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Abstract

We analyze the effect of handicaps on turnout. A handicap is a difference in the vote tally between alternatives that strategic voters take as predetermined when they decide whether to turn out for voting. Handicaps are implicit in many existing democratic procedures. Within a costly voting framework with private values, we show that turnout incentives diminish considerably across the board if handicaps are large, while low handicaps yield more mixed predictions. The results extend beyond the baseline model - e.g. by including uncertainty and behavioral motivations - and can be applied to the optimal design of Assessment Voting. This is a new voting procedure where (i) some randomly-selected citizens vote for one of two alternatives, and the results are published; (ii) the remaining citizens vote or abstain, and (iii) the final outcome is obtained by applying the majority rule to all votes combined. If the size of the first voting group is appropriate, large electorates will choose the majority's preferred alternative with high probability and average participation costs will be moderate or low.

JEL Classification: C72, D70, D72

Keywords: Turnout - Referenda - Elections - Pivotal voting - Private value

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The Effect of Handicaps on Turnout for Large Electorates: An Application to Assessment Voting*

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Keywords: turnout; referenda; elections; pivotal voting; private value

JEL Classification: C72; D70; D72

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1 Introduction

Motivation and background: The (manifold) incentives to turn out

Do standard democratic voting procedures map the preferences of the electorate into outcomes representatively (and efficiently)? Turnout in elections where participation is voluntary tends to be significantly lower than the size of the electorate, so it is unclear whether those citizens who vote represent the distribution of preferences in the entire population. In the 2016 US presidential elections, for instance, the voting rate over the total population was 56%, and the figure varied substantially across sex, age, and origins.[1] In Switzerland, a direct democracy, the average participation rate in federal votes has dropped below 50% in recent years, with many young individuals participating only if the decision at stake is easy to understand.[2] As has been widely argued in the literature, some citizens may not exercise their right to vote when doing so is costly, and these costs will potentially affect election outcomes (Palfrey and Rosenthal, 1983; Ledyard, 1984; Palfrey and Rosenthal, 1985). There are many reasons why voting may be costly for an individual: going to the polling station requires time and effort, understanding the voting process is sometimes a far from trivial task, the fear of being among the losers of a vote can be a psychological burden, rain on the election day can make things inconvenient (Gomez et al., 2007), or there may be bureaucratic hurdles or troubles, to name but a few.[3]

When voting is costly and voluntary, a rational individual will compare such costs with the expected benefit of casting his/her vote, which is proportional to the (endogenous) probability of being pivotal. Focusing on this motivation to vote, three stylized facts are predicted by a significant strand of the literature on costly voting with private values and two alternatives (see e.g. Campbell, 1999; Börgers, 2004; Krasa and Polborn, 2009; Taylor and Yildirim, 2010a,b; Myatt, 2015). These stylized facts set out our baseline model and are the starting point for our analysis. First, if citizens vote at all, they will do so for their preferred alternative, and hence the only decision voters take is whether or not to turn out. Second, individual turnout rates decrease with the size of the electorate; this is called the *size effect*. It captures the intuitive idea that pivotal probabilities are lower, the larger the number of citizens who turn out to vote. Third, *ceteris paribus*, supporters of the minority alternative display higher relative turnout than supporters of the majority alternative; this is called the *underdog effect*. Since voting for one's preferred alternative is a public good for all of its supporters, the free-riding incentives are larger for the majority than for the minority.

The (perceived) probability of being pivotal is mediated by a number of factors we consider

[1]See `https://www.census.gov/data/tables/time-series/demo/voting-and-registration/voting-historical-time-series.html`, retrieved 20 November 2018. Using 2004 US presidential election data, Kawai et al. (2015) claim that minority, low-income and less-educated voters were underrepresented and that the result in eight states would have been different had all voters turned out.

[2]See `https://www.swissinfo.ch/eng/democratic-duty_should-we-worry-about-low-voter-turnouts-in-switzerland-/44248880`, retrieved 29 November 2018.

[3]In the 2019 Spanish general elections, the turnout rate among the citizens living abroad was 6%, while the overall turnout rate was 76 %, see `https://www.publico.es/politica/elecciones-generales-6-16-espanoles-extranjero-logrado-votar.html`, retrieved 14 May 2019. The administrative burden is higher—and the hurdles to vote are more numerous—for those who live abroad, see `https://magnet.xataka.com/en-diez-minutos/como-votan-los-espanoles-en-el-extranjero`, retrieved 14 May 2019.

in our analysis. They include voting cost advantage, aggregate and/or individual uncertainty, and poll manipulations, as well as the use of heuristics and behavioral rules such as a pivotality overestimation, sense of civic duty, or the will to conform to voting for the alternative leading in the polls. These elements mitigate or aggravate the extent of the underdog effect. They account for some of the most relevant factors affecting turnout incentives and thus determining how elections and referenda map preferences into outcomes. A citizen with lower voting costs, for instance, typically displays higher relative turnout all else being equal; this will be called the *cost effect* (Campbell, 1999; Taylor and Yildirim, 2010b). Considering voting motivations that differ from a calculus based on pivotality will largely expand the scope of our analysis and results.

The effect of handicaps on turnout: A new voting procedure

In the present paper, we adopt (as a baseline set-up) a costly voting framework with private values and investigate the supplementary effect of handicaps on turnout incentives. A *handicap* is a difference in the vote tally between alternatives that strategic citizens take as predetermined when they decide whether to turn out. Handicaps arise in one-round voluntary voting when some assumptions of the standard model are perturbed, e.g., when some votes are manipulated, some citizens can publicly commit their vote ahead of voting day, or information about the ongoing voting outcome is released. In either case, the perceived difference in the vote count between alternatives before voting starts takes the form of a (possibly stochastic) handicap. Handicaps are also explicitly at work in sequential voting, in voting procedures where a qualified majority is required, as well as in the signature gathering procedures that regulate popular initiatives.

Although our analysis applies to *all* the above cases, we will focus on a new voting procedure, which we call *Assessment Voting* (AV).[4] There are two reasons for considering AV—which is *merely* one example where handicaps are relevant—the default focus of our analysis and application of our results. First, AV will feature handicaps in the simplest way. This will make our analysis and results transparent. Second, in AV, the equilibrium effect of handicaps can be adjusted by design, thereby offering a potential way of resolving the underdog effect and the cost effect, as well as other turnout distortions that occur in standard voting procedures. AV specifies the following course of events:

1. A number of citizens are randomly selected from the entire population; these constitute the *Assessment Group (AG)*.
2. All members of AG cast their vote (simultaneously) for one of the alternatives at hand or abstain.
3. The number of votes cast for either of the alternatives in the first round is made public.
4. All citizens who do *not* belong to AG decide (simultaneously) whether to abstain or to vote for one of the alternatives, and thus the second voting round takes place.
5. The alternative with the most votes in the two rounds combined is implemented. Ties are broken by the toss of a fair coin.

[4]For a verbal description, see Gersbach (2015).

Main results about Assessment Voting: The case of high handicaps

We show that for large electorates, if the size of AG is chosen appropriately and no member of this group abstains, AV has the following properties in the case of two alternatives: First, the alternative preferred by the majority of the population will be chosen with a probability arbitrarily close to one. Second, expected average participation costs will be similar in extent to the average participation costs in the one-round voluntary voting procedure. Although participation costs are private, they can also be included in the societal calculus: from a utilitarian viewpoint, correct decisions should be taken at the lowest possible average cost of participating in the voting procedure—see Section 5 for a rationale on this approach. From this perspective, AV is superior *within* our framework to one-round voluntary voting, since it yields better decisions at a comparable average participation cost. At the same time, AV is superior to compulsory one-round voting, where the alternative favored by the majority is also chosen with high probability—in fact, with probability one *ex post*—but at the highest possible participation cost. It will transpire that AV is an efficient mixture of voluntary and compulsory one-round voting schemes. The latter are natural benchmarks. In fact, AV will implement the socially optimal solution asymptotically as the (expected) size of the electorate goes to infinity.

Because AV is compatible with the basic democratic principle that every citizen be granted exactly one vote, our results put forward the possibility to experiment with this new procedure for (electronic) voting by the entire citizenry. This is particularly pertinent in the case of binary decisions and in contexts where frequent voting might place a strain on the correct functioning of democracy by generating inefficiencies linked to the existence of participation costs. In Switzerland, for instance, citizens were called four times in 2018 alone to decide on a total of ten public initiatives at the federal level.[5] One possibility that could have made voting less demanding for the Swiss citizens, while enabling them to retain a right to vote on all initiatives, would have been the following: split the Swiss population into ten representative subpopulations, and recognize exactly one of them as the first voting group when adopting AV to vote upon the initiatives.[6]

Baseline set-up

To assess the properties of AV, and hence of handicaps, we consider a society composed of risk-neutral citizens called upon to choose one of two alternatives, say A and B. These alternatives can be the two options at stake in a referendum—say, a proposal, on the one hand, and the status quo or a counter-proposal, on the other—or two candidates in a runoff election. Each citizen's preferred alternative is private and independently drawn from a common Bernoulli distribution. We assume that *ex ante* it is more likely for a citizen to prefer A to B than B to A. This means that from an *ex-ante* utilitarian perspective A is the desirable alternative. For each citizen, there

[5]See `https://en.wikipedia.org/wiki/2018_Swiss_referendums`, retrieved 31 December 2018.

[6]Given the unequal power granted to members of the two voting groups, *any* real-world implementation of AV should guarantee that the identity of both groups cannot be manipulated or biased—we refer to Section 7.1. This concern also applies to Random Sample Voting (RSV, Chaum, 2016), where only a randomly selected group of citizens has a right to vote. Although RSV (critically) fails to grant every citizen one vote as AV does, we will show that both procedures yield the same outcome in equilibrium—provided that voting is compulsory in RSV and in the first round of AV. This provides further justification for experimenting with AV in real-world environments, by drawing upon the features of, and knowledge about, RSV (see e.g. Amar, 1984; Fishkin, 2018).

is also a *common* cost $c > 0$ that materializes if and only if s/he votes. The probabilities of preferring A to B and vice versa, as well as the cost parameter c, are common knowledge. Since we are considering large societies, it will be convenient to assume that the number of citizens follows a Poisson distribution, meaning that our political game will be a Poisson game (Myerson, 1998, 2000). The difference in the vote tally in favor of alternative A obtained in the first voting round of AV (i.e. total votes for A minus total votes for B) is what we will call the *handicap*.

Equilibrium and welfare results

Characterizing the equilibria of the sequential game underlying AV is a complex task, even if we focus on the customary type-symmetric, totally mixed strategy equilibria. This complexity derives from the fact that the strategies of second-round citizens have to take the outcome of the first round—i.e., the handicap—into account. The first-round voters for their part face two sources of uncertainty: within-round uncertainty (How will the other members of AG vote, if at all?), and across-round uncertainty (How will the second-round citizens vote—if at all—in response to the handicap and to the predicted votes of all other members of the second group?).

For general handicaps, the above features may yield a multiplicity of equilibria, which we shall describe and (partially) characterize. The most relevant result for the implementation of AV, however, is that if the size of AG is sufficiently large and we assume that voting for its members is compulsory—or incentivized through subsidies—, with an arbitrarily high probability only one equilibrium exists for the game starting after the publication of the first-round vote tally: no citizen will cast a vote in the second voting round (referred to hereafter as the *no-show* equilibrium).[7] This is a fairly general result since it will hold even if there is aggregate and/or individual uncertainty—and asymmetry of information, in particular—, polls are manipulated, reckoning of pivotal probabilities is biased, or some minority voters conform to voting for the alternative leading in the polls, among other extensions of our baseline model.

Under the no-show equilibrium, the outcome (i.e., the alternative chosen and the costs of voting incurred by all citizens) is thus fully determined in the first voting round. Since the composition of AG is adequately representative of the entire citizenry, socially optimal alternatives will typically be chosen at low societal cost without depriving citizens of their right to vote: the low level of turnout in the second voting round will simply arise as the result of a cost-benefit analysis made by the citizens participating in this voting round.

As a consequence, in the case of AV, the two components of utilitarian welfare—viz., average participation costs and average utility from the alternative implemented—will also be determined entirely by the outcome of the first voting round. On the one hand, the alternative will be resolved by the (random) composition of AG, and hence as a result of the law of large numbers the probability that the socially desirable alternative A will be chosen goes to one with the size of such group. In fact, the probability distribution of the first-round handicap shifts to the right with the size of AG. Increasing AG size, in turn, will reduce the number of cases where any fixed group of B-supporters can change the final outcome by voting in the second round, thereby

[7] Voting is sincere for AG members, and this will be compatible with equilibrium behavior. That is, given that everybody votes for their preferred alternative, in (our) equilibrium, no citizen will have any incentives to deviate.

cutting down the individual incentives for them to vote in that round. These incentives actually disappear as soon as the handicap exceeds a threshold, whose absolute magnitude depends *only* on the extent of the participation costs. Specifically, the threshold is of the order of $1/c^2$. As they are already in the lead, A-supporters neither have incentives to turn out.

On the other hand, there will be no other costs associated with voting except the costs (or the subsidies) necessary to make all members of AG vote. If the citizenry is large enough, it is possible to set the size of AG such that alternative A is chosen with high probability and the average voting costs remain comparatively low. This property holds because the threshold for handicaps discouraging participation in the second round does not change as the size of the entire electorate increases. From a purely positive perspective, this result about welfare sheds light on the optimal size of a representative electorate when the following two objectives are pursued: on the one hand, maximizing the probability of choosing the socially optimal alternative; on the other, minimizing the participation costs. The size (of AG) yielded by our results could then be used as benchmark, for comparison with (inefficient) turnout levels in real elections and referenda.

Further results about low handicaps

Besides the insights just described, we also provide further findings on the equilibrium structure of the second-round voting when the handicap yielded by the first round has not reached the threshold above which only the no-show equilibrium exists. Analyzing this case helps us to understand the complications that may arise under AV if the size of AG is not appropriate for a given scenario, and is thus particularly relevant for the performance of AV in (small) committees. Such an analysis is also insightful for an array of democratic procedures that (implicitly) feature handicaps—see Section 7.2. First, we prove that partially mixed and totally mixed equilibria will typically exist when the handicap is below such vote threshold. These equilibria yield different outcomes, which means that mixed predictions cannot be avoided for low handicap values. Second, we show that in addition to well-known turnout distortions such as the underdog effect, a further equilibrium effect occurs in the second round of AV if there has not been a tie in the first one; we call this the *handicap effect*. To be specific, all else being equal, the marginal value of voting will be larger for the supporters of the handicapped alternative, i.e., the alternative that comes second in the vote-count from the first round. When A has obtained more votes than B in the first round, in particular, both the underdog effect and the handicap effect will reduce the marginal value of a vote for all A-supporters. In some equilibria these two effects combined can be strong enough to make alternative B more likely to win than alternative A. By contrast, if alternative B has attracted more votes in the first round, both effects work in opposite directions, and each may be dominant at different times.

Model extensions

We also analyze the robustness of our results about AV, as well as those about one-round voluntary voting as benchmark, by extending the baseline set-up in four main directions. First, as a rather technical exercise, we relax the assumption that equilibria of the second voting round have to be type-symmetric. Second, we allow some citizens to incur no voting costs (say, because they are partisan) or even like to vote (say, because of some sense of moral duty). We distinguish two polar cases, depending on whether or not partisanship is correlated with preferences. Third, we allow voting costs to differ across supporters of the different alternatives. This generates what has been called the cost effect. Fourth, we consider the case of three or more alternatives. All these extensions not only enhance the relevance of the results about AV, but they also expand (theoretical) knowledge about standard elections and referenda—as well as other democratic procedures—by extending the boundaries of the costly voting paradigm. In particular, we will argue that there is not necessarily a contradiction between having (a share of) the citizens vote rationally and observing a high turnout. This is paramount to the appeal of AV as an actual voting procedure, since its main (theoretical) strength is that if designed properly, it can provide the citizenry with the right incentives to turn out. Since AV can guarantee that AG represents the entire society, a low level of turnout could be more acceptable for citizens in the case of AV compared to one-round voluntary voting, all else being equal.

Organization of the paper

The paper is organized as follows: In Section 2 we discuss our contribution to several strands of the literature. In Section 3 the model is introduced. In Section 4 we analyze the voting equilibria under AV. In Section 5 we explore if AV improves welfare compared to one-round voting, whether compulsory or voluntary. In Section 6 we analyze some extensions of our baseline model—the proofs are in Appendix B. In Section 7 we do two things. First, we discuss the real-world implementation of AV, with an emphasis on some protocols developed in computer science. Second, we reinterpret our results in the framework of existing democratic procedures. Section 8 concludes. The proofs pertaining to the main body of the paper are in Appendix A.

2 More Related Literature

To examine the performance of AV, in our baseline model we consider a society populated by rational individuals who decide whether to turn out by comparing voting costs with the benefits linked to pivotality. Explaining why citizens vote is central to the study of democracy, and a number of theories have been proposed. The case for rational theories of turnout that build on pivotality—such as the one that assumes that voting is costly—can be made in at least two ways. First, such theories (including ours) can explain some of the observed phenomena underlying elections and referenda, which can be—and have been—decisive *in the margin* (see e.g. the examples in Campbell, 1999; Taylor and Yildirim, 2010a).[8] As a recent example of the importance of turnout incentives, PSOE's main concern in the 2019 Spanish elections was that its electoral base would not turn out in sufficiently great numbers, as the party was clearly leading in the polls. During the political campaign, party leaders focused greatly on mobilizing these votes

[8]Also worthy of mention are other theories of rational turnout that aim at replicating empirical observations. Feddersen and Sandroni (2006) (see also Coate and Conlin, 2004) consider ethical voters who derive additional utility by complying with some self-determined social norm. Within a framework of incomplete information, Matsusaka (1995) argues that voters who are very confident that they have voted for the right candidate derive higher utility. DellaVigna et al. (2016) argue that if lying is costly, people vote to be able to tell others.

by triggering the fear that PSOE would not obtain enough votes to form a government.[9] Beyond democracies, voting also takes place in (private) environments. The use of new technologies has made voting much more accessible, and there are manifold examples in recent times where some form of electronic voting has been adopted as a policy-making procedure with substantial money at stake.[10] These voting instances exhibit some of the phenomena featured in elections and referenda that can be explained within the costly voting paradigm. Second, the hypothesis that there is often a strategic component to voting based on pivotality has been substantiated to some extent in some (small-scale) laboratory experiments (see Levine and Palfrey, 2007; Palfrey, 2009), though there is also evidence against some of these predictions (see e.g. Duffy and Tavits, 2008; Faravelli et al., 2017; Agranov et al., 2018). The former experiments show that the size effect, the underdog effect, and the competition effect (according to which turnout is higher when polls are tight) are all consistent with the observed data.[11]

Proceeding against the background provided by the existing literature, we show that the main appeal of AV is that in a cost-efficient way it corrects a handful of turnout distortions, and very particularly the underdog effect. These distortions rationalize the frequent randomness observed in voting outcomes or, more generally, in set-ups where participation is costly (Osborne et al., 2000).[12] While with a common voting cost both alternatives are expected to win with equal probability in one-round voluntary voting due to the underdog effect, the extent of such effect generally depends on the cost distributions and the electorate size, and it is typically weaker (see Taylor and Yildirim, 2010b, for the case of different costs). That is, although supporters of alternative B all turn out individually more than supporters of alternative A, the majority alternative is expected to win more often than the minority. This is the case if the support of the (common) continuous cost distribution comprises the zero cost (Herrera et al., 2014).[13]

Within this latter framework, Krishna and Morgan (2015) have subsequently shown that in fact, the majority alternative will be chosen with probability one in the limit as the expected number of citizens goes to infinity. This is possible because, unlike in our baseline set-up, expected total turnout also grows unboundedly with the size of the electorate. Though some technicalities are different in our model, the logic underlying our main result about high handicaps enables us to obtain the same efficiency result when we extend our model to include partisan voters—i.e., voters who have zero voting costs—and when we assume that the existence and proportion of these voters are uncorrelated with preferences. For this extension of our baseline set-up, expected

[9] See `https://elpais.com/politica/2019/03/30/actualidad/1553970287_749277.html`, retrieved 22 May 2019. PSOE stands for Partido Socialista Obrero Español.

[10] See e.g. `https://vote.makerdao.com/polling`, `https://www.etherchain.org/charts/progpow`, or `https://eosauthority.com/producers_chart`, retrieved 11 July 2019.

[11] The idea that, conditional on turnout, citizens will vote for the strategy they prefer most has empirical support (see e.g. Bhattacharya et al., 2014), although it may not hold with three or more alternatives if voting costs are heterogeneous across citizens (Xefteris, 2019). Regarding the question whether tied outcomes are mainly driven by tied polls, there is both negative (Gerber et al., 2017) and positive (Bursztyn et al., 2017) evidence.

[12] With preferences being gauged by pre-election polls, the standard costly voting theory predicts that narrow victory margins are very likely. In fact, (unexpected) tied outcomes in real elections and referenda are not rare. Examples are the 2000 U.S. presidential election, the vote on Brexit held in 2016 in the UK, or the independence referendum held in Quebec in 1995. See `https://en.wikipedia.org/wiki/List_of_close_election_results`, retrieved on 11 November 2018, for a comprehensive list of tied elections.

[13] The underdog effect is completely offset when there is full aggregate uncertainty about the true preferences of the electorate (Taylor and Yildirim, 2010a) or citizens are very risk-averse (Grillo, 2017).

total turnout is not bounded and the underdog effect has no implication for outcomes in the limit as the expected size of the electorate goes to infinity, in which case the majority rule implements the utilitarian optimal solution.[14] As a consequence, AV does not improve on the majority rule under these circumstances. From this perspective, our contribution is to identify *one* fundamental mechanism—namely, handicaps created by partisan voters—how the non-partisan voters' turnout incentives vanish, and to study scenarios that differ from the limit case where partisan voters come in arbitrarily large numbers.

If, in contrast to the above case, partisanship is correlated with preferences so that the difference in votes cast for either of the two alternatives is distributed symmetrically around zero, one-round voluntary voting will still yield divided decisions (e.g., each alternative will be chosen with probability 1/2 if costs are equal across voting types). AV will thus remains attractive in this second extension of the baseline model including partisan voters. Assuming a correlation between partisanship and preferences offers *one* non-trivial way to circumvent the paradox of voting in the presence of participation costs: Expected total turnout can be made arbitrarily high and this is compatible with citizens casting votes on the basis of a cost-benefit analysis. Moreover, this does not require that these citizens must have arbitrarily small voting costs. It suffices for any non-partisan voter to have voting costs that are in extent never higher than the (perceived) probability of partisan voters' net effect yielding either a tie or one vote fewer for his/her preferred alternative. The correlation between preferences and voting costs may be plausible in real-world scenarios and we discuss possible micro-foundations involving contest functions and political disaffection in Section 6.2. As it happens, one contribution of our results on partisan voters is highlighting the potential importance of the (lack of) correlation between preferences and voting costs in explaining (lack of) randomness in decisions for standard elections and referenda. This link has not received much scholarly attention until now.

Another strand of the literature on costly voting has analyzed sequential procedures with a focus on information aggregation about a public-value component of citizen utility (see e.g. Dekel and Piccione, 2000; Battaglini, 2005). These papers assume that the entire citizenry is divided into groups that queue up to vote. Taking this perspective, our result that turnout incentives disappear as soon as the handicap reaches a certain threshold suggests an optimal stopping rule for sequential voting procedures. On the other hand, our twist is to consider private values. This enables us to expand knowledge about the role of sequentiality in the context of costly voting by showing how a particular two-round voting mechanism can lead to efficient decisions.[15]

The costly voting literature has also shown that Poisson games characterize the limit scenario where the non-stochastic number of citizens goes to infinity (see e.g. Taylor and Yildirim, 2010b). Considering a Poisson game simplifies the analysis greatly, but does not come at a price of loss generality in the case of large electorates. Recent papers that study Poisson voting models are McMurray (2012), Hughes (2016), and Arzumanyan and Polborn (2017). Also very recently, Meroni and Pimienta (2017) have analyzed the structure and number of Nash equilibria in Pois-

[14]Numerous papers argue that democracies based on the majority rule yield (Pareto) efficient outcomes (see e.g. Wittman, 1989). Unlike here, these papers tend to focus on principal-agent and informational problems.

[15]For other sequential voting procedures with purely private values, see Hummel (2011) and Bognar et al. (2015), who take a mechanism-design approach to minimizing the costs of voting.

son games under different voting schemes. Our paper adds to this strand of the literature by (partially) characterizing the equilibrium set of a particular voting game and by finding an interesting property of Poisson games and pivotal probabilities in general: large handicaps diminish these probabilities drastically.

For our exercise on welfare, we compare AV with one-round voting procedures. A justification for this is offered by Kartal (2014), who undertakes a comparative welfare analysis of one-round voting schemes in a model with endogenous turnout. If all voters incur the same cost, all voting schemes that satisfy some minimal regularity properties (including one-round voluntary voting with the majority rule) yield the same level of welfare and thus establish a natural benchmark. In the case of (small) committees where voting is costly, Grüner and Tröger (2019) have recently characterized the (second-best) utilitarian optimal voting rules among those that prescribe voluntary participation. Our paper complements their results by implying that the first-best utilitarian optimal solution can be attained asymptotically for large electorates, provided that voting is made compulsory for a small share of the population.

Finally, to ensure that AV yields in general the socially optimal alternative—albeit with higher voting costs than in the utilitarian optimal solution—, we assume that voting is compulsory (or subsidized) in the first round but voluntary in the second. There is a rich literature on the advantages and drawbacks of compulsory voting (see e.g. Birch, 2016). On the one hand, it increases turnout, reduces inequalities, and solves free-riding problems. On the other hand, it may also increase support for leftist policies in referenda (Bechtel et al., 2016), have a negative impact on the level of civic participation (Lundell, 2012), pose ethical concerns (Lever, 2010), or be ineffective due to habit formation (Bechtel et al., 2018). Our results suggest that making voting compulsory for everybody may not be necessary in general to implement the alternative preferred by the majority, and they also identify the parameter constellations for which requiring voting to be compulsory instead of voluntary would be beneficial for society. These insights add particularly to the costly voting literature that has compared voluntary and compulsory one-round voting (see e.g. Börgers, 2004; Krasa and Polborn, 2009; Krishna and Morgan, 2012).

3 Baseline Model

3.1 Set-up

We consider a society whose citizens have a right to vote for one of two alternatives (or candidates), say A and B. Citizens are risk-neutral and are indexed by i or j ($i, j \in \mathbb{N}$). There is a number p, with $1/2 < p < 1$, such that citizen i's preferred alternative is A with probability $p =: p_A$ and B with probability $1 - p =: p_B$. Individual preferences are stochastically independent and private, whereas the value of p is common knowledge. If citizen i's preferred alternative is chosen, s/he derives utility 1; s/he derives utility 0 if the other alternative is chosen. This normalization is standard and has no bearing on equilibrium outcomes from a qualitative perspective. As for welfare, this normalization implies that we disregard intensities of preferences. On occasion, we may also say that citizen i's type is $t_i = A$ ($t_i = B$) if his/her preferred alternative is A (B). Additionally, if i exercises his/her right to vote, s/he incurs a cost c, which is subtracted from his/her utility. We consider that[16]

$$0 < c < 1/2. \tag{1}$$

We summarize the citizen utility profile in Table 1.

	i's preferred alternative is chosen	i's preferred alternative is *not* chosen
i votes	$1-c$	$-c$
i does *not* vote	1	0

Table 1: Voter Utilities.

3.2 A new two-round voting procedure

Under *Assessment Voting* (*AV*), there are two voting rounds. In the first round, some citizens are chosen to participate by fair randomization.[17] All of them together constitute the so-called *Assessment Group* (*AG*). We use N_1, a positive integer, to denote the size of AG. In principle, all members of AG could decide (simultaneously) whether to exercise their right to vote or not, and if so, which alternative to vote for. However, we shall assume that all members of AG will vote. One possibility is to make voting compulsory. Alternatively, all members of AG could be given a subsidy slightly above c, their cost of voting, so that participating in the first round involves no cost for them. Whether members of the first round exercise their right to vote or not, they cannot vote in the second round. In the second round, only citizens who are *not* members of AG have a right to vote. Before the voting in the second round takes place, the number of votes that each alternative received in the first voting round is disclosed and becomes common knowledge. Henceforth, we use d to denote the vote difference between alternatives A and B in the first round, which we refer as the *handicap*. In particular, if $d > 0$, alternative A received d more votes than alternative B from the members of AG. Thus we say that B is *handicapped* with respect to A. The alternative that receives more votes within the two voting rounds combined is implemented, with ties being broken by fair randomization.

We assume that the *total* number of citizens with a right to vote is $N = N_1 + N_2$, where N_2 follows a Poisson distribution with parameter $n_2 \in \mathbb{R}_+$.[18] Then, we use $n = N_1 + n_2$ to denote the expected number of citizens. Following Myerson (2000), the number of citizens of type t

[16]If $c > 1/2$, no citizen has incentives to vote at all. Assuming a degenerate distribution for voting costs is a simplification but is not critical since we are considering large populations, in which case the incentives to vote are very small for those citizens with costs higher than the lowest (Taylor and Yildirim, 2010b). Assuming that c is common across types of citizens will enable us to focus on the differential effect of AV with respect to standard one-round voting procedures. Similar results nonetheless obtain in the case where the two types of citizens incur different voting costs—see Section 6.3.

[17]That is, each citizen has the same likelihood of being a member of AG. Citizen preferences do not change across rounds and they are independent of whether they are selected for AG or not.

[18]This assumption on the census is made for convenience, but it does not qualitatively affect our results. It guarantees that there are enough citizens to make up for the members of AG.

($t \in \{A, B\}$) in the second round follows a Poisson distribution with parameter $n_2 \cdot p_t$. The properties of the Poisson distribution together with the stochastic independence of individual types ensure that from the perspective of a voter of type t, the number of voters of his/her same type also follows a Poisson distribution with parameter $n_2 \cdot p_t$. This will simplify the analysis greatly. Finally, we use Ω_1 (Ω_2) to denote the set of citizens of the first (second) voting round.

3.3 Equilibrium concept and information

We study the existence and multiplicity of type-symmetric perfect Nash equilibria in our voting game. By *type-symmetric* we mean that within each round all citizens of the same type use the same strategy. Moreover, we assume that if they *do* turn out, they will vote *sincerely*, i.e., we assume that they will either vote in favor of their preferred alternative or abstain. In the second round, sincere voting arises endogenously as in the one-round voting procedures already analyzed in the literature (see e.g. Taylor and Yildirim, 2010b). This follows from the fact that once the results of the first round become common knowledge, voting for the alternative that is not the preferred one is a weakly-dominated strategy for any citizen. With regard to the first round, we impose sincere voting as an assumption of our model but it will still turn out to be compatible with equilibrium behavior. In combination with the subsidies given to members of AG, this means that the first-round outcome, and hence handicap d, follows mechanically from the size of AG and the value of p. The reason is that every member of AG will vote, and they will choose the alternative they prefer. Accordingly, let citizen i be a member of AG and consider the following random variable:

$$\mathcal{X}_i = \begin{cases} +1 & \text{if } t_i = A, \\ -1 & \text{if } t_i = B. \end{cases} = \begin{cases} +1 & \text{with probability } p_A, \\ -1 & \text{with probability } p_B. \end{cases} \tag{2}$$

Then handicap d is the outcome of the random variable D defined by

$$D := \sum_{i \in \Omega_1} \mathcal{X}_i. \tag{3}$$

As far as the citizens' strategic choices are concerned, we can thus focus on the game starting after the first voting round and after the value of d has been made public, which we denote by $\mathcal{G}^2(d)$. We assume that the citizens who vote in the second round can only condition their vote on their type and the observed value of d, since nothing else is payoff-relevant. Accordingly, for each $d \in \{-N_1, \ldots, 0, \ldots, N_1\}$, a strategy for citizen i is a mapping

$$\alpha_i : \{A, B\} \times \{d\} \to [0, 1].$$

That is, $\alpha_i(t, d)$ indicates the probability of citizen i voting for his/her preferred alternative if s/he is of type t and the vote difference between the two alternatives in the first round (i.e., the handicap) is d. As is standard, we assume that for each $d \in \{-N_1, \ldots, 0, \ldots, N_1\}$, there are numbers $\alpha_A(d) \in [0, 1]$ and $\alpha_B(d) \in [0, 1]$ such that $\alpha_i(A, d) = \alpha_A(d)$ if $t_i = A$ and $\alpha_i(B, d) =$

$\alpha_B(d)$ if $t_i = B$. That is, citizens of the same type vote with the same probability. A strategy profile is denoted by $\alpha = (\alpha_A, \alpha_B)$. Finally, we define $d_A = d$ and $d_B = -d$.

4 Analysis of Assessment Voting

We start by analyzing the second round of AV described by $\mathcal{G}^2(d)$, and then focus on the analysis of the entire voting procedure.[19]

4.1 Second voting round

In the second voting round of AV, citizen i's vote will make a difference in the final outcome only if the votes of the remaining citizens that have a right to vote in this round are such that:

- in the second round, i's preferred alternative obtains $d_{t_i} + 1$ fewer votes than the other alternative, or
- in the second round, i's preferred alternative obtains d_{t_i} fewer votes than the other alternative.

In the first case, i's vote in favor of the preferred alternative t_i will turn a defeat of t_i into a tie, while in the second case, i's vote in favor of t_i will turn a tie into a win for t_i. In both cases, utility derived from the alternative that is eventually implemented increases by $1/2$ if citizen i turns out and votes in favor of the preferred alternative.

In the following, we prove a series of results that describe and (partially) characterize the set of equilibria of game $\mathcal{G}^2(d)$. This characterization will prepare the ground for an extensive description of AV's performance, which will take place in Section 4.2. The results shown in this section have nonetheless value in their own right since they can be applied to a variety of existing democratic procedures in which there may be a difference *a priori* between the votes for the different alternatives—we have called this the handicap. These applications have been spelled out in the Introduction and are discussed in Section 7.2 in more detail, and they enable a reinterpretation of our insights from a purely positive perspective—the (default) application of our results to AV is also normative in nature.

It will be convenient to use $x_A := n_2 p_A \alpha_A$ and $x_B := n_2 p_B \alpha_B$ to denote the expected number of votes for each alternative given strategy profile α. Note that determining the pair $(\alpha_A(d), \alpha_B(d))$ is equivalent to determining the pair $(x_A, x_B) = (x_A(d), x_B(d))$. We focus initially on *totally*

[19] We do not consider the proposal-making process. Without aggregate uncertainty, a benevolent social planner—say, a benevolent parliament—could always implement the alternative preferred by the majority. While our results on AV hold even if there is aggregate uncertainty—in which case the social planner may not be sufficiently informed about what the optimal decision should be—, it is worth noting that in democracy, voting is often needed to ratify some decisions regardless of whether there is aggregate uncertainty or not. This is the case in Switzerland, where many decisions have to be supported by a popular vote no matter whether pre-referendum polls are very precise or not. As mentioned, one-round voluntary voting outcomes are prone to suffer from a range of turnout distortions, which AV aims at correcting.

mixed (strategy) equilibria of $\mathcal{G}^2(d)$, i.e. we assume that $0 < \alpha_i(d) < 1$ for $i \in \{A, B\}$. This type of equilibrium is of central importance in the costly voting literature (see e.g. Taylor and Yildirim, 2010b). Accordingly, we next derive the conditions that make both types of citizen indifferent between abstaining and voting in favor of their preferred alternative, thereby incurring cost c. These conditions pin down all the possible totally mixed equilibria of game $\mathcal{G}^2(d)$.

First, we assume that $d = 0$ and use n_A (n_B) to denote the votes in favor of A (B) in the second round of AV. Then we obtain the following two equilibrium conditions:

$$c = \frac{1}{2}\left(P[n_A = n_B] + P[n_A = n_B - 1]\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!}\frac{x_B^k}{e^{x_B} k!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!}\frac{x_B^{k+1}}{e^{x_B}(k+1)!}, \tag{4}$$

$$c = \frac{1}{2}\left(P[n_A = n_B] + P[n_A = n_B + 1]\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!}\frac{x_B^k}{e^{x_B} k!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^{k+1}}{e^{x_A}(k+1)!}\frac{x_B^k}{e^{x_B} k!}. \tag{5}$$

The first equation corresponds to the indifference condition for any voter i of type $t_i = A$, the second is the indifference condition for any voter i of type $t_i = B$. One can directly observe that *no* solution (x_A, x_B) of the above system of equations depends on n_2, the expected number of citizens in the second voting round. As a consequence, the expected level of absolute turnout in that round, which is equal to $x_A + x_B$, does not depend on n_2 either; only the probabilities according to which each citizen votes—namely, α_A and α_B—actually *do* depend on n_2. This is a property of the Poisson probability distribution. As we said in Section 2, by increasing the size of a finite population where each voter's type is drawn independently according to a Bernoulli distribution, we obtain in the limit a variable-size Poisson distribution for both voter types.[20] Mathematically, the case where $d = 0$ corresponds to the case of one-round voluntary voting.[21] By simple algebraic manipulations, we obtain $x_A = x_B = x$, where

$$x \cdot \sum_{k=0}^{\infty} \frac{x^{2k}}{k!(k+1)!} = 2ce^{2x} - \sum_{k=0}^{\infty} \frac{x^{2k}}{k!k!}. \tag{6}$$

The above equation has a unique solution in the unknown x (see Lemma 1 in Arzumanyan and Polborn, 2017). Given that $x_A = x_B$, it then follows from $p_A > p_B$ that $\alpha_A < \alpha_B$, i.e., members of the majority (viz. A-supporters) will each turn out with a lower probability than members of the minority (viz. B-supporters). The reason is that the marginal value of casting a vote is lower for the former than for the latter: if A-supporters turned out with the same probability as B-supporters, the former (latter) would expect to be pivotal in fewer (more) cases, but this contradicts equilibrium behavior since all citizens have the same voting costs. In fact, by symmetry it follows that the probability that each alternative will be chosen by AV is 1/2 if $d = 0$ after the first round, regardless of the exact values of p_A and p_B (provided that voting costs are equal across types). This is the manifestation of the well-known *(full) underdog effect*,

[20] With our focus on large electorates, the property that absolute turnout does not depend on the size of the electorate, of technical nature, does not come at the price of loss generality. As we do in Section 6.2, one can always assume that beyond the citizens performing a cost-benefit analysis for turnout, there are other citizens with zero voting costs that always vote. When these latter voters are present, absolute turnout can depend on the electorate size and even grow unboundedly with the latter.

[21] When voting is compulsory, the theoretical analysis is trivial: all citizens vote for their preferred alternative and they incur the cost of voting.

a bedrock feature in most models of one-round costly voting, as we noted in the Introduction and in Section 2.

Second, we assume that $d \geq 1$.[22] Recall that n_A (n_B) denotes the votes in favor of A (B) in the second round of AV. Then, to pin down all the potential totally mixed equilibria of game $\mathcal{G}^2(d)$, we obtain the following system of equations:[23]

$$\begin{aligned} c &= \frac{1}{2}\left(P[n_A = n_B - d] + P[n_A = n_B - 1 - d]\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^{k+d}}{e^{x_B}(k+d)!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^{k+d+1}}{e^{x_B}(k+d+1)!}, \end{aligned} \tag{7}$$

$$\begin{aligned} c &= \frac{1}{2}\left(P[n_A = n_B - d] + P[n_A = n_B + 1 - d]\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^{k+d}}{e^{x_B}(k+d)!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^{k+d-1}}{e^{x_B}(k+d-1)!}. \end{aligned} \tag{8}$$

As with $d = 0$, no solution (x_A, x_B) of the above system of equations depends on n_2. There is a new feature, however. If the first round of AV has not yielded a tie, then, *ceteris paribus*, the electoral support in the second round for the alternative that obtained fewer votes in the first round will be expected to be higher than the electoral support for the other alternative. This is shown in the following proposition:

Proposition 1. *Assume $d > 0$. Then, if (x_A, x_B) is a solution of the system of equations defined by (7) and (8), it must be that $x_A < x_B$.*

The above result implies that provided that a totally mixed strategy equilibrium of $\mathcal{G}^2(d)$ exists, if A needs fewer votes than B in the second round of AV to be eventually chosen (i.e., $d > 0$), then A-supporters will vote with a lower probability than B-supporters (i.e., $\alpha_A < \alpha_B$). This follows from the fact that

$$\frac{\alpha_A}{\alpha_B} = \frac{x_A}{x_B} \cdot \frac{p_B}{p_A} < 1. \tag{9}$$

Hence, when $d > 0$, there may be two effects that reduce the marginal value of voting for the A-supporters relative to the B-supporters. The first effect is captured by the term p_B/p_A in Equation (9), and is merely the underdog effect, which is based on the *ex-ante* distribution of preferences. The second effect is captured by the term x_A/x_B in Equation (9) and arises only because alternative B is handicapped with respect to alternative A. That is, alternative A needs fewer votes than alternative B in the second voting round to be chosen as the final outcome. This second effect is therefore called the *handicap effect*. Remarkably, examples show that for some handicaps $d > 0$, game $\mathcal{G}^2(d)$ can sometimes have the following two equilibria: one in which alternative A is expected to win with a probability higher than 1/2 and another in which alternative A is expected to win with a probability lower than 1/2.[24] That is, both effects

[22]The case $d \leq -1$ can be proven analogously by symmetry. This enables us to put the focus of this section on the case $d \geq 0$ without any loss of generality.

[23]The right-hand side of Equations (7) and (8) are modified Bessel functions of the first kind.

[24]First, let $d = 5$ and $c = 0.169185$. Then, $(x_A, x_B) = (0, 5.4)$ is an equilibrium of $\mathcal{G}^2(d)$, and the probability that A will be eventually chosen is 0.467359. Second, let $d = 5$ and $c = 0.182668$. Then, $(x_A, x_B) = (0, 4.4)$ is an equilibrium of $\mathcal{G}^2(d)$, and the probability that A will be eventually chosen is 0.644138.

combined can be so strong as to overturn the expected advantage that alternative A has over alternative B in terms of both the support within the population (with a right to vote in the second round) and the handicap (from the first round). When $d < 0$, both effects work in opposite directions. The underdog effect increases the probability that alternative B will be eventually chosen, the handicap effect favors the chances of alternative A. Each of the two effects can be more dominant than the other at different times.

Take now *any* integer $d \geq 0$ and some pair (x_A, x_B) that is a solution to the corresponding system of equilibrium equations. Then, the turnout probability in the totally mixed equilibrium determined by (x_A, x_B) of a citizen with preferred alternative $t \in \{A, B\}$ is proportional to x_t and inversely proportional both to the (perceived) support p_t that his/her preferred alternative gains in the entire population and to the expected size of the electorate participating in the second voting round, n_2. This follows from

$$\alpha_t = \frac{x_t}{n_2 \cdot p_t}. \tag{10}$$

Although this is a trivial observation, it captures some of the main comparative statics of our model of costly voting (with and without handicaps) such as the underdog effect and the size effect (whereby individual turnout rates decrease with the size of the electorate).[25] The observation in (10) also has important implications when there is aggregate uncertainty about p_A, and hence about p_B, or when such parameters can be manipulated or affected by behavioral or heuristic rules. From (10) it is clear, for instance, that political parties or lobbies supporting either alternative would like to convince their supporters that their share in the society is less than what it really is. One way to do so is for parties or lobbies to manipulate the polls through which their constituency informs itself. Deviations in the perceived support can also be equal across voter types: first, when there are more polls available to the citizens; second, when in pre-election polls the number of citizens who report that they have not yet decided their preferences increases. The latter figure can be very high at times, particularly in volatile scenarios and at the beginning of the campaign—when many citizens make up their mind about whether or not they will be voting.[26]

Once supporters of some alternative are convinced that their share in the society is lower than what it really is, they overestimate the probability that they will be pivotal, and as a result they are more likely to turn out. From an empirical viewpoint, the perceived citizen support for one's own preferred alternative could be considered a proxy of the perceived probability of being pivotal.[27] There is substantial evidence that individuals tend to overestimate the probability of being pivotal (see e.g. Levine and Palfrey, 2007; Duffy and Tavits, 2008; Faravelli et al., 2017).

[25]The observation in (10) also explains the *weak competition effect* (whereby the individual turnout rates of supporters of either alternative become closer when pre-election polls are tighter). Since the total turnout $x_A + x_B$ is independent of p_A and p_B, the competition effect (whereby the total turnout rate is higher when pre-election polls are tight) cannot be directly replicated in our model.

[26]In Spain, for instance, the ratio of citizens who reported they had not yet decided who to vote for was above 36% prior to the 2015 general elections, and has generally been above 25% for the last decades, see `https://www.eldiario.es/piedrasdepapel/20D-CIS-indecisos_6_460613961.html`, retrieved 6 December 2018.

[27]The empirical literature has considered various proxies for pivotality, such as closeness in the election (see e.g. Foster, 1983) or the size of the electorate (see e.g. Hansen et al., 1987).

This indicates a potential way in which a society populated by citizens actively performing a cost-benefit analysis based on pivotality and participation costs can sustain levels of turnout that are higher than the ones predicted by a narrow interpretation of the theory.

Above and beyond aggregate uncertainty or poll manipulations, some citizens may use heuristics to estimate this probability (Myatt, 2015) or follow other behavioral rules—such as conforming to voting for the alternative leading in the polls—which, in turn, affect the pivotality calculus of other citizens. For example, suppose that a share of citizens in the minority group decides to vote for the winner, say, because they like to be part of the winning coalition. If this is not anticipated by the remaining citizens—who vote according to their perceived pivotal probabilities—, the final outcome will display a higher relative voting share for the alternative favored in the polls compared to the case where no citizen will vote according to winner confirmation. This (reduced-form) *bandwagon effect* counteracts the underdog effect—in particular, both effects may coexist—and it offers a rationale within the costly voting theory for (more frequent) lopsided elections.[28]

4.1.1 High handicaps

Having described some properties of the totally mixed strategy equilibria of $\mathcal{G}^2(d)$, we investigate in the remainder of this section the general existence of equilibria of this game and, more particularly, we analyze the system of equations defined by (7) and (8). The first result demonstrates that this system of equations is incompatible if the handicap yielded by the first round of AV is large enough (in absolute value).

Proposition 2. *There exists a positive integer $d^*(c)$ such that the system of equations defined by (7) and (8) has no solution for all $d \geq d^*(c)$. Moreover, $d^*(c)$ increases as c decreases.*[29]

Hence, game $\mathcal{G}^2(d)$ has no totally mixed strategy equilibria if $d \geq d^*(c)$. The reason is that all citizens will prefer to save the cost of voting rather than casting a ballot and then expecting their vote to be pivotal given the votes in *both* rounds. This property of $\mathcal{G}^2(d)$ simply requires that citizens perform a standard cost-benefit analysis. Most importantly, $d^*(c)$ depends only on c, the cost parameter. Specifically, $d^*(c)$ is of the order of $1/c^2$. In particular, this threshold is independent of n_2 and hence of the expected size of the electorate. The threshold $d^*(c)$ does not depend on p_A and p_B either (nor on any beliefs citizens may have about these parameters). The latter implies that changes in the perceived support of the alternatives within the electorate will not make it possible that equilibria different from the no-show equilibrium exist.

The negative result identified by Proposition 2 does not follow from the fact that the two equations of the system are incompatible, but rather from the fact that neither of them can separately hold for values of d that are large enough. This implies, also remarkably, that the result that voting is discouraged for all voters of the second round extends to scenarios where the voting

[28] The bandwagon effect (whereby the majority obtains more votes than the minority relative to the electorate's true preferences) has received much scholarly attention (see Klor and Winter, 2007; Duffy and Tavits, 2008; Großer and Schram, 2010; Morton et al., 2015, among others). Beyond the need to conform to voting for the (ongoing) majority (see Callander, 2007, for multi-stage voting schemes), the bandwagon effect can also be rationalized in a costly voting set-up through very concave utility functions (Grillo, 2017).

[29] The increase of $d^*(c)$ as function of c is weak, since $d^*(c)$ must be an integer number.

costs differ between A-supporters and B-supporters, as well as to scenarios where voters are uncertain about their own preferences. To state this second impossibility result formally, we now assume that $d \geq 2$ and focus on equilibria of the following type: citizens of type A vote with probability zero, while citizens of type B randomize. Note that because $d \geq 2$, a profile where only citizens of type A vote with positive probability cannot be an equilibrium, as alternative A will be chosen with certainty in the absence of any further votes. Hence we assume that $\alpha_A = 0$ and $0 < \alpha_B < 1$, in which case we have the following two equilibrium conditions:

$$2c \geq \frac{x_B^d}{e^{x_B} d!} + \frac{x_B^{d+1}}{e^{x_B}(d+1)!} \tag{11}$$

and

$$2c = \frac{x_B^d}{e^{x_B} d!} + \frac{x_B^{d-1}}{e^{x_B}(d-1)!}. \tag{12}$$

The inequality guarantees that citizens i of type $t_i = A$ are content with their decision not to vote, while the second equation is the indifference condition for any voter i of type $t_i = B$. We can prove the following proposition:

Proposition 3. *There exists a positive integer $d^*(c)$ such that Equation (12) does not have a solution for all $d \geq d^*(c)$.*

The proof of Proposition 3 is technical in nature and shows that the right-hand side of Equation (12) is lower than the left-hand side for all $x_B > 0$, provided that handicap d is sufficiently large—i.e., provided that $d \geq d^*(c)$. As a consequence, if $d \geq d^*(c)$, there are also no *partially mixed equilibria of* $\mathcal{G}^2(d)$ in which one voter type randomizes (between voting and abstaining) and the other voter type does not vote at all. Given the asymmetry introduced by d in the game describing the second round of AV, it is natural to examine equilibria of this type besides the more customary totally mixed equilibria. We point out that the threshold $d^*(c)$ of Proposition 3 is precisely the threshold used in Proposition 2 and that we will use this same notation throughout the paper, including the Appendices. On the other hand, it is straightforward to see that if $d \geq 2$, there is an equilibrium in which no citizen will vote, so the problem of the existence of equilibria of $\mathcal{G}^2(d)$ is trivial. This latter equilibrium—we call it the *no-show equilibrium*—involves only pure strategies, and, in particular, it is neither partially mixed nor totally mixed. The combination of Proposition 2 and Proposition 3 leads to the first main result of the paper.[30]

Theorem 1. *If $d \geq d^*(c)$, the only equilibrium of $\mathcal{G}^2(d)$ is the no-show equilibrium.*

Theorem 1 refers to partially mixed and totally mixed equilibria, as well as to equilibria in pure strategies. This result indicates that if the absolute vote difference between the two alternatives in the first voting round, namely $|d|$, is large enough, there are no incentives for any citizen to participate in the second voting round. This can be seen as a property of the Poisson distribution—and hence of large electorates—, and it holds regardless of the (expected) size of the second-round voting group.

[30]Proposition 3 also shows that no profile in pure strategies can be an equilibrium of $\mathcal{G}^2(d)$.

Remarkably, Theorem 1 continues to hold in the presence of aggregate and/or individual uncertainty (in particular, when there is asymmetry of information), polls manipulation, cost differences across types, and also if there are minority voters who conform to voting for the alternative leading in the polls.[31] As developed in Faravelli et al. (2017), the incentive to vote with the majority might be stronger in larger groups—e.g., the entire society—compared to the smaller groups—e.g., AG in AV. The robustness of Theorem 1 will be the basis for the (potential) appeal of AV as a voting procedure and is in contrast with voluntary one-round voting, which is much more sensitive to the above perturbations.[32] In the case of AV, citizens could thus save the costs of learning the true values of p_A and p_B, because any concerns about manipulation or strategic transmission of these parameters are eliminated. As we will see in Section 7.2, the generality of Theorem 1 has implications in voting beyond AV and may accordingly constitute a general technical contribution to the literature.

It is important to stress that, in AV, all members of AG will exercise their right to vote and that this *hard fact*—namely, actual votes—is very different from reported opinions in pre-election polls, in which case cheap talk or other strategic behavior may lead to a biased revelation of preferences (Goeree and Grosser, 2007; Agranov et al., 2018). The first round of AV cannot be substituted either by a pre-voting stage with an information market that can deliver information about the aggregate preferences, since the latter does not affect the vote tally directly as handicaps do.

4.1.2 Low handicaps

The above analysis has focused on the case where handicap d is large enough (in absolute value). An ensuing question is then whether there are equilibria different from the no-show equilibrium when $|d|$ is moderately low. To answer this question, we proceed in two steps. We first focus on partially mixed equilibria and then on totally mixed equilibria. In the first case, we obtain the following result:

Proposition 4. *Given $d \geq 1$, there is $c^*(d) \in (0, 1/2)$ such that for all $c < c^*(d)$ an equilibrium $(0, x_B)$ of $\mathcal{G}^2(d)$ exists.*

The above proposition complements the result of Theorem 1. While $d^* = d^*(c)$ determines the threshold on handicaps above which the no-show equilibrium is the only equilibrium, $c^* = c^*(d)$ determines the cost level below which equilibria that are different from the no-show equilibrium exist. It turns out—see the proof of Proposition 2—that for any given $c \in (0, 1/2)$, there exist constants K_1 and K_2, with $K_2 < K_1$, such that

(i) if $d > \frac{K_1}{c^2}$, the only equilibrium of $\mathcal{G}^2(d)$ is the no-show equilibrium, and

(ii) if $d < \frac{K_2}{c^2}$, then $\mathcal{G}^2(d)$ has equilibria that differ from the no-show equilibrium.

[31] In Section 6.5 we discuss the case of individual uncertainty in more detail.

[32] Theorem 1 will also remain valid if the number of citizens with a right to vote follows some distribution—not necessarily a Poisson distribution—satisfying the counterpart of Propositions 2 and 3. Some examples of (non-parametric) distributions can be found in Krishna and Morgan (2015).

Hence, both thresholds, $d^*(c)$ and $c^*(d)$, are (asymptotically) tight, in the sense that $d^* \sim \frac{1}{(c^*)^2}$. In addition, it can be verified numerically that uniqueness of equilibria of $\mathcal{G}^2(d)$ is not guaranteed within all admissible parameter ranges, even if we only consider equilibria of the type $(0, x_B)$.[33]

As to the second case, it turns out that totally mixed equilibria also exist if d is low. Given the trivial existence of the no-show equilibrium when $|d| \geq 2$, standard arguments based on fixed-point theorems fail to show the existence of totally mixed equilibria of our voting game. In the proof of the next proposition, we develop a (technical) approach to show the existence—but not the uniqueness—of a solution to the equilibrium system of equations. The technique we develop here has a potential value in its own right, since it may be helpful for other (voting) settings where fixed-point theorems are not helpful.

Proposition 5. *Given $d \geq 1$, there is $c^{**}(d) \in (0, 1/2)$ such that for all $c < c^{**}(d)$ an equilibrium (x_A, x_B) of $\mathcal{G}^2(d)$ exists.*

The threshold $c^{**}(d)$ of Proposition 5 is very close to the threshold $c^*(d)$ obtained in Proposition 4. In particular, both thresholds are of the same order in d.[34] Propositions 4 and 5 combined establish that there are at least two different equilibria (with different expected turnout levels and winning probabilities) besides the no-show equilibrium if the first-round handicap does not achieve the required threshold. There is one partially mixed equilibrium—in which the handicap effect is so strong that only supporters of the alternative that is handicapped will turn out—and there is one totally mixed equilibrium. As a consequence, when the handicap is low (in absolute value), multiplicity of equilibria cannot be avoided. This makes it difficult to predict the outcome of AV, unless conditions are imposed on the size of AG guaranteeing that threshold $d^*(c)$ will be reached with a probability that is sufficiently high. This is crucial, since the handicap effect could provide incentives for citizens to vote against their true preferences in the first round. These incentives vanish when citizens anticipate that the above threshold will almost surely be reached. Remarkably, the multiplicity of equilibria survives even if there is aggregate uncertainty and, in particular, even if there is asymmetry of information across voter types.[35]

We summarize all the results of this section in Table 2.

	$0 \leq d \leq 1$	$2 \leq d \leq \frac{K_2}{c^2}$	$\frac{K_1}{c^2} \leq d$
(Multiple) equilibria with positive expected turnout	✓	✓	✗
Equilibrium with zero turnout	✗	✓	✓

Table 2: A (partial) characterization of equilibria of game $\mathcal{G}^2(d)$ for $d \geq 0$.

[33] Multiplicity of partially mixed equilibria occurs if we consider $c = 0.2$ and $d = 3$. In this case, $(0, y_1)$ and $(0, y_2)$ are equilibria of $\mathcal{G}^2(3)$, where $y_1 \approx 3.17$ and $y_2 \approx 3.76$ are positive solutions of the equation $0.4e^y = \frac{y^3}{6} + \frac{y^4}{24}$, both of which additionally satisfy the inequality $0.4e^y > \frac{y^4}{24} + \frac{y^5}{120}$. Multiplicity of equilibria also occurs if we restrict ourselves to totally mixed strategy equilibria. Numerical examples for this case can be provided upon request.

[34] From the proofs of Proposition 4 and 5, $c^*(d) = \frac{1}{2}\left(\frac{d^{d-1}}{e^d(d-1)!} + \frac{d^d}{e^d d!}\right) > \frac{1}{2}\left(\frac{d^d}{e^d d!} + \frac{d^{d+1}}{e^d(d+1)!}\right) = c^{**}(d)$.

[35] The proofs of Propositions 4 and 5 can be easily adapted to the case where the ex ante probability of being a supporter for either alternative is uniform.

4.2 First voting round

Theorem 1 yields a very strong and fairly robust prediction. If handicap d is above a certain threshold depending *only* on the cost parameter c, no citizen will vote in the second round of AV. It turns out that by making N_1, the size of AG, large enough, the probability that d will be larger than this threshold converges to one. This is proved in the following result, which characterizes the outcome of AV (almost surely).

Theorem 2. *For every $\varepsilon > 0$, there is $N_1^* = N_1^*(\varepsilon, c, p_A - p_B)$ such that for all $N_1 \geq N_1^*$ the outcome of AV satisfies the following properties with probability at least $1 - \varepsilon$:*

- *All citizens of the first voting round vote for their preferred alternative.*
- *No citizen of the second voting round votes.*
- *Alternative A is chosen.*

According to the above theorem, citizens who have a right to vote in the second round are all discouraged from voting if AG is large enough. The logic behind this result hinges on the analysis of Section 4.1 and the law of large numbers. Recall that *(i)* alternative A is *ex ante* more highly preferred in the society than alternative B, *(ii)* members of AG are selected randomly, and *(iii)* voting is compulsory or subsidized for members of AG.[36] As a consequence, with high probability the handicap yielded by the first voting round of AV will be higher as we increase the size of *AG*, until the threshold for the first-round handicap is reached, above which only the no-show equilibrium exists.

It is important to point out that provided that there is little aggregate uncertainty about p_A and p_B, it is immaterial for the main thrust of Theorem 2 to hold whether voting in the first round *does* happen before voting in the second round. As long as it is common knowledge that voting will be made compulsory for a group of citizens of a certain size—namely, those who belong to AG—, the remaining citizens taking part in one-round voluntary voting would anticipate that a handicap above the threshold $d^*(c)$ would be very likely, and then they would abstain.[37] This would save the potential costs of organizing two separate voting rounds. With one voting round only, having a sufficiently large number of announced early votes can therefore deter costly participation.

The next corollary follows from Theorem 2 and reveals how the size of AG should vary with respect to the most important parameters of the model. From a purely positive perspective, this corollary provides a benchmark for the (desired) direction of change in turnout when some parameters change.

[36] If voting can neither be made compulsory for AG members nor subsidies transferred to them, the following possibility could be used instead, especially if voting is done electronically: keep selecting citizens randomly until the desired number of citizens voting has been reached. As long as the decisions to vote in the first round of AV are not correlated with preferences and kept secret, this additional option would be equivalent to making voting compulsory or subsidizing it. Also note that instead of being guaranteed a monetary subsidy to vote, members of AG could be given a half-day off from work to go the ballot box.

[37] With high aggregate uncertainty about preferences, these results would be qualitatively similar, but the probability that the threshold $d^*(c)$ would be attained could be much lower. This could prevent the no-show equilibrium from being unique.

Corollary 1. *Let $N_1^* = N_1^*(\varepsilon, c, p_A - p_B)$ as defined in Theorem 2, where $\varepsilon > 0$. Then,*

- *N_1^* increases if ε decreases, with $\lim_{\varepsilon \to 0} N_1^* = \infty$,*
- *N_1^* increases if $p_A - p_B$ decreases, with $\lim_{p_A - p_B \to 0} N_1^* = \infty$,*
- *N_1^* increases if c decreases, with $\lim_{c \to 0} N_1^* = \infty$.*

The behavior of N_1^* with respect to changes in ε and $p_A - p_B$ is self-evident. If the society is more divided (i.e., lower $p_A - p_B$), or if we want to be more certain that the voting outcome will be dictated entirely by AG members (i.e., lower ε), the size of AG has to be greater. Corollary 1 also implies that, *ceteris paribus*, a lower size of AG suffices when c increases. This is not obvious, since the direct effect of a higher participation cost is to make voting not only more costly for a given citizen, but also for everybody else. The latter makes voting by an average citizen less likely, thereby increasing the probability that a single vote will be pivotal. Nevertheless, the net effect of increasing c does indeed disincentivize voting for all citizens in the second round of AV for a larger set of handicaps—see Proposition 2. This implies that the size of AG can be reduced if participation costs increase, without reducing the probability that the outcome will be as described by Theorem 2.

Two final remarks and some illustrations are in order. First, N_1^* must be generally large enough to satisfy two objectives: on the one hand, second-round citizens' incentives to turn out must disappear; on the other, alternative A must be chosen with at least probability $1 - \varepsilon$. In specifications where N_1^* is very large, in particular, one possible interpretation is that voting should be made compulsory.[38] Second, although N_1^* provides a sufficient condition with regard to the size of AG for the outcome of AV to be as described by Theorem 2, the discussion at the end of Section 4.1 has shown that this required size is (approximately) tight, in the sense that the desired outcome may fail to hold for lower AG sizes. Finally, Table 3 depicts the value of N_1^* for some parameter constellations.[39]

	$p_A - p_B = 0.01$	$p_A - p_B = 0.05$
$\varepsilon = 0.1$	16,922,427	3,271,649
$\varepsilon = 0.01$	17,268,220	3,301,394

$c = 0.001$ (with $d^*(c) = 159,155$)

	$p_A - p_B = 0.01$	$p_A - p_B = 0.05$
$\varepsilon = 0.1$	864,256	146,049
$\varepsilon = 0.01$	954,780	152,788

$c = 0.005$ (with $d^*(c) = 6,367$)

	$p_A - p_B = 0.01$	$p_A - p_B = 0.05$
$\varepsilon = 0.1$	291,314	41,856
$\varepsilon = 0.01$	352,459	45,769

$c = 0.01$ (with $d^*(c) = 1,592$)

	$p_A - p_B = 0.01$	$p_A - p_B = 0.05$
$\varepsilon = 0.1$	63,075	3,003
$\varepsilon = 0.01$	109,143	4,858

$c = 0.1$ (with $d^*(c) = 16$)

Table 3: The (optimal) size of the Assessment Group (AG).

[38] We stress that N_1 (the size of the first voting group) and n_2 (the size of the second group) are independent parameters of our model.

[39] The values depicted in Table 3 do *not* depend on the total number of citizens in the population.

5 Welfare Analysis

After the characterization of the equilibrium outcome under AV, an ensuing question is what the evaluation of such a voting procedure looks like from a welfare perspective. There are two one-round voting procedures that are natural benchmarks for this evaluation: first, voting may be voluntary; second, voting may be compulsory. For the analysis of either of the two procedures, it will be convenient to assume that the total number of citizens follows a Poisson probability distribution of parameter $N_1 + n_2$. As for our notion of social valuation, we focus on expected average utilitarian welfare (henceforth *welfare*). The societal calculus thus takes into account both the utilities derived from the alternative being implemented and the participation costs that accrue in the process of voting. The former needs no justification. As for the latter, it requires that, *ceteris paribus*, turnout be as low as possible. To quote Feddersen and Sandroni (2006)—see p. 1273: *"Expressions of concern about low turnout need not represent a concern for turnout, per se. Instead, people may be concerned by what low turnout signals about society, e.g., lack of civic mindedness."* Our analysis builds on this premise, i.e., subject to implementing the socially desirable alternative with a given probability, voting costs should be minimized.

Under one-round voluntary voting, one can easily verify from Section 4 that welfare is

$$W^{vol} := \frac{1}{2} - \frac{x}{N_1 + n_2} \cdot c, \tag{13}$$

where x is the solution to Equation (6). Under one-round compulsory voting, it is also easy to verify that welfare amounts to

$$W^{com} := w^{com} - c, \tag{14}$$

where w^{com} is the welfare obtained from the alternative eventually implemented when the entire population, which has an expected size equal to $N_1 + n_2$, votes sincerely. It is easy to verify that $w^{com} = p_A \cdot (1 - z^{com}(N))$ for some function $z^{com}(N)$ that satisfies $\lim_{N\to\infty} z^{com}(N) = 0$. Moreover, according to Theorem 2, there is $N_1^* = N_1^*(\varepsilon, c, p_A - p_B)$ such that for all $N_1 \geq N_1^*$, with probability at least $1 - \varepsilon$ we have

$$W^{AV} := p_A - \frac{N_1}{N} \cdot c, \tag{15}$$

where p_A coincides with the expected welfare obtained from the alternative being eventually implemented when all members of AG (which has a certain size N_1) vote sincerely. Finally, a lower bound for welfare is

$$\underline{W} := p_B - c. \tag{16}$$

We will use $\underline{W}$ to estimate welfare when our analysis does not yield clear-cut predictions as to the outcome of AV. Our main result regarding welfare is the following:

Theorem 3. *For every $\varepsilon > 0$, there exist positive integers $N_1^{**}(\varepsilon, c, p_A - p_B)$ and $n_2^*(c)$ such that if $N_1 \geq N_1^{**}(\varepsilon, c, p_A - p_B)$ and $n_2 \geq n_2^*(c)$,*

$$W^{AV} > \max\{W^{vol}, W^{com}\}.$$

The logic behind Theorem 3 can be explained as follows: On the one hand, if the size of AG is large enough ($N_1 \geq N_1^{**}(\varepsilon, c, p_A - p_B)$), Theorem 2 guarantees that only members of AG will vote. These will represent the society's preferences very accurately. On the other, if the entire electorate is large enough ($n_2 \geq n_2^*(c)$), participation costs will weigh less in welfare than the utility obtained from the alternative chosen, but they will still matter.[40] This means that in large societies where participation costs are of first-order importance, AV will perform better on average than standard one-round voting, whether voting in the latter is voluntary or compulsory. In comparison with one-round voluntary voting, average participation costs in AV will be similar in extent, but decisions will represent the population preferences much more accurately. Compared to compulsory one-round voting, decisions will represent the population preferences equally well, but participation costs will be much lower in AV. Hence, within our set-up, AV simultaneously exhibits the most desirable properties of voluntary and compulsory one-round voting, and it can thus be seen as an appropriate mixture of both approaches.

As a matter of fact, the same logic behind Theorem 3 can be applied to show that AV implements the optimal solution from a utilitarian perspective, which we denote by $W^{opt} := p_A$, asymptotically as the expected size of the electorate goes to infinity. When the number of citizens grows unbounded, the absolute size of AG can be made arbitrary large and, at the same time, its relative size can be made arbitrarily small. This asymptotic result is formalized next.

Corollary 2. *The following holds:*

$$\lim_{n_2 \to \infty} W^{AV} = W^{opt}.$$

Spanning all values for the relative size of AG with respect to total population, we find at the two extremes of the spectrum one-round voluntary voting (when AG's relative size is 0) and one-round compulsory voting (when AG's relative size is 1).[41] From this perspective, Theorem 3 suggests the optimal size of AG for this family of schemes. Another option is to pose this question without reference to AV. What is the optimal size of a representative electorate of expected size $N_1 + n_2$ when we balance the maximization of the probability of choosing the socially optimal solution against the minimization of the voters' participation costs? The answer to this question is provided by Theorems 2 and 3, which can be used as benchmark for positive turnout analyses in elections and referenda.

[40] This follows from the fact that *(i)* the (expected) absolute turnout is independent of the (expected) size of the group of citizens who have the right to vote in the second round, and *(ii)* all citizens derive utility from the alternative chosen, no matter whether they have voted or not.

[41] Another possible benchmark for AV is the following: Choose the level of a fine, $\phi \geq 0$, to be imposed on those who do not turn out in one-round voting. If $\phi \geq c$, the outcome (i.e., the alternative chosen and the average voting costs incurred) will be the same as in compulsory one-round voting with cost of voting equal to c. If $\phi < c$, the alternative will be chosen as in one-round voluntary voting with cost of voting equal to $c - \phi$. That is, both alternatives will be chosen with the same probability. Moreover, individual turnout probabilities will be higher for citizens of both types compared to the case where voting costs are equal to c. This follows from the proof of Proposition 1 in Arzumanyan and Polborn (2017), which shows that the solution of Equation (6)—namely, the total turnout rate for either voter type, denoted by x—is a decreasing function of c. Accordingly, when $\phi < c$ average participation costs will be higher compared to one-round voluntary voting with voting cost equal to c, and welfare will be lower. As a consequence, Theorem 3 shows that AV with AG size equal to $N_1^{**}(\varepsilon, c, p_A - p_B)$ will also be superior to this benchmark based on fines.

Theorem 3's statement is valid for given parameters. From a constitutional perspective, however, neither $N_1^{**}(\varepsilon, c, p_A - p_B)$ nor $n_2^*(c)$ could depend on the particular instances of voting that would take place. These voting instances would be characterized by different parameters, and more particularly by different values of $p_A - p_B$ and c. One possibility would be to take the maximum values of N_1^{**} and n_2^* for a certain constellation of parameters, which could then be drafted in the constitution. If such a constellation is broad enough to encompass (almost) all plausible scenarios, AV continues to be preferable to standard one-round voting procedures.

We conclude the welfare analysis with three further remarks. First, alternatives that find little support in the citizenry are bound to be defeated in equilibrium when AV is used. In direct democracies such as Switzerland and California, this may reduce the incentives to initiate popular voting on issues that are only supported by a small minority. In Switzerland, in particular, the 100,000-signature threshold for popular initiatives has become easier to attain with the help of social media. With more popular votes in the form of referenda, organization and opportunity costs represent an increasingly important factor to be taken into account. Facing more popular votes also demands more effort from the citizens themselves, especially when they are poorly informed *ex ante*. AV constitutes a new democratic tool that may help to solve some of these problems, thereby adding to the appeal of this voting procedure from a welfare perspective.

Second, if costs associated with participation do not enter the societal welfare calculus, a trivial (theoretical) solution to the problem of implementing the socially optimal alternative through voting is to make it compulsory in one round. Quite often, however, large and convex compliance costs, caps on fines or ethical concerns (see e.g. Lever, 2010) make it impossible to attain turnout rates close to 100%.[42] This makes it easier for adopted decisions to be unrepresentative of the preferences of the whole electorate, as in voluntary voting. Our results indicate that making voting compulsory for all citizens may not actually be necessary to implement the alternative preferred by the majority, so the turnout rates of around 80% typically observed where voting is compulsory may not be a matter for concern.

Third and last, whether AG members participate in AV because voting is compulsory or because they receive a subsidy compensating for their voting costs is immaterial for outcomes. As for welfare, it will make a difference only if voting costs are heterogeneous. In such case, making voting compulsory will mean less costs from a welfare perspective since every citizen will pay *exactly* their costs, while subsidies should in principle be made equal to the highest possible cost to ensure participation of all AG members. Of course, one could consider lower subsidies and target a larger number of individuals to be members of AG (those with higher costs may not vote). If costs are uncorrelated with preferences, outcomes would not change and average participation costs may be lower.

[42]See `https://www.idea.int/data-tools/data/voter-turnout/compulsory-voting`, retrieved 3 November 2018.

6 Robustness and Model Extensions

The baseline set-up can be extended in at least four sensible ways. First, as an exercise of a rather technical nature, one can ask whether the prediction that citizens of the second round will (almost) never vote if the size of AG is large enough hinges on our equilibrium concept, and very particularly on the assumption that citizens of the same type *all* use the same strategy. Second, it is clear that in real elections and referenda, a cost-benefit analysis is not the only motivation to vote. For instance, some citizens are committed to parties or issues, while other individuals may feel a moral obligation to exercise their right to vote. Hence, it is reasonable to ask whether our results extend to a set-up where a fraction of the citizens has zero voting costs. Third, there may exist differences across alternatives regarding how costly it is for its supporters to cast a vote. Fourth, one may speculate about the performance of AV when three or more alternatives exist. We discuss these issues next. The formal results (viz. Propositions 6–13) and their proofs can be found in Appendix B.

6.1 More citizen types

We start by inquiring whether the assumption that all citizens with the same preferences will use the same strategy does drive the result that no equilibria differing from the no-show equilibrium exist if the handicap yielded by the first round of AV is large enough in absolute terms. Given the nature of this negative result, investigating this issue transcends a mere technical robustness check. The answer is negative, i.e., with regard to the second round of AV, the no-show equilibrium remains unique if the handicap is above a certain threshold, even if we consider different (sub)types of citizens who have the same preferences, with each subtype potentially using a different strategy. This is shown in Proposition 6 (see Appendix B), the proof of which is based on properties of the Poisson distribution and the multinomial theorem. This adds to the robustness of our prediction regarding the outcome of AV, because it indicates that more freedom for choosing strategies is not enough to incentivize citizens to vote.

6.2 Partisan voters

The results of Section 4, and hence of Section 5, rest on the assumption that voting is costly for *all* voters. In this section, we study the robustness of these results with respect to the presence of voters who experience no costs of voting or even actually enjoy voting. Such voters are generically referred to as *partisan voters* and are assumed to have zero participation costs. They act out of duty or obligation, be it to their party or to society and democracy (see e.g. Riker and Ordeshook, 1968; Blais, 2000; Feddersen, 2004).[43] We will proceed on the assumption that partisan voters will vote in the AV round they are allocated to—independently of whether

[43]See Herrera et al. (2014) for a recent analysis of how partisan voters affect turnout rates in elections.

they expect to influence the final outcome—and that they will do so sincerely.[44] All this will be common knowledge.[45]

As a consequence, in terms of voting outcome partisan voters will make no difference in the first round of AV. All members of this group are *incentivized* to vote and thus everybody in the group will vote, no matter whether their voting costs are positive or zero. For the second round, by contrast, partisan voters *do* matter, since they will vote independently of the outcome in the first round. This will affect the non-partisans' decision whether or not to vote in this second round. A crucial observation is that as far as strategic concerns of the latter voters are regarded, the partisan voters voting in the second round are indistinguishable from the members of AG. The only difference with respect to the baseline set-up where there are no partisans is that the size of this *compound set* (the set made up of members of AG plus partisan voters with a right to vote in the second round) is now stochastic. With appropriate modification of thresholds, we will be able to build on the analysis in Sections 4 and 5.

6.2.1 The uncorrelated case

To explore the consequences when partisan voters are present, further assumptions are needed about the share of these citizens. There are at least two distinct possibilities. First, whether a citizen is partisan or not may be independent of his/her own preferences. This is the standard assumption in the literature, despite there is no conclusive empirical evidence behind it. For this case, we use $\mu \in [0, 1]$ to denote the expected share of partisan voters, which we assume to be common knowledge. Since each voter is equally likely to be partisan, μ is also the probability that an arbitrary voter will have zero voting costs. When $\mu = 0$, we recover our baseline set-up.

When partisanship is not correlated with preferences, we prove (see Proposition 7 in Appendix B) that for a given cost c and a given AG size N_1, the size of the compound set will be above the threshold $N_1^*(c)$ of Theorem 2 with a probability close to one if both the expected size of the second voting group (n_2) and the share of partisans (μ) are large enough. Also with a very high probability, citizens who are not partisan will then have no incentives to vote in the second round of AV. The latter property follows from Theorem 1, while Proposition 7 is an application of tail-bound analysis for the Poisson distribution. We note that the (sufficient) lower bound for μ converges to zero as n_2 tends to infinity and that it decreases with N_1. Taking $N_1 = 0$, in particular, yields a bound for the share of partisans above which only such voters will vote in one-round voluntary voting.

The above results have the following implications for welfare: If the electorate is large and the share of partisan voters is also sufficiently large, AV yields the majority's preferred outcome with probability close to one. One-round voting, whether compulsory or voluntary, also yields

[44]Another possibility is that a partisan voter has a participation cost $c > 0$, but votes nevertheless with probability one by not including such a cost in his/her calculus. Whether or not partisan voters experience a cost if they vote has no bearing on the equilibrium outcome, provided that they vote (sincerely) in all cases. The only difference between the two approaches will affect welfare considerations.

[45]If the fact that there are partisan voters is not known by non-partisan voters, the equilibrium behavior of the baseline set-up without partisan voters remains valid in essence. Nevertheless, the outcome may be different, as we will need to add the votes from the partisan voters.

this outcome. The main difference across voting procedures lies in the average participation costs. They are highest for compulsory voting and higher for AV than for voluntary voting.[46] The reason is that, in AV, the non-partisan members of AG will (be forced to) vote, but these citizens will abstain in one-round voluntary voting. In such cases, incentivizing citizens to vote in the first round of AV is actually socially undesirable. This result rationalizes the use of the majority rule when the share of partisans and the size of the electorate are both large enough, provided that preferences are *not* correlated with partisanship. The social optimum decision will be implemented trivially in such cases. Though some technicalities are different, this same result is obtained by Krishna and Morgan (2015).

Now that we have analyzed the case of a sufficiently large share of partisan voters within a large electorate, one ensuing question is what are the consequences when partisans are present in other scenarios. We demonstrate (see Proposition 8 in Appendix B) that for a given n_2, the vote difference yielded by partisan voters in the second round—and, hence, in one-round voluntary voting, where there is no first round—is smaller than $d^*(c)$ with probability close to one if the share of partisan voters is itself low enough. In such cases, the participation of non-partisan voters in the first round of AV is essential to reach the critical threshold for handicaps above which only the no-show equilibrium exists. If this threshold is not reached, we know from Proposition 4 and Proposition 5 that there are typically at least two equilibria besides the no-show equilibrium. In particular, there is no assurance that the alternative preferred by the majority will be selected with high probability. This is, in particular, the case with one-round voluntary voting. Under these circumstances, AV could still be preferable to one-round voting from a welfare perspective.

6.2.2 The correlated case

In the above discussion, we have assumed independence between partisanship and preferences. The main implication is that the expected difference between the number of A-supporters and B-supporters who are partisan is positive and increases unboundedly with the expected number of citizens. An alternative and equally plausible assumption is that the number of partisan voters favoring alternative A is similar to the number of partisan voters favoring alternative B. In the US, for instance, the percentage of registered voters identifying themselves as Republicans is close to the percentage of registered voters identifying themselves as Democrats.[47]

This second approach to partisanship is based on the assumption that the difference between the number of A-supporters and B-supporters who are partisan is distributed according to a probability distribution that is symmetric around zero. Since $p_A > p_B$, this implies a degree of correlation between preferences and partisanship (i.e., voting costs). It is more likely for B-supporters to be partisan than A-supporters.[48] One possibility is that political parties need

[46]The difference between AV and one-round voluntary voting disappears if partisan voters incur voting cost c, which they disregard in their decision to vote but which has to be computed for the societal welfare calculus.

[47]See `http://www.people-press.org/2018/03/20/1-trends-in-party-affiliation-among-demographic-groups/`, retrieved 30 November 2018.

[48]As seen in the Introduction, turnout rates in the US (and hence its determinants) differ across sex, age, and origins, and so do preferences. Kawai et al. (2015) have recently found for the US that, in fact, preferences and voting costs are correlated through socioeconomic and socioeconomic characteristics.

to exert some effort or money to persuade citizens to become partisan in their favor, with the outcome whether or not a citizen becomes partisan for either party being determined according to some contest function. If parties have similar (limited) budgets and use the same technology to persuade citizens, one plausible assumption is that they will be successful in equal numbers (at least, in expectation).[49]

An additional rationale for our second approach to examine partisan voters has to do with political disaffection (see e.g. Torcal and Montero, 2006). One can model this negative attitude towards the political system as a random shock that may be privately observed by partisan citizens, who will become non-partisan before the election as a result of the shock. Although their preferred alternative will be the same no matter whether they have incurred a shock or not, these citizens may no longer vote when performing a cost-benefit analysis. Our assumption here is simply to consider that members of the majority are more likely to receive such shocks than members of the minority. This is a plausible assumption in the case of (runoff) elections in which the majority candidate is the incumbent representing the establishment and, as has happened recently in many countries, there is a wake of populist movements undermining the political institutions.

When preferences are correlated with partisanship, we show (see Proposition 9 in Appendix B) that the probability of either alternative winning in one-round voluntary voting is the same as in the case without partisans, namely 1/2 in the case of common, fixed voting costs. This is a remarkable result that transcends AV and expands knowledge on standard elections and referenda. From a theoretical perspective, a non-partisan citizen voting on a cost-benefit basis will decide to turn out if s/he expects to be pivotal in certain cases. In most of the costly voting literature, this property requires randomness in outcomes and that the expected total turnout be bounded regardless of the electorate size.[50] Our result shows that the latter property can be dispensed with. In our reduced-form approach to partisanship, it suffices for non-partisan voters to turn out with a positive probability that the net effect on the vote-count generated by partisan voters is distributed according to some probability distribution that is symmetric around zero. Because the total number of partisan voters can nonetheless be made arbitrarily large, so can total turnout. Roughly speaking, the two groups of partisan voters cancel each other out (in expectation). Randomness in decisions can therefore originate from two different sources, namely, from partisan and from non-partisan voters. Remarkably, voting costs of non-partisan voters need not be arbitrarily low as the (expected) turnout level goes to infinity, provided that the probability distribution of the vote-count difference yielded by partisan voters has enough concentration around zero, or at least so is perceived by non-partisan voters.

One can then prove that by choosing an appropriate size for AG, AV will be preferable to one-round voting in this second framework with partisan voters. While in the baseline set-up we

[49] If we reinterpret partisan voters as (non-rationally) uninformed voters, assuming that the vote tally these citizens yield is symmetrically distributed around zero follows from assuming that these voters follow the most simple heuristic possible: they vote for either alternative with equal probability, if they vote at all. The role of uninformed voters is therefore simply to provide some noise to the voting outcome.

[50] Randomness of the outcome and low participation are also connected in more general settings where participation is costly (Osborne et al., 2000)

choose N_1 such that the difference $d^*(c)$ will be achieved with high probability, with partisan voters we need to impose an even higher difference in the first round of AV.[51] This is due to the uncertainty regarding the number of partisan voters participating in the second round, which makes the vote-count difference created by partisan voters in this second round stochastic, albeit with zero expected value.

6.3 Cost difference across types

For the most part of our analysis thus far—with and without handicaps—, we have assumed no difference across citizen types regarding voting costs. The only exception is when partisanship is correlated with preferences, in which case the cost distributions are different for supporters of either alternative, albeit with the same support—see Section 6.2. The assumption of equal voting costs has allowed us to isolate two main equilibrium effects on turnout: the underdog effect—which is generated by preferences—and the handicap effect—which is generated by handicaps. We next explore the robustness of our results when voter types have positive voting costs drawn from different degenerated distributions. Cost differences may occur in the political process. For example, an incumbent majority government could try to influence voting costs through executive action or, alternatively, the members of the minority could be more motivated or feel they have more at stake, to name but two examples. In either case, one group—be it the majority or the minority—would be advantaged over the other in terms of how costly it is for them to turn out to vote.

We have already mentioned in Section 4.1.2 that Theorem 1—which describes AV's performance when the handicap attained in the first voting round is large enough relative to voting costs—is robust against differences in voting costs. We can also show (see Propositions 10–12 in Appendix B) that when handicaps are low—and, in particular, when the handicap is zero—, there is a further equilibrium effect that is neither linked to preferences nor to handicaps. As one would expect, *ceteris paribus*, the incentives to turn out are greater for those with a lower cost. Moreover, if we consider that no group will turn out with probability one, this effect—which we call the *cost effect*—can offset the underdog effect and the handicap effect completely if the relative difference between voting costs exceeds a threshold that depends on handicap d. In such cases, only the citizens with the lowest voting costs will turn out with positive probability in the second voting round of AV.[52]

6.4 Three or more alternatives

A setting with three or more alternatives enables us to extend the application of AV from binary decisions (above all, referenda) to other decisions, say elections for executive offices or primaries, where several candidates typically compete. The case of multiple alternatives has been recently studied by Arzumanyan and Polborn (2017) for homogeneous voting costs and later generalized by

[51] Proofs can be provided upon request.

[52] With no handicaps, a similar result has been proved by Taylor and Yildirim (2010b).

Xefteris (2019) to account for heterogeneity in such costs. Building on their set-up, we show that sincere voting in the first round is consistent with equilibrium behavior, although other equilibria may also exist (possibly involving strategic voting).[53] In doing so, we prove the counterpart of Theorem 1 for three or more alternatives. Consider the alternative ranked first in the first round of AV according to the plurality rule. Then, there is a threshold $d^{**}(c)$ guaranteeing that if this alternative has received at least $d^{**}(c)$ more votes than any other alternative, no citizen will vote in the second round of AV. This is shown in Proposition 13 (see Appendix B), the proof of which is based on an induction argument on the number of alternatives.[54] Our analysis is nonetheless silent with respect to the general welfare evaluation of AV in this case. The reason is that with three alternatives, the alternative that wins according to the plurality rule (under sincere voting) may lose in a pairwise voting against any other alternative. The extent to which AV favors coordination on certain alternatives over standard one-round voting is crucial for establishing a comprehensive welfare comparison between both voting procedures. Such an analysis is beyond the scope of this paper.

6.5 Remarks and future research

We conclude this section with two remarks that add to the robustness of our results. First, in the case of large electorates, our assumption of homogeneous voting costs covers all relevant cases of costs distributions with a strictly positive support and possibly a mass point at zero. If, as considered for example in Herrera et al. (2014) and Krishna and Morgan (2015), the (common) cost distribution is differentiable within some non-negative, compact and convex set that includes zero, some technicalities change, but one can verify that the main thrust of Theorem 1 remains valid. If the vote-count difference from the first round of AG is above a certain threshold, with high probability total turnout in the second voting round will be low enough not to overturn the outcome from the first stage. In fact, expected total turnout in the second voting round converges to zero if the first-round handicap goes to infinity. This implies that our welfare analysis of AV remains valid with arbitrary cost distributions, provided that there are enough citizens (in absolute terms). Note that with arbitrary cost distributions, AG members have random voting costs, while in one-round voluntary voting only the citizens why the lowest costs will turn out. Yet, the benefits of AV from implementing the right alternative with higher probability will compensate the benefits of having lower voting costs in one-round voluntary voting. As for the robustness of our results about lower handicaps, the main mechanisms from our set-up will also be at work in a set-up based on Herrera et al. (2014). A comprehensive analysis of such cases is nonetheless beyond the scope of our paper.

[53] As shown by Myatt (2007) and Bouton and Castanheira (2012), strategic voting and coordination may occur when there are three or more alternatives.

[54] Proposition 13 (see Appendix B) bears some resemblance to Hummel (2012), who also considers voting over several alternatives taking place in two stages, with the outcome of the first stage made public before the second voting stage takes place. While we show that turnout incentives completely disappear in the second round if the vote-count difference from the first voting round is large enough, Hummel (2012) shows that with three candidates, incentives to vote (sincerely) for the candidate who obtained the least votes in the first round also disappear in the second round if the difference in votes with respect to the first two candidates is large enough. The latter happens with arbitrarily large probability if citizens vote sincerely in the first round and the electorate is itself arbitrarily large. Hummel (2012) differs from our model in that voting is not costly.

Second, in our analysis (with high and low handicaps) we have explicitly proceeded on the assumption that every citizen has full information about their own preferences. Often, however, there is also a public-value component over and above the private-value component in the decision to be taken. This public-value component depends on the unknown state of the world, about which each citizen receives a noisy signal. Such a set-up is considered by Ghosal and Lockwood (2009) for the case of one-round voluntary voting, but with some further differences from to our setting. Ghosal and Lockwood (2009) prove a perfect separation result: The information from the private signals will be used if and only if the public-value component is sufficiently important in utility terms. If not, the private-value component will dictate all voting decisions.

Our insights about high handicaps translate into an extension of our set-up based on Ghosal and Lockwood (2009). This is because regardless of citizens' utilities and beliefs, if the handicap obtained in the first voting round of AG is large enough, no citizen will cast a vote in the second round. In particular, no information aggregation will occur in that round—see Theorem 1. Suppose now that AG members anticipate that turnout in the second round of AG will be zero. If voting in the first round is according to the (certain) private-value component when this is more important than the (uncertain) private-value component, no information will be aggregated either. If voting takes place in accordance with the signals when the public-value component is more important, by contrast, dispersed information will be aggregated, as in Condorcet jury models (see e.g. Boland, 1989; Austen-Smith and Banks, 1996). This would lead to the implementation of the socially optimal alternative with very high probability. By making voting compulsory in the first round, AV could also solve the problem of turnout being too low locally (see Proposition 3 in Ghosal and Lockwood, 2009). This is another potential advantage of AV with respect to one-round voluntary voting in welfare terms (at least in some equilibria). A thorough analysis of this setting for low handicaps remains for further research.

7 Implementation, Reinterpretation, and Remarks

In this section we do two things. First, we discuss the potential implementation of AV as an electronic voting procedure. Second, we reinterpret some of our (mathematical) results for existing democratic procedures, with special emphasis on one-round voluntary voting.

7.1 Real-world implementation of Assessment Voting

As far as our previous theoretical analysis is concerned, it is immaterial whether AV is to be implemented with paper ballots or to be introduced as an electronic voting procedure. This feature does not necessarily hold if we consider the implementation of AV in real-world environments. Since the implementation with paper ballots does not pose specific challenges on its own, we focus here mainly on electronic voting. The latter type of voting may reduce participation costs, make voting easier, enable citizens living abroad to exercise their rights, and eliminate invalid votes, the share of which can sometimes be very significant.[55] These features open up the possibility of improving the design of the current electoral systems (see Demange, 2018). At the same time, however, electronic voting involves a variety of concerns related to privacy, integrity, transparency, affordability, and accessibility.[56] Thus, it comes as no surprise that many efforts within one strand of computer science research are currently directed at solving these problems. Doing so would very particularly enhance the possibility of implementing AV as an electronic voting procedure, both in direct democracies and in those representative democracies that allow referenda. Chaum (2016), for instance, has proposed an initial protocol for Random Sample Voting (RSV)—see Footnote 6—, which was already used on a small scale and is of clear interest for AV.[57] Chaum (2016) argues that randomness of voter selection, non-manipulability, verifiability, and anonymity can be guaranteed.

In research conducted parallel to our paper, a provably secure protocol for RSV has been developed by Basin et al. (2018). The protocol guarantees individual verifiability and receipt freeness, as long as some cryptographic assumptions are fulfilled. The latter paper, a complement to Chaum (2016), considers a strong adversary model. A (malevolent) adversary can intercept all messages sent via the network, construct new messages and deliver them, and even compromise a (small) number of citizens in such a way that the adversary can control their behavior. By using additional uncompromised devices, Basin et al. (2018) construct a protocol that could deliver a trustworthy outcome for the first round of AV.

Guaranteeing uniform random choice for the subset of players in the first round is clearly an important requirement for the correct functioning of AV. Micali and Cheng (2017) argue that a verifiable random function can also be implemented in the blockchain environment. This environment is of double interest for AV, since this voting procedure could be implemented using blockchain protocols and even be used for the governance of the entire blockchain.[58]

Another aspect of the implementation of AV that requires careful inspection, whether implementation is electronic or not, is the size of AG. As already discussed, one can easily envision situations where there is aggregate uncertainty about the support for each alternative within the citizenry, in which case p_A and p_B are not known with full precision. For example, suppose that p_A is believed—by the citizens and the social planner—to be distributed in the interval $I := [\underline{p_A}, \overline{p_A}]$ with some cumulative distribution function $F(\cdot)$, where

$$\frac{1}{2} < \underline{p_A} < \overline{p_A} \leq 1. \tag{17}$$

[55] The share of invalid votes is usually around 4% worldwide, but at some elections it has been much higher. This was the case in some elections for the local parliament in Zurich, in which the share of invalid votes was 26%. See `http://aceproject.org/electoral-advice/archive/questions/replies/864793780` and `https://www.swissinfo.ch/eng/politics/electronic-voting_ten-arguments-for-and-against-e-voting/43959200`, retrieved 2 November 2018.

[56] See `https://icorating.com/upload/whitepaper/PlwK2rH1BpHDSzd56OZTRlzjJo4ffOyNSO41kOKJ.pdf`, retrieved 1 November 2018.

[57] See `github.com/rsvoting/publications/blob/master/trials/crypto-2015-demo-report.md` and `github.com/rsvoting/publications/blob/master/trials/rwc-2016-demo-report.md`, retrieved 1 November 2018.

[58] It is crucial to have an efficient voting procedure for achieving true decentralization in the pubic ledger. For blockchains, how changes in governance should be decided is one of the central themes in the blockchain research area. See Goodman (2014), for an example of a blockchain that has a built-in voting procedure.

First, the social planner can set the size of AG such that AV will yield the no-show equilibrium with probability close to one for the worst possible realization of p_A, namely $\underline{p_A}$. Second, the first round of AV reveals information about the value of p_A. This may help in taking better informed—though not necessarily more efficient—decisions in the second voting round if threshold $d^*(c)$ has not been attained. Importantly, AV can still perform well even if the interval I contains 1/2. When $p_A = 1/2$, both alternatives are equally likely to gather the largest support within the citizenry. In this second scenario, consider some small $\delta > 0$, and then set the size of AG equal to $N_1 = N_1^*(c, \epsilon, 2\delta)$. Then, the result of Theorem 2 holds with probability $(1 - (F(1/2 + \delta) - F(1/2 - \delta)) \cdot (1 - \epsilon)$. That is, as long as the distribution of p_A is not highly concentrated around 1/2, the probability that the outcome of AV will still be as described by Theorem 2 is high. The possibility that $p_A \approx p_B$ then simply sets a lower bound for AG size. Moreover, as far as expected welfare is concerned, this knife-edge case can be neglected if it is ex ante not very likely. A more thorough investigation must nonetheless be left to future research.

7.2 Reinterpreting our results

The procedure we have suggested in this paper, AV, consists of two voting rounds. Voting is compulsory in the first round and voluntary in the second. Mathematically, the game underlying the second round is the game underlying one-round voluntary voting in large electorates, with one fundamental difference: in AV, the handicap d taken as given before the vote takes place may differ from zero. For the game describing the second round of AV, we have proved a series of results "on the equilibrium path" (i.e. for large values of d) and "off the equilibrium path" (i.e. for low values of d). From a broader perspective, one can conceive of handicap d (and of other parameters) as capturing some elements of one-round voting and other related procedures in democracy, in which case our results provide some interesting insights. As already mentioned, AV is just one *natural* application of our analysis.

In Sections 6.2 and 6.3, we have already shown three results on one-round voluntary voting (with two alternatives) that follow from our analysis. First, the majority rule will implement the utilitarian optimal solution if partisanship is uncorrelated with preferences and both the share of partisan voters and the electorate are sufficiently large. Second, decisions yielded by the majority rule will be highly uncertain if, in contrast to the previous case, partisanship is correlated with preferences in a specific way. Third, an advantage in voting costs typically translates into an electoral advantage, even in the presence of handicaps. We next provide six additional examples of how to reinterpret our set-up and enable an application of our results to democratic procedures that differ from one-round voluntary voting. These insights display the property that (future) costly participation can be deterred if some example-specific form of handicap is sufficiently large.

First, let one of the alternatives, $s \in \{A, B\}$, represent the status quo. Then, two requirements are imposed: *(i)* a share $\rho \in [1/2, 1)$ of total votes is needed for the alternative $-s \in \{A, B\} \setminus \{s\}$ to be implemented; *(ii)* a participation quorum requirement $q \in \mathbb{N}$ needs to be reached (if the total number of voters that turn out is less than q, the status quo will prevail). These conditions were imposed for example on the independence referendum in Montenegro.[59] Straightforward algebraic manipulations indicate that

$$x_{-s} - x_s \geq q \cdot (2\rho - 1),$$

where x_s (x_{-s}) is the number of votes for s ($-s$). Magnitude $q\cdot(2\rho-1)$ plays the role of handicap d in game $\mathcal{G}^2(d)$, and it converges to q as ρ approaches one.[60] In the light of Theorem 1, large qualified majorities in combination with large quorum requirements may disincentivize turnout dramatically, and hence protect the status quo even if there is a large majority in favor of the reform.[61]

Second, we have already mentioned that in some countries such as Switzerland one can initiate political action by gathering a minimum number of signatures, say $\sigma \in \mathbb{N}$. By identifying σ with handicap d and focusing on partially mixed equilibria of $\mathcal{G}^2(d)$, Proposition 3 shows that there is a threshold for σ that may block popular initiatives altogether. This threshold, which depends asymptotically on the inverse of the square of the cost of signing the initiative, may in absolute terms be much lower than the support this initiative gathers in the population.[62]

Third, sometimes true information about voting outcomes is revealed before all polling stations are closed. This is the case in Spain, for example, where the official turnout rate at each municipality is revealed at three points in time during election day. In principle, one can extract from these reported rates information about the development of the voting outcome. Our analysis with an arbitrary number of alternatives—see Section 6.4—has revealed that, if we leave aside the strategic incentives for choosing the moment for going to the polls, early release of information of this type may have a significant impact on voting outcomes, particularly if voters infer that one alternative is leading the other(s) by more than $d^*(c)$. This insight also applies if voting results are leaked, a possibility enhanced by the extensive use of social networks (see e.g. the examples in Morton et al., 2015). This may be relevant for US federal elections, because citizens of some states cast their vote well ahead citizens of other states. Such a time window offers the possibility for strategic behavior by different political parties, including the diffusion of information concerning the ongoing actual voting outcome that can take the form of handicaps. These practices are forbidden by law in many countries.[63]

Fourth, it is known that vote buying cannot be avoided completely in elections and referenda and that it is even common in certain countries (see e.g. Brusco et al., 2004; Finan and Schechter, 2012). At the same time, the possibility of vote buying is a major concern in electronic voting (see e.g. Parkes et al., 2017). Suppose the vote-count difference resulting from vote buying (by all parties involved)—i.e., the handicap(s)—can be assessed with some precision before election

[59] See `https://en.wikipedia.org/wiki/Montenegrin_independence_referendum,_2006`, retrieved 8 November 2018. We refer to Herrera and Mattozzi (2010) for a list of countries with participation quorum requirements.

[60] An equilibrium of $\mathcal{G}^2(d)$ with zero (positive) turnout will always be an equilibrium in this new setting (if the turnout level is above q).

[61] This "quorum paradox" also arises in the framework of a group turnout model (Herrera and Mattozzi, 2010).

[62] We refer to Battaglini (2017) and Ginzburg (2018), as well as the references therein, for related papers on the subject of initiatives (or petitions).

[63] See `https://en.wikipedia.org/wiki/Election_silence`, retrieved 11 November 2018.

day. Then, our result from Section 6.4 suggests that there may be a threshold for the number of votes to be bought by a single party relative to the others, namely $d^*(c)$ or $d^{**}(c)$, for which vote buying is a very effective tool in demobilizing voters. Trying to buy votes beyond this threshold may be futile.

It is worth noting that a further appeal of AV relative to one-round voluntary voting has to do with vote buying. In the latter voting procedure, vote buying can be targeted wherever possible, as only the absolute number of ballots bought matters for disincentivizing electoral participation. In AV, by contrast, this strategy is unlikely to be as effective, since AG members are chosen randomly, no matter whether they have committed themselves *ex ante* to sell their ballots or not. *Ex post*, the identification of AG members would also be costly.

Fifth, it is quite often the case that some parties or issues have a solid electoral base of citizens who vote for them regardless of any consideration; we have referred to them as partisan voters. Sometimes, these voters are concentrated in regions or towns, or more frequently in districts. When all other citizens are aware of the existence of partisan voters—who, by definition, are *committed* to vote—, the incentives for non-partisan citizens to turn out may be substantially affected by differences in the number of partisan voters across parties or issues within the entire jurisdiction. These differences take the form of handicaps. Especially if they are very large, these differences may be pivotal for the election outcome. This case is covered for two alternatives in Proposition 7 (see also Proposition 8) in Appendix B, which shows the importance for turnout incentives—and hence for outcomes—of the extent to which political parties can announce committed votes.

Sixth, assume that a parliament votes on a binary decision and that after the vote, citizens can decide in a referendum which of the two alternatives they prefer. Results from the parliament vote and the referendum will be aggregated to determine the final outcome—say, with parliament and the citizenry having equal weight. This procedure is called Co-voting (Gersbach, 2017). Our results show that if voting is costly for the citizens, a clear majority in parliament cannot be overturned later by an equally clear majority with opposite sign in the citizenry. That is, whether the initiative is given to parliament or to the citizenry could have dramatic consequences. If organizing a referendum is very costly for society, our results suggest that as far as Co-voting is concerned, there might be no need to call on a referendum to revise a parliament's decision if the latter has been lopsided. In actual democracies, if a referendum can be called after the parliament's decision, the two voting outcomes are not added. Rather, citizens have full power to revoke the parliament's decision if they reach some vote share that is independent of the parliament's outcome. Although this is different from Co-voting, some of our insights are applicable to actual democracies.

In the previous examples, we have for the most part reinterpreted the results from Propositions 2, 3, and 13 in different democratic frameworks. Nevertheless, the handicap effect identified by Proposition 1 and the equilibrium existence results from Propositions 4 and 5 are also insightful in such frameworks, as are some of the examples discussed in Section 4.2. That is, future costly participation might *not* be deterred in the above examples if the corresponding handicap is sufficiently low, but it is difficult to predict what the outcome will be in these circumstances.

If handicap d is below $d^*(c)$, for instance, the alternative lagging behind during election day may get a boost if results are released early, vote buying might backfire, and enlarging the partisan electoral base may have counterproductive effects.

The above considerations have illustrated how the results we have obtained for AV can generate relevant insights for an array of existing democratic procedures. For instance, we have implicitly offered a rationale for banning the publication of ongoing results before all polling stations are closed. Doing so may open a Pandora's box enabling many strategic behavior options, some of which, as our examples have shown, may lead to very inefficient outcomes. The latter observation is also insightful for the US primaries, as the delegate vote tally in the underlying sequential voting scheme plays a similar role as handicap d in AV.[64] In the same vein, our analysis and results cast some doubts on whether having the same referendum take place twice—and adding up the results of both instances—could be generally used to eliminate randomness in the decisions compared to one-shot referenda. A more comprehensive understanding of all the scenarios outlined in this section would require an analysis in its own right.

Finally, we note that despite our focus on one-shot decisions, our analysis is also valid for a series of decisions, as long as they are independent of each other. In this case, one can interpret the (fixed) Assessment Group as "legislature by lot," with the citizens who do not belong to this legislature having the right to undo the legislature's decisions at a later stage.

8 Conclusion

We have examined the role of early information about the vote tally in voting, and applied our results to the optimal design of Assessment Voting (AV). This a new voting procedure that fulfills all standard democratic requirements (e.g. one person, one vote) and is very simple. Within our framework of costly voting with private values and two alternatives, AV lowers the participation costs of popular votes and ensures that the majority/minority relation in the citizenry is better reflected in the voting outcome. AV could therefore be a partial remedy to some of the problems often associated with standard one-round voting procedures, which serve as a natural benchmark. Given the current availability of computer-science protocols capable of implementing this voting procedure, AV may thus be suitable for experimentation in democracies (and also in private voting environments). This is clearly pertinent for direct democracies such as Switzerland and California, but also for those representative democracies where there exists a growing demand for more frequent consultation of the citizenry. AV offers a potentially efficient way to do so.

Although our main analysis has focused on binary decisions (primarily referenda, but also runoff elections), the main mechanisms can also take effect with three or more alternatives, which is the case of multi-candidate elections. Remarkably, our results regarding equilibrium outcomes and welfare continue to hold if we either allow subgroups of the electorate to use voting strategies different from those of other subgroups even if they have the same preferences, if voting costs

[64] The literature on US primaries is vast (see e.g. Morton and Williams, 2000; Klumpp and Polborn, 2006; Knight and Schiff, 2010).

differ across citizen types, or if there is a significant, but neither extremely large nor very biased, share of citizens who vote out of a sense of duty. This is because the (mathematical) result that turnout incentives completely disappear if there is public asymmetry between alternatives in the form of a large vote-count difference (i.e., in the form of a large handicap) relies solely on one assumption: the only requirement is that voters do a cost-benefit analysis based on participation costs and pivotality. This calculus is further immune to aggregate and/or individual uncertainty, poll manipulation, and the impact of citizens who vote according to heuristics or behavioral rules, and it thus yields a fairly general result, which is arguably our main technical contribution. As it happens, this result can be applied to a variety of (voting) settings, AV being just one natural application. We have also identified a turnout compensation effect linked to low first-round handicaps, which we have called the handicap effect and is relevant both for large elections and small committees. In proving all of our results, we have contributed to the understanding of the costly voting paradigm, and thus to the general understanding of the incentives to turn out in democratic procedures. The costly voting models can replicate many phenomena observed in real-world elections and referenda, and are thus more versatile than usually assumed. This, in turn, enhances our insights on AV.

Our analysis could be extended in other ways beyond the ones considered in this paper. First, we could study in detail all circumstances in which citizens may only have partial knowledge of their own preferences, say, because there is a stochastic public-value component in the decision to be made. As we have seen, a sequential voting procedure such as AV enables the transmission of information from voters of the first voting group to voters of the second voting group, but it also opens up the possibility of using this transmission channel strategically. Second, in anticipation of the use of AV, proposal-making may change. For instance, proposals that have no chance under AV (but *do* have a chance in single-round voting) may be no longer made. Third, one could consider application of AV at the district level in multi-district elections with first-past-the-post electoral systems. Fourth, in the case of runoff elections, one can conceive of the handicap as some *hard* valence that provides one of the candidates with a number of votes more than the other candidate before the quest for the remaining votes takes place. Endogenizing this hard valence together with the candidate's ideological location is required for a more comprehensive theory of elections. Fifth, voting situations occur beyond the elections and referenda that take place in democracies. We have mentioned the importance of governance for blockchain technologies, which offer a real-world environment to test (a possibly tailored version of) AV. All these issues are subjects for future research.

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Appendix A

In this appendix we prove Propositions 1–5, as well as Theorems 2 and 3 and Corollary 1. For the proofs of the propositions, we do not follow the order established by their indices but a more constructive order. By doing so, we will be able to prove some results by building on previous proofs. This will facilitate reading. Accordingly, we start by proving Proposition 3, as it will be used in the proof of Proposition 2. In its turn, the proof of Proposition 1 precedes that of Theorem 2.

Proof of Proposition 3: The goal of the proof is to show that if d is sufficiently large, there does *not* exist a non-negative solution in y for the following equation:

$$2c = \frac{y^d}{e^y d!} + \frac{y^{d-1}}{e^y (d-1)!}. \tag{18}$$

We start by noting that the right-hand side of Equation (18) is equal to 0 for $y = 0$ and $d \geq 2$, and tends to 0 as y tends to ∞. Therefore, proving that Equation (18) does not have a non-negative solution is equivalent to proving that for all $y \in \mathbb{R}_+$, the left-hand side of Equation (18) is strictly larger than the right-hand side.[65] To that end, we prove two auxiliary results. First, for a given $d \geq 1$, we define

$$f_d(y) := ce^y - \frac{y^d}{d!}. \tag{19}$$

We claim that

$$f_d(y) > 0 \text{ for all } y \in \mathbb{R}_+ \Rightarrow f_{d+1}(y) > 0 \text{ for all } y \in \mathbb{R}_+. \tag{20}$$

For the proof of the claim, assume that the left-hand side of (20) is true. Then,

$$\frac{\partial f_{d+1}(y)}{\partial y} = f_d(y) > 0. \tag{21}$$

That is, $f_{d+1}(y)$ is increasing in $y \in \mathbb{R}_+$. Since $f_{d+1}(0) = c > 0$, it follows immediately that the claim in (20) is correct. Second, for a given $d \geq 2$, define

$$g_d(y) := \frac{f_d(y)}{e^y} = c - \frac{y^d}{e^y d!} \tag{22}$$

and note that

$$g_d(y) > 0 \Leftrightarrow f_d(y) > 0. \tag{23}$$

Consider now the following claim, which we will also prove:

$$g_{d^*}(y) > 0 \text{ for all } y \in \mathbb{R}_+ \text{ for some } d^* := d^*(c) \geq 1. \tag{24}$$

By straightforward calculations,

$$\frac{\partial g_d(y)}{\partial y} = -\frac{y^{d-1}(d-y)}{e^y d!}. \tag{25}$$

[65] We use $\mathbb{R}_+$ to denote the set of non-negative real numbers.

It then follows that $y^* = d$ is the (global) minimum of $g_d(y)$ in $\mathbb{R}_+$, since

$$\frac{\partial g_d(y)}{\partial y}|_{y=d} = 0$$

and $\frac{\partial g_d(y)}{\partial y}$ is negative for all $y < d$ and positive for all $y > d$. We accordingly obtain that, for all $y \in \mathbb{R}_+$,

$$g_d(y) \geq g_d(d) = c - \frac{d^d}{e^d d!} \geq c - \frac{1}{\sqrt{2\pi d}e^{\frac{1}{12d}}},$$

where the last inequality holds by an improved Stirling's approximation (Robbins, 1955). Hence, a sufficient condition for the claim of (24) to hold is that

$$c > \frac{1}{\sqrt{2\pi d}e^{\frac{1}{12d}}}.$$

It is straightforward to verify that the right-hand side of the above inequality is a decreasing function of d, provided that $d \geq 1$. Moreover, the right-hand side converges to zero as d goes to infinity. Accordingly, we let $d^*(c)$ be (uniquely) defined as the smallest positive integer larger than one that satisfies

$$c > \frac{1}{\sqrt{2\pi(d^*(c)-1)}e^{\frac{1}{12(d^*(c)-1)}}}. \tag{26}$$

Note that, in particular,

$$d^*(c) = \Omega\left(\frac{1}{c^2}\right). \tag{27}$$

Now we have proved the claim in Equation (24). Finally, let $d \geq d^*(c)$. Then, for all $y \in \mathbb{R}_+$,

$$2c - \left(\frac{y^d}{e^y d!} + \frac{y^{d-1}}{e^y (d-1)!}\right) = g_d(y) + g_{d-1}(y) > 0,$$

where the strict inequality holds by the claims in (20) and (24). This completes the proof of the proposition. □

Proof of Proposition 2: The goal of the proof is to show that if d is sufficiently large the following system of equations in (x, y) does *not* have a solution with non-negative components:

$$2c = \sum_{k=0}^{\infty} \frac{x^k}{e^x k!}\left(\frac{y^{k+d}}{e^y (k+d)!} + \frac{y^{k+d+1}}{e^y (k+d+1)!}\right), \tag{28}$$

$$2c = \sum_{k=0}^{\infty} \frac{x^k}{e^x k!}\left(\frac{y^{k+d}}{e^y (k+d)!} + \frac{y^{k+d-1}}{e^y (k+d-1)!}\right). \tag{29}$$

The system of equations is obtained from (7) and (8) by algebraic manipulations and by setting $x_A = x$ and $x_B = y$. From the proof of Proposition 3, there is a positive integer $d^* = d^*(c)$ such that, for all $d \geq d^*$, $k \geq 0$ and $y \in \mathbb{R}_+$,

$$\frac{y^{k+d}}{e^y (k+d)!} + \frac{y^{k+d+1}}{e^y (k+d+1)!} < 2c. \tag{30}$$

Moreover, it is known from the properties of the Poisson probability distribution that

$$\sum_{k=0}^{\infty} \frac{x^k}{e^x k!} = 1. \tag{31}$$

Accordingly,

$$\sum_{k=0}^{\infty} \frac{x^k}{e^x k!} \left(\frac{y^{k+d}}{e^y (k+d)!} + \frac{y^{k+d+1}}{e^y (k+d+1)!} \right) < \sum_{k=0}^{\infty} \frac{x^k}{e^x k!} 2c = 2c,$$

where first inequality is due to (30) and the equality is due to (31). This completes the proof of the proposition, since (28) cannot be satisfied for any (x, y) with $x, y \in \mathbb{R}_+$. $\square$

Proof of Proposition 4: Throughout the proof, we have $d \geq 1$ fixed. First, we show that an equilibrium $(0, x_B)$ of $\mathcal{G}^2(d)$ exists if and only if Equation (12) has a solution. It suffices to prove sufficiency, i.e., if Equation (12) holds for a given $(0, x_B)$, this must be an equilibrium of $\mathcal{G}^2(d)$. Take the smallest positive root of Equation (12), which we denote by x_B^*. Additionally, consider

$$h_{d+1}(y) = 2ce^y - \frac{y^{d+1}}{(d+1)!} - \frac{y^d}{d!} \quad \text{and} \quad h_d(y) = 2ce^y - \frac{y^d}{d!} - \frac{y^{d-1}}{(d-1)!}.$$

That is, x_B^* is the smallest positive solution y of the equation $h_d(y) = 0$. In particular, it must be the case that

$$h_d(x_B^*) = 0$$

and, by continuity of h_d and the fact that $h_d(0) = 2c > 0$,

$$h_d(y) \geq 0 \text{ for all } y \leq x_B^*. \tag{32}$$

Next, note that from Equation (21) in Proposition 3, it follows that

$$\frac{\partial h_{d+1}(y)}{\partial y} = \frac{\partial}{\partial y} \left(f_{d+1}(y) + f_d(y) \right) = f_d(y) + f_{d-1}(y) = h_d(y), \tag{33}$$

where f_{d-1}, f_d, f_{d+1} were defined in (19). Hence, Equations (32) and (33) imply that

$$\frac{\partial h_{d+1}(y)}{\partial y} \geq 0 \text{ if } y \leq x_B^*. \tag{34}$$

Then, (34) implies that

$$2c - \frac{(x_B^*)^d}{e^{x_B^*} d!} - \frac{(x_B^*)^{d+1}}{e^{x_B^*} (d+1)!} = \frac{h_{d+1}(x_B^*)}{e^{x_B^*}} \geq \frac{h_{d+1}(0)}{e^{x_B^*}} = 2c > 0.$$

As a consequence, Inequality (11) is (strictly) satisfied for x_B^*, and hence $(0, x_B^*)$ is an equilibrium of $\mathcal{G}^2(d)$.

Second, we show that there is $c^*(d) > 0$ such that an equilibrium of $\mathcal{G}^2(d)$ of the type $(0, x_B)$ exists for all $c \leq c^*(d)$. By the first part of the proof, it is sufficient to prove that such $c^*(d)$ exists guaranteeing that there is x_B such that $h_d(x_B) = 0$, provided that $c \leq c^*(d)$. Indeed, let $c^* := c^*(d)$ be defined as follows:

$$2c^* = \frac{d^{d-1}}{e^d(d-1)!} + \frac{d^d}{e^d d!}.$$

Then, for $0 < c \leq c^*$,

$$h_d(0) = 2c > 0$$

and

$$h_d(d) = 2ce^d - \frac{d^d}{d!} - \frac{d^{d-1}}{(d-1)!} \leq 2c^* e^d - \frac{d^{d-1}}{(d-1)!} - \frac{d^d}{d!} = 0.$$

Hence, due to continuity of h_d, the equation $h_d(x_B) = 0$ must have a solution. This proves the result of the proposition.

We conclude the proof with a remark. If we apply Stirling's formula to $c^*(d) = \frac{d^d}{e^d d!}$, we obtain

$$c^*(d) = O\left(\frac{1}{\sqrt{d}}\right). \tag{35}$$

In combination with (27), Condition (35) implies that $d \geq d^*(c)$, with $d^*(c) = \Omega\left(\frac{1}{c^2}\right)$, is not only sufficient for the result in Proposition 3 to hold, it is in fact *also* necessary. In other words, the difference in the vote count obtained after the first voting round *must* reach threshold $d^*(c)$ in order for the no-show equilibrium to be the only equilibrium of $\mathcal{G}^2(d)$, where $\mathcal{G}^2(d)$ is the game in the second round. More specifically, for any given $c \in (0, 1/2)$, there exist constants K_1 and K_2, with $K_2 < K_1$, such that the following two statements hold: First, if $d > \frac{K_1}{c^2}$, the no-show equilibrium is the only equilibrium of $\mathcal{G}^2(d)$. Second, if $d < \frac{K_2}{c^2}$, then $\mathcal{G}^2(d)$ has equilibria that are different from the no-show equilibrium. The existence of K_1 and K_2 follows from Conditions (27) and (35). $\square$

Proof of Proposition 5: To establish the proposition, it will be sufficient to show that the system of equations composed of (28) and (29)—see the proof of Proposition 2—has a solution $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$. For that purpose, we henceforth let an integer $d \in \mathbb{N}$ with $d \geq 1$ be given, and then have $t_d(x, y)$ denote the right-hand side of Equation (28), i.e.,

$$t_d(x,y) = \sum_{k=0}^{\infty} \frac{x^k}{e^x k!} \left(\frac{y^{k+d}}{e^y (k+d')!} + \frac{y^{k+d+1}}{e^y(k+d+1)!} \right) = \sum_{k=0}^{\infty} \frac{x^k}{e^x k!} \left(2c - g_{k+d}(y) - g_{k+d+1}(y)\right), \tag{36}$$

where the second equality follows from the definition of $g_{d'}(y)$, with $d' \in \mathbb{N}$, in (22)—see the proof of Proposition 3. It is then straightforward to verify that finding a solution (x, y) to the above system of equations is equivalent to finding a solution (x, y) to

$$2c = t_d(x, y), \tag{37}$$

$$2c = t_{d-1}(x, y). \tag{38}$$

For a moment, let k be some given natural number. Using the expression for the derivative of $g_{d'}(y)$ in (25), also from the proof of Proposition 3, yields

$$\frac{\partial g_{k+d}(y)}{\partial y} = -\frac{(k+d)y^{k+d-1} - y^{k+d}}{e^y(k+d)!} = -\frac{y^{k+d-1}}{e^y(k+d-1)!} + \frac{y^{k+d}}{e^y(k+d)!} = g_{k+d-1}(y) - g_{k+d}(y). \tag{39}$$

We now take the derivative of $t_d(x, y)$, for which we rely on expressions (36) and (39). For this purpose, we further observe that the sums of $t_d(x, y)$ and the sums of its derivative uniformly converge in both variables, and thus

$$
\begin{aligned}
\frac{\partial t_d(x,y)}{\partial y} &= -\sum_{k=0}^{\infty} \frac{x^k}{e^x k!}\left(\frac{\partial g_{k+d}(y)}{\partial y} + \frac{\partial g_{k+d+1}(y)}{\partial y}\right) \\
&= -\sum_{k=0}^{\infty} \frac{x^k}{e^x k!}(g_{k+d-1}(y) - g_{k+d}(y) + g_{k+d}(y) - g_{k+d+1}(y)) \\
&= -\sum_{k=0}^{\infty} \frac{x^k}{e^x k!}(g_{k+d-1}(y) + g_{k+d}(y)) + \sum_{k=0}^{\infty} \frac{x^k}{e^x k!}(g_{k+d}(y) + g_{k+d+1}(y)) \\
&= \sum_{k=0}^{\infty} \frac{x^k}{e^x k!}(2c - g_{k+d-1}(y) - g_{k+d}(y)) - \sum_{k=0}^{\infty} \frac{x^k}{e^x k!}(2c - g_{k+d}(y) - g_{k+d+1}(y)) \\
&= t_{d-1}(x,y) - t_d(x,y). \qquad (40)
\end{aligned}
$$

Next, we claim that for any real number $\varepsilon > 0$, there exists another real number $z^*(\varepsilon)$, such that

$$\frac{z^k}{e^z k!} < \varepsilon, \text{ for all } z > z^*(\varepsilon) \text{ and all } k \in \mathbb{N}. \qquad (41)$$

The proof of this claim follows immediately from two observations. First, given ε, we know from Proposition 3 that there is $k^*(\varepsilon)$ such that

$$\frac{z^k}{e^z k!} < \varepsilon, \text{ for all } z \in \mathbb{R}_+ \text{ and any integer } k \text{ with } k \geq k^*(\varepsilon). \qquad (42)$$

Second, consider some $k \in \{0, \ldots, k^*(\varepsilon)\}$. Considering k a constant, $\frac{z^k}{k!}$ is a polynomial function of z, while e^z is an exponential function. Therefore, there is $z(\varepsilon, k)$ such that

$$\frac{z^k}{e^z k!} < \varepsilon, \text{ for all } z \geq z(\varepsilon, k). \qquad (43)$$

Then, the claim in (41) follows from (42) and (43) by taking

$$z^*(\varepsilon) = \max_{k \in \{0,\ldots,k^*(\varepsilon)\}} z(\varepsilon, k).$$

Now, note that for any fixed x, we have

$$t_d(x, 0) = 0 \qquad (44)$$

and

$$\lim_{y\to\infty} t_d(x,y) = 0. \qquad (45)$$

The first equality is straightforward since $d \geq 1$, the latter holds since (41) implies that

$$
\begin{aligned}
\lim_{y\to\infty} t_d(x,y) &= \lim_{\varepsilon\to 0} \lim_{\substack{y\to\infty \\ y \geq z^*(\varepsilon)}} \sum_{k=0}^{\infty} \frac{x^k}{e^x k!}\left(\frac{y^{k+d}}{e^y (k+d')!} + \frac{y^{k+d+1}}{e^y (k+d+1)!}\right) \\
&< \lim_{\varepsilon\to 0} \lim_{\substack{y\to\infty \\ y \geq z^*(\varepsilon)}} 2\varepsilon \cdot \sum_{k=0}^{\infty} \frac{x^k}{e^x k!} = \lim_{\varepsilon\to 0} 2\varepsilon = 0,
\end{aligned}
$$

where the penultimate equality follows from the fact that $\sum_{k=0}^{\infty} \frac{x^k}{e^x k!} = 1$. As a consequence, Equations (44) and (45), together with the fact $t_d(x,y)$ is a positive function, imply that there is $y^*(x)$ such that $t_d(x,y)$ is maximum at $(x,y^*(x))$. Since $t_d(x,y)$ is differentiable (and, hence, bounded) in both variables, the following first-order condition must hold at the maximum:

$$\frac{\partial t_d(x,y^*(x))}{\partial y}=0. \tag{46}$$

By (40), it then follows that

$$t_d(x,y^*(x))=t_{d-1}(x,y^*(x)). \tag{47}$$

Thanks to the equality in (47), we are now in a position to solve the system of equations made up of (37) and (38). To do this, let us define the following real-valued function:

$$s(x):=t_d(x,y^*(x))=t_{d-1}(x,y^*(x))=\max_{y\in\mathbb{R}_+}t_d(x,y). \tag{48}$$

We now study the properties of $s(x)$. First, from the definitions of $t_d(x,y)$ and $s(x)$, it is straightforward that

$$s(0)\geq t_d(0,d)=\frac{d^d}{e^d d!}+\frac{d^{d+1}}{e^d(d+1)!}.$$

Second,

$$\lim_{x\to\infty}s(x)=0.$$

The above limit can be proved in the same way as the limit in (45), if we use (41) and the fact that for all $y\in\mathbb{R}_+$,

$$\sum_{k=0}^{\infty}\left(\frac{y^{k+d}}{e^y(k+d)!}+\frac{y^{k+d-1}}{e^y(k+d-1)!}\right)\leq 2.$$

Third, $t_d(x,y)$ has bounded partial derivatives with respect to both variables x and y, since the sums uniformly converge to finite values. This implies that the function $t_d(x,y)$ is uniformly continuous in both variables x and y. Indeed, analogously to (40), it can be shown that

$$\frac{\partial t_d(x,y)}{\partial x}=t_{d-1}(x,y)-t_d(x,y).$$

Then, one can verify that the partial derivatives of $t_d(x,y)$ are in $[-1,1]$. This follows from noting that both $t_d(x,y)$ and $t_{d-1}(x,y)$ belong to $[0,1]$ when $x,y\geq 0$.

Fourth, we show by contradiction that $s(x)$ is continuous. Suppose that $s(x)$ is not continuous. Then, there exists x^0 and a sequence $(x_n)_{n\geq 1}$ converging to x^0, such that $(s(x_n))_{n\geq 1}$ does not converge to $s(x^0)$. We distinguish two cases:

- **Case 1:** There exists $\varepsilon>0$ and a subsequence of $(x_n)_{n\geq 1}$ that converges to x^0, say $(x'_n)_{n\geq 1}$, such that for each $i\in\mathbb{N}$,

  $$s(x'_i)<s(x^0)-\varepsilon. \tag{49}$$

  By the definition of function $s(x)$, we must have $s(x_i)\geq t(x_i,y(x^0))$, which together with (49), implies that for all $i\in\mathbb{N}$,

  $$t(x_i,y(x^0))<s(x^0)-\varepsilon. \tag{50}$$

  However, by uniform continuity of $t_d(x,y)$ in x, it must also be the case that

  $$\lim_{i\to\infty}t_d(x_i,y(x^0))=s(x^0).$$

This is in contradiction to (50).

- **Case 2:** There exists $\varepsilon > 0$ and a subsequence of $(x_n)_{n\geq 1}$ that converges to x^0, say $(x'_n)_{n\geq 1}$, such that for each $i \in \mathbb{N}$,
$$s(x^0) + \varepsilon < s(x'_i). \tag{51}$$
By the definition of function $s(x)$, we must have $s(x^0) \geq t(x^0, y(x'_i))$, which together with (51), implies that for all $i \in \mathbb{N}$,
$$t(x^0, y(x'_i)) < s(x'_i). \tag{52}$$
However, since $t_d(x, y)$ is uniformly continuous in x,
$$\lim_{i\to\infty} t_d(x_i, y(x^0)) = s(x^0).$$
This is a contradiction with (52).

Finally, note that $s(0) \geq 2c$ is equivalent to
$$c \leq \frac{1}{2}\left(\frac{d^d}{e^d d!} + \frac{d^{d+1}}{e^d (d+1)!}\right) := c^{**}(d). \tag{53}$$

Assuming (53), it follows from the properties of $s(x)$ outlined above that there is $x^* \in \mathbb{R}_+$ such that $s(x^*) = 2c$. In turn, by the definition of $s(x)$, this implies that the pair $(x^*, y^*(x^*))$ satisfies
$$t_{d-1}(x^*, y^*(x^*)) = t_d(x^*, y^*(x^*)) = 2c.$$

That is, $(x^*, y^*(x^*))$ solves the system of equations made up of Equations (37) and (38). This completes the proof of the proposition. □

Proof of Proposition 1. Throughout the proof, we have $d \geq 1$ denote a given positive integer. In addition, for any integer $h \geq 1$, define
$$l_h(x, y) := \sum_{k=0}^{\infty} \frac{x^k}{k!}\left(\frac{y^{k+h-1}}{(k+h-1)!} - \frac{y^{k+h+1}}{(k+h+1)!}\right). \tag{54}$$

Given a solution $(x_A, x_B) \in \mathbb{R}_+ \times \mathbb{R}_+$ of the system of equations made up of (28) and (29), one can obtain after straightforward algebraic manipulations that
$$l_d(x_A, x_B) = 0. \tag{55}$$

Next, we discuss some properties of $l_h(x, y)$ for all integers $h \geq 1$. We start by noting that for all $y > 0$,
$$\begin{aligned} l_h(y, y) &= \sum_{k=0}^{\infty}\left(\frac{y^{2k+h-1}}{k!(k+h-1)!} - \frac{y^{2k+h+1}}{k!(k+h+1)!}\right) \\ &= \frac{y^{h-1}}{(h-1)!} + \sum_{k=0}^{\infty} y^{2k+h+1}\left(\frac{1}{(k+1)!(k+h)!} - \frac{1}{k!(k+h+1)!}\right) > 0, \end{aligned} \tag{56}$$

where the inequality holds since for any integer $k \geq 0$,
$$\frac{k!(h+k+1)!}{(k+1)!(h+k)!} = \frac{k+1+h}{k+1} > 1.$$

Next, we show that function $l_h(x, y)$ is increasing in x. For this purpose, we first note that

$$\frac{\partial l_h(x, y)}{\partial x} = l_{h+1}(x, y). \tag{57}$$

Second, for all integers $k \geq 0$ and all $y > 0$,

$$\frac{y^{k+h-1}}{(k+h-1)!} - \frac{y^{k+h+1}}{(k+h+1)!} > 0 \Longleftrightarrow (k+h+1)(k+h) - y^2 > 0. \tag{58}$$

Third, take the smallest integer $h^* \geq 1$ such that

$$x_B^2 < (h^* + 1)h^*.$$

Trivially, for any integer $k \geq 0$,

$$(k + h^* + 1)(k + h^*) > (h^* + 1)h^*,$$

which, together with (58), implies that

$$\frac{{x_B}^{k+h^*-1}}{(k+h^*-1)!} - \frac{{x_B}^{k+h^*+1}}{(k+h^*+1)!} > 0. \tag{59}$$

It then follows from (57) and (59) that for all $x > 0$,

$$\frac{\partial l_{h^*}(x, x_B)}{\partial x} > 0. \tag{60}$$

From (56) and (60), we then see that for all $x \geq x_B$,

$$l_{h^*}(x, x_B) > 0. \tag{61}$$

Finally, by induction on $h = h^*, \ldots, 1$, if we use (56) and (57) repeatedly, we can similarly show—departing from (60) and (61) as the base case—that $l_h(x, x_B)$ is positive and increasing in x, given that $x \geq x_B$. In particular, if $x_A \geq x_B$, it must be the case that

$$l_d(x_A, x_B) > 0, \tag{62}$$

which is in contradiction to (55). This completes the proof of the proposition. □

Proof of Theorem 2: As already mentioned in the main body of the paper, we assume that all citizens in AG vote sincerely, i.e., that they vote for their preferred alternative. Below, we show that this assumption is also consistent with equilibrium behavior. Accordingly, the behavior of any such citizen i is described by the random variable $\mathcal{X}_i$—see (2)—, while the difference in vote count for alternative A with respect to alternative B obtained in the first voting round is described by the random variable

$$D = \sum_{i \in \Omega_1} \mathcal{X}_i,$$

which has been defined in (3), with Ω_1 denoting the set of citizens who belong to AG. Because $\mathbb{E}[\mathcal{X}_i] =$

$p_A - p_B$ and X_i are i.i.d., it follows that

$$E[D] = N_1 \cdot \mathbb{E}[\mathcal{X}_i] = N_1 \cdot (p_A - p_B).$$

Recall that $d^* = d^*(c)$ has been defined in Proposition 2. This integer guarantees that if d, the outcome associated with the random variable D, is at least d^*, the only equilibrium of game $\mathcal{G}_2^*(d)$ is the no-show equilibrium. In that case, the only votes are cast in the first round, and because $d > 0$, alternative A will be chosen. Now, let

$$N_1^* = N_1^*(c, \varepsilon, p_A - p_B) := \left\lceil \frac{d^*}{p_A - p_B} + \frac{\ln \frac{2}{\varepsilon}}{(p_A - p_B)^2} + \frac{\sqrt{2d^*(p_A - p_B) \ln \frac{2}{\varepsilon} + (\ln \frac{2}{\varepsilon})^2}}{(p_A - p_B)^2} \right\rceil. \tag{63}$$

Henceforth, we assume that

$$N_1 \geq N_1^*(c, \varepsilon, p_A - p_B). \tag{64}$$

Then, we obtain the following:

$$d^* - E[D] = d^* - N_1 \cdot (p_A - p_B) \leq d^* - N_1^* \cdot (p_A - p_B) < 0, \tag{65}$$

where the first inequality follows from Inequality (64) and the second inequality follows from the fact that $N_1^* \geq \frac{d^*}{p_A - p_B}$, as implied by the definition of N_1^* in (63). Then, the following chain of inequalities also holds:

$$P\left[D \leq d^*\right] = P[D - E[D] \leq d^* - E[D]] \leq P\left[|D - E[D]| \geq E[D] - d^*\right],$$

where the last inequality holds due to (65). Moreover, by Hoeffding's inequality (Hoeffding, 1963),

$$\begin{aligned} P\left[|D - E[D]| \geq E[D] - d^*\right] &\leq 2exp\left(- \frac{(E[D] - d^*)^2}{2N_1} \right) \\ &= 2exp\left(- \frac{(N_1(p_A - p_B) - d^*)^2}{2N_1} \right) \leq \varepsilon, \end{aligned}$$

where the last inequality holds by (63) and (64). Combining the last two chains of inequalities yields

$$P\left[D \geq d^*\right] \geq P\left[D > d^*\right] \geq 1 - \varepsilon.$$

Accordingly, with probability $1-\varepsilon$, no citizen will vote in the second round. Given this outcome, citizens in the first round will not want to change their sincere voting decision. On the one hand, all first-round citizens whose preferred alternative is A are content with their decision as their preferred outcome will be implemented. On the other, all first-round citizens whose preferred alternative is B would not obtain a better outcome by switching their vote to A in the first round, for this would only increase d. This completes the proof. □

Proof of Corollary 1: From the proof of Theorem 2—see (63)—, it follows that N_1^* will increase if either ε or $p_A - p_B$ decreases. We now focus on changes in c. From the proof of Proposition 3—see Equation (27)—, we know that $d^*(c)$ increases as c decreases. Since N_1^* decreases when d^* increases, the claim holds. □

Proof of Theorem 3: Under AV, the average per-capita social cost of subsidizing is $f \cdot c$, where f is the expected ratio of the AG size to the total number of voters

$$f = \mathbb{E}\left[\frac{N_1}{N_1 + N_2}\right]. \tag{66}$$

Since N_2 is a Poisson random variable with parameter n_2, we can easily obtain the following upper bound for f:

$$f = N_1 \cdot \sum_{k=0}^{\infty} \frac{1}{N_1 + k} \frac{n_2^k}{k! e^{n_2}} \leq N_1 \cdot \sum_{k=0}^{\infty} \frac{n_2^k}{(k+1)! e^{n_2}} = \frac{N_1}{n_2} \cdot \left(1 - \frac{1}{e^{n_2}}\right). \tag{67}$$

In particular, for a fixed N_1, we have $\lim_{n_2 \to \infty} f = 0$. Next, according to Theorem 2, if $N_1 \geq N_1^*(\varepsilon, c, p_A - p_B)$, the outcome will be fully determined by AG with probability $1 - \varepsilon$. Therefore,

$$W^{AV} \geq (1 - \varepsilon) \cdot (w_d(N_1, n_2) - cf) + \varepsilon \cdot (p_B - c), \tag{68}$$

where $\varepsilon > 0$ and $w_d(N_1, n_2)$ is the expected average welfare (in the entire population) obtained from the alternative implemented when members of AG, a group of size N_1, vote sincerely. One can see that

$$w_d(N_1, n_2) = p_A \cdot (1 - z_d(N_1, n_2)), \tag{69}$$

where $z_d(N_1, n_2)$ is some function that satisfies

$$\lim_{N_1 \to \infty} z_d(N_1, n_2) = 0. \tag{70}$$

Hence, there is $\varepsilon^* > 0$ such that for all $N_1 \geq N_1^*(\varepsilon^*, c, p_A - p_B)$ we derive from Inequality (68) that

$$W^{AV} > p_A - cf + \delta(N_1, n_2), \tag{71}$$

where

$$\lim_{N_1 \to \infty} \delta(N_1, n_2) = 0. \tag{72}$$

Finally, because $p_A - p_B > 0$ and due to (67) and (72), there must be $N_1^{**}(\varepsilon, c, p_A - p_B)$, with $N_1^{**}(\varepsilon, c, p_A - p_B) \geq N_1^*(\varepsilon^*, c, p_A - p_B)$ and $n_2^*(c)$ such that if $N_1 \geq N_1^{**}(\varepsilon^*, c, p_A - p_B)$ and $n_2 \geq n_2^*(c)$,

$$p_A - cf + \delta(N_1, n_2) > p_A - c = W^{com}$$

and

$$p_A - cf + \delta(N_1, n_2) > \frac{1}{2} - \frac{2x}{N_1 + n_2} \cdot c = W^{vol},$$

where x is the solution to Equation (6). In combination with (68) and (71), the latter two inequalities prove that $W^{AV} > \max\{W^{vol}, W^{com}\}$. □

Appendix B

In this appendix we extend our analysis in three directions. First, we analyze the robustness of Theorem 1 when citizens with the same preferences use different strategies; second, we analyze AV under the assumption that voters are of two types: non-partisan (with participation cost $c > 0$) and partisan (with zero participation costs); third, we investigate the performance of AV when there are more than two alternatives.

Multiple citizen types

In the main body of the paper, we have assumed that all agents who preferred alternative A to B used the same strategy. More particularly, in our analysis of game $\mathcal{G}^2(d)$, we assumed that all citizens played one of two strategies: α_A for citizens whose preferred alternative is A and α_B for citizens whose preferred alternative is B. In this section, we assume that citizens of type A and B may be of different (sub)types and that these are given exogenously.

More specifically, for a given integer $T \geq 1$, let $\mathbb{S}^T = \{(\rho_k)_{k=1}^T | \rho_1, \ldots, \rho_T \geq 0, \sum_{k=1}^T \rho_k = 1\}$ denote the T-simplex. Then, we assume that there exist $\rho_A = (\rho_A^k)_{k=1}^{T^A} \in \mathbb{S}^{T^A}$ and $\rho_B = (\rho_B^k)_{k=1}^{T^B} \in \mathbb{S}^{T^B}$, with $T^A, T^B \geq 1$, such that any citizen i's probability of being of (sub)type t_A^k is equal to $p_A \cdot \rho_A^k$. We assume that citizens of different (sub)types may use different strategies, i.e, they may randomize between voting or not, using different probabilities. Accordingly, we use $\alpha_{A,k}$, with $\alpha_{A,k} \in [0, 1]$, to denote the probability according to which citizens of type t_A^k will turn out (and then vote for alternative A).[66] In turn, $\alpha_{B,k}$ can be analogously defined for B-supporters. By the properties of the Poisson probability distribution, the number of citizens of each (sub)type t_A^k in the second round of AV is a Poisson random variable with parameter $n_2 \cdot p_A \cdot \rho_A^k \cdot \alpha_{A,k}$, which we denote by $x_{A,k}$. Similarly, the number of citizens of each (sub)type t_B^k is a Poisson random variable with average $n_2 \cdot p_B \cdot \rho_B^k \cdot \alpha_{B,k}$, which we denote by $x_{B,k}$. We recall that $d^*(c)$ has been defined as the (minimum) threshold guaranteeing that, if $d \geq d^*(c)$, no citizen will turn out in the second round of AV. We can prove the following result, which generalizes Theorem 1 to a setting with multiple citizen types.

Proposition 6. *Assume that there are T^A (sub)types of A-supporters and T^B (sub)types of B-supporters. For any cost c, with $0 < c < 1/2$, if $d \geq d^*(c) \geq 2$, the only equilibrium is the no-show equilibrium.*

Proof. Let $\mathbb{N}$ denote the set of non-negative integer numbers. The fact that the no-show strategy profile is an equilibrium is trivial, provided that $d^*(c) \geq 2$. To show that this is the unique equilibrium, we distinguish two cases.

Case I: $T^A \geq 1$ and $T^B = 1$

For all voters of type A, regardless of their subtype, the indifference condition between turning out and abstaining is the following:

$$2c = \sum_{(k_1,\ldots,k_{T^A}) \in \mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{{x_{A,r}}^{k_r}}{e^{x_{A,r}} k_r!} \cdot \left(\frac{{x_B}^{\sum_{s=1}^{T^A} k_s + d}}{e^{x_B} (\sum_{s=1}^{T^A} k_s + d)!} + \frac{{x_B}^{\sum_{s=1}^{T^A} k_s + d + 1}}{e^{x_B} (\sum_{s=1}^{T^A} k_s + d + 1)!} \right). \tag{73}$$

[66] To avoid cumbersome notation, we have dropped the dependence of strategies on handicap d.

Nevertheless, by Inequality (30)—see the proof of Proposition 2—, we obtain for all $d \geq d^*(c)$ and all $x_B \in \mathbb{R}_+$

$$
\begin{aligned}
&\sum_{(k_1,\dots,k_{T^A})\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{{x_{A,r}}^{k_r}}{e^{x_{A,r}}k_r!} \cdot \left(\frac{x_B^{\sum_{s=1}^{T^A} k_s + d}}{e^{x_B}(\sum_{s=1}^{T^A} k_s + d)!} + \frac{x_B^{\sum_{s=1}^{T^A} k_s + d+1}}{e^{x_B}(\sum_{s=1}^{T^A} k_s + d+1)!}\right) \\
< &\sum_{(k_1,\dots,k_{T^A})\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{{x_{A,r}}^{k_r}}{e^{x_{A,r}}k_r!} \cdot 2c = 2c, \qquad (74)
\end{aligned}
$$

where the second inequality holds from the following identity that

$$
\sum_{(k_1,\dots,k_{T^A})\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{x_{A,r}^{k_r}}{e^{x_{A,r}}k_r!} = 1. \qquad (75)
$$

Assuming Equation (96), it must be the case that Equation (73) does not have a solution, and hence there cannot be an equilibrium of game $\mathcal{G}^2(d)$ in which A-supporters are split into T^A (sub)types and each (sub)type t_A^r of citizen plays according to a totally mixed strategy $x_{A,r}$. Finally, it only remains to prove Equation (96). We prove the claim by induction on T^A. The case $T^A = 1$ holds directly from the properties of the Poisson probability distribution. Hence, assume that Equation (96) holds for some $T^A \geq 1$. Then,

$$
\begin{aligned}
\sum_{(k_1,\dots,k_{T^A+1})\in\mathbb{N}^{T^A+1}} \prod_{r=1}^{T^A+1} \frac{x_{A,r}^{k_r}}{e^{x_{A,r}}k_r!} &= \sum_{k=0}^{\infty}\left[\sum_{\substack{(k_1,\dots,k_{T^A+1})\in\mathbb{N}^{T^A+1},\\ k_{T^A+1}=k}} \left(\prod_{r=1}^{T^A} \frac{x_{A,r}^{k_r}}{e^{x_{A,r}}k_r!} \cdot \frac{x_{A,T^A+1}^{k}}{e^{x_{A,T^A+1}}k!}\right)\right] \\
&= \sum_{k=0}^{\infty}\left[\frac{x_{A,T^A+1}^{k}}{e^{x_{A,T^A+1}}k!} \sum_{(k_1,\dots,k_{T^A})\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{x_{A,r}^{k_r}}{e^{x_{A,r}}k_r!}\right] = \sum_{k=0}^{\infty} \frac{x_{A,T^A+1}^{k}}{e^{x_{A,T^A+1}}k!} = 1,
\end{aligned}
$$

where the penultimate equality holds by induction and the last equality holds due to the properties of the Poisson probability distribution.

Case II: $T^A \geq 1$ and $T^B > 1$

Let us assume T^A is given. We introduce further notation. Given $x_B = (x_{B,1}, ..., x_{B,T^B})$ and $k^B = (k_1^B, ..., k_{T^B}^B)$, we use $P(x_B, k^B)$ to denote the probability that, for each (sub)type t_s^B $(s = 1, \dots, T^B)$, there are exactly k_s^B citizens of this (sub)type that vote, provided that citizens of type t_B^s use strategy $\alpha_{B,s}$ (which leads to $x_{B,s}$). Because (sub)types are drawn independently, we obtain

$$
P(x_B, k^B) = \prod_{s=1}^{T^B} \frac{x_{B,s}^{k_s^B}}{e^{x_{B,s}}k_s^B!}.
$$

Moreover, because of the multinomial theorem we obtain that for all $m \geq 0$

$$
\sum_{\substack{(k_1^B,\dots,k_{T^B}^B)\in\mathbb{N}^{T^B},\\ \sum_{s=1}^{T^B} k_s^B = m}} P(x_B, k^B) = \frac{\left(\sum_{s=1}^{T^B} x_{B,s}\right)^m}{e^{\sum_{s=1}^{T^B} x_{B,s}}m!}. \qquad (76)
$$

For *all* voters of type A, the indifference condition between turning out and abstaining is

$$
\begin{aligned}
2c &= \sum_{\left(k_1^A,\ldots,k_{T^A}^A\right)\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{{x_{A,r}}^{k_r^A}}{e^{x_{A,r}} k_r^A!} \cdot \left(\sum_{\substack{k^B=\left(k_1^B,\ldots,k_{T^B}^B\right)\in\mathbb{N}^{T^B},\\ \sum_{s=1}^{T^B} k_s^B=\sum_{s=1}^{T^A} k_s^A+d}} P(x_B,k^B) + \sum_{\substack{k^B=\left(k_1^B,\ldots,k_{T^B}^B\right)\in\mathbb{N}^{T^B},\\ \sum_{s=1}^{T^B} k_s^B=\sum_{s=1}^{T^A} k_s^A+d+1}} P(x_B,k^B) \right) \\
&= \sum_{\left(k_1^A,\ldots,k_{T^A}^A\right)\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{{x_{A,r}}^{k_r^A}}{e^{x_{A,r}} k_r^A!} \cdot \left(\frac{\left(\sum_{s=1}^{T^B} x_{B,s}\right)^{\sum_{s=1}^{T^A} k_s^A+d}}{e^{\sum_{s=1}^{T^B} x_{B,s}} \left(\sum_{s=1}^{T^A} k_s^A+d\right)!} + \frac{\left(\sum_{s=1}^{T^B} x_{B,s}\right)^{\sum_{s=1}^{T^A} k_s^A+d+1}}{e^{\sum_{s=1}^{T^B} x_{B,s}} \left(\sum_{s=1}^{T^A} k_s^A+d+1\right)!} \right) \\
&= \sum_{\left(k_1^A,\ldots,k_{T^A}^A\right)\in\mathbb{N}^{T^A}} \prod_{r=1}^{T^A} \frac{{x_{A,r}}^{k_r^A}}{e^{x_{A,r}} k_r^A!} \cdot \left(\frac{\sigma_B^{\sum_{s=1}^{T^A} k_s^A+d}}{e^{\sigma_B} \left(\sum_{s=1}^{T^A} k_s^A+d\right)!} + \frac{\sigma_B^{\sum_{s=1}^{T^A} k_s^A+d+1}}{e^{\sigma_B} \left(\sum_{s=1}^{T^A} k_s^A+d+1\right)!} \right) < 2c,
\end{aligned}
$$

where $\sigma_B := \sum_{s=1}^{T^B} x_{B,s}$, the second equality follows from Equation (76), and the inequality follows from Inequality (74) if $d \geq d^*(c)$. Because this involves a contradiction, it must be the case that if $d \geq d^*(c)$, there cannot exist an equilibrium of game $\mathcal{G}^2(d)$ in which A-supporters are split into T^A (sub)types and each (sub)type t_A^r of citizen plays according to a totally mixed strategy $x_{A,r}$, and in which B-supporters are split into T^B (sub)types and each (sub)type t_B^r of citizen plays according to a totally mixed strategy $x_{B,r}$. This completes the proof.[67] □

Partisan voters

In this section, we assume that some voters are partisan, i.e., that they vote in favor of their preferred alternative regardless of any other considerations. One possibility is that they experience no voting costs. This is the approach we take in what follows. We consider two cases. First, we assume that partisanship is independent of preferences; second, we assume that the expected number of partisan voters is the same for both alternatives, so that partisanship is no longer independent of preferences.

No correlation between partisanship and preferences

We start by recalling that N_1 citizens vote in the first round of AV and that N_2 citizens vote in the second round, where N_1 is a given number (which will remain fixed throughout) and N_2 is drawn from a Poisson random variable with parameter n_2. By the properties of Poisson games, the number of partisan voters (non-partisan voters) in the second round is a Poisson random variable with parameter $\mu \cdot n_2$ ($(1-\mu)\cdot n_2$), where μ is the probability that a citizen is a partisan voter.

Our first result shows that if the (expected) number of partisan voters who have a right to vote in the second voting round is large enough (because the share of partisan voters and the size of the second voting group are large), such voters, together with AG members, will decide the outcome of AV alone. The reason is that non-partisan voters will have no incentives to vote in the second round of AV. We recall that the threshold $d^*(c)$ has been defined in Proposition 2.

[67] The case in which some (sub)types play according to pure strategies can be proved analogously to the case considered above. We also note that, although we have focused on the case where T_A and T_B are finite numbers, the claim in Proposition 6 can be extended to the case where T_A or T_B are infinite.

Proposition 7. *For each $\delta > 0$, there exists $N^* = N^*(c, p_A - p_B, \delta, N_1) \in \mathbb{R}_+$ such that if $n_2 \cdot \mu \geq N^*$ the vote-count difference (i.e., the handicap) yielded by AG members plus the partisan voters who have a right to vote in the second round of AV is larger than $d^*(c)$ with probability at least $1 - \delta$.*

Proof. First, let $\varepsilon > 0$, the precise value of which will be determined later. Then, given $N_1^* = N_1^*(\varepsilon, c, p_A - p_B)$ as introduced in Theorem 2, define

$$n^* := \max\{0, N_1^*(\varepsilon, c, p_A - p_B) - N_1\}.$$

Second, we show that the probability of a Poisson random variable X with parameter λ being larger than any given natural number converges to one exponentially as λ tends to infinity. This follows from noting that the tail bound of such a random variable for $x < \lambda$ (see Mitzenmacher and Upfal, 2005, p. 97) is

$$P(X \leq x) \leq \frac{e^{-\lambda}(e\lambda)^x}{x^x}. \tag{77}$$

Now set $\lambda = n_2 \cdot \mu$ and $x = \frac{n_2 \cdot \mu}{2}$. Then, by using (77), we obtain

$$P\left(\text{there are at most } \frac{n_2 \cdot \mu}{2} \text{ partisan voters in the second round of AV}\right) \leq \left(\frac{2}{e}\right)^{\frac{n_2 \cdot \mu}{2}}.$$

Therefore, the probability that the number of partisan voters in the second voting round is larger than or equal to $\frac{n_2 \cdot \mu}{2}$ is at least $1 - (\frac{2}{e})^{\frac{n_2 \cdot \mu}{2}}$. If we take n_2^* and μ^* such that

$$\left(\frac{2}{e}\right)^{\frac{n_2^* \cdot \mu^*}{2}} \leq \varepsilon$$

and

$$n_2^* \cdot \mu^* \geq n^*,$$

then the probability that the number of citizens from AG plus the number of partisan citizens participating in the second voting round of AV is larger than or equal to N_1^* is $1 - \varepsilon$. Moreover, because preferences are uncorrelated with partisanship and all members of AG vote (because they are incentivized to do so), as do all partisan voters of the second round (because they are partisan), the probability that the outcome of AV is described by Theorem 2 is $(1 - \varepsilon)^2$. Choosing ε as the smallest solution of

$$(1 - \varepsilon)^2 = (1 - \delta)$$

completes the proof. $\square$

We note that N^* is decreasing in N_1, and that by taking n_2^* to infinity, μ^* converges to zero. The above result thus warns us against the use of AV (with compulsory voting in the first round) when partisans are numerous enough to determine the outcome by themselves. Because this property also holds for one-round voluntary voting (it suffices to take the bound given by setting $N_1 = 0$), there is no underdog effect that AV can solve—as already discussed, one-round voting with the majority rule implements the utilitarian optimal solution in such circumstances.

Our second proposition complements Proposition 7 insofar as it proves that for fixed N_1 and n_2, if, by contrast to the latter result, there are not too many partisans, then the members of AG plus partisan voters with a right to vote in the second voting round will not yield a vote-count difference (i.e., a handicap) above the threshold above which only the no-show equilibrium exists with very high probability. Given the results in Propositions 5 and 4, this opens up the possibility that one-round voluntary voting decisions may often not reflect the will of the majority. As in the baseline model without partisans, AV is able to correct this drawback at low cost. For the sake of simplicity, we assume that $N_1 = 0$. The logic of the proof is the same for different values of N_1.

Proposition 8. *For each $\delta > 0$, there exists a threshold $\mu^{**} = \mu^{**}(c, p_A - p_B, \delta, n_2)$ such that for any $\mu < \mu^{**}$, the vote-count difference (i.e., the handicap) yielded by the partisan voters who have a right to vote in the second round of AV is smaller than $d^*(c)$ with probability at least $1 - \delta$.*

Proof. Again, we use a tail bound of a Poisson random variable with parameter λ. Specifically, this time we use the fact that the tail bound of such random variable for $x > \lambda$ (see Mitzenmacher and Upfal, 2005, p. 97) is

$$P(X \geq x) \leq \frac{e^{-\lambda}(e\lambda)^x}{x^x}. \tag{78}$$

Now set $\lambda = n_2 \cdot \mu$ and $x = d^*(c)$, where, as we recall, $d^*(c)$ has been defined in Proposition 2. Then, using (78), we obtain

$$P\left(\text{there are at least } d^*(c) \text{ partisan voters in the second round of AV}\right) \leq e^{d^*(c) - n_2 \cdot \mu} \cdot \left(\frac{n_2 \cdot \mu}{d^*(c)}\right)^{d^*(c)}.$$

The right-hand side of the above inequality is decreasing in μ and, moreover,

$$\lim_{\mu \to 0} \left[e^{d^*(c) - n_2 \cdot \mu} \cdot \left(\frac{n_2 \cdot \mu}{d^*(c)}\right)^{d^*(c)} \right] = 0.$$

Therefore, there exists $\mu'(c, p_A - p_B, \delta, n_2)$ such that for $\mu < \mu'(c, p_A - p_B, \delta, n_2)$,

$$e^{d^*(c) - n_2 \cdot \mu} \cdot \left(\frac{n_2 \cdot \mu}{d^*(c)}\right)^{d^*(c)} < \delta.$$

In turn, the probability that there are at most $d^*(c)$ partisan voters in the second round of AV is at least

$$1 - e^{d^*(c) - n_2 \cdot \mu} \cdot \left(\frac{n_2 \cdot \mu}{d^*(c)}\right)^{d^*(c)}.$$

Taking μ^{**} such that

$$\mu^{**} = \min\left\{ \mu'(c, p_A - p_B, \delta, n_2), \frac{d^*(c)}{n_2} \right\}$$

completes the proof. The reason is that even if all partisan voters with a right to vote in the second round vote alike, the vote-count difference (i.e., the handicap) they yield will not be above $d^*(c)$. $\square$

We point out that higher thresholds μ^{**} could have been attained had we considered more likely outcomes for the vote-count difference yielded by partisan voters, in which case bounds for the Skellam distribution would have to be used instead of bounds for the Poisson distribution. Qualitatively, the results would be the same, but the technicalities would be much more involved.

Correlation between partisanship and preferences

In the previous section, we assumed that partisanship and preferences were drawn independently for each citizen. One important consequence is that, as we increase n_2, we increase the (expected) difference between the number of partisan voters who are A-supporters and the number of partisan voters who are B-supporters. This follows because $p_A > p_B$, i.e., is more likely to be an A-supporter than a B-supporter. In the present section, we again focus on the second round of AV or, equal in essence, we assume $N_1 = 0$. That is, we consider one-round voluntary voting. Then, we take a reduced form modelling assumption on partisan voters and assume that the vote-count difference yielded by their votes (i.e., votes for A minus votes for B) follows a probability distribution that is symmetric around zero. We use $f(d)$, with $d \in \mathbb{Z}$, to denote this (discrete) density function. Note that we are not making any assumption about the expected total number of partisan voters, which can be arbitrarily chosen (in particular, it can be made arbitrarily high). One possibility is that both the number of A-supporters who are partisan and the number of B-supporters who are partisan each follow a Poisson distribution with parameter $(n_2 \cdot \mu)/2$, in which case $f(d)$ is a Skellam distribution. In the latter case, the number of non-partisan voters who are A-supporters (B-supporters) is in turn distributed according to a Poisson distribution of parameter $(1-\mu)p_A$ $((1-\mu)p_B)$. We proceed on this assumption regardless of whether $f(d)$ is a Skellam distribution or not. This can be done without loss of generality. As a consequence, A is the socially preferred alternative from an *ex-ante* perspective. For this setting, we obtain the following result:

Proposition 9. *In any totally mixed strategy equilibrium, the probability that either alternative will win is equal to* $\frac{1}{2}$*. Moreover, if*

$$c < \frac{1}{2} \cdot (f(0) + f(-1)) \tag{79}$$

there is always a totally mixed strategy equilibrium.

Proof. We start by showing that both alternatives are expected to be chosen with equal probability. The indifference condition for each non-partisan A-supporter is

$$c = \frac{1}{2} \sum_{d=-\infty}^{\infty} f(d) \sum_{a=0}^{\infty} \left[p_{x_A}(a)p_{x_B}(a+d) + p_{x_A}(a)p_{x_B}(a+d+1)\right], \tag{80}$$

while the indifference condition for each non-partisan B-supporter is

$$c = \frac{1}{2} \sum_{d=-\infty}^{\infty} f(d) \sum_{a=0}^{\infty} \left[p_{x_A}(a)p_{x_B}(a+d) + p_{x_A}(a)p_{x_B}(a+d-1)\right]. \tag{81}$$

In the above equations, x_A (x_B) is the expected number of A-supporters (B-supporters) voting in the second round, $x_A = \alpha_A p_A (1-\mu) n_2$ ($x_B = \alpha_B p_B (1-\mu) n_2$). This is analogous to our baseline set-up. In turn, $p_{x_A}(a)$ ($p_{x_B}(b)$) denotes the probability that a Poisson random variable with parameter x_A (x_B) is equal to a (b).

We claim—and will now prove—that $x_A = x_B$. By making the right-hand sides of (80) and (81) equal, we obtain after straightforward algebraic manipulations the following expression:

$$\sum_{d=-\infty}^{\infty} f(d) \sum_{a=0}^{\infty} p_{x_A}(a)p_{x_B}(a+d+1) = \sum_{d=-\infty}^{\infty} f(d) \sum_{a=0}^{\infty} p_{x_A}(a)p_{x_B}(a+d-1). \tag{82}$$

Let now some $d \geq 0$ be fixed momentarily. Then, consider the summand corresponding to d from the left-hand side (82) and the summand corresponding to $-d$ from the right-hand side. If we subtract both terms, we obtain

$$\sum_{a=0}^{\infty} p_{x_A}(a)p_{x_B}(a+d+1) - \sum_{a=0}^{\infty} p_{x_A}(a)p_{x_B}(a-d-1), \tag{83}$$

which can be rewritten as

$$\sum_{a=0}^{\infty} p_{x_A}(a)p_{x_B}(a+d+1) - \sum_{a=0}^{\infty} p_{x_A}(a+d+1)p_{x_B}(a) = \sum_{a=0}^{\infty} \frac{x_A^a x_B^a}{e^{x_A}e^{x_B}a!(a+d+1)!}\left(x_B^{d+1} - x_A^{d+1}\right).$$

On the one hand, the left-hand side of the above expression can be obtained from (83) by means of the change of variable $a := a+d+1$. Indeed, it suffices to note that $p_{x_B}(a-d-1) = 0$ whenever $a-d-1 < 0$. On the other hand, the right-hand side of the above expression can be obtained from the left-hand side if we note that $x_A^{d+1} - x_B^{d+1} = (x_A - x_B)\cdot g_d(x_A, x_B)$, where $g_d(x_A, x_B)$ is some polynomial in x_A and x_B with the property that all its coefficients are positive. In the case of $d < 0$, we can obtain the very same expressions. As a consequence, summing up such expressions for all $d \in \mathbb{Z}$ yields

$$(x_A - x_B)g(x_A, x_B) = 0,$$

where $g(x_A, x_B)$ is some strictly positive function. Therefore, $x_A = x_B$, as we claimed. Then, the probability that alternative A will win is equal to

$$P^A_{x_A,x_B} = \sum_{d=-\infty}^{\infty} f(d) \sum_{a=0}^{\infty} p_{x_A}(a) \sum_{b=0}^{a-d-1} \left[p_{x_B}(b) + \frac{1}{2}p_{x_B}(a-d)\right],$$

while the probability that alternative B will win is equal to

$$P^B_{x_A,x_B} = \sum_{d=-\infty}^{\infty} f(d) \sum_{b=0}^{\infty} p_{x_B}(b) \sum_{a=0}^{b+d-1} \left[p_{x_A}(a) + \frac{1}{2}p_{x_A}(b+d)\right].$$

It then remains to note that $P^A_{x_A,x_B} = P^B_{x_B,x_A}$, which together with $x_A = x_B$ implies that $P^A_{x_A,x_A} = \frac{1}{2}$, as we claimed.

Now that we have shown that the probability that either alternative will win is equal to $1/2$ in any totally mixed strategy equilibrium, we show the existence of such equilibria if (79) holds. Due to the symmetry of $f(d)$ and the fact that $x := x_A = x_B$, one can easily see that Equations (80) and (81) are the same and that they reduce to

$$c = \frac{1}{2} \sum_{d=-\infty}^{\infty} f(d) \sum_{a=0}^{\infty} \left[p_x(a)p_x(a+d) + p_x(a)p_x(a+d+1)\right]. \tag{84}$$

Let us focus for a moment on the right-hand side of Equation (84), which we denote as $G(x)$. On the one hand, one can easily see that if $y \neq 0$, then $p_0(y) = 0$. After some algebraic manipulations and simplifications, this property implies that

$$G(0) = \frac{1}{2}(f(0) + f(-1)). \tag{85}$$

On the other hand, along the lines of the proof of Proposition 5—see Equations (41), (42), and (43)—one can verify that

$$\lim_{x \to \infty} G(x) = 0. \tag{86}$$

This is very intuitive: if turnout yielded by partisan voters goes to infinity, the probability that their voting will make a difference goes to zero. Finally, given the continuity of $G(x)$, Equations (85) and (86) imply that there must be a positive real number x^* such that $c = G(x^*)$. This proves the existence of totally mixed strategy equilibria, thereby completing the proof of the proposition. $\square$

The above result shows that an equilibrium where non-partisan voters turn out with positive probability exists for one-round voluntary voting. It suffices for voting costs to be lower than half the (perceived) probability that the net vote effect of partisan voters' will yield either a tie or one vote fewer for their preferred alternative. This is just a sufficient condition, and weaker conditions will also guarantee the existence of such equilibria. Inquiring about such conditions is beyond the scope of our paper.

Different costs across types

In this section, we analyze the case where voting costs differ across citizen types. Specifically, we assume that voting costs are drawn from two degenerate distributions: all A-supporters have a voting cost equal to c_A, with $0 < c_A < 1/2$, while all B-supporters have a voting cost equal to c_B, with $0 < c_B < 1/2$. We consider all possible cases regarding the relation between c_A and c_B, and in particular it will be possible for the minority to enjoy a cost advantage or for the majority to do so. Recall that for a voter of type $t \in \{A, B\}$, total expected turnout in an equilibrium $(\alpha_A(d), \alpha_B(d))$ of game $\mathcal{G}^2(d)$ is given by $x_t = n_2 \cdot p_t \cdot \alpha_t$, and that it is equivalent to describe an equilibrium by (α_A, α_B) or by (x_A, x_B).

As a benchmark case, we focus on one-round voluntary voting, and thus we assume $d = 0$. The equilibrium conditions for a totally mixed equilibrium in this case are

$$c_A = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^k}{e^{x_B} k!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^{k+1}}{e^{x_B}(k+1)!}, \tag{87}$$

$$c_B = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^k}{e^{x_A} k!} \frac{x_B^k}{e^{x_B} k!} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{x_A^{k+1}}{e^{x_A}(k+1)!} \frac{x_B^k}{e^{x_B} k!}. \tag{88}$$

By subtracting Equations (87) and (88), we obtain

$$c_A - c_B = \frac{1}{2}(x_B - x_A)\sum_{k=0}^{\infty} \frac{x_A^k x_B^k}{e^{x_A + x_B}(k+1)!k!}.$$

That is, $c_A - c_B$ and $x_A - x_B$ have opposite signs, so expected turnout is higher for the alternative whose supporters have lower voting costs; we have have called this the *cost effect*. We note that if $d = 0$, there cannot be a handicap effect, and then the cost effect dominates over the underdog effect: If the majority has lower voting costs than the minority, the former displays higher aggregate—but not necessarily individual—turnout levels, despite any potential underdog effect that could favor the latter.

Next, we focus on positive handicaps, i.e., we assume $d > 0$.[68] We obtain the following result:

[68] The case $d < 0$ yields symmetric results.

Proposition 10. *Let* $(\alpha_A(d), \alpha_B(d)) \in (0,1) \times (0,1)$ *be a (totally mixed) equilibrium of game* $\mathcal{G}^2(d)$, *for some* $d > 0$. *Then,*

(i) $c_A > c_B$ *implies* $x_A < x_B$,

(ii) $c_A < c_B$ *does not necessarily imply* $x_A > x_B$.

Proof. First, we show Part *(i)*. It suffices to build on the proof of Proposition 1, and hence we can focus on the differences that arise when $c_A \neq c_B$.[69] After algebraic manipulations, one can verify that the expression defined in (54) reduces to

$$l_d(x_A, x_B) = 2(c_B - c_A) < 0. \tag{89}$$

In the proof of Proposition 1, where $c_A = c_B$, we have $l_d(x_A, x_B) = 0$—see Equation (55). Now, if we assume $x_A \geq x_B$, we obtain (62) as in the case where $d = 0$. This contradicts (89), and hence it must be that $x_A < x_B$.

Second, we show Part *(ii)* by providing a counterexample to the statement that $c_A < c_B$ must imply $x_A > x_B$. Indeed, let $d = 2$ and $c_A \approx 0.1577 < c_B \approx 0.1733$. Then, $x_A = 9.6$ and $x_B = 9.8$ define an equilibrium of game $\mathcal{G}^2(d)$.

□

According to Proposition 10, if the minority (namely, B-supporters) has lower costs than the majority (namely, A-supporters) and alternative B is handicapped with respect to alternative A, the expected turnout in favor of alternative B in the second round of AV is higher than that of alternative B in any totally mixed equilibrium. In this case, the cost effect, the handicap effect and the underdog effect all work towards increasing the marginal value of a vote for alternative B compared to the marginal value of a vote for alternative A. If $c_A < c_B$, by contrast, the cost effect works against the handicap effect (and against the underdog effect). This is reflected in the possibility that despite the cost disadvantage, total expected turnout is larger for alternative B than for alternative A. As discussed above, this is not possible if $d = 0$, in which case there is no handicap effect and the underdog effect alone does not suffice to counteract the cost effect. Which effect dominates in general is nonetheless ambiguous in the case of non-zero handicaps. While Part *(ii)* of Proposition 10 has shown that the handicap effect together with the underdog effect may dominate over the cost effect, the opposite holds for different cost specifications. One such example is $c_A \approx 0.192498$, $c_B \approx 0.180021$, and $d = 2$. It can be verified that $x_A = 6.8$ and $x_B = 10$ define an equilibrium of game $\mathcal{G}^2(d)$. A complete characterization of the interplay between the three equilibrium effects in the case of cost differences is beyond the scope of the present paper.

Up to this point, in our analysis in this section we have investigated some properties of totally mixed equilibria, but we have not tackled the issue whether such equilibria (or other equilibria) exist at all. We address this next. First, we show that if the difference across types regarding voting costs is sufficiently large in relative terms, the citizens with the highest costs will typically not turn out *at all* in any possible equilibrium of $\mathcal{G}^2(d)$, and hence their preferred alternative will obtain zero votes in the second voting round of AV. One possible exception is the case where the minority is very small—i.e., p_B is very small—and has arbitrarily lower relative voting costs than the majority (in particular, B-supporters

[69] A complete proof can be provided upon request.

could have zero voting costs). In such a scenario, despite cost differences, there is an equilibrium in which all members of the minority turn out with certainty and all members of the majority turn out with probability one.

Proposition 11. *Consider the game $\mathcal{G}^2(d)$. Then,*

(i) For a given c_A, there is $c_B^*(c_A,d) > 0$ such that if $c_B < c_B^*(c_A,d)$, then A-supporters abstain in any equilibrium where B-supporters do not vote with probability one.

(ii) For a given c_B, there is $c_A^*(c_B,d) > 0$ such that if $c_A < c_A^*(c_B,d)$ and $d > 1$, then B-supporters abstain in any equilibrium where A-supporters do not vote with probability one.

Proof. For $x, y \geq 0$, it will be convenient to define

$$q_d(x,y) = \sum_{k=0}^{\infty} \frac{x^k}{e^x k!} \frac{y^{k+d+1}}{e^y (k+d+1)!}. \tag{90}$$

We stress that we focus on equilibria in which at least one voter type does not turn out with probability one. Note that $\alpha_t < 1$ if and only if $x_t < n_2 \cdot p_t$, where $t \in \{A, B\}$. Then, the equilibrium conditions for an equilibrium to be defined by $(x_A, x_B) = (x, y)$, with $x > 0$ and $0 \leq y < n_2 \cdot p_B$, can be written as

$$2c_A \leq q_d(x,y) + q_{d-1}(x,y), \tag{91}$$

$$2c_B \geq q_{d-1}(x,y) + q_{d-2}(x,y). \tag{92}$$

In any equilibrium defined by inequalities (91) and (92), A-supporters turn out with positive probability, while B-supporters need not necessarily do so (and they do not turnout with probability one, by assumption). By contrast, the equilibrium conditions for an equilibrium to be defined by $(x_A, x_B) = (x, y)$, with $0 \leq x < n_2 \cdot p_A$ and $y > 0$, are

$$2c_A \geq q_d(x,y) + q_{d-1}(x,y), \tag{93}$$

$$2c_B \leq q_{d-1}(x,y) + q_{d-2}(x,y). \tag{94}$$

In any equilibrium defined by inequalities (93) and (94), B-supporters turn out with positive probability, while A-supporters need not necessarily do so (and they do not turnout with probability one, by assumption).

First, we show Part *(i)*. To this purpose, we have c_A fixed and proceed by contradiction, i.e., we assume that equilibria defined by (x_A, x_B) with $x_A > 0$ and $0 \leq x_B < n_2 \cdot p_B$ exist, even if c_B is arbitrarily close to zero. Then, consider any decreasing sequence $\{c_n\}_{n\geq 0}$ such that $\lim_{n\to\infty} c_n = 0$. For every such sequence, we let $x_n > 0$ and $0 \leq y_n < n_2 \cdot p_B$ be compatible with inequalities (91) and (92) when A-supporters have voting costs equal to c_A and B-supporters have voting costs equal to c_n. It suffices to consider any subsequence where such equilibria exist. On the one hand, assume that

$$z := \inf_{n\geq 0} q_{d-1}(x_n, y_n) > 0.$$

From inequality (92), it must necessarily be that

$$c_n \geq z.$$

However, this contradicts our assumptions on the sequence $\{c_n, x_n, y_n\}_{n\geq 0}$. Hence, there has to be $c_B^*(c_A, d)$, with $c_B^*(c_A, d) > 0$, such that if $c_B < c_B^*(c_A, d)$, then $x_A = 0$ in any (possible) equilibrium (x_A, x_B) of game $\mathcal{G}^2(d)$ where $x_B < n_2 \cdot p_B$. If, on the other hand, $z = 0$, it follows from $c_A > 0$ being fixed that, at least for some subsequence $\{c_n, x_n, y_n\}_{n\geq 0}$,

$$w := \inf_{n\geq 0} q_d(x_n, y_n) > 0. \tag{95}$$

Next, we claim that

$$q_{d-1}(x_n, y_n) \geq w \cdot \frac{d+1}{n_2 \cdot p_B}, \tag{96}$$

which, together with (95), implies that $z > 0$, a contradiction with the assumption that $z = 0$. In such case, i.e. if (96) holds, we obtain that $x_A = 0$ if c_B is below $c_B^*(c_A, d) = \frac{1}{n_2 \cdot p_B} w(d+1)$. Accordingly, it remains to prove the claim in (96) for all $n \geq 0$, which holds since

$$q_{d-1}(x_n, y_n) \geq \frac{d+1}{y_n} q_d(x_n, y_n) \geq \frac{d+1}{n_2 \cdot p_B} q_d(x_n, y_n) \geq w \cdot \frac{d+1}{n_2 \cdot p_B},$$

where the first inequality follows from some algebraic manipulations applied to the definition of $q_d(x, y)$—see expression (90)—, the second inequality holds because $y_n \leq n_2 \cdot p_B$ (by definition), and the third inequality follows from (95). This completes the proof of the claim in (96).

Second, we show Part *(ii)*. We recall that we are assuming $d > 1$. Similarly to the proof of Part *(i)*, we take $c_B > 0$ fixed and proceed by contradiction. That is, we assume that equilibria defined by (x_A, x_B) with $0 \leq x_A < n_2 \cdot p_A$ and $x_B > 0$ exist, even if c_A is arbitrarily close to zero. Consider any decreasing sequence $\{c_n\}_{n\geq 0}$ such that $\lim_{n\to\infty} c_n = 0$. For every such sequence, we let $0 \leq x_n < n_2 \cdot p_A$ and $y_n > 0$ be compatible with inequalities (93) and (94) when B-supporters have voting costs equal to c_B and A-supporters have voting costs equal to c_n. On the one hand, assume that

$$z := \inf_{n\geq 0} q_{d-1}(x_n, y_n) > 0,$$

From inequality (93), it must necessarily be that

$$c_n \geq z.$$

However, this contradicts our assumptions on the sequence $\{c_n, x_n, y_n\}_{n\geq 0}$. Hence, there has to be $c_A^*(c_B, d)$, with $c_A^*(c_B, d) > 0$, such that if $c_A < c_A^*(c_B, d)$, then $x_B = 0$ in any equilibrium of $\mathcal{G}^2(d)$ where $x_A < n_2 \cdot p_A$. If, on the other hand, $z = 0$, it follows from $c_B > 0$ being fixed that, at least for some subsequence $\{c_n, x_n, y_n\}_{n\geq 0}$,

$$w := \inf_{n\geq 0} q_{d-2}(x_n, y_n) > 0. \tag{97}$$

Next, we claim that there is a constant χ such that

$$q_{d-1}(x_n, y_n) \geq w^2 \cdot \chi, \tag{98}$$

which, together with (95), implies that $z > 0$, a contradiction with the assumption that $z = 0$. First, we note that for $d > 1$,

$$w \leq y_n. \tag{99}$$

The reason is that for any $x_n \geq 0$,

$$w \leq q_{d-2}(x_n, y_n) = \sum_{k=0}^{\infty} \frac{x^k}{e^{x_n} k!} \frac{y_n^{k+d-1}}{e^y (k+d-1)!} \leq y_n.$$

We stress that we are assuming $d > 1$. Let now $v^s(x,y)$ denote the partial sum of s terms in $q_{d-2}(x,y)$, i.e.

$$v^s(x,y) := \sum_{k=0}^{s} \frac{x^k}{e^x k!} \frac{y^{k+d-1}}{e^y (k+d-1)!}. \tag{100}$$

Take any $\varepsilon > 0$ sufficiently small. Then, there is some large enough integer $s(\varepsilon)$ such that

$$v^{s(\varepsilon)}(x_n, y_n) \geq w \cdot (1 - \varepsilon). \tag{101}$$

The latter inequality can be proved by using the concentration bound of a Poisson random variable given in (78) and the fact that, by definition, we have $y_n \leq n_2 \cdot p_B < \infty$. Therefore, we obtain that for all $n \geq 0$,

$$q_{d-1}(x_n, y_n) \geq v^{s(\varepsilon)}(x_n, y_n) \cdot \frac{y_n}{s(\varepsilon)} \geq w \cdot \frac{y_n \cdot (1-\varepsilon)}{s(\varepsilon)} \geq w^2 \cdot \frac{1-\varepsilon}{s(\varepsilon)}.$$

The first inequality follows from some algebraic manipulations applied to the definitions of $q_{d-1}(x,y)$—see expression (90)—and $v^s(x,y)$—see expression (100)—, the second inequality holds due to (101), and the third inequality follows from (99). This completes the proof of the claim in (98), and hence that of the proposition. □

We have to make four remarks. First, in the above result, we do not rule out the possibility that an equilibrium exists in which the citizens with the lowest voting costs turn out with probability one and the citizens with the highest voting costs turn out with positive probability, even if the relative difference in voting costs is arbitrarily large. This is because citizens cannot vote with a probability higher than one, and hence the variables x_A are x_B bounded from above (for fixed n_2, the size of the electorate). Second, Part *(ii)* of Proposition 11 does not extend to the case where $d = 1$. In the above proof, one can observe why. When $d = 1$, $q_{d-1}(x,y)$ has a term that does not depend on y, namely $\frac{x^0}{e^x 0!}$. This enables us to take a sequence $y_n \to 0$ and obtain $c_n = q_2(x, y_n) + q_1(x, y_n) \to 0$ for $c_B = q_1(x, y_n) + q_0(x, y_n)$. In equilibrium, the latter expression is lower bounded by $\frac{1}{e^x}$, for fixed x. Third, the same result as in Proposition 11 is obtained by Taylor and Yildirim (2010a) in the case where $d = 0$. It is important to point out that the authors are *implicitly* assuming that no citizen votes with certainty, analogously to what we assume. Fourth and last, Proposition 11 has identified sufficient conditions on the relative voting costs for the different citizen types that guarantee that no equilibria exist in which the citizens with the highest cost turn out with positive probability and the citizens with the lowest cost do not turn out with probability one. This is a consequence of the cost effect dominating completely over any other potential equilibrium effect (handicap or underdog). Reversely, one can ask when do partially mixed equilibria exist, in which only the citizens with the lowest voting costs participate, albeit with probability lower than one. We do this next.

Proposition 12. *Let $d \geq 1$ be given. There is a threshold $c^*(d)$, such that for any $c_B \leq c_A \leq c^*(d)$, there is a partially mixed equilibrium of type $(0, x_B)$.*

Proof. The result follows directly from the proof of the Proposition 4, with the same threshold $c^*(d)$. □

In the main section we have seen that our results about high values of handicaps do not hinge on the assumption that the majority and the minority have equal voting costs. The results in this section show that the main insights of the baseline model with regard to low values of handicaps are not knife-edged either on the assumption of equal voting costs across types, very particularly when alternative A has some initial advantage—namely, handicap d is positive—and is associated with higher voting costs than alternative B.

Three or more alternatives

In this section, we analyze the case of three or more alternatives. As already noted in Section 2, this case has been analyzed for one-round voluntary voting by Arzumanyan and Polborn (2017) (in the case of heterogeneous voting costs) and by Xefteris (2019) (in the case of homogeneous voting costs). Although for tractability we build on the approach of the former paper to analyze AV, at the end of this section we will briefly discuss how to extend our results to the set-up of the latter paper. We show that the negative result identified by Theorem 1 holds regardless of the number of alternatives at hand. That is, we show that there is a threshold—which coincides with $d^*(\frac{c}{2})$—such that there is an equilibrium of the second-round voting game in which no citizen will turn out, provided that the vote-count difference in the first voting round between the alternative with most votes and all other alternatives is sufficiently large.[70]

Accordingly, suppose there is a set of m alternatives $A_1, A_2, ..., A_m$, denoted by $\mathcal{A}$. Citizens are of one of $m!$ possible types $(A_1, A_2, ..., A_m), ..., (A_m, A_{m-1}, ..., A_1)$, where type $(A_{i_1} A_{i_2}, ..., A_{i_m})$ stands for the citizen whose most preferred alternative is A_{i_1}, the second most preferred alternative is A_{i_2}, and so on. Let $\mathcal{T}$ denote the set of all citizen types. We assume that there are $V_1, \ldots, V_m$ such that each citizen i derives a utility level V_j if his/her j^{th} best alternative wins. Without loss of generality, we impose the normalization $1 = V_1 \geq V_2 \geq ... \geq V_m = 0$. In addition, we use $(a_1, a_2, ..., a_m)$ to denote the number of votes collected by each alternative in the first voting round of AV.

As in the case of the two alternatives analyzed in the main body of the paper, we proceed on the assumption that the number of citizens of each type $(A_{i_1}, A_{i_2}, ..., A_{i_m})$ in the second voting round of AV is distributed according to a Poisson probability distribution with parameter $p_{i_1,i_2,...,i_m}$. As for the solution concept, we again consider type-symmetric perfect Nash equilibria and use the following notation: If a citizen of type $(A_{i_1}, ..., A_{i_m})$ decides to turn out in the second round of AV, s/he will vote for alternative A_{i_j} with probability $p^{i_j}_{i_1,i_2,...,i_m}$, where $1 \leq j \leq m$. It will suffice to assume momentarily that these probabilities are exogenously given and satisfy

$$p^{i_m}_{i_1,i_2,...,i_m} = 0 \quad \text{and} \quad \sum_{j=1}^{m-1} p^{i_j}_{i_1,i_2,...,i_m} = 1.$$

Note that we are assuming that citizens will never vote for their least preferred alternative—this assumption generalizes sincere voting in a framework with at least three alternatives. By using $x_{i_1,i_2,...,i_m} \in [0, 1]$ to denote the (equilibrium) probability that a voter of type $(A_{i_1}, ..., A_{i_m})$ will turn out at all, we find that in the second round of AV the number of votes in favor of alternative A_j when citizens vote according to the strategy profile $(x_{i_1,i_2,...,i_m})_{(A_{i_1},...,A_{i_m})\in\mathcal{T}}$ is distributed as a Poisson random variable with

[70] We cannot rule out the possibility that equilibria may also exist in which strategic voting will occur in the first voting round. One-round voting mechanisms have the same drawback.

parameter[71]

$$\eta_j := n_2 \cdot \sum_{(A_{i_1},\ldots,A_{i_m})\in\mathcal{T}:i_k=j} p_{i_1,i_2,\ldots,i_m} \cdot x_{i_1,i_2,\ldots,i_m} \cdot p^{i_k}_{i_1,i_2,\ldots,i_m}.$$

The reason is that a sum of independent Poisson random variables is itself a Poisson random variable. Together with the probabilities $(p^{i_j}_{i_1,i_2,\ldots,i_m})_{(A_{i_1},\ldots,A_{i_m})\in\mathcal{T},i_j\in\{1,\ldots,m\}}$, any profile $(x_{i_1,i_2,\ldots,i_m})_{(A_{i_1},\ldots,A_{i_m})\in\mathcal{T}}$ uniquely determines a vector $(\eta_1,\ldots,\eta_m)$. Because we want to prove that no equilibrium will exist where $x_{i_1,i_2,\ldots,i_m} > 0$ for some $(A_{i_1},\ldots,A_{i_m}) \in \mathcal{T}$, we can in fact assume that $(p^{i_j}_{i_1,i_2,\ldots,i_m})_{(A_{i_1},\ldots,A_{i_m})\in\mathcal{T},i_j\in\{1,\ldots,m\}}$ are taken as given. As a tie-breaking rule, we consider that if there are k alternatives with the same number of votes combined in the two voting rounds and if the remaining alternatives have strictly fewer votes, then the alternative that wins will be chosen among these k alternatives, each alternative having a probability $\frac{1}{k}$ of winning.

Next, suppose that alternative A_i has received a_i votes in the first voting round. We can assume that $a_1 \leq a_2 \leq \ldots \leq a_m$ without loss of generality. Finally, we use $\mathcal{G}^m(a_1,a_2,\ldots,a_m)$ to denote the modification of $\mathcal{G}^2(d)$, so that citizens can now vote for any of the m alternatives in any voting round. We have the following proposition:

Proposition 13. *For any $c > 0$ and any vector of votes $(a_1,a_2,\ldots,a_m)$ after the first voting round, there is a positive integer $d^{**}(c)$ such that, if $a_m - a_{m-1} \geq d^{**}(c)$, the only equilibrium of game $\mathcal{G}^m(a_1,a_2,\ldots,a_m)$ is the no-show equilibrium.*

Proof. The proof is based on an induction on m. The case $m = 2$ is proven in Theorem 1. Now assume that the claim of the proposition is true for the case of $m-1$ alternatives, and consider the case of m alternatives. In particular, we will show that no citizen will vote for alternative A_1 in any equilibrium of game $\mathcal{G}^m(d)$, where A_1 is the alternative receiving the lowest number of votes in the first voting round. This means that, instead of m alternatives, it is as if there were only $m-1$ alternatives, $A_2,\ldots,A_m$. Because $a_m - a_{m-1} \geq d^{**}(c)$, we find by induction that the only equilibrium that survives is the no-show equilibrium.

We distinguish two cases corresponding to the cases in which one more vote in the second voting round in favor of alternative A_1 will make a difference in the final outcome. In both cases, we let i be a citizen of type $(A_1,A_{i_2},\ldots,A_{i_n})$. It will suffice to consider this type of citizen.[72]

Case I: *In the two voting rounds combined, alternative A_1 received exactly the same number of votes as each of the alternatives of a given (non-empty) set $\mathcal{B}$, with all alternatives in $\mathcal{A} \setminus (\mathcal{B} \cup \{A_1\})$ receiving strictly fewer total votes than those in $\mathcal{B}$.*

In this case, with one additional vote in the second voting round, A_1 will win without ties. Accordingly, the expected gain that citizen i derives from voting for A_1 in the second round is equal to

$$H(\mathcal{B}) := 1 - \frac{1}{1+|\mathcal{B}|} \cdot \left(1 + \sum_{j\in\mathcal{B}} V_j\right).$$

Let x denote the total number of votes received by A_1 and alternatives from $\mathcal{B}$ in the two voting rounds combined. It is straightforward to verify that $x \geq a_m$, because alternative A_m already has a_m votes from the first round (the highest number among all alternatives). Then, the probability of having an

[71]To avoid cumbersome notation, we do not write the dependence of $x^{i_j}_{i_1,i_2,\ldots,i_m}$ on $(a_1,a_2,\ldots,a_m)$.
[72]The argument of the proof can be easily adapted for all other types of citizens.

alternative of set $\mathcal{B}$ winning the voting after both rounds (excluding i's vote) is

$$P_{equal}(\mathcal{B}) := \sum_{x=a_m}^{\infty} \left(\prod_{j \in \mathcal{B} \cup \{A_1\}} \frac{\eta_j^{x-a_j}}{e^{\eta_j}(x-a_j)!} \cdot P(x, \mathcal{A} \setminus (\mathcal{B} \cup \{A_1\})) \right),$$

where $P(x, \mathcal{S})$ denotes the probability that alternatives in set S will *all* receive strictly fewer votes than x. Let s denote the size of any arbitrary set $\mathcal{S}$. It is easy to verify the following:

$$P(x, \mathcal{S}) = \sum_{\substack{(l_1,\ldots,l_s) \in \mathbb{N}^s, \\ l_j + a_{s_j} < x, j=1,\ldots,s}} \prod_{r=1}^{s} \frac{\eta_{s_r}^{l_r}}{e^{\eta_{s_r}} l_r!}. \tag{102}$$

Case II: *In the two voting rounds combined, alternative A_1 received one vote less than each of the alternatives of a given (non-empty) set $\mathcal{C}$, with all alternatives in $\mathcal{A} \setminus (\mathcal{C} \cup \{A_1\})$ receiving strictly fewer total votes than those in $\mathcal{C}$.*

In this case, with one additional vote in the second voting round, there is a chance that A_1 will be chosen. Accordingly, the expected gain that citizen i derives from voting in the second round in favor of A_1 is equal to

$$F(\mathcal{B}) := \frac{1}{1+|\mathcal{C}|} \cdot \left(1 + \sum_{j \in \mathcal{C}} V_j \right) - \frac{1}{|\mathcal{C}|} \cdot \sum_{j \in \mathcal{C}} V_j,$$

which is always a non-negative number since $\max_{j \in \mathcal{C}} V_j \leq 1$. Let $x+1$ now denote the number of total votes received by each of the alternatives in set $\mathcal{C}$ in the two voting rounds combined. That is, alternative A_1 has received x votes in both rounds combined, and it must be the case that $x \geq a_m$. Then, the probability of having an alternative of set $\mathcal{C}$ winning the voting after both rounds (excluding i's vote) is

$$P_{low}(\mathcal{C}) := \sum_{x=a_m}^{\infty} \left(\frac{\eta_1^{x-a_1}}{e^{\eta_1}(x-a_1)!} \cdot \prod_{j \in \mathcal{C}} \frac{\eta_j^{x+1-a_j}}{e^{\eta_j}(x+1-a_j)!} \cdot P(x, \mathcal{A} \setminus (\mathcal{C} \cup \{A_1\})) \right),$$

where $P(x, S)$ has been defined in Equation (102).

Finally, let $2^{\mathcal{A}}$ denote the power set of $\mathcal{A}$. Then, the indifference condition for citizen i that equalizes the expected gain of voting for alternative A_1 and the cost of voting is

$$c = \sum_{\mathcal{B} \in 2^{\mathcal{A} \setminus \{A_1\}} \setminus \emptyset} P_{equal}(\mathcal{B}) \cdot H(\mathcal{B}) + \sum_{\mathcal{C} \in 2^{\mathcal{A} \setminus \{A_1\}} \setminus \emptyset} P_{low}(\mathcal{C}) \cdot F(\mathcal{C}). \tag{103}$$

By Inequality (30)—see the proof of Proposition 2—, if $d \geq d^{**}(c)$ and for all $y \in \mathbb{R}_+$ and $k \geq 0$, it holds that

$$\frac{y^{k+d}}{e^y (k+d)!} < \frac{c}{2}, \tag{104}$$

If we now assume that $a_m - a_{m-1} \geq d^{**}(c)$, which implies $a_m - a_1 \geq d^{**}(c)$, then because $H(\mathcal{B})$ and $F(\mathcal{C})$ are at most one and all the events described in the calculations of P_{equal} and P_{low} are disjoint, we have

$$\sum_{\mathcal{B}\in 2^{\mathcal{A}\setminus\{A_1\}}\setminus\emptyset} P_{equal}(\mathcal{B})\cdot H(\mathcal{B}) + \sum_{\mathcal{C}\in 2^{\mathcal{A}\setminus\{A_1\}}\setminus\emptyset} P_{low}(\mathcal{C})\cdot F(\mathcal{C})$$
$$\leq \left(\sum_{\mathcal{B}\in 2^{\mathcal{A}\setminus\{A_1\}}\setminus\emptyset} P_{equal}(\mathcal{B}) + \sum_{\mathcal{C}\in 2^{\mathcal{A}\setminus\{A_1\}}\setminus\emptyset} P_{low}(\mathcal{C})\right) \leq \left(\sum_{x=a_m}^{\infty} \frac{\eta_1^{x-a_1}}{e^{\eta_1}(x-a_1)!}\cdot (P_1(x)+P_2(x))\right) < c,$$

where the strict inequality holds by Inequality (104) and $P_1(x), P_2(x)$ are some functions satisfying

$$\sum_{x=a_m}^{\infty} P_1(x) \leq 1 \text{ and } \sum_{x=a_m}^{\infty} P_2(x) \leq 1.$$

That is, Equation (103) cannot hold if $a_m - a_1$ is above a certain threshold, which in fact coincides with $d^*(\frac{c}{2})$, and hence is approximately four times bigger than $d^*(c)$. Letting $d^{**}(c) = d^*(\frac{c}{2})$ concludes the proof. □

We conclude by noting that when dealing with three or more alternatives within a costly voting set-up, Xefteris (2019) considers the possibility that voting costs may vary across voters. Assuming heterogeneous voting costs in our set-up is equivalent to assuming a common voting cost $c > 0$, but then proceeding on the assumption that utilities V_i can no longer be normalized. One can verify that our main result in this section—Proposition 13—will remain valid in this generalized setting, albeit with a different threshold $d^{**}(c)$.