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AMBIGUITY ATTITUDES, LEVERAGE CYCLE AND ASSET PRICES

Marzio Bassanin, Ester Faia and Valeria Patella

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Abstract

Financial crises often originate in debt markets, where collateral constraints and opacity of asset values generate intrinsic instability. In such ambiguous contexts endogenous beliefs formation plays a crucial role in explaining asset price and leverage cycles. We introduce state-contingent ambiguity attitudes embedding ambiguity aversion and seeking, which endogenously induces pessimism (left-skewed beliefs) in recessions and optimism (rightskewed beliefs) in booms, in a model where borrowers face occasionally binding collateral constraints. We use GMM estimation with latent value functions to estimate the ambiguity attitudes process. By simulating a crisis scenario in our model we show that optimism in booms is responsible for higher asset price and leverage growth and pessimism in recessions is responsible for sharper de-leveraging and asset price bursts. Analytically and numerically (using global methods) we show that our state-contingent ambiguity attitudes coupled with the collateral constraints can explain relevant asset price and debt cycle facts around the unfolding of a financial crisis.

JEL Classification: E0, E5, G01

Keywords: ambiguity attitudes, occasionally binding constraints, kinked multiplier preferences, leverage cycle, asset price cycle

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Ambiguity Attitudes, Leverage Cycle and Asset Prices*

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Abstract

Financial crises often originate in debt markets, where collateral constraints and opacity of asset values generate intrinsic instability. In such ambiguous contexts endogenous beliefs formation plays a crucial role in explaining asset price and leverage cycles. We introduce state-contingent *ambiguity attitudes* embedding ambiguity aversion and seeking, which endogenously induces *pessimism* (left-skewed beliefs) in recessions and *optimism* (right-skewed beliefs) in booms, in a model where borrowers face occasionally binding collateral constraints. We use GMM estimation with latent value functions to estimate the ambiguity attitudes process. By simulating a crisis scenario in our model we show that optimism in booms is responsible for higher asset price and leverage growth and pessimism in recessions is responsible for sharper de-leveraging and asset price bursts. Analytically and numerically (using global methods) we show that our state-contingent ambiguity attitudes coupled with the collateral constraints can explain relevant asset price and debt cycle facts around the unfolding of a financial crisis.

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1 Introduction

Most financial crises originate in debt markets and asset price as well as leverage cycles have important effects on the real economy. Opacity and collateral constraints are the two most notable features of debt markets and both can be a source of instability¹. First, collateral constraints expose debt markets to fluctuations in collateral values, typically a risky asset, and the anticipatory effects associated to their endogenous changes trigger large reversal in debt and asset positions. Second, agents trading in debt markets hold doubts about the fundamental value of the collateral. In this context ambiguity attitudes and endogenous beliefs formation are crucial in determining the dynamic of asset values and, through the collateral constraint, of leverage. Finally, both in normal times and around crises events, pro-cyclicality of leverage, namely the loan-to-value ratio, emerges on top and above the pro-cyclicality of credit. Models of the credit cycle, with always or occasionally binding collateral constraints, induce pro-cyclical debt². However pro-cyclicality of leverage and counter-cyclicality of debt margins are two important facts of the leverage cycle³. For leverage pro-cyclicality to arise borrowers shall assign lower price of uncertainty to future contingencies, or else hold optimistic beliefs in booms, and vice versa in recessions. The resulting higher valuation of future collateral in booms makes the collateral constraint slack, or else raises the loan-to-value ratio, vice versa in recessions. In sum, we argue that the interaction between beliefs pro-cyclicality and occasionally binding collateral constraints can explain the main facts of asset price and leverage cycles.

Despite the relevance of both financial constraints and endogenous beliefs in explaining the unfolding of financial crises, as well as the dynamic of leverage cycles and asset prices, they have not been considered jointly in the literature and in dynamic macro models. We fill this gap by constructing and bringing to the data a model in which borrowers are subject to occasionally binding collateral constraints and whereby the valuation of collateral, a risky asset, depends upon a state-dependent endogenous beliefs formation. The latter induces waves of optimism in good times and pessimism in bad times⁴. Here an important novel part of our analysis lies. To generate those beliefs we extend the classical framework for decisions under robustness⁵, which features ambiguity aversion, by generalizing in a dynamic context the *bi-separable preferences* axiomatized in Ghirardato and Marinacci (2001) and Ghirardato, Maccheroni and Marinacci (2004). The latter convexify the decision maker problem of finding the optimal beliefs by combining both aversion and seeking behaviors. As in the standard multiplier framework a' la Hansen and Sargent (2001), our borrower makes decisions in two steps. First, he, endowed with a sequence of subjective beliefs, optimally chooses the posterior beliefs or the optimal likelihood ratio subject to an entropy constraint. Given the optimal beliefs, in the second step the borrower chooses debt and investment in risky assets. Importantly, in the first stage the penalty parameter on entropy, which defines the attitude of the agent towards model ambiguity,

¹See Holmstrom (2015).

²See contributions cited in the next session.

³See Gorton and Metrick (2012) among others, and our own statistics calculations provided in section 6.2.

⁴See Barberis (2013) for discussion on the role of over-confidence and under-confidence in particular for asset prices and leverage around the 2007-2008 financial crisis.

⁵See pioneering work by Hansen and Sargent (2001, 2007b) and Maccheroni, Marinacci and Rustichini (2006).

is contingent upon the state of the economy and depends on the borrower’s realized payoff. A sequence of negative shocks pushes the payoff below its historical mean and makes the borrower averse to ambiguity. This optimally results in pessimistic posterior beliefs, hence skewed toward the lower tail of the future events distribution. Subjective beliefs in turn affect the stochastic discount factor and, through it, the valuation of collateral. The ensuing fall in the value of collateral induces sharp de-leveraging and fall in asset demand. The opposite is true following a sequence of positive shocks. Optimism, which materializes in good times, raises the value of collateral, renders the debt constraint slack and favors the build-up of debt, risky investment and leverage.

To validate our beliefs formation process empirically, we determine the mapping between ambiguity attitudes and the expected utility through structural estimation of the model. Specifically, we develop a novel estimation method by adapting the method of moments featured in [Chen, Favilukis and Ludvigson \(2013\)](#) (CFL henceforth) to our model-based combined Euler equation, in debt and risky asset. CFL develop an estimation procedure for recursive preferences whereby the future value function, which appears in the pricing kernel, is treated as a latent factor with an unknown functional form. In our case, the value function is, instead, derived analytically and estimated by Kalman-filtering consumption data. We find that ambiguity aversion prevails when the value function is below its expected value, a case which we define as bad states, and vice versa. As argued above, those attitudes endogenously result in optimism or right-skewed beliefs in booms and pessimism in recessions⁶.

We substantiate our arguments further through a series of analytical and numerical exercises. Analytically, we derive expressions for the asset price, the equity premium, the Sharpe ratio and equilibrium leverage, showing that the stochastic discount factor emerging under our beliefs formation process tends to heighten their dynamics. Specifically, optimism reduces the price of future risk, or alternatively it tilts upward the stochastic discount factors, thereby increasing the value of all future discounted dividends. The opposite is true in recessions, when pessimism materializes. In this case agents also require an additional premium for uncertainty, as the higher weight on bad events raises the price of risk. The additional premium required in face of future downfalls in collateral values raises, the unconditional Sharpe ratio increases, also in comparison to the model with no ambiguity attitudes.

Next, we solve our model numerically by employing global non-linear methods⁷. Our solution algorithm is able to account for the different sources of non-linearity coming from the collateral constraint and the kink in borrowers’ beliefs. The model is calibrated under the estimated state-

⁶Our macro estimates are well in line with experimental evidence. [Abdellaoui et al. \(2011\)](#) provide foundations for S-shaped preferences with changing ambiguity attitudes and show through experimental evidence that pessimism (left-skewed beliefs) prevails in face of losses, while optimism prevails in face of gains. Further experimental evidence by [Boiney \(1993\)](#) [Kraus and Litzenberger \(1976\)](#) has associated ambiguity seeking (aversion) with right (left) skewed beliefs. On another front, survey evidence by [Rozsypal and Schlafmann \(2017\)](#), shows that low-income households hold pessimistic beliefs about the future, while the opposite is true for high-income households.

⁷We employ an endogenous grid approach ([Carroll \(2006\)](#)) accommodating for different regimes (portions of the state space) with binding or non-binding constraints ([Jeanne and Korinek \(2010\)](#)). Functions are approximated using piecewise linear interpolation and the exogenous state process is discretized with the [Tauchen and Hussey \(1991\)](#) method.

dependent penalty process. To appreciate the role of the belief formation process, we compare the model with and without ambiguity attitudes and assess its transmission mechanism through a simulated crisis event study (following [Bianchi and Mendoza \(2018\)](#)) and the policy functions. The first shows neatly that optimism fosters the build-up of debt prior to the crisis event, a realistic feature of the crisis unfolding. The policy functions confirm the leverage dynamic discussed so far. Beyond that, they, which are typically kinked around the point in which the constraint becomes binding, show that pessimistic beliefs tend to increase the amplitude of the debt constrained region. The opposite is true for optimistic beliefs. This also implies that the shadow price of debt, or the debt margin, raises in face of pessimistic beliefs, or after an history of negative shocks, and falls in face of optimistic beliefs, or after an history of positive shocks.

At last, to further validate our model empirically, we test its ability to match some relevant debt and asset price empirical moments. Under the optimized calibration, the model is able to match equity returns and debt volatility and pro-cyclicality⁸. The comparison with the model featuring solely the collateral constraint shows that state-contingent ambiguity attitudes significantly contribute to the data matching. Two elements are important. First, the switch from optimism to pessimism (and back) induces a kink in the value function. This, coupled with the occasionally binding nature of the collateral constraint, generates sharp non-linearities, thereby explaining high long run premia and excess volatility of asset prices and leverage (see [Cochrane \(2017\)](#)⁹). Ambiguity aversion alone, and the pessimistic beliefs associated with it, typically induce persistence through enhanced precautionary behavior, but produce little volatility in asset returns. In our model, the combination of right and left-skewed beliefs, bring the right amount of persistence and volatility needed to match asset price facts and debt dynamic. Second, the model endogenously generates an asymmetric distribution of asset returns as borrowers require an additional premium for uncertainty under pessimism. This specific feature brings the Sharpe ratio closer to the data. Third, the model with ambiguity attitudes obtains a pro-cyclical leverage, against a counter-cyclical one of the model with solely occasionally binding constraints. This is because of the counter-cyclical behavior of the borrowers' stochastic discount factor as induced by our endogenous and time-varying beliefs.

To fully verify robustness of our results, in [Appendix G](#) we extend the model to include an intermediation sector and an intermediation shock, interpreted as financial innovation or monitoring intensity. The latter affects the tightness of the borrowing limit, hence credit supply. This extension confirms the important role of ambiguity attitudes in presence of credit supply shocks. Moreover, the severity of the crisis is enhanced, as the credit supply restriction exacerbates the de-leveraging, and the debt pro-cyclicality is stronger, as a positive credit supply shock contributes to relax the debt constraint in booms and to tighten it in recessions.

The rest of the paper is structured as follows. [Section 2](#) compares the paper to the literature.

⁸Debt pro-cyclicality, confirmed by our data analysis, is also well documented by [Jorda, Schularick and Taylor \(2016\)](#) at aggregate level and using historical data. It is also well supported for consumer debt, see for instance [Fieldhouse, Livshits and MacGee \(2016\)](#) among others.

⁹As noted also in [Cochrane \(2017\)](#) the ability to match contemporaneously the long run equity premia and their cyclical properties is related to the agents' attitude toward losses. As agents become very afraid of bad times, they tend to shy away from risky investments and tend to over-react to the possibility of future bad events. And vice versa in good times.

Section 3 describes the model and the ambiguity attitudes specification. Section 4 presents the estimation procedure and results. Section 5 investigates the analytical results. Section 6 discusses quantitative findings. Section 7 concludes. Appendices and other results follow.

2 Comparison with Past Literature

Our paper is first and foremost related to models with occasionally binding collateral constraints, such as [Mendoza \(2010\)](#), [Lorenzoni \(2008\)](#) and [Bianchi and Mendoza \(2018\)](#). They, among others, point to the role of anticipatory effects, associated with occasionally binding debt constraints. The fact that borrowers leverage or de-leverage in anticipation of future movements in collateral values can explain sudden collapses in debt.

Second, our paper is connected to some literature studying the expectation formation process and the role of confidence shocks for waves of optimism and pessimism (see [Lorenzoni \(2009\)](#), [Angeletos, Collard and Dellas \(2018\)](#)). In those papers agents form expectations through higher order beliefs, hence by forecasting the others' forecast rather than choosing beliefs optimally under ambiguity attitudes like in our case. Also, all those papers assess the role of confidence for the business cycle, but not for the leverage cycle¹⁰.

Our model is also connected to the literature on robustness and ambiguity aversion (see [Hansen and Sargent \(2001, 2007b\)](#) and [Maccheroni, Marinacci and Rustichini \(2006\)](#))¹¹. A key difference is that we model ambiguity attitudes that encompass both aversion, which induces pessimistic beliefs, and seeking, that induces optimistic beliefs. We do so by building a dynamic generalization of the bi-separable preferences axiomatized in a static context by [Ghirardato and Marinacci \(2001\)](#) and [Ghirardato, Maccheroni and Marinacci \(2004\)](#)¹². Moreover, none of the papers using the robustness methodologies examines its role for the leverage cycle.

Some recent papers combine financial frictions and expectation formation processes. [Boz and Mendoza \(2014\)](#) insert parametric learning into a model with collateral constraints, while [Kozlowski, Veldkamp and Venkateswaran \(2018\)](#) introduces non-parametric learning into a model with firms' subject to bankruptcy shocks¹³. In our case beliefs are optimally chosen by solving a decision problem subject to an entropy constraint. The flexible specification that we adopt for the multiplier on the entropy constraint allows our model to nest both optimistic (right-skewed) and pessimistic beliefs (left-skewed). Furthermore, we bring our model to the data through GMM estimation and simulated moments' data-matching.

¹⁰Along similar lines [Schmitt-Grohé and Uribe \(2012\)](#) assess the role of news shocks for the business cycle.

¹¹Some papers have studied the role of ambiguity aversion, hence pessimism, into asset price model. See for instance [Barillas, Hansen and Sargent \(2007\)](#), [Epstein and Schneider \(2008\)](#), [Drechsler \(2013\)](#) and [Leippold, Trojani and Vanini \(2008\)](#) among others. None considers varying ambiguity attitudes, nor they study the role of that for the leverage cycle.

¹²Note that the state-contingent nature of ambiguity attitudes is also well documented in experimental studies. See for instance [Dimmock et al. \(2015\)](#), [Baillon et al. \(2017\)](#), [Trautmann and van de Kuilen \(2015\)](#), and [Roca, Hogarth and Maule \(2006\)](#) among others.

¹³[Geanakoplos \(2010\)](#) and [Simsek \(2013\)](#) introduced exogenous binomial beliefs, whereby optimists are those assigning all weight to the good state and vice versa for pessimists, into a static model with collateral constraints. Their goal is mainly to assess the role of optimist/pessimist heterogeneity into the scope for trade and the determination of the margin.

At last, our paper relates to the extensive literature on the estimation of the stochastic discount factors (occasionally defined as SDF henceforth) with behavioral elements. More closely, we build upon the latent factor estimation method of [Chen, Favilukis and Ludvigson \(2013\)](#). We depart from them along the following dimensions. First, we adapt their estimation procedure to value functions with state-contingent ambiguity attitudes and in presence of an occasionally binding collateral constraint. Secondly, our latent factor is derived analytically, while in their case it is estimated semi-parametrically. To the best of our knowledge, this is the first attempt to test ambiguity attitudes with time series analysis. Among other things, the estimation allows us to pin down the exact form of the state-contingency in the multiplier, which turns out to be negative in the gain domain and positive in the loss domain. This result is also intuitive as it naturally leads to optimistic beliefs in booms and pessimistic ones in recessions and as we argued above those shifts are crucial in explaining the leverage cycle.

3 A Model of Ambiguous Leverage Cycle

Our baseline model economy is an otherwise standard framework with borrowers facing occasionally binding collateral constraints. Debt supply is fully elastic with an exogenous interest rate, as largely employed in some recent literature.¹⁴ Collateral in this economy is provided by the value of the risky asset.

The novelty concerns the interaction between agents' beliefs formation and debt capacity. Indeed, we endow borrowers with state-contingent ambiguity attitudes, which include both ambiguity aversion and seeking. The underlying logic is similar to the one pioneered by the game-theoretic approach à la [Hansen and Sargent \(2007a\)](#) in which ambiguity avers agents fear model mis-specification and thus explore the fragility of their decision rules with respect to various perturbations of the objective probability distribution. In our framework, agents can be also endowed, depending on the state of the economy, with an ambiguity seeking attitude, and thus look for utility gains generated by deviations from the objective model. Ambiguity aversion results endogenously in pessimistic beliefs, relatively to the case in which objective and subjective beliefs coincide, while ambiguity seeking generates optimism.

The state-contingent nature of the ambiguity attitudes heightens the occasionally binding nature of the collateral constraint. As agents become optimist their demand for risky assets boosts collateral values and expands debt capacity. The opposite is true with pessimism. The causality also runs the other way around. As debt capacity expands, the accumulation of wealth affects their future value function, hence their optimal choice of beliefs from the first stage.

3.1 Beliefs Formation

The source of uncertainty in the model is a shock to aggregate income y_t , which is our exogenous state and follows a finite-space stationary Markov process. We define the state space as S_t , the

¹⁴This model economy corresponds to a limiting case in which lenders are risk-neutral. Alternatively the model can be interpreted as a small open economy with debt supplied from the rest of the world.

realization of the state at time t as s_t and its history as $s^t = \{s_0, s_1, \dots, s_t\}$, with associated probability $\pi(s^t)$. The initial condition of the shock is known and defined with s_{-1} .

Borrowers are endowed with the approximating model $\pi(s^t)$ over the history s^t , but they also consider alternative nearby models $\tilde{\pi}(s^t)$.¹⁵ Following the relevant literature, we introduce the non-negative measurable function $M(s^t) = \tilde{\pi}(s^t)/\pi(s^t)$, defined as the likelihood ratio and capturing the distortions with respect to the approximating model. Given the history, the conditional likelihood ratio emerges as $m(s_{t+1}|s^t) = \tilde{\pi}(s_{t+1}|s^t)/\pi(s_{t+1}|s^t)$. For ease of notation, since now onward we use the following convention: $M_t = M(s^t)$, $M_{t+1} = M(s^{t+1})$ and $m_{t+1} = m(s_{t+1}|s^t)$. The above definition of M_t allows to represent the subjective expectation of a random variable x_t in terms of the approximating probability model, $\tilde{\mathbb{E}}_t\{x_t\} = \mathbb{E}_t\{M_t x_t\}$, where $\tilde{\mathbb{E}}_t$ is the subjective expectation operator, while \mathbb{E}_t is the objective expectation operator. The function M_t follows a martingale process and as such it satisfies the following condition $\mathbb{E}\{M_{t+1}\} = M_t$. We then can reformulate the conditional likelihood ratio in terms of martingale increments as follows:

$$m_{t+1} \equiv \frac{M_{t+1}}{M_t} \quad \text{for } M_t > 0, \quad \text{with } \mathbb{E}_t\{m_{t+1}\} = 1 \quad (1)$$

and $m_{t+1} = 1$ otherwise. Moreover, the discrepancy between the approximating and the subjective models is measured by the relative entropy, $\varepsilon(m_{t+1}) = \mathbb{E}_t\{m_{t+1} \log m_{t+1}\}$, that is a positive-valued, convex function of π and is uniquely minimized at zero when $m_{t+1} = 1$, which identifies the case with no beliefs distortions.

Within this framework, we endow agents with both ambiguity aversion and seeking attitudes. They are ambiguity averse when they fear that deviations from the objective model would imply utility losses. As a consequence, they form subjective expectations according to a worst-case scenario evaluation in order to define a lower bound for the potential losses. This attitude is coherent with a period of strong economic uncertainty in which agents are not able to make precise economic forecasts. Contrary, agents are ambiguity seekers when they look for potential utility gains deviating from the approximated model. This is a situation that characterizes periods of markets' exuberance. In this scenario agents form subjective expectations according to a best-case scenario evaluation. In other words, borrowers have different degrees of trust in their own subjective beliefs, so they act as ambiguity averse when they fear deviations from the approximated model and as ambiguity seekers when they hold high confidence in their beliefs.

The coexistence of these two attitudes is introduced in the model with a new structure of beliefs, which we define *kinked beliefs*. They are derived as a dynamic extension of the bi-separable preferences axiomatized in [Ghirardato and Marinacci \(2001\)](#) and [Ghirardato, Maccheroni and](#)

¹⁵The alternative probability measure $\tilde{\pi}$ is absolutely continuous with respect to π . This means that events with positive probability under the alternative model, hold positive probability under the approximating model.

Marinacci (2004), and are described by the following utility function:

$$\begin{aligned}
V_t = \alpha(s^t) & \left\{ \underbrace{\min_{\substack{\{m_{t+1}, M_t\}_{t=0}^{\infty} \\ \mathbb{E}\{m_{t+1}\}=1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t M_t u(c_t) + \beta \theta^+(s^t) \varepsilon(m_{t+1}) \right]}_{\substack{\text{ambiguity aversion side:} \\ \text{worst-case scenario evaluation}}} \right\} + \\
[1 - \alpha(s^t)] & \left\{ \underbrace{\max_{\substack{\{m_{t+1}, M_t\}_{t=0}^{\infty} \\ \mathbb{E}\{m_{t+1}\}=1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t M_t u(c_t) + \beta \theta^-(s^t) \varepsilon(m_{t+1}) \right]}_{\substack{\text{ambiguity seeking side:} \\ \text{best-case scenario evaluation}}} \right\} \quad (2)
\end{aligned}$$

where $\alpha(s^t) = \{0, 1\}$ identifies the weight on the prevalent ambiguity attitude, $u(c_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$ and $\mathbb{E}_0 \sum_{t=0}^{\infty} \left[\beta^t M_t u(c_t) \right]$ is the expected discounted value of the utility flows from consumption. The last term in each state is the discounted entropy scaled by the penalty process $\theta \in \{\theta^+, \theta^-\} \in \mathbb{R}$ with $\theta^+ > 0$ and $\theta^- < 0$. The realizations of the penalty process control the degree of agents ambiguity. In absolute terms, higher values of θ imply less distorted beliefs and convergence to the reference objective model, while with $\theta \rightarrow 0$ agents ambiguity amplifies.

We do not make any ex-ante conjecture on the exact nature of the state-contingency dependence between the prevalent ambiguity side, identified by α , and the state of the economy s^t . The only restriction concerns the assumption that the process θ must shift from the positive (negative) to the negative (positive) domain, hence the implied shifts in ambiguity attitudes. Later on, our estimation results (see Section 4) will allow us to establish that ambiguity seeking (negative multiplier) prevails in the gain domain, and ambiguity aversion (positive multiplier) features the loss domain. Upon this, we assume the following functional form for α :

$$\alpha(s^t) = \mathbb{I}_{V_{t-1} \leq \bar{V}}, \quad \text{hence} \quad [1 - \alpha(s^t)] = \mathbb{I}_{V_{t-1} > \bar{V}} \quad (3)$$

The kinked nature of the value function is related to the state of the economy, defined on the distance of the agents' realized value function V_{t-1} from its historical mean \bar{V} . When $V_{t-1} \leq \bar{V}$ the economy is in bad states and agents behave as ambiguity averse. The opposite is true during good states, when $V_{t-1} > \bar{V}$.

3.2 Budget and Collateral Constraint

The rest of the model follows the standard specification of a borrower problem facing occasionally binding collateral constraints (see e.g. Mendoza (2010)). The representative agent holds an infinitely lived asset x_t , which pays every period a stochastic dividend d_t and is available in fixed unit supply. The asset can be traded across borrowers at the price Q_t . In order to reduce the dimension of the state space, we assume that the dividend is a fraction α of the income realization. Therefore, we indicate with $(1 - \alpha)y_t$ the labor income and with $d_t = \alpha y_t$ the financial income. Agents can borrow using one-period non-state-contingent bonds, paying

an exogenous real interest rate R . The budget constraint faced by the representative agent is:

$$c_t + Q_t x_t + \frac{b_t}{R} = (1 - \alpha)y_t + x_{t-1}[Q_t + d_t] + b_{t-1} \quad (4)$$

where c_t indicates consumption and b_t bond holdings. We use the convention that positive values of b denote assets. Agents' ability to borrow is restricted to a fraction ϕ of the value of asset holding:

$$-\frac{b_t}{R} \leq \phi Q_t x_t \quad (5)$$

Appendix B provides a micro-founded derivation of this constraint, based on a limited enforcement problem.

3.3 Recursive Formulation

Following Hansen and Sargent (2007b), we rely on the recursive formulation of the problem, which allows us to re-write everything only in terms of m_{t+1} , taking into account the changing nature of the ambiguity attitudes.

We now partition the state space \mathbf{S}_t in two blocks, given by the endogenous and the exogenous states, $\mathbf{S}_t = \{B_t, y_t\}$, where B_t is the aggregate bond holdings and y_t the income realization. Note that the aggregate asset holding is not a state variable because it is in fixed supply. Moreover, the problem is also characterized by the two individual state variables (b_t, x_t) . The borrowers' recursive optimization problem reads as follows:

$$\begin{aligned} & V(b_t, x_t, \mathbf{S}_t) = \\ & \max_{c_t, x_{t+1}, b_{t+1}} \left\{ \begin{aligned} & \mathbb{I}_{V_{t-1} \leq \bar{V}} \min_{m_{t+1}} \left[u(c_t) + \beta \mathbb{E}_t \left(m_{t+1} V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) + \theta^+(\mathbf{S}_t) m_{t+1} \log m_{t+1} \right) \right] \\ & + \mathbb{I}_{V_{t-1} > \bar{V}} \max_{m_{t+1}} \left[u(c_t) + \beta \mathbb{E}_t \left(m_{t+1} V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) + \theta^-(\mathbf{S}_t) m_{t+1} \log m_{t+1} \right) \right] \\ & + \lambda_t \left[y_t + Q(\mathbf{S}_t)(x_t + \alpha y_t) + b_t - Q(\mathbf{S}_t)x_{t+1} - c_t - \frac{b_{t+1}}{R} \right] \\ & + \mu_t \left[\phi Q(\mathbf{S}_t)x_{t+1} + \frac{b_{t+1}}{R} \right] + \beta \theta(\mathbf{S}_t) \psi_t [1 - \mathbb{E}_t m_{t+1}] \end{aligned} \right\} \end{aligned}$$

where the aggregate states follow the law of motion $\mathbf{S}_{t+1} = \Gamma(\mathbf{S}_t)$. In the above problem λ_t and μ_t are the multipliers associated to the budget and collateral constraints respectively, while the term $\beta \theta_t \psi_t$ is the multiplier attached to the constraint $\mathbb{E}_t[m_{t+1}] = 1$.

The above optimization problem is solved as follows. First an inner optimization and then an outer optimization problem are derived sequentially. In the first stage agents choose the optimal incremental probability distortion for given saving and portfolio choices. In the second stage, for given optimal likelihood ratio, they solve the consumption/saving problem and choose the optimal amount of bonds and risky assets. Intuitively, the problem is modelled as a game of strategic interactions between the maximizing agents, who face Knightian uncertainty¹⁶, and a

¹⁶Knight (1921) advanced the distinction between risk, namely the known probability of tail events, and uncertainty, namely the case in which such probabilities are not known. Ambiguity usually refers to cases of uncertainty where the state space is well defined, but objective probabilities are not available.

malevolent/benevolent agent that draws the distribution (see Hansen and Sargent (2007b) who proposed this reading).

3.3.1 The Inner Problem

Through the inner optimization problem, the borrower chooses the optimal entropy or conditional likelihood ratio, namely the optimal deviation between his own subjective beliefs and the objective probability distribution. In order to simplify the notation, in the next sections we use $\theta(\mathbf{S}_t) = \theta_t$. The first order condition with respect to m_{t+1} , which is functionally equivalent under the two cases, is given by:

$$V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) + \theta(\mathbf{S}_t)[\log m_{t+1} + 1] - \theta(\mathbf{S}_t)\psi_t = 0 \quad (6)$$

Rearranging terms, we obtain:

$$\begin{aligned} 1 + \log m_{t+1} &= -\frac{V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})}{\theta(\mathbf{S}_t)} + \psi_t \\ m_{t+1} &= \exp \left\{ \frac{-V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})}{\theta(\mathbf{S}_t)} \right\} \exp\{\psi_t - 1\} \end{aligned} \quad (7)$$

Finally, imposing the constraint $\mathbb{E}_t[m_{t+1}] = 1$, and defining $\sigma(\mathbf{S}_t) = -1/\theta(\mathbf{S}_t)$, the optimal conditional likelihood ratio is:

$$m_{t+1} = \frac{\exp \{ \sigma(\mathbf{S}_t) V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) \}}{\mathbb{E}_t [\exp \{ \sigma(\mathbf{S}_t) V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) \}]} \quad (8)$$

Equation (8) defines the state-contingent incremental probability distortion. The magnitude and direction of this discrepancy depends upon the agents' value function and the value of the ambiguity process $\theta(\mathbf{S}_t)$.

3.3.2 The Outer Problem

For a given optimal LR m_{t+1} , upon substituting it into the value function, the borrower solves an outer optimization problem in consumption, risky assets and debt. The resulting recursive problem is:

$$\begin{aligned} V(b_t, x_t, \mathbf{S}_t) &= \max_{c_t, x_{t+1}, b_{t+1}} \left\{ u(c_t) + \frac{\beta}{\sigma(\mathbf{S}_t)} \log [\mathbb{E}_t \exp \{ \sigma(\mathbf{S}_t) V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) \}] \right. \\ &\quad + \lambda_t \left[y_t + Q(\mathbf{S}_t)(x_t + d_t) + b_t - Q(\mathbf{S}_t)x_{t+1} - c_t - \frac{b_{t+1}}{R} \right] \\ &\quad \left. + \mu_t \left[\phi Q(\mathbf{S}_t)x_t + \frac{b_{t+1}}{R} \right] \right\} \end{aligned} \quad (9)$$

The borrower's first order conditions with respect to bond holdings and risky assets read as follows:

$$u_c(c_t) = \beta R \mathbb{E}_t \{m_{t+1} u_c(c_{t+1})\} + \mu_t \quad (10)$$

$$Q_t(\mathbf{S}_t) = \beta \frac{\mathbb{E}_t \{m_{t+1} u_c(c_{t+1}) [Q_{t+1}(\mathbf{S}_{t+1}) + \alpha y_{t+1}]\}}{u_c(c_t) - \phi \mu_t} \quad (11)$$

where u_c indicates the marginal utility of consumption. Equation (10) is the Euler equation for bonds and displays the typical feature of models with occasionally binding collateral constraints. In particular, when the constraint binds there is a wedge between the current and the expected future consumption marginal utility, given by the shadow value of relaxing the collateral constraint. Equation (11) is the asset price condition. Ambiguity attitudes affect both asset prices and borrowing decisions, since the optimal m_{t+1} tilts the stochastic discount factor and through this it affects the pricing of all assets in the economy.

The model closes with the complementarity slackness condition associated to the collateral constraint:

$$\mu_t \left[\frac{b_{t+1}}{R} + \phi Q_t(\mathbf{S}_t) \right] = 0 \quad (12)$$

and with the goods and stock markets clearing conditions:

$$c_t + \frac{b_{t+1}}{R} = y_t + b_t \quad (13)$$

$$x_t = 1 \quad (14)$$

Definition 3.1 (Recursive Competitive Equilibrium). *A Recursive Competitive Equilibrium is given by the value function V_t , allocations (c_t, b_{t+1}) , probability distortions m_{t+1} and prices Q_t , such that:*

- *given prices and allocations, the probability distortions solve the inner problem;*
- *given prices and probability distortions, the allocations and the value function solve the outer problem;*
- *the allocations are feasible, satisfying (13) and (14);*
- *the aggregate states' law of motion is consistent with the agents' optimization;*

3.4 Pessimism and Optimism

To determine under which states the multiplier $\theta(\mathbf{S}_t)$ turns positive or negative, in the next section we estimate our model-implied Euler equations through GMM methods. Before that we discuss how the ambiguity attitudes generate endogenous waves of optimism and pessimism. For simplicity of exposition, we report the optimal condition for m_{t+1} :

$$m_{t+1} = \frac{\exp \{ \sigma(\mathbf{S}_t) V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) \}}{\mathbb{E}_t \{ \exp \{ \sigma(\mathbf{S}_t) V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) \} \}} \quad (15)$$

The conditional deviation m_{t+1} affects how agents assign different subjective probabilities, relatively to the objective ones, to future events. In particular, if for some states $m_{t+1} > 1$ agents are assigning higher subjective probabilities to those states, while if $m_{t+1} < 1$ the opposite holds. Given this, the ambiguity process $\sigma(\mathbf{S}_t)$ captures the degree to which agents are uncertain about the probability measure, while its sign and time-varying nature jointly tell how those conditions on m_{t+1} are linked to positive or negative state realizations. The following lemma summarizes this consideration and defines optimism and pessimism in the agents' attitude.

Lemma 3.2. *When $\theta(\mathbf{S}_t) < 0$, $m_{t+1} > 1$ for future good states and $m_{t+1} < 1$ for future bad states. Hence, beliefs endogenously emerge as right-skewed and agents act with optimism. When $\theta(\mathbf{S}_t) > 0$, the opposite is true and agents act with pessimism.*

Proof. When $\theta(\mathbf{S}_t) < 0$ ($\sigma(\mathbf{S}_t) > 0$), for future high utility states ($V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) > \mathbb{E}_t\{V(b_{t+1}, x_{t+1})\}$) the condition $\exp\{\sigma(\mathbf{S}_t)V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})\} > \mathbb{E}_t\{\exp\{\sigma(\mathbf{S}_t)V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})\}\}$ holds, namely the risk-adjusted value function for future good states is larger than the average one. This implies that $m_{t+1} > 1$ for those states and thereby that agents overweight the probability to end up tomorrow in a good state. For future low-utility states ($V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1}) < \mathbb{E}_t\{V(b_{t+1}, x_{t+1})\}$), instead, $\exp\{\sigma(\mathbf{S}_t)V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})\} < \mathbb{E}_t\{\exp\{\sigma(\mathbf{S}_t)V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})\}\}$ holds, implying a $m_{t+1} < 1$. Agents are under-weighting the probability that tomorrow a negative state will realize. The opposite is true for $\theta(\mathbf{S}_t) > 0$ ($\sigma(\mathbf{S}_t) < 0$).

3.4.1 Beliefs Formation: A binomial state space example

To gain some further intuition we discuss a particular case with only two income states, which we define as high, with a sup-index h , and low, with a sup-index l . We also consider only two periods which we label as $t = 0, 1$. By assumption, the collateral constraint is slack in the high state, while it binds under the low state. The states have a binomial probability structure such that state h realizes with the objective probability π , while state l with its complement $1 - \pi$. We recall the notation for the subjective probabilities $\tilde{\pi}$. Equipped with these assumptions, we characterize the dynamic between time 0 and time 1. The conditional likelihood ratios can be specified as follows:

$$m_1^j = \frac{\exp\{\sigma_0 V_1^j\}}{\pi \exp\{\sigma_0 V_1^h\} + (1 - \pi) \exp\{\sigma_0 V_1^l\}}, \quad j = \{h, l\} \quad (16)$$

where $V_1^h > \mathbb{E}_0\{V_1\}$ and $V_1^l \leq \mathbb{E}_0\{V_1\}$ are the state conditions. Depending on the time zero realization of the state, we assume two different (in sign) values of the inverse of the penalty parameter, σ_0 .

To fix ideas imagine that the income realization at time zero is the low state l , hence $\sigma_0^l < 0$. Given Lemma 3.2, we have that $\exp\{\sigma_0^l V_1^h\} < \mathbb{E}_0\{\exp\{\sigma_0^l V_1\}\}$ and $\exp\{\sigma_0^l V_1^l\} > \mathbb{E}_0\{\exp\{\sigma_0^l V_1\}\}$. Therefore, the marginal likelihood ratios are $m_1^h < 1$ and $m_1^l > 1$. As a consequence, we define the following subjective probabilities as:

$$\tilde{\pi}^h = \pi m_1^h < \pi \quad \tilde{\pi}^l = (1 - \pi) m_1^l > (1 - \pi) \quad (17)$$

As we can see, in the low state agents assign an higher (lower) subjective probability - with respect to the objective one - to future negative (positive) events, typical of a pessimistic attitude. The opposite is true in the high state h , when $\sigma_o^h > 0$. In this case, $\exp\{\sigma_0^h V_1^h\} > \mathbb{E}_0 \{\exp\{\sigma_0^h V_1\}\}$ and $\exp\{\sigma_0^h V_1^l\} < \mathbb{E}_0 \{\exp\{\sigma_0^h V_1\}\}$ producing $m_1^h > 1$ and $m_1^l < 1$. Here agents assign higher (lower) subjective probability to future positive (negative) events, showing an optimistic attitude:

$$\tilde{\pi}^h = \pi m_1^h > \pi \quad \tilde{\pi}^l = (1 - \pi) m_1^l < (1 - \pi) \quad (18)$$

4 Estimation of the Model-implied SDF

The exact dependence of the ambiguity process on the state of the economy requires to be substantiated on empirical grounds. To this purpose we engage into a structural and model-based estimation of the penalty process. Specifically we estimate the model-implied Euler equations. This delivers a process for θ_t , whose state-contingent nature empirically supports our decision problem and value function specification.

We devise a novel estimation method apt to a model with collateral constraints and kinked beliefs. It is based on adapting the minimum distance estimation conditional on latent variables to our modelling environment. In a nutshell we derive a moment condition by using the combined non-linear expression for the Euler equations (10) and (11). As we show in Appendix C, the latter depends upon the value function. We follow the approach in [Chen, Favilukis and Ludvigson \(2013\)](#), who write the Euler moment condition as function of the estimated value function. A crucial difference between our method and theirs is that their value function has an unknown functional form, which is estimated semi-parametrically, while ours can be derived analytically, given our beliefs formation process. Specifically, following [Hansen, Heaton and Li \(2008\)](#), we derive its functional form, which is then estimated using maximum likelihood.

The estimation procedure, whose detailed derivations are contained in Appendix C, can be described as follows. First, one shall re-write the value function in terms of an ambiguity factor. For this, we adapt the steps used in the recursive preference literature to the case of our kinked beliefs (see Appendix C.1). Next, the implied SDF is derived (see Appendix C.2) and the value function is estimated (see Appendix C.3). Substitution of the derived SDF into the combined Euler equations for debt and risky assets, (10) and (11), delivers the final moment condition (see Appendix C.4). At last, as it is common for GMM estimation, we condition on a set of instruments, \mathbf{z}_t . The resulting moment condition reads as follows:

$$\mathbb{E}_t \left\{ \left[\underbrace{\beta \left(\frac{c_{t+1}}{c_t} \right)^{-1} \left(\frac{\exp(V_{t+1}) c_{t+1}}{c_{t+1} \frac{c_t}{\beta \sqrt{\exp(V_t)}}} \right)^{\sigma_t}}_{\Lambda_{t,t+1}} (R_{t+1}^s - \phi R_{t+1}) + \phi - 1 \right] \mathbf{z}_t \right\} = 0 \quad (19)$$

where $R_{t+1}^s = \frac{q_{t+1} + d_{t+1}}{q_t}$ is the cum-dividend return on risky asset and R_{t+1} is the risk-free

Table 1: Estimation Results

Sample	Estimated parameters ¹				
	β	θ	$\theta(\tilde{v}_t > E\tilde{v}_t)$	$\theta(\tilde{v}_t \leq E\tilde{v}_t)$	$J - test$
1980-2018	0.985 (.020)		-1.907 (.043)	2.500 (.063)	4.079 (.538)
1985:Q1-2007:Q2	0.906 (.017)	-3.987 (.164)			3.713 (.715)
2007:Q3-2015:Q1	0.841 (.015)	17.536 (.009)			2.117 (.909)

In parenthesis: the HAC standard errors for the parameter estimates and the p-values for the J-test.

interest rate, which is time-varying in the data. Note that the expression for the SDF can be decomposed into two factors, $\Lambda_{t,t+1}^1 = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1}$ and $\Lambda_{t,t+1}^2 = \left(\frac{\exp(V_{t+1}) c_{t+1}}{\beta \sqrt{\frac{c_{t+1}}{\exp(V_t)}}} \right)^{\sigma_t}$, where the second captures the role of ambiguity attitudes and collapses to 1 in the benchmark case. Equation (19) is estimated fully nonlinearly with GMM methods. Note that tight restrictions are placed on asset returns and consumption data since our moment condition embodies both financial and ambiguity attitudes. For the estimation we use a loan to value ratio at $\phi = 0.2$ and, given that $\theta_t = -1/\sigma_t$, we estimate the beliefs parameters, the discount factor β and the ambiguity process θ_t .

Regarding the data, we use real per capita expenditures on non-durables and services as a measure of aggregate consumption. For R we use the three-month T-bill rate, while R^s is proxied by the Standard & Poor 500 equity return¹⁷. The choice of the instruments follows the literature on time-series estimation of the Euler equations¹⁸. They are grouped into internal variables, namely consumption growth and interest rates two-quarters lagged, and external variables, namely the value and size spreads, the long-short yield spread and the dividend-price ratio (see also Yogo (2006)). A constant is additionally included in order to restrict model errors to have zero mean. Finally, the model’s over-identifying restrictions are tested through the J-test (test of over-identifying restrictions, Hansen (1982))¹⁹.

Table 1 presents the results. The estimated values of θ_t are conditioned to the logarithm of the continuation value ratio, defined as $\tilde{v}_t = V_t - \log(c_t)$. Consistently with our previous definition, good states are those for which the current latent value function is higher than its mean and vice versa for bad states. Column 3 shows results conditioned upon the relation $\tilde{v}_t > \mathbb{E}\{\tilde{v}_t\}$, while column 4 reports the results for the complementary condition. We find that a negative value (-1.907) prevails over good states, namely those for which $\tilde{v}_t > \mathbb{E}\{\tilde{v}_t\}$, and

¹⁷Data sources are NIPA Tables https://www.bea.gov/iTable/index_nipa.cfm, CRSP Indices database <http://www.crsp.com/products/research-products/crsp-historical-indexes>, and the Shiller database <http://www.econ.yale.edu/~shiller/data.htm>, respectively.

¹⁸See Stock, Wright and Yogo (2002) for a survey on the relevance of instruments choice in a GMM setting.

¹⁹This is a specification test of the model itself and it verifies whether the moment conditions are enough close to zero at some level of statistical confidence, if the model is true and the population moment restrictions satisfied.

Table 2: Pricing Kernel estimated Moments

Moments(1980-2018)	Mnemonics	$\Lambda_{t,t+1}$	$\Lambda_{t,t+1}^1$	$\Lambda_{t,t+1}^2$
SDF mean	$\mathbb{E}_t(SDF_t)$	0.814	0.980	0.830
SDF volatility	σ^{SDF_t}	0.379	0.005	0.384
SDF cyclicality	$Corr(SDF_t, \Delta c_t)$	-0.211	-0.999	-0.194
SDF and Debt	$Corr(SDF_t, \Delta b_t)$	-0.070	-0.365	-0.064
SDF and Equity return	$Corr(SDF_t, R_{t+1}^s)$	-0.104	-0.302	-0.099

that a positive value (2.500) prevails in bad states, namely those for which $\tilde{v}_t \leq \mathbb{E}\{\tilde{v}_t\}$. This gives clear indication on the state-contingent nature of the ambiguity attitudes, being averse to entropy deviations in bad states and opportunistic toward them in good states. According to Lemma 3.2 above we know that $\theta_t < 0$, which according to our estimation prevails in good states, implies that agents act optimistically, or alternatively that they assign higher weights to future good state. Similarly a $\theta_t > 0$ speaks in favor of pessimism.

To further test robustness of our result we run unconditional estimation over two different historical periods. We choose the first to be the Great Moderation sample (1985:Q1-2007:Q2), which captures the boom phase preceding the 2007-2008 financial crisis. The sub-sample representing the recessionary states is the period following the crisis, namely the (2007:Q3-2015:Q1). Estimations, reported in the last two rows of Table 1, confirm the same state-contingent nature uncovered in the conditional estimates. Finally note that for each sample the J -test fails to reject model in equation (19) at conventional significance levels.

As a final check of the soundness of our estimation results, given the estimated beliefs parameters, we investigate the cyclical properties of the pricing kernel, namely the estimated SDF, and through them, those of the risk premia. To this purpose we use the decomposition of the SDF in $\Lambda_{t,t+1}^1$ and $\Lambda_{t,t+1}^2$ in order to isolate the contribution arising from the ambiguity attitudes. The empirical moments of the SDF are listed in Table 2. They interestingly show that the high volatility in the SDF is totally driven by the ambiguity attitudes component, which, for the same reason, contributes less to the SDF clear countercyclical properties.

5 Analytical Results

The relevance of our model and expectation formation process is best verified by assessing its quantitative properties through model simulations. However, prior to that we also wish to highlight the economic channel through which the interaction of beliefs and occasionally binding constraints can explain facts about asset prices and leverage, such as heightened volatility and pro-cyclicalities. To this purpose we derive analytical expressions for asset price, equity premia and Sharpe ratio and show their dependence on the optimal LR, m_{t+1} , and on the shadow price of debt, μ_t . Finally, we show how our pro-cyclical beliefs induce a pro-cyclical equilibrium leverage ratio and a counter-cyclical margin, namely the Lagrange multiplier on the collateral

constraint.

Proposition 5.1 (Asset Price Recursion). *The recursive formula for the asset price over the infinite horizon in our model reads as follows:*

$$q_t = \lim_{T \rightarrow \infty} \mathbb{E}_t \left\{ \sum_{i=1}^T d_{t+i} \prod_{j=1}^i K_{t+j-1, t+j} \right\} \quad (20)$$

where $K_{t,t+1} = \frac{\Lambda_{t,t+1}}{1 - \phi \mu'_t}$ with $\Lambda_{t,t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}$ and $\mu'_t = \frac{\mu_t}{u_c(c_t)}$.

Proof is described in Appendix D.1. The asset price clearly depends on m_{t+1} . Consider first good states in which, according to Lemma 3.2, endogenous beliefs are right-skewed toward the upper tails ($m_{t+1} > 1$ for future positive states). This implies that $\Lambda_{t,t+1}$ and hence $K_{t,t+1}$ over-weights future states with high dividends payments. As a consequence, the asset price is higher with respect to the case without beliefs distortions. Asset price grows in good states, but it does so more under optimistic beliefs. In bad states, instead, left-skewed beliefs ($m_{t+1} < 1$ for future negative states) imply that both $\Lambda_{t,t+1}$ and $K_{t,t+1}$ are tilted toward the lower tails and thereby the asset price is lower. In this sense ambiguity attitudes contribute to the heightened dynamic of the asset price boom and bust cycles.

The asset price also depends upon the shadow price of debt μ_t . In bad states, when the collateral constraint is binding, this factor interacts with pessimist attitudes and exacerbates the asset price decline. Indeed, when the agents' borrowing ability is constrained, the current marginal utility $u_c(c_t)$ increases, while the marginal utility of consumption tomorrow $\beta u_c(c_{t+1})$ declines. This effect reduces both $\Lambda_{t,t+1}$ and $K_{t,t+1}$ and thereby the asset price. This effect is only partially mitigated by the presence of the shadow price of debt in the denominator of $K_{t,t+1}$, which represents the additional value of the risky asset as a collateral.

Proposition 5.2 (Equity Premium). *The equity premia reads as follows:*

$$\Psi_t = \mathbb{E}_t \{ R_{t+1}^s \} - R = \frac{R \left(1 - \text{cov}(\Lambda_{t,t+1}, R_{t+1}^s) - \phi \mu'_t \right)}{1 - \mu'_t} - R \quad (21)$$

where $\Lambda_{t,t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}$ and $\mu'_t = \frac{\mu_t}{u_c(c_t)}$.

See Appendix D.2 for the proof. The above proposition also shows unequivocally the dependence of the premia over the beliefs distortions m_{t+1} , and the shadow price of debt μ_t . While the exact dynamic of the equity premium depends on the solution of the full-model and upon its general equilibrium effects, we can still draw some general conclusions on the interactions.

Beliefs affect the SDF, hence its covariance with asset returns. Independently of the sign of the $\text{cov}(\Lambda_{t,t+1}, R_{t+1}^s)$ ²⁰, we can conjecture that optimism and pessimism increase the covariance between consumption and asset returns. One way to see this is by looking at the upper bound for the covariance, which according to the Cauchy-Schwarz inequality is given by $\text{cov}(\Lambda_{t,t+1}, R_{t+1}^s) \leq$

²⁰This depends on whether $\mathbb{E}_t(\Lambda_{t,t+1}, R_{t+1}^s) > \mathbb{E}_t(\Lambda_{t,t+1})\mathbb{E}_t(R_{t+1}^s)$ or $\mathbb{E}_t(\Lambda_{t,t+1}, R_{t+1}^s) < \mathbb{E}_t(\Lambda_{t,t+1})\mathbb{E}_t(R_{t+1}^s)$.

$\sqrt{\text{Var}(\Lambda_{t,t+1})\text{Var}(R_{t+1}^s)}$. Therefore any factor that increases the variance of $\Lambda_{t,t+1}$ or of R_{t+1}^s will increase their covariance. In our case, endogenous beliefs formation, by inducing fluctuations in m_{t+1} , tend to increase the SDF variance, given by $\text{Var}(\Lambda_{t,t+1}) = \text{Var}\left(\beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}\right)$, determining the covariance with asset returns. To fix ideas, consider the case of a negative covariance between the SDF and the risky asset returns. In this case borrowers are less hedged, hence they require a premium. They however ask for an even higher premium in presence of uncertainty, that is when agents are unsure about the exact distribution of events. Fluctuations in the uncertainty premium are heightened in face of time-varying beliefs.

The premium also depends upon the shadow price of debt. For given covariance between the SDF and the risky return, the following derivative holds: $\frac{\partial \Psi_t}{\partial \mu_t} = R \frac{(1-\phi) - \text{cov}(\Lambda_{t,t+1}, R_{t+1}^s)}{(1-\mu_t)^2}$. If the $\text{cov}(\Lambda_{t,t+1}, R_{t+1}^s)$ is negative the derivative is certainly positive²¹, suggesting that, even without uncertainty, less-hedged borrowers require higher premia when the constraint tightens, as they are more exposed to the risk of collateral fluctuations. In presence of uncertainty, borrowers require additional premia that depend upon the time-varying attitude toward it. Fluctuations in beliefs generally raise the absolute value of the covariance. This in turn raises the sensitivity of the premia to the shadow price of debt, $\frac{\partial \Psi_t}{\partial \mu_t}$. When borrowers are uncertain about the future asset value, they require marginally higher premia when the constraint tightens.

Proposition 5.3 (Sharpe Ratio). *The Sharpe ratio in our model reads as follows:*

$$SR = \frac{\Psi_t}{\sigma_z} = \left[\frac{\sigma_{\Lambda_t^*}^2}{\bar{\Lambda}^{*2}} - 2\mu_t' \frac{(\phi-1)\Psi_t}{\sigma_{\Psi}^2} - \frac{\mu_t'^2 (\phi-1)^2}{\bar{\Lambda}^{*2} \sigma_{\Psi}^2} \right]^{\frac{1}{2}} \quad (22)$$

where $\Psi_t = \mathbb{E}_t\{R_{t+1}^s\}$ is the asset excess return, σ_{Ψ}^2 the excess return volatility, $\bar{\Lambda}$ the long run SDF value, and $\sigma_{\Lambda_t^*}^2$ the SDF volatility.

Proof is given in Appendix D.3. Matching the Sharpe ratios empirical values is typically hard for models with asset pricing and/or financial frictions²². The reason being that typically an increase in the excess returns of the risky assets is accompanied by an increase in its volatility. Our analytical derivation in (22) clarifies the channels through which our model generates the empirically-valid dynamic for the Sharpe ratio. First, fluctuations in m_{t+1} drive increased amplifications in the stochastic discount factor, Λ_t^* , hence fostering its variance, $\sigma_{\Lambda_t^*}^2$. Second, the kinked nature of the value function steepens fluctuations in m_{t+1} since marginal utilities tend to infinity around the kink. This in turn raises the SDF variance, hence the Sharpe ratio. Third, as argued earlier, in our model borrowers require an additional premium for uncertainty²³. The latter raises the excess return volatility, σ_{Ψ}^2 , which in turn boosts the SR as per equation (22).

At last, note that the Sharpe ratio depends negatively upon the shadow value of debt. When the constraint binds borrowers engage in de-leveraging. In turn they reduce the demand of risky

²¹If the $\text{cov}(\Lambda_{t,t+1}, R_{t+1}^s) > 0$, then whether $\frac{\partial \Psi_t}{\partial \mu_t}$ is positive or negative depends upon whether the $\text{cov}(\Lambda_{t,t+1}, R_{t+1}^s) < (1-\phi)$ or not.

²²In past literature it was noted that the model-implied Sharpe ratio can match the empirical counterpart by assuming implausibly large values for the risk-aversion parameter (see Cochrane (2005), chapter 13). In the numerical simulations below we show that this is not the case for our model.

²³Here we refer to the distinction between uncertainty and *risk* introduced by Knight (1921).

asset. This dampens the expected excess returns relatively to the return on debt. Interestingly, this channel is compatible with the pro-cyclical nature of the returns on risky assets observed in the data.

Proposition 5.4 (Leverage). *The equilibrium leverage ratio in our model can be easily derived by combining equations 5 and 10. This delivers:*

$$-\frac{b_t}{Q_t} \leq \phi \frac{1 - \mu'_t}{\mathbb{E}_t \Lambda_{t,t+1}} \quad (23)$$

If positive shocks persist into the future agents assign higher weights to future good states. Given the current value of the margin, namely the Lagrange multiplier on the collateral constraint, this implies an increase in leverage. The raise of debt, relative to asset value, also implies that for the slackness condition, 12, to remain valid into the future the Lagrange multiplier or the margin shall decline. Interestingly our numerical simulations in section 6.2 show that only the model with our ambiguity attitudes delivers pro-cyclical leverage, while the equivalent featuring solely occasionally binding constraints obtains counter-cyclicality.

6 Quantitative Results

To verify the quantitative relevance of our model we solve it numerically employing a global solution method (see Appendix E). We group our results in three sections. First, we search for the optimal model calibration using a mixed strategy, which employs external information, the GGM estimates for the degree of ambiguity attitudes and a moment matching routine. The latter minimizes the distance between the targets and the model-implied moments. This gives further empirical validation to our model. Second, under the optimal calibration we verify if the model can match several volatilities and correlations for equity returns, and debt. Third, under this optimal calibration we examine policy functions and conduct a crisis event study. The latter allows us to provide grounds on the ability of the model to generate meaningful crisis dynamics.

6.1 Calibration Strategy

This section describes the calibration strategy. We divide the set of structural parameters in three groups, as Table 3 shows. The first group includes parameters calibrated using external information. Those are the risk-free rate, the loan-to-value ratio, the fraction of financial wealth over total wealth and the income shock auto-correlation parameter. The second group relates to the penalty process θ , calibrated using the GMM results shown above. The last group, instead, includes the remaining parameters which are derived using a moments matching routine.

In order to calibrate the third group of parameters, we choose to match six empirical moments, whose results are shown in Table 4. The targeted moments include the debt volatility σ^b and auto-correlation ρ^b , the correlation between debt and consumption growth $Corr(\Delta b^t, \Delta c^t)$, the expected return on risky assets $\mathbb{E}_t(R_i^s)$, the volatility of risky asset returns σ^{R^s} , and the

Table 3: Values for the calibrated parameters

Parameter	Meaning	Strategy	Value
R	<i>Risk-free rate</i>	3month T-bill rate	1.0114
ϕ	<i>Loan-to-value ratio</i>	Crises Probability (4%)	0.20
α	<i>Share of dividend</i>	Fraction of financial wealth	0.11
ρ_y	<i>Income Persistence</i>	Curatola and Faia (2018)	0.634
θ^+	<i>Pessimism</i>	GMM estimation	2.500
θ^-	<i>Optimism</i>	GMM estimation	-1.907
γ	<i>Risk aversion</i>	Matching Moments ¹	2.075
β	<i>Discount factor</i>	Matching Moments	0.930
σ_y	<i>Income Volatility</i>	Matching Moments	0.0415

¹ The moment matching routine is based on the following grid: $\sigma^y \in [0.02, 0.07]$, $\beta = [0.92, 0.98]$, and $\gamma = [1, 2.2]$. For each parameter we check that the optimal values do not hit the bounds.

correlation between them and consumption growth $Corr(R_t^s, \Delta c_t)$. To compute the empirical equivalent we focus on the data sample 1980-2018, which captures a period of both large debt growth and subsequent de-leverage. More details on the data sources are in Appendix 3. Results for equity premium would be redundant in this context, given an exogenous risk-free rate in the model.

6.2 Empirical Moments Matching

In this section we evaluate the model’s ability to match some empirical moments under the optimal calibration determined above. To better gauge the role of the ambiguity attitudes we compare the theoretical moments with state-contingent ambiguity attitudes (labelled AA since now on) and without them, that is when $m_{t+1} = 1$ for all future states (labelled BE, that stands for benchmark, since now on)²⁴. Table 4 summarizes the main results. Both models exhibit amplification induced by the occasionally binding collateral constraint and match the pro-cyclicality of debt and equity returns, which is well documented in the data.

However, state-contingent ambiguity attitudes help the model-based moments to get closer to the data along several dimensions.

First, it increases the long run equity premium and the asset price volatility. It has already been noted (see Cochrane (2005)) that the ability to match contemporaneously the long run equity premia and their cyclical properties is related to the agents’ attitude toward events on the tails. For instance, if agents sensitivity toward future downturns increases, the conditional volatility of asset prices raises, and this translates in higher long run averages. Our model fea-

²⁴More specifically, we compare two identical models with the same parameters specification (see Table 3 fore reference) which differ only for the beliefs specification: the AA (Ambiguity Attitudes) model where agents are endowed with kinked beliefs and the BE model where agents display beliefs without distortions

Table 4: Empirical (1980-2018) and model-based moments

Moments	Mnemonics	Empirical	Model AA ¹	Model BE
Debt volatility	σ^b	13.381	13.821	7.326
Debt persistence	ρ^b	0.855	0.394	0.323
Debt cyclicality	$Corr(\Delta b_t, \Delta c_t)$	0.478	0.394	0.841
Leverage cyclicality	$Corr(\Delta \frac{b_t}{Q_t}, \Delta c_t)$	0.300	0.574	-0.769
Equity return	$E_t(R_t^s)$	12.184	11.940	7.438
Equity return volatility	$\sigma^{R_t^s}$	15.801	17.672	11.822
Equity return cyclicality	$Corr(\Delta R_t^s, \Delta c_t)$	0.411	0.415	0.508
Sharpe Ratio	SR	0.558	0.539	0.427

¹ Column 3 and 4 compare theoretical moments for the models with (AA) and without (BE) ambiguity.

tures precisely this property because borrowers are endogenously optimistic on the upper tail, and pessimistic on the lower tail. Second, the kink in borrowers' beliefs makes debt volatility more in line with the empirical evidence. Debt dynamics are indeed amplified by the combination of exuberance and lax debt constraints in booms and the combination of pessimism and binding constraints in recessions.

Third, the model with ambiguity attitudes is capable to match leverage pro-cyclicalities, while the baseline model cannot. This is a crucial aspect of our model. As discussed earlier, models of the credit cycle have been successful in reproducing pro-cyclicalities of debt, but fail to induce leverage pro-cyclicalities. For the latter to materialize the counter-cyclicalities of borrowers' SDF is needed jointly with occasionally binding constraints. A significant build-up of leverage, namely the ratio between debt and asset value, materializes if, after a sequence of positive shocks, borrowers assign lower weight to future downfalls in asset values, or conversely higher weight to future raises. This reduces the overall price of risk the borrower assigns to future contingencies and in turn raises the evaluation of future collateral. The latter renders the constraint slack and raises the equilibrium level of debt relatively to that of asset values.

Finally, the contemporaneous matching of both leverage and asset price statistics is usually hard to obtain in models of the financial accelerator families unless one introduces exogenous risk shocks (Christiano, Motto and Rostagno (2014)). Our model can achieve this joint matching through the combination of credit frictions and time-varying beliefs.

6.3 Model Simulations

To provide further insights on the transmission mechanism through which our model can account for the dynamic of asset prices and debt, we present two additional exercises. First, we simulate a crisis event and verify whether our model can generate meaningful dynamics around it. We focus more specifically on the model's ability, relative to the benchmark (BE), to induce the build-up of leverage and asset price booms generally observed prior to a crisis event (Reinhart and Rogoff (2011)), and the burst following it. We find that the model generates a debt build-up before the crises in line with the last two US credit boom episodes. Second, we investigate the

behavior of the policy functions. The latter speak on the role of optimism and pessimism in affecting the slackness of the debt constraint.

6.3.1 Simulated Crises Event Study

The crisis event study displayed in [Figure 1](#) highlights the model’s ability to endogenously generate financial crises. The event analysis is realized using model-simulated data. Crises are defined as events in which the collateral constraint binds and a de-leveraging of significant relevance (two standard deviations above the ergodic mean) occurs. A seven-periods event window is constructed around the crisis event, which materializes at time zero. The plots show the dynamic of selected variables prior and after the event. The first four panels of [Figure 1a](#) display the path of the main macroeconomic variables, namely debt, asset price, equity premium and consumption, for the AA model.

The pre-crisis period is characterized by a strong leverage build-up, a significant increase in asset prices and a decline in the equity return. Those are well-documented dynamics observed prior to crises. In order to identify how the switching in agents beliefs affects the above results, [Figure 1b](#) replicates the exercise by comparing the dynamics of the AA and the BE model. Interestingly in the AA model those patterns are much more pronounced compared to the BE model, where they are almost absent²⁵. In particular, in the BE model debt remains close to the ergodic mean before the crises materialization, confirming the crucial role of optimism in generating the debt growth. Agents assign higher weights to future good events and this boosts asset prices as argued also in [Proposition 5.1](#). The resulting increase in the value of collateral makes the constraint slack and favors the leveraging. Interestingly the magnitude of this build-up is close to the percentage deviation of the aggregate credit from its long-term trend registered in the last two US credit boom (see [figure 3](#) in [Appendix F](#)). At the peak-of-the-cycle US debt was 8.4% in 1988 and 32.3% in 2007 higher than the long-term trend (20.2 % on average), while our simulations predict a deviation from the ergodic mean of about 25%.

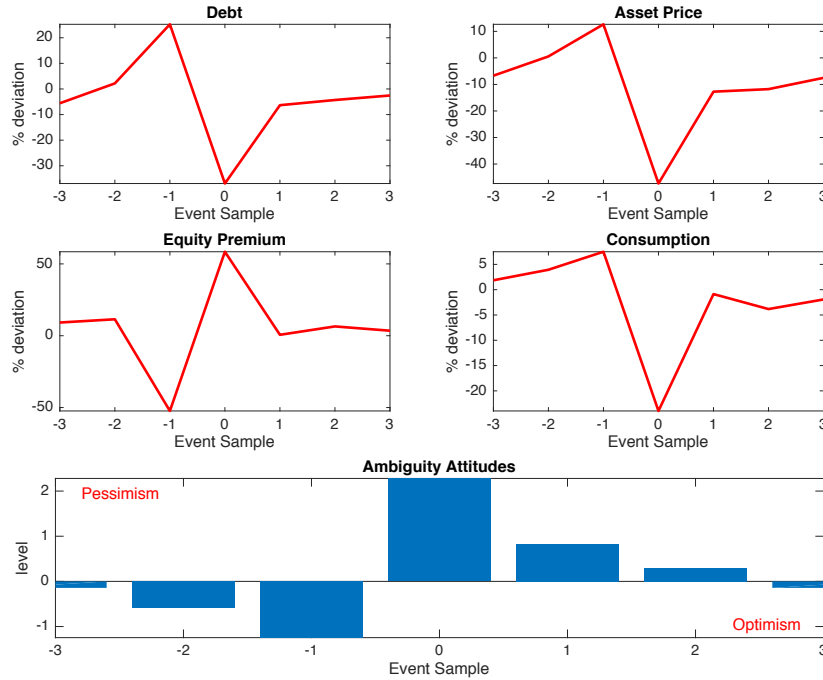
Further, [Figure 1b](#) shows that during the crisis the combination of Fisherian debt deflation and agents’ time-varying beliefs generates large declines in debt, asset prices and consumption, as well as a strong increases in the equity premium. Once more the role of ambiguity attitudes emerges neatly from the panels. While the model with occasionally binding constraints is able to generate alone a large de-leveraging, the latter is more pronounced in the AA model. The pessimistic attitudes are responsible for the sharper decline.

Finally, the last panel of [Figure 1a](#) shows the evolution of agents’ beliefs, where the values of θ interestingly fluctuate over the leverage cycle reproducing waves of optimism and pessimism. Coherently with our empirical evidence, agents are optimistic in booms ($\theta < 0$), but switch to pessimism after the large tail event materializes ($\theta > 0$).

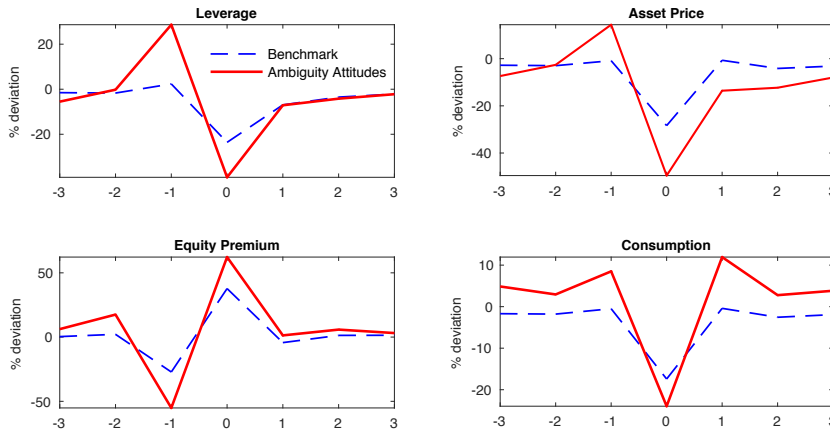
²⁵This shortcoming associated to the agents’ rationality of this class of models is known and not specifically related to our model (see the crises event studies in [Bianchi \(2011\)](#) and [Bianchi and Mendoza \(2018\)](#) among many others.

Figure 1: Simulated Crises Event Study

(a) Ambiguity Attitudes



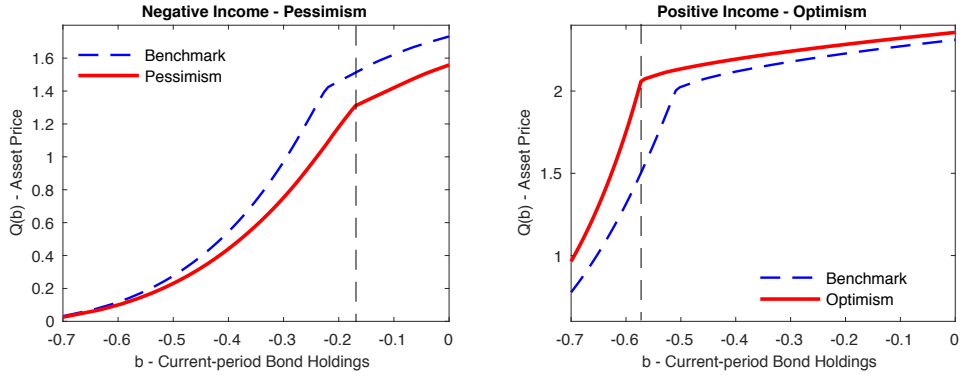
(b) Ambiguity Attitudes vs Benchmark



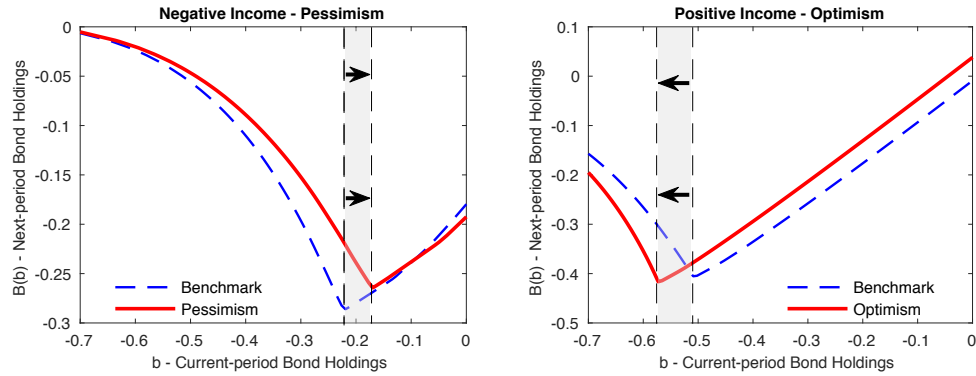
Note: The simulated crisis event is performed following *Bianchi and Mendoza (2018)*. First, we run the model unconditional simulation long path (100.000 periods). Second, over the simulated path a crisis is defined as the situation in which the collateral constraint is binding and there is a massive capital outflows (the current account is at least two standard deviations greater than its ergodic mean). Then, we study the average behavior of the main model variables around (three period before and three after) the identified crisis events. The path of the endogenous macroeconomic variables (debt, asset price, equity premium and consumption) is expressed in terms of deviations from the respective ergodic means.

Figure 2: Policy Functions Analysis

(a) Asset Price



(b) Bond holdings



6.3.2 Policy Functions Analysis

At last, we further investigate the model transmission mechanism by inspecting policy function dynamics. First, we comment further on the role of ambiguity attitudes in enhancing asset price and leverage fluctuations, also in comparison to the benchmark model. Second, given the kinked nature of the policy functions induced by the debt constraint, we are able to derive results also in terms of shifts of the constrained/unconstrained regions. Third, we comment further on why time-varying beliefs in our model crucially matter in reproducing pro-cyclicality of leverage on top and beyond pro-cyclicality of credit.

Figure 2 shows the decision rules for debt and asset prices, comparing the model with ambiguity attitudes (red lines) to the one without (blue-dotted lines). In order to appreciate the non-linearity coming from the kinked beliefs, we show the policy functions associated to a positive income realization (+5% from the ergodic mean; right panels) and those associated to a negative realization (−5% from the ergodic mean; left panels). The extreme income realizations allow us to better compare the decision rules of optimistic attitudes, namely those arising for sure along the upper tail of the income distribution, and pessimistic ones, namely those arising for sure along the lower tail.

Figure 2a plots the asset price decision rule as a function of the current-period debt holdings, our endogenous state variable. As expected, the policy functions are highly non-linear. The kink corresponds to the level of current-period bond holdings that makes the collateral constraint marginally binding. On the left of the vertical line, namely when current-period debt is high, the collateral constraint is binding. Financial amplification is stronger in this region, as shown by the large swings in asset prices. The opposite is true when the constraint is slack, that is on the right of the vertical line. Time-varying beliefs also contribute to heighten model's non-linearities. For positive realizations, hence optimistic beliefs, the policy function of the AA model lies above the corresponding BE one. Given the same exogenous shocks, optimistic agents demand more of the risky asset, hence boost its price. The opposite is true for negative realizations, hence pessimistic attitudes. These results are in line with Proposition 5.1.

Figure 2b depicts the dynamic of debt. Once again the role of non-linearities is evident. The collateral constraint generates a *V-shaped* bond holdings decision rule²⁶, with the collateral constraint being binding on the left and slack on the right of the kink. The comparison between the AA and BE models highlights two things. First, agents' attitude affects asymmetrically the size of the constrained region. It becomes smaller in good states, hence with optimistic borrowers, and larger in bad states, hence with pessimistic borrowers. This implies that optimistic agents have a larger current and perceived (future) debt capacity in good states, which contributes to the sharper leverage build-up, relatively to the BE model²⁷. From equation 10 the Lagrange multiplier or the debt margin, μ_t , when positive would read as follows:

$$1 - \mu'_t = \beta R \frac{\mathbb{E}_t \{m_{t+1} u_c(c_{t+1})\}}{u_c(c_t)}, \quad \mu'_t = \frac{\mu_t}{u_c(c_t)} \quad (24)$$

The above equation, which characterizes the constrained region, provides a good guide to understand the role of beliefs in the shift of the kink. Pessimist beliefs, by over-weighting future low consumption states, tilt the stochastic discount factor towards the left tail and thereby increase the value of μ_t . This explains well why the kink in bottom left panel of Figure 2b shifts to the right. The opposite happens under optimistic beliefs. Agents set $m_{t+1} < 1$ to the emergence of a lower tail in the distribution for future events. This in turn raises the SDF and lowers the debt margin, μ_t . Interestingly, the debt margin, μ_t , tends to behave counter-cyclically, raising in bad states and falling in good states. This is well in line with empirical evidence and provides a crucial intersection in the transmission characterizing a leverage cycle, on top and above a credit cycle. In credit cycle models, which assume an always binding constraint, the association between the value of collateral and debt, also leads to pro-cyclicality of debt, but not to the pro-cyclicality of leverage. The latter arises only if debt margins behave counter-cyclically. For this to happen the model needs to feature time-varying attitudes toward uncertainty that tend to place more weight on the tails as argued so far.

²⁶This is re-assuring as those dynamics are common to models with high deleveraging and financial crises (see Bianchi (2011) and Bianchi and Mendoza (2018) among many others)

²⁷Under optimism or solely positive shocks, panel on the bottom right, the level of debt is lower under the AA model than under the baseline model for each shock realization. However, the unconstrained region is larger under the AA model. Hence in aggregate there is larger leverage build-up under optimism.

Second, in the binding region of the left panel (negative states), the fall in next-period debt is higher in the AA model relatively to the BE one. Once more, the left-skewed beliefs are responsible for the sharper de-leveraging relatively to the BE model. Third, the kink in the policy function shifts to the right under pessimistic beliefs and to the left under optimistic ones.

7 Conclusions

Two compelling motivations guide our work. First, financial crisis are most often triggered by endogenous instability in debt markets. The latter are typically characterized by collateral constraints and opacity in asset values. Under lack of transparency, the beliefs formation process acquires an important role since eventually it affects the value of collateral and with it debt capacity. Models combining financial frictions, in the form of collateral constraints, and information frictions are still very rare. Second, the narrative of most financial crises depicts sharp increases in debt and asset prices prior to them and sharp reversal afterwards. A leverage cycle, more than a credit cycle, seems to be at work, particularly so around crises events. For a leverage cycle to emerge the model needs to account for time-varying loan-to-value ratios or debt margins, namely the collateral constraint multiplier. To achieve such an outcome uncertainty shall be introduced in the model. Time-varying beliefs which endogenously produce waves of optimism in good times reduce the contemporaneous loan-to-value ratio, debt margin relative to collateral value, making the constraint slack and allowing leverage to build up. The opposite is true in bad times.

Motivated by the above considerations, we introduce endogenous time-varying optimizing beliefs in a model in which borrowers fund risky assets through debt and are subject to occasionally binding collateral constraints. A strength of the model is that beliefs are not exogenously imposed, but arise from a well-founded decision problem subject to an entropy constraint. A second strength is that we provide strong empirical ground to the process governing the penalty parameter on the entropy constraint. We do so through a novel GMM estimation strategy. In the model waves of optimism emerge in booms, while waves of pessimism arise in recessions. Time-varying beliefs in turn affect the debt margins in a way that produces sharp leverage cycles. Intuitively, in booms optimistic borrowers demand more risky assets, which results in higher asset price growth compared to the case with only collateral constraints, and lever up more. In recessions pessimistic borrowers de-leverage sharply and off load risky assets. At the same time the shadow price of debt or the debt margin is lowered under optimistic beliefs, hence after an history of positive shocks, and is increased under pessimistic beliefs, hence after an history of negative shocks. This helps to reproduce a leverage cycle on top and above a credit cycle.

Finally, we show through numerical simulations, based on non-linear global methods, that the two main elements of our model, namely occasionally binding constraints and endogenous beliefs, provide not only a good accounting of the crises unfolding, but are also capable of matching a number of asset price and debt facts, both in the long run and over the business cycle.

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A Kinked beliefs versus Ghirardato, Maccheroni and Marinacci (2004)'s biseparable preferences

One of the paper contributions consists in the generalization to a dynamic context of the biseparable preferences a' la Ghirardato and Marinacci (2001) and Ghirardato, Maccheroni and Marinacci (2004). The latter can also be framed in terms of *kinked multiplier beliefs*, but allows us to account for state-contingent attitudes changing from averse to seeking, and the other way round. These preferences are defined as follows:

$$V(c_t) = \begin{cases} \min_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^+ \varepsilon(m_{t+1}) \right\} & \text{if } V_{t-1} \leq \mathbb{E}_{t-1}\{V_t\} \\ \max_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^- \varepsilon(m_{t+1}) \right\} & \text{if } V_{t-1} > \mathbb{E}_{t-1}\{V_t\} \end{cases} \quad (25)$$

Given the specification in equation 25, we can represent our kinked beliefs as follows:

$$V_t(c_t) = \mathbb{I}_{\theta_t \geq 0} \min_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta \varepsilon(m_{t+1}) \right\} + \mathbb{I}_{\theta_t < 0} \max_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta \varepsilon(m_{t+1}) \right\} \quad (26)$$

As noted in Ghirardato, Maccheroni and Marinacci (2004) the indicator function shall depend only upon expected utility mapping. We design the following expected utility mapping. We pose that $\theta_t < 0$ whenever $V_{t-1} > \mathbb{E}_{t-1}\{V_t\}$, which since now we often refer as the gain domain, and viceversa for the loss domain. We can therefore re-write our preferences as:

$$V_t(c_t) = \mathbb{I}_{V_{t-1} \leq \mathbb{E}_{t-1}\{V_t\}} \min_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^+ \varepsilon(m_{t+1}) \right\} + \mathbb{I}_{V_{t-1} > \mathbb{E}_{t-1}\{V_t\}} \max_{\{m_{t+1}, M_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \left\{ \beta^t M_t u(c_t) + \beta \theta^- \varepsilon(m_{t+1}) \right\} \quad (27)$$

B Micro-foundation of the collateral constraint

In this section we provide micro-foundations for a delegated monitoring problem in which the collateral constraint emerges as result of an incentive-compatible debt contract enforced through a bank. The micro-foundations follow Bianchi and Mendoza (2018). Debt contracts are signed by a bank that must enforce debtor incentives. Between periods borrowers can divert revenues for an amount \tilde{d}_t . At the end of the period the diversion is no longer possible and payment is enforced. Banks can monitor financial diversion due to special relationship lending abilities²⁸. If the bank detects the diversion, the asset posed as collateral can be seized up to a percentage

²⁸We assume zero monitoring costs for simplicity. Extending it to the case with positive monitoring costs is rather straightforward.

ϕ . As common in dynamic economies we assume that the contract is done under no memory, so that in the next period borrowers can re-enter the debt agreement even if they defaulted in the previous period. This assumption allows us to preserve the Markov structure of the contracting/intermediation problem.

The collateral constraint can be derived from an incentive-compatibility constraint on borrowers if limited enforcement prevents banks from redeploying more than a fraction ϕ of the value of the assets owned by a defaulting borrower. Define V^R and V^D respectively as the value of repayment and default and define as V the continuation value.

If the borrower defaults, the diverted resources enter his budget constraint and the recursive problem reads as follows (for notational convenience we skip the beliefs constraints for the purpose of this derivation):

$$\begin{aligned} V^D(b_t, x_t, \mathbf{S}_t) = & \max_{c_t, x_{t+1}, b_{t+1}} \{u(c_t) + \beta \mathbb{E}_t [m_{t+1} V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})]\} + \\ & + \lambda_t \left[y_t + Q(\mathbf{S}_t)(x_t + \alpha y_t) + \tilde{d}_t + b_t - Q(\mathbf{S}_t)x_{t+1} - c_t - \frac{b_{t+1}}{R} \right] + \\ & + \mu_t \left[\phi Q(\mathbf{S}_t)x_{t+1} + \frac{b_{t+1}}{R} \right] \end{aligned} \quad (28)$$

On the other side, if the borrower repays his value function reads as follows:

$$\begin{aligned} V^R(b_t, x_t, \mathbf{S}_t) = & \max_{c_t, x_{t+1}, b_{t+1}} \{u(c_t) + \beta \mathbb{E}_t [m_{t+1} V(b_{t+1}, x_{t+1}, \mathbf{S}_{t+1})]\} + \\ & + \lambda_t \left[y_t + Q(\mathbf{S}_t)(x_t + \alpha y_t) + b_t - Q(\mathbf{S}_t)x_{t+1} - c_t - \frac{b_{t+1}}{R} \right] + \\ & + \mu_t \left[\phi Q(\mathbf{S}_t)x_{t+1} + \frac{b_{t+1}}{R} \right] \end{aligned} \quad (29)$$

The comparison of the two easily shows that the households repay if and only if $\tilde{d}_{t+1} < \phi Q(\mathbf{S})x_{t+1}$. In this case debt can be rolled-over across periods.

C GMM Estimation of the Ambiguity Process

In this section we detail the derivations needed to achieve the moment condition that is the object of our GMM estimation. Further below we also provide a description of the dataset used in the estimation.

C.1 General Approach

We use a GMM estimation procedure based on the moment condition obtained from the combined Euler equation for debt and risky assets. The methodology is a variant of the techniques developed for asset pricing models with recursive preferences, pioneered by [Epstein and Zin \(1989\)](#) and [Kreps and Porteus \(1978\)](#). Hence, the starting point is to reformulate our value function, capturing multiplier beliefs, in terms of an ambiguity term. The latter is achieved by

mapping the multiplier preferences to a special case of the recursive ones. This can be done by assuming a logarithmic continuation value, a logarithmic utility function and an ambiguity adjustment factor, Q , which accounts for waves of optimism and pessimism. We depart from the well-known equivalence between multiplier and recursive preferences by embedding state-contingent ambiguity attitudes. We start by reporting the value function derived after substituting the solution of the inner problem, presented in Section 3.3.1:

$$V_t = u(c_t) - \beta \theta_t \log \left[\mathbb{E}_t \left\{ \exp \left(-\frac{V_{t+1}}{\theta_t} \right) \right\} \right] \quad (30)$$

The above equation embeds a logarithmic ambiguity-adjusted component, defined as $Q_t(V_{t+1})$, which maps future continuation values into current realizations. We can re-write (30) as follows:

$$\begin{aligned} V_t &= u(c_t) + \beta h^{-1} \mathbb{E}_t \{ h(V_{t+1}) \} \\ &= u(c_t) + \beta Q_t(V_{t+1}) \end{aligned} \quad (31)$$

where $h(V_{t+1}) = \exp \left(-\frac{V_{t+1}}{\theta_t} \right)$, as implied by the equivalence between specifications under recursive and multiplier preferences (see Hansen et al. (2007)). It then follows that the ambiguity adjustment component reads as follows:

$$Q_t(V_{t+1}) = h^{-1} \mathbb{E}_t \{ h(V_{t+1}) \} = -\theta_t \log \left[\mathbb{E}_t \left\{ \exp \left(-\frac{V_{t+1}}{\theta_t} \right) \right\} \right] \quad (32)$$

C.2 Pricing Kernel-SDF

The next step to obtain our moment condition is to derive an expression for the stochastic discount factor as function of $Q_t(V_{t+1})$. To this purpose, we shall compute the marginal utility of consumption and the derivative of the current value function with respect to the next period one, which we define as MV_{t+1} and which reads as follows:

$$\begin{aligned} MV_{t+1} &= \frac{\partial V_t}{\partial Q_t(V_{t+1})} \frac{\partial Q_t(V_{t+1})}{\partial V_{t+1}} = \beta \frac{\exp(-\frac{V_{t+1}}{\theta_t})}{\mathbb{E}_t \left\{ \exp(-\frac{V_{t+1}}{\theta_t}) \right\}} \\ &= \beta \exp \left(-\frac{1}{\theta_t} (V_{t+1} - Q_t(V_{t+1})) \right) \end{aligned} \quad (33)$$

Given a logarithmic utility function $u(c_t) = \log(c_t)$, the marginal utility of consumption is $MC_t = c_t^{-1}$. Using the above expressions we can derive the SDF as function of Q_t :

$$\Lambda_{t,t+1} = \frac{MV_{t+1} MC_{t+1}}{MC_t} = \beta \frac{c_{t+1}^{-1}}{c_t} \underbrace{\exp \left(-\frac{1}{\theta_t} (V_{t+1} - Q_t(V_{t+1})) \right)}_{m_{t+1}} \quad (34)$$

where $m_{t+1} = \exp \left(-\frac{1}{\theta_t} (V_{t+1} - Q_t(V_{t+1})) \right)$ is the optimal likelihood ratio. Equation (34) shows that the SDF has a two-factor structure. The first is the standard consumption growth, while the second is the ambiguity factor. The latter depends upon the distance between the future

value function and its certainty equivalent, namely the future insurance premium. Under no uncertainty this premium vanishes²⁹.

C.3 Estimation of the Continuation Value Ratio

Since estimation requires strictly stationary variables, we shall re-scale the value function (31) by consumption (see Hansen, Heaton and Li (2008) (HHL henceforth). Subtracting the log of consumption, $\tilde{c}_t = \log(c_t)$, on both sides we have that:

$$\tilde{v}_t = \beta Q_t(\tilde{v}_{t+1} + \Delta\tilde{c}_{t+1}) \quad (35)$$

where we define $\tilde{v}_t = V_t - \tilde{c}_t$ as the *continuation value ratio*, scaled by the log of consumption. Next substituting (32) into (35) we obtain:

$$\tilde{v}_t = -\beta\theta_t \log(\mathbb{E}_t \{\exp[\sigma_t(\tilde{v}_{t+1} + \Delta\tilde{c}_{t+1})]\}) \quad (36)$$

where $\sigma_t = -1/\theta_t$, and it is negative when $\theta_t > 0$ and positive when $\theta_t < 0$. An expression for equation (36) can be derived analytically along the lines of HHL. Indeed, since \tilde{v}_t is a function of states governing the dynamic behaviour of consumption growth, g_{t+1}^c , we can guess it as a function of a Markov process, defined as ξ_t :

$$g_{t+1}^c = \tilde{c}_{t+1} - \tilde{c}_t = \mu_c + H\xi_t + \mathbf{A}\epsilon_{t+1} \quad (37)$$

$$\xi_{t+1} = F\xi_t + \mathbf{B}\epsilon_{t+1} \quad (38)$$

where ϵ_{t+1} is a $(2x1)$ i.i.d. vector with zero mean and covariance matrix I . A and B are $(2x1)$ vectors. The exogenous states, ϵ_{t+1} , income shocks in our case, have an impact on consumption directly and through the states, ξ_t . Its estimated value, defined as $\hat{\xi}_t$, is obtained through Kalman filtering of consumption data. Then, given (37), we guess the continuation value ratio as depending only upon the estimated states, $\hat{\xi}_t$:

$$\tilde{v}_t = \mu_v + U_v \hat{\xi}_t \quad (39)$$

where $U_v \hat{\xi}_t$ is the discounted sum of expected future growth rates of consumption. After some derivations we can write U_v and μ_v as follows:

$$U_v \equiv \beta(I - \beta F)^{-1} H \quad (40)$$

$$\mu_v \equiv \frac{\beta}{1 - \beta} \left(\mu_c + \frac{\sigma_t}{2} |A + U_v B|^2 \right)$$

where the term $A + U_v B$ captures the dependence between the the continuation value and the exogenous shocks.

²⁹Indeed the continuation value would be perfectly predictable $\left(\exp\left(-\frac{V_{t+1}}{\theta}\right) = \mathbb{E}_t \exp\left(-\frac{V_{t+1}}{\theta}\right), m_{t+1}^* = 1 \right)$ with zero adjustment $(Q_t(V_{t+1}) = V_{t+1})$.

C.4 SDF and the Euler Equation

Next, given the estimated \tilde{v}_t from (39), substituting (36) into (34) delivers:

$$\Lambda_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1} \left(\frac{\frac{\exp(V_{t+1}) c_{t+1}}{c_{t+1}} \frac{c_t}{c_t}}{\exp(Q_t (\tilde{v}_{t+1} + \Delta \tilde{c}_{t+1}))} \right)^\sigma \quad (41)$$

Note that equation (41) is equivalent to the SDF obtained under Epstein and Zin (1989) preferences under the assumption of unitary EIS. At last, upon using (35) into (41) and upon substituting the resulting SDF into the combined Euler for debt and risky assets, namely equations (10) and (11), we obtain:

$$\mathbb{E}_t \left\{ \underbrace{\beta \left(\frac{c_{t+1}}{c_t} \right)^{-1} \left(\frac{\exp(V_{t+1}) c_{t+1}}{c_{t+1}} \frac{c_t}{c_t} \right)^{\sigma_t}}_{\Lambda_{t,t+1}} (R_{t+1}^s - \phi R_{t+1}) + \phi - 1 \right\} = 0 \quad (42)$$

where $R_{t+1}^s = \frac{d_{t+1} + q_{t+1}}{q_t}$. For the estimation we shall write the debt rate as time-varying.

D Analytical Derivations

This appendix derives analytical expressions for asset prices and returns presented in the propositions of the main text.

D.1 Asset Price

From the borrowers' optimality condition on risky assets we can write the asset price at time t as follows:

$$\begin{aligned} q_t &= \beta \mathbb{E}_t \left\{ \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1} (q_{t+1} + d_{t+1}) \right\} + \phi \mu'_t q_t \\ &= \beta \mathbb{E}_t \{ \Lambda_{t,t+1} (d_{t+1} + q_{t+1}) \} + \phi \mu'_t q_t \end{aligned} \quad (43)$$

where we have used the following definitions for the SDF $\Lambda_{t,t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}$ and the adjusted Lagrange multiplier on the collateral constraint, which we defined as $\mu'_t = \frac{\mu_t}{u_c(c_t)}$. Then denoting $K_{t,t+1} = \frac{\Lambda_{t,t+1}}{1 - \phi \mu'_t}$, we derive the following expression for the asset price:

$$q_t = \mathbb{E}_t \{ K_{t,t+1} (d_{t+1} + q_{t+1}) \} \quad (44)$$

Proceeding by forward substitution:

$$\begin{aligned}
q_t &= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}(d_{t+2} + q_{t+2}))\} \\
&= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2})\} + \mathbb{E}_t\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}(d_{t+3} + q_{t+3})\} \\
&= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2} + K_{t+1,t+2}K_{t+2,t+3}d_{t+3})\} + \\
&\quad + \mathbb{E}_t\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}(d_{t+4} + q_{t+4})\} \\
&= \mathbb{E}_t\{K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2} + \\
&\quad + K_{t+1,t+2}K_{t+2,t+3}d_{t+3} + K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}d_{t+4})\} + \\
&\quad + \mathbb{E}_t\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}q_{t+4}\}
\end{aligned} \tag{45}$$

At the final step, the solution for the asset price reads as follows:

$$q_t = \mathbb{E}_t \left\{ \sum_{i=1}^T d_{t+i} \prod_{j=1}^i K_{t+j-1,t+j} \right\} + \mathbb{E}_t \left\{ \prod_{i=0}^T K_{t+i,t+i+1} q_{t+T} \right\} \tag{46}$$

Taking the limit for $T \rightarrow \infty$ of the above condition delivers equation (20).

D.2 The Equity Premium

Expanding the borrower's FOC for the risky asset and plugging in it the derivation for $\mathbb{E}_t\{\Lambda_{t,t+1}\}$ and the definition $R_{t+1}^s = \frac{q_{t+1} + d_{t+1}}{q_t}$ we get:

$$\begin{aligned}
1 &= \mathbb{E}_t\left\{\Lambda_{t,t+1} \frac{q_{t+1} + d_{t+1}}{q_t}\right\} + \phi\mu'_t \\
&= \mathbb{E}_t\{\Lambda_{t,t+1}\}\mathbb{E}_t\left\{\frac{q_{t+1} + d_{t+1}}{q_t}\right\} + \text{Cov}\left(\Lambda_{t,t+1}, \frac{q_{t+1} + d_{t+1}}{q_t}\right) + \phi\mu'_t \\
&= \left(\frac{1 - \mu'_t}{R}\right)\mathbb{E}_t\{R_{t+1}^s\} + \text{Cov}(\Lambda_{t,t+1}, R_{t+1}^s) + \phi\mu'_t
\end{aligned} \tag{47}$$

Rearranging the expression above, we obtain the return on risky assets as:

$$\mathbb{E}_t\{R_{t+1}^s\} = \frac{R(1 - \text{cov}(\Lambda_{t,t+1}, R_{t+1}^s) - \phi\mu'_t)}{1 - \mu'_t} \tag{48}$$

Upon subtracting the risk-free rate, the premium between the return on the risky asset and the risk-free rate can be derived as follows:

$$\Psi_t = \frac{R(1 - \text{cov}(\Lambda_{t,t+1}, R_{t+1}^s) - \phi\mu'_t)}{1 - \mu'_t} - R. \tag{49}$$

D.3 The Sharpe Ratio and the Hansen and Jagannathan (1991) Bounds

Writing down the two borrowers' optimal conditions for the risk-free and risky assets, respectively:

$$1 = \mathbb{E}_t\{\Lambda_{t,t+1}R\} + \mu'_t \quad (50)$$

$$1 = \mathbb{E}_t\{\Lambda_{t,t+1}R_{t+1}^s\} + \phi\mu'_t \quad (51)$$

where $\mu'_t = \frac{\mu_t}{u_c(c_t)}$, $\Lambda_{t,t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}$ and $R_{t+1}^s = \frac{q_{t+1} + d_{t+1}}{q_t}$. In order to derive the excess return between the risky asset and the risk-free asset, we subtract (50) from (51), obtaining:

$$0 = \mathbb{E}_t\{\Lambda_{t,t+1}(R_{t+1}^s - R)\} + \mu'_t(\phi - 1). \quad (52)$$

Then, we define the excess return as $z_{t+1} = R_{t+1}^s - R$. Assuming a linear general form for the stochastic discount factor $\Lambda_{t,t+1}$:

$$\Lambda_{t,t+1}^* = \bar{\Lambda}^* + \tilde{\beta}^m(z_{t+1} - E_t z_{t+1}) \quad (53)$$

The above shall satisfy the following condition:

$$0 = \mathbb{E}_t\{\Lambda_{t,t+1}^* z_{t+1}\} + \mu'_t(\phi - 1), \quad (54)$$

which, once expanded, gives:

$$\begin{aligned} 0 &= \mathbb{E}_t\{\Lambda_{t,t+1}^*\} \mathbb{E}_t\{z_{t+1}\} + \text{cov}(\Lambda_{t,t+1}^*, z_{t+1}) + \mu'_t(\phi - 1) \\ &= \mathbb{E}_t\{\Lambda_{t,t+1}^*\} \mathbb{E}_t\{z_{t+1}\} + \mathbb{E}_t\{(z_{t+1} - \bar{z})(\Lambda_{t,t+1}^* - \bar{\Lambda}^*)\} + \mu'_t(\phi - 1) \\ &= \mathbb{E}_t\{\Lambda_{t,t+1}^*\} \mathbb{E}_t\{z_{t+1}\} + \mathbb{E}_t\{(z_{t+1} - \bar{z})(z_{t+1} - \bar{z})\tilde{\beta}^m\} + \mu'_t(\phi - 1) \\ &= \mathbb{E}_t\{\Lambda_{t,t+1}^*\} \mathbb{E}_t\{z_{t+1}\} + \sigma_z^2 \tilde{\beta}^m + \mu'_t(\phi - 1). \end{aligned} \quad (55)$$

Hence:

$$\tilde{\beta}^m = -(\sigma_z^2)^{-1} \mathbb{E}_t\{\Lambda_{t,t+1}^*\} \mathbb{E}_t\{z_{t+1}\} - (\sigma_z^2)^{-1} \mu'_t(\phi - 1) \quad (56)$$

The variance of the stochastic discount factor is then obtained as follows:

$$\begin{aligned} \text{Var}(\Lambda_{t,t+1}^*) &= \text{Var}((z_{t+1} - \mathbb{E}_t\{z_{t+1}\})' \tilde{\beta}^m) \\ &= \tilde{\beta}^m \sigma_z^2 \tilde{\beta}^m \\ &= (-(\sigma_z^2)^{-1} \bar{\Lambda}_t^* \mathbb{E}\{z_{t+1}\} - (\sigma_z^2)^{-1} \mu'_t(\phi - 1))' \sigma_z^2 \\ &\quad (-(\sigma_z^2)^{-1} \bar{\Lambda}_t^* \mathbb{E}\{z_{t+1}\} - (\sigma_z^2)^{-1} \mu'_t(\phi - 1)) \\ &= (\sigma_z^2)^{-1} (\bar{\Lambda}^*)^2 (\mathbb{E}_t\{z_{t+1}\})^2 + \\ &\quad + 2\mu'_t(\phi - 1) ((\sigma_z^2)^{-1} \bar{\Lambda}_t^* \mathbb{E}\{z_{t+1}\}) + ((\sigma_z^2)^{-1} \mu'_t(\phi - 1))^2. \end{aligned} \quad (57)$$

Rearranging the expression above delivers:

$$\frac{\sigma_{\Lambda_t^*}^2}{\Lambda^{*2}} = \frac{(\mathbb{E}_t\{z_{t+1}\})^2}{\sigma_z^2} + 2\mu_t' \frac{(\phi - 1)\mathbb{E}_t\{z_{t+1}\}}{\sigma_z^2} + \frac{\mu_t'^2 (\phi - 1)^2}{\Lambda^{*2} \sigma_z^2}. \quad (58)$$

The Sharpe Ratio (SR hereafter) on stock asset returns over bonds results to be:

$$SR = \frac{(\mathbb{E}_t\{z_{t+1}\})^2}{\sigma_z^2} = \frac{\sigma_{\Lambda_t^*}^2}{\Lambda^{*2}} - 2\mu_t' \frac{(\phi - 1)\mathbb{E}_t\{z_{t+1}\}}{\sigma_z^2} - \frac{\mu_t'^2 (\phi - 1)^2}{\Lambda^{*2} \sigma_z^2} \quad (59)$$

Thus, the SR depends upon the variance of the stochastic discount factor, adjusted for distorted beliefs, and upon μ_t' .

E Numerical Method

Our numerical method extends the algorithm of [Jeanne and Korinek \(2010\)](#) to persistent shocks and state-contingent ambiguity attitudes. The method, following the endogenous grid points approach of [Carroll \(2006\)](#), performs backward time iteration on the agent's optimality conditions. We derive the set of policy functions $\{c(b, s), b'(b, s), q(b, s), \mu(b, s), V(b, s)\}$ that solve the competitive equilibrium as described by the system:

$$c(b, s)^{-\gamma} = \beta R \mathbb{E} \{m(b', s')c(b', s'^{-\gamma})\} + \mu(b, s) \quad (60)$$

$$q(b, s) = \beta \frac{\mathbb{E} \{m(b', s')c(b', s'^{-\gamma}[q(b', s') + \alpha y']\}}{c(b, s)^{-\gamma} - \phi \mu(b, s)} \quad (61)$$

$$\mu(b, s) \left[\frac{b'(b, s)}{R} + \phi q(b, s) \right] = 0 \quad (62)$$

$$c(b, s) + \frac{b'(b, s)}{R} = y + b \quad (63)$$

$$V(b, s) = \frac{c(b, s)^{1-\gamma} - 1}{1-\gamma} + \frac{\beta}{\sigma} \ln \mathbb{E} \{ \exp\{\sigma V(b', y')\} \} \quad (64)$$

where $m(b, s)$ is the expectation distortion increment. The solution method proceeds according to the following steps:

1. We set a grid $G_b = \{b_1, b_2, \dots, b_H\}$ for the next-period bond holding b' and a grid $G_s = \{s_1, s_2, \dots, s_N\}$ for the shock state space $s = \{y, \sigma\}$. The income process y , is discretized with [Tauchen and Hussey \(1991\)](#) method, while the grid for the *inverse* of the penalty parameter σ (recall that θ is the inverse of σ) follows a simple two-state rule:³⁰

$$\sigma = \begin{cases} \sigma^+ & \text{if } V < \mathbb{E}\{V\} \\ \sigma^- & \text{if } V \geq \mathbb{E}\{V\} \end{cases} \quad (65)$$

³⁰We use 800 grids point for bonds and 45 grid points for the exogenous shocks; we implement linea interpolation in order to approximate the policy functions outside the grids

2. In iteration step k , we start with a set of policy functions $c_k(b, s)$, $q_k(b, s)$, $\mu_k(b, s)$ and $V_k(b, s)$. For each $b' \in G_b$ and $s' \in G_s$:

a) we derive the expectation distortion increment:

$$m_k(b', s') = \frac{\exp\{\sigma V_k(b', s')\}}{\mathbb{E}[\exp\{\sigma V_k(b', s')\}]} \quad (66)$$

and then, the distorted expectations in the Euler equation for bonds and for the risky assets (equations (1) and (2)).

b) we solve the system of optimality conditions under the assumption that the collateral constraint is slack:

$$\mu^u(b', s) = 0 \quad (67)$$

As a result, $c^u(b', s)$, $q^u(b', s)$, $\mu^u(b', s)$, $V^u(b', s)$ and $b^u(b', s)$ are the policy functions for the unconstrained region;

c) in the same way, we solve the system for the constrained region of the state space, where the following condition holds:

$$q^c(b', s) = -\frac{b'/R}{\phi} \quad (68)$$

$c^c(b', s)$, $q^c(b', s)$, $\mu^c(b', s)$, $V^c(b', s)$ and $b^c(b', s)$ are the respective policy functions.

d) we derive the next period bond holding threshold \bar{b}' such that the borrowing constraint is marginally binding. For each $s \in G_s$ it satisfies the following condition:

$$\bar{b}'^c(\bar{b}', s) + \frac{\bar{b}'(s)}{R} = 0 \quad (69)$$

When this point is out of the grid we use linear interpolation. Given this value, we can derive for each policy function the frontier between the binding and non-binding region: $x^u(\bar{b}'^c(\bar{b}', s))$ for $x = \{c, b, q, \mu, V\}$.

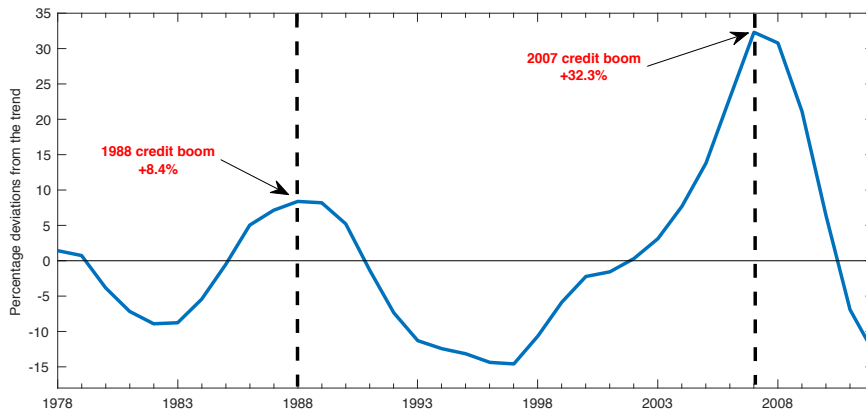
3. In order to construct the step $k+1$ policy function, $x_{k+1}(b, s)$, we interpolate on the pairs $(x^c(b'^c(b', s)))$ in the constrained region and on the pairs $(x^u(b'^u(b', s)))$ in the unconstrained region. As a result we find: $c_{k+1}(b, s)$, $q_{k+1}(b, s)$, $\mu_{k+1}(b, s)$ and $V_{k+1}(b, s)$.

4. We then evaluate convergence. When:

$$\sup \|x_{k+1} - x_k\| < \epsilon \quad \text{for} \quad x = c, q, \mu, V \quad (70)$$

we establish that the competitive equilibrium has been reached. Otherwise, we set $x_k(b, s) = (1 - \delta)x_{k+1}(b, s) + \delta x_k(b, s)$ and continue the iterations from point 2. We use a value of δ close to 1.

Figure 3: US leverage cycle and credit booms



Note: In the chart we plot the deviation of the aggregate credit from its long term trend, obtained by HP-filtering debt data with a smoothing parameter λ equal to 400.000. In the leverage cycle literature this parameter is set to account its higher average duration with respect to business cycles. The two credit booms, indicated with the dotted vertical lines, are identified following ?.

F Data Description for Empirical Moments

In this section we describe the data employed for the computation of the empirical target moments used for model matching. We compute several moments for asset prices, returns and debt data. Data are for US. The sample spans 1980:Q1 to 2018:Q4, since this corresponds to the period of rapid debt growth and decline. Debt is given by private non-financial sector for all sectors taken from BIS: http://www.bis.org/publ/qtrpdf/r_qt1403g.pdf, consumption is given by Personal Consumption Expenditure taken from the NIPA Tables³¹, GDP is also taken from the NIPA Tables, the risk-free rate is the 3month T-bill rate taken from the CRSP Indices database³², risky returns are proxied by the S&P500 equity return with dividends from the Shiller Database³³. All variables are deflated by CPI index. Note that HP-filtered series are computed as deviations from a long-term trend. Therefore, we work with a much larger smoothing parameter ($\lambda = 400,000$) than the one employed in the business cycle literature, to pick up the higher expected duration of the credit cycle (see <http://www.bis.org/publ/bcbs187.pdf>).

G Intermediation Sector and Intermediation Shocks

Lack of transparency and ambiguity play an important role in crises developments as we showed so far, but the instability in the intermediation sector

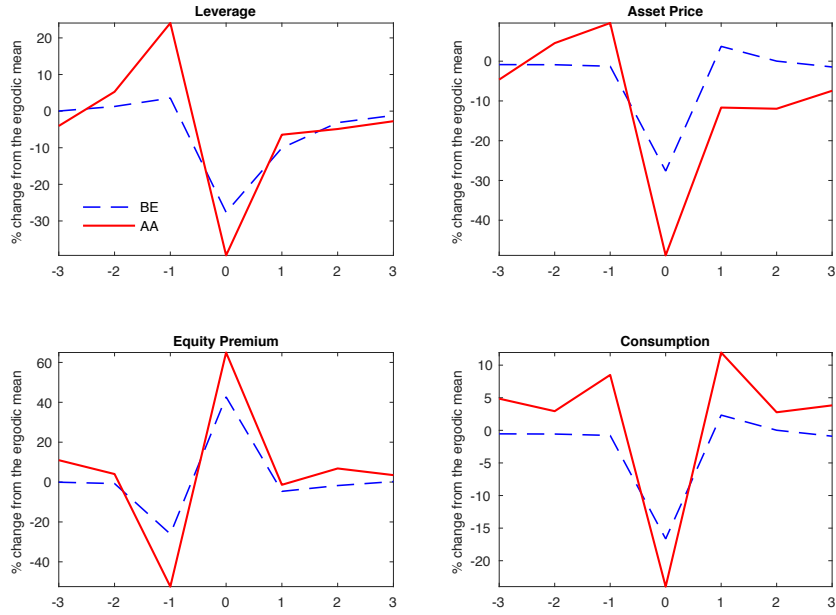
can also play a role. For this reason we assess the role of the intermediation channel by adding a credit supply shock. Our goal is mostly to verify whether the channels we highlighted so far remain important. We find that the role of ambiguity attitudes is preserved and, if

³¹See https://www.bea.gov/iTable/index_nipa.cfm.

³²See <http://www.crsp.com/products/research-products/crsp-historical-indexes>.

³³See <http://www.econ.yale.edu/~shiller/data.htm>

Figure 4: Crises Event Study with income and intermediation shock



anything, is amplified through the interaction with the intermediation shock.

We introduce intermediation by assigning the role of debt monitoring to a bank. This adds realism since atomistic lenders do not monitor or screen debtors individually, but largely assign this function to an intermediary. In this context the collateral constraint still emerges from an incentive compatible debt contract (see Appendix B), though the latter is now enforced by a bank. In this context an intermediation shock can be rationalized as a shock on the parameter governing the loan-to-value ratio, ϕ . Such a shock, by steepening the incentive constraint, tightens credit supply. Prior to the crisis financial innovation, in the form of derivatives and/or asset back securities issuance, allowed banks to off-load credit risk. A sudden freeze of the asset backed market liquidity would then result in a sudden fall in the value of ϕ , hence in a credit supply shock. We re-examine policy functions, crisis events and second moments of our model in response to such a credit supply shock. The shock is calibrated as follows. We define a high and a low level of the loan-to-value ratio, respectively $\phi_l = 0.22$ and $\phi_h = 0.28$. Those values are chosen so as to match the empirical volatility of debt. The credit shock follows a two-state regime-switching Markov process, with a transition matrix calibrated to replicate the empirical probability and duration of the crises events, as in Bianchi and Mendoza [Bianchi and Mendoza \(2018\)](#). Specifically, the probability to remain in a high state, π_{hh} is set equal to 0.955. The latter allows us to match a frequency of crises close to 4%. The transition probability from a low to a high state, π_{lh} , is equal to one, implying a one year duration of the crises. The remaining transition probabilities are set as complements of the previous ones, i.e $\pi_{hl} = 1 - \pi_{hh}$ and $\pi_{ll} = 1 - \pi_{lh}$.

[Figure 4](#) compares the crisis event in the model with ambiguity attitudes and without it. The crisis event is defined as before, but now it is triggered by a combination of income and intermediation shocks. The crisis now originates exactly when both shocks turn negative. The

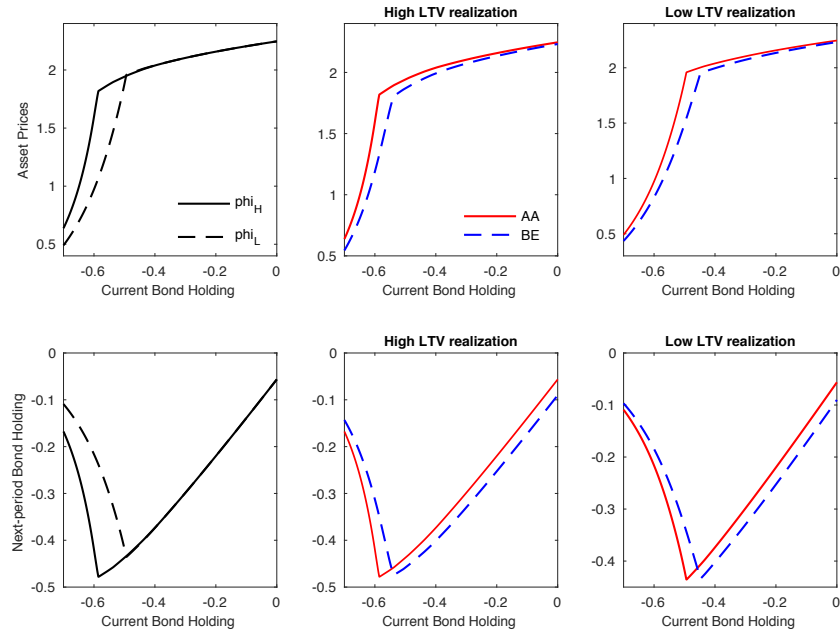
model with ambiguity attitudes induces sharper leverage build-up and de-leveraging compared to the benchmark model even when subject to credit supply shocks.

Figure 5 below shows the policy functions conditional to positive realizations of the income shock for asset prices and debt by comparing various scenarios. In the first column we compare the model with ambiguity attitudes for two values of ϕ . This case allows us to isolate only the contribution of credit supply. As before the kink represents the turn in which the constraint shifts from binding to non-binding. The comparison shows that a low ϕ , namely tight credit due to high monitoring standards, has two effects. On the one side, it enlarges the constrained region. On the other side, it reduces leverage, and this effect can be beneficial in the medium to long run. The second and the third columns compare the models with and without ambiguity attitudes, respectively for low levels of ϕ (second column) and high levels of ϕ (third column). Two interesting observations emerge. First, as before under the model with ambiguity attitudes asset prices are higher. Debt follows a pattern similar to the one discussed in absence of the credit supply shock. Second, the comparison between a high and a low level of ϕ shows that the constrained region now becomes larger under the low loan to value ratio. Overall the mechanisms induced by the ambiguity channel remain unaltered. The credit supply shock mainly affects the size of the constrained region. Figure 5b shows the results for the policy functions conditional on negative income realizations. The message is largely symmetric to the one described above.

At last, we ask whether the introduction of the intermediation shock can improve upon the moment matching and if so along which dimensions. Table 5 below shows the comparison of a selected number of second moments between the data and the models, with and without ambiguity attitudes, but with intermediation shocks. Results are preserved and partly improved along two dimensions. First, debt pro-cyclicality is enhanced. This is so since the double occurrence of the negative income and credit supply shock tightens leverage more. The volatility of debt is somewhat higher, mostly so in the model with ambiguity attitudes, and is closer to the data value. On the other side, the introduction of the intermediation shock worsens the volatility of risky returns, which now goes above the one detected in the data. This is due to the fact that our simple characterization of the intermediation sector does not include a choice for equity capital. The presence of the latter would indeed limit the extent of fire sales in risky assets when credit supply tightens, hence it would reduce fluctuations in asset prices.

Figure 5: Policy Functions for the model with intermediation

(a) Positive income shock realization



(b) Negative income shock realization

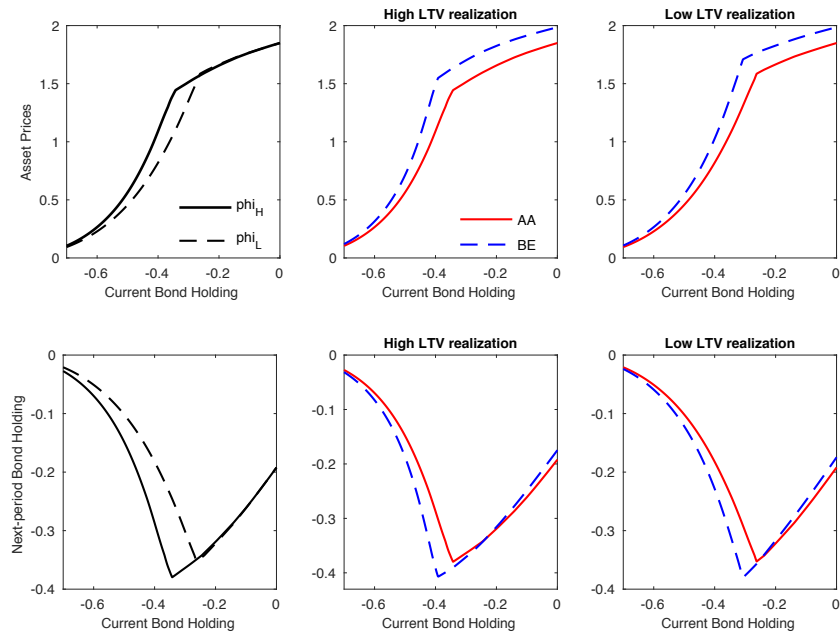


Table 5: Empirical and model-based moments

Moments	Mnemonics	Empirical	AA	AA + shock ¹	BE + shock
Debt volatility	σ^b	12.52	12.37	11.55	9.78
Debt persistence	ρ^b	0.846	0.539	0.432	0.385
Debt cyclicalities	$Corr(\Delta b_t, \Delta c_t)$	0.668	0.378	0.792	0.795
Equity return	$E_t(R_t^s)$	9.38	8.19	8.67	7.88
Equity return volatility	$\sigma^{R_t^s}$	16.21	17.46	23.45	19.40
Equity return cyclicalities	$Corr(\Delta R_t^s, \Delta c_t)$	0.474	0.989	0.983	0.992

¹ Column 4 shows the theoretical moments of the AA model without the intermediation shock. Instead, columns 5 and 6 show the theoretical moments for the AA and BE models with the intermediation shocks. For the AA and BE specifications different moment matching exercises are run, then the two models might differ in the parameter values.