the two examples, despite the apparent similarity of the settings. This suggests that market structure matters very much for welfare effects in multi-sector models, and that this goes beyond a simple 'perfectly–imperfectly competitive' dichotomy. A better understanding of the role of market structure for applied welfare analysis in multisector models thus seems important. Providing elements of answer is the key objective of our analysis.

Common elements. We first describe what both examples have in common. There are two sectors and L identical consumers. Without loss of generality, per-capita income is normalized to one. Preferences are defined over bundles of products supplied by the two sectors. We assume homothetic subutilities so that well-defined price indices P_1 and P_2 exist in both sectors. The elasticity of substitution is constant and given by $\gamma > 1$, i.e., goods produced by the two sectors are *gross substitutes*. It is well known that the indirect utility is then $V \equiv (P_1^{1-\gamma} + P_2^{1-\gamma})^{-\frac{1}{1-\gamma}}$.

Sector 1 is perfectly competitive. All firms in that sector share the same technology, with constant marginal cost $1/\theta$, where $\theta > 0$ is a productivity parameter. We focus on the welfare consequences of a productivity-enhancing shock in sector 1, i.e., an increase in θ. Sector 2 is imperfectly competitive, and we will make precise the market structure below.

Let $\alpha \geq 0$ and $1 - \alpha \geq 0$ denote consumer's budget shares allocated to sectors 1 and 2. Due to perfect competition in sector 1, the equilibrium price level P_1 is independent of the budget allocation and is given by

$$\widehat{P}_1(\theta) = 1/\theta. \tag{1}$$

In contrast, the utility-maximizing budget allocation $a(\cdot,\cdot)$ depends on P_1 and P_2 as follows:

$$a(P_1, P_2) = \frac{P_1^{1-\gamma}}{P_1^{1-\gamma} + P_2^{1-\gamma}}, \qquad 1 - a(P_1, P_2) = \frac{P_2^{1-\gamma}}{P_1^{1-\gamma} + P_2^{1-\gamma}}. \tag{2}$$

In what follows, we will often derive expressions *conditional on some given budget allocation* $(\alpha, 1-\alpha)$. The latter does not need to maximize utility off equilibrium. Distinguishing between α and $a(P_1, P_2)$ serves two purposes. First, it captures the fact that firms take consumers' budget shares as given. Second, as we will see in Section 3, this way of proceeding allows for a clear decomposition of the aggregate consequences of sector-specific shocks.

Let us now show in the simplest possible way that the welfare effects of a rise in θ can go either way. We relegate all derivations to Appendix 6.

Example 1: CES monopolistic competition. Assume that sector 2 is monopolistically competitive. The elasticity of substitution σ across varieties in that sector is constant and such that $\sigma > \gamma$, i.e., any two varieties of good 2 are better substitutes than goods 1 and 2. In the monopolistically competitive sector, the price index depends on that allocaton and is given by (up to a positive constant, see Appendix A.1):

$$\widehat{P}_2(1-\alpha) = (1-\alpha)^{\frac{1}{1-\sigma}}. \tag{3}$$

Plugging (1) and (3) into (2), the equilibrium budget share $\alpha^*(\theta)$ is a solution to:

$$\alpha = \frac{\theta^{\gamma-1}}{\theta^{\gamma-1} + (1-\alpha)^{\frac{\gamma-1}{\sigma-1}}}. \tag{4}$$

This equation has a unique stable solution and the share of budget allocated to sector 1 naturally increases as P_1 falls in response to an increase in productivity.

Inserting (1)–(4) into the indirect utility yields $V^*(\theta) = [1-\alpha^*(\theta)]^{\frac{\sigma-\gamma}{1-\sigma}}$. Since $\sigma > \gamma > 1$, $V^*(\theta)$ increases with $\alpha^*(\theta)$, which, in turn, increases in θ. Thus, $\mathrm{d}V^*/\mathrm{d}\theta > 0$. In words, a 'good' sector-specific shock yields economy-wide welfare gains—a somewhat expected and reassuring result.[3] However, this result is not robust. We now show that even a slight change in assumptions may dramatically change the outcomes.

[3] If we assume that $\sigma < \gamma$, an increase in θ may lead to welfare losses (see, e.g., Matsuyama, 1995). This assumption is not overly realistic. It is also not in line with the existing literature that naturally assumes that the elasticity of substitution is—almost by definition of a sector—larger within than across sectors.

Example 2: Cournot competition. Assume that sector 2 produces a homogeneous good in a Cournot oligopoly, the number of firms being determined by free entry. Let $c > 0$ be the constant marginal cost shared by all firms while $f > 0$ is the fixed cost required to set up a firm in sector 2.

The price level in sector 1 is still given by (1), whereas the price index in sector 2 now depends on the budget allocaton and is given by (see Appendix A.2 for the derivation):

$$\widehat{P}_2(1-\alpha) = \frac{c}{1-(f/L)/\sqrt{1-\alpha}}. \tag{5}$$

Sector 2 operates only if $1-\alpha > (f/L)^2$, i.e. either the market is suficiently large, or entry into sector 2 is suficiently free, or both. In that case, (5) decreases in $1-\alpha$.[4] Using (2), (1) and (5)—and assuming that sector 2 operates—the equilibrium budget share $\alpha^*(\theta)$ is a solution to

$$\alpha = \frac{(c\theta)^{\gamma-1}}{(c\theta)^{\gamma-1} + \left(1-(f/L)/\sqrt{1-\alpha}\right)^{\gamma-1}}, \tag{6}$$

and there is a unique stable interior equilibrium $\alpha^* \in (0,1)$ associated with (6).[5] Provided sector 1 is not too productive, i.e., $\theta < \overline{\theta}$, the budget share of sector 1 increases with productivity improvements: $\mathrm{d}\alpha^*/\mathrm{d}\theta > 0$. Thus, the behavior of α^* is as in our first example.

However, the economy-wide welfare effect may be the opposite. To see this, note that the expression for the welfare level V is given by

$$V = \frac{1}{c}\left(1 - \frac{f/L}{\sqrt{1-\alpha}}\right)(1-\alpha)^{\frac{1}{1-\gamma}},$$

which increases with $1-\alpha$ (hence decreases with α) when the following inequality holds:

$$2\sqrt{1-\alpha} < (1+\gamma)\frac{f}{L}.$$

Hence, when the oligopolistic sector is small enough, welfare decreases with a productivity improvement in the competitive sector: $\mathrm{d}V^* < 0$.

The intuition for this result is as follows. Due to its oligopolistic nature and small size, there are only a 'small' number of firms operating in sector 2. A positive shock to sector 1—followed by a reallocation of budget towards that sector, i.e., an increase in α^*—then forces firms out of sector 2. This in turn leads to an increase in prices and markups as fewer firms now compete. Indeed, $\widehat{P}_2$ is increasing in α, as can be seen from (5). Hence, positive shocks to one sector can exacerbate distortions in other sectors. These general equilibrium effects

[4] When $1-\alpha \leq f/L$, we set $P_2 = \infty$ as sector 2 ceases to operate.

[5] We relegate a comprehensive analysis of all possible equilibria—including the corner equilibrium where sector 2 does not produce—to Appendix A. We also provide the proofs of all statements there.

are often disregarded—by, e.g., using Cobb-Douglas preferences—and better understanding under which conditions the propagation of good shocks through the economy can lead to bad outcomes appears to be an important issue.

Note that these results are driven by differences in imperfectly competitive market structures. Example 1 predicts unambiguously welfare gains, whereas example 2 shows that a good shock may lead to bad outcomes. The main objective in the remainder of this paper is too better understand what drives such differences and to highlight the implications of specific modeling choices—monopolistic competition, oligopoly, substitutes vs complements—for qualitative and quantitative welfare statements in applied economic analysis.

3 Model and general results

Consider an economy with S sectors, each of them producing a differentiated good. At this stage, we are fully agnostic about the market structure and technology in each sector, except that we assume them to be compatible with free entry.[6] We assume that there is a unit mass of identical consumers who have *weakly separable* preferences across the differentiated goods (Blackorby, Primont, and Russell, 1978; Varian, 1983). This means that the utility function can be represented as $U(u_1, u_2, \ldots, u_S)$, where $U : \mathbb{R}^S_+ \to \mathbb{R}$ is an *upper-tier* utility and u_s for $s = 1, 2, \ldots, S$ are *subutilities* defined over the set of varieties produced in each sector s. We assume that U is continuous, strictly increasing, and strictly quasi-concave. As to the subutilities, we assume that they are homothetic. Hence, the indirect utility u^*_s is a special case of the Gorman polar form and can therefore be represented by an ideal price index P_s (see Gorman, 1961; Jehle and Reny, 2011). In what follows, we let $\mathbf{P} \equiv (P_1, P_2, \ldots, P_S)$ denote the vector of the S sectoral price indices. When required we also adopt standard notation and let $\mathbf{P}_{-i} \equiv (P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_S)$ denote the vector of the sectoral price indices other than sector i.

Let Δ_{S-1} be the $(S-1)$-dimensional simplex and $\mathbf{a} : \mathbf{P} \in \mathbb{R}^S_+ \to \Delta_{S-1}$ be the *budget-share mapping*. The s-th component of $\mathbf{a}(\mathbf{P})$, denoted by $\alpha_s \equiv a_s(\mathbf{P})$, is the *budget share* of sector s for a given vector of price indices.[7] As in Matsuyama and Ushchev (2017), the mapping $\mathbf{a}(\cdot)$ summarizes all the relevant information about the upper-tier utility U, i.e., we can determine equilibrium and sign welfare changes without knowing U.[8] We denote the budget share

[6]We will provide more structure when required in specific applications in Section 5.

[7]The budget-share mapping is obtained by solving the consumer's first-stage budgeting problem. Normalizing income to one without loss of generality, this problem can be equivalently reformulated as follows: $\mathbf{a}(\mathbf{P}) \equiv \arg\max_{\boldsymbol{\alpha} \in \Delta_{S-1}} U\left(\alpha^*_1/P^*_1, \alpha^*_2/P^*_2, \ldots, \alpha^*_S/P^*_S\right)$, where α^*_s/P^*_s is the representation of the indirect utility in sector s.

[8]When $S = 2$ and the goods are gross substitutes, every $\mathbf{a}(\cdot)$ such that $\partial a_s/\partial P_s \leq 0$ for $s = 1, 2$ actually defines a well-behaved utility function U, i.e., it is a true primitive of the model. See Appendix A for the proof.

mapping by $\mathbf{a}(\cdot)$ and specific budget shares—values that the mapping takes—by $\boldsymbol{\alpha}$.

Finally, let $\theta \in \Theta$ be a scalar parameter that has a direct impact *only on prices in sector* $s = 1$, whereas it affects the other sectors $s = 2, 3, \ldots, S$ only indirectly. We henceforth refer to changes in θ as 'shocks'. The set Θ of possible shocks is assumed to be a non-empty open interval.

3.1 Equilibrium

We can first determine an *intrasectoral equilibrium* in sector s for a given α_s and $\boldsymbol{\alpha}_{-s}$.[9] Provided such an equilibrium exists, we obtain price indices, $\widehat{P}_1(\alpha_1, \theta)$ and $\widehat{P}_s(\alpha_s)$ for $s = 2, 3, \ldots, S$ that are functions of the sectoral budget share α_s (and θ in sector 1). We need to impose some minimum assumptions about how $\widehat{P}_1(\alpha_1, \theta)$ and $\widehat{P}_s(\alpha_s)$ vary with α_s and θ. In what follows, we assume that:

(A1). The direct effect of an increase in θ on sector 1's price index is given by: $\partial \widehat{P}_1 / \partial \theta < 0$.

Assumption (**A1**) states that—conditional on the budget structure—an increase in θ makes the products of the first sector less expensive. In other words, the direct effect of a larger θ is welfare improving.

(A2). For each sector $s = 1, 2, \ldots, S$, the direct effect of an increase in the budget share α_s reduces the sectoral price index: $\partial \widehat{P}_s / \partial \alpha_s < 0$.

Assumption (**A2**) states that prices fall in a sector as the budget share allocated to that sector increases. The intuition is that when consumers spend more on the goods produced by a particular sector, this invites entry of new firms and eventually reduces the price level. Let us emphasize that, at this level of generality, (**A2**) is an ad hoc assumption. Yet, one can show that (**A2**) holds for free-entry market structures such as monopolistic competition and differentiated oligopoly with both increasing and constant elasticity of demand, as well as with decreasing elasticity of demand when anti-competitive effects are mild. These assumptions are in line with the empirical evidence that documents the existence of pro-competitive effects (e.g., Bellone et al., 2016). Note that the price indices in our examples of Section 2 satisfy the two assumption (**A1**) and (**A2**). We also provide specific examples that micro-found such a behavior of the price indices in Section 5.

[9] Depending on the context we consider, the intrasectoral equilibrium may be: (i) a monopolistically competitive equilibrium; (ii) a Cournot-Nash equilibrium with entry; (iii) a Bertrand-Nash equilibrium with entry; or (iv) a more complex equilibrium concept, e.g., with varying toughness of competition à la d'Aspremont and Dos Santos Ferreira (2009).

We are now equipped to formally define the *equilibrium of the economy*.

Definition (Equilibrium). *An equilibrium is a bundle* $(\mathbf{P}^*, \boldsymbol{\alpha}^*) \in \mathbb{R}^S_+ \times \Delta_{S-1}$ *that satisfies the following conditions*:

$$P_1^* = \widehat{P}_1(\alpha_1^*, \theta); \tag{7}$$

$$P_s^* = \widehat{P}_s(\alpha_s^*), \quad s = 2, 3, \ldots, S; \tag{8}$$

$$\boldsymbol{\alpha}^* = \mathbf{a}(\mathbf{P}^*). \tag{9}$$

Conditions (7) and (8) impose consistency: the equilibrium price levels are the intrasectoral equilibrium price indices evaluated at the equilibrium budget shares. Condition (9) states that the budget structure $\boldsymbol{\alpha}^*$ is consistent with rational consumer behavior as summarized by the budget-share mapping $\mathbf{a}(\cdot)$. To alleviate notation, let

$$\mathbf{b}(\boldsymbol{\alpha}, \theta) \equiv \mathbf{a}(\widehat{P}_1(\boldsymbol{\alpha}, \theta), \widehat{\mathbf{P}}_{-1}(\boldsymbol{\alpha})). \tag{10}$$

We can now show that an equilibrium exists.

Proposition 1 (Existence of equilibrium) *When (**A1**)–(**A2**) hold, then:*

(i) an equilibrium $(\mathbf{P}^*, \boldsymbol{\alpha}^*)$ *as defined by (7)–(9) always exists;*

(ii) there is a one-to-one correspondence between the set of equilibria and the set of solutions to the fixed point condition

$$\boldsymbol{\alpha} = \mathbf{b}(\boldsymbol{\alpha}, \theta). \tag{11}$$

Proof. To prove the result, we proceed in reverse order. First, part (ii) is obtained by substituting $\mathbf{P}^*$, given by (7)–(8), into (9). To prove part (i), observe that for any $\theta \in \Theta$ the mapping $\mathbf{b}(\cdot, \theta)$ is continuous and maps Δ_{S-1} into itself (by **(A1)–(A2)**, the price indices are continuous in α and θ). Hence, it has at least one fixed point by Brouwer's Theorem, i.e., (11) has at least one solution $\boldsymbol{\alpha}^*$. Plugging $\boldsymbol{\alpha}^*$ into (7) and (8) yields $\mathbf{P}^*$. Finally, condition (9) is satisfied by the definition of $\mathbf{b}(\cdot, \theta)$ in (10) and because α^* solves the fixed-point condition (11). □

3.2 The welfare effects of a positive intrasectoral shock

We first derive general results on the welfare effects of a positive intrasectoral shock for the case of S sectors. Let $E(\mathbf{P}, U)$ be the expenditure function associated with the upper-tier utility $U(\cdot)$. Thus, welfare gains due to an increase in θ are equivalent to a decrease in the equilibrium value E^* of the expenditure function in response to changes $\mathrm{d}P_s^*$ of the sectoral price indices. By Shephard's lemma, we have:

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \sum_{s=1}^{S} u_s^* \frac{\mathrm{d}P_s^*}{\mathrm{d}\theta} = \sum_{s=1}^{S} \frac{\alpha_s^*}{P_s^*} \frac{\mathrm{d}P_s^*}{\mathrm{d}\theta}, \tag{12}$$

where we make use of the relationship $u_s^* = \alpha_s^*/P_s^*$ linking a homothetic utility to its ideal price index. Unless indicated otherwise, here and everywhere below all derivatives and elasticities are evaluated at equilibrium: $\boldsymbol{\alpha} = \boldsymbol{\alpha}^*$ and $\mathbf{P} = \mathbf{P}^*$. Differentiating (7)–(8) with respect to θ yields a linear system of equations in $\mathrm{d}\mathbf{P}^*/\mathrm{d}\theta$, which can be substituted into (12) to obtain after simplifications (see Appendix B for all derivations):

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\alpha_1^*}{P_1^*}\frac{\partial \widehat{P}_1}{\partial \theta} + \sum_{s=1}^{S} \mathcal{E}_{\alpha_s}(\widehat{P}_s)\,\frac{\mathrm{d}\alpha_s^*}{\mathrm{d}\theta}, \tag{13}$$

where $\mathcal{E}_{\alpha_s}(\widehat{P}_s) \equiv (\partial \widehat{P}_s/\partial \alpha_s)/(\alpha_s/\widehat{P}_s)$ is the elasticity of the price index in sector s with respect to α_s. As budget shares sum to one, we have $\sum_s \mathrm{d}\alpha_s^*/\mathrm{d}\theta = 0$ and (13) becomes:

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\alpha_1^*}{P_1^*}\frac{\partial \widehat{P}_1}{\partial \theta} + \mathcal{S}, \quad \text{where} \quad \mathcal{S} \equiv \sum_{s=2}^{S} \left[\mathcal{E}_{\alpha_s}(\widehat{P}_s) - \mathcal{E}_{\alpha_1}(\widehat{P}_1)\right] \frac{\mathrm{d}\alpha_s^*}{\mathrm{d}\theta}. \tag{14}$$

The intuition behind the decomposition (14) is as follows. The first term in $\mathrm{d}E^*/\mathrm{d}\theta$ captures the within-sector welfare gains which occur in sector 1 as a result of an increase in θ. The impact of the redistribution of budget shares between sectors is captured by the second term $\mathcal{S}$ in $\mathrm{d}E^*/\mathrm{d}\theta$, which we refer to as the *welfare shifter*. The nature of this shifter can be understood by looking at its sth component. To fix ideas, assume that an increase in θ leads to an increase in the equilibrium budget share of sector $s \neq 1$, i.e., $\mathrm{d}\alpha_s^*/\mathrm{d}\theta > 0$. Then, the interaction between sectors 1 and s amplifies the welfare gains—the decrease in expenditure—originating in sector 1 if and only if the price index P_s is more sensitive to a change in the sectoral budget share than the price index P_1, i.e., $\left|\mathcal{E}_{\alpha_s}(\widehat{P}_s)\right| \geq \left|\mathcal{E}_{\alpha_1}(\widehat{P}_1)\right|$. Of course, the welfare gains originating in sector 1 are dampened if the reverse inequality holds. We can summarize these findings as follows:

	$\mathrm{d}\alpha_s^*/\mathrm{d}\theta > 0$	$\mathrm{d}\alpha_s^*/\mathrm{d}\theta < 0$
$\lvert\mathcal{E}_{\alpha_1}(\widehat{P}_1)\rvert > \lvert\mathcal{E}_{\alpha_s}(\widehat{P}_s)\rvert$	Dampened	Amplified
$\lvert\mathcal{E}_{\alpha_1}(\widehat{P}_1)\rvert < \lvert\mathcal{E}_{\alpha_s}(\widehat{P}_s)\rvert$	Amplified	Dampened

While expression (14) is useful to provide intuition, an alternative decomposition will prove more convenient. To obtain it, we compute the vector $\mathrm{d}\boldsymbol{\alpha}^*/\mathrm{d}\theta$ by differentiating (10)–(11) with respect to θ. Plugging the result into (13) we then obtain (see Appendix B for all derivations):

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\alpha_1^*}{P_1^*}\frac{\partial \widehat{P}_1}{\partial \theta} \cdot \mathcal{M}, \quad \text{where} \quad \mathcal{M} \equiv 1 + \sum_{r=1}^{S} \left[\sum_{s=1}^{S} \frac{\Delta_{rs}}{\Delta} \mathcal{E}_{\alpha_s}(\widehat{P}_s)\right] \frac{\alpha_r^*}{\alpha_1^*} \mathcal{E}_{P_1}(a_r), \tag{15}$$

where $\mathcal{E}_{P_1}(a_r)$ is the elasticity of the budget share function a_r of sector r with respect to the price index P_1 of sector 1. In equation (15), $\Delta \equiv \det\left(\mathbf{I} - \frac{\partial \mathbf{a}}{\partial \mathbf{P}}\frac{\partial \widehat{\mathbf{P}}}{\partial \boldsymbol{\alpha}}\right)$ is the determinant of the matrix of price-index elasticities, and Δ_{rs} is the algebraic complement of its rs-th entry. The decomposition (15) has a similar flavor to (14), with one important difference: the welfare multiplier $\mathcal{M}$ highlights the fact that the changes $\mathrm{d}\boldsymbol{\alpha}^*/\mathrm{d}\theta$ depend, in a rather complicated way, on the linkages in the demand system. The latter are captured both by the interplay between the elasticities $\mathcal{E}_{P_1}(a_r)$ of the budget-share functions with respect to the price index of the first sector; and by the sensitivity $\mathcal{E}_{\alpha_s}(\widehat{P}_s)$ of sectoral price indices to shifts in the budget structure.

The following proposition, which readily follows from (15), summarizes the role played by the welfare multiplier $\mathcal{M}$:

Proposition 2 (Welfare multiplier) *When (**A1**)–(**A2**) hold, then:*

(i) the interactions between sectors amplify (resp., dampen) the welfare effects of an increase in θ if and only if $\mathcal{M} > 1$ (resp., $\mathcal{M} < 1$);

(ii) an increase in θ leads to welfare gains (resp., losses) if and only if $\mathcal{M} > 0$ (resp., $\mathcal{M} < 0$);

(iii) the global welfare effect is always equivalent to the direct effect of an increase in θ (i.e., $\mathcal{M} \equiv 1$) if the upper-tier utility $U(\cdot)$ is Cobb-Douglas.

Proof. Part (i) follows immediately by observing that setting $\mathcal{M} = 1$ in (15) we obtain the welfare change in the one-sector economy. Observe that by (**A1**) we have

$$\text{sign}\left(\frac{\mathrm{d}E^*}{\mathrm{d}\theta}\right) = -\text{sign}(\mathcal{M}), \tag{16}$$

which proves part (ii). Finally, part (iii) follows from the fact that with Cobb-Douglas upper-tier utility ($U \equiv \prod_s u_s^{\beta_s}$, with $\beta_s > 0$ and $\sum_s \beta_s = 1$) the budget-share mapping is a constant function: $\mathbf{a}(\mathbf{P}) = (\beta_1, ..., \beta_n)$. Hence, $\mathrm{d}\alpha_s^*/\mathrm{d}\theta = 0$ for $s = 1, 2, \ldots, n$. This immediately implies that $\mathcal{M} = 1$ (and $\mathcal{S} = 0$). In other words, the welfare effect of a sectoral shock θ does not contain any between-sector component. $\square$

Proposition 2 shows that the sign of $\mathcal{M}$ is a *sufficient statistic* for determining whether a shock in θ gives rise to welfare gains or welfare losses. Furthermore, when $0 < \mathcal{M} < 1$, there are welfare gains but they are smaller than those in a single-sector economy or under a Cobb-Douglas upper-tier utility. The latter is a borderline case in which the direct welfare effect of a sectoral shock is equal to the welfare effect in the whole economy. This is obviously a very special case, and one has to keep in mind that multisector models that rely on Cobb-Douglas upper-tier utilities and homothetic subutilities—despite their widespread use—provide welfare statements that do not capture intersectoral effects.

4 Two-sector economy

With more than two sectors, $\mathcal{M}$ can be neither signed nor compared to 1 unambiguously. Sharper predictions can be made for the case of two sectors, which we hence study in the rest of the paper.[10] For the sake of convenience, we introduce the following notation: $a_1(P_1, P_2) \equiv a(P_1, P_2)$ and $\alpha_1 \equiv \alpha$, which implies $a_2(P_1, P_2) = 1 - a(P_1, P_2)$ and $\alpha_2 = 1 - \alpha$ and will sometimes simply refer to the goods produced by sectors 1 and 2 as 'goods 1 and 2'. Before starting our analysis, we establish the following lemma.

Lemma 1 *Let $a(P_1, P_2)$ be the budget share of good 1, and let $\gamma(P_1, P_2)$ be the elasticity of substitution between goods 1 and 2. Then, the following identity holds:*

$$\gamma(P_1, P_2) = 1 - \left[\mathcal{E}_{P_1}(a) + \mathcal{E}_{P_2}(1-a)\right]. \tag{17}$$

Proof. See Appendix C. □

We now first look at the comparative statics of an increase in θ. We then examine its welfare implications.

4.1 Comparative statics

The sign of the welfare shifter $\mathcal{S}$ in (14) depends on the direction in which the sectoral budget shares move in response to a shock. This motivates our attention to comparative statics of equilibrium in a two-sector economy. With $S = 2$, the fixed-point condition (11) becomes

$$\alpha = b(\alpha, \theta), \tag{18}$$

where $b(\cdot, \theta)$ is defined by

$$b(\alpha, \theta) \equiv a\left(\widehat{P}_1(\alpha, \theta), \widehat{P}_2(1-\alpha)\right). \tag{19}$$

By Proposition 1, an equilibrium exists, and every solution α^* to (18) uniquely defines an equilibrium. Thus, how α^* responds to an increase in θ is fully determined by the behavior of the function $b(\cdot, \theta)$. What determines its properties?

[10]The assumption $S = 2$ is not as restrictive as one may think. Indeed, the case of two sectors is isomorphic to a seemingly more general case when the set of sectors can be split into two groups, such that: (i) preferences for goods produced in each group of sectors are homothetic, so that one can construct an ideal price index per group; and (ii) the parameter θ only affects sectors in the first group. An additional advantage of studying the two-sector case is that we can prove that the budget share mapping is a primitive of the model in the case where goods are gross substitutes: any budget share function has a one-to-one mapping with a well-defined upper-tier utility function (see Appendix A).

To answer this question, recall the following standard definitions: two goods are called gross complements (resp., gross substitutes) if an increase in the price for one good leads to a decrease (resp., an increase) in the demand for the other good. Since u_s^* is the consumption aggregator in the case of homothetic preferences, the goods 1 and 2 are gross complements if and only if $\partial u_s^*/\partial P_r > 0$, $s, r = 1, 2$ and $s \neq r$. Otherwise, goods 1 and 2 are gross substitutes. Formally, these properties can be reformulated in terms of the budget-share elasticities:

$$\text{gross complements} \iff \mathcal{E}_{P_1}(a) > 0 \quad \text{and} \quad \mathcal{E}_{P_2}(1-a) > 0; \tag{20}$$

$$\text{gross substitutes} \iff \mathcal{E}_{P_1}(a) < 0 \quad \text{and} \quad \mathcal{E}_{P_2}(1-a) < 0. \tag{21}$$

As is well known—and as seen from (17)—under gross complementarity $\gamma(P_1, P_2) < 1$, whereas under gross substitutability $\gamma(P_1, P_2) > 1$. The properties of the b-locus can then be summarized as follows:

Lemma 2 (Comparative statics) *Assume that **(A1)–(A2)** hold. Then:*

(i) the function $b(\cdot, \theta)$ is downward sloping (resp., upward sloping) if and only if the goods produced by the two sectors are gross complements (resp., gross substitutes);

(ii) an increase in θ leads to a downward (resp., an upward) shift of the b-locus if and only if the goods produced by the two sectors are gross complements (resp., gross substitutes).

Proof. Using (19), we have

$$\frac{\partial b}{\partial \alpha} = \frac{\partial a}{\partial P_1}\frac{\partial \widehat{P}_1}{\partial \alpha} + \frac{\partial (1-a)}{\partial P_2}\frac{\partial \widehat{P}_2}{\partial (1-\alpha)}.$$

Combining this expression with **(A2)** and (20)–(21) shows that $\partial b/\partial \alpha < 0$ under gross complements, while the opposite holds under gross substitutes. This proves part (i). To prove part (ii), we use (19) to compute

$$\frac{\partial b}{\partial \theta} = \frac{\partial a}{\partial P_1}\frac{\partial \widehat{P}_1}{\partial \theta}.$$

Combining this expression with **(A1)** and (20)–(21) shows that $\partial b/\partial \theta < 0$ under gross complements, while the opposite holds under gross substitutes. □

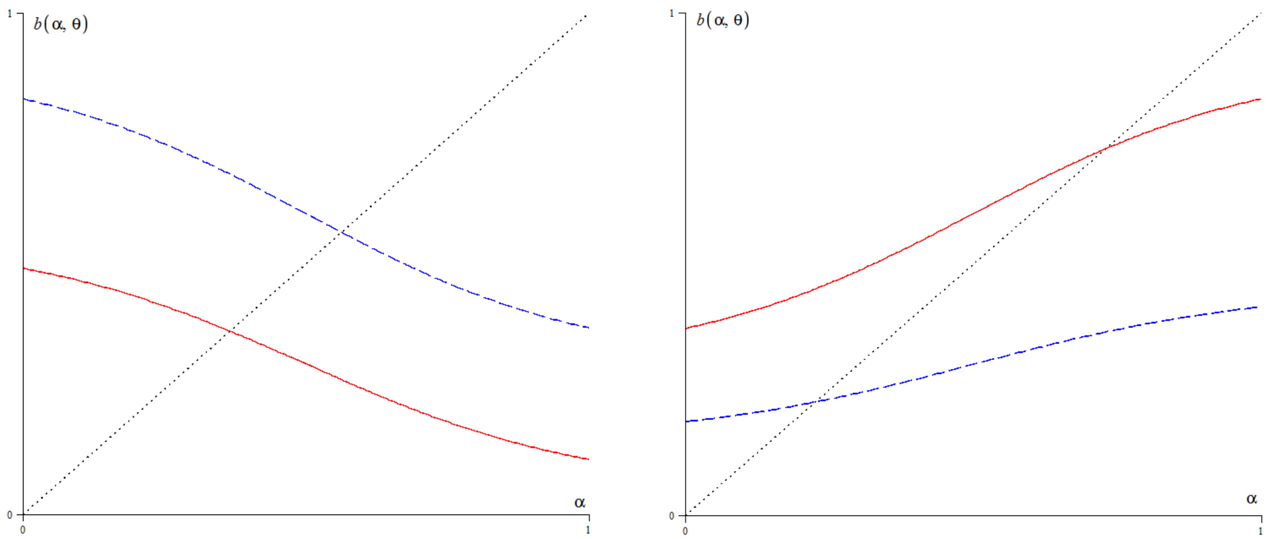
Lemma 2 has two important implications. First, when goods are gross complements, the equilibrium is always unique and interior. Indeed, in this case α^* is the intersection of the 45°-line with the downward-sloping b-curve (see panel (a) of Figure 1 below). Turning to gross substitutes, the equilibrium is still unique provided that the b-locus is not too steep (see panel (b) of Figure 1 below). In other words, for the equilibrium to be unique, the b-locus must be such that it only intersects the 45°-line 'from above' and never does so 'from below'. This holds when

$$\left.\frac{\partial b(\alpha, \theta)}{\partial \alpha}\right|_{\alpha=\alpha^*} = \mathcal{E}_{\alpha}(\widehat{P}_1)\mathcal{E}_{P_1}(a) + \mathcal{E}_{1-\alpha}(\widehat{P}_2)\mathcal{E}_{P_2}(1-a) < 1 \tag{22}$$

at any equilibrium.[11] In what follows, we assume that (22) is always satisfied. By doing so, we guarantee that the equilibrium is unique. This leaves no room for discussions that may arise under multiple equilibria, qualifying those with counterintuitive welfare properties as 'implausible' or 'abnormal' cases.

Figure 1: Comparative statics of equilibria.

(a) Gross complements. (b) Gross substitutes.



Note: The figure depicts the locus $b(\alpha, \theta)$ before (blue dashed) and after (red solid) a shock to θ.

Second, whether the demand system satisfies gross substitutability or gross complementarity fully determines the direction in which an increase in θ shifts the budget structure. Namely, a larger share of income is spent on good 1 under gross substitutes, while under gross complements a higher θ has the opposite effect. Figure 1 illustrates this result.

4.2 Welfare changes

We now can provide a sharper answer to the central question of the paper: when do the interactions between sectors amplify or dampen the positive effect from a higher θ? Moreover, we can also investigate whether a larger θ necessarily translate into overall welfare gains. As in the comparative statics, the answer depends critically on whether the goods produced by the two sectors are gross substitutes or gross complements. In a two-sector economy, the answer

[11] Formally, the expressions $\partial b(\alpha^*, \theta)/\partial \alpha$ and $\mathcal{E}_\alpha(\widehat{P}_1)\mathcal{E}_{P_1}(a) + \mathcal{E}_{1-\alpha}(\widehat{P}_2)\mathcal{E}_{P_2}(1-a)$ are only well defined when α^* is an interior equilibrium. To evaluate them at a corner equilibrium, $\alpha^* = 0$ or $\alpha^* = 1$, one must replace the usual derivatives in (22) with appropriate one-sided derivatives.

for the first part is straightforward. Whether a higher θ amplifies or dampens the direct welfare gains can be summarized as follows:

	Gross complements	Gross substitutes
$\lvert\mathcal{E}_{\alpha}(\widehat{P}_1)\rvert > \lvert\mathcal{E}_{1-\alpha}(\widehat{P}_2)\rvert$	Dampened	Amplified
$\lvert\mathcal{E}_{\alpha}(\widehat{P}_1)\rvert < \lvert\mathcal{E}_{1-\alpha}(\widehat{P}_2)\rvert$	Amplified	Dampened

This result follows immediately from the comparative statics of α^* with respect to θ (see Figure 1 above) and the analysis in Subsection 3.2. The intuition is as follows. Consider the case where the goods supplied by the two sectors are gross substitutes, and the price index of sector 1 is more sensitive to changes in its budget share than that of sector 2. As θ rises, good 1 gets cheaper. As a consequence, consumers spend relatively more on good 1 than on good 2, which amplifies the direct positive effect $\partial\widehat{P}_1/\partial\theta$ of the sector-specific shock. Although the budget reallocation implies that welfare from sector 2 decreases because of less product diversity, the positive welfare effect of the budget share reallocation to sector 1 dominates, so that total welfare increases even more than in the case of fixed budget shares. The intuition behind the other cases can be explained along the same lines.

The second part of our question is whether an increase in θ can, in spite of its direct welfare-improving effect in sector 1, give rise to welfare losses. To answer this question, we look closer at the welfare multiplier $\mathcal{M}$ in (15). With $S = 2$ sectors, it reduces to (see Appendix D for the derivations):

$$\mathcal{M} = 1 + \frac{\left[\mathcal{E}_{\alpha}(\widehat{P}_1) - \mathcal{E}_{1-\alpha}(\widehat{P}_2)\right]\mathcal{E}_{P_1}(a)}{1 - \left[\mathcal{E}_{1-\alpha}(\widehat{P}_2)\mathcal{E}_{P_2}(1-a) + \mathcal{E}_{\alpha}(\widehat{P}_1)\mathcal{E}_{P_1}(a)\right]}. \tag{23}$$

The numerator $\left[\mathcal{E}_{\alpha}(\widehat{P}_1) - \mathcal{E}_{1-\alpha}(\widehat{P}_2)\right]\mathcal{E}_{P_1}(a)$ of the second term in (23) captures the dampening/amplifying effect of interactions between sectors, while the denominator captures the strength of these interactions reflected by the slope (22) of the b-locus. We are now equipped to prove the following result.

Proposition 3 (Welfare with gross complements) *Consider a two-sector economy where goods 1 and 2 are gross complements, and assume that* **(A1)–(A2)** *hold. Then:*

(i) the equilibrium is always unique and interior;

(ii) a higher θ always yields welfare gains, i.e., $\mathcal{M} > 0$.

Proof. Part (i) has already been proven above (see the discussion right after Lemma 2). To prove part (ii), restate (23) as follows:

$$\mathcal{M} = \frac{1 + \mathcal{E}_{1-\alpha}(\widehat{P}_2)\left[\gamma(P_1, P_2) - 1\right]}{1 - \left[\mathcal{E}_{1-\alpha}(\widehat{P}_2)\mathcal{E}_{P_2}(1-a) + \mathcal{E}_{\alpha}(\widehat{P}_1)\mathcal{E}_{P_1}(a)\right]}, \tag{24}$$

where we have used (17). The denominator in (24) is positive by (22). Furthermore, using **(A2)** and $\gamma(P_1, P_2) < 1$, we find that the numerator in (24) is also positive, whence $\mathcal{M} > 0$. Combining this with Proposition 2 completes the proof. □

The intuition behind Proposition 3 is that, although the direct welfare-improving effect of the shock may be dampened by intersectoral linkages, under gross complements this dampening is never sufficiently strong to turn gains into losses. By contrast, such a situation may occur with gross substitutes. More precisely, the following result holds.

Proposition 4 (Welfare with gross substitutes) *Consider a two-sector economy in which goods 1 and 2 are gross substitutes, and assume that* **(A1)–(A2)** *and* (22) *hold. Then:*

(i) the equilibrium is always unique and interior;

(ii) an increase in θ leads to a welfare loss in the vicinity of the equilibrium (P_1^*, P_2^*, α^*) if the following inequality holds:

$$\gamma(P_1^*, P_2^*) - 1 > -\frac{1}{\mathcal{E}_{1-\alpha}(\widehat{P}_2)}, \tag{25}$$

(iii) an increase in θ leads to a welfare gain otherwise.

Proof. A unique interior equilibrium guarantees that the denominator in (24) is always positive by condition (22). Thus, condition (25) is equivalent to $\mathcal{M} < 0$, i.e. welfare losses by Proposition 2. □

Proposition 4 shows that the case of welfare losses—due to shocks that are a priori beneficial at the sectoral level—is neither redundant nor of zero measure. Indeed, any budget-share function that satisfies (25) will do the job. Simple analytically solvable examples are provided in Section 5, where we illustrate our general results. More specific applications are relegated to Appendix F.

5 Applications

Unlike our simple motivating examples in Section 2, we now investigate how likely our results are to occur in more complex models that are usually used in applied economic problems. In particular, we now present two applications that allow for product differentiation. First, we show that welfare gains are amplified or mitigated by intersectoral effects in the standard models used in the literature, which combine CES lower-tier utilities, Cobb-Douglas or CES upper-tier utilities, and monopolistic competition. However, under the (plausible) assumption that the upper-tier elasticity of substitution is smaller than the lower-tier elasticities of substitution, the effects are never strong enough to induce welfare losses. Put differently, using

CES preferences, combined with monopolistic competition in all sectors, is almost equivalent to assuming that there are always welfare gains. Second, we show that small departures from those assumptions are enough to generate situations in which welfare-improving intra-sectoral shocks can generate adverse welfare effects in the aggregate. We illustrate this point using CES sub-utilities and Cournot competition with differentiated goods.

5.1 CES monopolistic competition

In what follows, we assume that goods 1 and 2 are gross substitutes and that the subutilities are CES with elasticities of substitution $\sigma_1 > 1$ and $\sigma_2 > 1$. This specification has been so widely used that it is worth considering separately. We further assume that the market structure is monopolistic competition, so that the sectoral price indices are—up to a normalization—given by

$$\widehat{P}_1(\alpha, \theta) = \phi(\theta)\alpha^{\frac{1}{1-\sigma_1}}, \qquad \widehat{P}_2(1-\alpha) = (1-\alpha)^{\frac{1}{1-\sigma_2}}, \tag{26}$$

where $\phi(\theta)$ is a decreasing and continuous function. One can think of θ as representing average productitivity in the sector, so that a shock corresponds to technological progress in sector 1 leads to a drop in sectoral prices.

Because $\mathcal{E}_\alpha(\widehat{P}_1) = 1/(1-\sigma_1)$ and $\mathcal{E}_{1-\alpha}(\widehat{P}_2) = 1/(1-\sigma_2)$, (23) becomes

$$\mathcal{M} = 1 + \frac{\frac{\sigma_1-\sigma_2}{(\sigma_1-1)(\sigma_2-1)}\mathcal{E}_{P_1}(a)}{1 + \frac{1}{\sigma_2-1}\mathcal{E}_{P_2}(1-a) + \frac{1}{\sigma_1-1}\mathcal{E}_{P_1}(a)}. \tag{27}$$

In what follows, we assume that $1 < \sigma_2 < \sigma_1$, i.e., the price index in sector 2 is more elastic than the price index in sector 1. If that condition does not hold, $\mathcal{M} \geq 1$ from (27). Furthermore, it is now easy to see that $\mathcal{M} \equiv 1$ if and only if: (i) the upper-tier utility is Cobb-Douglas ($\mathcal{E}_{P_1}(a) \equiv 0$); or (ii) the elasticities of substitution are identical in both sectors ($\sigma_1 = \sigma_2$). Although both of these are knife-edge cases, they have been widely used in the literature.

The next proposition shows that welfare losses with CES lower-tier utilities and monopolistic competition can never arise when goods are poorer substitutes than varieties within sectors.

Proposition 5 (Welfare with CES subutilities) *Assume that goods 1 and 2 are gross substitutes and that the subutilities are CES, so that the sectoral price indices are given by (26). Then, for any gap $\sigma_1 - \sigma_2 > 0$:*

(i) welfare losses can occur if and only if the upper-tier elasticity is greater than at least one of the lower-tier elasticities: $\gamma(P_1^*, P_2^*) > \sigma_2$;

(ii) welfare gains always arise otherwise.

Proof. Substituting $\mathcal{E}_{1-\alpha}(\widehat{P}_2) = 1/(1-\sigma_2)$ into (25), we directly obtain $\gamma > \sigma_2$. By Proposition 4, we thus have welfare losses in the vicinity of an interior equilibrium. □

Proposition 5 shows that with CES lower-tier preferences and monopolistic competition, losses can only arise if the elasticity of substitution of the upper-tier utility exceeds that of at least one of the lower-tier utilities. Although, the assumption that different goods are closer substitutes than varieties of the same good is not very plausible, we first provide an intuition for this result and then discuss extentions beyond the CES monopolistic competition framework. Firms move from sector 2 to sector 1 following a positive shock to prices in sector 1. With product differentiation, there is the standard market failure that producers do not capture the whole consumer surplus. Hence, they do not internalize the negative effect their reallocation creates—consumers value variety in sector 2 more strongly than in sector 1—which may result in welfare losses. This is an illustration where product selection is not optimal (see, e.g., Dixit and Norman, 1980, Ch.9). With CES preferences and monopolistic competition, this effect can only arise if goods are closer substitutes than the varieties are among each other, which makes sure that the negative effect in sector 2 dominates the positive effect in sector 1. We provide two 'classical examples'—international trade and heterogeneous firms—to illustrate this point further in Appendix F.

The reason why the two-sector CES monopolistic competition model always yields gains due to the shock θ—at least when $\gamma < \sigma_2$—is that consumers only loose due to one effect: less product diversity in sector 2. However, a second negative effect also arises in general: an increase in prices in sector 2 due to a smaller market and less competition, which increases markups. Hence, if markups are variable in sector 2, welfare losses may arise even if $\gamma < \sigma_2$ when prices are 'elastic enough' in that sector. There are at least two ways that this can occur: (i) if we depart from CES preferences in sector 2 and have increasing elasticity of demand; or (ii) if we depart from monopolistic competition in sector 2 and have, e.g., oligopolistic competition. In those cases, a shrinking market for sector 2 leads to both smaller product diversity and higher prices in this sector. The combination of these two sources of welfare losses can dominate the positive direct effect of the shock when prices in sector 2 are elastic enough to the shock in sector 1.

To illustrate this case, we now develop an example where the market structure in sector 2 is oligopolistically competitive whereas that in sector 1 is monopolistically competitivite. Although preferences are CES in both sectors, welfare losses can arise even if $\gamma < \sigma_2$. This shows that market structure matters substantially for assessing welfare changes in multisector models with imperfect competition (see also d'Aspremont and Dos Santos Ferreira, 2016).

5.2 CES oligopolistic competition

We continue to assume that goods 1 and 2 are gross substitutes and that there is a productivity-enhancing shock in sector 1. To make our point most simply, we assume that the upper-tier price index is given by:

$$\mathcal{P}(P_1, P_2) = \begin{cases} kP_1^{\underline{\alpha}}P_2^{1-\underline{\alpha}}, & \frac{P_2}{P_1} \leq \left(\frac{\underline{\alpha}}{1-\underline{\alpha}}\right)^{1/(\gamma-1)}, \\ \left(P_1^{1-\gamma} + P_2^{1-\gamma}\right)^{1/(1-\gamma)}, & \left(\frac{\underline{\alpha}}{1-\underline{\alpha}}\right)^{1/(\gamma-1)} < \frac{P_2}{P_1} < \left(\frac{\overline{\alpha}}{1-\overline{\alpha}}\right)^{1/(\gamma-1)}, \\ KP_1^{\overline{\alpha}}P_2^{1-\overline{\alpha}}, & \frac{P_2}{P_1} \geq \left(\frac{\overline{\alpha}}{1-\overline{\alpha}}\right)^{1/(\gamma-1)}, \end{cases}$$

where the constants k and K are defined, respectively, as follows:

$$k \equiv \left[\underline{\alpha}^{\underline{\alpha}}(1-\underline{\alpha})^{1-\underline{\alpha}}\right]^{1/(\gamma-1)}, \qquad K \equiv \left[\overline{\alpha}^{\overline{\alpha}}(1-\overline{\alpha})^{1-\overline{\alpha}}\right]^{1/(\gamma-1)},$$

and where $0 < \underline{\alpha} < \overline{\alpha} < 1$. This correponds to the case where upper-tier preferences are CES provided that the differences in sectoral prices are not too large, whereas upper-tier preferences are Cobb-Douglas when sectoral price differences are large enough.[12]

We assume that sector 1 is monopolistically competitive with CES preferences and constant elasticity σ_1.[13] Hence, from (26) we have $\widehat{P}_1(\alpha, \theta) = (1/\theta)\alpha^{1/(1-\sigma_1)}$, where we set $\theta = 1/c_1$ (c_1 is the marginal cost in sector 1) so that $\phi(\theta) = 1/\theta$. Therefore, we have $\mathcal{E}_\theta(\widehat{P}_1) = -1 < 0$, which means that Assumption **(A1)** is satisfied. As to sector 2, we assume that it is oligopolistically competitive with CES preferences given by

$$u_2 = \left[\sum_{i \in \mathcal{I}_2} x_i^{(\sigma_2-1)/\sigma_2}\right]^{\sigma_2/(\sigma_2-1)}, \tag{28}$$

where $\mathcal{I}_2$ is the set of varieties produced in sector 2, with $\sigma_1 > \sigma_2 > \gamma$. We denote by $n_2 \equiv |\mathcal{I}_2|$ the number of oligopolistic firms in sector 2. We derive all equilibrium expressions for that sector in Appendix E and show that Assumption **(A2)** holds. We also show that condition (25) for welfare losses can be expressed as follows:

$$n_2^* < \overline{n}_2 \equiv 1 + \left(\sigma_2 - \frac{1}{2}\right)\left[\sqrt{1 + \frac{\sigma_2(\sigma_2-1)}{(\sigma_2-1/2)^2}\frac{\gamma-1}{\sigma_2-\gamma}} - 1\right], \tag{29}$$

[12] This preference structure guarantees that we exclude the case of corner solutions where only one sector produces. It is not required for our argument but simplifies the analysis.

[13] Although we assume CES monopolistic competition in sector 1, this is not required for our result. Indeed, we could also have Cournot oligopoly in sector 1 and obtain similar results.

where n_2^* is the equilibrium number of firms in sector 2. The latter is pinned-down by the free-entry condition $\widetilde{\pi}_2(n_2^*) = f_2$, which is given by

$$\frac{n_2^2}{n_2 + \sigma_2 - 1} = \frac{1-\alpha}{f_2 \sigma_2}. \tag{30}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{n^2}{n + \sigma_2 - 1}\right) = \frac{n^2 + 2(\sigma_2 - 1)n}{(n + \sigma_2 - 1)^2} > 0,$$

the left-hand side of (30) is an increasing function of n_2^*. Since the right-hand side of (30) goes to zero as f_2 becomes large, this implies that n_2^* goes to zero too. Hence, $n_2^* < \overline{n}_2$ holds, i.e., *if the fixed cost in sector 2 is sufficiently high, a technological improvement in sector 1 may lead to welfare losses economywide*.[14]

The intuition for the foregoing results is as follows. First, when fixed costs are large, there are only a limited number of firms operating in sector 2. A positive shock to sector 1—followed by a reallocation of budget towards that sector—forces firms out of sector 2. Since there are only few firms operating in that sector, any shock that reduces the mass of firms is likely to cause a substantial increase in prices. Second, there is a variety effect. The positive shock in sector 1 triggers entry and thus increases welfare because consumers love variety. Yet, this is at least partly offset by the loss of variety in sector 2. As can be seen from (27), when $\sigma_1 > \sigma_2$, $\mathcal{M} < 1$ (since $\mathcal{E}_{P_1}(a) < 0$): more preferred varieties in sector 2 are displaced by less preferred varieties in sector 1. The combination of both effects may be strong enough to reduce welfare following a positive shock to sector 1.

To summarize, welfare losses are likely in economies where some industries are strongly concentrated, i.e., 'granular'. Given the recent literature that emphasizes the aspect of granularity in macroeconomics (e.g., Gabaix, 2011), industrial organization (e.g., Shimomura and Thisse, 2012), and trade (e.g., Di Giovanni et al., 2014; Parenti, 2018), this seems important to keep in mind when thinking about welfare.

We now provide a numerical illustration of the foregoing results. To this end, we set the parameter values of the model as follows: $\underline{\alpha} = 0.1$, $\overline{\alpha} = 0.4$, $\gamma = 4$, $\sigma_1 = 6$, $\sigma_2 = 5$, $f_2 = 0.1$.[15] As explained before, we also set $\theta = 1/c_1$, and shocks to θ can be interpreted as a decrease in the constant marginal cost in sector 1.

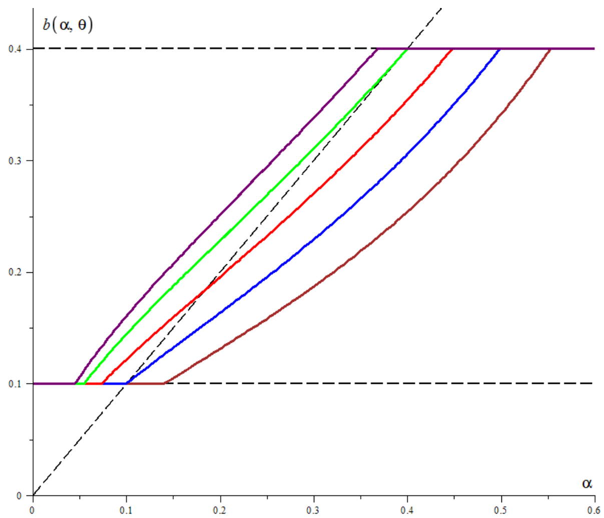
Panel (a) of Figure 2 illustrates the equilibrium for different values of θ. The dashed line is the 45°-line and the colored lines correspond to the function $b(\cdot, \theta)$ for different values of θ. Because preferences are CES for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ but Cobb-Douglas otherwise, the function becomes horizontal for low or high expenditure shares.

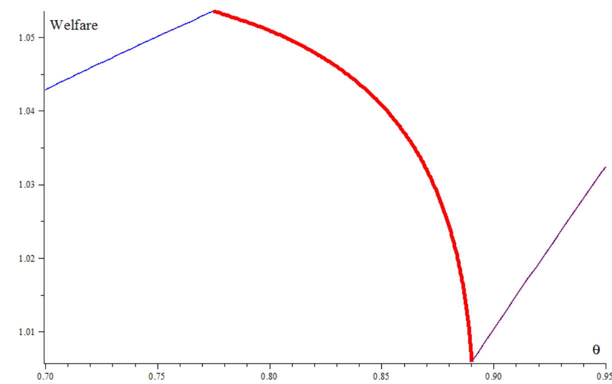
[14] Formally, the threshold is given by $f_2 \geq \overline{f}_2 \equiv (\overline{n}_2 + \sigma_2 - 1) \,/\, (\sigma_2 \overline{n}_2^2)$, where $\overline{n}_2$ is defined by (29).

[15] These parameter values are not crucial, but they allow us to focus on cases where there are no multiple equilibria.

Figure 2: Numerical illustration of welfare losses.

(a) Equilibrium structure. (b) Welfare changes.





The equilibria are at the intersection of the 45°-line and the b-loci.[16] As panel (a) of Figure 2 shows, positive shocks—increasing θ—shifts the sector 1 budget share upwards by moving the $b(\cdot, \theta)$ function to the north-west (e.g., going from the red to the green curve). Negative shocks have, of course, the opposite effects.

Panel (b) illustrates the welfare effects of the shocks. There are two regimes. First, when θ is either small or large, we are on the 'Cobb-Douglas part' of the preferences (the blue lines in panel (b)) and a positive shock is welfare improving. Second, when θ takes intermediate values, we are on the 'CES part' of the preferences (the thick red line in panel (b)) and a positive shock is welfare worsening.

6 Conclusions

The aim of this paper was twofold. First, we proposed a multisector model that nests many of the approaches used in the applied literature. We derived a 'welfare multiplier' and established precise conditions under which welfare-improving intra-sectoral shocks are magnified or dampenend in the aggregate. The magnitude and the sign of the welfare multiplier crucially hinge on complementarity or substitutability between goods in consumers' preferences: when goods are gross complements, the welfare effects are always positive, whereas they may be negative when goods are gross substitutes. The case of Cobb-Douglas preferences is a limiting

[16]There are several equilibria on the figure since there is one for each $b(\cdot, \theta)$ function. However, for a given set of parameter values, the equilibrium is unique.

case for which a welfare improving shock in one sector is always welfare improving in general and of the same magnitude as the sectoral shock itself.

Second, we showed that a specific combination of assumptions that is widely used in the literature—namely CES preferences and monopolistic competition as the market structure—give rise to very specific results. In particular, the combination of these assumptions guarantees that welfare gains always occur under 'reasonable' assumptions on the elasticities of substitution of different tiers of preferences. This result is, however, not robust as there exists a large class of homothetic preferences under which it need not hold.

As argued in this paper, both qualitative and—especially— quantitative welfare statements based on multisector Cobb-Douglas-CES models with monopolistic competition should be taken with a grain of salt. Our results are useful in that they provide a simple way to assess the robustness of welfare changes due to sectoral shocks. All sectoral elasticities are easy to compute for CES monopolistic competition models. Assuming different values for the intersectoral elasticities of substitution, we can then compute the welfare multiplier and estimate by how much intrasectoral effects map into aggregate welfare changes.

Acknowledgements. We thank Yasusada Murata, Jacques-François Thisse and seminar and conference participants at HSE Moscow, HSE Saint Petersburg, Ifo, Aarhus, EEA-ESEM 2017 in Lisbon, SAET 2019 in Ischia, and the 2019 Australasian economic theory workshop in Sydney for very helpful comments and suggestions. This article was prepared within the framework of the HSE University Basic Research Program and funded by the Russian Academic Excellence Project '5-100'. Behrens gratefully acknowledge financial support from the CRC Program of the Social Sciences and Humanities Research Council (SSHRC) of Canada for the funding of the *Canada Research Chair in Regional Impacts of Globalization*.

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Online appendix

A. Motivating examples: detailed analysis

In this Appendix, we provide a detailed analysis of the two motivating examples discussed in Section 2.

A.1. CES monopolistic competition

Let $[0, N]$ be a continuum of monopolistically competitive firms operating in sector 2 and σ be the elasticity of substitution across varieties. Each firm $i \in [0, N]$ produces a single variety (henceforth variety i) of good 2, and each variety is produced by a single firm. Each firm incurs a constant fixed cost, $f > 0$, and a constant marginal cost, $c > 0$.

Deriving (3). We now show that the price level $\widehat{P}_2(1-\alpha)$ conditional on the budget allocation $(\alpha, 1-\alpha)$ satisfies (3). Firms take the budget allocation $(\alpha, 1-\alpha)$ as given. The market demand faced by firm $i \in [0, N]$ is as follows:

$$q_i = \frac{E}{P_2}\left(\frac{p_i}{P_2}\right)^{-\sigma}, \tag{A.1}$$

where $E \equiv (1-\alpha)L$ is total expenditure on good 2, p_i is the price for variety i, and P_2 is the price index given by

$$P_2 \equiv \left(\int_0^N p_i^{1-\sigma}\mathrm{d}i\right)^{\frac{1}{1-\sigma}}. \tag{A.2}$$

The demand system (A.1)–(A.2) yields the standard symmetric free-entry equilibrium:

$$p_i^* = p^* \equiv \frac{c\sigma}{\sigma-1}, \qquad q_i^* = q^* \equiv (\sigma-1)\frac{f}{c}, \qquad N^* = \frac{E}{\sigma f}, \tag{A.3}$$

for all $i \in [0, N^*]$. Combining (A.3) with (A.2) and taking into account that $E = (1-\alpha)L$, we obtain:

$$\widehat{P}_2(1-\alpha) = K(1-\alpha)^{\frac{1}{1-\sigma}}, \tag{A.4}$$

where $K > 0$ is a constant defined by

$$K \equiv \frac{c\sigma}{\sigma-1}\left(\frac{L}{\sigma f}\right)^{1/(1-\sigma)}.$$

Up to the coefficient K, (A.4) coincides with (3). One can show that equation (A.4) keeps holding if we account for Melitz-type heterogeneity across firms (as in Appendix G).

Uniqueness and comparative statics of the stable solution to (4). In accordance with (18) – (19), denote by $b(\alpha,\theta)$ the right-hand side of (4). For all $\theta > 0$, we have:

$$0 < b(0,\theta) < 1, \qquad b(1,\theta) = 1, \qquad \left.\frac{\partial b(\alpha,\theta)}{\partial \alpha}\right|_{\alpha=1} = \infty.$$

By continuity, and because the b-locus is very steep in the vicinity of $\alpha = 1$, we infer that $b(\alpha,\theta) > \alpha$ when α is close to zero, while the opposite holds when α is close to one. By the intermediate value theorem, there exists an interior solution $\alpha^*(\theta)$ to (4). To establish uniqueness, we use routine calculus to get:

$$\frac{\partial b(\alpha,\theta)}{\partial \alpha} = \frac{\gamma-1}{\sigma-1}\frac{\theta^{\gamma-1}(1-\alpha)^{\frac{\gamma-\sigma}{\sigma-1}}}{\left(\theta^{\gamma-1}+(1-\alpha)^{\frac{\gamma-1}{\sigma-1}}\right)^2} > 0,$$

$$\frac{\partial^2 b(\alpha,\theta)}{\partial \alpha^2} = \frac{(\gamma-1)}{(\sigma-1)^2}\cdot\theta^{\gamma-1}\cdot\frac{(\sigma-\gamma)\theta^{\gamma-1}+(\sigma+\gamma-2)(1-\alpha)^{\frac{\gamma-1}{\sigma-1}}}{\left(\theta^{\gamma-1}+(1-\alpha)^{\frac{\gamma-1}{\sigma-1}}\right)^3}\cdot(1-\alpha)^{\frac{1+\gamma-2\sigma}{\sigma-1}} > 0.$$

Both inequalities rely on the assumption that $\sigma > \gamma > 1$. Thus, $b(\alpha,\theta)$ is strictly increasing and strictly convex in α for all $(\alpha,\theta) \in [0,1]\times(0,+\infty)$. Recalling that $b(\alpha,\theta) > \alpha$ for very small α, while $b(\alpha,\theta) < \alpha$ when α is very close to one, we conclude that the b-locus can only have one interior intersection point $\alpha^*(\theta)$ with the 45°-line. Furthermore, as $b(\alpha,\theta) < \alpha$ for small values of α, the b-locus intersects the 45°-line from above. Hence, the stability condition (22) holds at $\alpha = \alpha^*(\theta)$. In contrast, the corner solution, $\alpha = 1$, is unstable, as it does not satisfy (22).

Finally, it is readily verified that $\partial b(\alpha,\theta)/\partial\theta > 0$, i.e. an increase in θ shifts upwards the b-locus. Because the b-locus intersects the 45°-line from above, we conclude that $\mathrm{d}\alpha^*(\theta)/\mathrm{d}\theta > 0$.

Figure 3 below illustrates these results for the case when $\gamma = 2$, $\sigma = 3$, and $c = f = 1$. The solid curves on Figure 3 represent the b-loci for different levels of productivity θ, while the dotted line is the 45°-line.

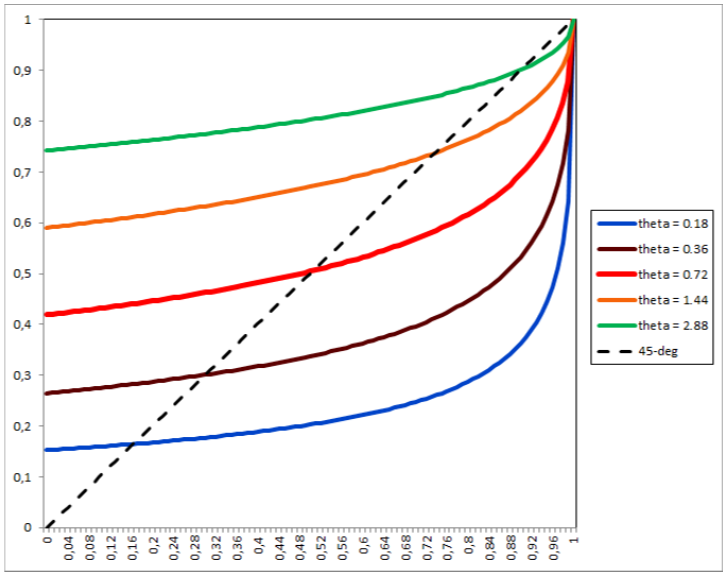
A.2. Cournot oligopoly, homogeneous good

Each firm $i = 1,2,\ldots,n$ operating in sector 2 takes the budget allocation $(\alpha, 1-\alpha)$ as given. Hence, the inverse market demand curve faced by all firms in sector 2 is given by

$$P_2 = L\frac{1-\alpha}{Q}, \tag{A.5}$$

where Q is the aggregate quantity consumed.

Figure 3: The b-loci: CES monopolistic competition.



Derivation of equation (5). We now show that the price level $\widehat{P}_2(1-\alpha)$ conditional on the budget allocation $(\alpha, 1-\alpha)$ is given by equation (5). Due to (A.5), firm i's profit function is given by

$$\pi_i(q_i, Q_{-i}) = \left(L\frac{1-\alpha}{q_i + Q_{-i}} - c \right) q_i, \tag{A.6}$$

where q_i is firm i's output, while $Q_{-i} \equiv Q - q_i = \sum_{k \neq i} q_k$ is the total output of all firms but i. As implied by (A.6), firm i's first-order condition is given by

$$L\frac{1-\alpha}{q_i + Q_{-i}} - L\frac{(1-\alpha)q_i}{(q_i + Q_{-i})^2} - c = 0,$$

which becomes after simplifications:

$$q_i = Q - \frac{c}{L(1-\alpha)}Q^2. \tag{A.7}$$

Summation across i on both sides of (A.7) yields:

$$Q = nQ - \frac{nc}{L(1-\alpha)}Q^2,$$

solving which for Q, we determine the aggregate output:

$$Q = \frac{n-1}{n}\frac{L(1-\alpha)}{c}. \tag{A.8}$$

Furthermore, (A.7) implies that the Cournot-Nash equilibrium must be symmetric, combining which with (A.8) yields firm i's output at the unique interior Cournot-Nash equilibrium:

$$q_i = \frac{n-1}{n^2}\frac{L(1-\alpha)}{c}, \quad i = 1, 2, \dots, n. \tag{A.9}$$

Plugging (A.8)–(A.9) into (A.5)–(A.6), we obtain the equilibrium price level and the equilibrium profit as functions of the number n of firms in sector 2:

$$P_2 = \frac{cn}{n-1}, \qquad \pi_i = \frac{L(1-\alpha)}{n^2}. \tag{A.10}$$

It remains to determine the equilibrium number n^* of firms. To do so, we equate the operating profit to the fixed cost f, so that the free-entry condition is $L(1-\alpha)/n^2 = f$. Solving the free-entry condition for n yields:

$$n^* = \frac{\sqrt{1-\alpha}}{f/L}. \tag{A.11}$$

Plugging $n = n^*$ into (A.10) yields equation (5). Clearly, when $1-\alpha > (f/L)^2$, the price-level function given by (5) satisfies (**A2**). Otherwise, we set $P_2 = \infty$, as sector 2 simply does not operate in this case.

Welfare conditional on budget allocation. Using the expressions for the utility-maximizing budget allocation, $(a(P_1, P_2), 1 - a(P_1, P_2))$, and the price index (5) of sector 2, we obtain the following expression for the welfare level V:

$$V = \frac{1}{c}\left(1 - \frac{f/L}{\sqrt{1-\alpha}}\right)(1-\alpha)^{\frac{1}{1-\gamma}}. \tag{A.12}$$

Note, however, that (A.12) only holds when sector 2 operates at a positive scale, i.e. when $1-\alpha > f/L$.

It can be verified that the right-hand side of (A.12) increases with $1-\alpha$ if and only if the following inequality holds:

$$2\sqrt{1-\alpha} < (1+\gamma)\frac{f}{L}. \tag{A.13}$$

Furthermore, the right-hand side of (A.12) does not depend directly on θ, whence the impact of θ on the equilibrium welfare level $V^*(\theta)$ is fully channelled by changes in $\alpha^*(\theta)$. This, in turn, implies that

$$\text{sign}\left(\frac{\mathrm{d}V^*}{\mathrm{d}\theta}\right) = \text{sign}\left(\frac{\mathrm{d}\alpha^*}{\mathrm{d}\theta}\right) \tag{A.14}$$

if and only if $\alpha = \alpha^*(\theta)$ satisfies (A.13). Otherwise, $\alpha^*(\theta)$ and $V^*(\theta)$ are driven in the opposite directions. Thus, studying the welfare consequences of the "good shock" in the perfectly competitive sector amounts to studying the comparative statics of the consumer's budget structure with respect to the productivity level θ in that sector.

The equilibrium expenditure share $\alpha^*(\theta)$ allocated to sector 1 is determined as a solution to the following fixed point condition:

$$\alpha = b(\alpha, \theta) \equiv \begin{cases} \dfrac{(c\theta)^{\gamma-1}}{(c\theta)^{\gamma-1} + \left(1 - (f/L)/\sqrt{1-\alpha}\right)^{\gamma-1}}, & \alpha \leq 1 - (f/L)^2, \\ \\ 1, & \text{otherwise.} \end{cases} \tag{A.15}$$

Clearly, equation (A.15) implies that

$$\frac{\partial b(\alpha, \theta)}{\partial \theta} \geq 0 \qquad \text{and} \qquad \frac{\partial b(\alpha, \theta)}{\partial \alpha} \geq 0,$$

for all $\theta > 0$ and for all $\alpha \neq (f/L)^2$.

Equilibrium and comparative statics. We now come to the characterization of equilibria and their comparative statics with respect to the productivity level θ in sector 1. Both are given by the following Proposition.

Proposition 6 (Equilibria and comparative statics) *In the two-sector economy described above, there exists a threshold productivity level $\overline{\theta} > 0$, such that:*

(i) if $\theta < \overline{\theta}$, there exist three different equilibria, including two interior equilibria, $\alpha = \alpha^*(\theta)$ and $\alpha = \alpha^{**}(\theta)$, such that $0 < \alpha^*(\theta) < \alpha^{**}(\theta) < 1 - f/L$, and a corner equilibrium in which $\alpha = 1$, i.e. the Cournot sector does not operate;

(ii) if $\theta = \overline{\theta}$, then there exist two equilibria: the "tangency equilibrium" $\alpha = \overline{\alpha}$, where

$$\lim_{\theta \uparrow \overline{\theta}} \alpha^*(\theta) = \overline{\alpha} = \lim_{\theta \uparrow \overline{\theta}} \alpha^{**}(\theta),$$

and the corner equilibrium where $\alpha = 1$;

(iii) if $\theta > \overline{\theta}$, the only equilibrium is the corner equilibrium in which $\alpha = 1$;

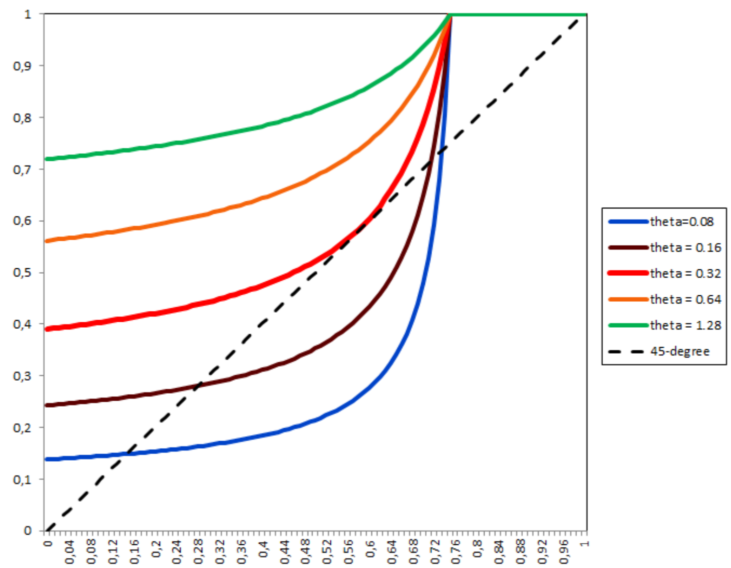
(iv) for all $\theta < \overline{\theta}$, we have:

$$\frac{\mathrm{d}\alpha^*(\theta)}{\mathrm{d}\theta} > 0, \quad \frac{\mathrm{d}\alpha^{**}(\theta)}{\mathrm{d}\theta} < 0.$$

Figure 4 illustrates these results for the case when $\gamma = 2$, $c = 1$, $f/L = 0.25$.

In this case, we have $\overline{\theta} \approx 0.32$. The solid curves on Figure 2 plot the b-locus described by the right-hand side of (6) for different values of θ. Clearly, the b-locus is shifted upwards by a productivity hike in sector 1.

Figure 4: The b-loci: Cournot, homogeneous good.



Proof of Proposition 6. Denote by $A(\theta)$ the set of zeros of the function $g : (0, 1-(f/L)^2) \times \mathbb{R}_+ \to [-1, 1]$ defined as follows:

$$g(\alpha,\theta) \equiv b(\alpha,\theta) - \alpha, \qquad A(\theta) \equiv \{\alpha \mid 0 < \alpha < 1-(f/L)^2, \text{ and } g(\alpha,\theta) = 0\},$$

where $b(\cdot,\theta)$ is the b-locus defined by (A.15). Clearly, $A(\theta)$ coincides with the set of interior equilibria. Consider the following family of optimization problems parameterized by $\theta \in \mathbb{R}_+$:

$$\min_{\alpha} g(\alpha,\theta) \qquad \text{s.t.} \qquad 0 \le \alpha \le 1-(f/L)^2. \tag{A.16}$$

Denote by $m(\theta)$ the value function of (A.16), and call this function the *m-function*. Clearly, an interior equilibrium exists if and only if $m(\theta) \le 0$. Therefore, the proof relies on the properties of the m-function. The following Lemma summarizes these properties.

Lemma 3 (Properties of the m-function) *The m-function is well-defined, strictly increasing, differentiable, and such that $m(0) < 0 < m(\infty)$.*

Proof. By the continuity of $b(\cdot,\theta)$ and by the Weierstrass theorem, the m-function is well defined by (A.16). Differentiability and monotonicity follow from the envelope theorem, applying which yields:

$$\frac{\mathrm{d}m(\theta)}{\mathrm{d}\theta} = \frac{\partial b(\alpha^*,\theta)}{\partial\theta} > 0,$$

where α^* is a minimizer in (A.16). It remains to find the boundary values, $m(0)$ and $m(\infty)$, of the m-function. To do so, we look at what happens to the b-locus when θ takes on the extreme values: $\theta = 0$ and $\theta = \infty$. Using (A.15), we find that, under $\theta = 0$ (respectively, $\theta = \infty$) the b-locus becomes a horizontal line at the zero (respectively, unity) level. In other words, for all $\alpha \in [\,0,\, 1 - (f/L)^2\,)$, we have:

$$\lim_{\theta \to 0} b(\alpha, \theta) = 0, \qquad \lim_{\theta \to \infty} b(\alpha, \theta) = 1,$$

which, in turn, implies

$$m(0) = -(1 - (f/L)^2) < 0, \qquad m(\infty) = (f/L)^2 > 0.$$

This completes the proof of Lemma 3. □

Combining Lemma 3 with the intermediate value theorem, we infer that there exists a unique $\overline{\theta} > 0$, such that $m(\overline{\theta}) = 0$. This immediately implies part (iii) of Proposition 6: when $\theta > \overline{\theta}$, we have $m(\overline{\theta}) > 0$, which means $b(\alpha, \theta) > \alpha$ for all $\alpha \in \left[0, 1 - (f/L)^2\right]$.

To prove part (ii), we provide a geometric characterization of $\theta = \overline{\theta}$ by establishing the following Lemma.

Lemma 4 (The tangency property of $\boldsymbol{\theta} = \overline{\theta}$) *Assume that the b-locus is given by* (A.15). *Then:*
(i) when $\theta = \overline{\theta}$, the optimization problem (A.16) *has a unique interior solution $\overline{\alpha}$;*
(ii) geometrically, $\overline{\alpha}$ is the unique tangency point of $b(\cdot, \overline{\theta})$ to the 45°-line;
(iii) when $\theta \neq \overline{\theta}$, the b-locus has no tangency points to the 45°-line.

Proof. By the Weierstrass theorem, the optimization problem obtained from (A.16) by setting $\theta = \overline{\theta}$ has a minimizer $\overline{\alpha}$. Furthermore, it is readily verified using (A.15) that, for any $\theta > 0$ (hence for $\theta = \overline{\theta}$) the strict inequality $b(\alpha, \theta) - \alpha > 0$ holds when α is sufficiently close to either bound of the interval $[0,\, 1 - (f/L)^2]$. Combining this with $m(\overline{\theta}) = b(\overline{\alpha}, \overline{\theta}) - \overline{\alpha} = 0$, we infer that $\overline{\alpha}$ has to be an interior minimizer. Hence, $\overline{\alpha}$ has to satisfy the FOC of the optimization problem (A.16): $\partial b(\alpha, \overline{\theta})/\partial \alpha = 1$. Furthermore, because $\overline{\alpha}$ is a minimizer, it has to satisfy (A.15): $b(\overline{\alpha}, \overline{\theta}) - \overline{\alpha} = m(\overline{\theta}) = 0$. To sum up, $\left(\overline{\alpha}, \overline{\theta}\right)$ must be a solution to the following system:

$$b(\alpha, \theta) = \alpha, \qquad \frac{\partial b(\alpha, \theta)}{\partial \alpha} = 1. \tag{A.17}$$

where $b(\alpha, \theta)$ is defined by (A.15).

To prove Lemma 4, it suffices to show that (A.17) has the unique solution. That this immediately implies uniqueness of an interior minimizer $\overline{\alpha}$ in (A.16) is straightforward. To see why uniqueness of the solution to (A.17) also proves (ii) and (iii), we need to understand better what the system (A.17) means geometrically. Let (α_0, θ_0) be a solution to (A.17). The first equation in (A.17) says that the point (α_0, α_0) belongs to both the b-locus, $b(\cdot, \theta_0)$, and the 45°-line. The second equation in (A.17) that the slope of $b(\cdot, \theta_0)$ equals one at $\alpha = \alpha_0$. Hence, both equations taken together hold if and only if $\alpha = \alpha_0$ is the tangency point of $b(\cdot, \theta_0)$ to the 45°-line. Thus, it remains to show that $(\overline{\alpha}, \overline{\theta})$ is the only solution to (A.17).

Using (A.15), the system (A.17) can be equivalently restated as follows:

$$\left(1 - \frac{f/L}{\sqrt{1-\alpha}}\right)^{\gamma-1} \cdot \frac{\alpha}{1-\alpha} = (c\theta)^{\gamma-1},$$

$$(1-\alpha)^{-3/2} \cdot \left(1 - \frac{f/L}{\sqrt{1-\alpha}}\right)^{\gamma-2} \cdot \left[(c\theta)^{\gamma-1} + \left(1 - \frac{f/L}{\sqrt{1-\alpha}}\right)^{\gamma-1}\right]^{-2} = -\frac{(c\theta)^{1-\gamma}}{1-\gamma} \cdot \frac{2}{f/L}.$$

By eliminating $(c\theta)^{\gamma-1}$, we obtain after simplifications:

$$(\gamma - 1)\left(\sqrt{1-\alpha}\right)^2 + (2L/f)\sqrt{1-\alpha} - (1+\gamma) = 0. \tag{A.18}$$

Equation (A.18) is a quadratic equation in $\sqrt{1-\alpha}$, hence it can be solved in closed form. The inequalities $0 < \overline{\alpha} < 1 - (f/L)^2$ can be readily verified by observing that the left-hand side of equation (A.18) increases in $\sqrt{1-\alpha}$, and is negative (resp., positive) when evaluated at $\sqrt{1-\alpha} = f/L$ (resp., at $\sqrt{1-\alpha} = 1$). Hence, by the intermediate value theorem, (A.18) has a unique positive solution w.r.t. $\sqrt{1-\alpha}$ over $(f/L, 1)$. Computing this solution leads to:

$$\overline{\alpha} \equiv 1 - \left[\frac{\sqrt{1 + (\gamma^2 - 1)f/L} - 1}{(\gamma - 1)f/L}\right]^2. \tag{A.19}$$

Plugging $\alpha = \overline{\alpha}$ into (6), we uniquely determine $\overline{\theta}$:

$$\overline{\theta} \equiv \frac{1}{c}\left(1 - \frac{f/L}{\sqrt{1-\alpha}}\right)\left(\frac{\overline{\alpha}}{1-\overline{\alpha}}\right)^{\frac{1}{\gamma-1}}. \tag{A.20}$$

Equations (A.19)–(A.20) define the unique solution to the system (A.17). This completes the proof of Lemma 4. ☐

Lemma 4 immediately implies part (ii) of Proposition 6. Indeed, if $\theta = \overline{\theta}$, then, by (i) of Lemma 4, we have: $g(\alpha, \overline{\theta}) > 0$ for all $\alpha \in \{z \in \mathbb{R} \,|\, 0 \leq z \leq 1 - (f/L)^2,\ z \neq \overline{\alpha}\}$. Hence, no interior equilibria (except, of course, the tangency equilibrium $\alpha = \overline{\alpha}$) exist.

It remains to prove part (i) of Proposition 6. To do so, we observe that, by strict monotonicity of the m-function stated in Lemma 3, we have $m(\theta) < 0$ for all $\theta < \overline{\theta}$. Thus, there exists at least one $\widetilde{\alpha}(\theta) \in \left(0, 1 - (f/L)^2\right)$, such that $b\left(\widetilde{\alpha}(\theta), \theta\right) - \widetilde{\alpha}(\theta) < 0$. Combining this with $b(0, \theta) > 0$ and $b(1 - (f/L)^2, \theta) = 1 > 1 - (f/L)^2$ and using the intermediate value theorem, we infer that the fixed-point condition (6) has at least two interior solutions, $\alpha^*(\theta)$ and $\alpha^{**}(\theta)$, satisfying the inequalities:

$$0 < \alpha^*(\theta) < \widetilde{\alpha}(\theta) < \alpha^{**}(\theta) < 1 - (f/L)^2.$$

To complete the proof, we need the following Lemma.

Lemma 5 (No three interior solutions) *Equation* (6) *never has more than two interior solutions.*

Proof. Assume that, on the contrary, for some value $\theta_0 < \overline{\theta}$, there exist at least three interior equilibria. Define:

$$\alpha_1 \equiv \inf A(\theta), \qquad \alpha_2 \equiv \inf\left[A(\theta) \setminus \{\alpha_1\}\right], \qquad \alpha_3 \equiv \inf\left[A(\theta) \setminus \{\alpha_1, \alpha_2\}\right]. \tag{A.21}$$

Using (6), it is readily verified that $g(\cdot, \theta_0)$ is a real-analytic function over $\left(0, 1-(f/L)^2\right)$, i.e. the Taylor series of $g(\alpha, \theta_0)$ in α converges to $g(\alpha, \theta_0)$ in an open neighborhood of each $\alpha_0 \in (0, 1-(f/L)^2)$:

$$g(\alpha, \theta_0) = \sum_{k=0}^{\infty} c_k \left(\alpha - \alpha_0\right)^k, \qquad \text{where} \qquad c_k \equiv \frac{1}{k!} \left.\frac{\partial^k g(\alpha, \theta_0)}{\partial \alpha^k}\right|_{\alpha=\alpha_0}.$$

It is well known (Courant and Fritz, 2012, Ch. 7, p. 545) that zeros of a real-analytic function are isolated. Hence, for any $\alpha_0 \in A(\theta_0)$ there exists $\varepsilon > 0$, such that

$$A(\theta_0) \cup (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) = \alpha_0.$$

This, in turn, implies that $A(\theta_0)$ is a closed set, and it remains closed after removing finitely many points from it. Combining this with (A.21), we infer that $\alpha_k \in A(\theta_0)$, $k = 1, 2, 3$, and there are no zeros of $g(\cdot, \theta_0)$ in between. Furthermore, by Lemma 3, none of $\alpha = \alpha_k$, $k = 1, 2, 3$, can be a tangency point of the b-locus, $b(\cdot, \theta_0)$, to the 45°-line. Thus, the only remaining possibility is that b-locus intersects the 45°-line from above at $\alpha = \alpha_1$ and $\alpha = \alpha_3$, while it does so from below at $\alpha = \alpha_2$. In other words, the following inequalities hold:

$$\left.\frac{\partial b(\alpha, \theta_0)}{\partial \alpha}\right|_{\alpha=\alpha_1} < 1, \qquad \left.\frac{\partial b(\alpha, \theta_0)}{\partial \alpha}\right|_{\alpha=\alpha_2} > 1, \qquad \left.\frac{\partial b(\alpha, \theta_0)}{\partial \alpha}\right|_{\alpha=\alpha_3} < 1.$$

As $b\left(1-(f/L)^2, \theta_0\right) = 1$, there must exist, by continuity, at least one more interior solution, $\alpha_4 > \alpha_3$. Following the same logic as in (A.21), define it by

$$\alpha_4 \equiv \inf\left[A(\theta) \setminus \{\alpha_1, \alpha_2, \alpha_3\}\right].$$

By analogy with (A.16), define the following two families of optimization problems parameterized by $\theta \in \mathbb{R}_+$:

$$\min_{\alpha} g(\alpha, \theta) \qquad \text{s. t.} \qquad \alpha_{2k-1} \leq \alpha \leq \alpha_{2k}, \qquad k = 1, 2. \tag{A.22}$$

Let m_k-function stand for the value function of the kth problem in (A.22), $k = 1, 2$. Along the same lines as in the proofs of Lemmas 3 and 4, it can be shown that there exist $\theta_k \in (\theta_0, \overline{\theta}\,]$,

$k = 1, 2$ such that $m_k(\theta_k) = 0$, and the kth problem in (A.22) has a unique interior minimizer $\overline{\alpha}_k$ when $\theta = \theta_k$. Furthermore, repeating verbatim the argument of Lemma 4, one can show that $\alpha = \overline{\alpha}_k$, $k = 1, 2$, are tangency points of $b(\cdot, \theta)$ to the 45°-line. By Lemma 4 (iii), it must be that $\theta_1 = \theta_2 = \overline{\theta}$. Because $\alpha_1 < \alpha_2$, we arrive to a contradiction with Lemma 4 (ii). Thus, our conjecture—that at least three interior solutions to (6) may exist—was wrong. This completes the proof. □.

Part (i) of Proposition 6 follows immediately from Lemma 5. This completes the proof of Proposition 6. □

Welfare effects. We are now equipped to studying the welfare effects triggered by a productivity shock in the perfectly competitive sector. Denote by V^*, V^{**} and V^{corn} the equilibrium welfare levels associated with the two interior equilibria (when they exist) and with the corner equilibrium, respectively (see Proposition 6 above).

Proposition 7 (Welfare effects) *Let $\overline{\theta}$ be the threshold productivity introduced in Proposition 6. Then:*

(i) for all $\theta \in (0, \overline{\theta})$, we have

$$V^*(\theta) > V^{**}(\theta) > V^{\text{corn}}(\theta);$$

(ii) if γ is sufficiently large,[17] then welfare varies with θ as follows:

$$\frac{\mathrm{d}V^*(\theta)}{\mathrm{d}\theta} < 0, \quad \frac{\mathrm{d}V^{**}(\theta)}{\mathrm{d}\theta} > 0, \quad \frac{\mathrm{d}V^{\text{corn}}(\theta)}{\mathrm{d}\theta} > 0;$$

(iii) as θ reaches the threshold level $\overline{\theta}$, both $V^*(\theta)$ and $V^{**}(\theta)$ discontinuously drop;

(iv) as θ grows beyond $\overline{\theta}$, the welfare level V^{corn} in the only remaining equilibrium grows proportionately to θ.

Proof of Proposition 7. Consider first the case when the perfectly competitive sector is not very productive: $\theta < \overline{\theta}$. Then, no matter in which equilibrium the economy ends up being, the price level in the perfectly competitive sector is always given by

$$P_1^* = P_1^{**} = P_1^{\text{corn}} = 1/\theta.$$

Hence, to prove part (i), all we need to show is that $P_2^* < P_2^{**} < P_2^{\text{corn}}$. This is readily verified by combining (**A2**) with the fact that $\alpha^*(\theta) < \alpha^{**}(\theta) < 1$. So, the proof of part (i) is complete.

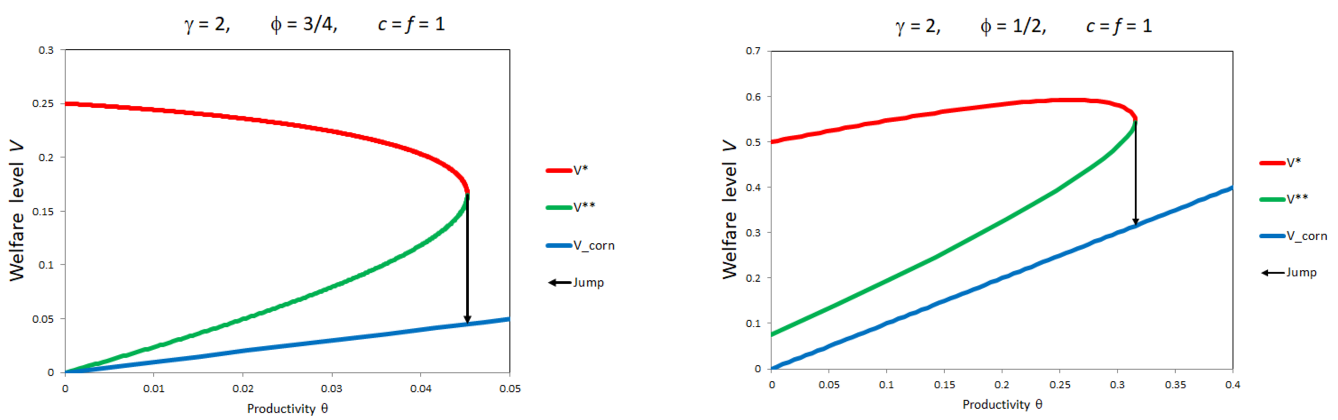
[17] To be precise, it must be that $\gamma \geq 2L/f - 1$. If this condition fails to hold, i.e. when either goods produced by the two sectors are poor substitutes or the relative market size L/f is large, then all the results of Proposition 2 still hold, except that $V^*(\theta)$ varies non-monotonically. Figure 3b provides an illustration.

Part (ii) follows immediately from combining (A.13) with (A.14). Part (iii) is due to the fact that interior equilibria swiftly vanish as θ reaches $\widehat{\theta}$. To prove part (iv), observe that, by Proposition 6 ., the corner equilibrium $\alpha = 1$ is the only equilibrium when θ exceeds $\overline{\theta}$. In this equilibrium, only sector 1 operates. Hence, we have: $V = 1/P_1 = \theta$, i.e. the equilibrium welfare level is growing proportionately with θ. This completes the proof of Proposition 7 . □

Figure 5-a illustrates the foregoing results for $\gamma = 2$, $f/L = 3/4$, and $c = f = 1$.

Figure 5: Welfare losses: Cournot, homogeneous good.

(a) $\gamma \geq 2L/f - 1$ (b) $\gamma < 2L/f - 1$



B. The budget share as a primitive with two sectors and gross substitutes

With two sectors, and if the goods produced by the two sectors are gross substitutes, the budget share function $a(P_1, P_2) \equiv a_1(P_1, P_2)$ of good 1 can actually be viewed *as a primitive of the model*. To be precise, the following result holds.

Lemma 6 (Primitive of the model) *Any differentiable function $a(P_1, P_2)$ that: (i) decreases in P_1; (ii) increases in P_2; and (iii) maps all price vectors (P_1, P_2) into $[0, 1]$ is a budget share of good 1 generated by some monotonic, continuous, and strictly quasi-concave utility function over $\mathbb{R}^2_+$.*

Proof. It suffices to prove that the demand functions

$$u_1(y, P_1, P_2) \equiv \frac{y}{P_1} \alpha \left(\frac{P_1}{y}, \frac{P_2}{y} \right), \quad \text{and} \quad u_2(y, P_1, P_2) \equiv \frac{y}{P_2} \left[1 - \alpha \left(\frac{P_1}{y}, \frac{P_2}{y} \right) \right], \tag{B.1}$$

where $y > 0$ is income, satisfy the following properties: (i) $P_1 u_1(y, P_1, P_2) + P_2 u_2(y, P_1, P_2) = y$, i.e., the budget constraint holds; and (ii) the Slutsky matrix

$$\mathbf{S}(P_1, P_2, y) \equiv \begin{pmatrix} \frac{\partial u_1}{\partial P_1} + u_1 \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial P_2} + u_2 \frac{\partial u_1}{\partial y} \\ \\ \frac{\partial u_2}{\partial P_1} + u_1 \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial P_2} + u_2 \frac{\partial 2}{\partial y} \end{pmatrix}$$

of the demand system (B.1) is symmetric and negative semidefinite.

By Antonelli's (1952) integrability theorem, equation (B.1) describes true Marshallian demands generated by some continuous, monotonic, and strictly quasi-concave utility if and only if (i) and (ii) hold. That the budget constraint is satisfied follows immediately from (B.1). Moreover, the demands (B.1) are homogeneous of degree zero in (P_1, P_2, y). For the case of two goods, this is sufficient for $\mathbf{S}(P_1, P_2, y)$ to be symmetric (see, e.g., Jehle and Reny, 2011, Ch. 2). To prove that $\mathbf{S}(P_1, P_2, y)$ is negative semidefinite, observe that the price vector $\mathbf{p} \equiv (P_1, P_2)$ lies in the nullspace of the Slutsky matrix due to the budget constraint. Furthermore, the vector $\mathbf{e} \equiv (1, 0)$ always renders the quadratic form induced by $\mathbf{S}$ negative. Indeed, the $(1, 1)$-entry of the Slutsky matrix is given by

$$s_{11} \equiv \frac{\partial u}{\partial P_1} + u \cdot \frac{\partial u}{\partial y} = -(1-\alpha)\left(\frac{\alpha y}{P_1^2} - \frac{1}{P_1}\frac{\partial \alpha}{\partial (P_1/y)}\right) - \frac{P_2}{P_1^2}\alpha\frac{\partial \alpha}{\partial (P_2/y)} < 0,$$

so we get

$$\mathbf{e}^T \mathbf{S} \mathbf{e} = s_{11} < 0. \tag{B.2}$$

Because the vectors $\mathbf{e} = (1, 0)$ and $\mathbf{P} = (P_1, P_2) \in \mathbb{R}^2_{++}$ form a basis in $\mathbb{R}^2$, there must exist coefficients θ_1 and θ_2 for any vector $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$ such that $\mathbf{h} = \theta_1 \mathbf{e} + \theta_2 \mathbf{P}$. Computing $\mathbf{h}^T \mathbf{S} \mathbf{h}$, we get:

$$\mathbf{h}^T \mathbf{S} \mathbf{h} = \theta_1^2 \mathbf{e}^T \mathbf{S} \mathbf{e} + 2\theta_1\theta_2 \mathbf{e}^T \mathbf{S} \mathbf{P} + \theta_1^2 \mathbf{P}^T \mathbf{S} \mathbf{P} = \theta_1^2 \mathbf{e}^T \mathbf{S} \mathbf{e} = \theta_1^2 s_{11}.$$

Due to (B.2), we always have $\theta_1^2 s_{11} \leq 0$, whence $\mathbf{S}(P, P_2, y)$ is negative semidefinite. □

Note that Lemma 6 holds only for the case of gross substitutes. Although $a(\cdot, \cdot)$ may not play the role of a primitive of the model when goods are gross complements, the analysis for that case is much more straightforward.

C. Establishing expressions (13)–(15)

The equilibrium conditions are given by:

$$\mathbf{P}^* = \widehat{\mathbf{P}}(\boldsymbol{\alpha}, \theta), \tag{C.1}$$

$$\boldsymbol{\alpha}^* = \mathbf{a}(\mathbf{P}^*). \tag{C.2}$$

The change in the expenditure function due to a shock θ is computed as follows:

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \sum_{s=1}^{S} \frac{\alpha_s^*}{P_s^*} \frac{\mathrm{d}P_s^*}{\mathrm{d}\theta}. \tag{C.3}$$

Differentiating both parts of (C.1) with respect to θ, we get

$$\frac{\mathrm{d}\mathbf{P}^*}{\mathrm{d}\theta} = \frac{\partial \widehat{P}_1}{\partial \theta} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{\partial \widehat{\mathbf{P}}}{\partial \boldsymbol{\alpha}} \frac{\mathrm{d}\boldsymbol{\alpha}^*}{\mathrm{d}\theta},$$

which we can plug into (C.3) to obtain

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\alpha_1^*}{P_1^*} \frac{\partial \widehat{P}_1}{\partial \theta} + \sum_{s=1}^{S} \frac{\alpha_s^*}{P_s^*} \frac{\partial \widehat{P}_s}{\partial \alpha_s} \frac{\mathrm{d}\alpha_s^*}{\mathrm{d}\theta} = \frac{\alpha_1^*}{P_1^*} \frac{\partial \widehat{P}_1}{\partial \theta} + \sum_{s=1}^{S} \mathcal{E}_{\alpha_s}(\widehat{P}_s) \frac{\mathrm{d}\alpha_s^*}{\mathrm{d}\theta}. \tag{C.4}$$

Because $\sum_{s=1}^{S} \mathrm{d}\alpha_s^* / \mathrm{d}\theta = 0$ by definition, we can further restate (C.4) as follows:

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\alpha_1^*}{P_1^*} \frac{\partial \widehat{P}_1}{\partial \theta} + \mathcal{S}, \tag{C.5}$$

where the shifter $\mathcal{S}$ is given by

$$\mathcal{S} \equiv \sum_{s=2}^{S} \left(\mathcal{E}_{\alpha_s}(\widehat{P}_s) - \mathcal{E}_{\alpha_1}(\widehat{P}_1) \right) \frac{\mathrm{d}\alpha_s^*}{\mathrm{d}\theta}. \tag{C.6}$$

To compute the vector $\mathrm{d}\boldsymbol{\alpha}^* / \mathrm{d}\theta$, we differentiate (C.2) with respect to θ to obtain

$$\frac{\mathrm{d}\boldsymbol{\alpha}^*}{\mathrm{d}\theta} = \frac{\partial \widehat{P}_1}{\partial \theta} \frac{\partial \mathbf{a}}{\partial \mathbf{P}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{\partial \mathbf{a}}{\partial \mathbf{P}} \frac{\partial \widehat{\mathbf{P}}}{\partial \boldsymbol{\alpha}} \frac{\mathrm{d}\boldsymbol{\alpha}^*}{\mathrm{d}\theta}.$$

Some algebraic manipulations then yield

$$\left(\mathbf{I} - \frac{\partial \mathbf{a}}{\partial \mathbf{P}} \frac{\partial \widehat{\mathbf{P}}}{\partial \boldsymbol{\alpha}} \right) \frac{\mathrm{d}\boldsymbol{\alpha}^*}{\mathrm{d}\theta} = \frac{\partial \widehat{P}_1}{\partial \theta} \frac{\partial \mathbf{a}}{\partial P_1} \quad \Rightarrow \quad \frac{\mathrm{d}\alpha_s^*}{\mathrm{d}\theta} = \frac{\partial \widehat{P}_1}{\partial \theta} \sum_{r=1}^{S} \frac{\Delta_{rs}}{\Delta} \frac{\partial a_r}{\partial P_1}, \tag{C.7}$$

where $\Delta \equiv \det \left(\mathbf{I} - \frac{\partial \mathbf{a}}{\partial \mathbf{P}} \frac{\partial \widehat{\mathbf{P}}}{\partial \text{ff}} \right)$ and Δ_{rs} is the algebraic complement of the rs-th entry of the matrix $\mathbf{I} - \frac{\partial \mathbf{a}}{\partial \mathbf{P}} \frac{\partial \widehat{\mathbf{P}}}{\partial \boldsymbol{\alpha}}$. Plugging (C.7) into (C.4) finally yields

$$\begin{aligned} \frac{\mathrm{d}E^*}{\mathrm{d}\theta} &= \frac{\partial \widehat{P}_1}{\partial \theta} \frac{\alpha_1^*}{P_1^*} \left(1 + \sum_{s=1}^{S} \frac{\alpha_s^*}{P_s^*} \frac{\partial \widehat{P}_s}{\partial \alpha_s} \sum_{r=1}^{S} \frac{\Delta_{rs}}{\Delta} \frac{\partial a_r}{\partial P_1} \frac{P_1^*}{\alpha_r^*} \frac{\alpha_r^*}{\alpha_1^*} \right) \\ &= \frac{\partial \widehat{P}_1}{\partial \theta} \frac{\alpha_1^*}{P_1^*} \left[1 + \sum_{r=1}^{S} \left(\sum_{s=1}^{S} \frac{\Delta_{rs}}{\Delta} \mathcal{E}_{\alpha_s}(\widehat{P}_s) \right) \frac{\alpha_r^*}{\alpha_1^*} \mathcal{E}_{P_1}(a_r) \right]. \end{aligned}$$

D. Proof of Lemma 1

Proof. The elasticity of substitution $\gamma(P_1, P_2)$ between two goods is defined as follows:

$$\gamma(P_1, P_2) \equiv - \left. \frac{\mathrm{d} \ln(X_1/X_2)}{\mathrm{d} \ln(P_1/P_2)} \right|_{V(P_1,P_2)=\text{const}}, \tag{D.1}$$

where $V(P_1, P_2)$ is the indirect utility function (recall that income is normalized to one), and X_1 and X_2 are the Marshallian demands for goods 1 and 2, respectively:

$$X_1(P_1, P_2) \equiv \frac{a(P_1, P_2)}{P_1}, \quad \text{and} \quad X_2(P_1, P_2) \equiv \frac{1 - a(P_1, P_2)}{P_2}. \tag{D.2}$$

Thus, (D.1) can be equivalently rewritten as

$$\gamma(P_1, P_2) \equiv 1 - \left. \frac{\mathrm{d} \ln[a/(1-a)]}{\mathrm{d} \ln(P_1/P_2)} \right|_{V(P_1,P_2)=\text{const}}. \tag{D.3}$$

Due to the requirement that $V(P_1, P_2)$ be constant, it must be that $(\mathrm{d}P_1,\ \mathrm{d}P_2) \perp \nabla V$, i.e., the prices changes are orthogonal to the gradient ∇V. Furthermore, Roy's identity implies that $\nabla V \parallel (X_1, X_2)$, i.e., the gradient is parallel to the Marshallian demands. Combining these observations with (D.2) yields:

$$(\mathrm{d}P_1,\ \mathrm{d}P_2) \parallel \left(\frac{a(P_1, P_2)}{P_1}, \frac{1 - a(P_1, P_2)}{P_2} \right),$$

which, in turn, implies

$$\begin{aligned}
\left. \frac{\mathrm{d} \ln[a/(1-a)]}{\mathrm{d} \ln(P_1/P_2)} \right|_{V(P_1,P_2)=\text{const}} &= \frac{P_1/P_2}{a/(1-a)} \left. \frac{\mathrm{d}\,(a/(1-a))}{\mathrm{d}(P_1/P_2)} \right|_{V(P_1,P_2)=\text{const}} \\
&= \frac{\frac{1}{(1-a)^2} \frac{\partial a}{\partial P_1} \mathrm{d}P_1 - \frac{1}{(1-a)^2} \frac{\partial (1-a)}{\partial P_2} \mathrm{d}P_2}{\frac{1}{P_2} \mathrm{d}P_1 - \frac{P_1}{P_2^2} \mathrm{d}P_2} \frac{P_1}{P_2} \frac{1 - a(P_1, P_2)}{a(P_1, P_2)} \\
&= \frac{\frac{1}{(1-a)^2} \frac{\partial a}{\partial P_1} \frac{1-a}{P_2} + \frac{1}{(1-a)^2} \frac{\partial (1-a)}{\partial P_2} \frac{a}{P_1}}{\frac{1-a}{P_2} \frac{1}{P_2} + \frac{P_1}{P_2^2} \frac{a}{P_1}} \frac{P_1}{P_2} \frac{1 - a(P_1, P_2)}{a(P_1, P_2)}.
\end{aligned}$$

After simplifications, we thus obtain

$$\frac{\mathrm{d} \ln[a/(1-a)]}{\mathrm{d} \ln(P_1/P_2)} = \mathcal{E}_{P_1}(a) + \mathcal{E}_{P_2}(1-a),$$

which we can plug into (D.3) to obtain (17). □

E. Establishing expression (23)

Although expression (23) is a special case of expression (15), it is more convenient to establish it differently. To obtain it, we first apply the implicit function theorem to (7)–(8) to get:

$$\begin{pmatrix} 1-\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P_1} & -\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P_2} \\ -\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_1} & 1-\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2} \end{pmatrix}\begin{pmatrix} \frac{\mathrm{d}P_1^*}{\mathrm{d}\theta} \\ \frac{\mathrm{d}P_2^*}{\mathrm{d}\theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial\widehat{P}_1}{\partial\theta} \\ 0 \end{pmatrix}.$$

Solving this linear system for $(\mathrm{d}P^*/\mathrm{d}\theta,\ \mathrm{d}P_2^*/\mathrm{d}\theta)^T$ yields:

$$\begin{pmatrix} \frac{\mathrm{d}P^*}{\mathrm{d}\theta} \\ \frac{\mathrm{d}P_2^*}{\mathrm{d}\theta} \end{pmatrix} = \begin{pmatrix} 1-\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P_1} & -\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P_2} \\ -\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_1} & 1-\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2} \end{pmatrix}^{-1}\begin{pmatrix} \frac{\partial\widehat{P}_1}{\partial\theta} \\ 0 \end{pmatrix}.$$

By inverting the matrix, we obtain:

$$\begin{pmatrix} \frac{\mathrm{d}P_1^*}{\mathrm{d}\theta} \\ \frac{\mathrm{d}P_2^*}{\mathrm{d}\theta} \end{pmatrix} = \frac{\frac{\partial\widehat{P}_1}{\partial\theta}}{1-\left(\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P_1}+\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2}\right)}\begin{pmatrix} 1-\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2} \\ \frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_1} \end{pmatrix}. \tag{E.1}$$

Recall next that, by homotheticity of the lower-tier utilities, we have: $u = \alpha/P_1$ and $v = (1-\alpha)/P_2$. Using this relationship, and plugging the expressions (E.1) for $\mathrm{d}P_1^*/\mathrm{d}\theta$ and $\mathrm{d}P_2^*/\mathrm{d}\theta$ into (C.3) and, we get

$$\frac{\mathrm{d}E^*}{\mathrm{d}\theta} = \frac{\partial\widehat{P}_1}{\partial\theta}\frac{\frac{\alpha}{P_1}\left(1-\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2}\right)+\frac{1-\alpha}{P_2}\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P}}{1-\left(\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P}+\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2}\right)}. \tag{E.2}$$

Simplifying the numerator of the fraction in the right-hand side of (E.2) yields:

$$\begin{aligned} &\frac{\alpha}{P_1}\left(1-\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2}\right)+\frac{1-\alpha}{P_2}\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_1} \\ &\quad=\frac{\alpha}{P_1}\left(1-\frac{1-\alpha}{P_2}\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2}\frac{P_2}{1-\alpha}+\frac{1-\alpha}{P_2}\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_1}\frac{P_1}{1-\alpha}\frac{1-\alpha}{\alpha}\right) \\ &\quad=\frac{\alpha}{P_1}\left(1-\mathcal{E}_{P_2}(1-a)\mathcal{E}_{1-\alpha}(\widehat{P}_2)-\mathcal{E}_{1-\alpha}(\widehat{P}_2)\mathcal{E}_{P_1}(a)\right). \end{aligned} \tag{E.3}$$

Note also that in equilibrium we have

$$\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P_1} = \mathcal{E}_\alpha(\widehat{P}_1)\mathcal{E}_{P_1}(a), \quad \text{and} \quad \frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2} = \mathcal{E}_{P_2}(1-a)\mathcal{E}_{1-\alpha}(\widehat{P}_2), \tag{E.4}$$

so that the denominator in (E.2) becomes

$$1-\frac{\partial\widehat{P}_1}{\partial\alpha}\frac{\partial a}{\partial P}-\frac{\partial\widehat{P}_2}{\partial(1-\alpha)}\frac{\partial(1-a)}{\partial P_2}=1-\mathcal{E}_\alpha(\widehat{P}_1)\mathcal{E}_P(a)-\mathcal{E}_{P_2}(1-a)\mathcal{E}_{1-\alpha}(\widehat{P}_2). \tag{E.5}$$

Plugging (E.3)–(E.5) into (E.2) completes the proof. □

F. Cournot oligopoly

In this appendix, we provide details for the derivations of the Cournot oligopoly equilibrium case in Section 5.2. We start with consumers' utility maximization problem. The first-order conditions with respect to variety $i \in \mathcal{I}_2$ are given by:

$$p_i=\frac{x_i^{-1/\sigma_s}}{\lambda_2}, \tag{F.1}$$

where λ_2 is the marginal utility of additional spending on good 2:

$$\lambda_2\equiv\frac{1}{1-\alpha}\sum_{j=1}^{n_2}x_j^{1-1/\sigma_2}. \tag{F.2}$$

Turning next to profit maximization, the profit π_i of firm $i \in \mathcal{I}_2$ is given by

$$\pi_i=\frac{x_i^{1-1/\sigma_2}}{\lambda_2}-c_2x_i, \tag{F.3}$$

where $c_2 > 0$ is the constant marginal cost that we assume identical across all firms operating in sector 2. Firm i's first-order condition follows the 'marginal revenue = marginal cost' principle. Taking into account that—unlike in monopolistic competition—each firm has an impact on the market, we get:

$$\left(1-\frac{1}{\sigma_2}\right)\frac{x_i^{-1/\sigma_2}}{\lambda_2}-\frac{x_i^{1-1/\sigma_2}}{\lambda_2^2}\frac{\partial\lambda_2}{\partial x_i}=c_2. \tag{F.4}$$

Combining (F.4) with (F.2), we obtain after simplifications:

$$\left(1-\frac{x_i^{1-1/\sigma_2}}{\sum_{j=1}^{n}x_j^{1-1/\sigma_2}}\right)\left(1-\frac{1}{\sigma_2}\right)\frac{x_i^{-1/\sigma_2}}{\lambda_2}=c_2.$$

This implies that, together with (F.1), the price set by all n_2 firms at a symmetric Cournot-Nash equilibrium (where $x_1 = x_2 = \ldots = x_{n_2}$) is given by

$$\widetilde{p}_2(n_2)=\frac{c_2\sigma_2}{\sigma_2-1}\frac{n_2}{n_2-1}. \tag{F.5}$$

Because of the budget constraint, at a symmetric outcome we have $n_2 p_2 x_2 = 1 - \alpha$, whence the Cournot-Nash equilibrium output $\widetilde{x}_2(n_2)$ of each firm $i \in \mathcal{I}_2$ is given by

$$\widetilde{x}_2(n_2) = \frac{1-\alpha}{n_2 \widetilde{p}_2(n_2)} = \frac{1-\alpha}{c_2}\frac{\sigma_2 - 1}{\sigma_2}\frac{n_2 - 1}{n_2^2}. \tag{F.6}$$

Combining (F.5) with (F.6), we find that the Cournot-Nash equilibrium profit $\widetilde{\pi}_2(n_2)$ of each firm $i \in \mathcal{I}_2$ is given by

$$\widetilde{\pi}_2(n_2) = \frac{1-\alpha}{n_2}\left(1 - \frac{\sigma_2 - 1}{\sigma_2}\frac{n_2 - 1}{n_2}\right). \tag{F.7}$$

We are now equipped to determine the number $\widehat{n}_2(1-\alpha)$ of firms operating in sector 2 and the price index $\widehat{P}_2(1-\alpha)$. Let $f_2 > 0$ be the fixed cost required to start a business in sector 2. Then, $\widehat{n}_2(1-\alpha)$ is pinned down by the free-entry condition: $\widetilde{\pi}_2(n_2) = f_2$. Combining it with (F.7) yields after simplifications:

$$\frac{f_2 \sigma_2}{1-\alpha} n_2^2 = n_2 + \sigma_2 - 1. \tag{F.8}$$

Solving the quadratic equation (F.8) for n_2 yields a unique positive solution

$$\widehat{n}_2(1-\alpha) = \left(1 + \sqrt{1 + 4 f_2 \sigma_2 \frac{\sigma_2 - 1}{1-\alpha}}\right)\frac{1-\alpha}{2 f_2 \sigma_2}. \tag{F.9}$$

At a symmetric outcome, the CES price index is given by $\widehat{P}_2(1-\alpha) = \widehat{p}_2(1-\alpha)\left[\widehat{n}_2(1-\alpha)\right]^{1/(1-\sigma_2)}$, so that the elasticity $\mathcal{E}_{1-\alpha}(\widehat{P}_2)$ can be expressed as

$$\mathcal{E}_{1-\alpha}(\widehat{P}_2) = \frac{1}{1-\sigma_2}\mathcal{E}_{1-\alpha}(\widehat{n}_2) + \mathcal{E}_{1-\alpha}(\widehat{p}_2). \tag{F.10}$$

Using expressions (F.5) and (F.9), we obtain the elasticities:

$$\mathcal{E}_{1-\alpha}(\widehat{n}_2) = \frac{\widehat{n}_2(1-\alpha) + \sigma_2 - 1}{\widehat{n}_2(1-\alpha) + 2(\sigma_2 - 1)}, \qquad \mathcal{E}_{1-\alpha}(\widehat{p}_2) = -\frac{1}{\widehat{n}_2(1-\alpha) - 1}\mathcal{E}_{1-\alpha}(\widehat{n}_2). \tag{F.11}$$

As shown by (F.11), the following inequalities are always satisfied: $\mathcal{E}_{1-\alpha}(\widehat{n}_2) > 0 > \mathcal{E}_{1-\alpha}(\widehat{p}_2)$. Combining this with (F.10) we conclude that $\mathcal{E}_{1-\alpha}(\widehat{P}_2) < 0$, i.e., Assumption **(A2)** holds. Using (F.10) and (F.11) in (25), the condition for welfare losses is as follows:

$$n_2^* < \overline{n}_2 \equiv 1 + \left(\sigma_2 - \frac{1}{2}\right)\left[\sqrt{1 + \frac{\sigma_2(\sigma_2 - 1)}{(\sigma_2 - 1/2)^2}\frac{\gamma - 1}{\sigma_2 - \gamma}} - 1\right]. \tag{F.12}$$

G. CES monopolistic competition examples

The sectoral price indices (26) are relevant for CES models of trade and firm heterogeneity à la Melitz (2003). We now develop one example of each and show that welfare losses can arise for a broad set of parameter values in both types of models. However, as stated in Proposition (5), it requires a smaller intersectoral elascticity of substitution than the elasticity of substitution within the second sector.[18]

We start with a trade model with homogeneous firms in Appendix F.1, and then present a two-sector closed economy model with firm heterogeneity in Appendix F.2. We show that both sectoral trade liberalization and positive sectoral productivity shocks can generate welfare losses in the aggregate economy.

G.1. Trade liberalization

We first study trade between two countries. Each country has two sectors that both produce horizontally differentiated varieties. The market structure in each sector is monopolistic competition among symmetric firms. We assume that countries are symmetric in all respects, including preferences, production costs, and population sizes. Hence, wages are equalized in equilibrium.[19] Each country hosts L consumers who have identical CES subutilities.

Firms in both sectors incur a constant fixed cost, f, and a constant marginal cost, c, both paid in terms of labor. The shipping costs are of the iceberg form: $\tau_1 \geq 1$ and $\tau_2 \geq 1$ units of the first and second goods have to be dispatched for one unit to arrive.

We assume that the freeness of trade, $\tau_1^{1-\sigma_1}$, stands for the parameter $\theta \in \Theta \equiv (0, 1]$. Therefore, the CES price indices (26) in each sector are given by

$$\widehat{P_1}(\alpha, \tau_1) = K_1 \alpha^{\frac{1}{1-\sigma_1}} \left(1 + \tau_1^{1-\sigma_1}\right)^{\frac{1}{1-\sigma_1}}, \qquad \widehat{P_2}(1-\alpha) = K_2 (1-\alpha)^{\frac{1}{1-\sigma_2}} \left(1 + \tau_2^{1-\sigma_2}\right)^{\frac{1}{1-\sigma_2}}, \quad \text{(G.1)}$$

where $K_i \equiv [f(\sigma_i - 1)/(c_i L)]^{\frac{1}{\sigma_i - 1}}$, $i = 1, 2$, is a positive coefficient independent of α and $\phi(\theta) \equiv (1 + \tau_1^{1-\sigma_1})^{\frac{1}{1-\sigma_1}}$ is a function of trade freeness. In this setting, a shock to θ is a shock to the freeness of trade. Note that equation (G.1) immediately implies that (A1)-(A2) hold.

The analysis of Section 3 yields strong welfare results for this class of models. First, as implied by Proposition 3, when goods produced by the two sectors are gross complements, trade liberalization, i.e., a reduction in τ_1 in the first sector, always leads to welfare gains. Second, as implied by Proposition 4, when goods are gross substitutes, trade liberalization

[18] See Matsuyama (1995) for a discussion of that assumption and possible interpretations.

[19] An alternative setting is asymmetric in population countries and free trade in the second sector. The foregoing analysis for both settings is valid.

in the first sector may lead to welfare losses under well-behaved upper-tier preferences. This result is to be contrasted with those typically obtained in one-sector settings, where assumption **(A1)** would *always* guarantee gains from trade (Krugman, 1980). In the CES case, love for variety associated with a particular sector is inversely related to the elasticity of substitution in this sector. Therefore, losses from trade arise when the love for variety of the good produced by the first sector, i.e., the one in which trade liberalization takes place, is lower than love for variety for the good produced by the second sector. The intuition behind this result is as follows. Trade liberalization in the first sector results in a higher budget share α of this sector, thus reducing product diversity in the second sector. As a result, even though product diversity increases in the first sector, consumers end up with a lower welfare level. To sum up, standard arguments used in industrial organization that lower costs lead to lower prices which always make consumers better off need not hold in the multi-sectoral settings with endogenous entry.

When upper-tier preferences $U(\cdot,\cdot)$ are Cobb-Douglas, losses from trade never arise. However, many studies on multi-sectoral models employ Cobb-Douglas upper-tier preferences because they are easy to handle. This could at least partially explain why in these various settings trade liberalization always lead to welfare improvement.

To show that losses from trade is not a zero-measure case, let us develop a concrete example. Consider CES preferences in each of the two sectors so that the equilibrium price indices are given by (G.1). We parametrize the upper-tier preferences using the following budget share function

$$a(P_1, P_2) = e^{-\zeta\frac{P_1}{P_2}}, \tag{G.2}$$

where $\zeta > 0$ is a parameter that reflects the intersectoral elasticity of substitution. This budget share function corresponds to a well-defined upper-tier utility function (see Appendix A). Furthermore, it captures the case of gross substitutes. To see this, note that the elasticities of the budget share function with respect to the price indices are given by

$$\mathcal{E}_{P_1}(a) = -\zeta\frac{P_1}{P_2} < 0, \quad \text{and} \quad \mathcal{E}_{P_2}(1-a) = \frac{1}{1-a(P_1,P_2)^{-1}}\zeta\frac{P_1}{P_2} < 0. \tag{G.3}$$

The elasticities of the price indices (G.1) with respect to budget shares are given by

$$\mathcal{E}_{\alpha}(\widehat{P}_1) = \frac{1}{1-\sigma_1}, \qquad \mathcal{E}_{1-\alpha}(\widehat{P}_2) = \frac{1}{1-\sigma_2}. \tag{G.4}$$

In what follows, we assume that $1 < \sigma_1 < \sigma_2$. Combining (G.1) and (G.2), we obtain the following fixed point condition:

$$\alpha^* = \exp\left[-\zeta\frac{(1-\alpha^*)^{\frac{1}{\sigma_2-1}}}{(\alpha^*)^{\frac{1}{\sigma_1-1}}}\frac{\left(1+\tau_2^{1-\sigma_2}\right)^{\frac{1}{\sigma_2-1}}}{\left(1+\tau_1^{1-\sigma_1}\right)^{\frac{1}{\sigma_1-1}}}\right] \quad \Rightarrow \quad \ln(1/\alpha^*) = \zeta\frac{(1-\alpha^*)^{\frac{1}{\sigma_2-1}}}{(\alpha^*)^{\frac{1}{\sigma_1-1}}}\frac{\left(1+\tau_2^{1-\sigma_2}\right)^{\frac{1}{\sigma_2-1}}}{\left(1+\tau_1^{1-\sigma_1}\right)^{\frac{1}{\sigma_1-1}}}. \tag{G.5}$$

One can show that the right-hand side of the second expression in (G.5) is an increasing function of α^*, with only one inflection point between 0 and 1. Moreover, it is concave and larger than the left-hand side when $\alpha^* \to 0$, while it is convex and reaches 0 when $\alpha^* = 1$. Therefore, (G.5) pins down a unique $\alpha^* \in (0, 1)$. This unique interior equilibrium is stable since the right-hand side of the second expression in (G.5) intersects the 45° line from above.

Plugging (G.3)–(G.4) into (23), using (G.2), and simplifying, we get

$$\mathcal{M} = \frac{1 - \frac{1}{\sigma_2 - 1}\frac{\ln(1/\alpha^*)}{1-\alpha^*}}{1 - \frac{\ln(1/\alpha^*)}{\sigma_1 - 1} - \frac{\ln(1/\alpha^*)}{\sigma_2 - 1}\frac{\alpha^*}{1-\alpha^*}}. \tag{G.6}$$

The Proposition XX of equilibrium guarantees that the denominator of expression (G.6) is positive. Thus, losses from trade arise when the numerator is positive:

$$\frac{\ln(1/\alpha^*)}{1-\alpha^*} > \sigma_2 - 1. \tag{G.7}$$

Using the expression of $\ln(1/\alpha^*)$ from (G.5), we thus have the condition

$$\frac{\ln(1/\alpha^*)}{1-\alpha^*} = \zeta \frac{(1-\alpha^*)^{\frac{\sigma_2}{\sigma_2-1}}}{(\alpha^*)^{\frac{1}{\sigma_1-1}}} \frac{\left(1+\tau_2^{1-\sigma_2}\right)^{\frac{1}{\sigma_2-1}}}{\left(1+\tau_1^{1-\sigma_1}\right)^{\frac{1}{\sigma_1-1}}}. \tag{G.8}$$

The left-hand side of (G.7), as given by (G.8), is larger than 1 for $\alpha^* \in (0, 1)$. Therefore, in this example losses from trade always arise when the varieties of sector 2 are highly differentiated ($1 < \sigma_2 < 2$). Moreover, expression (G.8) is an increasing function of σ_1 and a decreasing function of σ_2. Hence, when $\sigma_2 > 2$, losses from trade arise when the gap between two sectoral elasticities is large enough. Put differently, for any given σ_2 there exists a threshold value $\overline{\sigma}_1$ such that for any $\sigma_1 > \overline{\sigma}_1$ (G.7) holds. In that case, a positive trade shock to sector 1 reduces welfare in the economy.

G.2. Productivity improvements

Consider next a closed two-sector economy where both sectors are monopolistically competitive and firms are heterogeneous in productivity à la Melitz (2003). The lower-tier utilities are CES, with elasticities of substitution that satisfy $1 < \sigma_2 < \sigma_1$. Let $\Gamma_0(\cdot)$ and $\Gamma_1(\cdot)$ be two cumultative distribution functions (CDF) of firms' marginal cost—the inverse of productivity—defined over $\mathbb{R}_+$ such that $\Gamma_0(\cdot)$ first-order stochastically dominates $\Gamma_1(\cdot)$. Assume that the CDF of the marginal cost distribution c in the first sector is given by $\Gamma_\theta(c) \equiv (1-\theta)\Gamma_0(c) + \theta\Gamma_1(c)$, $\theta \in (0, 1)$. Hence, an increase in $\theta \in (0, 1)$ corresponds to a left-shift of the cost distribution, i.e., a positive productivity shock.

We start by fixing the budget share $\alpha \in (0, 1)$ of sector 1 and consider it as if it were a one-sector closed economy with per-capita income equal to α.[20] Following Melitz (2003), the cutoff cost $\widehat{c}_1$ and the cutoff firm output $\widehat{q}_1$ are uniquely determined by the cutoff condition

$$\widehat{c}_1\widehat{q}_1 = (\sigma_1 - 1)F \tag{G.9}$$

and the zero expected profit condition

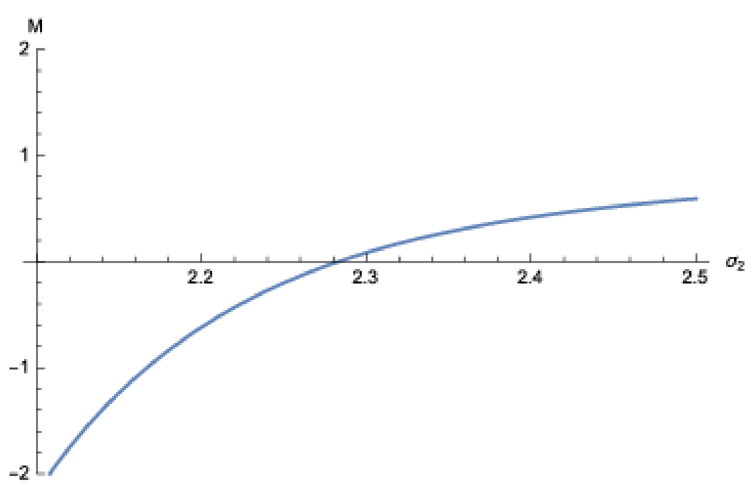
$$\int_0^{\widehat{c}_1} \left[\left(\frac{\widehat{c}_1}{c} \right)^{\sigma_1 - 1} - 1 \right] \mathrm{d}\Gamma_\theta(c) = \frac{F_e}{F}, \tag{G.10}$$

where F is a fixed cost of production and F_e is a fixed cost of entry.[21] Because the lower-tier utilities are CES, the market clearing condition and the optimal pricing rule imply the following equalities: $\widehat{q} = \frac{\alpha}{P_1} (\widehat{p}_1 / P_1)^{-\sigma_1}$ and $\widehat{p}_1 = \frac{\widehat{c}_1 \sigma_1}{\sigma_1 - 1}$, where $\widehat{p}_1$ is the price of the cutoff firm. Plugging these expressions into (G.9) and solving the resulting equation for P_1, we obtain

$$\widehat{P}_1(\theta, \alpha) = k_1 \widehat{c}_1(\theta) \alpha^{1/(1-\sigma_1)}, \tag{G.11}$$

where $k_1 \equiv \frac{\sigma_1}{\sigma_1 - 1} (\sigma_1 F)^{1/(\sigma_1 - 1)} > 0$ is a positive bundle of parameters that depends neither on α nor on θ. Setting $\phi(\theta) \equiv k_1 \widehat{c}_1(\theta)$, (G.11) is structurally identical to (26). Thus, Proposition 5 is applicable. We therefore conclude that there exist well-behaved upper-tier utility functions exhibiting gross substitutability such that a decrease in θ—an improvement in sector-1 productivity—leads to welfare losses.

Figure 6: Values of $\mathcal{M}$ for different values of σ_2.



[20]In a closed economy, we normalize the wage to one without loss of generality.
[21]Because the left-hand side of (G.10) is an increasing function of $\widehat{c}$, (G.10) has a unique solution $\widehat{c} = \widehat{c}(\theta)$. Furthermore, using the expression for $\Gamma_\theta(\cdot)$ and (G.10) yields that $\widehat{c}(\theta)$ increases in θ.

Numerical example. To fix ideas, let us develop a concrete example. Assume again that the budget share function is given by (G.2), with elasticities (G.3). From (G.11), we further have

$$\mathcal{E}_\alpha(P_1) = -\frac{1}{\sigma_1 - 1} < 0 \quad \text{and} \quad \mathcal{E}_{1-\alpha}(P_2) = -\frac{1}{\sigma_2 - 1} < 0.$$

By assumption $\sigma_1 > \sigma_2$ so that $\mathcal{E}_\alpha(P_1) < \mathcal{E}_{1-\alpha}(P_2)$. Hence, at any unique interior equilibrium, it must be that $\mathcal{M} < 1$, i.e., part of the gains in sector 1 are offset by losses in sector 2.

We can readily verify that $\mathcal{M} < 0$ can occur, i.e., the gains in sector 1 are more than offset by the losses in sector 2. To see this, let $\sigma_1 = 6$ and $\zeta = \ln 2$ (so that budget shares are equal for equal price indices). Assume further that

$$\Gamma_1(c) = (1-\theta) LogNormal(2,1) + \theta LogNormal(3,1), \quad \text{and} \quad \Gamma_2(c) = LogNormal(3,1), \tag{G.12}$$

where $LogNormal(\mu, \sigma)$ is the lognormal distribution with mean μ and standard deviation σ.[22] Starting with $\theta = 1$, both sectors have the same underlying productivity distribution, and as θ decreases sector 1 progressively has lower costs (i.e., becomes more productive). Finally, we set $F_e = F = 100$.

Figure 6 depicts the equilibrium values of $\mathcal{M}$ as we vary σ_2 between 2.1 and 2.5. As shown, over that range of parameter values, there are two regimes. First, when σ_2 exceeds about 2.3, we have $0 < \mathcal{M} < 1$. In words, we have a regime in which a decrease in θ—a productivity improvement in sector 1—increases welfare, but less than it would do in a single-sector economy. Part of the gains in sector 1 are offset by losses in sector 2. Second, when $\sigma_2 < 2.3$, $\mathcal{M} < 0$. In that regime, the losses in sector 2 exceed the gains in sector 1 so that the productivity improvement in sector 1 actually reduces welfare in the economy.

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[22] Using Pareto distributions with the same shape parameter but with different supports—where the support of sector 1 is parametrized by θ—allows to solve for the cutoffs in a closed form. Yet, these cutoffs depend in a highly non-linear way on the elasticities σ_1 and σ_2, which does not allow to get simple results.

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