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## **THE TERM STRUCTURE OF GOVERNMENT DEBT UNCERTAINTY**

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## Abstract

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JEL Classification: G12, G13, E43, E44

Keywords: fixed income volatility, information content of government bond volatility, government bond variance swaps, Treasury markets

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# The Term Structure of Government Debt Uncertainty\*

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*Keywords:* fixed income volatility; expected government bond volatility, government bond volatility futures and options; volatility of volatility; U.S. Treasury markets.

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# 1. Introduction

Two key milestones in modern asset pricing are the acknowledgment that capital market volatility is random (Engle, 2004) and that dedicated portfolios of derivatives could be used to index and trade forward-looking measures of this volatility. While asset volatility is now understood as a distinct asset class, most of the insights into its measurement and pricing have been made in the equity space. In comparison, relatively little is known regarding indexing and pricing interest rate volatility and derivatives based thereupon. How do markets price uncertainty around interest rate movements? How does benchmarking work in the fixed income volatility space? How is asset evaluation affected in these markets by, or in anticipation of, tail events such as the Lehman's bankruptcy or Fed rate hikes? What type of derivative instruments would be able to mitigate losses arising from these events? This paper addresses these questions in relation to government bond markets.

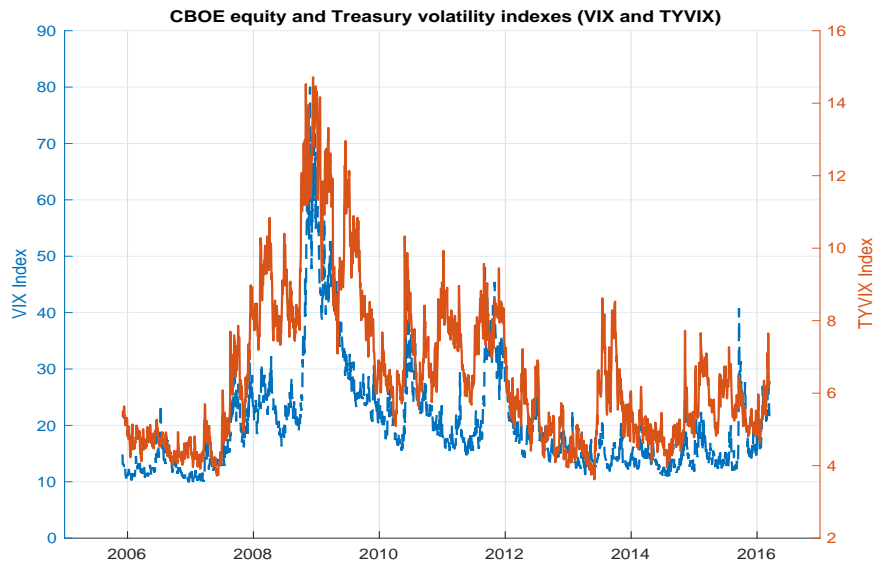
Relative to the existing literature, reviewed in more detail in Section 2, our contribution is to construct a model with random government bond volatility and evaluate derivatives written on expected Treasury volatility, such as futures and options on futures. These instruments provide valuable information regarding the term structure of expectations and risk premiums around uncertainty in government bond volatility, which go well beyond the information contained by variance swap strikes, reviewed in a moment.

Recent history provides us with an interesting laboratory where to test the predictions of our model. Figure 1 plots the Cboe/CBOT 10-Year Treasury Note Volatility Index (TYVIX), calculated by Cboe Exchange, Inc. (Cboe) since May 2013, which is the government bond counterpart to Cboe's popular VIX index for equity market volatility. Both indexes are linked to the fair value of variance swaps, which are contracts with payoffs based directly on the volatility of the underlying assets (see Section 3). In particular, in a government bond variance swap, one party purchases insurance from another for protecting against exposure to realized government bond volatility over a given horizon. To illustrate, the higher the expected uncertainty in Treasury markets in the next month, the higher TYVIX. Figure 1 reveals that, while the two indexes sometimes display similar dynamics, there are several episodes in which they exhibit divergent behavior. For example, TYVIX is more reactive than VIX to the initial phases of the 2008 subprime mortgage crisis. As a second example, during Spring 2009, good signals about the U.S. economy would trigger a reverse flight-to-quality: the ensuing

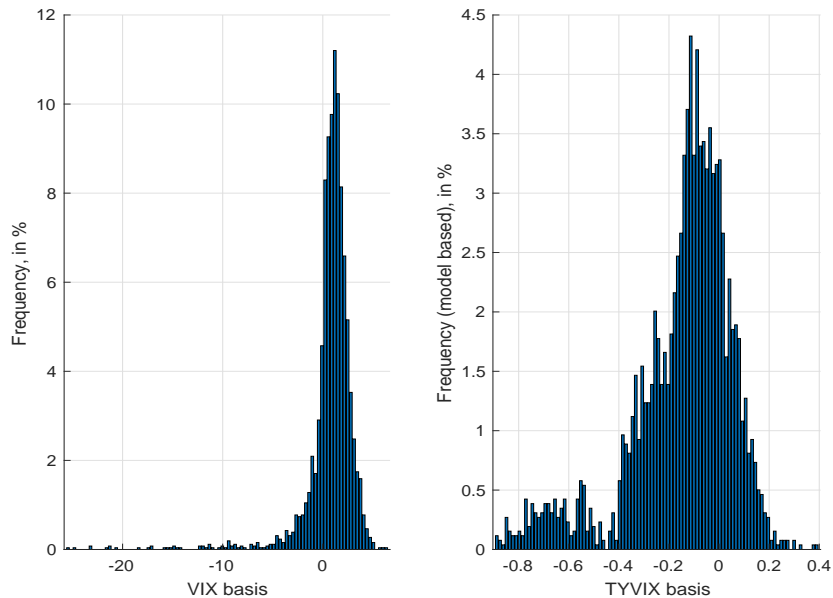
bond sell-off and equity rally would lead VIX to decrease and TYVIX to spike. As a third example, the TYVIX almost doubles during the 2013 “taper tantrum,” when the Fed signaled the markets a tightening in its policy stance; in this same period, the VIX is range-bounded, seemingly unaffected by the events of the time.

As explained, our model aims to explain the value of derivatives on variance swap rates such as TYVIX, in addition to VIX. The scope of our paper is practically relevant because, as we know, indices of expected volatility such as VIX or TYVIX are not investable. Instead, VIX futures and options are traded; and, TYVIX futures have been introduced already while options have not yet. However, to date, there are no models available to price derivatives on expected government bond volatility. Furthermore, the episodes in Figure 1 suggest that fixed income volatility has economic content beyond that of equity volatility; accordingly, derivatives on fixed income expected volatility might well exhibit different behavior than those on VIX. To illustrate, a portfolio comprised of fixed income securities might not be adequately protected while only relying on VIX-related instruments: fixed income volatility is likely to incorporate information on economic developments that may not be captured by equity. Finally, it has been known for long time that it would be problematic to price fixed income volatility only based on fixed income securities, due the incomplete nature of these markets (Collin-Dufresne and Goldstein, 2002).

Our model is calibrated to nearly fifteen years of data to assess the potential role of derivatives on expected government bond volatility as both hedging instruments and sources of information. Figure 2 contains model predictions regarding market expectations and aversion to the timing of uncertainty in Treasury markets and compares them with properties in equity markets. A well-known feature of VIX is that its futures curve is often in contango: the VIX futures curve is often increasing, perhaps reflecting risk aversion of the bid side of the market for volatility. Our calibrated model implies that the U.S. Treasury market works quite differently: Figure 2 suggests that TYVIX futures markets may be more often in backwardation, i.e., they may frequently come with an inverted future curve. A frequent backwardation is not an arbitrage opportunity, but it is puzzling, meaning as it does that insurance providers are not being paid for their role. The model predicts that this property results due to the following mechanism. Interest rate uncertainty is driven by two factors: a short-term and a long-term factor. In episodes of increased uncertainty, the term structure of government bond



**Figure 1.** Indexes of equity and government bond volatility maintained by Chicago Board Options Exchange, VIX and TYVIX. Data are sampled daily and span the period from November 2005 to February 2016.



**Figure 2.** Frequency distribution of the basis in the equity and Treasury volatility spaces. *Left panel:* Distribution of the equity VIX basis, defined as the difference between 1-month future contracts and VIX levels. *Right panel:* Distribution of the Treasury VIX basis, defined as the difference between 1-month future contracts implied by the model of this paper and TYVIX levels. Data are sampled daily and span the period from November 2005 to February 2016.

volatility futures shifts up as a result of an increase in both uncertainty factors; however, the curve enters and, then, remains in backwardation for a while, because the long-term uncertainty factor mean-reverts at a very fast pace.

We rely on the model predictions and investigate how these properties may be exploited to help protect portfolios exposed to interest rate volatility. We provide evidence that overlaying fixed income portfolios with derivatives on government bond volatility may significantly mitigate drawdowns occurring in such periods of acute volatility as those mentioned above. The model also helps understand episodes of historical importance such as the timing leading to inversions in the volatility term structure vis-à-vis the increase in the Fed rate targets in 2015, the reaction of the volatility curve to persistent or transitory shocks and, finally, the behavior of the “volatility of volatility,” that is, the price of being insured against fluctuations in the volatility of Treasury markets throughout hypothetical options on TYVIX futures. We show that the volatility of volatility may help anticipate historically important episodes, such as the 2007 subprime crisis or the 2013 taper tantrum.

The basic assumption of our model is that the dynamics of the yield curve and volatility are driven by three factors: the short-term rate, its instantaneous volatility and a long-run uncertainty factor. Note that Mele and Obayashi (2015) provide a theoretical framework around the pricing of government bond variance swaps in a “model-free” fashion that does not hinge upon any assumptions other than absence of arbitrage and standard price dynamics based on Brownian motions. Our focus in this paper is on modeling derivatives on the fair values of such variance swaps; we, thus, need to rely on a parametric model that describes the random movements in the underlying risks. Our model features two important properties. First, it accounts for the endogenous nature of government bond volatility, as further elaborated below. Second, the model is formulated in a way that government bond volatility futures and options can be priced and hedged while ensuring that the entire yield curve is fitted without error. The second property is particularly appealing for the purpose of using the model consistent with the pricing of other securities; but fitting the entire yield curve is also appealing because it enables the very same curve to feed information regarding the direction of future volatility, as explained in a moment.

Due to the distinct nature of interest rate volatility, our model relies on an analytical framework that is separate from that available in the equity case. To illustrate, government bond variance swaps



are priced under a market numéraire corresponding to the forward probability, not the risk-neutral one. Moreover, the assets underlying our contracts are short-lived (i.e., futures on government bonds), and this finite-horizon expiration complicates the analysis of the variance strike. The next section reviews the state of the literature on these delicate topics. Last, but not least, we are pricing variance swaps on futures that are, in turn, referenced to coupon bearing bonds: duration and, then, the entire yield curve does impart information onto future volatility movements, as anticipated.

Not only do numéraires matter while pricing fixed income volatility. It is the very same concept of volatility that needs to be carefully defined in our context. Government bond volatility needs to satisfy no-arbitrage restrictions jointly with the underlying assets. Intuitively, bond price volatility depends on the sensitivities of the bond price with respect to the variables that drive the whole yield curve. Because these sensitivities are endogenous, bond price volatility cannot be modeled separately from the underlying bond price. In the next section, we discuss two papers in which fixed income volatility is modeled while taking into account its endogenous nature, although with a different scope than ours. Note that equity volatility is more straightforward to model. For example, options written on stocks with stochastic volatility can be evaluated through arbitrary specifications of volatility, as the latter could be merely juxtaposed to the dynamics of stock returns for practical derivative evaluation purposes (see, e.g., Heston, 1993).<sup>1</sup> Unlike equity volatility dynamics, which may be specified exogenously without violating no-arbitrage conditions, government bond volatility must be consistent with the dynamics of the whole yield curve. Our model takes these features into account, and also provides quasi-closed form solutions that can be readily implemented to evaluate futures and options on volatility indices while also ensuring that the underlying indices are priced without error.

The plan of the paper is as follows. The next section contrasts our paper with the existing literature. Section 3 provides a succinct description of the basic framework underlying government bond volatility: the dynamics of the underlying assets, the value of dedicated government bond variance swaps, and an index of forward-looking volatility arising therefrom. Sections 4 and 5 develop a parametric model of bond price volatility that leads to predictions on this government bond volatility index as well as the price of futures and options referenced to the index. Section 6 contains our

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<sup>1</sup>General equilibrium models of equity evaluation lead to a similar kind of endogeneity though (see, e.g., Mele, 2007).

empirical findings. Section 7 concludes. Three Appendixes contain technical details omitted from the main text.

## 2. Related literature

The literature on contingent claim theory for fixed income volatility is much more recent than in the equity space. A well-known accomplishment in the equity literature is to have identified conditions under which variance swaps may be priced in a model-free manner.<sup>2</sup> In an analogous effort, Mele and Obayashi (2015) provide a systematic account of expected volatility that spans a variety of segments in the fixed income universe such as, for example, interest rate swaps (IRS), government bonds, time-deposits and credit. They identify the appropriate numéraire and probability to be used to evaluate variance swaps in each of these markets. For example, the fair value of a government bond variance swap links to the expected variance under the forward probability; in the case of a variance swap on IRS, the pricing probability is the annuity probability; in the credit space, the appropriate probability is the risky annuity. To illustrate, the Cboe SRVIX index launched in 2012 is a measure of expected basis point realized volatility in the IRS space under the annuity probability.<sup>3</sup>

This paper makes a natural step forward, and considers derivatives written on such variance swap strikes in the government bond market. To achieve this task, we need to formulate a model for the whole term structure of interest rates and, hence, expected volatility, which takes into account no-arbitrage restrictions across all maturities. Note that Cieślak and Povala (2016) also model the term structure of expected volatility of yields (a model distinct from ours), although they do not consider the pricing of derivatives written on this volatility. To illustrate, the authors examine properties of expected realized yield volatility under the risk-neutral probability. Their focus is certainly important but different from ours because, as noted, we need to price a government bond variance swap rate under the forward probability. Our focus is precisely on modeling the value of future and option derivatives referenced to this variance swap rate. Moreover, we are interested in bond price volatility, not yield volatility, as derivative instruments on price volatility seem to be more suited for market adoption, as also witnessed by Cboe launch of futures on TYVIX.

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<sup>2</sup>See, e.g., Demeterfi, Derman, Kamal and Zou (1999,a,b), Bakshi and Madan (2000), Britten-Jones and Neuberger (2000), and Carr and Madan (2001).

<sup>3</sup>See Chapter 2, Section 2.5 in Mele and Obayashi (2015).

Our paper also relates to Choi, Mueller and Vedolin (2017). The authors also consider variance swaps in Treasury markets and, consistent with Mele and Obayashi (2015), they price them under the forward probability. However, their formula is less general than that in Mele and Obayashi (2015), as the authors do not model the discrepancy between the maturity of the future underlying government debt and the maturity of the variance swap (see Proposition 1 in Section 3 of the present paper). More fundamentally, the focus of Choi, Mueller and Vedolin is the pricing of variance swaps, which is, instead, one of the starting points of our analysis. We actually show that futures and options on TYVIX provide information that goes well beyond TYVIX. For example, Choi, Mueller and Vedolin show that Treasury markets command a variance risk premium and that selling realized volatility against a fixed strike may be profitable. In other words, gamma trading may be profitable on the sell side. Instead, we consider vega strategies, and we show that the term structure of government bond uncertainty matters; now, selling futures on expected volatility may be far from being a profitable strategy, due to backwardation (see Figure 2). Furthermore, our model is also used to investigate higher order properties of expected volatility: we predict values of European-style options written on TYVIX futures, and investigate empirical properties of the resulting “volatility of volatility.”

Finally, our paper relates to work by Trolle and Schwartz (2014) in the IRS space. The authors consider variance risk premiums in the IRS market and, following Mele and Obayashi (2015), evaluate an IRS variance swap by re-scaling the variance payoff by the annuity factor. Parallel work on variance risk premiums in the IRS market, and relying on the the SRVIX index, includes Mele, Obayashi and Shalen (2015). Fornari (2010) is an early paper on variance risk premiums in the IRS market. To our knowledge, there are no attempts in the literature to model the value of future and option derivatives on SRVIX. The task of the present paper is confined to government bond markets.

### 3. Government bond volatility: the underlying assets

#### 3.1. Forward bond prices

Let  $F_t(S, \mathbb{T})$  be the forward price at  $t$ , for delivery at  $S$ , of a coupon bearing bond expiring at  $\mathbb{T}$ , with  $t \leq S \leq \mathbb{T}$ . We assume the bond pays off coupons  $\frac{C_i}{n}$  over the sequence of dates  $T_i, i = 1, \dots, N$ , where  $n$  is the frequency of coupon payments, and  $\mathbb{T} \equiv T_N$ . The forward price satisfies  $F_t(S, \mathbb{T}) = \frac{P_t^c(\mathbb{T})}{P_t(S)}$ ,

where  $P_t(S)$  is the price at  $t$  of a zero coupon bond expiring at  $S \geq t$ ,  $P_t^c(\mathbb{T})$  is the price at  $t$  of the underlying bond,

$$P_t^c(\mathbb{T}) \equiv \sum_{i=i_t}^N \frac{C_i}{n} P_t(T_i) + P_t(\mathbb{T}), \quad (1)$$

and  $T_{i_t}$  is the first available coupon payment date a bondholder would have access to.

Let  $r_t$  be the instantaneous short-term rate at time  $t$  and  $Q$  the risk-neutral probability. It is well-known<sup>4</sup> that in a diffusion setting,  $F_t(S, \mathbb{T})$  satisfies

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = V_\tau(S, \mathbb{T}) \cdot dW_\tau^S, \quad \tau \in (t, S), \quad (2)$$

where  $V_\tau(S, \mathbb{T})$  is the instantaneous volatility process adapted to  $W_\tau^S$ , a multidimensional Brownian motion under  $Q_S$  (the  $S$ -forward probability), defined through the Radon-Nikodym derivative

$$\left. \frac{dQ_S}{dQ} \right|_{\mathbb{F}_S} = \frac{e^{-\int_t^S r_\tau d\tau}}{P_t(S)}, \quad (3)$$

and  $\mathbb{F}_S$  denotes the information set at time  $S$ . Note that  $V_\tau(S, \mathbb{T}) = \sigma_\tau^B(\mathbb{T}) - \sigma_\tau(S)$ , where  $\sigma_\tau^B(\mathbb{T})$  denotes the instantaneous volatility of the coupon bearing bond price,  $P_t^c(\mathbb{T})$ , and  $\sigma_\tau(S)$  is the instantaneous volatility on a zero coupon bond expiring at  $S$ . We now describe the contracts that help investors hedge against fluctuations in the realized volatility of  $F_\tau(S, \mathbb{T})$ .

### 3.2. Government bond variance swaps

Consider the following payoff of a bond variance swap at time  $T > t$

$$\pi(T, \mathbb{T}) \equiv \bar{V}_t(T, S, \mathbb{T}) - \mathbb{P}(t, T, S, \mathbb{T}), \quad T \leq S,$$

where  $\bar{V}_t(T, S, \mathbb{T}) \equiv \int_t^T \|V_\tau(S, \mathbb{T})\|^2 d\tau$  is the bond price integrated variance up to  $T$ , and  $\mathbb{P}(t, T, \mathbb{T})$  is the variance swap strike. At inception, time- $t$ , the value of this contract is zero, such that

$$\mathbb{P}(t, T, S, \mathbb{T}) = \frac{1}{P_t(T)} \mathbb{E}_t \left[ e^{-\int_t^T r_\tau d\tau} \bar{V}_t(T, S, \mathbb{T}) \right] = \mathbb{E}_t^{Q^T}(\bar{V}_t(T, S, \mathbb{T})), \quad (4)$$

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<sup>4</sup>See, e.g., Brigo and Mercurio (2006).

where the second equality follows by a change of probability, and  $\mathbb{E}_t$  and  $\mathbb{E}_t^{Q_T}$  denote time- $t$  conditional expectations under the risk-neutral probability and under the  $T$ -forward probability.

That is, the fair value of the variance swap is the expected realized variance under the  $T$ -forward probability. It is a notable point of departure from the standard equity case in which the fair value of a variance swap is the risk-neutral expectation assuming interest rates are constant. For the case at hand, we obviously cannot assume constant interest rates. Moreover, we still need to evaluate the R.H.S. of the second equality in (4), which necessitates a theoretical treatment different from that leading to the standard equity VIX methodology. This difference is due to a “maturity mismatch”: the expectation in Eq. (4) is taken under the  $T$ -forward probability, but the maturity of the forward price is  $S \geq T$ .

Formally, the dynamics of  $F_T(S, \mathbb{T})$  under the “variance swap pricing” probability (see Eq. (4)) are those under  $Q_T$ , not under  $Q_S$  as in Eq. (2), such that by Girsanov theorem,

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = \underbrace{V_\tau(S, \mathbb{T})(V_\tau(S, \mathbb{T}) - V_\tau(T, \mathbb{T}))}_{\equiv z_\tau(T, S, \mathbb{T})} d\tau + V_\tau(S, \mathbb{T}) \cdot dW_\tau^T, \quad \tau \in (t, T), \quad (5)$$

where  $W_\tau^T$  is a multidimensional Brownian motion defined under  $Q_T$ . To illustrate, a variance swap may have a maturity equal to one month, but be referenced to a forward maturing in 2 months. The presence of a non-zero drift in Eq. (5) leads to a theoretical issue absent in the standard literature on equity variance swaps. In particular, Mele and Obayashi (2015; Chapter 4) prove the following result:

**Proposition 1.** *The fair value of the variance swap in Eq. (4) is*

$$\begin{aligned} \mathbb{P}(t, T, S, \mathbb{T}) = & 2 \left( 1 - \mathbb{E}_t^{Q_T} \left( e^{\bar{z}(t, T, S, \mathbb{T})} - \bar{z}(t, T, S, \mathbb{T}) \right) \right) \\ & + \frac{2}{P_t(T)} \left( \int_0^{F_t(S, \mathbb{T})} \text{Put}_t(K) \frac{1}{K^2} dK + \int_{F_t(S, \mathbb{T})}^\infty \text{Call}_t(K) \frac{1}{K^2} dK \right), \end{aligned} \quad (6)$$

where  $\text{Put}_t(K)$  and  $\text{Call}_t(K)$  denote the prices of European puts and calls, written on bond forwards, struck at  $K$ ,  $\bar{z}(t, T, S, \mathbb{T}) \equiv \int_t^T z_\tau(T, S, \mathbb{T}) d\tau$ , and  $z_\tau(T, S, \mathbb{T})$  is defined as in Eq. (5).

A forward looking gauge of government bond volatility can then be based on the following index

$$\text{GB-VI}(t, T, S, \mathbb{T}) \equiv \sqrt{\frac{1}{T-t} \mathbb{P}(t, T, S, \mathbb{T})}, \quad (7)$$

where  $\mathbb{P}(t, T, S, \mathbb{T})$  is as in Eq. (6).

Note that the first term in Eq. (6) is model-dependent: in general, the term  $\bar{z}(t, T, S, \mathbb{T})$  is random and a model is needed to determine its expectation under  $Q_T$ . Even when  $\bar{z}(t, T, S, \mathbb{T})$  is deterministic, we would need to estimate the volatilities  $V_\tau(\cdot, \cdot)$  while at the same time relying on a model that ensures no arbitrage restrictions on the volatility curve,  $V_\tau(\cdot, \cdot)$ .

Eq. (B.11) in Appendix B provides the expression for  $\bar{z}(t, T, S, \mathbb{T})$  predicted by the three-factor model with random volatility introduced in Section 3. Mele and Obayashi (2015; Chapter 4) rely on Vasicek’s model (a special case of the model in Section 3), and document that, in practice, the contribution of the first term on the R.H.S. of Eq. (6) is very limited when  $S - T$  is small. For example, Cboe maintains its TYVIX index relying on near- and next-month options on quarterly futures as underlying, which generates a negligible impact on  $\mathbb{P}(t, T, S, \mathbb{T})$ .

Note, finally, that the TYVIX index is calculated based on American options, not European. Mele and Obayashi (2015; Chapter 4) show that the presence of an early exercise premium might inflate the “true” index based on Eq. (7) by no more than one relative percentage point under realistic market conditions. We shall henceforth refer to the government bond volatility index in Eq. (7) as TYVIX for conciseness.

## 4. A model of forward bond price volatility

In the equity derivative literature, it is standard to model return volatility separately from the underlying stock returns, as explained in the Introduction. By contrast, government bond volatility cannot be taken to be exogenous. To illustrate, bond volatility converges to zero as time to maturity goes to zero, regardless of whether volatility is random. This section develops a model in which government bond volatility is determined while taking account of no-arbitrage restrictions on bond price dynamics. First, we provide preliminary background to evaluate the whole yield curve we reference upon in this paper (see Proposition 2). Second, we use predictions within this background and

determine TYVIX values that are consistent with no-arbitrage restrictions across all maturities (see Propositions 3 and 4). Section 5 relies on this outcome and provides values of futures and options referenced to TYVIX.

#### 4.1. *Fluctuating uncertainties*

We consider a three-factor extension of the Ho and Lee (1986) model: one in which the short-term rate  $r_\tau$  at any point in time  $\tau \geq t$  is a mean-reverting process with mean-reverting stochastic volatility; and we also allow the *expected* short-term basis point variance to be time-varying, mean-reverting, and driven by an independent factor. That is, we assume that

$$\begin{cases} dr_\tau &= \kappa_r (\theta_\tau - r_\tau) d\tau + v_\tau dW_{1\tau} \\ dv_\tau^2 &= \kappa_v (\mu_\tau - v_\tau^2) d\tau + \xi v_\tau \left( \rho dW_{1\tau} + \sqrt{1 - \rho^2} dW_{2\tau} \right) \\ d\mu_\tau &= \kappa_\mu (m - \mu_\tau) d\tau + \gamma \sqrt{\mu_\tau} dW_{3\tau} \end{cases} \quad (8)$$

where  $W_j$  are standard Brownian motions under the risk-neutral probability.

A standard assumption in the literature is that the short-term rate is mean-reverting around another mean-reverting long-term trend (see, e.g., Chapters 12-13 in Mele, 2019). In this model, basis point variance,  $v_\tau^2$ , is mean-reverting around a long-term, mean-reverting uncertainty factor,  $\mu_\tau$ ; finally,  $\theta_\tau$  is an “infinite-dimensional” parameter that one may rely upon to fit the initial yield curve at the initial date  $t$  without errors. Remaining notation for the constant parameters should be straightforward. Note that we are assuming that the increments  $d\mu_\tau$  are uncorrelated with  $dr_\tau$  and  $dv_\tau^2$ : this assumption makes the model tractable, leading to exponential affine prices (see Proposition 2 below), which facilitates the modeling of variance contracts and the price of futures and options based thereon.

#### 4.2. *Discussion*

In Eqs. (8),  $\theta_\tau$  is a time-varying albeit deterministic parameter. Our model is affine, as anticipated, and precisely because the conditional characteristic function of our state vector can be shown to be exponential-affine in the state  $(r_t, v_t^2, \mu_t)$ , as in Duffie, Filipović and Schachermayer (2003). However,

while we rely on an affine model, pricing volatility turns out to be challenging: the instantaneous variances are affine in the state, but values of future and options on expected volatility are not. These complications arise not only due to the presence of non-linear payoffs (e.g., square-root, in the futures case), but also because the very same state variables follow complicated dynamics under the relevant pricing probability (see, e.g., Eqs. (12) below). Therefore, in Section 6, we need to rely on approximations and numerical methods to implement our pricing models for volatility derivatives that are contained in Section 5.

Our motivation behind the two uncertainty factors is the following. Preliminary empirical evidence gathered while calibrating a two-factor model (described in Appendix A) suggests that fluctuations in long-term interest rate uncertainty affect developments in near-term expected rate volatility. This preliminary evidence is at the basis of the formulation in (8), in which interest rate volatility fluctuates around a long-term uncertainty factor,  $\mu_\tau$ . Trolle and Schwartz (2014) adopt a similar formulation while modeling the volatility of swap rates. Note that the assumption that volatility mean-reverts around a random term parallels similar assumptions made in the equity literature (e.g., Egloff, Leipold and Wu, 2010; Bates, 2012; Mencía and Sentana, 2013). The complication in this paper is that we price bond price volatility; while the latter is driven by both  $v_t^2$  and  $\mu_t$ , it also links, endogenously, to the elasticities of the bond price with respect to changes in  $v_t^2$  and  $\mu_t$ , as we explain below.

### 4.3. The reference yield curve

We begin with summarizing the key properties of a zero coupon bond price predicted by the model.

**Proposition 2.** *If the short-term rate is solution to Eqs. (8), the price at  $\tau \geq t$  of a zero coupon bond expiring at  $S$  when the state is  $(r_\tau, v_\tau, \mu_\tau)$ , is*

$$P_\tau(r_\tau, v_\tau^2, \mu_\tau, S) \equiv e^{A_S(\tau) - B_S(\tau)r_\tau + C_S(\tau)v_\tau^2 + D_S(\tau)\mu_\tau}, \quad (9)$$

for three functions  $A_S(\cdot)$ ,  $B_S(\cdot)$  and  $C_S(\cdot)$  defined in Appendix B (see Eqs. (B.1), (B.2) and (B.3)).

Eq. (9) holds for any value taken by the time-varying parameter  $\theta_\tau$  (e.g., a constant); however,



Appendix B provides the functional form for  $\theta_\tau$  (see Eq. (B.4)) such that the model predictions fit the initial yield curve without errors (see Eq. (B.5)). This feature allows us to make predictions regarding the future yield curve and, as discussed below, expected government bond volatility, while feeding the model with all the bond prices observed at time  $t$ , not only the short-term rate. It is standard market practice to allow a model for such a perfect fit (see, e.g., Mele and Obayashi, 2016), a procedure we have adapted to our environment with stochastic volatilities.

An interesting property of the model is that it allows the current yield curve to affect expected volatility. How is this property possible? Intuitively, the yield curve determines how future coupons are discounted, thereby affecting both the coupon bearing bond price and its volatility. For this reason, these “price feedbacks” only apply to coupon-bearing bonds, as explained next.

#### 4.4. Forward prices and volatility

This section identifies the forward price volatility  $V_\tau(S, \mathbb{T})$  in Eq. (2) predicted by the model of this section. Consider, first, the price of the forward on the coupon bearing bond; by Eq. (1), it is

$$F_t(S, \mathbb{T}) \equiv F_t(r_t, v_t^2, \mu_t, S, \mathbb{T}) \equiv \sum_{i=i_t}^N \frac{C_i}{n} F_t^z(r_t, v_t^2, \mu_t, S, T_i) + F_t^z(r_t, v_t^2, \mu_t, S, \mathbb{T}), \quad (10)$$

where

$$F_t^z(S, T_o) \equiv F_t^z(r_t, v_t^2, \mu_t, S, T_o) \equiv \frac{P_t(r_t, v_t^2, \mu_t, T_o)}{P_t(r_t, v_t^2, \mu_t, S)} \quad (11)$$

is the price of a forward expiring at  $S \leq T_o$ , written on a zero coupon bond expiring at some  $T_o$ , and  $P_t(r_t, v_t^2, \mu_t, \cdot)$  is the price of a zero coupon bond predicted by Eq. (9). We now rely on Eqs. (10)-(11) and derive model-based predictions of volatility.

##### 4.4.1. Zero coupon bonds

We first deal with zero coupon bonds. This case allows us to derive closed-form solutions for expected volatility, which we shall use in Section 4.4.2 while pricing the volatility of coupon bearing bonds. In

Appendix B, we show that, under  $Q_T$ , a forward price  $F_\tau^z(S, T_o)$  is solution to

$$\begin{cases} \frac{dF_\tau^z(S, T_o)}{F_\tau^z(S, T_o)} &= m_\tau^F d\tau + v_\tau (v_{1\tau}^z(S, T_o) dW_{1\tau}^T + v_{2\tau}^z(S, T_o) dW_{2\tau}^T) + \sqrt{\mu_\tau} v_{3\tau}^z(S, T_o) dW_{3\tau}^T \\ dv_\tau^2 &= (\kappa_v \mu_\tau - G_T(\tau) v_\tau^2) d\tau + \xi v_\tau (\rho dW_{1\tau}^T + \sqrt{1 - \rho^2} dW_{2\tau}^T) \\ d\mu_\tau &= (\kappa_\mu m - H_T(\tau) \mu_\tau) d\tau + \gamma \sqrt{\mu_\tau} dW_{3\tau}^T \end{cases} \quad (12)$$

where the three volatility coefficients are, with  $B(\cdot)$ ,  $C(\cdot)$  and  $D(\cdot)$  defined as in Proposition 2,

$$\begin{aligned} v_{i\tau}^z(S, T_o) &= \varphi_{i\tau}(T_o) - \varphi_{i\tau}(S), \quad i = 1, 2, 3, \\ \varphi_{1\tau}(\mathcal{X}) &\equiv -B_{\mathcal{X}}(\tau) + \xi \rho C_{\mathcal{X}}(\tau), \quad \varphi_{2\tau}(\mathcal{X}) \equiv \xi \sqrt{1 - \rho^2} C_{\mathcal{X}}(\tau), \quad \varphi_{3\tau}(\mathcal{X}) \equiv \gamma D_{\mathcal{X}}(\tau), \end{aligned} \quad (13)$$

and  $G_T(\tau)$  and  $H_T(\tau)$  are time-varying speeds of mean-reversion of the basis point variance  $v_\tau^2$  and long-term uncertainty  $\mu_\tau$ , and equal to

$$G_T(\tau) \equiv \kappa_v + \xi (\rho B_T(\tau) - \xi C_T(\tau)), \quad H_T(\tau) \equiv \kappa_\mu - \gamma^2 D_T(\tau), \quad \tau \in [t, T]. \quad (14)$$

Finally, Appendix B provides the expression for  $m_\tau^F$  (see Eq. (B.6)).

Denote the volatility vector of the forward price in Eqs. (12) with  $V_\tau^z(v_\tau, \mu_t; S, T_o) \equiv [v_{1\tau}^z(S, T_o) v_\tau \quad v_{2\tau}^z(S, T_o) v_\tau \quad v_{3\tau}^z(S, T_o) \sqrt{\mu_\tau}]$ . The next proposition provides an expression for the realized variance of the forward on a zero coupon bond price, and an ensuing forward looking volatility index.

**Proposition 3.** *Assume the short-term rate is solution to Eqs. (8). Then, the instantaneous realized variance of the forward price in Eqs. (12) is*

$$\|V_\tau^z(v_\tau, \mu_t; S, T_o)\|^2 = \phi_{1\tau}^z(S, T_o) v_\tau^2 + \phi_{2\tau}^z(S, T_o) \mu_\tau, \quad (15)$$

and the index of percentage volatility in Eq. (7) and, hence, the time  $t$  (square-root of the) fair value of a variance swap expiring at  $T$ , on a forward expiring at  $S$ , on a government zero coupon bond

expiring at  $T_o$  is, for  $t \leq T \leq S \leq T_o$ ,

$$\text{GB-VI}^z(v_t^2, \mu_t; t, T, S, T_o) \equiv \sqrt{\kappa_m m \mathcal{B}_1^z(t, T, S, T_o) + \mathcal{B}_2^z(t, T, S, T_o) v_t^2 + \mathcal{B}_3^z(t, T, S, T_o) \mu_t}, \quad (16)$$

where  $\phi_{j\tau}^z(S, T_o)$  are two deterministic functions of calendar time  $\tau$ , and  $\mathcal{B}_j^z(t, T, S, T_o)$  are three constants, all given in Appendix B (Eqs. (B.7) and (B.9)).

Eq. (15) tells us that the drivers of government bond *realized* variance are the short-term basis point variance  $v_\tau^2$  and the long-term uncertainty factor  $\mu_\tau$ . While the magnitude of  $v_\tau^2$  and  $\mu_\tau$  is too small to explain the level of realized government bond volatility, the two “multipliers”  $\phi_{j\tau}^z(S, T_o)$  have the potential to increase this volatility by several orders of magnitude, depending on model parameters. In particular, Section 6 provides evidence that a “long-run risk” mechanism underlies these multipliers’ effects: the higher the persistence of the short-term rate, its variance and long-term uncertainty, the higher the bond price exposures;<sup>5</sup> the higher these exposures, the higher the values of  $\phi_{j\tau}^z(S, T_o)$ . These magnifying effects operate in a nearly identical way on *expected* volatility, notably through the functions  $\mathcal{B}_j^z(t, T, S, T_o)$  in Eq. (16). Naturally, these properties extend to the coupon bearing bond volatility case dealt with below. We shall return to these points in much more detail while discussing the model empirical implications (see Section 6).

Finally, note that the yield curve does not provide information regarding developments in government bond volatility: the government bond volatility index in Eq. (16) is independent of the yield curve at time- $t$ . By contrast, we now show that the yield curve may well affect expected volatility of forwards on coupon bearing bonds.

#### 4.4.2. Coupon bearing bonds and volatility feedbacks

By Eq. (10), the price of the forward on a coupon-bearing bond satisfies

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = \sum_{i=i_t}^N \omega_\tau^i(S, \mathbb{T}) \frac{dF_\tau^z(S, T_i)}{F_\tau^z(S, T_i)}, \quad \omega_\tau^i(S, \mathbb{T}) \equiv \bar{C}_i \frac{F_\tau^z(S, T_i)}{F_\tau(S, \mathbb{T})}, \quad t < \tau < T < T_{i_t+1}, \quad (17)$$

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<sup>5</sup>By Eq. (9), the bond price exposures are  $B_S(\tau)$ ,  $C_S(\tau)$  and  $D_S(\tau)$ .

where  $\bar{C}_i \equiv \frac{C_i}{n}$  for  $i = i_t, \dots, N-1$ ,  $\bar{C}_N \equiv \frac{C_N}{n} + 1$ , and  $(C_i)_{i=1}^N$  denotes the series of coupons in Eq. (1).

In Appendix B, we provide the expression for the integrated drift that the model predicts for the forward price in Eq. (5).<sup>6</sup> Regarding the instantaneous realized variance of the forward price in Eq. (17), a direct calculation leaves

$$\|V_\tau(v_\tau, \mu_\tau; S, \mathbb{T})\|^2 \equiv \left( \sum_{i=1}^2 \left( \sum_{j=i_t}^N \omega_\tau^j(S, \mathbb{T}) v_{i\tau}^z(S, T_j) \right)^2 \right) v_\tau^2 + \left( \sum_{j=i_t}^N \omega_\tau^j(S, \mathbb{T}) v_{3\tau}^z(S, T_j) \right)^2 \mu_\tau. \quad (18)$$

where  $v_{i\tau}^z(S, T_j)$  are the zero coupon bond exposures in (13).

While summing up to one, the coupon bearing bond weights  $\omega_\tau^j(S, \mathbb{T})$  are random, and therefore affect both current volatility and the whole future realized variance and its expectation under the forward probability. This feature of the model is a price feedback: the path of government bond volatility is driven by basis point variance and long-term uncertainty ( $v_\tau^2$  and  $\mu_t$ ), but also by the entire yield curve, through the stochastic weights  $\omega_\tau^j(S, \mathbb{T})$ .

We wish to determine the fair value of a government bond variance swap based on the path of  $\|V_\tau(v_\tau, \mu_\tau; S, \mathbb{T})\|^2$  and the ensuing government bond volatility index matching the TYVIX in Eq. (7). Because the weights  $\omega_\tau^j(S, \mathbb{T})$  are random, closed-form solutions are not available that match Eq. (16) in Proposition 3. While Monte Carlo simulations could be used to approximate the model predictions on TYVIX, this approach is computationally prohibitive, and becomes particularly so when the objective is to price derivatives on TYVIX on a large sample period (see Section 6). We rely on a different approximation: we replace the random weights  $\omega_\tau^j(S, \mathbb{T})$  with their values taken at the evaluation time  $t$ , and proceed with pricing while relying on the “frozen” weights,  $\omega_t^j(S, \mathbb{T})$ .<sup>7</sup>

Let  $y_t^\mathbb{S}(\tau) \equiv -\frac{1}{\tau} \ln P_t^\mathbb{S}(t + \tau)$  denote the yield at time  $t$  and maturity  $\tau$ , where  $P_t^\mathbb{S}(\cdot)$  is the observed counterpart zero to  $P_t(\cdot)$ , and let  $\mathbf{Y}_{t, \mathbb{T}}^\mathbb{S} \equiv \{y_t^\mathbb{S}(\tau)\}_{\tau \in (0, \mathbb{T}-t)}$  denote the entire yield curve observed at  $t$  and up to time to maturity  $\mathbb{T} - t$ . We have:

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<sup>6</sup>That is,  $\bar{z}$  in Proposition 1; see Eq. (B.11).

<sup>7</sup>Similar approximations are utilized in the literature and market practice in other contexts. For example, these approximations are known as Rebonato’s (1998) approximations in the context of the calculation of swaption volatilities (see, e.g., Brigo and Mercurio, 2006).

**Proposition 4.** *Assume the short-term rate is solution to Eqs. (8), and approximate  $\omega_\tau^j(S, \mathbb{T})$  in Eq. (18) with  $\omega_t^j(S, \mathbb{T})$ . Then, the TYVIX index predicted by the model for time  $t$ , that is, the (square-root of the) fair value of a variance swap expiring at  $T$ , on a forward expiring at  $S$ , on a coupon bearing bond expiring at  $\mathbb{T}$ , is*

$$\text{GB-VI}(\mathbf{Y}_{t, \mathbb{T}}^{\$}, v_t^2, \mu_t; t, T, S, \mathbb{T}) = \sqrt{\kappa_{\mu} m \mathcal{B}_1(\mathbf{Y}_{t, \mathbb{T}}^{\$}, T, S, \mathbb{T}) + \mathcal{B}_2(\mathbf{Y}_{t, \mathbb{T}}^{\$}, T, S, \mathbb{T}) v_t^2 + \mathcal{B}_3(\mathbf{Y}_{t, \mathbb{T}}^{\$}, T, S, \mathbb{T}) \mu_t}, \quad (19)$$

where  $\mathcal{B}_i(\mathbf{Y}_{t, \mathbb{T}}^{\$}, \cdot, \cdot, \cdot)$  are three functions given in Appendix B (see Eqs. (B.12)).

The model predicts that the government bond volatility index depends on the initial yield curve,  $\mathbf{Y}_{t, \mathbb{T}}^{\$}$ , because the instantaneous bond price realized variance at  $\tau$ ,  $\|V_\tau(v_\tau, \mu_\tau; S, \mathbb{T})\|^2$ , depends on the coupon bearing bond weights  $\omega_\tau^j(S, \mathbb{T})$  (see Eq. (18)) and, hence, the initial position of the yield curve. As explained, our approximation to the true government bond volatility index relies on freezing the weights  $\omega_\tau^j(S, \mathbb{T})$  at  $\omega_t^j(S, \mathbb{T})$ .

## 5. Evaluation of government bond volatility derivatives

This section derives pricing implications regarding futures and options referenced to the TYVIX index predicted by the model (Eq. (19)). Section 6 explains that our calibration procedure is such that the model-based index exactly matches its empirical counterpart. As a by-product of this matching, we obtain model-implied estimates of the unobservable realized variances  $v_t^2$  and long-term uncertainties  $\mu_t$ , which we reconstruct through Eq. (19). These estimates are used to feed the future and option pricing equations derived in this section.

### 5.1. Futures

The value at time  $t$  of a future contract, referenced to TYVIX, and expiring at time  $t + \delta$ , is

$$\mathcal{F}(\mathbf{Y}_{t, \mathbb{T}}^{\$}, v_t^2, \mu_t; t + \delta, T, S, \mathbb{T}) \equiv \mathbb{E}_t(\text{GB-VI}(\mathbf{Y}_{t+\delta, \mathbb{T}+\delta}, v_{t+\delta}^2, \mu_{t+\delta}; t + \delta, T + \delta, S + \delta, \mathbb{T} + \delta)), \quad (20)$$

where  $\mathbb{E}_t$  denotes the expectation under the risk-neutral probability. The future value in Eq. (20) depends on the initial yield curve  $\mathbf{Y}_{t,\mathbb{T}}^{\$}$  because the model prediction on the whole yield curve at  $t + \delta$ ,  $\mathbf{Y}_{t+\delta,\mathbb{T}+\delta}$ , relies on the bond pricing model summarized by Proposition 2, and the latter pins down  $\mathbf{Y}_{t,\mathbb{T}}^{\$}$  exactly while predicting  $\mathbf{Y}_{t+\delta,\mathbb{T}+\delta}$ . The yield curve at  $t + \delta$  enters the TYVIX index predicted by the model through the weights  $\omega_{t+\delta}^i(S + \delta, \mathbb{T} + \delta)$  in Eq. (17), as explained after Proposition 4.

Relying on Proposition 4, the future price in Eq. (20) is

$$\begin{aligned} & \mathcal{F}(\mathbf{Y}_{t,\mathbb{T}}^{\$}, v_t^2, \mu_t; t + \delta, T, S, \mathbb{T}) \\ &= \iint \sqrt{\kappa_{\mu} m \mathcal{B}_1(\mathbf{Y}_{t_{\delta}, \mathbb{T}_{\delta}}, T_{\delta}, S_{\delta}, \mathbb{T}_{\delta}) + \mathcal{B}_2(\mathbf{Y}_{t_{\delta}, \mathbb{T}_{\delta}}, T_{\delta}, S_{\delta}, \mathbb{T}_{\delta}) v_{t+\delta}^2 + \mathcal{B}_3(\mathbf{Y}_{t_{\delta}, \mathbb{T}_{\delta}}, T_{\delta}, S_{\delta}, \mathbb{T}_{\delta}) \mu_{t+\delta}} \cdot \phi_t(x_{t+\delta}) dx_{t+\delta}, \end{aligned} \quad (21)$$

where  $\mathcal{B}_j(\mathbf{Y}_{t_{\delta}, \mathbb{T}_{\delta}}, T_{\delta}, S_{\delta}, \mathbb{T}_{\delta}) \equiv \mathcal{B}_j(\mathbf{Y}_{t+\delta, \mathbb{T}+\delta}, T + \delta, S + \delta, \mathbb{T} + \delta)$ ,  $\phi_t(x')$  denotes the risk-neutral joint density of the state  $x' \equiv (r', v^{2'}, \mu')$  conditional on (i) the initial yield curve,  $\mathbf{Y}_{t,\mathbb{T}}^{\$}$ , which includes the current short-term rate,  $r_t$ , and (ii) the basis point variance  $v_t^2$  and long-run uncertainty  $\mu_t$ .

Because the model in (8) is affine, the joint density  $\phi_t$  can be calculated in closed-form, up to the solution of a system of Riccati's equations. However, the expression for  $\mathcal{F}$  in Eq. (21) can be approximated by freezing the weights  $\omega_{t+\delta}^j(S + \delta, \mathbb{T} + \delta)$  at  $\omega_t^j(S, \mathbb{T})$ , similarly as with the approximations leading to Eq. (19) of Proposition 4. Accordingly, the value at  $t$  of a TYVIX future maturing at  $t + \delta$  can be approximated as

$$\begin{aligned} & \hat{\mathcal{F}}(\mathbf{Y}_{t,\mathbb{T}}^{\$}, v_t^2, \mu_t; t + \delta, T, S, \mathbb{T}) \\ & \equiv \iint \sqrt{\kappa_{\mu} m \mathcal{B}_1(\mathbf{Y}_{t,\mathbb{T}}^{\$}, T, S, \mathbb{T}) + \mathcal{B}_2(\mathbf{Y}_{t,\mathbb{T}}^{\$}, T, S, \mathbb{T}) x + \mathcal{B}_3(\mathbf{Y}_{t,\mathbb{T}}^{\$}, T, S, \mathbb{T}) y} \cdot f_{\delta}(x, y | v_t^2, \mu_t) dx dy, \end{aligned} \quad (22)$$

where the transition density  $f_{\delta}(x, y | v_t^2, \mu_t)$  is given in Appendix B (see Eq. (B.14)). This approximation considerably simplifies the numerical implementation of the model.

## 5.2. Options

The pricing of options is slightly more elaborated than that of the futures because the relevant transition densities are needed under the forward probability, rather than the risk-neutral. Consider

a European call option written on TYVIX, maturing at time  $t + \Delta$ , and struck at  $K$ . Its value is

$$\begin{aligned} & \mathcal{C}(\mathbf{Y}_{t,\mathbb{T}}^{\$}, v_t^2, \mu_t; t + \Delta, K, T, S, \mathbb{T}) \\ \equiv & P_t(T) \mathbb{E}_t^{Q^{t+\Delta}} (\text{GB-VI}(\mathbf{Y}_{t+\Delta, \mathbb{T}+\Delta}, v_{t+\Delta}^2, \mu_{t+\Delta}; t + \Delta, T + \Delta, S + \Delta, \mathbb{T} + \Delta) - K)^+. \end{aligned} \quad (23)$$

Note that the expectation in Eq. (23) is now under the forward probability. Therefore, and similar to the futures evaluation (see Eq. (21)), the time- $t$  joint density of the state  $(r_{t+\Delta}, v_{t+\Delta}^2, \mu_{t+\Delta})$  is needed to determine  $\mathcal{C}$ , albeit under the forward probability. It can be shown that  $(r_t, v_t^2, \mu_t)$  is still an affine process under the forward probability, and so its conditional density is known in closed form, up to the solution of Riccati's equations. One approximation to Eq. (23) obtains while freezing the weights  $\omega_{t+\Delta}^j(S + \Delta, \mathbb{T} + \Delta)$  at  $\omega_t^j(S, \mathbb{T})$ , similarly as with the futures evaluation equation (22). This yields the following approximation to the value at  $t$  of a TYVIX European call option maturing at  $t + \Delta$  and struck at  $K$ :

$$\begin{aligned} \hat{\mathcal{C}}(\mathbf{Y}_{t,\mathbb{T}}^{\$}, v_t^2, \mu_t; t + \Delta, K, T, S, \mathbb{T}) & \equiv P_t(T) \\ & \times \iiint \left( \sqrt{\kappa_{\mu} m \mathcal{B}_1(\mathbf{Y}_{t,\mathbb{T}}^{\$}, T, S, \mathbb{T}) + \mathcal{B}_2(\mathbf{Y}_{t,\mathbb{T}}^{\$}, T, S, \mathbb{T})x + \mathcal{B}_3(\mathbf{Y}_{t,\mathbb{T}}^{\$}, T, S, \mathbb{T})y - K} \right)^+ f_{\Delta}^F(x, y | v_t^2, \mu_t) dx dy, \end{aligned} \quad (24)$$

where the transition density  $f_{\Delta}^F(x, y | v_t^2, \mu_t)$  is given in Appendix B (see Eq. (B.19)).<sup>8</sup>

While modeling options on TYVIX *levels* leads to the previous analytical solution, options on TYVIX *futures* are more likely candidates for the purpose of trading or listing. Options on futures can also be evaluated through a representation similar to Eq. (23). However, Section 6.2.4 explains that options on futures are computationally difficult to evaluate, and proposes an approximation scheme that is used to implement their pricing on our dataset. Eq. (24) is only provided for completeness.

## 6. Model predictions

We implement the model and reconstruct an hypothetical history of TYVIX futures and option values that could have traded on each day of our sample. We aim to shed light into a number of characteristics in this Treasury volatility market.

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<sup>8</sup>Note that, in Section 6, we estimate the price of options written on TYVIX *futures*. For completeness, we provide Eq. (24), which regards option prices on TYVIX *levels*.

Firstly, we explore the main properties of TYVIX futures values in light of their trading implications: how would a long position in TYVIX futures drift over time? Secondly, we investigate the ability of TYVIX futures to hedge fixed income instruments: how would overlaying existing fixed income portfolios help mitigate drawdowns during high volatility periods? Thirdly, we examine the model predictions in periods marked by tail events: how would the TYVIX futures curve react to an unanticipated monetary policy decision? Would this decision affect the whole futures curve? How would this reaction change following smaller shocks? Finally, we explore the behavior of hypothetical options values on TYVIX futures. We reconstruct TYVIX option values based on parameters calibrated to predict TYVIX futures, and analyze the behavior of this “volatility of volatility” during important historical events.

The next subsection succinctly describes the model calibration strategy, with all technical details left in Appendix C. Our findings are in Section 6.2.

## 6.1. Calibration strategy

### 6.1.1. Data

The ideal situation to estimate our model would be one in which data could be available regarding both TYVIX and TYVIX derivatives (e.g., futures or options). Because futures on TYVIX were only launched in November 2014, our calibration relies on a proxy for the expected volatility in Treasury markets, namely at-the-money implied volatilities for CBOT’s 10-year Treasury note options (TY options, for brevity); below, we shall explain how we use TY implied volatilities in our calibration procedure.

We use the three-month Treasury Bill rate as a proxy for the instantaneous interest rate  $r_t$ . Finally, note that our government bond volatility model also relies on market data for the yield curve up to the maturity of the underlying coupon-bearing bond;<sup>9</sup> therefore, our dataset also comprises the term structure of US government bonds up to the relevant maturities, which we take to equal  $7\frac{1}{2}$  years, as further discussed below. Our sample, including TYVIX data, covers daily data available from Bloomberg from January 2, 2003 to February 19, 2016.

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<sup>9</sup>That is,  $Y_{t,T}^{\$}$  in Eqs. (19), (22) and (24); see also Eq. (31).



### 6.1.2. Risk premiums

We assume that the basis point variance of the short-term rate has square root dynamics under the physical probability  $P$ , just as it does under the risk-neutral  $Q$  (see Eqs. (8)). Precisely, we assume that the Radon-Nikodym derivative of  $P$  against  $Q$  is

$$\left. \frac{dP}{dQ} \right|_{\mathbb{F}_T} = e^{-\frac{1}{2} \int_t^T \|\Lambda_\tau\|^2 d\tau} - \int_t^T \Lambda_\tau dW_\tau^P,$$

where  $W_\tau^P$  is a vector Brownian motion under  $P$  and, by Girsanov's theorem, its components satisfy:  $dW_{i\tau}^P = dW_{i\tau} - \Lambda_{i\tau} d\tau$ , with  $\Lambda_{i\tau}$  denoting the  $i$ -th component of the risk premium vector  $\Lambda_\tau$  and, finally,  $\Lambda_{i\tau} = \lambda_i^A v_\tau + \lambda_i^B \frac{1}{v_\tau}$  for  $i = 1, 2$ , and  $\Lambda_{3\tau} = \lambda_3^A \sqrt{\mu_\tau} + \lambda_3^B \frac{1}{\sqrt{\mu_\tau}}$ , for six constants  $(\lambda_i^X)_{i \in \{1,2,3\}; X \in \{A,B\}}$ .

This specification of the risk premiums makes the model “essentially affine” (Duffee, 2002). All in all, the coefficients  $\lambda_i^B$  accommodate the situation in which risk compensation relates to both the volatility of the state variables *and* some constant component; for example, compensation for the exposure to basis point volatility is  $\Lambda_{2\tau} v_\tau = \lambda_2^A v_\tau^2 + \lambda_2^B$ . Note that, based on assumptions made below, our calibration strategy does not necessitate making inference on any of the risk premium coefficients. However, this discussion is instructive for the economic interpretation of our calibrated parameters.

Given our assumptions on risk premiums, we have that under  $P$ , the basis point variance process  $v_\tau^2$  is solution to

$$dv_\tau^2 = \kappa_v^* (\mu_\tau^* - v_\tau^2) d\tau + \xi v_\tau \left( \rho dW_{1\tau}^P + \sqrt{1 - \rho^2} dW_{2\tau}^P \right), \quad (25)$$

where

$$\kappa_v^* \equiv \kappa_v - \xi \lambda^A, \quad \mu_\tau^* \equiv \frac{\kappa_v}{\kappa_v^*} \mu_\tau + \frac{\xi}{\kappa_v^*} \lambda^B, \quad \lambda^X \equiv \rho \lambda_1^X + \sqrt{1 - \rho^2} \lambda_2^X, \quad X \in \{A, B\}. \quad (26)$$

The (variance) persistence parameter under the physical probability  $P$ ,  $\kappa_v^*$ , may differ from that under the risk-neutral  $Q$ ,  $\kappa_v$ , namely due to  $\lambda^A$ ; accordingly, we interpret  $\lambda^A$  as a “variance persistence risk premium.” Similarly, the “pull of attraction” of  $v_\tau^2$  under  $P$ ,  $\mu_\tau^*$ , may well have different dynamics under  $Q$ , due to risk-adjustments regarding both the short-term variance dynamics ( $\lambda^B$  in (26)) and

its own long-term expected value ( $m^*$  below), i.e.,

$$d\mu_\tau^* = \kappa_\mu^* (m^* - \mu_\tau^*) d\tau + \gamma^* \sqrt{\mu_\tau^*} dW_{3\tau}^P, \quad (27)$$

where

$$\kappa_\mu^* \equiv \kappa_\mu - \gamma \lambda_3^A, \quad m^* \equiv \left( \frac{\kappa_\mu}{\kappa_\mu^*} m + \frac{\gamma}{\kappa_\mu^*} \lambda_3^B \right) \frac{\kappa_v}{\kappa_v^*} + \frac{\xi}{\kappa_v^*} \lambda^B, \quad \gamma^* \equiv \gamma \frac{\kappa_v}{\kappa_v^*}.$$

Accordingly, we interpret  $\lambda^B$  and  $\lambda_3^B$  as “long-run variance risk premiums,” and  $\lambda_3^A$  as a “long-term variance persistence risk premium.”

### 6.1.3. Model implementation

We calibrate the model using a sample period covering nearly fifteen years of daily data and we do then make a few simplifying assumptions. Namely, we assume that  $\rho = 0$  and that the variance persistence risk premiums are zero (i.e.,  $\lambda^A = \lambda_3^A = 0$ ), implying that  $\kappa_v = \kappa_v^*$  and  $\kappa_\mu \equiv \kappa_\mu^*$ . We, then, create rolling window estimates  $\hat{\sigma}_t^2$  (say) of the basis point variance  $v_t^2$  of the short-term rate, relying on the 3-month Treasury Bill rate as a proxy of the short-term rate. We calibrate  $\kappa_v$  and  $\xi$ , by treating  $\hat{\sigma}_t^2$  as if it were the true basis point variance, and by matching the first and second moments of  $\hat{\sigma}_t^2$  to the theoretical counterparts implied by the model. Table 1 reports our calibrated values for  $\kappa_v$ , annualized, and  $\xi$ . The parameters  $\kappa_\mu$  and  $\gamma$  are calibrated similarly as  $\kappa_v$  and  $\xi$ , but are based on model-implied values of  $\mu_t$ .

**Table 1.** Calibrated parameter values and model-implied basis point uncertainties

$\kappa_r$	$\kappa_v$	$\xi$	$\text{avg}(v^2)$	$\text{avg}(\mu)$	$\kappa_\mu$	$m$	$\gamma$
0.0500	1.6057	0.0200	0.0132 <sup>2</sup>	0.0236 <sup>2</sup>	40.0235	0.0141 <sup>2</sup>	0.0759

We calibrate  $\kappa_r$  and daily values of  $\mu_t$  to ensure that the model (i) matches the TYVIX without errors, (ii) fits the term structure of short positions in unhedged calendar spreads, a proxy for hypothetical futures values calculated on ATM option values, (iii) generates a basis point interest rate variance that has minimum distance from the sample average of  $\hat{\sigma}_t^2$ . The proxies for the futures values in (ii) are calculated throughout a non-parametric procedure known in the equity space since at least Carr and Wu (2006) and Dupire (2006): for the approximation to a future value as of day  $t$

and expiring on day  $t + \delta$ , we have

$$\mathcal{F}_t^{\$}(\delta) \equiv \sqrt{(\delta + 1) \cdot \text{atm}_t^2(\delta + 1) - \delta \cdot \text{atm}_t^2(\delta)}, \quad (28)$$

where  $\text{atm}_t(\delta)$  is the TY option ATM implied volatility for maturity  $\delta$  (see Eq. (C.6) in Appendix C). Regarding the model prediction on TYVIX, we assume that the coupon bearing bond has time-to-maturity equal to  $7\frac{1}{2}$  years and pays \$3 semi-annually; that is, in terms of Eq. (1),  $C_i = \$6$ ,  $n = 2$ , and  $\mathbb{T} - t = 7.5$ .

We set  $m$  equal to the average estimates of long-run uncertainties,  $\mu_t$ , obtained calibrating a two-factor version of the model in which  $\mu_t$  is treated as a constant that is calibrated every day to produce minimum distance between the model-implied future values and the non-parametric estimates in Eq. (28) (see Eq. (C.4) in Appendix C). Finally, the parameter values in Table 1 are used to calibrate the three factor model-implied uncertainties  $v_t^2$  and  $\mu_t$  for each day, which then feed the model to produce predictions regarding future and option values.

Note that, in the model,  $m$  is the ergodic mean of both  $v_t^2$  and  $\mu_t$  under the risk-neutral probability  $Q$ . Precisely, and based on Eqs. (25) and (27), the ergodic means under the physical probability are  $E_{\infty}^p(v_t^2) = m^*$  and  $E_{\infty}^p(\mu_t) = m^* - \frac{\xi}{\kappa_v} \lambda^B$ .<sup>10</sup> Given the average values of the model-implied times series of  $v_t^2$  and  $\mu_t$  in Table 1, we have the following estimates:  $E_{\infty}^p(v_t^2) \approx 0.0132^2$  and  $E_{\infty}^p(\mu_t) \approx 0.0236^2$ ; this situation is consistent with risk premium coefficients satisfying  $\lambda_3^B > 0$  and  $\lambda^B < 0$ .

Table 1 reports the parameter values for  $\kappa_r$  and summary statistics for the model-implied short-term variance  $v_t^2$  and the model-implied expected basis point variance,  $\mu_t$ . The model matches the data when the short-term rate under the risk-neutral probability is quite persistent, i.e., when  $\kappa_r$  is small. The main reason underlying this finding is that our model is calibrated over a sample period in which unprecedented monetary policy interventions occurred in response to the 2007-2009 global financial crisis, which kept both interest rates *and* interest rate volatility extraordinarily low. In this context, a persistent short-term rate helps our model predict relatively high values for risk-adjusted expected Treasury *volatility* on long-term bonds, acting as a risk for the “long run,” in analogy with similar mechanisms put forward in the equity literature (see, e.g., Bansal and Yaron, 2004).

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<sup>10</sup>That is, for example,  $E_{\infty}^p(\mu_t) \equiv \lim_{T \rightarrow \infty} E^p(\mu_T | \mu_0)$ , where  $E^p(\cdot | \cdot)$  denotes conditional expectation taken under the physical probability.

Note that the model-implied basis point realized variance ( $v_t^2$  from Eq. (C.7) in Appendix C) is  $0.0132^2$  on average, which is higher than its statistical counterpart,  $\text{avg}(\hat{\sigma}_t^2) = 0.0101^2$ , i.e., an average volatility of about 100 basis points. Importantly, the model-implied realized basis point variance is on average only slightly lower than its long-term value under  $Q$ , i.e., the difference  $m - \text{avg}(v^2)$  is positive but small. This feature of the calibrated model leads to predictions of markets that are likely to experience frequent oscillations between contango and backwardation depending on the specific values taken by the realized short-term variance compared to its long-term risk-neutral expectation,  $m$ . This property leads to a variety of important implications, as we now explain.

## 6.2. Model implications

This section contains our empirical results. Sections 6.2.1 and 6.2.2 discuss our finding on backwardation of the futures volatility curve, assess properties of futures on expected volatility as hedging instruments for a variety of portfolios and, finally, provide comparisons with alternative outcomes, resulting while relying on VIX futures. Section 6.2.3 studies the effects of transitory and permanent shocks on the futures volatility curve, and analyzes details regarding the persistent inversion of this curve that occurred when the Fed raised interest rates in 2015. Finally, Section 6.2.4 contains model predictions on option evaluation—the volatility of volatility.

### 6.2.1. The calibrated futures term structure

The top panel of Figure 3 plots historical TYVIX levels against one-month futures prices based on the calibrated parameters and filtered values for both the short-term and long-term basis point variance,  $v_t^2$  and  $\mu_t$ . The two series track each other closely as one would reasonably expect for a short-dated futures price, while still exhibiting rich dynamics. For a closer examination of the difference, the bottom panel shows the future value minus the index level, or “TYVIX basis.” It indicates that the market oscillates between contango and backwardation with sustained periods of backwardation. This observation stands in contrast to the analogous basis in equity markets, as measured by VIX and traded one-month VIX futures prices, which displays persistent contango during the same time period (see Figure 2 in the Introduction). In particular, our model suggests that hypothetical markets for TYVIX futures would have spent more time in backwardation than in contango, especially during

the period 2007-2012.

The frequent backwardation is largely driven by the fact that we calibrate the model to fit the option-implied non-parametric estimate  $\mathcal{F}_t^{\mathfrak{S}}(\delta)$  in Eq. (28). By design, the model-implied one-month TYVIX futures closely track the non-parametric estimate based on ATM volatilities of TY options. This aspect of the calibration strategy is especially material when generating hypothetical historical futures prices as it imparts information on the volatility term structure from the TY options market to the shape of the estimated futures curve.<sup>11</sup>

Based on conversations with interest rate derivatives traders and strategists at numerous major dealer banks, the backwardation in the non-parametric estimates may be explained by the richness of the nearest month options that are often bid up going into scheduled macroeconomic announcements, such as non-farm payroll, GDP, and Federal Open Market Committee (FOMC) rate decisions.<sup>12</sup> Note that these features of the non-parametric estimates is *not* a possible artifact of how Bloomberg calculates generic one-month ATM implied volatilities for TY. Indeed, when options become extremely short-dated, e.g., within one week to expiry, traders cease to think in terms of implied volatilities and instead solely think in terms of prices because even one tick may correspond to large differences in implied volatilities. If the one-month implied volatilities are based in part on such short-dated options, it could then upwardly bias the one-month TY implied volatility data. However, Bloomberg does not use short-dated options while calculating implied volatilities in our dataset.

Section 6.2.3 provides explanations of this property based on the relative speed of mean-reversion in the short-term and long-term volatility factors,  $v_t^2$  and  $\mu_t$ . Intuitively, the higher the speed of mean reversion of  $\mu_t$  (relative to  $v_t^2$ ), the higher the impact of adverse economic shocks on the short-end of the futures volatility curve.

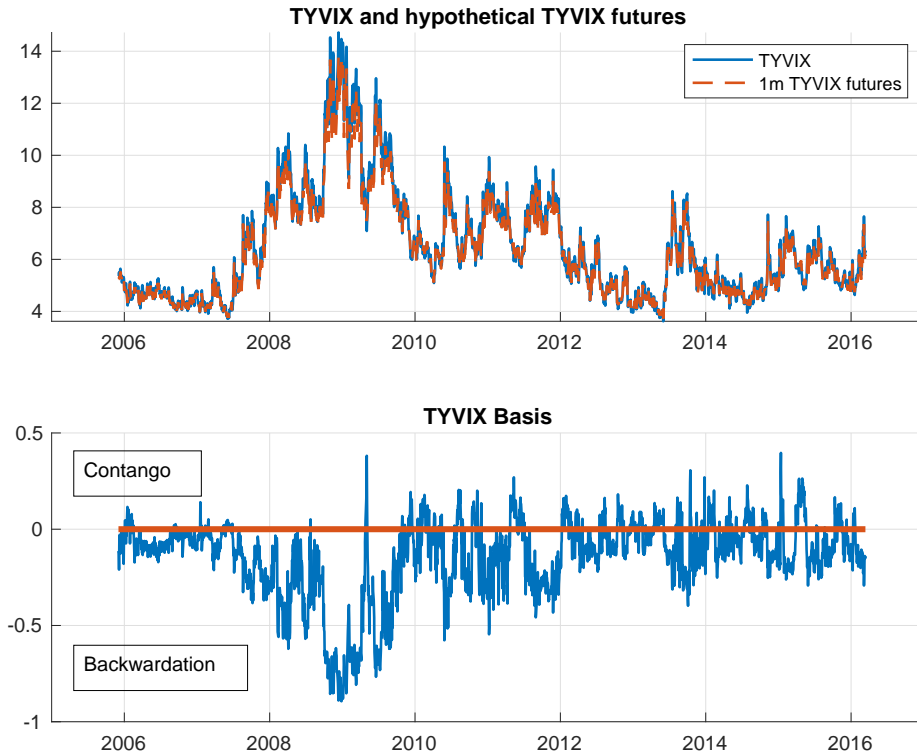
How does backwardation translate in terms of hypothetical P&L exercises from being long futures on expected government bond volatility? The blue line in Figure 4 shows non-compounded cumulative returns from being long the TYVIX index as if one could trade the index like a stock. In practice, however, one cannot trade the index like a stock and futures do not have constant maturity. Assuming a monthly futures listing schedule, the green line shows non-compounded cumulative returns from

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<sup>11</sup>Section 6.2.3 provides more intuition regarding conditions under which the model leads to backwardation.

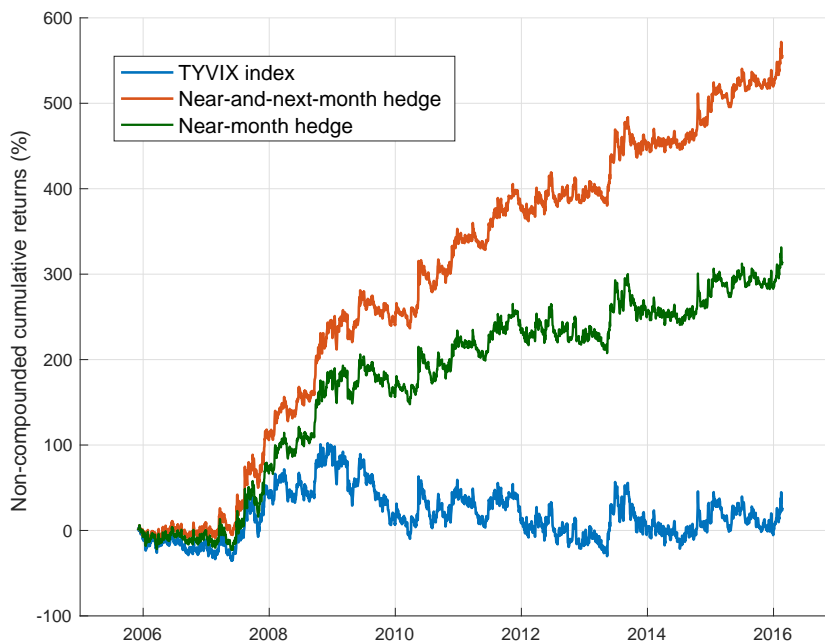
<sup>12</sup>We have conducted a separate event study of the behavior of TYVIX around major macroeconomic announcements, and found the results consistent with this anecdotal explanation. Our results are available upon request.

always being long the nearest-month futures contract that has at least five days to expiry; similarly, the red line shows returns from being long the near- and next-month contracts with weights set such that the weighted time to maturity is always 21 trading days. There is a pronounced upward drift in the P&L for being constantly long volatility using the simple rolling strategies, which is consistent with the frequent backwardation observed in Figure 3, and highlights the significant trading implications of the futures term structure.



**Figure 3.** *Top panel:* Cboe-CBOT TYVIX index of Treasury volatility and model-based calibrated one month TYVIX futures. *Bottom panel:* TYVIX basis, defined as the difference between one month future values and TYVIX values.

Table 2 provides more precise details on this drifting properties of the rolling strategies, while zooming the period beginning from 2008. In this period, near-and-next month hedge strategies have comparable average returns, although the former have better performance, due to lower performance variability and lower drawdowns. Below, we use near-term strategies to hedge various ETFs both in the fixed income and the equity space.



**Figure 4.** Cumulative returns from being long government bond volatility based on model-implied futures on the Cboe-CBOT TYVIX index of Treasury volatility. Red and green lines depict the P&L of strategies based on trading futures: the green is the P&L from holding the nearest-month future up to five days to expiry, and the red is the P&L from holding a portfolio of the near- and next-month futures with time-varying weights ensuring the average portfolio position is 21 trading days. The blue line tracks returns from being long the TYVIX as it was a tradeable asset.

**Table 2.** Performance of two volatility rolling strategies, the near-and-next and the near-month displayed in Figure 4. Mean and vol columns are the average returns and standard deviations of the strategies, annualized; Sharpe is the Sharpe ratio defined as the ratio of annualized means to standard deviations; Skewn. is the coefficient of skewness calculated on the strategies' returns; Drawdn. is the maximum drawdown on the strategies. Sample period is from January 2, 2008 to February 19, 2016.

	Rolling strategies				
	mean	vol	Sharpe	Skewn.	Drawdn.
near-next	0.2169	0.2252	0.9634	1.5527	0.1814
near	0.2299	0.3415	0.6733	0.9587	0.2831

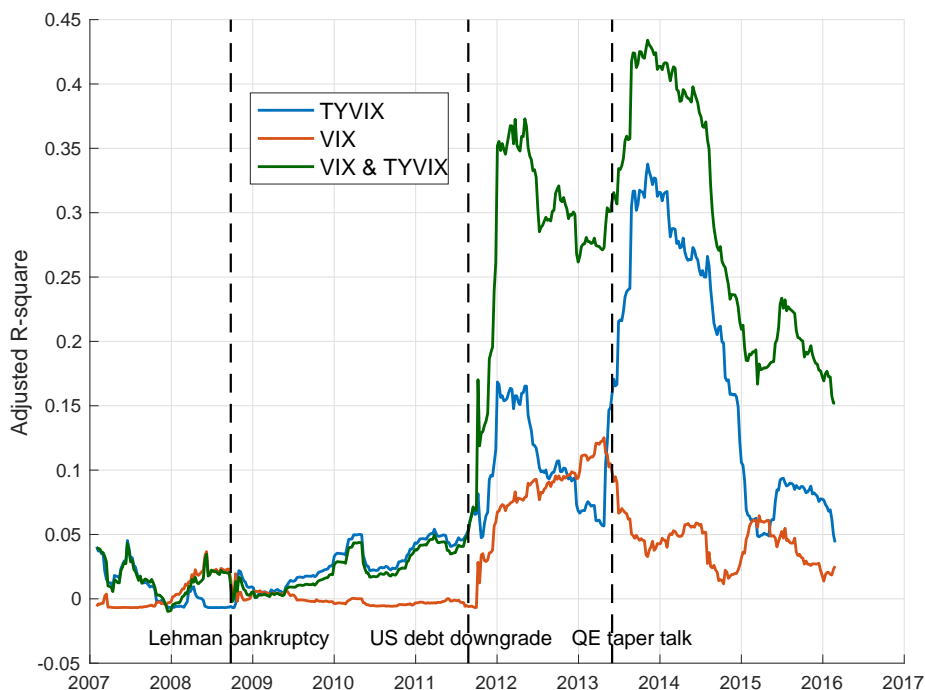
### 6.2.2. Applications to hedging bond portfolios

Given the prevalent use of equity volatility derivatives as equity portfolio hedging tools, a natural question that arises is whether interest rate volatility derivatives may be used for hedging bond portfolios. To address this question, we first examine the relation between returns on the near-term rolling hedging strategy explained in the previous section (the 21 day fixed maturity strategy depicted as the red line in Figure 4) and returns on the iShares Core U.S. Aggregate Bond ETF (ticker AGG), which is one of the most commonly-used diversified bond portfolio benchmarks.

Figure 5 shows adjusted- $R^2$  statistics from rolling OLS weekly regressions (and regression window equal to three years), with AGG returns on the left-hand-side and the TYVIX log-changes on the right-hand side. We include analogous results for VIX for the sake of comparison. The plot reveals two spikes in the correlation between the bond portfolio and TYVIX futures that correspond to the US debt rating downgrade in 2011 and the “QE Taper Talk” in mid-2013; to the contrary, the correlation was nearly zero at the depth of the financial crisis. One interpretation is that bond portfolio performance and interest rate volatility become tighter linked when interest rate risk is a significant concern in the market; the correlation during the 2008-2009 period was low because interest rate risk did not play a key role in the crisis.

Naturally, TYVIX and VIX are not investable. For example, it is well-known that log changes in the VIX index cannot necessarily be replicated by log changes on traded VIX futures due to effects relating to the volatility term structure: the VIX future curve is in contango more often than not, as explained in the Introduction. Therefore, we re-calculate rolling regressions using AGG returns on the left-hand-side and the rolling near-term TYVIX futures strategy depicted in Figure 4 on the right-hand-side. We perform the same regressions relying on a similar rolling near-term VIX future strategy. Then, we overlay AGG returns based on the sign and magnitude of the coefficient estimates in these regressions.





**Figure 5.** Adjusted- $R^2$  statistics calculated through weekly rolling regressions of the returns on the iShares Core U.S. Aggregate Bond ETF onto returns on model-implied one-month futures on TYVIX (blue line), returns on one-month futures on VIX (red), and returns on both TYVIX and VIX one-month futures (green). Estimation window is 3 years. One-month future returns on VIX and TYVIX are calculated through a portfolio of near- and next-month futures with time-varying weights ensuring the average portfolio position is 21 trading days (as in the TYVIX case depicted through the red line in Figure 4).

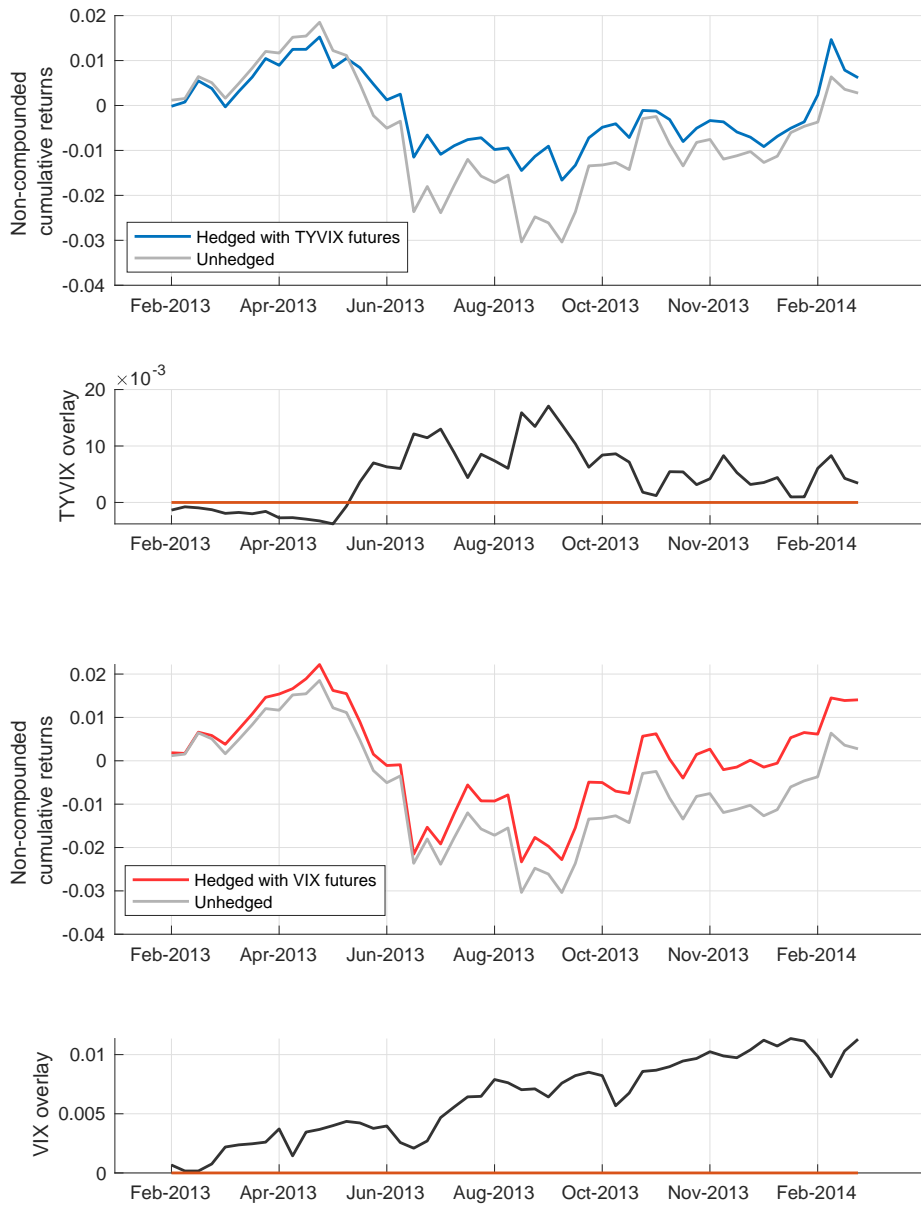
Figure 6 depicts unhedged and overlaid returns during the period comprising the QE Taper Talk events in 2013, from February 12, 2013 to February 18, 2014. The blue line shows the TYVIX hedged returns, which outperform the unhedged (in grey), by smoothing out two pronounced drawdowns. Figure 6 shows that the VIX-hedged returns generally outperform the unhedged returns. The reason for this outperformance is that in this sampling period, VIX futures returns and AGG are positively correlated, and the hedging portfolio then prescribes selling VIX futures, which leads to monetizing a contango premium that is typical in VIX futures market. Note, however, that it would be quite unusual to protect a portfolio from spikes in market volatility while being short VIX futures. In fact, the VIX-hedged strategy is not as effective as the TYVIX-hedged strategy during the 2013 rate-driven events. Hedging AGG with VIX futures does lead indeed to one quite pronounced drawdown in this

period.

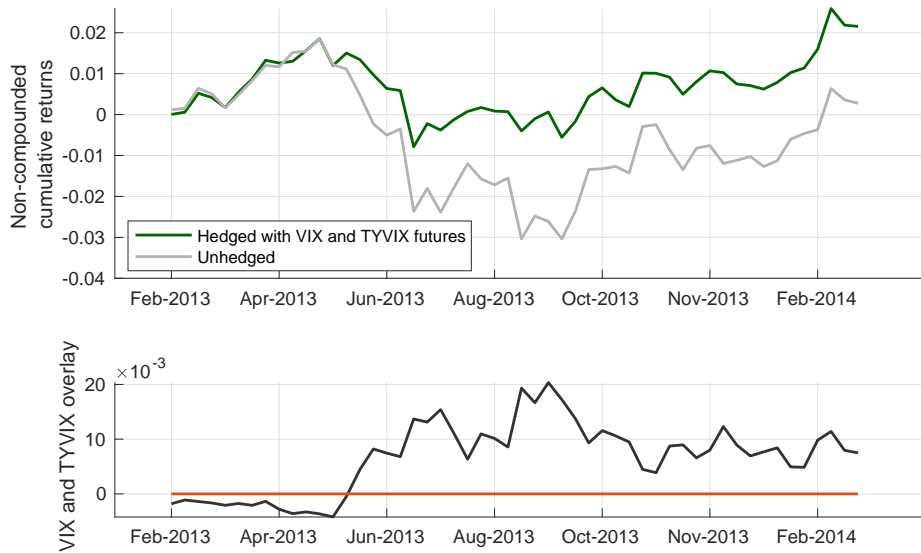
The bottom panels of Figure 6 show how AGG returns are hedged reasonably well when we rely on both TYVIX and VIX rolling strategies, evidencing the role the TYVIX hedging strategy plays while insuring against rate-driven events. These very simple hedging results suggest that TYVIX futures may be an effective hedging tool for bond portfolios, given the tendency of implied volatilities to spike more than usual during pronounced bond market sell-offs.

Table 3 contains a quantitative assessment of our findings that relate to the whole sampling period. We find that, while TYVIX futures help improve the performance of AGG in terms of Sharpe ratios, VIX futures compare better in this dimension; at the same time, VIX futures would not help avoid the large AGG drawdowns occurred during the QE Taper Talk episode, as already noted. Importantly, overlaying AGG with VIX futures leads to a pronounced skewness, a measure of the overall sensitivity of the overlaid portfolio to drawdowns.

Table 3 reports results regarding additional ETFs and assets. Overall, TYVIX derivatives may help mitigate drawdowns on fixed income related instruments, especially when used in conjunction with VIX derivatives. Note that a systematic use of VIX derivatives might even lead to deteriorate performance in the equity space: in this space, the overlay strategy prescribes to be long VIX futures, implying, then, large insurance costs arising through contango. The last two pieces of Table 3 reveal that a systematic use of TYVIX might slightly improve the overall performance, although the differences with the unhedged equity returns appear to be both economically and statistically very negligible.



**Figure 6**—Continued on the next page.



**Figure 6**—Continued from the previous page. Overlays with equity and interest rate volatility derivatives during the Taper Talk. Volatility derivatives are one-month future returns on VIX and TYVIX, and are calculated through portfolios that have an average position of 21 trading days (as in the TYVIX case depicted through the red line in Figure 4). Positioning into derivatives is determined through the coefficient of weekly rolling regressions of AGG returns onto derivative returns, with a fixed window equal to 3 years. *Top panels:* Cumulative returns from being long the iShares Core U.S. Aggregate Bond ETF, unhedged (grey line) and hedged with one month TYVIX futures (blue). *Middle panels:* Cumulative returns are as in the top panels, but hedging ratios calculated based on one month VIX futures. *Bottom panels:* Cumulative returns are as in the top panels, with hedging ratios regarding both one-month TYVIX futures and one-month VIX futures.

**Table 3.** Overlay performance on selected ETFs achieved through equity (vix) and interest rate (tyvix) volatility derivatives, or both (tyvix & vix). Volatility derivatives are one-month future returns on VIX and TYVIX, and are calculated through portfolios that have an average position of 21 trading days. Mean and vol columns are the average returns and standard deviations of the investment strategies, annualized; Sharpe is the Sharpe ratio defined as the ratio of annualized means to standard deviations; Skewn. is the coefficient of skewness calculated on the strategies returns; Drawdn. is the maximum drawdown on the investment strategies. Sampling period is from January 2, 2003 to February 19, 2016.

<i>AGG.US.Equity:</i> Treas, gvt, corp, MBS, ABS and CMBS					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	0.0026	0.0398	0.0671	-0.9144	-0.0477
tyvix	0.0059	0.0303	0.1972	-0.4082	-0.0313
vix	0.0136	0.0405	0.3363	-0.9712	-0.0444
tyvix & vix	0.0208	0.0279	0.7464	-0.6265	-0.0260
<i>LQD.US.Equity:</i> Investment-grade corporate bonds					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	0.0162	0.0729	0.2227	-1.2645	-0.0844
tyvix	0.0217	0.0559	0.3881	-0.8243	-0.0481
vix	0.0166	0.0731	0.2281	-1.2655	-0.0844
tyvix & vix	0.0308	0.0543	0.5664	-0.9050	-0.0478
<i>TIP.US.Equity:</i> Inflation protected T-notes					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0609	0.0704	-0.8637	-0.9247	-0.0986
tyvix	-0.0603	0.0551	-1.0929	-0.5730	-0.0885
vix	-0.0482	0.0725	-0.6648	-0.9264	-0.0944
tyvix & vix	-0.0422	0.0547	-0.7702	-0.6413	-0.0752
<i>TY1.Comdty:</i> Ten Year T-note					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0396	0.0573	-0.6925	-1.5188	-0.0797
tyvix	-0.0342	0.0473	-0.7214	-1.3031	-0.0729
vix	-0.0126	0.0588	-0.2148	-1.5298	-0.0624
tyvix & vix	-0.0026	0.0434	-0.0602	-1.6840	-0.0487
<i>TLT.US.Equity:</i> Treasury bonds (>20 years)					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0503	0.1257	-0.4003	-0.3346	-0.1666
tyvix	-0.0321	0.1067	-0.3007	-0.1014	-0.1408
vix	0.0505	0.1305	0.3870	-0.3681	-0.1403
tyvix & vix	0.0797	0.0904	0.8810	-0.1783	-0.0805
<i>SHY.US.Equity:</i> Treasury bonds ( $\in [1, 3]$ years)					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	0.0040	0.0049	0.8151	-0.5283	-0.0031
tyvix	0.0049	0.0044	1.1254	-0.5862	-0.0021
vix	0.0060	0.0052	1.1626	-0.6315	-0.0030
tyvix & vix	0.0072	0.0045	1.5924	-0.8048	-0.0020

Table 3 continued

<i>PZA.US.Equity</i> : Muni bonds (>20 years)					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0409	0.0786	-0.5210	-0.9045	-0.1187
tyvix	-0.0385	0.0742	-0.5187	-0.8962	-0.1027
vix	-0.0288	0.0795	-0.3621	-0.9568	-0.1153
tyvix & vix	-0.0214	0.0736	-0.2913	-1.0031	-0.0948
<i>MUB.US.Equity</i> : Investment grade muni bonds					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0238	0.0749	-0.3173	-1.5570	-0.0846
tyvix	-0.0212	0.0641	-0.3298	-1.3012	-0.0632
vix	-0.0221	0.0751	-0.2945	-1.5702	-0.0844
tyvix & vix	-0.0137	0.0636	-0.2162	-1.3119	-0.0633
<i>IEI.US.Equity</i> : Treasury bonds ( $\in [3, 7]$ years)					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0010	0.0310	-0.0322	-0.8102	-0.0383
tyvix	0.0040	0.0245	0.1657	-0.3682	-0.0240
vix	0.0133	0.0321	0.4160	-0.8825	-0.0336
tyvix & vix	0.0213	0.0228	0.9332	-0.6697	-0.0167
<i>IEF.US.Equity</i> : Treasury bonds ( $\in [7, 10]$ years)					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	-0.0181	0.0653	-0.2768	-0.7270	-0.0875
tyvix	-0.0090	0.0505	-0.1791	-0.2550	-0.0650
vix	0.0186	0.0677	0.2741	-0.7824	-0.0755
tyvix & vix	0.0339	0.0444	0.7639	-0.5026	-0.0385
<i>SPX.Index</i> : S&P 500 index					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	0.1911	0.1131	1.6891	-0.3388	-0.0545
tyvix	0.1801	0.1021	1.7630	-0.3443	-0.0501
vix	0.0327	0.0514	0.6358	-0.3224	-0.0490
tyvix & vix	0.0232	0.0563	0.4115	-0.5482	-0.0556
<i>DIA.US.Equity</i> : Dow Jones Industrial Average					
	mean	vol	Sharpe	Skewn.	Drawdn.
unhedged	0.1640	0.1097	1.4945	-0.5932	-0.0657
tyvix	0.1539	0.1014	1.5176	-0.5753	-0.0623
vix	0.0289	0.0607	0.4766	-0.3859	-0.0553
tyvix & vix	0.0205	0.0638	0.3220	-0.4650	-0.0604

### 6.2.3. Scenarios

This section describes a number of hypothetical developments in the government volatility curve that follow the occurrence of shocks affecting both interest rate and interest rate volatility. A prominent instance of an event leading to such shocks is the Fed decision on December 16, 2015 to raise interest rates for the first time in nearly a decade. We investigate two issues. Firstly, we examine the model predictions regarding the reaction of the TYVIX future curve to *hypothetical* interest rate volatility shocks; this type of analysis is useful for the purpose of stress-testing the model in risk-management experiments, but also for evaluating trading theses. Secondly, we examine the model predictions on the TYVIX future curve at dates of particular historical interest; our objective is to use our model and reverse-engineer from the data the magnitude of the economic shocks underlying interest rate volatility developments occurring on such dates of interest as that in December 2015.

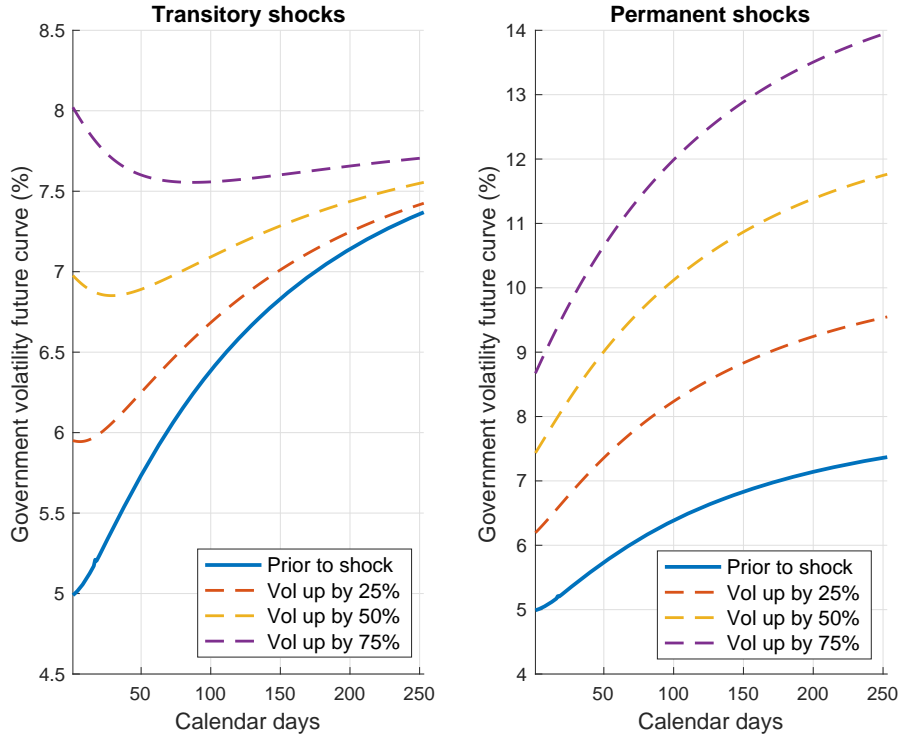
As for the futures curve reaction to given shocks, we consider experiments in which changes in current volatility might be either transitory or permanent. For example, we study how a permanent monetary policy shift may affect market expectations, by assessing how the TYVIX futures curve reacts to a permanent higher volatility of the short-term rate. We implement the experiments while feeding the model with information available up to December 4, 2015, about two weeks before the Fed announced an increase in the target range for the federal funds rate. In all the experiments, we assume that there is a 1% upward parallel shift in the yield curve accompanied by shocks of various nature that we will describe in a moment.<sup>13</sup>

The left panel of Figure 7 shows the impact of shocks to short-term basis point volatility while long-term expected volatility is kept constant and equal to the estimate of  $m$  in Table 1. We interpret this shock as being a “transitory” one. We consider 25%, 50% and finally 75% increases in short-term volatility. In these scenarios, the shorter end of the futures curve is the most affected, while the longer end reverts to the long-term mean. In times of extreme uncertainty, volatility curves have been known to flatten or even invert, and this experiment shows the model predictions are consistent with such dynamics of the term structure. The right panel shows how the model futures curve reacts when we also change  $m$  upwards by the same amount as the shock in the realized variance. This scenario may be interpreted as one of a significant change in long-term monetary policy uncertainty.

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<sup>13</sup>We repeat the experiments while assuming that no shifts occur in the yield curve, and find that results are qualitatively the same as those reported below.

The model predicts that, in this case, changes in the futures curve resemble parallel shifts with a slight counterclockwise twist.

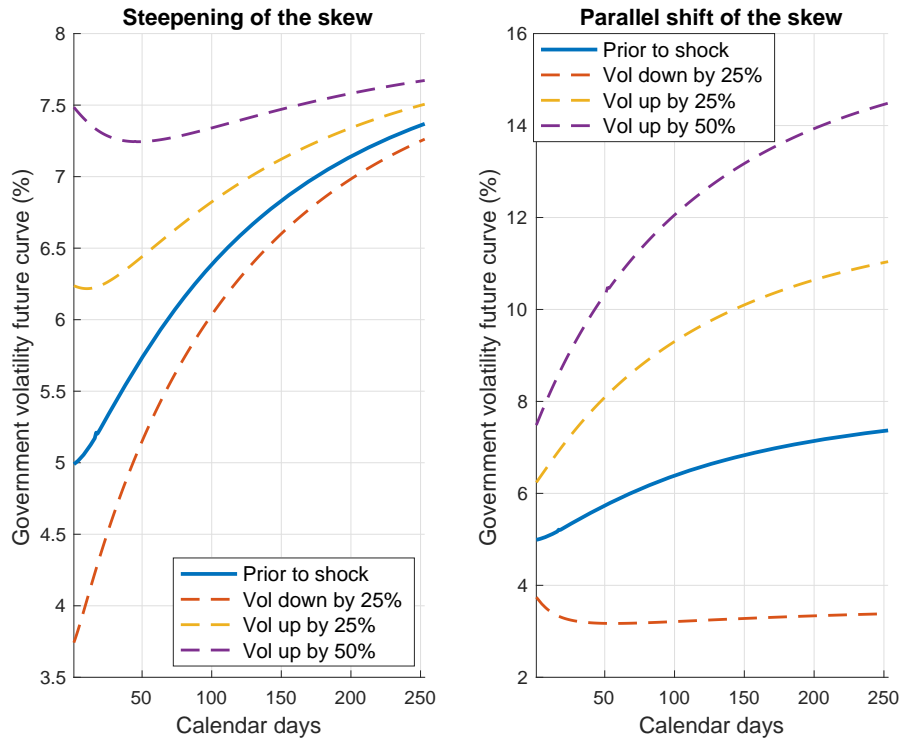


**Figure 7.** Changes in the TYVIX futures curve in reaction to shocks in the current short-term volatility of 25%, 50% and 75%. *Left panel:* Reaction to transitory shocks, i.e., shocks in short-term volatility that are not accompanied by changes in long-term expected volatility. *Right panel:* Reaction to permanent shocks, i.e., shocks in short-term volatility accompanied by a same magnitude change in long-term expected volatility.

The left panel of Figure 8 summarizes how the future curve reacts to a steepening or flattening of the volatility skew. We simulate such changes in the volatility skew by increasing or decreasing TYVIX while keeping ATM volatility constant, and calculating futures values through our calibration procedure (see Eqs. (C.4)-(C.5) in Appendix C). We consider three scenarios: a TYVIX reduction by 25%, and two increases of 25% and 50%. We interpret a steepening of the skew as a transitory shock, which corresponds to a “fear wave” hitting the markets: we interpret it to be transitory precisely because it merely affects out-of-the money options while keeping at-the-money option markets intact. The implications are indeed qualitatively the same as those in the left panel of Figure 7. In contrast, consider the right panel of Figure 8, in which we assume that at-the-money options are also affected



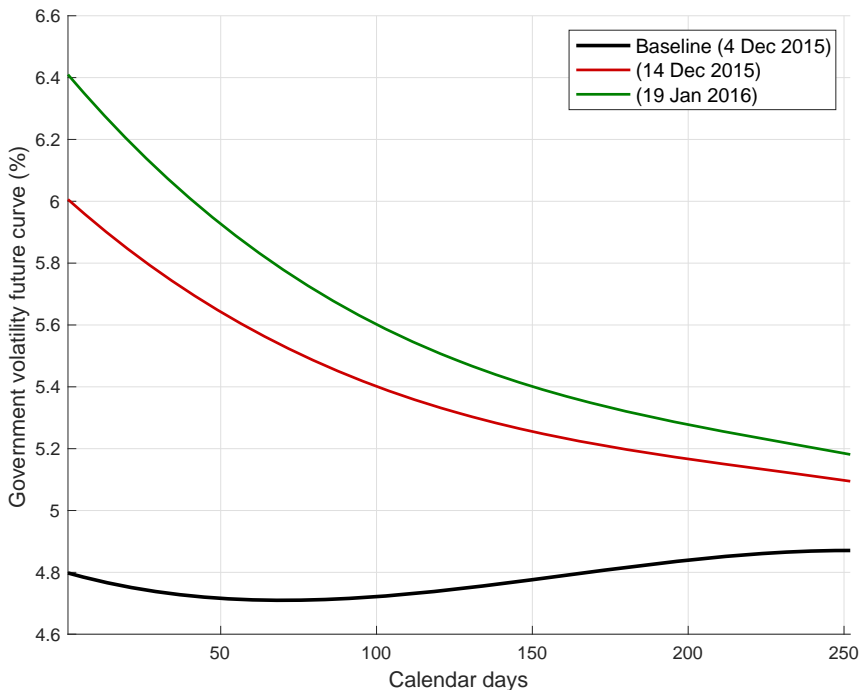
by a fear wave, in that ATM volatilities now increase by the same percentage amount experienced by TYVIX: we interpret this experiment as one illustrating how a permanent fear in option markets translates in terms of the TYVIX futures curve. The effects are qualitatively similar as those depicted in the right panel of Figure 7, with effects on the futures curve that increase with the futures expiry.



**Figure 8.** Changes in the TYVIX futures curve in reaction to shocks in the option skew. *Left panel:* Reaction to a steepening of the option skew, defined as an increase in TYVIX by  $-25\%$ ,  $25\%$  and  $50\%$  occurring while the ATM volatility remains constant. *Right panel:* Reaction to a parallel shift of the option skew, defined as an increase in TYVIX by  $-25\%$ ,  $25\%$  and  $50\%$  occurring while the ATM volatility changes by the same percentage amount.

Our second objective is to analyze the model predictions around December 16, 2015, when the Fed announced to modify its targets for the federal funds rates after almost a decade of inaction. While anecdotal evidence suggests that the outcome of this decision seemed to be largely anticipated by the market, Figure 9 shows that the TYVIX curve predicted by the model goes from contango to a sharp backwardation in a matter of a few days (from the 4th of December to just two days before the announcement). What is the order of magnitude of the shocks in the midst of such highly

volatile market conditions? During these days, the model-implied short-term variance ( $v_t^2$ ) rose by 44% whereas long-term uncertainty ( $\mu_t$ ) went through a 4-fold increase. Yet the model predicts a volatility market in backwardation, a situation persisting for several weeks in January 2016 (as exemplified by the TYVIX curve on January 19), a month marked by additional concerns on the weakening of data in emerging countries, oil prices and the stability of the global banking sector.



**Figure 9.** Model-based TYVIX futures curve on selected dates: (i) the baseline date, December 4, 2015; (ii) December 14, 2015, i.e., two days prior to the Fed announcement regarding decisions on interest rates; (iii) January 19, 2016.

Why does the model predict backwardation during these episodes? The intuitive reason is that long-run Treasury uncertainty,  $\mu_t$ , is also a mean-reverting process, and, according to our calibrated model, a low-persistent one: even large shocks on  $\mu_t$  are short-lived. With shocks on  $\mu_t$  absorbed relatively rapidly, the short- and medium-term future curve increases more than the long-term, producing backwardation in the volatility market during these months of heightened volatility.

To illustrate these properties throughout a simple case, consider a one-month future on the fair value of a hypothetical variance swap on the basis-point short-term rate. The fair value of this

contract is

$$\text{VS}_t(T) \equiv \frac{1}{T-t} \int_t^T \mathbb{E}(v_s^2 | v_t^2, \mu_t) ds.$$

Assume for simplicity that  $t = T - t = \Delta = \frac{1}{12}$ ; it can be shown that the “variance basis” at time-0 is

$$\mathbf{B}_0 \equiv F_0 - \text{VS}_0(t) = \underbrace{\bar{\beta} (\mathbb{E}_0(v_t^2) - v_0^2)}_{\equiv \mathbb{E}_0(\Delta v_t^2)} + \underbrace{\bar{\gamma} (\mathbb{E}_0(\mu_t) - \mu_0)}_{\equiv \mathbb{E}_0(\Delta \mu_t)}, \quad (29)$$

where  $F_0 = \mathbb{E}_0(\text{VS}_t(T))$  denotes the value of a variance future, and

$$\bar{\beta} = \frac{b_v}{\kappa_v \Delta}, \quad \bar{\gamma} = \frac{\kappa_v}{\kappa_v - \kappa_\mu} \left( \frac{b_\mu}{\kappa_\mu \Delta} - \frac{b_v}{\kappa_v \Delta} \right),$$

with  $b_X = 1 - e^{-\kappa_X \Delta}$ ,  $X \in \{v, \mu\}$ .

Eq. (29) tells us that, in this market, backwardation and contango are driven by mean-reversion in the short-term ( $v^2$ ) and long-term ( $\mu^2$ ) uncertainties. For example, the market is in contango when both current variances are “anemic” (i.e., lower than their conditionally expected values for the expiry date), and is in backwardation when they are both higher than their conditionally expected values.

Now, consider a large and positive shock in both the current  $v_0^2$  and  $\mu_0$ , similarly as in the Treasury volatility market subsequent to the Fed decisions in December 2015. It is possible to show that

$$\mathbb{E}_0(\Delta v_t^2) = b_v (m - v_0^2) - \frac{\kappa_v (b_v - b_\mu)}{\kappa_v - \kappa_\mu} (m - \mu_0), \quad \mathbb{E}_0(\Delta \mu_t) = b_\mu (m - \mu_0). \quad (30)$$

The first terms of both the R.H.S. in the previous expressions capture a standard mean-reversion effect. For example,  $\mathbb{E}_0(\Delta v_t^2)$  tends to be negative as  $v_0$  gets larger. Eqs. (30) also show that a large positive shock in the long-term uncertainty (i.e., one that drives  $\mu_0$  above  $m$ ) has two effects: (i) it leads  $\mathbb{E}_0(\Delta \mu_t)$  to negative values, thereby being a source of backwardation (see Eq. (29)); (ii) it raises expected realized variance,  $\mathbb{E}_0(\Delta v_t^2)$ , thereby acting as a source of contango. Therefore, the ultimate effect of a shock in  $\mu_0$  on the variance basis  $\mathbf{B}_0$  is parameter dependent. In a market hit by large and positive volatility shocks (both short-term and long-term), as during December 2015 and January 2016, the effects of  $\mu_0$  are less important than those in  $v_0^2$  when  $\mu_t$  is not persistent, as in

the case of our calibrated model.<sup>14</sup>

Naturally, the market we analyze in this paper is more complex than the variance market on the short-term rate conveyed in this example, especially due to the endogenous nature of bond price volatility. But the mechanism is similar: due to mean-reversion, our model predicts that because  $\mu_t$  is a low persistent process, the TYVIX curve enters in backwardation even when long-term uncertainty,  $\mu_t$ , raises more than the short-term,  $v_t^2$ .

#### 6.2.4. Option evaluation and the volatility of volatility

We use the calibrated model and predict hypothetical values for European-style options on TYVIX futures. Note that the option pricing model in Section 5 is referenced to options written on TYVIX, not TYVIX futures (see Eq. (24)). However, options on TYVIX futures seem to be more natural candidates than options on TYVIX as potential listed products. Their pricing is more complex, though: by standard arguments, the TYVIX future option price counterpart to the TYVIX option in Eq. (24) is

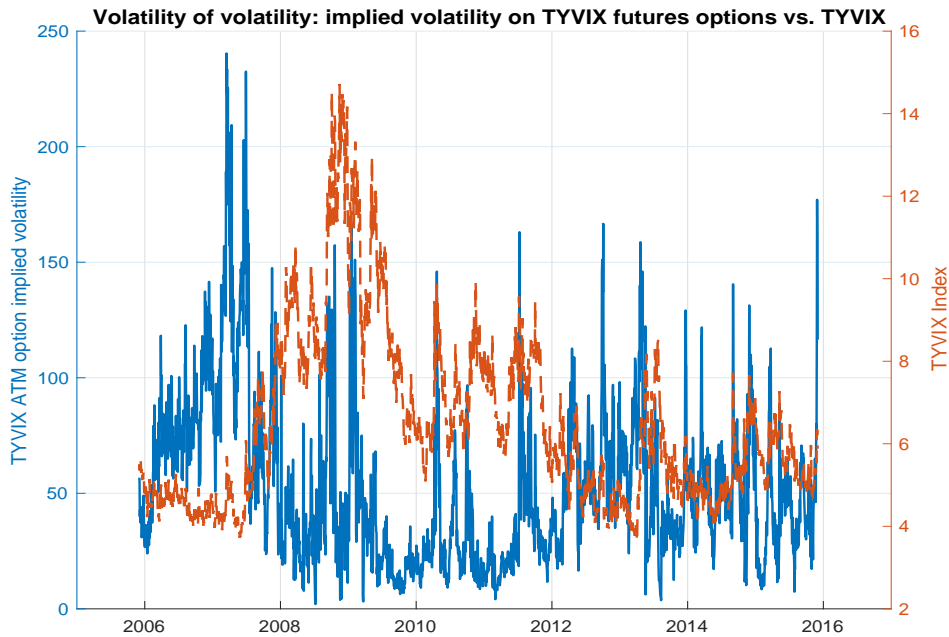
$$\begin{aligned} & \hat{C}_t^f(t, v_t^2, \mu_t; t + \Delta, K, T, S, \mathbb{T}) \\ & \equiv P_t(T) \iint \left( \hat{\mathcal{F}}(\mathcal{Y}_{T+\Delta, \mathbb{T}+\Delta}, x, y; T + \delta, T, S, \mathbb{T}) - K \right)^+ f_\Delta^F(x, y | v_t^2, \mu_t) dx dy, \end{aligned} \quad (31)$$

where  $\hat{\mathcal{F}}(\cdot)$  (and remaining notation) is as in Eq. (22) and  $f_\Delta^F$  is as in (24). In particular, Eq. (31) reveals that, in addition to the double integration needed to determine  $\hat{\mathcal{F}}(\cdot)$  at time  $T + \delta$ , one needs to integrate twice again to obtain the option value. It is computationally prohibitive, because the densities  $f_\delta$  and  $f_\Delta^F$  are also only known up to the solution to Riccati's equations. Appendix C.2 described an approximation to  $\hat{C}_t^f(t, v_t^2, \mu_t; t + \Delta, K, T, S, \mathbb{T})$ , which we employ in this section (see Eq. (C.8)).

We consider options maturing in one month, with underlying futures expiring in two months. Figure 10 depicts estimates of the at-the-money Black's implied volatility on these TYVIX future option values, where the at-the-money prices are taken to be the benchmark future values (i.e.,  $\mathcal{F}_n^\$$  in Eq. (C.6) of Appendix C.1).

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<sup>14</sup>To illustrate based on the parameter values in Table 1, the two loadings on  $(m - \mu_0)$  in Eqs. (30) are  $-\frac{\kappa_v(b_v - b_\mu)}{\kappa_v - \kappa_\mu} = -3.5073 \cdot 10^{-2}$  and  $b_\mu = 0.9644$ . The former is one order of magnitude lower than the latter.



**Figure 10.** *Solid line:* Implied volatilities calculated on hypothetical at-the-money one-month options on TYVIX futures, with futures maturing in two months. *Dashed line:* TYVIX Index.

The dynamics of this *volatility of volatility* (vol-of-vol) provide useful information. Note that the vol-of-vol is a proxy for expected volatility in TYVIX corrected by risk, that is, a price for being insured against imminent (one month) volatility in TYVIX. Figure 10 contrasts this vol-of-vol with the dynamics of TYVIX. According to the model, the market evaluation of this type of insurance had anticipated the occurrence of the subprime mortgage turmoil erupting in 2007: signs of future distress could be read in the sudden increase of the vol-of-vol since 2006, an increase likely driven by a rapid succession of increases in the Fed funds rate targets. Remarkably, the vol-of-vol also spikes during episodes of acute uncertainty such as the Lehman’s bankruptcy, although with limited persistence. Episodes of high and persistent vol-of-vol occur during the taper tantrum of 2013: not only does TYVIX increase during this episode, the vol-of-vol fluctuates quite considerably around and prior to statements of Fed officials in May 2013. Finally the Fed increase in the target is preceded by one of the most dramatic increases in the vol-of-vol since its inception. Our model, albeit only calibrated to match expected interest rate volatility, suggests that the volatility of fixed income market volatility may contain useful information on important market developments.

## 7. Conclusions

Fixed income volatility is being the focus of renewed interest in recent literature reviewed in Section 2. How would this uncertainty be mitigated by dedicated hedging instruments? While both literature and market practice have identified models and methods to monitor and assess realized and market-adjusted gauges of fixed income volatility, our contribution aims to a natural step forward. We develop a model to evaluate dedicated instruments referenced to market-adjusted expectations of volatility in government bond markets. Our model makes predictions regarding both expected volatility and hedging instruments such as futures and options written on this volatility.

Our model predicts that futures markets on U.S. Treasury expected volatility are likely to experience frequent oscillations between backwardation and contango. This property is in stark contrast to dynamics known in the equity space. In our model, volatility is driven by a short-term and a long-term factor. Mean-reversion in the long-term factor is rather quick; then, bad times raise the whole future volatility curve, but the long-term raises less, and this backwardation persists because the long-term factor mean-reverts relatively quickly. Due to frequent and persistent backwardation, our model suggests that hedging fixed income portfolios with government volatility futures leads to mitigate the occurrence of drawdowns while avoiding large insurance costs. We also use our model to ask whether derivatives on expected government bond volatility may help summarize or anticipate short-term or long-term uncertainty in monetary policy and other macroeconomic developments. Consistent with backwardation, we find that shocks on the government volatility curve tend to affect the short-end of the curve more than the long-term, just as it happened at the time the Federal Reserve raised interest rates in December 2015, the first time after many years of inaction. These conclusions obtain while calibrating our model to derivative data on interest rate volatility. Moreover, we use the very same calibrated model to predict the behavior of the volatility of volatility, that is, the implied volatility of hypothetical options referenced to government bond volatility futures. We provide evidence that this volatility of volatility has anticipatory power regarding market distress, being then a potential candidate for monitoring adverse market movements. Important topics for future research include the use of our model predictions to understand how government bond volatility connects to broader macroeconomic developments.

## Appendix A: Preliminary results regarding a two-factor model

Our analysis relies on the three-factor model in Sections 3 and 4, but it is useful to initially base our derivations on a simpler, two-factor model. This two-factor model helps illustrate properties of the three-factor, and does actually play an auxiliary role in our calibration strategy (see Appendix C). Consider the following dynamics of the short-term rate:

$$\begin{cases} dr_\tau &= \kappa_r (\theta_\tau - r_\tau) d\tau + v_\tau dW_{1\tau} \\ dv_\tau^2 &= \kappa_v (\mu - v_\tau^2) d\tau + \xi v_\tau \left( \rho dW_{1\tau} + \sqrt{1 - \rho^2} dW_{2\tau} \right) \end{cases} \quad (\text{A.1})$$

These dynamics are easily seen to be special case of Eqs. (8) in the main text (notably when the long-term uncertainty factor  $\mu_\tau$  is constant and equal to  $\mu$ ). We have:

**PROPOSITION A.1.** *Assume that the short-term rate is as in Eqs. (A.1). Then, the price at time  $\tau \geq t$  of a zero coupon bond expiring at time  $S$ , when the state is  $(r_\tau, v_\tau)$ , is*

$$P_\tau(r_\tau, v_\tau^2, S) \equiv e^{A_S(\tau) - B_S(\tau)r_\tau + C_S(\tau)v_\tau^2}, \quad (\text{A.2})$$

where

$$A_S(\tau) = - \int_\tau^S B_S(u) \kappa_r \theta_u du + \kappa_v \mu \int_\tau^S C_S(u) du, \quad B_S(\tau) = \frac{1 - e^{-\kappa_r(S-\tau)}}{\kappa_r}, \quad (\text{A.3})$$

and  $C_S(\tau)$  is the solution to the following Riccati's equation,

$$\dot{C}_S(\tau) = (\kappa_v + \rho \xi B_S(\tau)) C_S(\tau) - \frac{1}{2} (B_S^2(\tau) + \xi^2 C_S^2(\tau)), \quad C_S(S) = 0, \quad (\text{A.4})$$

and the dot indicates differentiation with respect to time  $\tau$ . Finally, set the parameter  $\theta_\tau$  to

$$\kappa_r \theta_\tau = \frac{\partial f_\S(t, \tau)}{\partial \tau} + \kappa_r \left( f_\S(t, \tau) + \frac{\partial C_\tau(t)}{\partial \tau} v_t^2 + \kappa_v \mu \int_t^\tau \frac{\partial C_\tau(u)}{\partial \tau} du \right) + \frac{\partial^2 C_\tau(t)}{\partial \tau^2} v_t^2 + \kappa_v \mu \int_t^\tau \frac{\partial^2 C_\tau(u)}{\partial \tau^2} du, \quad (\text{A.5})$$

where  $f_\S(t, \tau)$  denotes an hypothetically observed instantaneous forward rate at time  $t$  for maturity  $\tau$ . Then, the price predicted by the model matches the yield curve observed at  $t$ , viz

$$P_t(r_t, v_t^2, S) = e^{-\int_t^S f_\S(t, \tau) d\tau} \equiv P_t^\S(S) \quad \text{for all } S,$$

where  $P_t^\S(S)$  denotes the market price of a zero coupon bond at  $t$  and expiring at  $S$ .

**PROOF OF PROPOSITION A.1.** By standard arguments, the pricing function in Eq. (A.2) holds with the three functions,  $A_S(\cdot)$ ,  $B_S(\cdot)$  and  $C_S(\cdot)$  given in the proposition. To determine the infinite dimensional parameter  $\theta_\tau$  in Eq. (A.5), we match the instantaneous forward rate predicted by the

model at  $t$ , say  $f_t(r_t, v_t^2, S) \equiv -\frac{\partial \ln P_t(r_t, v_t^2, S)}{\partial S}$ , to  $f_{\S}(t, S)$ , leaving

$$\begin{aligned} f_{\S}(t, S) &= f_t(r_t, v_t^2, S) \\ &= -\frac{\partial A_S(t)}{\partial S} + \frac{\partial B_S(t)}{\partial S} r_t - \frac{\partial C_S(t)}{\partial S} v_t^2, \\ &= \kappa_r \int_t^S e^{-\kappa_r(S-u)} \theta_u du - \kappa_v \mu \int_t^S \frac{\partial C_S(u)}{\partial S} du + e^{-\kappa_r(S-t)} r_t - \frac{\partial C_S(t)}{\partial S} v_t^2, \end{aligned} \quad (\text{A.6})$$

where the last equality follows by differentiating the expressions for  $A_S(\tau)$  and  $B_S(\tau)$  in Eqs. (A.3).

Let us differentiate both sides of Eq. (A.6) with respect to  $S$ , leaving,

$$\frac{\partial f_{\S}(t, S)}{\partial S} = \kappa_r \left( \theta_S - \kappa_r \int_t^S e^{-\kappa_r(S-u)} \theta_u du \right) - \kappa_v \mu \int_t^S \frac{\partial^2 C_S(u)}{\partial S^2} du - \kappa_r e^{-\kappa_r(S-t)} r_t - \frac{\partial^2 C_S(t)}{\partial S^2} v_t^2. \quad (\text{A.7})$$

By replacing Eq. (A.6) into the R.H.S. of Eq. (A.7), and rearranging terms, leads to Eq. (A.5).

Next, we verify  $\theta_\tau$  in Eq. (A.5) is indeed the infinite dimensional parameter we are searching for. Replacing Eq. (A.5) into the first term on the R.H.S. of Eq. (A.6),

$$\begin{aligned} \int_t^S e^{-\kappa_r(S-u)} \kappa_r \theta_u du &= \underbrace{\int_t^S e^{-\kappa_r(S-u)} \left( \frac{\partial f_{\S}(t, u)}{\partial u} + \kappa_r f_{\S}(t, u) \right) du}_{= f_{\S}(t, S) - e^{-\kappa_r(S-t)} f_{\S}(t, t)} \\ &+ v_t^2 \int_t^S e^{-\kappa_r(S-u)} \left( \kappa_r \frac{\partial C_u(t)}{\partial u} + \frac{\partial^2 C_u(t)}{\partial u^2} \right) du \\ &+ \kappa_v \mu \int_t^S e^{-\kappa_r(S-u)} \left( \kappa_r \int_t^u \frac{\partial C_u(x)}{\partial u} dx + \int_t^u \frac{\partial^2 C_u(x)}{\partial u^2} dx \right) du, \end{aligned} \quad (\text{A.8})$$

where the equality below the first term of the R.H.S. of Eq. (A.8) follows by integrating by parts. By plugging Eq. (A.8) into Eq. (A.7), using that  $f_{\S}(t, t) = r_t$ , and rearranging terms,

$$\begin{aligned} f_t(r_t, v_t^2, S) &= f_{\S}(t, S) + \left( \int_t^S e^{-\kappa_r(S-u)} \left( \kappa_r \frac{\partial C_u(t)}{\partial u} + \frac{\partial^2 C_u(t)}{\partial u^2} \right) du - \frac{\partial C_S(t)}{\partial S} \right) v_t^2 \\ &+ \kappa_v \mu \left( \int_t^S e^{-\kappa_r(S-u)} \left( \kappa_r \int_t^u \frac{\partial C_u(x)}{\partial u} dx + \int_t^u \frac{\partial^2 C_u(x)}{\partial u^2} dx \right) du - \int_t^S \frac{\partial C_S(x)}{\partial S} dx \right). \end{aligned} \quad (\text{A.9})$$

We show that the second and the third terms of Eq. (A.9) are zero. As for the second term, we have that by an integration by parts,

$$\int_t^S e^{-\kappa_r(S-u)} \left( \kappa_r \frac{\partial C_u(t)}{\partial u} + \frac{\partial^2 C_u(t)}{\partial u^2} \right) du = \left[ e^{-\kappa_r(S-u)} \frac{\partial C_u(t)}{\partial u} \right]_{u=t}^{u=S} = \frac{\partial C_S(t)}{\partial S}, \quad (\text{A.10})$$



because by evaluating the Riccati's equation (A.4) in  $\tau = S$  delivers  $\frac{\partial C_y(y)}{\partial y} \equiv \dot{C}_S(\tau) \Big|_{\tau=S=y} = 0$  for all  $y$ . As for the third term,

$$\begin{aligned} & \int_t^S \left( \int_t^u e^{-\kappa_r(S-u)} \left( \kappa_r \frac{\partial C_u(x)}{\partial u} + \frac{\partial^2 C_u(x)}{\partial u^2} \right) dx \right) du \\ &= \int_t^S \left( \int_x^S e^{-\kappa_r(S-u)} \left( \kappa_r \frac{\partial C_u(x)}{\partial u} + \frac{\partial^2 C_u(x)}{\partial u^2} \right) du \right) dx \\ &= \int_t^S \frac{\partial C_S(x)}{\partial S} dx \end{aligned}$$

where the first equality follows by Fubini, and the second by Eq. (A.10). ■

An Internet Appendix provides reports theoretical and observed yield curves on selected dates of our sample. Next, we derive results on the volatility predicted by the model based on Proposition A.1. By Eq. (1), the price of a forward on a coupon bearing bond is

$$F_t(S, \mathbb{T}) \equiv F_t(r_t, v_t^2, S, \mathbb{T}) \equiv \sum_{i=i_t}^N \frac{C_i}{n} F_t^z(r_t, v_t^2, S, T_i) + F_t^z(r_t, v_t^2, S, \mathbb{T}), \quad (\text{A.11})$$

where

$$F_t^z(S, T_o) \equiv F_t^z(r_t, v_t^2, S, T_o) \equiv \frac{P_t(r_t, v_t^2, T_o)}{P_t(r_t, v_t^2, S)}. \quad (\text{A.12})$$

We develop predictions regarding the volatility of this two-factor model based on Eqs. (A.11)-(A.12). First, we focus on the zero coupon bond case and determine its realized variance and a forward looking volatility index (see Eqs. (A.17) and (A.21)); and second, we deal with the coupon bearing case (see Eq. (A.26)).

DYNAMICS OF FORWARDS ON ZERO COUPON BONDS. Below, we show that

$$\begin{cases} \frac{dF_\tau^z(S, T_o)}{F_\tau^z(S, T_o)} &= v_\tau^2 \Psi_\tau(T, S, T_o) d\tau + v_\tau (v_{1\tau}^z(S, T_o) dW_{1\tau}^T + v_{2\tau}^z(S, T_o) dW_{2\tau}^T) \\ dv_\tau^2 &= (\kappa_v \mu - G_T(\tau) v_\tau^2) d\tau + \xi v_\tau \left( \rho dW_{1\tau}^T + \sqrt{1 - \rho^2} dW_{2\tau}^T \right) \end{cases} \quad (\text{A.13})$$

where  $dW_{1\tau}^T$  and  $dW_{2\tau}^T$  are two independent Brownian motions under the forward probability  $Q_T$ ,  $\Psi_\tau(T, S, T_o)$  is a deterministic function (see Eq. (A.23) below), and

$$v_{i\tau}^z(S, T_o) = \varphi_{i\tau}(T_o) - \varphi_{i\tau}(S), \quad i = 1, 2, \quad (\text{A.14})$$

and for a generic  $\mathcal{X}$ ,

$$\varphi_{1\tau}(\mathcal{X}) \equiv -B_{\mathcal{X}}(\tau) + \xi\rho C_{\mathcal{X}}(\tau), \quad \varphi_{2\tau}(\mathcal{X}) \equiv \xi\sqrt{1-\rho^2}C_{\mathcal{X}}(\tau), \quad (\text{A.15})$$

and, finally,  $G_T(\tau)$  is a time-varying speed of mean-reversion of basis point variance  $v_\tau^2$  under  $Q_T$ ,

$$G_T(\tau) \equiv \kappa_v + \xi(\rho B_T(\tau) - \xi C_T(\tau)), \quad \tau \in [t, T]. \quad (\text{A.16})$$

**REALIZED AND RISK-ADJUSTED EXPECTED VARIANCE ON ZERO COUPON BONDS.** Next, define the volatility vector of the forward price in Eqs. (A.13),  $V_\tau^z(v_\tau; S, T_o) \equiv [v_{1\tau}^z(S, T_o) \quad v_{2\tau}^z(S, T_o)] \cdot v_\tau^2$ . By a direct calculation,

$$\|V_\tau^z(v_\tau; S, T_o)\|^2 = \phi_\tau^z(S, T_o) \cdot v_\tau^2, \quad (\text{A.17})$$

where

$$\phi_\tau^z(S, T_o) \equiv (B_{T_o}(\tau) - B_S(\tau))^2 + \xi^2(C_{T_o}(\tau) - C_S(\tau))^2 - 2\rho\xi(B_{T_o}(\tau) - B_S(\tau))(C_{T_o}(\tau) - C_S(\tau)), \quad (\text{A.18})$$

which follows by calculating the instantaneous variance of the forward price in Eq. (A.13).

Next, by taking expectations of the basis point variance process  $v_\tau^2$  in the second of Eqs. (A.13), we obtain that for all  $\tau$ ,

$$\mathbb{E}_t^{Q_T}(v_\tau^2) = \kappa_v \mu \int_t^\tau e^{-\int_s^\tau G_T(u) du} ds + e^{-\int_t^\tau G_T(u) du} \cdot v_t^2.$$

Replacing the previous expectation into the expectation of the annualized integrated variance under  $Q_T$  leaves,

$$\frac{1}{T-t} \int_t^T \mathbb{E}_t^{Q_T} \left( \|V_\tau^z(v_\tau; S, T_o)\|^2 \right) d\tau = \kappa_v \mu \mathcal{A}_1^z(t, T, S, T_o) + \mathcal{A}_2^z(t, T, S, T_o) \cdot v_t^2, \quad (\text{A.19})$$

where we have used Eq. (A.17) to evaluate  $\|V_\tau^z(v_\tau; S, T_o)\|^2$ , and

$$\begin{aligned} \mathcal{A}_1^z(t, T, S, T_o) &\equiv \frac{1}{T-t} \int_t^T \left( \int_t^\tau e^{-\int_s^\tau G_T(u) du} ds \right) \phi_\tau^z(S, T_o) d\tau, \\ \mathcal{A}_2^z(t, T, S, T_o) &\equiv \frac{1}{T-t} \int_t^T e^{-\int_t^\tau G_T(u) du} \phi_\tau^z(S, T_o) d\tau. \end{aligned} \quad (\text{A.20})$$

Therefore, in this two-factor model, the index of percentage volatility in Eq. (7) predicted by

Proposition 3 specializes to

$$\text{GB-VI}^z(v_t^2; t, T, S, T_o) \equiv \sqrt{\kappa_v \mu \mathcal{A}_1^z(t, T, S, T_o) + \mathcal{A}_2^z(t, T, S, T_o) \cdot v_t^2}, \quad t \leq T \leq S \leq T_o, \quad (\text{A.21})$$

where  $\phi_\tau^z(S, T_o)$  and  $A_j^z(t, T, S, T_o)$  are the two constants given in Eqs. (A.18) and (A.20).

PROOF OF EQS. (A.13). By Eq. (A.2), and Itô's lemma, the price of a zero coupon bond expiring at any fixed  $\mathcal{X}$ ,  $P_\tau(\mathcal{X}) \equiv P_\tau(r_\tau, v_\tau, \mathcal{X})$ , satisfies

$$\frac{dP_\tau(\mathcal{X})}{P_\tau(\mathcal{X})} = r_\tau d\tau + v_\tau (\varphi_{1\tau}(\mathcal{X}) dW_{1\tau} + \varphi_{2\tau}(\mathcal{X}) dW_{2\tau}),$$

where  $\varphi_{1\tau}(\mathcal{X})$  and  $\varphi_{2\tau}(\mathcal{X})$ . By a standard argument, we can define two standard Brownian motions under the forward probability  $Q_T$ ,  $dW_{i\tau}^T = dW_{i\tau} - v_\tau \varphi_{i\tau}(T) d\tau$ ,  $i = 1, 2$ , such that by Girsanov theorem, one has that under  $Q_T$ ,

$$\frac{dP_\tau(\mathcal{X})}{P_\tau(\mathcal{X})} = (r_\tau + v_\tau^2 (\varphi_{1\tau}(\mathcal{X}) \varphi_{1\tau}(T) + \varphi_{2\tau}(\mathcal{X}) \varphi_{2\tau}(T))) d\tau + v_\tau (\varphi_{1\tau}(\mathcal{X}) dW_{1\tau}^T + \varphi_{2\tau}(\mathcal{X}) dW_{2\tau}^T), \quad (\text{A.22})$$

where the two functions,  $\varphi_{i\tau}$ , are as in Eqs. (A.15). Applying Itô's lemma to the forward price predicted by the model,

$$F_t^z(S, T_o) = \frac{P_t(r_t, v_t^2, \mathcal{X})|_{\mathcal{X}=T_o}}{P_t(r_t, v_t^2, \mathcal{X})|_{\mathcal{X}=S}},$$

and utilizing Eq. (A.22), yields the first of Eqs. (A.13), with

$$\Psi_\tau(T, S, T_o) \equiv \sum_{i=1}^2 v_{i\tau}^z(S, T_o) v_{i\tau}^z(S, T). \quad (\text{A.23})$$

Finally, the second of Eqs. (A.13) follows by a change of probability in Eq. (A.1), leading to the time-varying speed of mean-reversion in Eq. (A.16). ■

GOVERNMENT BOND VOLATILITY INDEX. Next, we determine the government bond index based on the two-factor model. By Eqs. (17) and (A.13), the instantaneous variance of the forward returns is

$$\|V_\tau(v_\tau; S, \mathbb{T})\|^2 \equiv \left( \left( \sum_{j=i}^N \omega_\tau^j(S, \mathbb{T}) v_{1\tau}^z(S, T_j) \right)^2 + \left( \sum_{j=i}^N \omega_\tau^j(S, \mathbb{T}) v_{2\tau}^z(S, T_j) \right)^2 \right) \cdot v_\tau^2, \quad (\text{A.24})$$

where the functions  $v_{i\tau}^z(S, T_j)$  are defined as in Eq. (A.14). Set  $\omega_\tau^j(S, \mathbb{T}) \approx \omega_t^j(S, \mathbb{T})$  in Eq. (A.24), or,

$$\|V_\tau(v_\tau; S, \mathbb{T})\|^2 \approx \alpha_\tau(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) \cdot v_\tau^2, \quad (\text{A.25})$$

where

$$\alpha_\tau(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) \equiv \left( \sum_{j=i_t}^N \omega_t^j(S, \mathbb{T}) v_{1\tau}^z(S, T_j) \right)^2 + \left( \sum_{j=i_t}^N \omega_t^j(S, \mathbb{T}) v_{2\tau}^z(S, T_j) \right)^2,$$

and the dependence on  $\mathbf{Y}_{t, \mathbb{T}}^\$$  arises through the weights  $\omega_t^j(S, \mathbb{T})$  in Eq. (17).

Based on (A.25), we can now determine the government bond volatility index by calculations similar to those leading to Eq. (A.19):

$$\text{GB-VI}(\mathbf{Y}_{t, \mathbb{T}}^\$, v_t^2; t, T, S, \mathbb{T}) = \sqrt{\kappa_v \mu \mathcal{A}_1(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) + \mathcal{A}_2(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) v_t^2}, \quad (\text{A.26})$$

where

$$\begin{aligned} \mathcal{A}_1(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) &= \frac{1}{T-t} \int_t^T \left( \int_t^\tau e^{-\int_s^\tau G_T(u) du} ds \right) \alpha_\tau(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) d\tau, \\ \mathcal{A}_2(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) &= \frac{1}{T-t} \int_t^T e^{-\int_t^\tau G_T(u) du} \alpha_\tau(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) d\tau. \end{aligned}$$

**EVALUATION OF DERIVATIVES.** Similarly as in the three-factor model in the main text, we approximate the value at  $t$  of a TYVIX future maturing at  $t + \delta$  by freezing  $\omega_{t+\delta}^j(S + \delta, \mathbb{T} + \delta)$  at  $\omega_t^j(S, \mathbb{T})$ , obtaining

$$\hat{\mathcal{F}}(\mathbf{Y}_{t, \mathbb{T}}^\$, v_t^2; t + \delta, T, S, \mathbb{T}) \equiv \int_0^\infty \sqrt{\kappa_v \mu \mathcal{A}_1(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) + \mathcal{A}_2(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) x} \cdot \bar{f}_\delta(x | v_t^2) dx, \quad (\text{A.27})$$

where  $\bar{f}_\delta(v_{t+\delta}^2 | v_t^2)$  is the transition density of  $v_{t+\delta}^2$  given  $v_t^2$  under  $Q$ . It is well-known by Eqs. (A.1), this transition density is a non-central chi-square with  $2(q+1)$  degrees of freedom and non-centrality parameter  $2u_1$ , viz

$$\bar{f}_\delta(v_{t+\delta}^2 | v_t^2) = c e^{-u_1 - u_2} \left( \frac{u_2}{u_1} \right)^{q/2} I_q(2\sqrt{u_1 u_2}),$$

where

$$c \equiv \frac{2\kappa_v}{\xi^2 (1 - e^{-\kappa_v \delta})}, \quad u_1 \equiv c e^{-\kappa_v \delta} v_t^2, \quad u_2 \equiv c v_{t+\delta}^2, \quad q \equiv \frac{2\kappa_v \mu}{\xi^2} - 1,$$

and  $I_q(\cdot)$  is the modified Bessel function of the first kind of order  $q$ .

Next, we approximate the value at  $t$  of a TYVIX European call option maturing at  $t + \Delta$  and struck at  $K$ , by freezing  $\omega_{t+\delta}^j(S + \delta, \mathbb{T} + \delta)$  at  $\omega_t^j(S, \mathbb{T})$ , obtaining

$$\hat{\mathcal{C}}(\mathbf{Y}_{t, \mathbb{T}}^\$, v_t^2; t + \Delta, K, T, S, \mathbb{T})$$

$$\equiv P_t(T) \int_0^\infty \left( \sqrt{\kappa_v \mu \mathcal{A}_1(\mathbf{Y}_{t,\mathbb{T}}^\$, T, S, \mathbb{T}) + \mathcal{A}_2(\mathbf{Y}_{t,\mathbb{T}}^\$, T, S, \mathbb{T})x - K} \right)^+ \cdot \bar{f}_\Delta^F(x|v_t^2) dx, \quad (\text{A.28})$$

where  $\bar{f}_\Delta^F(v_{t+\Delta}^2|v_t^2)$  denotes the transition density of the basis point variance under the forward probability  $Q_{t+\Delta}$ , determined as follows. Consider, first, the conditional characteristic function of  $v_{t+\Delta}^2$  given  $v_t^2$  under the forward probability  $Q_{t+\Delta}$ , viz

$$c_\Delta^F(t, v_t^2; \vartheta) = \mathbb{E}_t^{Q_{t+\Delta}} \left( e^{i\vartheta v_{t+\Delta}^2} \right) = e^{\chi_1(t, \Delta, \vartheta) + \chi_2(t, \Delta, \vartheta)v_t^2}, \quad (\text{A.29})$$

for two functions  $\chi_1(t, \Delta, \vartheta)$  and  $\chi_2(t, \Delta, \vartheta)$  to be determined below (see Eqs. (A.30)-(A.31)).

Note that by Eq. (A.13), and under  $Q_{t+\Delta}$ , the basis point variance  $v_\tau^2$  is solution to

$$dv_\tau^2 = (\kappa_v \mu - G_{t+\Delta}(\tau) v_\tau^2) d\tau + \xi v_\tau \left( \rho dW_{1\tau}^{t+\Delta} + \sqrt{1 - \rho^2} dW_{2\tau}^{t+\Delta} \right), \quad \tau \in [t, t + \Delta],$$

where  $G_{t+\Delta}(\tau)$  is the time-varying persistence parameter defined in Eq. (A.16). By standard arguments, we find that the two functions  $\chi_1(t, \Delta, \vartheta)$  and  $\chi_2(t, \Delta, \vartheta)$  in Eq. (A.29) satisfy, for  $\tau \in [t, t + \Delta]$ ,

$$\begin{cases} 0 = \dot{\chi}_1(\tau, \Delta, \vartheta) + \kappa_v \mu \chi_2(\tau, \Delta, \vartheta) \\ 0 = \dot{\chi}_2(\tau, \Delta, \vartheta) - G_{t+\Delta}(\tau) \chi_2(\tau, \Delta, \vartheta) + \frac{1}{2} \xi^2 \chi_2^2(\tau, \Delta, \vartheta) \end{cases} \quad (\text{A.30})$$

with boundary conditions

$$\chi_1(t + \Delta, \Delta, \vartheta) = 0 \quad \text{and} \quad \chi_2(t + \Delta, \Delta, \vartheta) = i\vartheta. \quad (\text{A.31})$$

The transition density  $\bar{f}_\Delta^F(v_{t+\Delta}^2|v_t^2)$  is the Fourier's inverse transform of  $c_\Delta^F(t, v_t^2; \vartheta)$  in Eq. (A.29):

$$\bar{f}_\Delta^F(v_{t+\Delta}^2|v_t^2) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\vartheta v_{t+\Delta}^2} c_\Delta^F(t, v_t^2; \vartheta) d\vartheta. \quad (\text{A.32})$$

## Appendix B: Analysis of the three-factor model

This appendix extends results in Appendix A and provides analytical details regarding the three-factor model of the main text.

**PROOF OF PROPOSITION 2.** Follows by generalizing the arguments made to prove Proposition A.1. In particular, the functions  $A_S(\cdot)$ ,  $B_S(\cdot)$  and  $A_S(\cdot)$  in the pricing function of Proposition 2 (see Eq. (9)) are

$$A_S(\tau) = - \int_\tau^S B_S(u) \kappa_r \theta_u du + \kappa_\mu m \int_\tau^S D_S(u) du, \quad B_S(\tau) = \frac{1 - e^{-\kappa_r(S-\tau)}}{\kappa_r}, \quad (\text{B.1})$$

where  $C_S(\tau)$  and  $C_D(\tau)$  are solutions to the following Riccati's equations

$$\dot{C}_S(\tau) = (\kappa_v + \rho\xi B_S(\tau)) C_S(\tau) - \frac{1}{2} (B_S^2(\tau) + \xi^2 C_S^2(\tau)), \quad C_S(S) = 0, \quad (\text{B.2})$$

and

$$\dot{D}_S(\tau) = \kappa_\mu D_S(\tau) - \left( \kappa_v C_S(\tau) + \frac{1}{2} \gamma^2 D_S^2(\tau) \right), \quad D_S(S) = 0, \quad (\text{B.3})$$

and the dot indicates differentiation with respect to time  $\tau$ . Moreover, set the parameter  $\theta_\tau$  equal to

$$\begin{aligned} \kappa_r \theta_\tau &= \frac{\partial f_{\S}(t, \tau)}{\partial \tau} + \kappa_r \left( f_{\S}(t, \tau) + \frac{\partial C_\tau(t)}{\partial \tau} v_t^2 + \frac{\partial D_\tau(t)}{\partial \tau} \mu_t + \kappa_\mu m \int_t^\tau \frac{\partial D_\tau(u)}{\partial \tau} du \right) \\ &+ \frac{\partial^2 C_\tau(t)}{\partial \tau^2} v_t^2 + \frac{\partial^2 D_\tau(t)}{\partial \tau^2} \mu_t + \kappa_\mu m \int_t^\tau \frac{\partial^2 D_\tau(u)}{\partial \tau^2} du, \end{aligned} \quad (\text{B.4})$$

where  $f_{\S}(t, \tau)$  denotes an hypothetically observed instantaneous forward rate at time  $t$  for maturity  $\tau$ . Then, the price predicted by the model matches the yield curve observed at  $t$ , viz

$$P_t(r_t, v_t^2, \mu_t, S) = e^{-\int_t^S f_{\S}(t, \tau) d\tau} \equiv P_t^{\S}(S) \quad \text{for all } S, \quad (\text{B.5})$$

where  $P_t^{\S}(S)$  denotes the market price of a zero coupon bond at  $t$  and expiring at  $S$ .

Note that the functions  $B_S$  and  $C_S$  in (B.1) and (B.2) are the same as  $B_S$  and  $C_S$  in the two-factor model (see (A.3) and (A.4)). Moreover, if  $\mu_\tau$  is constant (i.e.  $\kappa_\mu = \gamma = 0$ ) and equal to  $\mu \equiv m$ , the function  $D_S$  in Eq. (B.3) is solution to  $\dot{D}_S(\tau) = -\kappa_v C_S(\tau)$ , such that  $A_S(\tau) + D_S(\tau) \mu$  in (9) collapses to  $A_S(\tau)$  in (A.2). Likewise, the infinite dimensional parameter  $\theta_\tau$  in Eq. (B.4) collapses to that in the two-factor model in Eq. (A.5) under the same conditions ■

**PROOF OF EQS. (12) AND PROPOSITION 3.** First, we generalize Eq. (A.22) in Appendix A and determine the dynamics of a zero coupon bond price  $P_\tau(\mathcal{X}) \equiv P_\tau(r_\tau, v_\tau, \mu_\tau, \mathcal{X})$ . Second, we apply Itô's lemma to  $F_\tau^z(S, T_o)$  in Eq. (11) and obtain Eqs. (12) relying on the thusly determined dynamics of  $P_\tau(\mathcal{X})$ . We find that the drift of  $\frac{dF_\tau^z}{F_\tau^z}$  in Eq. (12) is

$$m_\tau^F d\tau \equiv m_\tau^F(S, T_o) d\tau = \mathbb{E}_\tau^{Q_T} \left( \frac{dF_\tau^z(S, T_o)}{F_\tau^z(S, T_o)} \right) = (v_\tau^2 \Psi_{A\tau}(T, S, T_o) + \mu_\tau \Psi_{B\tau}(T, S, T_o)) d\tau, \quad (\text{B.6})$$

where

$$\Psi_{A\tau}(T, S, T_o) = \Psi_\tau(T, S, T_o), \quad \Psi_{B\tau}(T, S, T_o) = v_{3\tau}^z(S, T_o) v_{3\tau}^z(S, T),$$

and  $\Psi_\tau(T, S, T_o)$  is as in Eq. (A.23). The Brownian motions under  $Q_T$  are defined as  $dW_{i\tau}^T = dW_{i\tau} - v_\tau \varphi_{i\tau}(T) d\tau$ ,  $i = 1, 2$ , and  $dW_{3\tau}^T = dW_{3\tau} - \sqrt{\mu_\tau} \varphi_{3\tau}(T) d\tau$ . Note that the two exposures in (13),  $v_{1\tau}^z(S, T_o)$  and  $v_{2\tau}^z(S, T_o)$ , are the same as those in the two-factor version of the model (see Eqs.

(A.14)), and that the function  $G_T$  in (14) is the same as in Eq. (A.16)).

The expression for the realized variance in Eq. (15) in Proposition 3 follows by a direct calculation, and the two functions  $\phi_{i\tau}^z(S, T_o)$  are given by

$$\phi_{1\tau}^z(S, T_o) = \phi_\tau^z(S, T_o), \quad \phi_{2\tau}^z(S, T_o) \equiv \gamma^2 (D_{T_o}(\tau) - D_S(\tau))^2, \quad (\text{B.7})$$

where  $\phi_\tau^z(S, T_o)$  is as in Eq. (A.18).

Finally, we derive Eq. (16) in Proposition 3. Note that by Eq. (15), the expectation of the annualized integrated government bond variance under  $Q_T$  is

$$\begin{aligned} & \frac{1}{T-t} \int_t^T \mathbb{E}_t^{Q_T} \left( \|V_\tau^z(v_\tau, \mu_\tau; S, T_o)\|^2 d\tau \right) \\ &= \frac{1}{T-t} \int_t^T \left( \phi_{1\tau}^z(S, T_o) \cdot \mathbb{E}_t^{Q_T}(v_\tau^2) + \phi_{2\tau}^z(S, T_o) \cdot \mathbb{E}_t^{Q_T}(\mu_\tau) \right) d\tau. \end{aligned} \quad (\text{B.8})$$

The expectations under  $Q_T$  of the basis point variance  $v_\tau^2$  and long-term uncertainty  $\mu_\tau$  in (12) are, for all  $\tau$ ,

$$\begin{cases} \mathbb{E}_t^{Q_T}(v_\tau^2) = \kappa_v \int_t^\tau e^{-\int_s^\tau G_T(u) du} \mathbb{E}_t^{Q_T}(\mu_s) ds + e^{-\int_t^\tau G_T(u) du} \cdot v_t^2 \\ \mathbb{E}_t^{Q_T}(\mu_\tau) = \kappa_m m \int_t^\tau e^{-\int_\ell^\tau H_T(u) du} d\ell + e^{-\int_t^\tau H_T(u) du} \cdot \mu_t \end{cases}$$

Replacing the previous expectations into Eq. (B.8), and rearranging, yields Eq. (16), where the functions  $\mathcal{B}_j^z(t, T, S, T_o)$  are given by

$$\begin{aligned} \mathcal{B}_1^z(t, T, S, T_o) &\equiv \frac{1}{T-t} \int_t^T (\kappa_v b_{1\tau}(t, T) \phi_{1\tau}^z(S, T_o) + b_{2\tau}(t, T) \phi_{2\tau}^z(S, T_o)) d\tau, \\ \mathcal{B}_2^z(t, T, S, T_o) &\equiv \frac{1}{T-t} \int_t^T e^{-\int_t^\tau G_T(u) du} \phi_{1\tau}^z(S, T_o) d\tau, \\ \mathcal{B}_3^z(t, T, S, T_o) &\equiv \frac{1}{T-t} \int_t^T \left( \kappa_v b_{3\tau}(t, T) \phi_{1\tau}^z(S, T_o) + e^{-\int_t^\tau H_T(u) du} \phi_{2\tau}^z(S, T_o) \right) d\tau, \end{aligned} \quad (\text{B.9})$$

and

$$\begin{aligned} b_{1\tau}(t, T) &\equiv \int_t^\tau e^{-\int_s^\tau G_T(u) du} \left( \int_t^s e^{-\int_\ell^s H_T(u) du} d\ell \right) ds \\ b_{2\tau}(t, T) &\equiv \int_t^\tau e^{-\int_\ell^\tau H_T(u) du} d\ell, \\ b_{3\tau}(t, T) &\equiv \int_t^\tau e^{-\int_s^\tau G_T(u) du} e^{-\int_t^s H_T(u) du} ds, \end{aligned} \quad (\text{B.10})$$

and the two functions  $G_T$  and  $H_T$  are as in Eqs. (14). ■

COUPON BEARING BONDS: THE INTEGRATED DRIFT  $\bar{z}(t, T, S, \mathbb{T})$  PREDICTED BY THE MODEL.

By setting  $T_i \equiv T_o$  and plugging the first of Eqs. (12) into Eq. (17) leaves

$$\begin{aligned} \frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} &= \sum_{i=i_t}^N \omega_\tau^i(S, \mathbb{T}) m_\tau^F(S, T_i) d\tau \\ &+ \sum_{i=i_t}^N \omega_\tau^i(S, \mathbb{T}) (v_\tau (v_{1\tau}^z(S, T_i) dW_{1\tau}^T + v_{2\tau}^z(S, T_i) dW_{2\tau}^T) + \sqrt{\mu_\tau} v_{3\tau}^z(S, T_i) dW_{3\tau}^T), \end{aligned}$$

where  $m_\tau^F(S, \cdot)$  is defined as in Eq. (B.6),

$$m_\tau^F(S, T_i) = v_\tau^2 \Psi_{A\tau}(T, S, T_i) + \mu_\tau \Psi_{B\tau}(T, S, T_i),$$

such that by the definition of  $\bar{z}$  in Proposition 1, the integrated drift predicted by the model is

$$\bar{z}(t, T, S, \mathbb{T}) = \int_t^T \sum_{i=i_t}^N \omega_\tau^i(S, \mathbb{T}) m_\tau^F(S, T_i) d\tau. \quad \blacksquare \quad (\text{B.11})$$

PROOF OF PROPOSITION 4. Eq. (19) follows by setting  $\omega_\tau^j(S, \mathbb{T}) \approx \omega_t^j(S, \mathbb{T})$  in Eq. (18), such that

$$\|V_\tau(v_\tau, \mu_\tau; S, \mathbb{T})\|^2 \approx \alpha_{1\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) v_\tau^2 + \alpha_{2\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) \mu_\tau,$$

where

$$\begin{aligned} \alpha_{1\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) &\equiv \left( \sum_{j=i_t}^N \omega_t^j(S, \mathbb{T}) v_{1\tau}^z(S, T_j) \right)^2 + \left( \sum_{j=i_t}^N \omega_t^j(S, \mathbb{T}) v_{2\tau}^z(S, T_j) \right)^2, \\ \alpha_{2\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) &\equiv \left( \sum_{j=i_t}^N \omega_t^j(S, \mathbb{T}) v_{3\tau}^z(S, T_j) \right)^2. \end{aligned}$$

The expressions for the three functions in Eq. (19) follow by the same arguments made to derive (B.9). They are

$$\begin{aligned} \mathcal{B}_1(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) &\equiv \frac{1}{T-t} \int_t^T \left( \kappa_v b_{1\tau}(t, T) \alpha_{1\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) + b_{2\tau}(t, T) \alpha_{2\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) \right) d\tau \\ \mathcal{B}_2(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) &\equiv \frac{1}{T-t} \int_t^T e^{-\int_t^\tau G_T(u) du} \alpha_{1\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) d\tau, \\ \mathcal{B}_3(\mathbf{Y}_{t, \mathbb{T}}^\$, T, S, \mathbb{T}) &\equiv \frac{1}{T-t} \int_t^T \left( \kappa_v b_{3\tau}(t, T) \alpha_{1\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) + e^{-\int_t^\tau H_T(u) du} \alpha_{2\tau}(\mathbf{Y}_{t, \mathbb{T}}^\$, S, \mathbb{T}) \right) d\tau \end{aligned} \quad (\text{B.12})$$

where the three function  $b_{jt}(t, T)$  are as in Eqs. (B.10) and the two functions  $G_T$  and  $H_T$  are as in



Eqs. (14).

DERIVATION OF THE TRANSITION DENSITIES IN (22) AND (24). We first determine  $f_\delta(x, y | v_t^2, \mu_t)$  in (22)—the transition density  $(v_{t+\Delta}^2, \mu_{t+\Delta})$  given  $(v_t^2, \mu_t)$  under the risk-neutral probability  $Q$ . Define the conditional characteristic function of  $(v_{t+\Delta}^2, \mu_{t+\Delta})$  given  $(v_t^2, \mu_t)$  under  $Q$ ,

$$\bar{c}_\delta(t, v_t^2, \mu_t; \vartheta_1, \vartheta_2) = \mathbb{E}_t \left( e^{i\vartheta_1 v_{t+\Delta}^2 + i\vartheta_2 \mu_{t+\Delta}} \right) = e^{\varsigma_1(t, \delta, \vartheta_1, \vartheta_2) + \varsigma_2(t, \delta, \vartheta_1, \vartheta_2) v_t^2 + \varsigma_3(t, \delta, \vartheta_1, \vartheta_2) \mu_t}, \quad (\text{B.13})$$

where the three functions  $\varsigma_j(t, \Delta, \vartheta_1, \vartheta_2)$  satisfy

$$\begin{cases} 0 = \dot{\varsigma}_1(\tau, \cdot) + \kappa_\mu m \varsigma_3(\tau, \cdot) \\ 0 = \dot{\varsigma}_2(\tau, \cdot) - \kappa_v \varsigma_2(\tau, \cdot) + \frac{1}{2} \xi^2 \varsigma_2^2(\tau, \cdot) \\ 0 = \dot{\varsigma}_3(\tau, \cdot) + \kappa_v \varsigma_2(\tau, \cdot) - \kappa_\mu \varsigma_3(\tau, \cdot) + \frac{1}{2} \gamma^2 \varsigma_3^2(\tau, \cdot) \end{cases}$$

subject to the boundary conditions

$$\varsigma_1(t + \delta, \delta; \vartheta_1, \vartheta_2) = 0 \quad \text{and} \quad \varsigma_{j+1}(t + \delta, \delta; \vartheta_1, \vartheta_2) = i\vartheta_j, \quad j = 1, 2.$$

The transition density  $f_\Delta^F(v_{t+\Delta}^2, \mu_{t+\Delta} | v_t^2, \mu_t)$  is obtained as the Fourier's inverse transform of  $\bar{c}_\delta(t, v_t^2, \mu_t; \vartheta_1, \vartheta_2)$  in (B.13), viz

$$f_\delta(v_{t+\Delta}^2, \mu_{t+\Delta} | v_t^2, \mu_t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\vartheta_1 v_{t+\Delta}^2 - i\vartheta_2 \mu_{t+\Delta}} \bar{c}_\delta(t, v_t^2, \mu_t; \vartheta_1, \vartheta_2) d\vartheta_1 d\vartheta_2. \quad (\text{B.14})$$

Next, we determine  $f_\Delta^F(x, y | v_t^2, \mu_t)$  in (24)—the transition density of  $(v_{t+\Delta}^2, \mu_{t+\Delta})$  given  $(v_t^2, \mu_t)$  under the forward probability  $Q_{t+\Delta}$ . The conditional characteristic function under  $Q_{t+\Delta}$  is defined as

$$\bar{c}_\Delta^F(t, v_t^2, \mu_t; \vartheta_1, \vartheta_2) = \mathbb{E}_t^{Q_{t+\Delta}} \left( e^{i\vartheta_1 v_{t+\Delta}^2 + i\vartheta_2 \mu_{t+\Delta}} \right) = e^{\zeta_1(t, \Delta, \vartheta_1, \vartheta_2) + \zeta_2(t, \Delta, \vartheta_1, \vartheta_2) v_t^2 + \zeta_3(t, \Delta, \vartheta_1, \vartheta_2) \mu_t}, \quad (\text{B.15})$$

for three functions  $\zeta_j(t, \Delta, \vartheta_1, \vartheta_2)$  to be determined below (see Eqs. (B.16)-(B.17)). By the last two equations of (12), we know that  $(v_t^2, \mu_t)$  satisfy

$$\begin{cases} dv_\tau^2 &= (\kappa_v \mu_\tau - G_{t+\Delta}(\tau) v_\tau^2) d\tau + \xi v_\tau \left( \rho dW_{1\tau}^{t+\Delta} + \sqrt{1 - \rho^2} dW_{2\tau}^{t+\Delta} \right) \\ d\mu_\tau &= (\kappa_\mu m - H_{t+\Delta}(\tau) \mu_\tau) d\tau + \gamma \sqrt{\mu_\tau} dW_{3\tau}^{t+\Delta} \end{cases}$$

where  $W_{j\tau}^{t+\Delta}$  are standard Brownian motions under  $Q_{t+\Delta}$ , and the two functions  $G_X(\tau)$  and  $H_X(\tau)$  are defined in Eqs. (14). By standard arguments, similar to those leading to Eqs. (A.30) in the

two-factor case, we find that the three functions  $\zeta_j(t, \Delta, \vartheta)$  in Eq. (B.15) satisfy, for  $\tau \in [t, t + \Delta)$ ,

$$\begin{cases} 0 = \dot{\zeta}_1(\tau, \cdot) + \kappa_\mu m \zeta_3(\tau, \cdot) \\ 0 = \dot{\zeta}_2(\tau, \cdot) - G_{t+\Delta}(\tau) \zeta_2(\tau, \cdot) + \frac{1}{2} \xi^2 \zeta_2^2(\tau, \cdot) \\ 0 = \dot{\zeta}_3(\tau, \cdot) + \kappa_v \zeta_2(\tau, \cdot) - H_{t+\Delta}(\tau) \zeta_3(\tau, \cdot) + \frac{1}{2} \gamma^2 \zeta_3^2(\tau, \cdot) \end{cases} \quad (\text{B.16})$$

subject to the boundary conditions

$$\zeta_1(t + \Delta, \Delta; \vartheta_1, \vartheta_2) = 0 \quad \text{and} \quad \zeta_{j+1}(t + \Delta, \Delta; \vartheta_1, \vartheta_2) = i\vartheta_j, \quad j = 1, 2. \quad (\text{B.17})$$

The transition density  $f_\Delta^F(v_{t+\Delta}^2, \mu_{t+\Delta} | v_t^2, \mu_t)$  is the Fourier's inverse transform of  $\bar{c}_\Delta^F(t, v_t^2, \mu_t; \vartheta_1, \vartheta_2)$  in Eq. (B.15):

$$f_\Delta^F(v_{t+\Delta}^2, \mu_{t+\Delta} | v_t^2, \mu_t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\vartheta_1 v_{t+\Delta}^2 - i\vartheta_2 \mu_{t+\Delta}} \bar{c}_\Delta^F(t, v_t^2, \mu_t; \vartheta_1, \vartheta_2) d\vartheta_1 d\vartheta_2. \quad (\text{B.19})$$

## Appendix C: Details regarding model calibration

### C.1. Futures

The model calibration relies on TY option implied volatilities. We use these implied volatilities to implement nonparametric estimates of hypothetical TYVIX future values, the benchmark in our optimization procedures. Accordingly, firstly, we provide a comprehensive derivation of these benchmarks as well as the approximations involved into them; and secondly, we provide details regarding the calibration of our model.

**NON-PARAMETRIC ESTIMATES OF FUTURES VALUES.** We derive non-parametric benchmarks for our model-based TYVIX futures values that rely on calendar spreads, which are originally proposed by Carr and Wu (2006) and Dupire (2006) in the equity case. Consider a generic  $\Delta$ -forward looking variance gauge (e.g.,  $\Delta = \frac{1}{12}$ ) for a generic risk:

$$V_{x_{t,\Delta}}^2 \equiv \mathbb{E}_t^* \left( \frac{1}{\Delta} \int_t^{t+\Delta} \sigma_\tau^2 d\tau \right),$$

where  $\sigma_\tau^2$  denotes the instantaneous variance of this risk, and the conditional expectation  $\mathbb{E}_t^*$  is taken under some numéraire probability (Mele and Obayashi, 2015).

Consider the price of a future on  $V_x$  expiring at  $t + \delta$ :

$$F_{t,t+\delta} \equiv \mathbb{E}_t(V_{x_{t+\delta,\Delta}}),$$

where now  $\mathbb{E}_t$  denotes the expectation under the risk-neutral probability. Let us, first, determine the expected squared Vx at  $t + \delta$ ,

$$\begin{aligned}
\mathbb{E}_t (V_{x_{t+\delta,\Delta}}^2) &= \mathbb{E}_t \left( \mathbb{E}_{t+\delta}^* \left( \frac{1}{\Delta} \int_{t+\delta}^{t+\delta+\Delta} \sigma_\tau^2 d\tau \right) \right) \\
&\approx \mathbb{E}_t \left( \mathbb{E}_{t+\delta} \left( \frac{1}{\Delta} \int_{t+\delta}^{t+\delta+\Delta} \sigma_\tau^2 d\tau \right) \right) \\
&= \mathbb{E}_t \left( \frac{1}{\Delta} \int_{t+\delta}^{t+\delta+\Delta} \sigma_\tau^2 d\tau \right) \\
&= \frac{1}{\Delta} \mathbb{E}_t \left( \int_t^{t+\delta+\Delta} \sigma_\tau^2 d\tau - \int_t^{t+\delta} \sigma_\tau^2 d\tau \right) \\
&= \frac{\delta + \Delta}{\Delta} V_{x_{t,\delta+\Delta}}^2 - \frac{\delta}{\Delta} V_{x_{t,\delta}}^2.
\end{aligned} \tag{C.1}$$

The second line is only an approximation because the risk-neutral probability and the numéraire probability are generically not the same. For example, in our context, the numéraire probability is the forward probability.

Next, fix  $\Delta = \frac{1}{12}$ , and set the futures month maturity equal to  $\delta = q\Delta$ , for integer  $q$ , such that the approximation in (C.1) is:

$$\mathbb{E}_t (V_{x_{t+q\Delta,\Delta}}^2) \approx (q+1) V_{x_{t,(q+1)\Delta}}^2 - q V_{x_{t,q\Delta}}^2. \tag{C.2}$$

Note that  $F_{t+q\Delta,t+q\Delta} = V_{x_{t+q\Delta,\Delta}}$ , such that:

$$\begin{aligned}
\mathbb{E}_t (F_{t+q\Delta,t+q\Delta})^2 &= \mathbb{E}_t (V_{x_{t+q\Delta,\Delta}}^2) - \text{var}_t (F_{t+q\Delta,t+q\Delta}) \\
&\approx (q+1) V_{x_{t,(q+1)\Delta}}^2 - q V_{x_{t,q\Delta}}^2 - \text{var}_t (F_{t+q\Delta,t+q\Delta}),
\end{aligned}$$

where the second line follows by the approximation in (C.2). Therefore, we have the following approximations:

$$\begin{aligned}
F_{t,t+q\Delta} &= \mathbb{E}_t (F_{t+q\Delta,t+q\Delta}) \\
&\approx \sqrt{(q+1) V_{x_{t,(q+1)\Delta}}^2 - q V_{x_{t,q\Delta}}^2 - \text{var}_t (F_{t+q\Delta,t+q\Delta})} \\
&\approx \sqrt{(q+1) V_{x_{t,(q+1)\Delta}}^2 - q V_{x_{t,q\Delta}}^2}.
\end{aligned} \tag{C.3}$$

In the calibration, we use the CBOT 10-Year Treasury Note option ATM implied volatility at day  $n$  for maturity  $\delta$ ,  $\text{atm}_n(\delta)$ , as a proxy for  $V_{x_{n,\delta}}$ , to calculate the previous expression and use it as a non-parametric estimate of hypothetical TYVIX future values; see Eq. (C.6) below.

CALIBRATION AND OTHER OPTIMIZATION PROCEDURES. Consider the following estimate of the basis point variance  $v_n^2$  of the short-term rate on each day  $n$  in our sample:

$$\hat{\sigma}_n^2 \equiv \frac{252}{30} \sum_{i=1}^{30} (r_{n-i+1} - r_{n-i})^2,$$

where  $r_n$  is the 3-month Treasury Bill rate as of day  $n$ . We calibrate  $\kappa_v$  in (8) while assuming  $\hat{\sigma}_n^2$  has the same autocovariance function as the true basis point variance, thereby estimating  $\kappa_v$  while fitting an autoregressive model to the  $\hat{\sigma}_n^2$  series. Likewise, we calibrate the volatility of variance parameter  $\xi$  in (8) to match the average volatility of changes in the basis point variance,  $\sqrt{\frac{\Delta^{-1} \text{var}(\Delta \hat{\sigma}_n^2)}{\text{avg}(\hat{\sigma}_n^2)}}$ , where  $\Delta^{-1} = \frac{1}{252}$ , and  $\text{var}(\Delta \hat{\sigma}_n^2)$  and  $\text{avg}(\hat{\sigma}_n^2)$  denote the sample variance of the daily changes in  $\hat{\sigma}_n^2$  and the sample average of  $\hat{\sigma}_n^2$ , respectively. As explained in the main text,  $\kappa_\mu$  and  $\gamma$  are calibrated in the same way, based on model-implied values of  $\mu$  (see Steps 1 to 3 below).

To estimate TYVIX future values, we proceed through four steps. In the first three steps, we calibrate the two-factor model in Appendix A and obtain model-implied values of the long-term basis point parameter  $\mu$  for each day of the sample. We utilize these values to calibrate the parameters  $\kappa_\mu$ ,  $m$  and  $\gamma$ . In the fourth and last step, we use all the parameter estimates in Table 1 and re-calibrate the full three-factor model to estimate TYVIX futures values. In the following,  $\mathcal{V}_n$  denotes the value of the TYVIX index as of day  $n$ , and  $N$  is the sample size.

*Step 1* Fix  $\kappa_r$  at some value  $\hat{\kappa}_r$ .

*Step 2* For each day  $n$ , calibrate  $\mu$  in the two-factor model so as to fit the two-month term structure of the non-parametric estimate of TYVIX futures based on Eq. (C.3):

$$\hat{\mu}_{\text{ini},n}(\hat{\kappa}_r) \equiv \arg \min_{\mu} \sum_{j=1}^2 \left( \hat{\mathcal{F}}(\mathcal{Y}_{n,\mathbb{T}}^{\$}, \hat{v}_{\text{ini},n}^2; n + \delta_j, T, S, \mathbb{T}; \hat{\kappa}_r) - \mathcal{F}_n^{\$}(\delta_j) \right)^2, \quad (\text{C.4})$$

under the constraint that

$$\hat{v}_{\text{ini},n}^2 = \hat{v}_{\text{ini},n}^2(\mu) \equiv \frac{\mathcal{V}_n^2 - \kappa_v \mu \mathcal{A}_1(\mathcal{Y}_{n,\mathbb{T}}^{\$}, T, S, \mathbb{T})}{\mathcal{A}_2(\mathcal{Y}_{n,\mathbb{T}}^{\$}, T, S, \mathbb{T})}, \quad (\text{C.5})$$

where  $\hat{\mathcal{F}}(\mathcal{Y}_{n,\mathbb{T}}^{\$}, \hat{v}_{\text{ini},n}^2; n + \delta_j, T, S, \mathbb{T}; \hat{\kappa}_r)$  is  $\hat{\mathcal{F}}(\cdot)$  in Eq. (A.27) when the parameter  $\kappa_r = \hat{\kappa}_r$ , and

$$\mathcal{F}_n^{\$}(\delta_j) = \sqrt{(\delta_j + 1) \cdot \text{atm}_n^2(\delta_j + 1) - \delta_j \cdot \text{atm}_n^2(\delta_j)} \quad (\text{C.6})$$

is a non-parametric estimate of the future value at day  $n$ , expiring at day  $n + \delta_j$ , with  $\delta_j$  equal to one and two months. The constraint (C.5) guarantees that the two-factor model prediction

(Eq. (A.26)) exactly matches the TYVIX. The rationale underlying the non-parametric estimate  $\mathcal{F}_n^{\mathbb{S}}(\delta_j)$  relies on the derivations leading to Eq. (C.3). The next step describes how values  $\hat{\kappa}_r$  in Step 1 are updated given the calibration of  $\mu_n$  in this step.

*Step 3* Iterate on  $\hat{\kappa}_r$  and the calibrated values  $(\hat{\mu}_n)_{n=1}^N$  in Steps 1 and 2, to ensure that

$$\hat{\kappa}_r((\hat{\mu}_n)_{n=1}^N) \equiv \arg \min_{\kappa_r} \left( \text{avg}(\hat{v}_{\text{ini},n}^2(\hat{\mu}_{\text{ini},n}(\kappa_r))) - \text{avg}(\hat{\sigma}_n^2) \right)^2.$$

*Step 4* Estimate  $\kappa_\mu$  by fitting an autoregressive model to the calibrated values of  $(\hat{\mu}_{\text{ini},n})_{n=1}^N$  obtained in the previous steps, and set  $m = \text{avg}(\hat{\mu}_{\text{ini},n})$  and  $\gamma = \sqrt{\frac{\Delta^{-1} \text{var}(\Delta \hat{\mu}_{\text{ini},n})}{\text{avg}(\hat{\mu}_{\text{ini},n})}}$ . Revise the estimates of the series  $(\hat{v}_{\text{ini},n}^2, \hat{\mu}_{\text{ini},n})_{n=1}^N$  while minimizing the distance of model-implied TYVIX futures to their non-parametric counterparts:

$$\hat{\mu}_n \equiv \arg \min_{\mu_n} \sum_{j=1}^2 \left( \hat{\mathcal{F}}(\mathbf{Y}_{n,\mathbb{T}}^{\mathbb{S}}, \hat{v}_n^2, \mu_n; n + \delta_j, T, S, \mathbb{T}; \hat{\kappa}_r) - \mathcal{F}_n^{\mathbb{S}}(\delta_j) \right)^2,$$

under the constraint that the model-implied realized variance makes the three-factor model-implied TYVIX equal to TYVIX for each day, relying on Eq. (19),

$$\hat{v}_n^2 = \frac{\mathcal{V}_n^2 - \kappa_\mu m \mathcal{B}_1(\mathbf{Y}_{n,\mathbb{T}}^{\mathbb{S}}, T, S, \mathbb{T}) - \mathcal{B}_3(\mathbf{Y}_{n,\mathbb{T}}^{\mathbb{S}}, T, S, \mathbb{T}) \hat{\mu}_n}{\mathcal{B}_2(\mathbf{Y}_{n,\mathbb{T}}^{\mathbb{S}}, T, S, \mathbb{T})}, \quad (\text{C.7})$$

and where  $\hat{\mathcal{F}}(\mathbf{Y}_{t,\mathbb{T}}^{\mathbb{S}}, v_t^2, \mu_t; t + \delta, T, S, \mathbb{T})$  is  $\hat{\mathcal{F}}(\cdot)$  in Eq. (22).

## C.2. Options

We approximate  $\hat{\mathcal{C}}_t^f(t, v_t^2, \mu_t; t + \Delta, K, T, S, \mathbb{T})$  in Eq. (31) as follows. Given the history of calibrated futures values obtained in Section 6.2.1, we perform a non-parametric regression of these calibrated values on to the model-implied  $v_t^2$  and  $\mu_t$  and  $\mathcal{B}_{jt} \equiv \mathcal{B}_j(\mathbf{Y}_{t,\mathbb{T}}^{\mathbb{S}}, T, S, \mathbb{T})$ , so as to learn about the functional relation linking these variables to the calibrated one. This functional relation is model-based, thereby leading to small error margins. Denote with  $\hat{\mathbb{F}}(v_t^2, \mu_t, \mathcal{B}_{1t}, \mathcal{B}_{2t}, \mathcal{B}_{3t}; t + \delta, T, S, \mathbb{T})$  the predictive part of this non-parametric regression. We approximate the price  $\hat{\mathcal{C}}_t^f$  in Eq. (31) using this estimate, and freezing  $\mathbf{Y}_{T,\mathbb{T}}$  at  $\mathbf{Y}_{t,\mathbb{T}}^{\mathbb{S}}$ , thereby freezing  $\mathcal{B}_{jT}$  at  $\mathcal{B}_{jt}$ :

$$\begin{aligned} & \hat{\mathcal{C}}_t^f(\mathbf{Y}_{t,\mathbb{T}}^{\mathbb{S}}, v_t^2, \mu_t; t + \Delta, K, T, S, \mathbb{T}) \\ & \equiv P_t(T) \iint \left( \hat{\mathbb{F}}(x, y, \mathcal{B}_{1t}, \mathcal{B}_{2t}, \mathcal{B}_{3t}; T + \delta, T, S, \mathbb{T}) - K \right)^+ f_\Delta^F(x, y | v_t^2, \mu_t) dx dy. \end{aligned} \quad (\text{C.8})$$

Note that, while  $\hat{\mathcal{C}}_t^f$  in (31) depends on a generic  $t$ ,  $\hat{\mathcal{C}}^f$  in (C.8) now only depends on  $\mathbf{Y}_{t,\mathbb{T}}^{\mathbb{S}}$ .

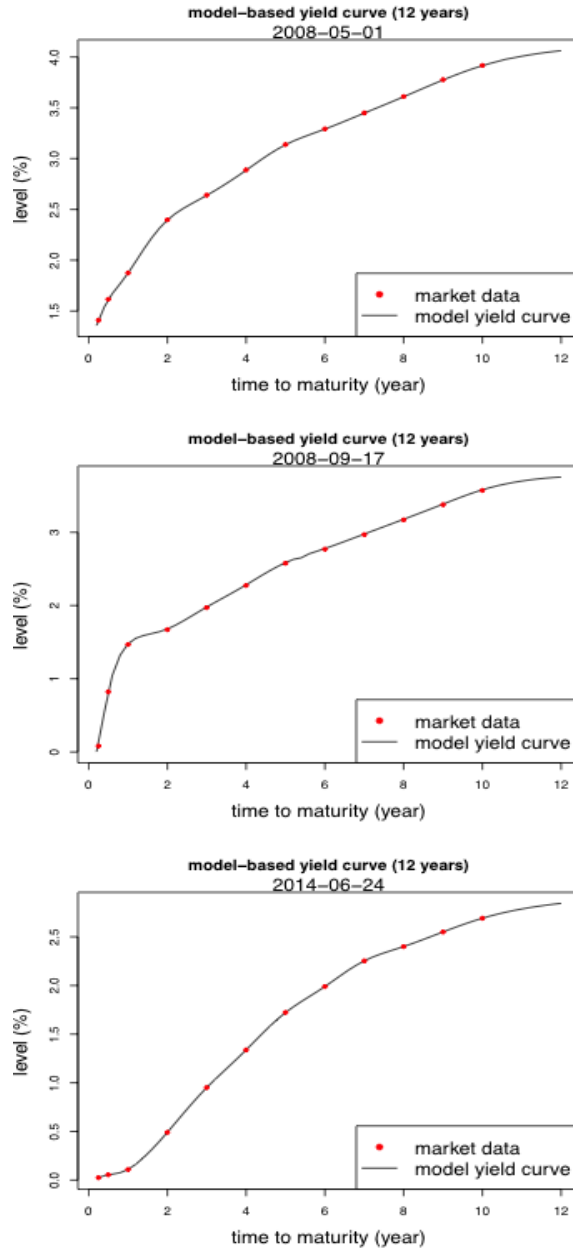
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## Internet Appendix

Figure A.1 reports theoretical and observed yield curves on selected dates of our sample. Theoretical values are obtained with the parameter values in Table 1 of the main text (see Section 6), and relying on Eq. (A.5) of Proposition 2. By design, the infinite-dimensional parameter  $\theta_\tau$  allows an exact fit of the yield curve on any given date as illustrated by Figure A.1, ensuring internal consistency between market-observed yields and the instantaneous variance of coupon-bearing government bond returns.



**Figure A.1.** Yield curves on selected days and model's reconstructions based on the calibration of the infinite dimensional parameter  $\theta_\tau$  through Eq. (A.5).