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## Abstract

This paper introduces a structural model for the coevolution of networks and behavior. The microfoundation of our model is a network game where agents adjust actions and network links in a stochastic best-response dynamics with a utility function allowing for both strategic externalities and unobserved heterogeneity. We show the network game admits a potential function and the coevolution process converges to a unique stationary distribution characterized by a Gibbs measure. To bypass the evaluation of the intractable normalizing constant in the Gibbs measure, we adopt the Double Metropolis-Hastings algorithm to sample from the posterior distribution of the structural parameters. To illustrate the empirical relevance of our structural model, we apply it to study R&D investment and collaboration decisions in the chemicals and pharmaceutical industry and find a positive knowledge spillover effect. Finally, our structural model provides a tractable framework for a long-run key player analysis.

JEL Classification: C11, C31, C63, C73, L22

Keywords: strategic network formation, network interactions, stochastic best-response dynamics, Unobserved heterogeneity, Double Metropolis-Hastings algorithm, R&D collaboration networks, Key players

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# A Structural Model for the Coevolution of Networks and Behavior\*

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## Abstract

This paper introduces a structural model for the coevolution of networks and behavior. The microfoundation of our model is a network game where agents adjust actions and network links in a stochastic best-response dynamics with a utility function allowing for both strategic externalities and unobserved heterogeneity. We show the network game admits a potential function and the coevolution process converges to a unique stationary distribution characterized by a Gibbs measure. To bypass the evaluation of the intractable normalizing constant in the Gibbs measure, we adopt the Double Metropolis-Hastings algorithm to sample from the posterior distribution of the structural parameters. To illustrate the empirical relevance of our structural model, we apply it to study R&D investment and collaboration decisions in the chemicals and pharmaceutical industry and find a positive knowledge spillover effect. Finally, our structural model provides a tractable framework for a long-run key player analysis.

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\*This has grown out ideas in a manuscript circulated under the title “Network Formation with Local Complements and Global Substitutes: The Case of R&D Networks”, which is now split into two papers. In this paper, we propose a general econometric model for the coevolution of networks and behavior incorporating both network formation externalities and unobserved heterogeneity. In the second paper, we abstract away from network formation externalities and focus on the equilibrium characterization, estimation and policy implications of dynamic R&D network models.

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## 1 Introduction

Since the seminal paper by Manski (1993), substantial progress has been made in the econometrics of networks following two research threads. The first thread studies the identification and estimation of behavioral interdependence of individuals in a network under the assumption that the network structure is exogenously given. A popular model in this literature is the linear social-interaction model (see, e.g., Bramoullé et al., 2009; Lee et al., 2010; Liu and Lee, 2010; Blume et al., 2015). The second thread focuses on the modeling and estimation of the network formation process, with some recent developments including Christakis et al. (2010), Snijders (2011), Leung (2015), Graham (2017), Mele (2017), Menzel (2017), Sheng (2017), Chandrasekhar and Jackson (2018), de Paula et al. (2018) and Dzemski (2018). To link the two research threads, we introduce a unified framework to model the coevolution of networks and behavior in this paper.

The microfoundation of our structural model is a network game where agents make decisions on actions and network links to maximize their utilities. The utility function is a generalization of the linear-quadratic utility function in Ballester et al. (2006) by incorporating direct payoffs from the network structure captured by *homophily/heterophily*, *popularity*, *congestion* and *cyclic triangle* effects (see Section 2.1 for more discussion on these effects).<sup>1</sup> We show that, under some mild assumptions, the utility function admits a potential game (Monderer and Shapley, 1996), where the change in the utility of an agent from adjusting an action or link is identical to the corresponding change in the potential function. The potential function thus aggregates individual incentives to change from the status quo and greatly simplifies the equilibrium analysis. This is crucial for deriving the likelihood function for the estimation.

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<sup>1</sup>The linear-quadratic utility function in Ballester et al. (2006) admits a best response function that underlies many well known linear social-interaction models in the literature (Blume et al., 2015).

The coevolution of networks and behavior is modeled as a stochastic best-response dynamics (Blume, 1993). In each period, a randomly selected agent gets a chance to change his action or meet with another agent to adjust the link between them to maximize his utility, taking the actions and links of the rest of the network as given. We show that this process follows a Markov chain that converges to a unique stationary distribution of actions and networks characterized by a Gibbs measure (i.e., an exponential family distribution that depends on the potential function).

Given an observation from the stationary distribution, the structural parameters can be identified and estimated based on the maximum likelihood principle. However, as pointed out in Mele (2017), the frequentist maximum likelihood method or Bayesian Metropolis-Hastings (MH) algorithm (Chib and Greenberg, 1995) are computationally infeasible due to the intractable normalizing constant in the Gibbs measure. To bypass the evaluation of the intractable normalizing constant, we adopt the Double Metropolis-Hastings (DMH) algorithm (Liang, 2010; Mele, 2017) to sample from the posterior distribution of the structural parameters. Compared with Mele (2017), we face the additional complication as we need to simulate actions as well as networks to generate auxiliary data in the DMH algorithm. We propose a computationally simple MH algorithm for this purpose. Another contribution relative to Mele (2017) is that we incorporate unobserved heterogeneity in the coevolution process of networks and behavior. This captures the potential correlation between action and network formation decisions. In Monte Carlo simulations, we find that unobserved heterogeneity is confounded with network interactions and ignoring it tends to overestimate the network spillover effect.

To illustrate the empirical relevance of our structural model, we apply it to study R&D investment and collaboration decisions in the chemicals and pharmaceutical industries. Using a unique dataset on R&D collaborations matched to firms' balance sheets, we find a positively significant knowledge spillover effect on firms' R&D investment decisions. When unobserved heterogeneity is controlled for, the estimated spillover effect becomes weaker but remains statistically significant. We also find that an R&D collaboration is more likely to form between firms in the same sub-sector (the *homophily* effect), firms with different productivities (the *heterophily* effect) and firms with a common collaboration partner (the *cyclic triangle* effect). Furthermore, firms benefit from forming

an R&D collaboration with a firm having other collaboration partners (the *popularity* effect), but the marginal benefit reduces as the number of other collaboration partners increases (the *congestion* effect).

Finally, the proposed structural model has important policy implications as it allows the policy maker to identify the *key player* whose exit would have the largest impact on welfare in the long run. Conventional key player analysis assumes the remaining network is unchanged after the key player is removed from the network (Ballester et al., 2006). This assumption might be reasonable if the key player analysis is considered as a short-run policy analysis and the network exhibits inertia in its evolution. However, in the long run, it is difficult to justify this assumption. Our structural model provides a tractable framework for a long-run key player analysis. We find that the key player rankings in the short run and in the long run do not coincide with each other. Therefore, it is important to choose the right analysis (short run or long run) for policy evaluation needs. We also show that the most important firms are not necessarily the ones with the highest market share, largest R&D expenditure, or most R&D alliances.

Our paper is related to recent papers that study the identification and estimation of network interactions with endogenous networks (Goldsmith-Pinkham and Imbens, 2013; ?; Auerbach, 2019; Johnsson and Moon, 2019). The focus of these papers is to consistently estimate network interaction effects controlling for the unobserved heterogeneity in the network formation process. However, network formation externalities (e.g., popularity, congestion and cyclic triangle effects) are ruled out in these models.<sup>2</sup> By contrast, our model incorporates both strategic externalities and unobserved heterogeneity. Hsieh and Lee (2017) also propose a network formation game with both network formation externalities and unobserved heterogeneity. But to guarantee the existence of a unique equilibrium, the game is assumed to be *cooperative* in that paper. The cooperative strategy assumption is difficult to justify when the number of agents is large. The closest work to our paper is Badev (2018), which proposes a network formation game where agents make decisions on *binary* actions and network links in the absence of unobserved heterogeneity. Our model, on the other hand, considers continuous actions and allows for unobserved heterogeneity.

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<sup>2</sup>Goldsmith-Pinkham and Imbens (2013) allow for time-lagged externalities in the endogenous network formation model.

The rest of the paper is organized as follows. The structural model is introduced in Section 2, where Section 2.1 derives the potential function, and Section 2.2 defines the stochastic coevolution process of networks and behavior and characterizes the stationary distribution. The estimation strategy is presented in Section 3, where Section 3.1 discusses the computational difficulty of conventional estimation methods, Section 3.2 outlines the DMH algorithm, Section 3.3 explains how to control for unobserved heterogeneity, and Section 3.4 provides some Monte Carlo simulation results. The empirical study is given in Section 4, where Section 4.1 presents the empirical model, Section 4.2 describes the data, Section 4.3 reports the estimation results, and Section 4.4 conducts a key player analysis. Finally, Section 5 concludes. All proofs are relegated to Appendix A. Implementation details of the DMH algorithm can be found in Appendix B.

## 2 Structural Model

### 2.1 Utility and the Potential Game

Consider a network  $g \in \mathcal{G}$  consisting a set of agents  $\mathcal{N} \equiv \{1, \dots, n\}$ , where  $\mathcal{G}$  is the set of all networks with  $n$  nodes. The topology of the network is represented by an  $n \times n$  adjacency matrix  $G = [g_{ij}]$ , where  $g_{ij} = 1$  if agents  $i$  and  $j$  form a link and  $g_{ij} = 0$  otherwise. The network links are reciprocal, that is,  $g_{ij} = 1$  implies  $g_{ji} = 1$ . As a normalization, we set  $g_{ii} = 0$  for all  $i \in \mathcal{N}$ . Let  $\mathcal{N}_i \equiv \{j \in \mathcal{N} | g_{ij} = 1\}$  denote the set of agent  $i$ 's peers (or, loosely speaking, "friends").

Agent  $i$ , with his exogenous characteristics given by a (row) vector  $X_i$ , makes decisions on network links  $g_{ij}$  and the effort level  $y_i$  in an activity to maximize utility. We assume  $X_i$  can be observed by all the agents. To introduce unobserved heterogeneity in the econometric model, we allow some components of  $X_i$  to be unobservable to the econometrician. Let  $Y = (y_1, \dots, y_n)'$ , and let  $Y_{-i}$  denote the effort levels of all agents but  $i$ . The utility of agent  $i$  follows a linear-quadratic function given by

$$U_i(g, Y, X) = a_i(g, X) + b(X_i)y_i + \lambda \sum_{j \in \mathcal{N}} g_{ij}y_i y_j - \frac{1}{2}y_i^2, \quad (1)$$



where

$$a_i(g, X) = \sum_{j \in \mathcal{N}} g_{ij} \left\{ \delta_0 + h(X_i, X_j, \delta_1) + \delta_2 d_{ij} + \delta_3 d_{ij}^2 + \delta_4 \sum_{k \in \mathcal{N} \setminus \{i, j\}} g_{ik} g_{jk} \right\}. \quad (2)$$

The first term of Equation (1),  $a_i(g, X)$ , captures the *direct* utility from the network topology. In particular,  $\delta_0$  is the fixed cost of maintaining links, and  $h(X_i, X_j, \delta_1)$  captures the (dis)similarity between agents  $i$  and  $j$  in exogenous characteristics, with the coefficient vector  $\delta_1$  representing the *homophily* or *heterophily* effect.  $d_{ij} = \sum_{k \in \mathcal{N} \setminus \{i, j\}} (g_{ik} + g_{jk})$  is the total number of links of agents  $i$  and  $j$  excluding the link  $g_{ij}$ . If  $\delta_2 > 0$  and  $\delta_3 < 0$ , then the coefficients  $\delta_2$  and  $\delta_3$  can be interpreted as the *popularity* and *congestion* effects respectively. That is, an agent may benefit from indirect links via linking to an agent with some “friends” (i.e., the popularity effect) but the marginal utility decreases as the number of indirect links increases (i.e., the congestion effect).  $\sum_{k \in \mathcal{N} \setminus \{i, j\}} g_{ik} g_{jk}$  is the number of common “friends” between agents  $i$  and  $j$ , with the coefficient  $\delta_4$  representing the *cyclic triangle* effect. We impose the following assumption on the functional form of  $h(X_i, X_j, \delta_1)$  to guarantee the existence of a potential game.

**Assumption 1.**  $h(X_i, X_j, \delta_1) = h(X_j, X_i, \delta_1)$  for any  $i, j \in \mathcal{N}$ .

The second term of Equation (1),  $b(X_i)y_i$ , represents the *direct* utility from the effort, with the marginal utility of effort given by  $b(X_i)$ . The third term,  $\lambda \sum_{j \in \mathcal{N}} g_{ij} y_i y_j$ , characterizes the *social* utility, with the spillover effect given by the coefficient  $\lambda$ . Finally, we assume the cost of exerting effort is given by the last term of Equation (1),  $\frac{1}{2} y_i^2$ , which exhibits increasing marginal cost. Maximizing Equation (1) with respect to  $y_i$  gives the best response function for the effort choice

$$y_i = \lambda \sum_{j \in \mathcal{N}} g_{ij} y_j + b(X_i), \quad (3)$$

which coincides with the one in Ballester et al. (2006).

**Remark 1.** In the network formation game considered in Mele (2017), agents only make decisions on links  $g_{ij}$  to maximize the *direct* utility from the network topology.<sup>3</sup> In our model, agents make

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<sup>3</sup>In Mele (2017), the specification of the direct utility from the network topology is slightly different from  $a_i(g, X)$  as the links are assumed to be non-reciprocal (directed) in that paper.

decisions on links  $g_{ij}$  as well as actions  $y_i$ , taking into account the *direct* utility from the network topology, the *direct* utility from the effort, and the *social* utility given by  $\lambda \sum_{j \in \mathcal{N}} g_{ij} y_i y_j$ . Badev (2018) considers a network formation game where agents decide on *binary* actions and network links. In our model, the action space is allowed to be continuous and the utility function implies a best response function given by Equation (3) that underlies many well known linear social-interaction models in the literature (e.g., Bramoullé et al., 2009; Liu and Lee, 2010; Blume et al., 2015).

The following proposition shows that the utility function defined in Equation (1) admits a potential game (Monderer and Shapley, 1996).

**Proposition 1.** *Under Assumption 1, the utility function defined in Equation (1) admits a potential game with the potential function given by*

$$Q(g, Y, X) = a(g, X) + \sum_{i \in \mathcal{N}} b(X_i) y_i + \frac{\lambda}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} g_{ij} y_i y_j - \frac{1}{2} \sum_{i \in \mathcal{N}} y_i^2, \quad (4)$$

where

$$a(g, X) = \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} g_{ij} \left\{ \delta_0 + h(X_i, X_j, \delta_1) + \delta_2 d_{ij} + \delta_3 d_{ij}^2 + \frac{1}{3} \delta_4 \sum_{k \in \mathcal{N} \setminus \{i, j\}} g_{ik} g_{jk} \right\}.$$

The potential function has the property that the change in the utility of an agent from adjusting a link or effort level is identical to the corresponding change in the potential function. The potential function thus aggregates the incentives to change from the status quo for every agent in the network. The existence of a potential function is crucial for the equilibrium characterization of the coevolution process introduced in the following section.

## 2.2 Coevolution of Networks and Behavior

Let the realization of the network in period  $t$  be denoted by  $g_t$  with the adjacency matrix  $G_t = [g_{ij,t}]$ , and let the network including all the current links but  $g_{ij,t}$  be denoted by  $g_{-ij,t}$ . Similarly, the effort profile of  $\mathcal{N}$  in period  $t$  is given by the vector  $Y_t = [y_{i,t}]$ , and the effort profile of  $\mathcal{N} \setminus \{i\}$  is written as  $Y_{-i,t}$ . To simplify notation, we drop  $X$  from  $U_i(g, Y, X)$  and  $Q(g, Y, X)$  henceforth.

The coevolution of networks and behavior is specified as a stochastic best-response dynamics (Blume, 1993). We assume time is discrete. Each time period is either a link-adjustment period (with probability  $0 < \rho_0 < 1$ ) or an effort-adjustment period (with probability  $1 - \rho_0$ ). In the following, we give details of these two adjustment periods and characterize the stationary distribution of the stochastic process.

**Link Adjustment** In a link-adjustment period, a pair of agents  $i$  and  $j$  is randomly selected from the population with probability  $\rho(g_{t-1}, X_i, X_j)$ . To make the equilibrium analysis feasible, we impose the following assumption on the selection rule characterized by  $\rho(g_{t-1}, X_i, X_j)$ .<sup>4</sup>

**Assumption 2.** (i)  $\rho(g_{t-1}, X_i, X_j) = \rho(g_{t-1}, X_j, X_i)$ ; (ii)  $\rho(g_{t-1}, X_i, X_j)$  does not depend on  $g_{ij,t-1}$ ; and (iii)  $\rho(g_{t-1}, X_i, X_j) > 0$  for all  $(i, j) \in \mathcal{N} \times \mathcal{N}$ .

Conditional on being selected, agents  $i$  and  $j$  update the link  $g_{ij}$  to maximize their current utilities taking the rest of the network and effort choices as given. Also, as in Mele (2017), we assume that agents do not take into account the effect of their decisions on the future effort choices and network evolution. To capture the uncertainty (from the perspective of the econometrician) in the link adjustment process, we introduce an idiosyncratic shock to the utility and assume that  $g_{ij,t} = 1$  if and only if both agents  $i$  and  $j$  find that  $g_{ij,t} = 1$  improves their utility, i.e.,

$$U_i(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1}) + \epsilon_{1t} \geq U_i(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1}) + \epsilon_{0t},$$

and

$$U_j(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1}) + \epsilon_{1t} \geq U_j(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1}) + \epsilon_{0t}.$$

As

$$\begin{aligned} & Q(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1}) - Q(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1}) \\ &= U_i(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1}) - U_i(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1}) \\ &= U_j(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1}) - U_j(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1}), \end{aligned}$$

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<sup>4</sup>See Mele (2017) for more discussion on the selection (or meeting) rule.

the above two inequalities can be written more compactly as

$$Q(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1}) + \epsilon_{1t} \geq Q(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1}) + \epsilon_{0t}.$$

Assuming the shocks  $\epsilon_{0t}$  and  $\epsilon_{1t}$  are independent from each other, i.i.d. across links and time periods and follow a Gumbel distribution with the distribution function  $F(\epsilon) = \exp[-\exp(-\epsilon/\sigma^2)]$ , conditional on the meeting of agents  $i$  and  $j$ , the probability for them to form a link is given by

$$\begin{aligned} & \Pr(g_{ij,t} = 1 | g_{-ij,t} = g_{-ij,t-1}, Y_t = Y_{t-1}) \\ &= \frac{\exp[\sigma^{-2}Q(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1})]}{\exp[\sigma^{-2}Q(g_{ij,t} = 0, g_{-ij,t-1}, Y_{t-1})] + \exp[\sigma^{-2}Q(g_{ij,t} = 1, g_{-ij,t-1}, Y_{t-1})]}, \end{aligned} \quad (5)$$

where the parameter  $\sigma^2$  captures the level of “noise” in link adjustment decisions.<sup>5</sup>

**Effort Adjustment** In an effort-adjustment period, an agent  $i$  is randomly selected from the population with probability  $\rho(X_i)$ . We assume any agent can be selected with positive probability in the following assumption.

**Assumption 3.**  $\rho(X_i) > 0$  for all  $i \in \mathcal{N}$ .

Conditional on being selected, agent  $i$  updates the effort level  $y_{it} \in \mathcal{Y}$  to maximize his current utility, where  $\mathcal{Y}$  is the set of all possible effort choices. We allow  $\mathcal{Y}$  to be continuous and assume that, taking the network  $g_{t-1}$  and the effort levels of the other agents  $Y_{-i,t-1}$  as given, the probability that agent  $i$  chooses an effort level in  $\mathcal{Z} \subset \mathcal{Y}$  in period  $t$  is given by

$$\Pr(y_{it} \in \mathcal{Z} | g_t = g_{t-1}, Y_{-i,t} = Y_{-i,t-1}) = \frac{\int_{\mathcal{Z}} \exp[\sigma^{-2}U_i(g_{t-1}, z, Y_{-i,t-1})] dz}{\int_{\mathcal{Y}} \exp[\sigma^{-2}U_i(g_{t-1}, y, Y_{-i,t-1})] dy}. \quad (6)$$

Similar to Equation (5) in the link adjustment period, the probability given in Equation (6) can be justified by an additive random utility model over a nonfinite choice set (McFadden, 1976), where the parameter  $\sigma^2$  captures the level of “noise” in effort adjustment decisions. Equation (6) admits

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<sup>5</sup>The parameter  $\sigma^2$  can be identified because the coefficient of  $y_i^2$  is normalized to  $-1/2$  in the utility (and potential) function.

the probability density function

$$p(y_{it}|g_t = g_{t-1}, Y_{-i,t} = Y_{-i,t-1}) = \frac{\exp[\sigma^{-2}U_i(g_{t-1}, y_{it}, Y_{-i,t-1})]}{\int_{\mathcal{Y}} \exp[\sigma^{-2}U_i(g_{t-1}, y, Y_{-i,t-1})]dy}. \quad (7)$$

**Equilibrium** In the stochastic process described above, the coevolution of the network  $g_t$  and effort choices  $Y_t$  follows a Markov chain. In the following proposition, we show that the Markov chain converges to a unique stationary distribution. Let  $\gamma$  denote the vector of all unknown parameters in the potential function and  $\theta = (\gamma', \sigma^2)'$ .

**Proposition 2.** *Under Assumptions 1-3, the coevolution process of the network and behavior converges to a unique stationary distribution characterized by the Gibbs measure*

$$\pi(g, Y|\theta) = c(\theta)^{-1} \exp[\sigma^{-2}Q(g, Y|\gamma)], \quad (8)$$

where  $c(\theta) = \sum_{g \in \mathcal{G}} \int_{\mathcal{Y}^n} \exp[\sigma^{-2}Q(g, Y|\gamma)]dY$ .

### 3 Estimation

In this section, we discuss how to estimate the structural parameters based on the Gibbs measure defined in Equation (8). We first assume all components in  $X_i$  can be observed by the econometrician in Sections 3.1 and 3.2. Then, to introduce unobserved heterogeneity, we allow some components of  $X_i$  to be unobservable in Section 3.3.

#### 3.1 Computational Problem and the Exchange Algorithm

Given an observation  $(g, Y)$  from the stationary distribution defined in Equation (8), we can estimate the parameter vector  $\theta$  based on the maximum likelihood principle. However, as pointed out in Mele (2017), the frequentist maximum likelihood method is impractical due to the computational difficulty in evaluating the normalizing constant  $c(\theta)$  in Equation (8), and a standard Bayesian method would encounter the same problem because, with the prior distribution  $p(\theta)$ , the posterior

distribution  $p(\theta|g, Y) \propto \pi(g, Y|\theta)p(\theta) = c(\theta)^{-1} \exp[\sigma^{-2}Q(g, Y|\gamma)]p(\theta)$  also contains the normalizing constant  $c(\theta)$ . To sample from the posterior using Markov Chain Monte Carlo (MCMC) method, a standard MH algorithm (Chib and Greenberg, 1995) updates  $\theta$  to  $\tilde{\theta}$ , a random draw from the proposal distribution  $q_{\theta}(\tilde{\theta}|\theta)$ , according to the acceptance probability

$$\alpha_{\theta, MH} = \min \left\{ 1, \frac{p(\tilde{\theta}|g, Y)q_{\theta}(\theta|\tilde{\theta})}{p(\theta|g, Y)q_{\theta}(\tilde{\theta}|\theta)} \right\} = \min \left\{ 1, \frac{c(\theta) \exp[\tilde{\sigma}^{-2}Q(g, Y|\tilde{\gamma})]p(\tilde{\theta})q_{\theta}(\theta|\tilde{\theta})}{c(\tilde{\theta}) \exp[\sigma^{-2}Q(g, Y|\gamma)]p(\theta)q_{\theta}(\tilde{\theta}|\theta)} \right\}.$$

The computational problem still exists as  $c(\theta)$  and  $c(\tilde{\theta})$  in the acceptance probability do not cancel each other.

A way to bypass the evaluation of the intractable normalizing constant  $c(\theta)$  is to use the exchange algorithm (?; Murray et al., 2006) as follows.

**Algorithm 1** (Exchange Algorithm). *At each iteration:*

**Step 1** Draw  $\tilde{\theta}$  from the proposal distribution  $q_{\theta}(\tilde{\theta}|\theta)$ .

**Step 2** Generate  $(\tilde{g}, \tilde{Y})$  from the distribution  $\pi(\tilde{g}, \tilde{Y}|\tilde{\theta})$  using a perfect sampler.

**Step 3** Accept  $\tilde{\theta}$  according to the acceptance probability

$$\begin{aligned} \alpha_{\theta, EX} &= \min \left\{ 1, \frac{p(\tilde{\theta}|g, Y)q_{\theta}(\theta|\tilde{\theta})\pi(\tilde{g}, \tilde{Y}|\theta)}{p(\theta|g, Y)q_{\theta}(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta})} \right\} \\ &= \min \left\{ 1, \frac{\exp[\tilde{\sigma}^{-2}Q(g, Y|\tilde{\gamma})]p(\tilde{\theta})q_{\theta}(\theta|\tilde{\theta}) \exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma)]}{\exp[\sigma^{-2}Q(g, Y|\gamma)]p(\theta)q_{\theta}(\tilde{\theta}|\theta) \exp[\tilde{\sigma}^{-2}Q(\tilde{g}, \tilde{Y}|\tilde{\gamma})]} \right\}. \end{aligned} \tag{9}$$

The main advantage of the exchange algorithm is that the acceptance probability does not contain the normalizing constant  $c(\theta)$  and thus can be evaluated. The following proposition shows that the unique stationary distribution of the above described exchange algorithm is the posterior distribution  $p(\theta|g, Y) \propto \pi(g, Y|\theta)p(\theta)$ .

**Proposition 3.** *The unique stationary distribution of Algorithm 1 is  $p(\theta|g, Y)$ .*

### 3.2 Double Metropolis-Hastings Algorithm

In the second step of the exchange algorithm, we need to generate auxiliary data using a perfect sampler (Propp and Wilson, 1996), which is computationally costly for our model and, more generally, exponential random graph models (ERGMs) (Wasserman and Pattison, 1996). To overcome this issue, Liang (2010) and Mele (2017) propose a DMH algorithm, which uses a finite run of the MH algorithm initialized at the observed  $(g, Y)$  to generate auxiliary data  $(\tilde{g}, \tilde{Y})$ . More specifically, at each iteration, the DMH algorithm follows the same steps as the exchange algorithm with the second step replaced by:

**Step 2\*** Generate  $(\tilde{g}, \tilde{Y})$  from the distribution  $\pi(\tilde{g}, \tilde{Y}|\tilde{\theta})$  using a finite run of the MH algorithm initialized at the observed  $(g, Y)$ .

Compared with Mele (2017), one additional complication is that we need to simulate both networks  $\tilde{g}$  and effort choices  $\tilde{Y}$  in Step 2\* of the DMH algorithm. To generate auxiliary data  $(\tilde{g}, \tilde{Y})$ , one could use a sampler following the process described in Section 2.2. However, the convergence of such a sampler could be slow in practice. To improve convergence and reduce computational burden, we propose a MH algorithm to generate  $(\tilde{g}, \tilde{Y})$  as follows:

**Algorithm 2** (Auxiliary Data Generation). *Given  $\theta$ , at each iteration:*

**Step 1** Draw  $\tilde{g}$  from the proposal distribution  $q_g(\tilde{g}|g)$ . Let  $\tilde{G}$  denote the adjacency matrix of  $\tilde{g}$ .

**Step 2** Generate  $\tilde{Y} \sim N(\tilde{Y}^*, \Sigma_{\tilde{Y}})$ , where  $\tilde{Y}^* \equiv (I_n - \lambda\tilde{G})^{-1}B(X)$ , with  $B(X) = [b(X_1), \dots, b(X_n)]'$ , is the equilibrium effort vector derived from the best response function (3), and  $\Sigma_{\tilde{Y}} = \sigma^2(I_n - \lambda\tilde{G})^{-1}$ .

**Step 3** Accept  $(\tilde{g}, \tilde{Y})$  according to the acceptance probability

$$\begin{aligned} \alpha_{(g,Y),MH} &= \min \left\{ 1, \frac{\pi(\tilde{g}, \tilde{Y}|\theta)p_Y(Y|g)q_g(g|\tilde{g})}{\pi(g, Y|\theta)p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g)} \right\} \\ &= \min \left\{ 1, \frac{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma)]p_Y(Y|g)q_g(g|\tilde{g})}{\exp[\sigma^{-2}Q(g, Y|\gamma)]p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g)} \right\}, \end{aligned}$$

where  $p_Y(\tilde{Y}|\tilde{g})$  denotes the density function of  $N(\tilde{Y}^*, \Sigma_{\tilde{Y}})$ .

In the following proposition, we show that the long run stationary distribution of the proposed MH algorithm is the Gibbs measure defined in Equation (8).

**Proposition 4.** *The unique stationary distribution of Algorithm 2 is  $\pi(g, Y|\theta)$ .*

**Remark 2.** If Step 1 of Algorithm 2 adopts a local sampler, where only one randomly selected link is updated at each iteration, the convergence can be slow as shown in Mele (2017). Therefore, we follow Mele's suggestion (see Appendix B of Mele, 2017) to allow for large steps, where multiple links are swapped at the same time, to improve convergence.

**Remark 3.** In Step 2 of Algorithm 2, we generate  $\tilde{Y}$  from a multivariate normal distribution because (i) it is computationally simple to sample from a normal distribution, and (ii) it resembles the effort adjustment process described in Section 2.2. To see the second point, we assume that link adjustment periods arrive much less frequent than effort adjustment periods (i.e.,  $\rho_0$  is very small) in the coevolution process. Given the network  $\tilde{g}$ , it follows a standard Gibbs sampler argument that the transition density defined in Equation (7) converges to

$$p_Y(\tilde{Y}|\tilde{g}) = \frac{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y})]}{\int_{\mathcal{Y}^n} \exp[\sigma^{-2}Q(\tilde{g}, Y)]dY}. \quad (10)$$

where

$$\begin{aligned} Q(g, Y) &= a(g) + \sum_{i \in \mathcal{N}} b(X_i)y_i + \frac{\lambda}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} g_{ij}y_i y_j - \frac{1}{2} \sum_{i \in \mathcal{N}} y_i^2 \\ &= a(g) + B(X)'Y - \frac{1}{2}Y'(I_n - \lambda G)Y. \end{aligned} \quad (11)$$

Inserting Equation (11) into Equation (10), it follows by the Gaussian integral formula (Bronshtein et al., 2015) that

$$\begin{aligned} p_Y(\tilde{Y}|\tilde{g}) &= \frac{\exp[\sigma^{-2}B(X)'\tilde{Y} - \frac{1}{2}\sigma^{-2}\tilde{Y}'(I_n - \lambda\tilde{G})\tilde{Y}]}{\int_{\mathcal{Y}^n} \exp[\sigma^{-2}B(X)'Y - \frac{1}{2}\sigma^{-2}Y'(I_n - \lambda\tilde{G})Y]dY} \\ &= (2\pi)^{-n/2} |\det \Sigma_{\tilde{Y}}|^{-1/2} \exp[-\frac{1}{2}(\tilde{Y} - \tilde{Y}^*)'\Sigma_{\tilde{Y}}^{-1}(\tilde{Y} - \tilde{Y}^*)] \end{aligned}$$



which is the density function of  $N(\tilde{Y}^*, \Sigma_{\tilde{Y}})$ .

**Remark 4.** In Algorithm 2, we often need to evaluate  $(I_n - \lambda\tilde{G})^{-1}$  and  $\det(I_n - \lambda\tilde{G})$ , where  $\tilde{G}$  is the adjacency matrix of the network  $\tilde{g}$  resulting from adding/removing a link to/from the network  $g$ . The computational cost of the inverse and determinant can be high when the network size is large. To alleviate the computational burden, we adopt a matrix perturbation technique detailed in Appendix B.2, and derive a result that facilitates the computation of  $(I_n - \lambda\tilde{G})^{-1}$  and  $\det(I_n - \lambda\tilde{G})$  when  $(I_n - \lambda G)^{-1}$  and  $\det(I_n - \lambda G)$  are known.

### 3.3 Unobserved Heterogeneity

The structural model introduced in Section 2 allows some component of exogenous characteristics  $X_i$  to be unobservable to the econometrician. More specifically, let  $X_i = [X_i^O, x_i^U]$ , where  $X_i^O = (x_{i1}^O, \dots, x_{iK}^O)'$  is a  $K$ -dimensional vector of exogenous characteristics observable to the econometrician and  $x_i^U \sim i.i.d.(0, \varsigma_x^2)$  is a scalar random variable capturing *unobserved heterogeneity*. Further, let  $Z_{ij}$  be a vector of dyad-specific exogenous characteristics based on  $X_i^O$  and  $X_j^O$ . For example, one could define the  $l$ th element of  $Z_{ij}$  as  $z_{ij,l} = |x_{il}^O - x_{jl}^O|$  if  $x_{il}^O$  is a continuous variable or  $z_{ij,l} = 1(x_{il}^O = x_{jl}^O)$  if  $x_{il}^O$  is a binary indicator variable. In the empirical model, we consider the following specification in the same spirit as Auerbach (2019) and Johnsson and Moon (2019), where  $b(X_i)$  in Equation (1) is given by

$$b(X_i) = \beta_0 + X_i^O \beta_1 + \beta_2 x_i^U \tag{12}$$

and  $h(X_i, X_j, \delta_1)$  in Equation (2) is given by

$$h(X_i, X_j, \delta_1) = Z_{ij} \delta_1 + x_i^U + x_j^U. \tag{13}$$

As the unobserved heterogeneity influences both the effort and link choices, failing to control for it would bias the estimation result.

We regard  $x^U = (x_1^U, \dots, x_n^U)'$  as individual random effects with a density function denoted by

$p(x^U)$ . Instead of sampling  $\theta$  from the marginal posterior distribution

$$p(\theta|g, Y) = \int p(\theta|g, Y, x^U)p_x(x^U)dx^U,$$

which does not have a closed form expression, we adopt the Bayesian data augmentation approach (Tanner and Wong, 1987; Albert and Chib, 1993) to sample  $\theta$  together with  $x^U$  from the joint posterior distribution  $p(\theta, x^U|g, Y) \propto \pi(g, Y|\theta, x^U)p(\theta)p(x^U)$  in the MCMC procedure. The details of the MCMC procedure can be found in Appendix B.

### 3.4 Monte Carlo Experiments

We conduct a Monte Carlo simulation with 100 repetitions to examine the performance of the proposed MCMC procedure. In each repetition, we generate a network  $g$  with the corresponding effort levels  $Y$  of  $n = 100$  individuals according to the Gibbs measure defined in Equation (8). The detailed data generating process (DGP) runs as follows. We first generate exogenous individual characteristics  $x_i^O$  and  $x_i^U$  in  $b(X_i)$  of Equation (12) from log-normal distribution,  $\ln x_i^O \sim N(1.5, 0.5)$ , and normal distribution  $x_i^U \sim N(0, 1)$ , respectively, with the coefficients  $(\beta_1, \beta_2) = (1, 1)$ . The dyad variable  $Z_{ij}$  in  $h(X_i, X_j, \delta_1)$  of Equation (13) is generated by  $Z_{ij} = |x_i^O - x_j^O|$  using the variable  $x_i^O$  previously produced. We set  $\delta_0 = -2.5$  and  $\delta_1 = -0.5$ . The popularity, congestion, and cyclic triangle effects are measured by the coefficients  $(\delta_2, \delta_3, \delta_4) = (0.5, -0.05, 0.5)$ . The spillover effect  $\lambda$  is set to 0.01, which is comparable to the empirical estimate in Section 4. Finally, we set the noise parameter  $\sigma^2$  to 0.5. Given the above generated variables, we generate the network and effort levels by Algorithm 2 in Section 3.2 with one million iterations and treat the realization of the last iteration as the generated sample. On average, the generated network has the average degree equals to 2.689, the density equals to 0.027, and the clustering coefficient equals to 0.081. The average effort level is 4.95.

To show the potential bias problem that may result from ignoring unobserved heterogeneity in the model, we perform the MCMC procedure with and without controlling for unobserved heterogeneity. We run MCMC for 20,000 iterations and drop the first 10,000 draws for burn-in and

Table 1: Monte Carlo simulation results.

	True value	Model 1		Model 2	
$\lambda$	0.0100	0.0238	(0.0049)	0.0114	(0.0024)
$\beta_1$	1.0000	0.9372	(0.0204)	0.9813	(0.0166)
$\beta_2$	1.0000			0.9671	(0.1636)
$\delta_0$	-2.5000	-3.2906	(1.2907)	-2.5169	(0.0885)
$\delta_1$	-0.5000	-1.0792	(0.2661)	-0.5145	(0.0561)
$\delta_2$	0.5000	0.5126	(0.3848)	0.4678	(0.0531)
$\delta_3$	-0.0500	-0.0385	(0.0325)	-0.0501	(0.0064)
$\delta_4$	0.5000	0.7484	(0.3692)	0.5000	(0.0582)
$\sigma^2$	0.5000	0.8843	(0.1667)	0.5106	(0.0399)
$\zeta_x^2$	1.0000			1.0893	(0.3244)

*Notes:* Model 1 ignores unobserved heterogeneity and Model 2 controls for unobserved heterogeneity. Standard deviations in parentheses.

use the rest draws for computing the posterior mean of each parameter. The simulation results are reported in Table 1 and the values reported are the mean and standard deviation of the 100 repetitions.

From the estimation results of Model 1 reported in the left panel of Table 1, we can see that the spillover effect parameter  $\lambda$  is overestimated by 238% when unobserved heterogeneity is ignored. The estimates of other parameters are also affected, for example, the estimate of the cyclic triangle effect  $\delta_4$  is upward biased by 50% and the noise parameter  $\sigma$  is also upward biased by 77%. The fixed cost of link formation  $\delta_0$  and the homophily effect  $\delta_1$  are downward biased by 32% and 116%, respectively. These numbers reveal that neglecting unobserved heterogeneity could cause severe biases in the estimation. On the other hand, the estimation results of Model 2 reported in the right panel of Table 1 show that the proposed MCMC estimation procedure can successfully recover the true model parameters under the correct model specification that takes unobserved heterogeneity into account.

## 4 Empirical Illustration

### 4.1 A Simple Model of R&D Collaboration

To illustrate the empirical relevance of our model and estimation strategy, we consider a simple model of R&D collaboration among a set of firms  $\mathcal{N} = \{1, \dots, n\}$  with their characteristics given by

vectors  $X_i$ . Firms can reduce their production costs by investing in R&D as well as by benefiting from an R&D collaboration with another firm. The amount of cost reduction depends on the R&D effort  $y_i$  of firm  $i$  and the R&D efforts of firm  $i$ 's collaboration partners. The marginal *production cost*  $c_i$  of firm  $i$  is given by

$$c_i = -b_1(X_i) - y_i - \lambda \sum_{j=1}^n g_{ij} y_j, \quad (14)$$

where  $b_1(X_i)$  captures firm heterogeneity with regard to productivity and  $g_{ij}$  indicates whether firms  $i$  and  $j$  have an R&D collaboration. The parameter  $\lambda$  captures the *knowledge spillover* effect. We assume that the cost of R&D effort is given by  $\frac{1}{2}y_i^2$ . We further assume it is costly to maintain R&D collaborations with the *collaboration cost*  $-a_i(g)$ . With output  $q_i$ , firm  $i$ 's profit is given by

$$\Pi_i = (p_i - c_i)q_i - \frac{1}{2}y_i^2 + a_i(g), \quad (15)$$

where  $p_i$  is the price of the good produced by firm  $i$ . We assume firms are local monopolies with the inverse demand function  $p_i = b_0 - q_i$ , where  $b_0$  represents the market size. Substitution of the inverse demand function and Equation (14) into Equation (15) yields

$$\Pi_i = [b_0 - q_i + b_1(X_i) + y_i + \lambda \sum_{j=1}^n g_{ij} y_j]q_i - \frac{1}{2}y_i^2 + a_i(g). \quad (16)$$

Profit maximization with respect to  $y_i$  gives  $q_i = y_i$ . Substitution of  $q_i = y_i$  into Equation (16) gives

$$\Pi_i = a_i(g) + b(X_i)y_i + \lambda \sum_{j=1}^n g_{ij} y_i y_j - \frac{1}{2}y_i^2, \quad (17)$$

where  $b(X_i) = b_0 + b_1(X_i)$ . Equation (17) conforms to the general linear quadratic payoff function defined in Equation (1). In the empirical study, we assume that  $a_i(g)$  is given by Equation (2) with  $h(X_i, X_j, \delta_1)$  defined in Equation (13) and that  $b(X_i)$  is given by Equation (12).

## 4.2 Data

In the empirical illustration, we focus on the sector “Chemicals and Allied Products” (with two-digit SIC code 28), as it is one of the most active sectors regarding R&D collaborations. Our data of interfirm R&D collaborations stems from two sources which have been widely used in the literature (Schilling, 2009). The first is the Cooperative Agreements and Technology Indicators (CATI) database (Hagedoorn, 2002). The database only records agreements for which a combined innovative activity or an exchange of technology is at least part of the agreement. The second is the Thomson Securities Data Company (SDC) alliance database. SDC collects data from the U.S. Securities and Exchange Commission (and their international counterparts) filings, trade publications, wires, and news sources. We include only alliances from SDC which are classified explicitly as R&D collaborations.<sup>6</sup> We then merged the CATI database with the Thomson SDC alliance database. For the matching of firms across datasets we adopted and extended the name matching algorithm developed as part of the NBER patent data project (Trajtenberg et al., 2009).<sup>7</sup> The systematic collection of inter-firm alliances in CATI started in 1987 and ended in 2006. We take 2006 as the base year and assume that an alliance lasts for 5 years (Rosenkopf and Padula, 2008). We construct the R&D collaboration network by coding  $g_{ij}$  as one if an alliance between firms  $i$  and  $j$  exists in 2006, and zero otherwise.

The combined CATI-SDC database only provides the names of the firms in an alliance. To obtain information about their balance sheets and income statements we matched the firms’ names in the CATI-SDC database with the firms’ names in Standard & Poor’s Compustat U.S. and Global Fundamentals databases, as well as Bureau van Dijk’s Orbis database (see, e.g., Bloom et al., 2013). For the purpose of matching firms across databases, we employ the above mentioned name matching algorithm. Compustat and Orbis databases only contain firms listed on the stock market, so they typically exclude small private firms. However, they should include most R&D intensive firms, as R&D is typically concentrated in publicly listed firms (Bloom et al., 2013).

We use a firm’s log-R&D expenditure to measure its R&D effort. Moreover, the firms’ pro-

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<sup>6</sup>For a comparison and summary of different alliance databases, including CATI and SDC, see Schilling (2009).

<sup>7</sup>See <https://sites.google.com/site/patentdatapoint>. We would like to thank Enghin Atalay and Ali Hortacsu for sharing their name matching algorithm with us.

Table 2: Descriptive statistics.

Sector	# of firms	log-R&D expenditures			productivity			# of R&D alliances		
		mean	min	max	mean	min	max	mean	min	max
SIC-28	347	9.6574	3.2109	15.2467	11.1018	5.0706	16.8160	0.8012	0	15
SIC-283	256	9.4861	3.2109	15.2467	10.8352	5.0706	16.8160	0.9297	0	15

*Notes:* R&D expenditure is measured by thousand U.S. dollars in 2006. A firm’s productivity is measured by its log-R&D capital stock (lagged by one year).

ductivities are measured by their log-R&D capital stocks (lagged by one year). As in Hall et al. (2000), Bloom et al. (2013) and König et al. (2018), the R&D capital stock is computed using a perpetual inventory method based on the firms’ R&D expenditures with a 15% depreciation rate. We drop firm observations with missing values on either R&D expenditure or R&D capital stock which results in a sample of 347 firms and 139 R&D alliances in the SIC-28 sector. The SIC-28 sector has eight sub-sectors coded with 3-digit SIC codes. Among them, the sub-sector “Drugs” (SIC-283) is the largest in our sample with 256 firms and 119 R&D alliances. Descriptive statistics of the sample are shown in Table 2.

### 4.3 Estimation Results

Assuming the observed R&D expenditures and collaborations follow the stationary distribution defined in Equation (8), we estimate the model parameters using the MCMC procedure described in Section 3. We run the MCMC algorithm for 35,000 iterations and drop the first 5,000 draws for burn-in and keep every 20th of the remaining draws to conduct the posterior analysis, i.e., compute the posterior mean (as a point estimate) and posterior variance for each parameter. To check the convergence of the MCMC algorithm, we provide the trace plot of draws for  $\lambda$  in Figure 1. The trace plot of MCMC draws for  $\lambda$  and its posterior distribution in the upper and middle panels show that the MCMC draws are stable and have good variations. The autocorrelation function (ACF) plotted in the bottom panel indicates that the correlation among draws decline gradually over iterations. The draws pass the convergence diagnostic test of Geweke (1992) with a p-value of 0.2216.<sup>8</sup>

<sup>8</sup>Geweke convergence diagnostic tests for an equal mean of the first 10% versus the last 50% of the draws. We also try different proportions (e.g., 30% versus 70%), and obtain similar results for the convergence of the MCMC algorithm.

Figure 1: Trace Plot for MCMC Draws of  $\lambda$

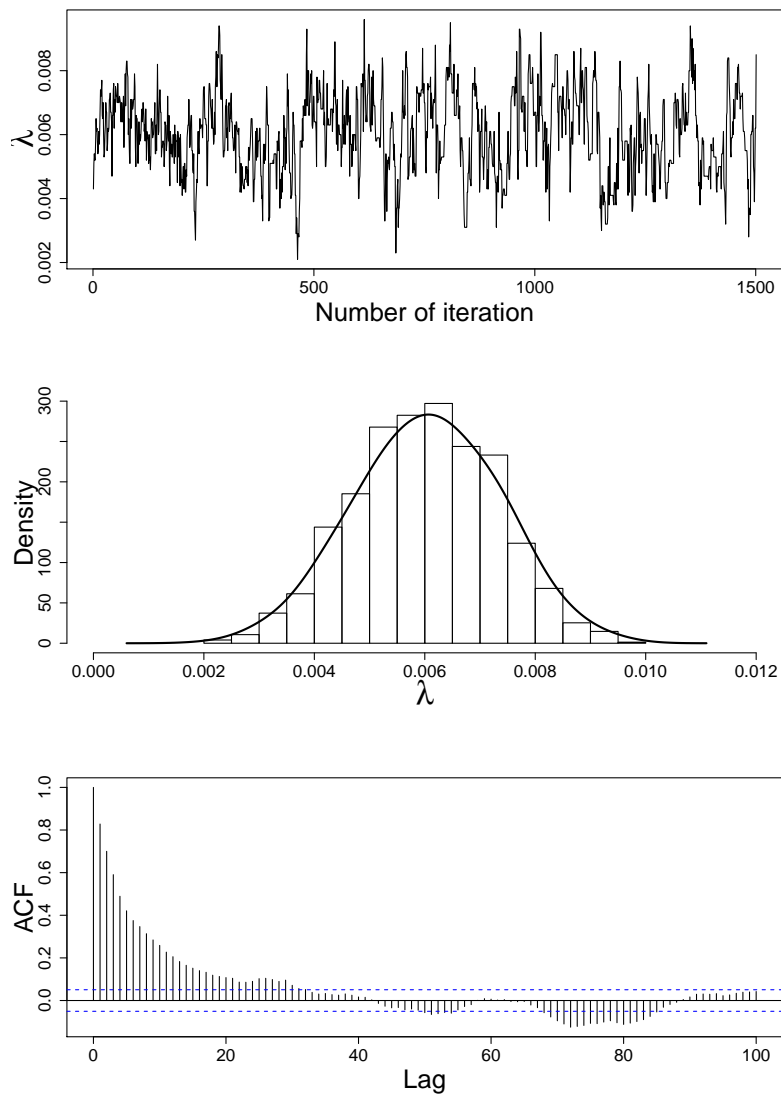


Table 3: Estimation results.

		Model 1		Model 2	
spillover effect	$(\lambda)$	0.0082	(0.0021)***	0.0060	(0.0013)***
<b>Production Cost</b>					
productivity	$(\beta_1)$	0.8662	(0.0192)***	0.8877	(0.0056)***
unobs. heterogeneity	$(\beta_2)$			0.0782	(0.0468)
sub-sector dummies		Yes		Yes	
<b>Collaboration Cost</b>					
constant	$(\delta_0)$	-4.7166	(0.0917)***	-5.1451	(0.1782)***
same sub-sector	$(\delta_{11})$	0.5322	(0.0275)***	0.8163	(0.1299)***
diff-in-productivity	$(\delta_{12})$	0.0708	(0.0192)***	0.0395	(0.0225)*
popularity	$(\delta_2)$	0.0011	(0.0318)	0.0608	(0.0349)*
congestion	$(\delta_3)$	-0.0036	(0.0014)**	-0.0056	(0.0015)***
cyclic triangle	$(\delta_4)$	0.4547	(0.0535)***	0.3531	(0.1088)***
<b>Noise Parameters</b>					
noise in decisions	$(\sigma^2)$	0.3835	(0.0228)***	0.4262	(0.0217)***
unobs. heterogeneity	$(\zeta_x^2)$			0.8609	(0.0760)***

*Notes:* Model 1 ignores unobserved heterogeneity and Model 2 controls for unobserved heterogeneity. Standard errors in parentheses. \*\*\*, \*\*, and \* indicate that the highest density range does not cover zero at 99%, 95%, and 90% levels.

The estimation results are reported in Table 3. Similar to what we observe in the Monte Carlo simulations in Section 3.4, the spillover effect is overestimated when unobserved heterogeneity is ignored. When unobserved heterogeneity is controlled for, the estimated spillover effect is weaker but remains positively significant.

To capture firm heterogeneity in the marginal production cost given by Equation (14), we include a productivity measure defined as a firm's one-year-lagged log-R&D capital stock and sub-sector dummies (defined at the 3-digit SIC level). As expected, the estimate of  $\beta_1$  shows that higher time-lagged R&D capital stock reduces the marginal production cost.

From the estimation of the collaboration cost  $-a_i(g)$  given by Equation (2), we find that the collaboration cost is lower between firms in the same sub-sector (reflected by  $\delta_{11}$ ; the *homophily* effect), firms with different productivities (reflected by  $\delta_{12}$ ; the *heterophily* effect) and firms with a common collaboration partner (reflected by  $\delta_4$ ; the *cyclic triangle* effect). Furthermore, firms benefit from forming an R&D collaboration with a firm having other collaboration partners (reflected by  $\delta_2$ ; the *popularity* effect), but the marginal benefit reduces as the number of other collaboration partners increases (reflected by  $\delta_3$ ; the *congestion* effect).



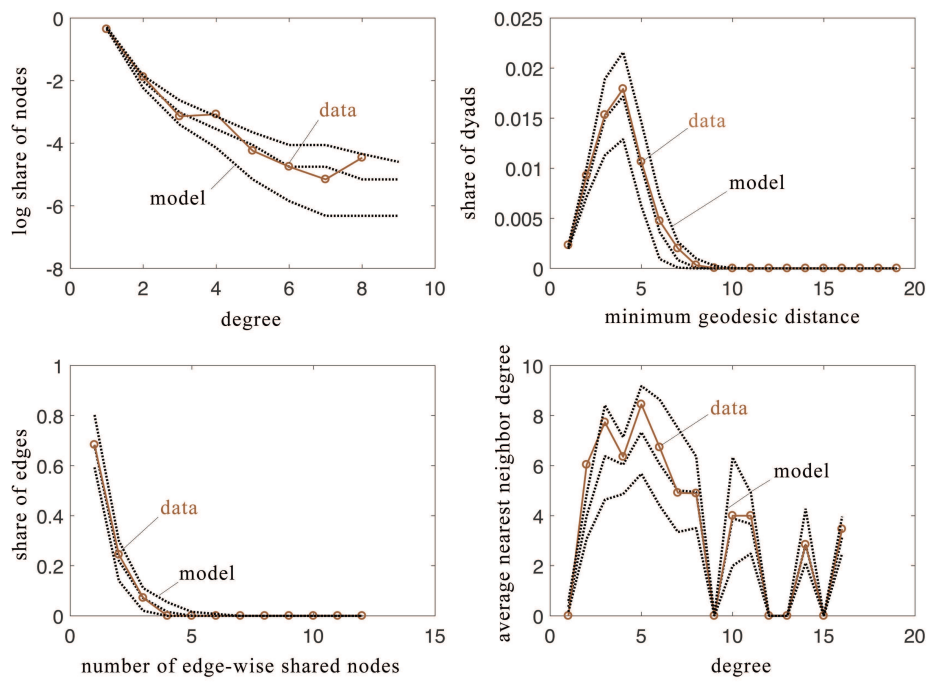
Finally, we evaluate the model’s goodness-of-fit following Hunter et al. (2008). We generate 100 networks from the Gibbs measure defined in Equation (8) with the estimates of Model 2 reported in the right panel of Table 3. The model’s goodness-of-fit is examined by comparing the 100 generated networks with the observed network in terms of four network statistics: the degree (the number of links of a firm), the minimum geodesic distance (the number of links in the shortest path between two firms), the number of edge-wise shared partners (the number of shared partners of two connected firms), and the average nearest neighbor connectivity (the average degree of the collaboration partners of a firm with a certain degree). Figure 2 shows the distributions of the four network statistics of the observed network (in solid lines) and the corresponding means and 95% confidence intervals of the 100 generated networks (in dashed lines). From the figure we find that the generated networks and the observed network display similar distributions over these four statistics. This shows that our estimated model is able to capture the underlying network generating process.

#### 4.4 Key Player Analysis

In this section we evaluate the expected total welfare loss from the exit of a firm from the network. The exit of a firm could either be due to financial reasons, such as the recession experienced by the American automobile manufacturing industry during the global financial downturn of 2007-2008, or legal reasons, such as the recent emission-fraud scandal of Volkswagen in 2015. In the former case, policy makers want to know the overall welfare gain of “bailing out” a bankrupting firm, while, in the latter case, policy makers want to know the overall welfare cost they impose on the economy by inflicting high penalties that might threaten the continued existence of a firm.

The firm whose exit results in the highest expected total welfare loss is termed the *key player* (Zenou, 2016). This counterfactual analysis is related to Ballester et al. (2006), who perform a key player analysis where agents are ranked according to the reduction in aggregate output when they are removed from the network, and König et al. (2018) who do this for the reduction in welfare similar to our setup. However, while these authors assume that the network is exogenously given and does not adapt to the exit of a firm, here we can relax this assumption and allow the

Figure 2: Goodness-of-fit Statistics



network to rewire to a new equilibrium after the exit of a firm. Formally, welfare is defined as  $W(g, Y) \equiv \sum_{i=1}^n U_i(g, Y)$  where  $U_i(g, Y)$  is utility (or, in this example, profit) given by Equation (1), and the key player is defined as  $i^* = \arg \max_{i \in \mathcal{N}} E[W(g_{-i}, Y_{-i})]$ , where  $g_{-i}$  and  $Y_{-i}$  denote the network and effort vector without agent  $i$  and the expectation is evaluated under the Gibbs measure  $\pi(g_{-i}, Y_{-i} | \theta)$  given by Equation (8).

We proceed by removing each firm from the network one at a time. Using the estimates of Model 2 reported in the right panel of Table 3, we then simulate the coevolution process of R&D investment and collaborations for the remaining  $n - 1$  firms using Algorithm 2 described in Section 3.2. We run the simulation for 10,000 iterations,<sup>9</sup> and use the observation of the last iteration to calculate the corresponding welfare value. We then repeat this procedure 200 times and report the average welfare loss.

The results for the key player analysis can be seen in Table 4. We report the welfare loss and key player ranking with and without allowing the network to rewire after the exit of a firm. One could interpret the key player analysis without network rewiring as a short-run analysis while the one with network rewiring as the long-run analysis. In general, the welfare loss is lower with network rewiring, because firms can mitigate the welfare loss from the exit of a firm by forming new links with the remaining firms. We also find that the key player rankings with and without network rewiring do not coincide with each other. Therefore, it is important to choose the right analysis (short run or long run) for policy evaluation needs. Finally, Table 4 shows that firms with the highest key player ranking are not necessarily the ones with the highest market share, largest R&D expenditure, or most R&D alliances. Rather, in order to identify the key player, we need to take every aspect of the structural model into account.

## 5 Conclusion

This paper proposes a structural model for the coevolution of networks and behavior. We provide a microfoundation for the model and characterize the equilibrium of the coevolution process. We

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<sup>9</sup>We also tried 15,000 and 20,000 iterations and get similar results.

Table 4: Key player ranking.

	SIC	mkt. share	R&D exp.	deg.	w/o network rewiring		with network rewiring	
					welfare loss	rank	welfare loss	rank
Johnson & Johnson Inc.	283	3.0547	15.1535	7	-0.5237	3	-0.3511	1
Wyeth	283	1.1686	14.2487	2	-0.5373	1	-0.3219	2
Schering-Plough Corp.	283	0.6057	13.8905	1	-0.5158	4	-0.3096	3
Bristol-Myers Squibb Co.	283	1.0287	14.2351	6	-0.5255	2	-0.2924	4
Pfizer Inc.	283	2.7679	15.2467	15	-0.4712	9	-0.2843	5
Unilever PLC	284	5.4914	13.2439	0	-0.5025	7	-0.2694	6
Abbott Laboratories Inc.	283	1.2907	14.5658	3	-0.5088	5	-0.2553	7
Merck & Co Inc.	283	1.2999	14.6794	10	-0.5046	6	-0.2515	8
Akzo Nobel NV	285	11.7496	13.2205	2	-0.4450	13	-0.2423	9
Bayer	280	3.8340	14.1742	10	-0.4562	11	-0.2361	10
Daiichi Sankyo Co. Ltd.	283	0.4590	13.4980	5	-0.4239	20	-0.2354	11
Elsai	283	0.3329	13.0432	1	-0.4673	10	-0.2297	12
L'Oreal SA	284	2.1873	12.7125	0	-0.4545	12	-0.2276	13
Novartis	283	2.0691	14.7913	15	-0.4715	8	-0.2190	14
Asahi Kasei Corp.	280	1.4715	12.3177	0	-0.4370	16	-0.2171	15
Merck KGaA	283	0.4515	13.0571	3	-0.4419	14	-0.2094	16
Henkel AG & Co. KGaA	284	1.7648	12.2638	0	-0.4243	19	-0.2022	17
Solvay SA	280	1.2445	12.7682	3	-0.3612	41	-0.1948	18
Kaocorp	284	1.1679	12.1513	0	-0.4272	18	-0.1948	19
Shionogi & Co. Ltd.	283	0.0986	11.9814	0	-0.4161	22	-0.1908	20
Chugai Pharma.	283	0.1610	12.3585	1	-0.4209	21	-0.1908	21
Takeda Pharma. Co. Ltd.	283	0.6445	13.6225	7	-0.4064	27	-0.1894	22
Amgen	283	0.8193	14.6398	13	-0.3903	33	-0.1852	23
Toray Industries Inc.	282	2.3563	12.1022	0	-0.4135	24	-0.1805	24
Syngenta AG	287	4.1430	12.8862	0	-0.4357	17	-0.1797	25
Colgate-Palmolive Co.	284	1.3493	11.6935	0	-0.3911	32	-0.1766	26
Monsanto Co.	287	3.7815	12.7928	0	-0.4376	15	-0.1710	27
Allergan Inc.	283	0.1759	13.1684	3	-0.3778	37	-0.1693	28
Human Genome Sci. Inc.	283	0.0015	11.5501	2	-0.3466	49	-0.1657	29
Mitsui Chemicals Inc.	282	2.5721	11.9677	0	-0.4015	28	-0.1649	30

*Notes:* The market share is defined as the percentage of a firm's sales in a sub-sector at the 3-digit SIC level. Welfare loss is measured in percentage.

show the model can be estimated using an MCMC algorithm and investigate the finite sample performance of the estimation procedure in a Monte Carlo simulation experiment. We then apply the model to study R&D investment and collaboration decisions in the chemicals and pharmaceutical industry and find a positive knowledge spillover effect. We also demonstrate how to use the model estimates to conduct a long-run key player analysis.

Due to the generality of the utility function we consider, we believe that our structural framework – from both theoretical and empirical perspectives – can be applied to a variety of related contexts, where externalities can be modelled in the form of an adaptive network. Examples include peer effects in education, crime, risk sharing, scientific coauthorship, etc. (Jackson and Zenou, 2014).

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## A Proofs

*Proof of Proposition 1.* First, we consider the case where the effort level of agent  $i$  is updated from  $y_0$  to  $y_1$ , while  $g$  and  $Y_{-i}$  remain unchanged. In this case,

$$\begin{aligned}
& Q(g, y_i = y_1, Y_{-i}, X) - Q(g, y_i = y_0, Y_{-i}, X) \\
&= b(X_i)(y_1 - y_0) + \lambda(y_1 - y_0) \sum_{j \in \mathcal{N}} g_{ij} y_j - \frac{1}{2}(y_1^2 - y_0^2) \\
&= U_i(g, y_i = y_1, Y_{-i}, X) - U_i(g, y_i = y_0, Y_{-i}, X).
\end{aligned}$$

Next, we consider the case where the network is updated from  $g_0 = \{g_{ij} = 0, g_{-ij}\}$  to  $g_1 = \{g_{ij} = 1, g_{-ij}\}$ , while  $Y$  remains unchanged. In this case,

$$Q(g_1, Y, X) - Q(g_0, Y, X) = a(g_1, X) - a(g_0, X) + \lambda y_i y_j.$$

As

$$\begin{aligned}
a(g_1, X) - a(g_0, X) &= \delta_0 + \delta_1 h_{ij} + \delta_2 d_{ij} + \delta_3 d_{ij}^2 + \delta_4 \sum_{k \in \mathcal{N} \setminus \{i, j\}} g_{ik} g_{jk} \\
&= a_i(g_1, X) - a_i(g_0, X),
\end{aligned}$$

we have

$$Q(g_1, Y, X) - Q(g_0, Y, X) = U_i(g_1, Y, X) - U_i(g_0, Y, X).$$

The desired result follows. ■

*Proof of Proposition 2.* The sequence  $\{(g_t, Y_t)\}$  is a Markov chain. A sufficient condition for the stationarity of the Gibbs measure given in Equation (8) is the detailed balance condition, i.e.,  $\pi(\omega_0)p(\omega_0, \omega_1) = \pi(\omega_1)p(\omega_1, \omega_0)$ , where  $p(\omega_0, \omega_1)$  is the transition density from state  $\omega_0$  to state  $\omega_1$ . Here, we only need to verify the detailed balance condition for (i)  $\omega_0$  and  $\omega_1$  differ by only one element of  $g$  and (ii)  $\omega_0$  and  $\omega_1$  differ by only one element of  $Y$ , since the transition density for other cases is zero.

(i)  $\omega_0$  and  $\omega_1$  differ by only one element of  $g$ . Let  $\omega_0 = (g_{ij} = 0, g_{-ij}, Y)$  and  $\omega_1 = (g_{ij} = 1, g_{-ij}, Y)$ . Then,

$$\begin{aligned}
& \pi(\omega_0)p(\omega_0, \omega_1) \\
= & \pi(\omega_0)\rho_0\rho(g_{t-1}, X_i, X_j) \frac{\exp[\sigma^{-2}Q(g_{ij} = 1, g_{-ij}, Y)]}{\exp[\sigma^{-2}Q(g_{ij} = 0, g_{-ij}, Y)] + \exp[\sigma^{-2}Q(g_{ij} = 1, g_{-ij}, Y)]} \\
= & \pi(\omega_1)\rho_0\rho(g_{t-1}, X_i, X_j) \frac{\exp[\sigma^{-2}Q(g_{ij} = 0, g_{-ij}, Y)]}{\exp[\sigma^{-2}Q(g_{ij} = 0, g_{-ij}, Y)] + \exp[\sigma^{-2}Q(g_{ij} = 1, g_{-ij}, Y)]} \\
= & \pi(\omega_1)p(\omega_1, \omega_0).
\end{aligned}$$

(ii)  $\omega_0$  and  $\omega_1$  differ by only one element of  $Y$ . Let  $\omega_0 = (g, y_0, Y_{-i})$  and  $\omega_1 = (g, y_1, Y_{-i})$ . Then,

$$\begin{aligned}
& \pi(\omega_0)p(\omega_0, \omega_1) \\
= & \pi(\omega_0)(1 - \rho_0)\rho(X_i) \frac{\exp[\sigma^{-2}U_i(g, y_1, Y_{-i})]}{\int_{\mathcal{Y}} \exp[\sigma^{-2}U_i(g, y, Y_{-i, t-1})]dy} \\
= & \pi(\omega_0)(1 - \rho_0)\rho(X_i) \frac{\exp[\sigma^{-2}Q(g, y_1, Y_{-i})]}{\int_{\mathcal{Y}} \exp[\sigma^{-2}Q(g, y, Y_{-i, t-1})]dy} \\
= & \pi(\omega_1)(1 - \rho_0)\rho(X_i) \frac{\exp[\sigma^{-2}Q(g, y_0, Y_{-i})]}{\int_{\mathcal{Y}} \exp[\sigma^{-2}Q(g, y, Y_{-i, t-1})]dy} \\
= & \pi(\omega_1)(1 - \rho_0)\rho(X_i) \frac{\exp[\sigma^{-2}U_i(g, y_0, Y_{-i})]}{\int_{\mathcal{Y}} \exp[\sigma^{-2}U_i(g, y, Y_{-i, t-1})]dy} \\
= & \pi(\omega_1)p(\omega_1, \omega_0).
\end{aligned}$$

The desired result follows by the reversibility, irreducibility, and Harris recurrence of the Markov chain. ■

*Proof of Proposition 3.* To show  $p(\theta|g, Y)$  is the stationary distribution, we need to check the detailed balance condition, i.e.,  $p(\theta|g, Y)p(\tilde{\theta}|\theta) = p(\tilde{\theta}|g, Y)p(\theta|\tilde{\theta})$  where

$$p(\tilde{\theta}|\theta) = q_\theta(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta}) \min \left\{ 1, \frac{p(\tilde{\theta}|g, Y)q_\theta(\theta|\tilde{\theta})\pi(\tilde{g}, \tilde{Y}|\theta)}{p(\theta|g, Y)q_\theta(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta})} \right\}.$$

Indeed,

$$\begin{aligned}
p(\theta|g, Y)p(\tilde{\theta}|\theta) &= p(\theta|g, Y)q_\theta(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta}) \min \left\{ 1, \frac{p(\tilde{\theta}|g, Y)q_\theta(\theta|\tilde{\theta})\pi(\tilde{g}, \tilde{Y}|\theta)}{p(\theta|g, Y)q_\theta(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta})} \right\} \\
&= \min \left\{ p(\theta|g, Y)q_\theta(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta}), p(\tilde{\theta}|g, Y)q_\theta(\theta|\tilde{\theta})\pi(\tilde{g}, \tilde{Y}|\theta) \right\} \\
&= \min \left\{ \frac{p(\theta|g, Y)q_\theta(\tilde{\theta}|\theta)\pi(\tilde{g}, \tilde{Y}|\tilde{\theta})}{p(\tilde{\theta}|g, Y)q_\theta(\theta|\tilde{\theta})\pi(\tilde{g}, \tilde{Y}|\theta)}, 1 \right\} p(\tilde{\theta}|g, Y)q_\theta(\theta|\tilde{\theta})\pi(\tilde{g}, \tilde{Y}|\theta) \\
&= p(\tilde{\theta}|g, Y)p(\theta|\tilde{\theta}).
\end{aligned}$$

The desired result follows by the reversibility, irreducibility, and Harris recurrence of the Markov chain. ■

*Proof of Proposition 4.* To show  $\pi(g, Y|\theta)$  is the stationary distribution, we need to check the detailed balance condition, i.e.,  $\pi(g, Y|\theta)p(\tilde{g}, \tilde{Y}|g, Y) = \pi(\tilde{g}, \tilde{Y}|\theta)p(g, Y|\tilde{g}, \tilde{Y})$  where

$$p(\tilde{g}, \tilde{Y}|g, Y) = p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g) \min \left\{ 1, \frac{\pi(\tilde{g}, \tilde{Y}|\theta)p_Y(Y|g)q_g(g|\tilde{g})}{\pi(g, Y|\theta)p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g)} \right\}.$$

Indeed,

$$\begin{aligned}
&\pi(g, Y|\theta)p(\tilde{g}, \tilde{Y}|g, Y) \\
&= c(\theta)^{-1} \exp[\sigma^{-2}Q(g, Y|\gamma)]p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g) \min \left\{ 1, \frac{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma)]p_Y(Y|g)q_g(g|\tilde{g})}{\exp[\sigma^{-2}Q(g, Y|\gamma)]p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g)} \right\} \\
&= c(\theta)^{-1} \min \left\{ \exp[\sigma^{-2}Q(g, Y|\gamma)]p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g), \exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma)]p_Y(Y|g)q_g(g|\tilde{g}) \right\} \\
&= \min \left\{ \frac{\exp[\sigma^{-2}Q(g, Y|\gamma)]p_Y(\tilde{Y}|\tilde{g})q_g(\tilde{g}|g)}{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma)]p_Y(Y|g)q_g(g|\tilde{g})}, 1 \right\} c(\theta)^{-1} \exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma)]p_Y(Y|g)q_g(g|\tilde{g}) \\
&= \pi(\tilde{g}, \tilde{Y}|\theta)p(g, Y|\tilde{g}, \tilde{Y}).
\end{aligned}$$

The desired result follows by the reversibility, irreducibility, and Harris recurrence of the Markov chain. ■

## B Implementation Details

### B.1 MCMC

In our empirical study of R&D collaboration networks, we want to estimate the spillover effect parameter  $\lambda$ , parameters in the marginal cost of production  $\beta = (\beta_0, \beta_1', \beta_2)'$  (with the dimension denoted by  $K$ ), parameters in the collaboration cost  $\delta = (\delta_0, \delta_1', \delta_2, \delta_3, \delta_4)'$  (with the dimension denoted by  $S$ ), and the noise parameter  $\sigma^2$ . These parameters are denoted by  $\theta = (\lambda, \beta', \delta', \sigma^2)'$ . Other than  $\theta$ , there are also unobservable individual-specific random variables  $x^U = (x_1^U, \dots, x_n^U)'$ , which are regarded as individual random effects, in the model. We assign the prior distributions of model parameters and unknown variables as follows:

1. Individual latent variable:  $x^U \sim N(0, \varsigma_x^2 I_n)$ , with  $\varsigma_x^2 \sim \kappa \text{Inv}\chi^2(\alpha)$ .
2. Spillover effect parameter:  $\lambda \sim U(-\|G\|_\infty^{-1}, \|G\|_\infty^{-1})$ .
3. Parameters in the marginal cost of production:  $\beta \sim N(\mu_\beta, \varsigma_\beta^2 I_K)$ .
4. Parameters in the collaboration cost:  $\delta \sim N(\mu_\delta, \varsigma_\delta^2 I_S)$ .
5. Noise parameter:  $\sigma^2 \sim N_{[0, \infty)}(\mu_\sigma, \varsigma_\sigma^2)$ .

The above prior distributions are conjugate priors commonly used in the Bayesian literature. First, treating  $x_i^U$  as an individual random effect, we specify a hierarchical prior for  $x_i^U$  with a prior for the variance given by  $\varsigma_x^2 \sim \kappa \text{Inv}\chi^2(\alpha)$ . The hyper-parameters  $\kappa$  and  $\alpha$  are to be specified by the user. The spillover effect parameter  $\lambda$  shares similar properties as the spatial lag parameter in the spatial econometrics literature and we use a uniform prior for  $\lambda$  following Smith and LeSage (2004) and assume  $\lambda \in (-\|G\|_\infty^{-1}, \|G\|_\infty^{-1})$  to guarantee that the best response function (3) has a unique equilibrium. Finally, to guarantee that  $\sigma^2$  is non-negative, we assume it follows a truncated normal distribution on  $[0, \infty)$ . We also assume independence across prior distributions of parameters and latent variables. We set  $\mu_\beta = 0$ ,  $\mu_\delta = 0$ ,  $\mu_\sigma = 0$ ,  $\varsigma_\beta^2 = \varsigma_\delta^2 = \varsigma_\sigma^2 = 100$ ,  $\kappa = 1$  and  $\alpha = 2$  to ensure our prior distributions cover a wide range of parameter spaces and thus be uninformative in our empirical analysis.

We adopt the Bayesian data augmentation approach (Tanner and Wong, 1987; Albert and Chib, 1993) to sample  $\theta$  together with  $x^U$  from the joint posterior distribution by the MCMC procedure. In an iteration of the MCMC procedure with the current values of the parameters and individual latent variables denoted by  $\theta$  and  $x^U$ , we perform the following steps sequentially. Let  $x_{-i}^U = (x_1^U, \dots, x_{i-1}^U, x_{i+1}^U, \dots, x_n^U)'$ .

Step I. Simulate  $\tilde{x}_i^U$  from  $p(\tilde{x}_i^U | g, Y, x_{-i}^U, \theta)$  by the DMH algorithm, for  $i = 1, \dots, n$ .

I.1. Propose  $\tilde{x}_i^U$  from a random walk proposal density  $q_x(\tilde{x}_i^U | x_i^U)$ .

I.2. Simulate auxiliary data  $(\tilde{g}, \tilde{Y})$  by  $R$  runs of Algorithm 2 defined in Section 3.2, starting from the observed  $(g, Y)$ . In the first step of Algorithm 2, we allow for both a “local update” (with probability  $1 - p_{global}$ ), where only one link is flipped, and a “global update” (with probability  $p_{global}$ ), where all links are flipped.<sup>10</sup>

I.3. Accept  $\tilde{x}_i^U$  according to the acceptance probability

$$\begin{aligned} \alpha_x &= \min \left\{ 1, \frac{\pi(g, Y | \theta, \tilde{x}^U)}{\pi(g, Y | \theta, x^U)} \cdot \frac{p(\tilde{x}_i^U | \zeta_x^2)}{p(x_i^U | \zeta_x^2)} \cdot \frac{\pi(\tilde{g}, \tilde{Y} | \theta, x^U)}{\pi(\tilde{g}, \tilde{Y} | \theta, \tilde{x}^U)} \right\} \\ &= \min \left\{ 1, \frac{\exp[\sigma^{-2}Q(g, Y | \gamma, \tilde{x}^U)]}{\exp[\sigma^{-2}Q(g, Y | \gamma, x^U)]} \cdot \frac{p(\tilde{x}_i^U | \zeta_x^2)}{p(x_i^U | \zeta_x^2)} \cdot \frac{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y} | \gamma, x^U)]}{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y} | \gamma, \tilde{x}^U)]} \right\}, \end{aligned}$$

where  $p(x_i^U | \zeta_x^2)$  denotes the density function of  $N(0, \zeta_x^2 I_n)$ . Otherwise, set  $\tilde{x}_i^U = x_i^U$ .

Step II. Simulate  $\zeta_x^2$  from  $[\kappa + \sum_{i=1}^n (\tilde{x}_i^U)^2] \text{Inv}\chi^2(\alpha + n)$  by a standard Gibbs sampler.

Step III. Simulate  $\tilde{\theta}$  from  $p(\tilde{\theta} | g, Y, \tilde{x}^U)$  by the DMH algorithm.

III.1. Propose  $\tilde{\theta}$  from a random walk proposal density  $q_\theta(\tilde{\theta} | \theta)$ .

III.2. Simulate auxiliary data  $(\tilde{g}, \tilde{Y})$  by  $R$  runs of Algorithm 2 defined in Section 3.2, starting from the observed  $(g, Y)$  and allowing for both local and global updates as described in Step I.2.

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<sup>10</sup>Similar local and global updates are suggested in Snijders (2002) and Mele (2017) to improve the convergence of graph sampling, particularly when the graph distribution exhibits a bimodal shape, one mode having low and the other high graph densities. In the simulation and empirical studies, we set the probability of global update  $p_{global} = 0.01$ .

III.3. Accept  $\tilde{\theta}$  according to the acceptance probability

$$\begin{aligned}\alpha_\theta &= \min \left\{ 1, \frac{\pi(g, Y|\tilde{\theta}, \tilde{x}^U)}{\pi(g, Y|\theta, \tilde{x}^U)} \cdot \frac{p(\tilde{\theta})}{p(\theta)} \cdot \frac{\pi(\tilde{g}, \tilde{Y}|\theta, \tilde{x}^U)}{\pi(\tilde{g}, \tilde{Y}|\tilde{\theta}, \tilde{x}^U)} \right\} \\ &= \min \left\{ 1, \frac{\exp[\tilde{\sigma}^{-2}Q(g, Y|\tilde{\gamma}, \tilde{x}^U)]}{\exp[\sigma^{-2}Q(g, Y|\gamma, \tilde{x}^U)]} \cdot \frac{p(\tilde{\theta})}{p(\theta)} \cdot \frac{\exp[\sigma^{-2}Q(\tilde{g}, \tilde{Y}|\gamma, \tilde{x}^U)]}{\exp[\tilde{\sigma}^{-2}Q(\tilde{g}, \tilde{Y}|\tilde{\gamma}, \tilde{x}^U)]} \right\},\end{aligned}$$

where  $\tilde{x}^U = (\tilde{x}_1^U, \dots, \tilde{x}_n^U)'$ . Otherwise, set  $\tilde{\theta} = \theta$ .

## B.2 Matrix Perturbation

In the MCMC algorithm, we need to evaluate  $(I_n - \lambda G)^{-1}$  and  $\det(I_n - \lambda G)$  whenever a link is added or removed in the network  $g$ . The following lemma is helpful for this purpose.

**Lemma 1.** *Let  $e_i$  be the  $i$ th unit basis vector in  $\mathbb{R}^n$ . Let  $A$  denote an  $n \times n$  matrix and*

$$B_{ij} = \frac{A^{-1}e_i e_j' A^{-1}}{1 + \alpha e_j' A^{-1} e_i}.$$

*Adding a perturbation  $\alpha$  to the matrix  $A$  in the  $(i, j)$ th and the  $(j, i)$ th position can be written as  $A + \alpha e_i e_j' + \rho e_j e_i'$ .*

(i) *The inverse of the perturbed matrix can be written as*

$$(A + \alpha e_i e_j' + \alpha e_j e_i')^{-1} = A^{-1} - \alpha B_{ij} - \alpha \frac{(A^{-1} - \alpha B_{ij})e_j e_i' (A^{-1} - \alpha B_{ij})}{1 + \alpha e_i' (A^{-1} - \alpha B_{ij})e_j}. \quad (18)$$

(ii) *The determinant of the perturbed matrix can be written as*

$$\det(A + \alpha e_i e_j' + \alpha e_j e_i') = [1 + \alpha e_i' (A^{-1} - \alpha B_{ij})e_j][1 + \alpha e_j' A^{-1} e_i] \det(A). \quad (19)$$

*Proof.* We first prove part (i) of Lemma 1. By the Sherman-Morrison formula (Meyer, 2000),

$$(A + \alpha e_i e_j')^{-1} = A^{-1} - \alpha \frac{A^{-1}e_i e_j' A^{-1}}{1 + \alpha e_j' A^{-1} e_i} = A^{-1} - \alpha B_{ij}.$$

Therefore,

$$(A + \alpha e_i e'_j + \alpha e_j e'_i)^{-1} = [(A^{-1} - \alpha B_{ij})^{-1} + \alpha e_j e'_i]^{-1} = A^{-1} - \alpha B_{ij} - \alpha \frac{(A^{-1} - \alpha B_{ij}) e_j e'_i (A^{-1} - \alpha B_{ij})}{1 + \alpha e'_i (A^{-1} - \alpha B_{ij}) e_j},$$

where the last equality holds by the Sherman-Morrison formula.

We next prove part (ii) of Lemma 1. By the matrix determinant lemma (Horn and Johnson, 1985),

$$\begin{aligned} \det(A + \alpha e_i e'_j + \alpha e_j e'_i) &= \det[(A + \alpha e_i e'_j) + \alpha e_j e'_i] \\ &= [1 + \alpha e'_i (A + \alpha e_i e'_j)^{-1} e_j] \det(A + \alpha e_i e'_j) \\ &= [1 + \alpha e'_i (A + \alpha e_i e'_j)^{-1} e_j] (1 + \alpha e'_j A^{-1} e_i) \det(A) \\ &= [1 + \alpha e'_i (A^{-1} - \alpha B_{ij}) e_j] (1 + \alpha e'_j A^{-1} e_i) \det(A), \end{aligned}$$

where the last equality holds by the Sherman-Morrison formula. ■

With Lemma 1 the inverse and determinant of the perturbed matrix  $A + \alpha e_i e'_j + \alpha e_j e'_i$  can be easily computed if the inverse and determinant of  $A$  are known.