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JEL Classification: D43, L41

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The Herfindahl-Hirschman Index and the distribution of social surplus [*]

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December 16, 2019

Abstract

I show that in a broad range of oligopoly models the Herfindahl-Hirschman index (HHI) reflects the ratio of producers' surplus and consumers' surplus and therefore the division of surplus between firms' owners and consumers.

JEL Classification: D43, L41

Keywords: HHI, MHHI, producers' surplus, consumers' surplus, oligopoly

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1 Introduction

The Herfindahl-Hirschman Index (HHI) of concentration, calculated as the sum of the squared market shares of firms, is by now a standard structural measurement tool for assessing the intensity of competition.[1] Although antitrust and merger-analysis policymakers often draw inferences about social welfare and changes in social welfare from this measurement, it is not entirely clear why the HHI an appropriate measure of social welfare and what exactly are its normative properties.

In this paper I show that in a broad range of oligopoly models, the HHI can be expressed as an increasing function of the ratio of producers' surplus to consumers' surplus and therefore reflects the distribution of total surplus between firms' owners and consumers, with higher values of HHI being associated with a lower share of consumers in the total surplus. Although this result pertains to the distribution of total welfare rather than its level, it is nonetheless interesting and policy relevant – especially in an era when income distribution issues have gained great prominence.[2]

More specifically, I begin by showing that in a Cournot model, where firms have (not necessarily identical) constant marginal costs, the HHI can be expressed as $H = \frac{1}{\eta(Q^*)} \frac{PS^*}{CS^*}$, where PS^* and CS^* are the equilibrium values of producers' and consumers' surplus, and $\eta(Q^*)$ is the elasticity of consumers' surplus with respect to the equilibrium output level. That is, the HHI is proportional to the ratio of producers' to consumers' surplus, and the factor of proportionality is equal to the inverse of $\eta(Q^*)$. This result generalizes to the case of common ownership with MHHI (the modified HHI as defined by O'brien and Salop (2000)) replacing HHI. The equation $H = \frac{1}{\eta(Q^*)} \frac{PS^*}{CS^*}$ can also be rewritten as $\frac{CS^*}{W^*} = \frac{1}{1+\eta(Q^*)H}$, where W^* is total surplus. Stated in this way, the HHI, along with $\eta(Q^*)$, determine the share of consumers in the total surplus. When at least one firm has an increasing marginal cost, the relationship between the HHI and $\frac{PS^*}{CS^*}$ becomes $H < \frac{1}{\eta(Q^*)} \frac{PS^*}{CS^*}$, which implies in turn that $\frac{CS^*}{W^*} < \frac{1}{1+\eta(Q^*)H}$. Now, the HHI, along with $\eta(Q^*)$,

[1] The index can be viewed as a weighted sum of the market shares of firms, where the weights are equal to the market shares. The index was independently developed by Hirschman (1945), who used it as a measure of a country's foreign trade concentration, and by Herfindahl (1950), who used it to measure "gross changes" in the concentration of the U.S. steel industry. The index was then used by Stigler (1964) in his seminal paper on collusion, and became popular after William Baxter introduced it in the Department of Justice when he served as the Assistant Attorney General in charge of the Antitrust Division in the early 1980's, and especially after it was included in the 1982 horizontal merger guidelines. For a history of HHI, see Calkins (1983).

[2] See for instance, Baker and Salop (2015), Hovenkamp (2017), and Lyons (2017). In particular Baker and Salop (2015) write "antitrust law and regulatory agencies could address inequality more broadly by treating the reduction of inequality as an explicit antitrust goal."

provide a lower bound on the share of consumers in the total surplus.

To illustrate the result that $\frac{CS^*}{W^*} = \frac{1}{1+\eta(Q^*)H}$, note that the 2010 horizontal merger guidelines of the DOJ and the FTC define markets as unconcentrated if HHI is below $1,500$ and state that "Mergers resulting in unconcentrated markets are unlikely to have adverse competitive effects and ordinarily require no further analysis." The guidelines also define markets as highly concentrated if HHI is above $2,500$ and state that "Mergers resulting in highly concentrated markets that involve an increase in the HHI of more than 200 points will be presumed to be likely to enhance market power."[3] If for example $\eta(Q^*) = 2$ (below I show that this is the case for instance when demand is linear), these thresholds can be interpreted as reflecting a willingness of the DOJ and FTC to tolerate mergers when consumers' surplus is at least $\frac{1}{1+2\times 0.15} = 77\%$ of the total surplus, but not tolerate relatively larger mergers when consumers' surplus is less than $\frac{1}{1+2\times 0.25} = 67\%$ of the total surplus.[4]

The result that $\frac{CS^*}{W^*} = \frac{1}{1+\eta(Q^*)H}$ implies that if we hold $\eta(Q^*)$ constant, an increase in HHI is associated with a decrease in the share of consumers in total surplus. It turns out that for a large class of inverse demand functions, including linear, constant elasticity, and log-linear inverse demand functions, $\eta(Q^*)$ is indeed a constant and is equal to the inverse of the cost pass-through rate. Consequently, an increase in HHI, due to demand or cost shocks or due to a decrease in the number of firms (say due to a merger or an exit), is associated with a decrease in the share of consumers in the total surplus. This conclusion does not change when $\eta(Q^*)$ is not constant, provided that $\eta(Q^*)H$ is increasing when H is increasing.

Turning to differentiated products, I show that in models with a linear demand system and constant marginal costs (e.g., Spence (1976), Dixit (1979), Singh and Vives (1984), Shubik and Levitan (1980), or the Vickery-Salop circular city model (Vickery (1964) and Salop (1979)) with either quantity or price competition, HHI can be expressed as an increasing function of the ratio of producers' to consumers' surplus. Hence, as in the Cournot case, larger values of HHI are associated with distributions of total surplus which are more favorable to firms' owners and less favorable to consumers. It should be noted however that since HHI is endogenous, demand or cost shocks or

[3] In the standard usage of the HHI in merger analysis, market shares are represented as percentage points, and as a result the HHI varies from 0 to $10,000$.

[4] At least in principle, antitrust policy in the U.S. is based on the consumer welfare standard, which considers only consumers'surplus. The tolerance of the merger guidelines to mergers when the HHI is below $1,500$ points, even if they raise the HHI and hence lower the share of consumers in the total surplus shows that in pratcice the DOJ and FTC do not assign all weight to consumers surplus.

changes in the number of firms, which cause an increase in HHI, may also affect the relationship between HHI and $\frac{PS^*}{CS^*}$ and hence it is not immediately obvious that an increase in HHI will be always associated with an increase in $\frac{PS^*}{CS^*}$. However I show that an increase in HHI due to demand or cost shocks is always associated with an increase in $\frac{PS^*}{CS^*}$. Moreover, an increase in HHI due to a change in the number of firms is also associated with an increase in $\frac{PS^*}{CS^*}$ when demand is given by the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, or when firms are symmetric and demand is given by the Shubik and Levitan (1980) specification.

The literature has already provided several interpretations of HHI. These interpretations are based on the Cournot model.[5] Cowling and Waterson (1976) show that when each firm has a constant marginal cost, HHI equals $\varepsilon \frac{PS^*}{R^*}$, where ε is the elasticity of demand and R^* is the equilibrium aggregate revenue.[6] Dansby and Willig (1979) consider a more general setting where firms do not necessarily have constant marginal costs and show that HHI equals $(\varepsilon\phi^*)^2$, where ϕ^* is the "industry performance gradient," which reflects the rate of change in welfare as output is adjusted by moving within a fixed distance from the equilibrium point. Kwoka (1985) considers a similar setting and shows that HHI equals εL^*, where $L^* \equiv \sum_{i=1}^{n} s_i^* L_i^*$ is a weighted average of the equilibrium Lerner indices of individual firms, with s_i^* being the equilibrium market share of firm i, and $L_i^* = \frac{p^* - c_i'}{p}$ is the equilibrium Lerner index of firm i. The three papers then imply that if we hold ε constant, an increase in HHI is associated with an increase in (i) the ratio of producers' surplus to aggregate revenues, (ii) the industry performance gradient, and (iii) the average price-cost margin in the industry. Farrell and Shapiro (1990) also consider a general Cournot model and show that an increase in HHI may be associated with an increase in welfare even when output falls. The reason is that in a Cournot equilibrium, larger firms have lower marginal costs, so if production shifts from small to large firms (and hence HHI increases), the cost savings from more efficient production may outweigh the negative effect of the reduction in total output.[7] While these

[5] An exception is Nocke and Schutz (2018) who study oligopoly with price competition and show, using a Taylor approximation, that HHI is proportional to the difference between consumer surplus and aggregate surplus under oligopoly and under monopolistic competition.

[6] Multiplying the individual first-order conditions for profit maximization, $p + p'q_i - k_i = 0$, by q_i, and summing up the product over all firms, yields $\sum_{i=1}^{n}\left(pq_i + p'(q_i)^2 - k_i q_i\right) = 0$, which can be rewritten as $\sum_{i=1}^{n}(p - k_i)\, q_i = -p' \sum_{i=1}^{n}(q_i)^2 \equiv -p'H$. Dividing both sides of the equation by $\sum_{i=1}^{n} pq_i = pQ$, using the fact that $\varepsilon \equiv -\frac{pQ}{p'}$, and rearranging, yields $\frac{\sum_{i=1}^{n}(pq_i - c_i)}{\sum_{i=1}^{n} pq_i} = \frac{-p'H}{pQ} \equiv \frac{H}{\varepsilon}$.

[7] Nocke and Whinston (2019) show that in a general Cournot model, the merger-induces efficiencies needed to ensure that the merger has no effect on consumers' surplus are independent of HHI, but are increasing with the naïve-computed change in HHI due to the merger (twice the product of the market shares of the merging firms).

results are all helpful, they do not tell us how HHI is related to the distribution of surplus between firms' owners and consumers, which is the main focus of this paper.

The rest of the paper is organized as follows. In Section 2, I present the main result, which I establish in the context of the Cournot model. In Section 3 I show that the main result generalizes to the case of common ownership; the only difference is that MHHI replaces HHI. In Section 4, I show that the main insight from Section 2 also generalizes to the case of differentiated products with linear demands. Concluding remarks are in Section 5. The Appendix contains technical proofs and derivations.

2 The main result

Consider a Cournot model with n firms. The cost of each firm i is $c_i(q_i) = F_i + k_i q_i$, where $F_i > 0$ is a fixed cost, $k_i > 0$ is firm i's constant marginal cost, and q_i is firm i's output. The inverse demand function is $p(Q)$, where $Q = \sum_{i=1}^{n} q_i$ is aggregate output, and $p'(Q) < 0$ and $p'(Q) + p''(Q)Q \leq 0$. These assumptions are standard (see e.g., Farrell and Shapiro, 1990) and ensure that the model is well behaved.[8] Each firm i chooses its output, q_i, to maximize its respective profit

$$\pi_i = p(Q)\, q_i - F_i - k_i q_i.$$

An interior Nash equilibrium is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:[9]

$$p(Q) + p'(Q)\, q_i - k_i = 0, \qquad i = 1, 2, , \ldots, n.$$

The price-cost margin of each firm i in an interior Nash equilibrium is given by

$$p(Q^*) - k_i = -p'(Q^*)\, q_i^*,$$

where $Q^* = \sum_{i=1}^{n} q_i^*$. Using this expression, the equilibrium producer surplus of each firm i (its profit gross of fixed cost) can be written as

$$PS_i^* = (p(Q^*) - k_i)\, q_i^* = -p'(Q^*)\, (q_i^*)^2. \tag{1}$$

[8] In particular, the latter assumption implies that the marginal revenue of each firm is downward sloping, i.e., $p'(Q) + p''(Q)\, q_i \leq 0$ for all q_i. If $p''(Q) \leq 0$, the result follows trivially and if $p''(Q) > 0$, then $p'(Q) + p''(Q)\, q_i < p'(Q) + p''(Q)\, Q \leq 0$.

[9] The equilibrium is interior if the price when the $n-1$ most efficient firms produce, is lower than k_i for the least efficient firm.

The equilibrium value of consumers' surplus is

$$CS^* = \int_0^{Q^*} p(z)\,dz - p(Q^*)\,Q^*. \tag{2}$$

Noting that $(CS^*)' = -p'(Q^*)\,Q^*$, and using (1), (aggregate) producers' surplus is given by

$$PS^* \equiv \sum_{i=1}^{n} PS_i^* = \frac{(CS^*)' \sum_{i=1}^{n} (q_i^*)^2}{Q^*}. \tag{3}$$

Given a Nash equilibrium, the market share of firm i is simply $\frac{q_i^*}{Q^*}$. Hence, HHI is given by

$$H = \sum_{i=1}^{n} \left(\frac{q_i^*}{Q^*}\right)^2 = \frac{\sum_{i=1}^{n} (q_i^*)^2}{(Q^*)^2}. \tag{4}$$

Substituting for $\sum_{i=1}^{n} (q_i^*)^2$ from (3) into (4) and rearranging, yields the following result (the proof is in the Appendix along with all other proofs):

Proposition 1: *In an n firms Cournot model, where firms have (possibly different) constant marginal costs,*

$$H = \frac{PS^*}{\eta(Q^*)\,CS^*}, \tag{5}$$

where $\eta(Q^*) \equiv \frac{Q^*(CS^*)'}{CS^*}$ is the elasticity of consumers' surplus with respect to output. Moreover, $\eta(Q^*) \geq 0$ if and only if $p'(Q) + p''(Q)\,Q \leq 0$.

Proposition 1 implies that HHI is proportional to $\frac{PS^*}{CS^*}$, which is the ratio of producers' to consumers' surplus;[10] the factor of proportionality is the inverse of the elasticity of consumers' surplus, $\eta(Q^*)$. That is, the value of HHI reflects the division of total surplus between firms' owners and consumers.

Another way to think about Proposition 1 is to denote the total surplus by $W^* = PS^* + CS^*$, in which case equation (5) can be rewritten as

$$\frac{CS^*}{W^*} = \frac{1}{1 + \eta(Q^*)\,H}. \tag{6}$$

Expressed in this way, HHI reflects the share of consumers in the total surplus: consumers obtain a larger share in the total surplus when $\eta(Q^*)$ is lower, i.e., when consumers' surplus becomes more inelastic with respect to output. Under the common assumption that $p'(Q) + p''(Q)\,Q \leq 0$, $\eta(Q^*) \geq 1$, so (6) implies that the share of consumers in the total surplus is bounded from above

[10] Weyl and Fabinger (2013) call $\frac{CS}{PS}$ (the inverse of $\frac{PS}{CS}$) the global incidence and show how it is releted to the pass-through rate of a per-unit tax.

by $\frac{1}{1+H}$. This implies in turn that under monopoly, where $H=1$, consumers obtain no more than 50% of the total surplus.

Figure 1 illustrates the share of consumers in the total surplus when $\eta(Q^*)=1$ and $\eta(Q^*)=2$, i.e., a 1% increase in output leads to a 1% or a 2% increase in consumers' surplus. Below, I show that $\eta(Q^*)=1$ when demand is log-linear and $\eta(Q^*)=2$ when demand is linear.

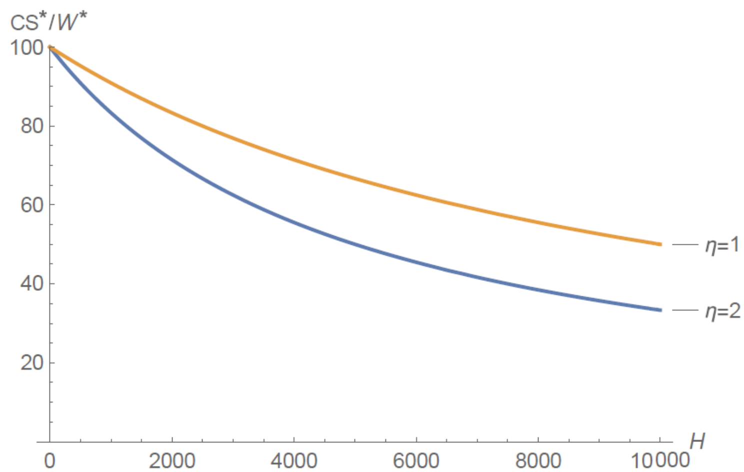

Figure 1: Consumers' share in the total surplus as a function of HHI when $\eta(Q^*)=1$ and $\eta(Q^*)=2$

To interpret Figure 1, recall from the Introduction that the 2010 horizontal merger guidelines of the DOJ and the FTC state that horizontal mergers are unlikely to have adverse competitive effects when the post-merger HHI is below $1,500$, but express concerns about horizontal merges when the post-merger HHI is above $2,500$. Equation (6) shows that if $\eta(Q^*)=1$, HHI below $1,500$ implies that consumers obtain more than $1/(1+0.15)=87\%$ of the total surplus, while HHI above $2,500$ implies that consumers obtain at most $1/(1+2\times 0.25)=80\%$ of the total surplus. When $\eta(Q^*)\geq 1$, these shares are an upper bound on the share of consumers in the total surplus. For instance, if $\eta(Q^*)=2$, HHI below $1,500$ implies that consumers obtain more than $1/(1+2\times 0.15)=77\%$ of the total surplus, while HHI above $2,500$ implies that consumers obtain at most $1/(1+2\times 0.25)=67\%$ of the total surplus. Viewed in this way, one can infer that whenever $\eta(Q^*)=2$, the DOJ and the FTC have competitive concerns when consumers obtain less than 67% of the total surplus, but not when they obtain more than 77% of the total surplus.[11]

[11] It should be noted that since the value of $\eta(Q^*)$ is likely to vary across industries (and over time), two industries

Proposition 1 is obtained under the assumption that all firms have constant marginal costs. One may wonder what happens when some, or even all, firms have increasing marginal costs. In this case, an interior Nash equilibrium is defined by the following system of first-order conditions:

$$p(Q) + p'(Q)\, q_i - c_i'(q_i) = 0, \qquad i = 1, 2, , \ldots, n,$$

where $c_i'(q_i)$ is the marginal cost of firm i (total cost is $F_i + c_i(q_i)$). Noting that $c_i''(q_i) \geq 0$ implies $c_i'(q_i) \geq \frac{c_i(q_i)}{q_i}$, the first-order conditions imply that the producer surplus of each firm i is such that

$$PS_i^* = \left(p(Q^*) - \frac{c_i(q_i^*)}{q_i^*} \right) q_i^* \geq \left(p(Q^*) - c_i'(q_i^*) \right) q_i^* = -p'(Q^*) (q_i^*)^2 .$$

Repeating the same steps as above, yields the following result:

Corollary 1: *In an n firms Cournot model, where firms have non-decreasing marginal costs,*

$$H \leq \frac{PS^*}{\eta(Q^*)\, CS^*},$$

or equivalently,

$$\frac{CS^*}{W^*} \leq \frac{1}{1 + \eta(Q^*)\, H}.$$

Corollary 1 implies that when at least one firm has an increasing marginal cost, $\eta(Q^*)\, H$ becomes the lower bound on the ratio of producers' to consumers' surplus, or equivalently, $\frac{1}{1+\eta(Q^*)H}$ becomes an upper bound on the share of consumers in the total surplus.[12]

Turning to changes in HHI, equation (5) implies that, holding $\eta(Q^*)$ constant, an increase in HHI is associated with an increase in $\frac{PS^*}{CS^*}$ and a reduction in the share of consumers in the total surplus. But since HHI is endogenous, changes in HHI due to demand or cost shocks, or changes in the number of firms following entry, exit, or mergers, are also likely to affect $\eta(Q^*)$ both directly (when the demand function changes) and indirectly (through their effect on Q^*). Consequently, equation (5) implies that an increase in H is associated with an increase in $\frac{PS^*}{CS^*}$ only when $H\eta(Q^*)$ moves in the same direction as H.

The next lemma, whose proof is in the Appendix, shows that a large family of demand functions has a constant $\eta(Q)$, in which case $H\eta(Q^*)$ surely increases with H.

with the same HHI could feature very different divisions of surplus between firms' owners and consumers. This suggests in turn that thresholds based only on HHI may be associated with very different distributional outcomes across industries.

[12] I thank Geert van Moer for pointing out this possibility to me.

Lemma 1: *An inverse demand function exhibits a constant elasticity of consumers' surplus if and only if it can be expressed as:*

$$p(Q) = A - bQ^{\delta}, \tag{7}$$

where $A \geq 0$ and $b\delta > 0$. The resulting elasticity of consumers' surplus, $\eta(Q)$, is then constant and given by $1 + \delta$.

Anderson and Renault (2003) refer to demand functions that satisfy (7) as ρ-linear.[13] The family of ρ-linear demand functions, which exhibit a constant $\eta(Q^*)$, is quite broad. It includes as special cases linear demand functions when $A, b > 0$ and $\delta = 1$; log-linear inverse demand functions when $A = \widetilde{A} + \frac{\widetilde{b}}{\delta}$, $b = \frac{\widetilde{b}}{\delta}$, and $\delta \to 0$, in which case the inverse demand function becomes $p = \widetilde{A} - \widetilde{b}\ln(Q)$;[14] and iso-elastic demand functions when $A = 0$, and $b, \delta < 0$, in which case the inverse demand function becomes $p = -bQ^{\delta}$. In the latter case, $-\frac{1}{\delta}$ represents the (constant) elasticity of demand. To ensure that the monopoly price is bounded from above, it must be that $\delta \in (-1, 0)$.

Lemma 1 implies that $\eta(Q^*) = 2$ under linear demand, $\eta(Q^*) = 1$ under log-linear demand, and $\eta(Q^*) = 1 + \delta < 1$ under iso-elastic demand. Note that when demand is iso-elastic, $p'(Q) + p''(Q)Q = -b\delta^2 Q^{\delta-1} > 0$, contrary to the assumption in Proposition 1. As a result, $(CS^*)'' < 0$, so $\eta(Q^*) \equiv \frac{Q^*(CS^*)'}{CS^*} < 1$.

Together with Proposition 1, Lemma 1 implies the following Corollary:

Corollary 2: *In an n firms Cournot model where firms have (possibly different) constant marginal costs and the inverse demand function is given by (7),*

$$H = \frac{PS^*}{(1+\delta)\, CS^*}, \tag{8}$$

or equivalently,

$$\frac{CS^*}{W^*} = \frac{1}{1 + (1+\delta)\, H}, \tag{9}$$

[13] A function f is called ρ-linear if f^{ρ} is linear. The name derives from the fact that the associated demand function, $Q(p) = \left(\frac{A-p}{b}\right)^{\frac{1}{\delta}}$, is ρ-linear when $\rho = \delta$. The family of ρ-linear demand functions was first used by Bulow and Pfleiderer (1983). The same functional form was also used by Genesove and Mullin (1998) to explore the methodology of using demand information to infer market conduct and unobserved cost components under static oligopoly behavior.

[14] To see this, note that since $A = \widetilde{A} + \frac{\widetilde{b}}{\delta}$ and $b = \frac{\widetilde{b}}{\delta}$, the inverse demand function is $p(Q) = \widetilde{A} + \frac{\widetilde{b}\left(1-Q^{\delta}\right)}{\delta}$. Using L'Hôpital's rule, $\lim_{\delta\to 0}\left(\widetilde{A} + \frac{\widetilde{b}\left(1-Q^{\delta}\right)}{\delta}\right) = \lim_{\delta\to 0}\left(\widetilde{A} - \frac{\widetilde{b}Q^{\delta}\ln(Q)}{1}\right) = \widetilde{A} - \widetilde{b}\ln(Q)$.

where $\delta \in (-1, 0)$ if demand has a constant elasticity $-\frac{1}{\delta}$, $\delta = 0$ if the inverse demand function is log-linear, and $\delta = 1$ if demand is linear.

Corollary 2 implies that when demand is ρ-linear, there is a constant relationship between HHI and the ratio of producers' to consumers' surplus: every 100 points increase in HHI is associated with an increase in producers' surplus relative to consumers' surplus by $0.1\,(1+\delta)$. Interestingly, Bulow and Pfleiderer (1983) show that when demand is given by (7), $1+\delta$ is the inverse of the cost pass-through rate, i.e., the rate at which a monopoly with a constant marginal cost k will raise its price in response to an increase in k.[15] $1+\delta$ is also related to the curvature of the demand function, $\sigma \equiv -\frac{p''(Q)Q}{p'(Q)}$, as $1+\delta = 2-\sigma$.

An interesting implication of Corollary 2 is that for a given value of HHI, the share of consumers in the total surplus is larger when demand is iso-elastic than when the inverse demand function is log-linear, and is even lower when demand is linear.

Another interesting implication of Corollary 2 is that when the inverse demand function is linear or log-linear, knowing H is sufficient to determine how total surplus is distributed between firms' owners and consumers. In either case, there is no need to know any other parameter to determine the relationship between HHI and $\frac{PS^*}{CS^*}$. In particular, equation (9) implies that consumers obtain $1/(1+H)$ of the total surplus when demand is log-linear and $1/(1+2H)$ when demand is linear. For example, when HHI is $1,500$, consumers obtain $1/(1+0.15) = 87\%$ of the total surplus when demand is log-linear and $1/(1+2\times 0.15) = 77\%$ when demand is linear; when HHI is $2,500$, consumers obtain $1/(1+0.25) = 80\%$ of the total surplus when demand is log-linear and $1/(1+2\times 0.25) = 67\%$ when demand is linear; and when HHI is $5,000$, they obtain $1/(1+0.5) = 67\%$ of the total surplus when demand is log-linear and $1/(1+2\times 0.5) = 50\%$ when demand is linear.

When the inverse demand function is iso-elastic, the relationship between HHI and $\frac{PS^*}{CS^*}$ also depends on the parameter δ, which is the inverse of the elasticity of demand. Equation (9) shows that holding HHI constant, consumers obtain a smaller share of the total surplus as demand becomes more elastic and δ increases from -1 to 0. To illustrate, suppose that $p(Q) = Q^{\delta}$ (i.e., $A = 0$ and $b = -1$), where $\delta \in (-1, 0)$ (the elasticity of demand, $-\frac{1}{\delta}$, grows in this case from 1 to ∞).

[15] The monopoly output of a firm with a constant marginal cost k is given by the following first-order condition: $p + p'Q - k = 0$. Fully differentiating this condition, yields $\frac{dQ}{dk} = \frac{1}{2p'+p"Q}$. Hence, $\frac{dp}{dk} = \frac{p'}{2p'+p"Q}$. When the inverse demand function if given by (7), the last equation becomes $\frac{dp}{dk} = \frac{1}{1+\delta}$.

Assuming that firms have the same marginal cost, k, the profit of each firm i is $\pi_i = \left(Q^{\delta} - k\right) q_i$.[16] In a symmetric Nash equilibrium, the quantity of each firm is $q^* = \frac{1}{n}\left(\frac{kn}{\delta+n}\right)^{\frac{1}{\delta}}$, so aggregate output is $Q^* = \left(\frac{kn}{\delta+n}\right)^{\frac{1}{\delta}}$, and the equilibrium price is $p\left(Q^*\right) = \frac{kn}{\delta+n}$. Hence, producers' and consumers' surplus are given by

$$PS^* = \left(\frac{kn}{\delta+n} - k\right)\left(\frac{kn}{\delta+n}\right)^{\frac{1}{\delta}} = -\delta n^{\frac{1}{\delta}}\left(\frac{k}{\delta+n}\right)^{\frac{1+\delta}{\delta}},$$

and

$$CS^* = \int_0^{Q^*} z^{\delta} dz - \left(Q^*\right)^{\delta} Q^* = -\frac{\delta\left(Q^*\right)^{1+\delta}}{1+\delta} = -\frac{\delta}{1+\delta}\left(\frac{kn}{\delta+n}\right)^{\frac{1+\delta}{\delta}}.$$

Although PS^* and CS^* may either increase or decrease with δ depending on the parameter values,[17] their ratio is linearly increasing with δ and given by $\frac{PS^*}{CS^*} = \frac{1+\delta}{n}$. Noting that when firms are symmetric, $H = \frac{1}{n}$, $\frac{PS^*}{CS^*} = \frac{1+\delta}{n}$ coincides with (8). Moreover, $\frac{PS^*}{CS^*}$ is higher as demand becomes more elastic and δ increases from -1 towards 0.

One may now wonder what happens when the demand function is not ρ-linear, in which case $\eta\left(Q\right)$ is no longer constant. The following result shows that, at least in the case where all firms have the same constant marginal cost k, an increase in H due to an infinitesimal merger that lowers the number of firms, n, slightly also increases $H\eta\left(Q^*\right)$.[18]

Proposition 2: *In an n firms Cournot model, where all firms have the same constant marginal cost, k, an infinitesimal merger leads to an increase in H and in $H\eta\left(Q^*\right)$, so following the merger the share of consumers in the total surplus falls.*

3 Common ownership

In recent years there is a growing concern about the potential anticompetitive effects of common ownership, i.e., the fact that a few large institutional investors such as Berkshire Hathaway, BlackRock, Vanguard, and State Street are the major shareholders of competing firms such as airlines or banks.[19] A common measure of concentration in the presence of common ownership is MHHI

[16] π_i is concave in q_i, because $\pi_i'' = \delta Q^{\delta-2}\left(2Q + (\delta-1)\, q_i\right) < 0$, where the inequality follows because $\delta \in (-1, 0)$ and $2Q + (\delta-1)\, q_i > 0$.

[17] For instance, when $n = 10$ and $k = 0.8$, PS^* is decreasing with δ for $-1 < \delta < -0.2185$ and increasing for $-0.2185 < \delta < 0$, while CS^* is decreasing with δ for $-1 < \delta < -0.1803$ and increasing for $-0.1803 < \delta < 0$.

[18] The concept of infinitesimal mergers is due to Farrell and Shapiro (1990).

[19] See Backus, Conlon, and Sinkinson (2019) for a recent paper that documents the rise of common ownership in the U.S. economy and the increased incentive of firms to internalize the negative competitive externality that they exert on rivals.

due to O'brien and Salop (2000). In this section I show that Proposition 1 above generalizes to the case of common ownership with MHHI replacing HHI.

Under common ownership, there are m shareholders who own shares in the various firms. Let α_{jk} be the stake that shareholder k owns in firm j. The wealth of shareholder k is equal to his combined stake in the n firms, $w_k = \sum_{i=j}^{n} \alpha_{jk}\pi_j$. The objective of the manager of each firm i is to maximize a weighted average of the wealth of the firm's shareholders, where the weight assigned to shareholder k's wealth is λ_{ik} (this weight may reflect the degree of control that shareholder k has over firm i):

$$O_i = \sum_{k=1}^{m} \lambda_{ik} w_k = \sum_{k=1}^{m} \lambda_{ik} \sum_{j=1}^{n} \alpha_{jk}\pi_j = \sum_{j=1}^{n}\sum_{k=1}^{m} \lambda_{ik}\alpha_{jk}\pi_j.$$

It is useful to rewrite the objective function of firm i's manager as

$$O_i = \sum_{j=1}^{n} \pi_j \left(\sum_{k=1}^{m} \lambda_{ik}\alpha_{jk}\right).$$

An interior Nash equilibrium when each manager i chooses his firm's output q_i to maximize his objective function O_i is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:

$$\left(p(Q) + p'(Q)\, q_i - k_i\right)\left(\sum_{k=1}^{m} \lambda_{ik}\alpha_{ik}\right) + \sum_{j\neq i}^{n} p'(Q)\, q_j \left(\sum_{k=1}^{m} \lambda_{ik}\alpha_{jk}\right) = 0, \qquad i = 1, 2, , \ldots, n.$$

The price-cost margin of each firm i in an interior Nash equilibrium is given by

$$p(Q^*) - k_i = -p'(Q^*) \sum_{j=1}^{n} q_j^* \underbrace{\left(\frac{\sum_{k=1}^{m} \lambda_{ik}\alpha_{jk}}{\sum_{k=1}^{m} \lambda_{ik}\alpha_{ik}}\right)}_{\kappa_{ij}},$$

where κ_{ij} is the weight that firm i's manager assigns to the profit of firm j relative to firm i (note that $\kappa_{ij} = 1$).[20] Using the last expression, the equilibrium producer surplus of each firm i can be written as

$$PS_i^* = (p(Q^*) - k_i)\, q_i^* = -p'(Q^*) \sum_{j=1}^{n} \kappa_{ij} q_j^* q_i^*.$$

[20] To see this, note that the objective function of firm i's manager can be rewritten as

$$\begin{aligned} O_i &= \pi_i \left(\sum_{k=1}^{m} \lambda_k \alpha_{ik}\right) + \sum_{j\neq i}^{n} \pi_j \left(\sum_{k=1}^{m} \lambda_k \alpha_{jk}\right) \\ &= \left(\sum_{k=1}^{m} \lambda_k \alpha_{ik}\right)\left[\pi_i + \sum_{j\neq i}^{n} \pi_j \underbrace{\left(\frac{\sum_{k=1}^{m} \lambda_k \alpha_{jk}}{\sum_{k=1}^{m} \lambda_k \alpha_{ik}}\right)}_{\kappa_{ij}}\right]. \end{aligned}$$

Hence, κ_{ij} is the weight that firm i's manager assigns to firm j's profit and $\kappa_{ii} = 1$.

Recalling that $(CS^*)' = -p'(Q^*)\,Q^*$, aggregate producers' surplus is given by

$$PS^* \equiv \sum_{i=1}^{n} PS_i^* = \frac{(CS^*)' \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} q_j^* q_i^*}{Q^*}.$$

Dividing and multiplying the right-hand side by Q^*, noting that $\frac{q_i^*}{Q^*} = s_i^*$ and $\frac{q_j^*}{Q^*} = s_j^*$ are the market shares of firms j and i, and recalling that $\eta(Q^*) \equiv \frac{Q^*(CS^*)'}{CS^*}$ is the elasticity of consumers' surplus with respect to output, yields

$$PS^* = \eta(Q^*)\, CS^* \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} s_j^* s_i^*,$$

where $\sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} s_j^* s_i^*$ is $MHHI$ as defined by O'brien and Salop (2000).[21] Hence,

Proposition 3: *In an n firms Cournot model, where firms have (possibly different) constant marginal costs, and the manager of each firm maximizes a weighted average of the wealth of the firm's shareholders (who also hold shares in rival firms),*

$$MHHI = \frac{PS^*}{\eta(Q^*)\, CS^*},$$

where $\eta(Q^*) \equiv \frac{Q^*(CS^*)'}{CS^*} > 0$ is the elasticity of consumers' surplus with respect to output.

Proposition 3 shows that Proposition 1 generalizes to the case of common ownership, with MHHI replacing HHI. The implication is that under common ownership, the value of MHHI reflects the division of the total surplus between firms' owners and consumers, with the share of consumers being larger the lower $\eta(Q^*)$ is.

4 Differentiated products

I now show that the key insight from the Cournot model carries over to models of differentiated products, provided that the demand system is linear. To this end, suppose that the n firms produce differentiated products and each firm i is facing an inverse demand function $p_i(q_1, \ldots, q_n)$ and has a cost function $c_i(q_i) = F_i + k_i q_i$, where $k_i < p_i(q_1, \ldots, q_n)$ when $q_i = 0$. The profit of each firm i is given by

$$\pi_i = (p_i(q_1, \ldots, q_n) - k_i)\, q_i - F_i.$$

[21] $\sum_{i=1}^{n} \sum_{j=1}^{n} \kappa_{ij} s_j^* s_i^*$ can be written as $\sum_{i=1}^{n} (s_i^*)^2 + \sum_{i=1}^{n} \sum_{j\neq i}^{n} \kappa_{ij} s_j^* s_i^*$, which is similar to the expression in equation (1) in O'brien and Salop (2000).

An interior Nash equilibrium when firms compete by setting quantities is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:

$$p_i(q_1, \ldots, q_n) + \frac{\partial p_i(q_1, \ldots, q_n)}{\partial q_i} q_i - k_i = 0, \qquad i = 1, 2, , \ldots, n. \tag{10}$$

Since in a Nash equilibrium, $p_i^* - k_i = -\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} q_i^*$, the equilibrium producer surplus of each firm i is

$$PS_i^* = (p_i^* - k_i) q_i^* = -\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i} (q_i^*)^2. \tag{11}$$

Hence, $PS^* \equiv \sum_{i=1}^n PS_i^*$ is a function of $\sum_{i=1}^n (q^*)^2$, which is the denominator of HHI, only if $\frac{\partial p_i(q_1^*, \ldots, q_n^*)}{\partial q_i}$ is constant. This holds however only if $p_i(q_1, \ldots, q_n)$ is linear in q_i.

Under price competition, the profit of each firm i is

$$\pi_i = (p_i - k_i) q_i(p_1, \ldots, p_n) - F_i,$$

where $q_i(p_1, \ldots, p_n)$ is the demand that firm i is facing. An interior Nash equilibrium is now a vector $(p_1^*, \ldots, p_n^*)$ that solves the following system of first-order conditions,

$$q_i(p_1, \ldots, p_n) + \frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i}(p_i - k_i) = 0, \qquad i = 1, 2, , \ldots, n. \tag{12}$$

Since in a Nash equilibrium, $p_i^* - k_i = -\frac{q_i^*}{\frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i}}$, the equilibrium producer surplus of each firm i is

$$PS_i^* = (p_i^* - k_i) q_i^* = \frac{(q_i^*)^2}{-\frac{\partial q_i(p_1, \ldots, p_n)}{\partial p_i}}. \tag{13}$$

Once again, $PS^* \equiv \sum_{i=1}^n PS_i^*$ is a function of $\sum_{i=1}^n (q^*)^2$ only if $q_i(p_1, \ldots, p_n)$ is linear in p_i.

In what follows, I will therefore assume that the inverse demand system is linear and given by

$$p_i = A_i - \beta q_i - \gamma \sum_{j \neq i}^{n} q_j, \qquad i = 1, 2, , \ldots, n, \tag{14}$$

where $A_1, \ldots, A_n$ and β, are positive parameters, and $0 < \gamma < \beta$ is a measure of the degree of product differentiation, with lower values of γ representing a larger degree of differentiation.[22] This inverse demand system corresponds to the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, but if $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, where $\tau > 0$, it corresponds to the Shubik and Levitan (1980) specification.[23] In the latter case, the parameter τ reflects the degree of product differentiation, with lower values of τ representing a larger degree of differentiation.

[22] Obviously, γ cannot be too low relative to β otherwise the products are not in the same market in which case HHI becomes meaningless.

[23] A third notable example for a differentiated products oligopoly model with linear demands is the Vickery-Salop circular city model (Vickery 1964 and Salop 1979)).

4.1 Quantity competition

With quantity competition, PS_i^* is given by (11), where (14) implies that $\frac{\partial p_i\left(q_1^*,\ldots,q_n^*\right)}{\partial q_i} = -\beta$ for all i. Hence $PS^* \equiv \sum_{i=1}^n PS_i^* = \beta \sum_{i=1}^n \left(q^*\right)^2$. In the Appendix, I show that, evaluated at the equilibrium quantities, consumers' surplus is given by

$$CS^* = \frac{(\beta-\gamma)\sum_{i=1}^n \left(q_i^*\right)^2 + \gamma\left(Q^*\right)^2}{2}. \tag{15}$$

Noting that $\sum_{i=1}^n \left(q^*\right)^2 = \frac{PS^*}{\beta}$ and substituting for Q^* from (15) into (4) and rearranging, yields the following result:

Proposition 4: *In an n firms differentiated products oligopoly with quantity competition, where firms have (possibly different) constant marginal costs and face the linear inverse demand system (14), HHI is given by*

$$H = \frac{\sum_{i=1}^n \left(\frac{PS_i^*}{\beta}\right)}{\frac{2CS^*}{\gamma} - \frac{\beta-\gamma}{\gamma}\sum_{i=1}^n \left(\frac{PS_i^*}{\beta}\right)} = \frac{\frac{PS^*}{CS^*}}{\frac{2\beta}{\gamma} - \frac{\beta-\gamma}{\gamma}\frac{PS^*}{CS^*}}. \tag{16}$$

In the Shubik-Levitan case, where $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, HHI is given by

$$H = \frac{\frac{PS^*}{CS^*}}{\frac{2(n+\tau)}{\tau} - \frac{n}{\tau}\frac{PS^*}{CS^*}}. \tag{17}$$

Proposition 4 implies that, similarly to the Cournot case, HHI is positively related to $\frac{PS^*}{CS^*}$, so higher values of HHI imply that consumers obtain a lower share in the total surplus. Notice from (16) that as $\gamma \to \beta$ (products become homogeneous), the right-hand side of (16) approaches $\frac{1}{2}\frac{PS^*}{CS^*}$, which by equation (8), is the value of H under Cournot competition when demand is linear (in which case $\delta = 1$).[24]

4.2 Price competition

To study the relationship between HHI and $\frac{PS^*}{CS^*}$ under price competition, we first need to invert the inverse demand system (14). In the Appendix, I show that the demand system associated with (14) is given by:

$$q_i = \mu\left(A_i - p_i\right) - \sigma\sum_{j\neq i}^n \left(A_i - p_j\right), \qquad i = 1, 2, , \ldots, n, \tag{18}$$

[24] In the Appendix I also verify that when the n firms are symmetric and have the same marginal cost, the right-hand sides of (8) and (17) are equal to $1/n$ which is the value of H when firms are symmetric.

where

$$\mu \equiv \frac{\beta + (n-2)\gamma}{(\beta - \gamma)(\beta + (n-1)\gamma)}, \qquad \sigma \equiv \frac{\gamma}{(\beta - \gamma)(\beta + (n-1)\gamma)}.$$

Now, PS_i^* is given by (13), where (18) implies that $-\frac{\partial q_i(p_1,\ldots,p_n)}{\partial p_i} = \mu$ for all i. Hence $PS^* \equiv \sum_{i=1}^n PS_i^* = \frac{1}{\mu}\sum_{i=1}^n (q^*)^2$. Substituting in (4), noting that consumers' surplus is still given by (15), and rearranging, yields the following result:

Proposition 5: *In an n firms differentiated products oligopoly with price competition, where firms have (possibly different) constant marginal costs and face a linear demand system (18), HHI is given by*

$$H = \frac{\gamma \sum_{i=1}^n (\mu \times PS_i^*)}{2CS^* - (\beta - \gamma)\sum_{i=1}^n (\mu \times PS_i^*)} = \frac{\frac{PS^*}{CS^*}}{\frac{\beta-\gamma}{\gamma}\left(\frac{2(\beta+(n-1)\gamma)}{\beta+(n-2)\gamma} - \frac{PS^*}{CS^*}\right)}. \tag{19}$$

In the Shubik-Levitan case, where $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, HHI is given by

$$H = \frac{\frac{PS^*}{CS^*}}{\frac{n}{\tau}\left(\frac{2n(1+\tau)}{n(1+\tau)-\tau} - \frac{PS^*}{CS^*}\right)}. \tag{20}$$

Proposition 5 shows that under price competition, HHI is also positively related to $\frac{PS^*}{CS^*}$, similarly to the case of quantity competition.

4.3 The normative implications of changes in HHI

To examine the normative implications of changes in HHI, it is important to bear in mind that HHI is endogenous, and hence it is not immediately obvious from Propositions 4 and 5 that an increase in HHI is necessarily associated with an increase in $\frac{PS^*}{CS^*}$, because the factors that cause an increase in HHI may also affect the relationship between HHI and $\frac{PS^*}{CS^*}$. To explore this issue further, note first that equations (16)-(17) and (19)-(20) are independent of the demand parameters $A_1, \ldots, A_n$, and the cost parameters $k_1, \ldots, k_n$. Hence, an increase in HHI due to changes in these parameters will be also associated with an increase in $\frac{PS^*}{CS^*}$, regardless of whether firms engage in quantity or price competition.

Turning to the demand parameters β and γ, I first prove in the Appendix that HHI is decreasing with $\frac{\gamma}{\beta}$ under both quantity and price competition. To examine how these changes affect the relationship between HHI and $\frac{PS^*}{CS^*}$, it is useful to rewrite (16) as $\frac{PS^*}{CS^*} = \frac{2}{1+\frac{\gamma}{\beta}\left(\frac{1}{H}-1\right)}$ and (19) as $\frac{PS^*}{CS^*} = \frac{2H\left(1-\frac{\gamma}{\beta}\right)\left(1+(n-1)\frac{\gamma}{\beta}\right)}{\left(H\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)\left(1+(n-2)\frac{\gamma}{\beta}\right)}$. The right-hand sides of the two equations are increasing with H and decreasing with $\frac{\gamma}{\beta}$.[25] Hence, an increase in HHI due to an increase in β or a decrease in γ (the n products become more differentiated) will be also associated with an increase in $\frac{PS^*}{CS^*}$ under both quantity and price competition.

In the Shubik-Levitan case, where $\beta = \frac{n+\tau}{1+\tau}$ and $\gamma = \frac{\tau}{1+\tau}$, the relationship between HHI and $\frac{PS^*}{CS^*}$ depends on the demand parameter, τ. Since $\frac{\gamma}{\beta} = \frac{\tau}{n+\tau}$ is increasing with τ, HHI which is decreasing with $\frac{\gamma}{\beta}$ under both quantity and price competition, is also decreasing with τ. Rewriting (17) as $\frac{PS^*}{CS^*} = \frac{2(n+\tau)}{\frac{\tau}{H}+n}$ and (20) as $\frac{PS^*}{CS^*} = \frac{2n^2(1+\tau)}{\left(n+\frac{\tau}{H}\right)(n(1+\tau)-\tau)}$, and noting that the right-hand sides of the two equations are increasing with H and decreasing with τ, it follows that an increase in HHI due to a decrease in τ (the n products become more differentiated) is associated with an increase in $\frac{PS^*}{CS^*}$ under both quantity and price competition.[26]

Finally, I consider an increase in HHI due to a change in the number of firms, n. Equation (16) is independent of n, implying that under quantity competition, an increase in HHI due to a change in n will be also associated with an increase in $\frac{PS^*}{CS^*}$. Equation (19) depends on n, but rewriting it as $\frac{PS^*}{CS^*} = \frac{2H\left(1-\frac{\gamma}{\beta}\right)\left(1+(n-1)\frac{\gamma}{\beta}\right)}{\left(H\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)\left(1+(n-2)\frac{\gamma}{\beta}\right)}$ and noting that the right-hand side is increasing with H and decreasing with n, it follows that an increase in H due to a decrease in n is associated with an increase in $\frac{PS^*}{CS^*}$.

As for the Shubik-Levitan case, recall that (17) and (20) can be rewritten as $\frac{PS^*}{CS^*} = \frac{2(n+\tau)}{\frac{\tau}{H}+n}$ and $\frac{PS^*}{CS^*} = \frac{2n^2(1+\tau)}{\left(\frac{\tau}{H}+n\right)(n(1+\tau)-\tau)}$. Since the right-hand sides of the two equations are increasing with both n and H, an increase in H due to a decrease in n can be associated with either an increase or decrease in $\frac{PS^*}{CS^*}$. However, in the symmetric case where $H = \frac{1}{n}$, the two equations become $\frac{PS^*}{CS^*} = \frac{2\left(1+\frac{\tau}{n}\right)}{1+\tau}$ and $\frac{PS^*}{CS^*} = \frac{2}{(1+\tau)\left(1+\tau-\frac{\tau}{n}\right)}$, and are clearly decreasing with n. Hence, an increase in H due to a decrease in n is associated with an increase in $\frac{PS^*}{CS^*}$.

Proposition 6: *An increase in HHI is associated with an increase in $\frac{PS^*}{CS^*}$ in all cases, except when demand is given by the Shubik-Levitan specification and the increase in HHI is due to a decrease in the number of firms, in which case the associated change in $\frac{PS^*}{CS^*}$ is in general ambiguous. However, in the symmetric case where $H = \frac{1}{n}$, an increase in HHI due to a decrease in the number of firms is associated with an increase in $\frac{PS^*}{CS^*}$.*

[25] The derivative of the right-hand side of the latter equation with respect to $\frac{\gamma}{\beta}$ is $-\frac{2H\left[1-H\left(1-\frac{\gamma}{\beta}\right)^2+2(n-2)\frac{\gamma}{\beta}+\left(3-3n+n^2\right)\left(\frac{\gamma}{\beta}\right)^2\right]}{\left(H\left(1-\frac{\gamma}{\beta}\right)+\frac{\gamma}{\beta}\right)^2\left(1+(n-2)\frac{\gamma}{\beta}\right)^2}$, which is negative since $n > 1 > H$ and since $3 - 3n + n^2 > 0$ for all n.

[26] The derivatives of the right-hand sides of the equations with respect to τ are $-\frac{2n(1-H)}{H\left(n+\frac{\tau}{H}\right)^2} < 0$ and $-\frac{2n^2[\tau(2+\tau)(n-1)+n(1-H)]}{H\left(n+\frac{\tau}{H}\right)^2(n(1+\tau)-\tau)^2} < 0$, where the inequalities follow because $n > 1 > H$.

Proposition 6 shows that when demand is given by the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, an increase in HHI is always associated with an increase in $\frac{PS^*}{CS^*}$, no matter whether the increase in HHI is driven by demand or cost shocks or a decrease in the number of firms. Hence, higher values of HHI are associated with a smaller share of consumers in the total surplus. The same conclusion also holds when demand is given by the Shubik and Levitan (1980) specification, provided that the increase in HHI is due to demand or cost shocks, or when firms are symmetric and the increase in HHI is due to a decrease in the number of firms.

5 Conclusion

I show that in either the Cournot model or a differentiated products model with linear demand, HHI reflects the division of total surplus between firms' owners and consumers. When all firms have constant marginal costs (not necessarily identical across firms) HHI is an increasing function of the ratio of producers' to consumers' surplus, implying that consumers get a lower share in the total surplus when the HHI is higher. This result generalizes to the case of common ownership with MHHI replacing HHI. When the marginal cost of at least one firm is increasing with output, the HHI is a lower bound on the ratio of producers' to consumers' surplus. These results imply that the HHI has a simple and intuitive normative interpretation.

6 Appendix

Following are the proofs of Propositions 1 and 2, and Lemma 1; the derivation of the demand system and consumers' surplus in the product differentiation case; a proof that in the product differentiation case, HHI is increasing with the demand parameter β and decreasing with the demand parameter γ; and checking that in the product differentiation case, $H = \frac{1}{n}$ when firms are symmetric.

Proof of Proposition 1: Substituting for $\sum_{i=1}^{n} (q_i^*)^2$ from (3) into (4), yields

$$H = \frac{PS^*}{(CS^*)' Q^*}.$$

Using the definition of $\eta(Q^*)$, yields (5). Note that $(CS^*)' = -p'(Q^*) Q^*$ and $(CS^*)'' = -(p'(Q^*) + p''(Q^*) Q^*)$. Hence, $\eta(Q^*) = \frac{Q^*(CS^*)'}{CS^*} \geq 1$ if and only if $(CS^*)'' = -(p'(Q^*) + p''(Q^*) Q^*) \geq 0$. ■

Proof of Lemma 1: The "if" part is straightforward. If the inverse demand function is given by (7), then

$$\begin{aligned} CS &= \int_0^Q \left(A - bz^\delta\right) dz - \left(A - bQ^\delta\right) Q \\ &= AQ - \frac{bQ^{1+\delta}}{1+\delta} - \left(A - bQ^\delta\right) Q \\ &= \frac{\delta b Q^{1+\delta}}{1+\delta}. \end{aligned} \tag{21}$$

The elasticity of CS with respect to output is

$$\eta(Q) = \left(\frac{\delta b Q^{1+\delta}}{\delta+1}\right)' \frac{Q}{\frac{\delta b Q^{\delta+1}}{\delta+1}} = 1 + \delta,$$

which is indeed a constant.

To prove the "only if" part, I first show that a constant $\eta(Q)$ implies a constant pass-through rate. To this end, suppose that $\eta(Q) \equiv \frac{QCS'}{CS} = \overline{\eta}$ for all Q. Since $\eta(Q)$ is constant,

$$\eta'(Q) = \frac{(CS' + QCS'') CS - Q (CS')^2}{(CS)^2} = 0,$$

which implies that

$$(CS' + QCS'') CS = Q (CS')^2.$$

Rewriting the equality,

$$\frac{CS' + QCS''}{CS'} = \frac{QCS'}{CS} = \overline{\eta}.$$

Recalling that $CS' = -p'(Q)\,Q$ and noting that $CS'' = -p'(Q) - p''(Q)\,Q$, yields

$$\begin{aligned}\overline{\eta} &= \frac{CS' + QCS''}{CS'} \\ &= \frac{-p'(Q)\,Q - Q\,(p'(Q) + p''(Q)\,Q)}{-p'(Q)\,Q} \\ &= \frac{2p'(Q) + p''(Q)\,Q}{p'(Q)}.\end{aligned} \tag{22}$$

The last expression however is just the inverse of the cost pass-through rate. To see why, note that if the market is served by a monopoly with a constant marginal cost k, the profit-maximizing output is implicitly defined by the first-order condition $p(Q) + p'(Q)\,Q - k = 0$. Fully differentiating the first-order condition with respect to Q and k and rearranging, yields

$$\frac{\partial Q}{\partial k} = \frac{1}{2p'(Q) + p''(Q)\,Q}.$$

Hence, the cost pass-through rate is

$$p'(k) \equiv p'(Q)\,\frac{\partial Q}{\partial k} = \frac{p'(Q)}{2p'(Q) + p''(Q)\,Q}. \tag{23}$$

Together with (22), this implies that $p'(k) = \frac{1}{\overline{\eta}}$. Bulow and Pfleiderer (1983) prove that an inverse demand function exhibits a constant cost pass-through rate if and only if it is represented by (7). Altogether then, a constant $\eta(Q)$ implies a constant $p'(k)$, which in turn implies that the inverse demand function is represented by (7).

Finally, it is easy to verify that when the inverse demand function is represented by (7), the cost pass-through rate is

$$p'(k) = \frac{-b\delta Q^{\delta-1}}{-2b\delta Q^{\delta-1} - b\delta(\delta-1)\,Q^{\delta-2}Q} = \frac{1}{1+\delta},$$

which is the inverse of $\eta(Q)$. ■

Proof of Proposition 2: When all firms have the same constant marginal cost k, the first-order condition for the output of each firm is given by

$$p(Q) + p'(Q)\,q_i - k = 0.$$

Since firms are symmetric, $q_i = \frac{Q}{n}$. Substituting in the first-order condition, fully differentiating with respect to Q^* and n, and rearranging terms,

$$\frac{\partial Q^*}{\partial n} = \frac{p'(Q^*)\,\frac{Q^*}{n^2}}{\frac{p'(Q^*)(n+1)}{n} + \frac{p''(Q^*)Q^*}{n}} = \frac{-(CS^*)'}{n\,[p'(Q^*)\,(n+1) + p''(Q^*)\,Q^*]},$$

where the last inequality follows because $(CS^*)' = -p'(Q^*)\,Q^*$.

Since firms are symmetric, $H = \frac{1}{n}$. The derivative of $\frac{1}{n}\eta(Q^*)$ with respect to n is given by

$$\begin{aligned}
\frac{\partial}{\partial n}\left(\frac{1}{n}\eta(Q^*)\right) &= -\frac{\eta(Q^*)}{n^2} + \frac{\eta'(Q^*)}{n}\frac{\partial Q^*}{\partial n} \\
&= -\frac{\frac{Q^*(CS^*)'}{CS^*}}{n^2} - \frac{\eta'(Q^*)(CS^*)'}{n^2\left[p'(Q^*)(n+1) + p''(Q^*)Q^*\right]} \\
&= \frac{(CS^*)'\left[\frac{Q^*[p'(Q^*)(n+1)+p''(Q^*)Q^*]}{CS^*} + \eta'(Q^*)\right]}{-n^2\left(p'(Q^*)(n+1) + p''(Q^*)Q^*\right)}.
\end{aligned}$$

The denominator in this expression is positive because the assumption that $p'(Q) + p''(Q)\,Q < 0$. Since $(CS^*)' = -p'(Q^*)\,Q^* > 0$, the sign of $\frac{\partial}{\partial n}\left(\frac{1}{n}\eta(Q^*)\right)$ depends on the sign of the square bracketed term in the numerator. Now, recall that $\eta(Q^*) = \frac{Q^*(CS^*)'}{CS^*}$ and note that $(CS^*)' = -p'(Q^*)\,Q^*$ and $(CS^*)'' = -p'(Q^*) - p''(Q^*)\,Q^*$. Hence,

$$\begin{aligned}
\eta'(Q^*) &= \frac{(CS^*)' + Q^*(CS^*)'' - \frac{Q^*\left((CS^*)'\right)^2}{CS}}{CS^*} \\
&= \frac{(CS^*)'(1-\eta(Q^*)) + Q^*(CS^*)''}{CS^*} \\
&= \frac{-p'(Q^*)\,Q^*(1-\eta(Q^*)) - Q^*\left(p'(Q^*) + p''(Q^*)\,Q^*\right)}{CS} \\
&= \frac{-Q^*\left[p'(Q^*)(2-\eta(Q^*)) + p''(Q^*)\,Q^*\right]}{CS^*}.
\end{aligned}$$

Substituting for $\eta(Q^*)$ and $\eta'(Q^*)$ in the square bracketed term in the numerator of $\frac{\partial}{\partial n}\left(\frac{1}{n}\eta(Q^*)\right)$,

$$\begin{aligned}
&\frac{Q^*\left[p'(Q^*)(n+1) + p''(Q^*)\,Q^*\right]}{CS^*} - \frac{Q^*\left[p'(Q^*)(2-\eta(Q^*)) + p''(Q^*)\,Q^*\right]}{CS^*} \\
= &\frac{Q^*p'(Q^*)\left[n-1+\eta(Q^*)\right]}{CS^*} < 0.
\end{aligned}$$

Hence, $\frac{\partial}{\partial n}\left(\frac{1}{n}\eta(Q^*)\right) < 0$. ■

The demand system in the product differentiation case: From (14) it follows that

$$q_i = \frac{1}{\beta}\left(A_i - p_i - \gamma\sum_{j\neq i}^{n} q_j\right).$$

Adding $-\frac{\gamma}{\beta}q_i$ to both sides of the equation, recalling that $Q = \sum_{i=1}^{n} q_i$, and rearranging, yields

$$q_i - \frac{\gamma}{\beta}q_i = \frac{1}{\beta}\left(A_i - p_i - \gamma\sum_{j\neq i}^{n} q_j - \gamma q_i\right), \qquad \Rightarrow \qquad q_i = \frac{A_i - p_i - \gamma Q}{\beta - \gamma}.$$

Summing over all firms and solving for Q, yields

$$Q = \frac{\sum_{i=1}^{n}(A_i - p_i) - \gamma n Q}{\beta - \gamma} \qquad \Rightarrow \qquad Q = \frac{\sum_{i=1}^{n}(A_i - p_i)}{\beta + (n-1)\gamma}.$$

Substituting for Q in q_i and rearranging, yields (18). $\blacksquare$

Consumers' surplus in the product differentiation case with linear demands: Starting with the Spence (1976), Dixit (1979), and Singh and Vives (1984) specification, the demand system is derived from the preferences of a representative consumer, whose utility function is quadratic:

$$u(q_1,\ldots,q_n)=\sum_{i=1}^{n}A_iq_i-\frac{\beta\sum_{i=1}^{n}q_i^2+\gamma\sum_{i=1}^{n}\sum_{j\neq i}^{n}q_iq_j}{2}+m, \tag{24}$$

where m is income spent on all other goods, $A_1,\ldots,A_n$ and β, are positive utility parameters, and $0<\gamma<\beta$. Maximizing $u(q_1,\ldots,q_n)$ subject to a budget constraint, $\sum_{i=1}^{n}p_iq_i+m=I$, where p_i is the prices of good i, and I is income, yields the system of inverse demand functions (14).

To express consumers' surplus, note first that the utility function of the representative consumer can now be written as:

$$\begin{aligned}u(q_1,\ldots,q_n) &= \sum_{i=1}^{n}A_iq_i-\frac{(\beta-\gamma)\sum_{i=1}^{n}q_i^2+\gamma\sum_{i=1}^{n}\sum_{j=1}^{n}q_iq_j}{2}+m\\ &= \sum_{i=1}^{n}A_iq_i-\frac{(\beta-\gamma)\sum_{i=1}^{n}q_i^2+\gamma Q^2}{2}+m,\end{aligned}$$

where the last equality follows since $\sum_{i=1}^{n}\sum_{j=1}^{n}q_iq_j=\left(\sum_{j=1}^{n}q_i\right)^2=Q^2$. Substituting for m from the budget constraint into (24) and using (14), consumers' surplus is given by

$$\begin{aligned}CS(q_1,\ldots,q_n) &= \sum_{i=1}^{n}A_iq_i-\frac{(\beta-\gamma)\sum_{i=1}^{n}q_i^2+\gamma Q^2}{2}-\sum_{i=1}^{n}p_iq_i\\ &= \sum_{i=1}^{n}A_iq_i-\frac{(\beta-\gamma)\sum_{i=1}^{n}q_i^*+\gamma Q^2}{2}-\sum_{i=1}^{n}\left(A_i-\beta q_i-\gamma\sum_{j\neq i}^{n}q_j\right)q_i\\ &= -\frac{(\beta-\gamma)\sum_{i=1}^{n}q_i^2+\gamma Q^2}{2}+(\beta-\gamma)\sum_{i=1}^{n}q_i^2+\gamma\sum_{i=1}^{n}\sum_{j=1}^{n}q_iq_j\\ &= \frac{(\beta-\gamma)\sum_{i=1}^{n}q_i^2+\gamma Q^2}{2}.\end{aligned}$$

Evaluating at the equilibrium quantities, yields $CS^*\equiv CS(q_1^*,\ldots,q_n^*)$, given by (15).

The Shubik-Levitan (1980) demand system is derived similarly, except that now $\beta=\left(\frac{n+\tau}{1+\tau}\right)$ and $\gamma=\frac{\tau}{1+\tau}$. Given these parameter values, consumers' surplus at the equilibrium quantities is

$$CS^*\equiv CS(q_1^*,\ldots,q_n^*)=\frac{n\sum_{i=1}^{n}(q_i^*)^2+\tau(Q^*)^2}{2(1+\tau)}.$$

$\blacksquare$

HHI in the product differentiation case is increasing with β and decreasing with γ: I begin by considering quantity competition. Using (14), the profit of each firm i is

$$\pi_i = \left(A_i - \beta q_i - \gamma \sum_{j \neq i}^{n} q_j - k_i \right) q_i.$$

An interior Nash equilibrium when firms set quantities is a vector $(q_1^*, \ldots, q_n^*)$ that solves the following system of first-order conditions:

$$A_i - 2\beta q_i - \gamma \sum_{j \neq i}^{n} q_j - k_i = 0, \qquad i = 1, 2, , \ldots, n.$$

Adding and subtracting γq_i from the left-hand side of the equation, recalling that $Q = \sum_{i=1}^{n} q_i$, and solving for q_i, yields the best-response function of each firm i against the aggregate quantity Q (which includes q_i):[27]

$$q_i = \frac{A_i - k_i - \gamma Q}{2\beta - \gamma}.$$

Summing over all $i = 1, 2, , \ldots, n$, and solving for Q, yields

$$Q^* = \frac{\sum_{i=1}^{n} (A_i - k_i)}{2\beta + (n-1)\gamma}.$$

Substituting Q^* in the best-response functions, yields

$$q_i^* = \frac{(2\beta + (n-1)\gamma)(A_i - k_i) - \gamma \sum_{i=1}^{n} (A_i - k_i)}{(2\beta - \gamma)(2\beta + (n-1)\gamma)}, \qquad i = 1, 2, , \ldots, n.$$

Given q_i^* and Q^*, the market share of each firm i, $s_i^* = \frac{q_i^*}{Q^*}$, is given by

$$\begin{aligned} s_i^* &= \frac{(2\beta + (n-1)\gamma)(A_i - k_i) - \gamma \sum_{i=1}^{n} (A_i - k_i)}{(2\beta - \gamma) \sum_{i=1}^{n} (A_i - k_i)} \\ &= \frac{\left(2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i) - \frac{\gamma}{\beta} \sum_{i=1}^{n} (A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right) \sum_{i=1}^{n} (A_i - k_i)} \\ &= \frac{\left(2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right) \sum_{i=1}^{n} (A_i - k_i)} - \frac{\frac{\gamma}{\beta}}{2 - \frac{\gamma}{\beta}}. \end{aligned}$$

It is straightforward to check that $\sum_{i=1}^{n} s_i^* = 1$ and that under symmetry where $A_i = A$ and $k_i = k$ for all i, $s_i^* = \frac{1}{n}$. Given the market shares, HHI is given by

$$H = \sum_{i=1}^{n} \underbrace{\left(\frac{\left(2 + (n-1)\frac{\gamma}{\beta}\right)(A_i - k_i)}{\left(2 - \frac{\gamma}{\beta}\right) \sum_{i=1}^{n} (A_i - k_i)} - \frac{\frac{\gamma}{\beta}}{2 - \frac{\gamma}{\beta}} \right)^2}_{s_i^*}$$

[27] The best-response of a firm against aggregate output was called by Selten (1973) the "fitting-in function."

Note that HHI depends on $\frac{\gamma}{\beta}$ rather than separately on β and γ. Differentiating HHI with respect to $\frac{\gamma}{\beta}$,

$$\begin{aligned}
\frac{\partial H}{\partial\left(\frac{\gamma}{\beta}\right)} &= 2\sum_{i=1}^{n} s_i^* \left(\frac{\left((n-1)\left(2-\frac{\gamma}{\beta}\right)+2+(n-1)\frac{\gamma}{\beta}\right)(A_i-k_i)}{\left(2-\frac{\gamma}{\beta}\right)^2\sum_{i=1}^{n}(A_i-k_i)} - \frac{2}{\left(2-\frac{\gamma}{\beta}\right)^2}\right) \\
&= \frac{4n}{\left(2-\frac{\gamma}{\beta}\right)^2}\sum_{i=1}^{n} s_i^*\left(\frac{A_i-k_i}{\sum_{i=1}^{n}(A_i-k_i)}-\frac{1}{n}\right) \\
&= \frac{4n}{\left(2-\frac{\gamma}{\beta}\right)^2}\left(\frac{\sum_{i=1}^{n} s_i^*(A_i-k_i)}{\sum_{i=1}^{n}(A_i-k_i)}-\frac{1}{n}\right).
\end{aligned}$$

To determine the sign of the inequality, note that the series $\frac{A_1-k_1}{\sum_{i=1}^{n}(A_i-k_i)},\ldots,\frac{A_n-k_n}{\sum_{i=1}^{n}(A_i-k_i)}$ can be ordered from large to small: $\frac{A_1-k_1}{\sum_{i=1}^{n}(A_i-k_i)} \geq \ldots \geq \frac{A_n-k_n}{\sum_{i=1}^{n}(A_i-k_i)}$. Since the market share of each firm i is increasing with $A_i - k_i$, it also follows that $s_1^* \geq \ldots \geq s_n^*$. By Chebyshev's sum inequality then, $\frac{1}{n}\sum_{i=1}^{n} s_i^* \times (A_i-k_i) \geq \left(\frac{1}{n}\sum_{i=1}^{n}(A_i-k_i)\right)\left(\frac{1}{n}\sum_{i=1}^{n} s_i\right)$. Noting that $\sum_{i=1}^{n} s_i^* = 1$, it follows that $\frac{\sum_{i=1}^{n} s_i^*(A_i-k_i)}{\sum_{i=1}^{n}(A_i-k_i)} \geq \frac{1}{n}$. Hence, the derivative is nonnegative and is strictly positive when firms are not symmetric.

Next, I turn to price competition. Using (18), the profit of each firm i is

$$\pi_i = \left(\mu(A_i-p_i) - \sigma\sum_{j\neq i}^{n}(A_i-p_j)\right)(p_i-k_i).$$

An interior Nash equilibrium when firms set prices is a vector $(p_1^*,\ldots,p_n^*)$ that solves the following system of first-order conditions:

$$\mu(A_i-p_i) - \sigma\sum_{j\neq i}^{n}(A_i-p_j) - \mu(p_i-k_i) = 0, \qquad i = 1,2,,\ldots,n.$$

Adding and subtracting $\sigma(A_i-p_i)$ from the left-hand side of the equation and reorganizing terms,

$$\mu(A_i+k_i) + \sigma A_i - (2\mu+\sigma)p_i - \sigma\sum_{i=1}^{n}(A_i-p_i) = 0, \qquad i = 1,2,,\ldots,n.$$

Solving for p_i, yields the best-response function of each firm i against the sum of the prices of all firms, $\sum_{i=1}^{n} p_i$ (including p_i):

$$p_i = \frac{\mu(A_i+k_i) + \sigma A_i - \sigma\sum_{i=1}^{n} A_i + \sigma\sum_{i=1}^{n} p_i}{2\mu+\sigma}.$$

Summing over all $i = 1,2,,\ldots,n$, and solving for $\sum_{i=1}^{n} p_i$, yields

$$\sum_{i=1}^{n} p_i^* = \frac{\mu\sum_{i=1}^{n}(A_i+k_i) - \sigma(n-1)\sum_{i=1}^{n} A_i}{2\mu-(n-1)\sigma}.$$

Substituting $\sum_{i=1}^{n} p_i^*$ in the best-response functions, yields

$$
\begin{aligned}
p_i^* &= \frac{\mu\left(A_i + k_i\right) + \sigma A_i - \sigma \sum_{i=1}^{n} A_i + \sigma \frac{\mu \sum_{i=1}^{n}(A_i+k_i) - \sigma(n-1)\sum_{i=1}^{n} A_i}{2\mu-(n-1)\sigma}}{2\mu+\sigma} \\
&= \frac{\left(\mu+\sigma\right) A_i + \mu k_i}{2\mu+\sigma} - \frac{\sigma\mu \sum_{i=1}^{n}\left(A_i - k_i\right)}{\left(2\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)}.
\end{aligned}
$$

Given p_i^* and $\sum_{i=1}^{n} p_i^*$, the quantity of firm i is given by

$$
\begin{aligned}
q_i^* &= \mu\left(A_i - p_i^*\right) - \sigma \sum_{j\neq i}^{n}\left(A_i - p_j^*\right) \\
&= \left(\mu+\sigma\right)\left(A_i - p_i^*\right) - \sigma \sum_{i=1}^{n} A_i + \sigma \sum_{i=1}^{n} p_i^* \\
&= \left(\mu+\sigma\right)\left(A_i - \frac{\left(\mu+\sigma\right)A_i + \mu k_i}{2\mu+\sigma} + \frac{\mu\sigma\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)}\right) \\
&\quad -\sigma \sum_{i=1}^{n} A_i + \sigma \frac{\mu\sum_{i=1}^{n}\left(A_i+k_i\right) - \sigma\left(n-1\right)\sum_{i=1}^{n} A_i}{2\mu-\left(n-1\right)\sigma} \\
&= \left(\mu+\sigma\right)\left(\frac{\mu\left(A_i-k_i\right)}{2\mu+\sigma} + \frac{\mu\sigma\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)}\right) - \frac{\mu\sigma\sum_{i=1}^{n}\left(A_i-k_i\right)}{2\mu-\left(n-1\right)\sigma} \\
&= \frac{\mu\left(\mu+\sigma\right)\left(A_i-k_i\right)}{2\mu+\sigma} - \frac{\mu^2\sigma\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)}.
\end{aligned}
$$

Summing over all firms,

$$
\begin{aligned}
Q^* &= \frac{\mu\left(\mu+\sigma\right)\sum_{i=1}^{n}\left(A_i-k_i\right)}{2\mu+\sigma} - \frac{n\mu^2\sigma\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)} \\
&= \frac{\mu\left(\left(\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right) - n\mu\sigma\right)\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)} \\
&= \frac{\mu\left(\mu-\left(n-1\right)\sigma\right)\sum_{i=1}^{n}\left(A_i-k_i\right)}{2\mu-\left(n-1\right)\sigma}.
\end{aligned}
$$

The market share of each firm i is then

$$
\begin{aligned}
s_i^* &= \frac{\frac{\mu(\mu+\sigma)(A_i-k_i)}{2\mu+\sigma} - \frac{\mu^2\sigma\sum_{i=1}^{n}(A_i-k_i)}{(2\mu+\sigma)(2\mu-(n-1)\sigma)}}{\frac{\mu(\mu-(n-1)\sigma)\sum_{i=1}^{n}(A_i-k_i)}{2\mu-(n-1)\sigma}} \\
&= \frac{\left(\mu+\sigma\right)\left(2\mu-\left(n-1\right)\sigma\right)\left(A_i-k_i\right) - \mu\sigma\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2\mu+\sigma\right)\left(\mu-\left(n-1\right)\sigma\right)\sum_{i=1}^{n}\left(A_i-k_i\right)} \\
&= \frac{\left(1+\frac{\sigma}{\mu}\right)\left(2-\left(n-1\right)\frac{\sigma}{\mu}\right)\left(A_i-k_i\right) - \frac{\sigma}{\mu}\sum_{i=1}^{n}\left(A_i-k_i\right)}{\left(2+\frac{\sigma}{\mu}\right)\left(1-\left(n-1\right)\frac{\sigma}{\mu}\right)\sum_{i=1}^{n}\left(A_i-k_i\right)} \\
&= \frac{\left(1+\frac{\sigma}{\mu}\right)\left(2-\left(n-1\right)\frac{\sigma}{\mu}\right)\left(A_i-k_i\right)}{\left(2+\frac{\sigma}{\mu}\right)\left(1-\left(n-1\right)\frac{\sigma}{\mu}\right)\sum_{i=1}^{n}\left(A_i-k_i\right)} - \frac{\frac{\sigma}{\mu}}{\left(2+\frac{\sigma}{\mu}\right)\left(1-\left(n-1\right)\frac{\sigma}{\mu}\right)}.
\end{aligned}
$$

The market shares then, and hence HHI, depend only on $\frac{\sigma}{\mu}$, but not separately on the parameters μ and σ. HHI is then given by

$$H=\sum_{i=1}^{n}\left[\frac{\left(1+\frac{\sigma}{\mu}\right)\left(2-(n-1)\frac{\sigma}{\mu}\right)(A_i-k_i)}{\left(2+\frac{\sigma}{\mu}\right)\left(1-(n-1)\frac{\sigma}{\mu}\right)\sum_{i=1}^{n}(A_i-k_i)}-\frac{\frac{\sigma}{\mu}}{\left(2+\frac{\sigma}{\mu}\right)\left(1-(n-1)\frac{\sigma}{\mu}\right)}\right]^2.$$

Differentiating HHI with respect to $\frac{\sigma}{\mu}$,

$$\begin{aligned}\frac{\partial H}{\partial\left(\frac{\sigma}{\mu}\right)} &= \frac{\left(2+(n-1)\left(\frac{\sigma}{\mu}\right)^2\right)n}{\left(2+\frac{\sigma}{\mu}\right)^2\left(1-(n-1)\frac{\sigma}{\mu}\right)^2}\sum_{i=1}^{n}s_i^*\left(\frac{A_i-k_i}{\sum_{i=1}^{n}(A_i-k_i)}-\frac{1}{n}\right)\\ &= \frac{\left(2+(n-1)\left(\frac{\sigma}{\mu}\right)^2\right)n}{\left(2+\frac{\sigma}{\mu}\right)^2\left(1-(n-1)\frac{\sigma}{\mu}\right)^2}\left(\frac{\sum_{i=1}^{n}s_i^*(A_i-k_i)}{\sum_{i=1}^{n}(A_i-k_i)}-\frac{1}{n}\right)\geq 0,\end{aligned}$$

where the inequality follows by Chebyshev's sum inequality. Since $\frac{\sigma}{\mu}=\frac{\gamma}{\beta+(n-2)\gamma}=\frac{\frac{\gamma}{\beta}}{1+(n-2)\frac{\gamma}{\beta}}$ is increasing with $\frac{\gamma}{\beta}$, so is HHI. ■

HHI in the product differentiation case with symmetric firms: When the n firms are identical, $q_i^*=q^*$ for all i. Substituting in (15), yields

$$CS^*=\frac{(\beta-\gamma)\,n\,(q_i^*)^2+\gamma\,(nq^*)^2}{2}=\frac{n\,(\beta+\gamma\,(n-1))\,(q_i^*)^2}{2}.$$

Noting that under symmetry and quantity competition, $PS^*=\beta n\,(q^*)^2$, it follows that $\frac{PS^*}{CS^*}=\frac{2\beta}{\beta+\gamma(n-1)}$. Substituting in (16) and simplifying, yields

$$H=\frac{\frac{2\beta}{\beta+\gamma(n-1)}}{\frac{2\beta}{\gamma}-\frac{\beta-\gamma}{\gamma}\frac{2\beta}{\beta+\gamma(n-1)}}=\frac{1}{n}.$$

Under price competition, $PS^*=\frac{n(q^*)^2}{\mu}$. Hence, $\frac{PS^*}{CS^*}=\frac{\frac{n(q^*)^2}{\mu}}{\frac{n(\beta+\gamma(n-1))\left(q_i^*\right)^2}{2}}=\frac{2}{\mu(\beta+\gamma(n-1))}$. Substituting in (19) and simplifying, yields

$$H=\frac{\frac{2}{\mu(\beta+\gamma(n-1))}}{\frac{2}{\mu\gamma}-\frac{\beta-\gamma}{\gamma}\frac{2}{\mu(\beta+\gamma(n-1))}}=\frac{1}{n}.$$

Since the result is independent of the parameters β and γ, it also holds for the Shubik and Levitan specification. ■

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