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Abstract

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JEL Classification: G11, G12, G13, G17

Keywords: correlation premium, correlation-risk premium, cross-section of returns, arbitrage pricing, string models, implied correlation

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Correlation Risk, Strings and Asset Prices

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Abstract

Standard asset pricing theories treat return volatility and correlations as two intimately related quantities, which hinders achieving a neat definition of a correlation premium. We introduce a model with a continuum of securities that have returns driven by a string. This model leads to new arbitrage pricing restrictions, according to which, holding any asset requires compensation for the granular exposure of this asset returns to changes in all other asset returns: an average *correlation* premium. We find that this correlation premium is both statistically and economically significant, and considerably fluctuates, driven by time-varying correlations and global market developments. The model explains the cross-section of expected returns and their counter-cyclicality without making reference to common factors affecting asset returns. It also explains the time-series behavior of the premium for the risk of changes in asset correlations (the *correlation-risk* premium), including its inverse relation with realized correlations.

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1. Introduction

The inability of the CAPM to explain the cross-section of expected returns has led to a proliferation of models driven by factors that have recently been the focus of criticism and re-newed rigorous statistical scrutiny (see, e.g., Harvey, Liu and Zhu, 2016). This paper proposes a new arbitrage pricing model in which the cross-section of expected returns links to arguably one amongst the simplest concepts in financial economics: correlation. The distinguishing feature of our approach is that we avoid making reference to factors while explaining asset correlations. Instead, correlations of each asset return with all remaining asset returns are the building block in our framework. That is, in our model, correlations do not result from the assumption of exogenously given "pricing factors." Rather, all correlations are the primitives of the model, and they jointly determine the whole set of no-arbitrage restrictions amongst all asset returns.

Correlation has a long history in asset pricing, although the typical approach has predominantly been to model asset returns in frameworks where correlation and volatility are intimately related. Consider, for example, the seminal Samuelson-Merton model, in which the assets correlations are pre-determined by the assumptions made on the assets betas; that is, in that model, the price of correlation risk is a function of the "lambdas." Ideally, instead, we would like to disentangle the price of correlation from these lambdas, that is, we would like to disentangle volatility from correlation.

In the Samuelson-Merton model, asset returns are driven by Brownian motions. An alternative model is one in which asset returns are driven by shocks that enable one to separate volatility from correlation. We rely on random field models, or stochastic string models, to think about correlation as being determined in this independent way.[1] Random field models were introduced in finance by Kennedy (1994, 1997) to model the term structure of interest rates, and Goldstein (2000) and Santa-Clara and Sornette (2001) provide extensions or a more general framework in this domain. Tsoulouvi

[1]If Brownian motions can be thought of as particles that move randomly over time, a two-parameter random field can be thought of as the random motion of a string. A three-parameter random field is also known as a membrane. This paper deals with strings.

(2005) applies random field models to derivative pricing. Our paper analyzes how random field models can be employed to explain the cross-section of the expected returns. Compared to other approaches, ours proposes, then, a new way to model asset returns. Our model is not built up around factors (be they observed or not). We direct our focus on correlations, as explained.

The model works as follows. Asset returns are driven by the realizations of a string. These realizations lead asset returns to co-move, and these co-movements become sources of priced risk: for any asset, the co-movements of its returns with all remaining asset returns receive a compensation. We derive the arbitrage restrictions amongst all asset returns and characterize this compensation: the expected excess return on each asset is the sum of the correlations of this return with all the remaining returns, weighted by some "premium function." The premium function is in common within the universe of all asset returns, in that the expected returns on all assets are averages of the assets correlations weighted with the same premium function.

Thus, the expected excess return on any asset reflects an average premium required to compensate for the asset returns granular exposure to all remaining returns. We term the result *correlation premium*. We test whether, indeed, the cross-section of the expected returns is explained by the cross-section of the correlation premiums. We find that the model provides a reasonable match of the cross-section of the expected returns, for a number of portfolios sorted through book-to-market, momentum and additional standard characteristics, at least comparable to the Fama-French benchmark. Furthermore, our model displays additional properties regarding returns predictability and the time-series of assets correlations, both realized and risk-adjusted, as we now explain.

In theory, our model does not require time-varying correlations: even if asset correlations were all constant, the cross-section of the expected excess returns would be a set of non-zero correlation premiums. However, in practice, correlations change over time. We model time-variation in these correlations as being driven by a pro-cyclical state variable,[2] such that correlations increase in bad times, i.e., for low realizations of this state variable. The cross-section of correlation premiums and,

[2] Thus, we rely on a factor model for *asset correlations*. The point of the paper is that we do not rely on a factor model for the cross-section of *asset returns*.

then, the expected excess returns is predictable, driven by the state variable. We reconstruct the dynamics of the state variable as a by-product of the model estimation method, based on moment conditions solved in closed-form (that is, a GMM). The model predicts that the cross-section of expected excess returns are countercyclical and asymmetrically related to market conditions: they increase more in bad times than they decrease in bad.

Our GMM consists of moment conditions on time-series properties of realized correlations, but also option-implied correlations. The model, then, provides additional predictions regarding the random nature of assets correlations. In particular, the risk of changing correlations may lead, and our empirical findings strongly suggest that they do lead, to a *correlation-risk premium*, the difference between risk-adjusted (i.e., option-implied) and realized correlations on S&P 500, a "global correlation-risk premium." Our model predicts realized correlations and correlation-risk premiums to be inversely related. In other words, risk-adjusted correlations move, on average, less than realized correlations in reaction to a changing market environment. This conclusion matches our empirical evidence and stands in sharp contrast with the evidence available from equity volatility markets, where volatility risk premiums are highly countercyclical. Remarkably, then, our model is able to fit both the correlation-risk premium resulting in derivative markets (on S&P 500), and cross-sections of asset returns that are not directly related to S&P 500. For example, the model is given a comfortable support within the international stock universe, such as the global ME-BTM 5x5 portfolio. Therefore, the model displays potential of being able to explain premiums for other asset classes, by just relying on our global correlation-risk premium.

The paper is organized as follows. The next section contains high level assumptions and general no-arbitrage restrictions. Section 3 provides model specifications for the purpose of empirical work. Section 4 develops cross-equation restrictions and contains our empirical results. Section 5 concludes. Appendix A contains technical details omitted from the main text and Appendix B develops model extensions.

2. Asset prices as strings

2.1. Primitives

We consider a market with a continuum of assets in $(0,1)$, and assume that each asset return is exposed to all remaining asset returns through the realization of a "string." Previous models with a continuum of assets include those formulated by Al-Najjar (1998) in a static exact factor framework and Gagliardini, Ossola and Scaillet (2016) in a conditional approximate factor setting, amongst others. Our approach is novel precisely because we are not relying on any factor structure, but on strings. Precisely, let $P_t(i)$ be the price of the i-th asset at t and $D_t(i)$ be its instantaneous dividend. We assume that the realized returns on each asset-i are solutions to

$$\frac{dP_t(i) + D_t(i)\,dt}{P_t(i)} = \mathcal{E}(\mathbf{y}_t, i)\,dt + \sigma(\mathbf{y}_t, i)\,dZ_t(i)\,, \quad i \in (0,1)\,, \tag{1}$$

where $Z_t(i)$, the string, is a process continuous in i and t, and such that $E(dZ_t(i)) = 0$, $var(dZ_t(i)) = dt$, and $cov(dZ_t(i)\,dZ_t(j)) = \rho(\mathbf{y}_t, i, j)\,dt$, for some function ρ taking values in $(-1,+1)$, and some state vector $\mathbf{y}_t$, to be introduced below; the volatility term, $\sigma(\mathbf{y}, i)$ is a continuous function of $\mathbf{y}$ and i, and $\rho(\mathbf{y}, i, j)$ is also continuous; finally, $\mathcal{E}(\mathbf{y}, i)$ is the expected return, determined below (see Proposition 1).[3]

Volatility, $\sigma(\mathbf{y}, i)$, summarizes the asset-i return exposure to how the very same asset return co-varies with all remaining asset returns. It, thus, plays a role similar to the standard "beta" in factor models. The notable feature of the model is that returns are risky because the realization of the string leads all asset returns to co-move; in standard models, instead, asset returns co-move, driven by the realization of common factors. In the next section, we shall explain how the random fluctuations of the string become priced sources of risk. A further important property of the model is that volatility is disentangled from correlation: the model relies on two distinct definitions of volatility and correlation. However, the two functions, $\sigma(\mathbf{y}, i)$ and $\rho(\mathbf{y}, i, j)$ may correlate, potentially driven

[3] Appendix B considers an extension of the model where asset-i return are driven by a "compound string," that is, by the realization of a convex functional of the whole string (i.e., not only by $dZ_t(i)$).

by the common state vector $\mathbf{y}$. Finally, one may formulate several assumptions on the state vector $\mathbf{y}$; in our empirical work, we shall assume it is a diffusion process, solution to

$$d\mathbf{y}_t = \mathbf{b}(\mathbf{y}_t)\, dt + \mathbf{\Sigma}(\mathbf{y}_t)\, d\mathbf{W}_t,$$

for some vector and diffusion matrix $\mathbf{b}$ and $\mathbf{\Sigma}$.

We now turn to the asset pricing implications of the assumptions we have made so far. That is, we describe a pricing kernel for this market and, then, derive cross-sectional restrictions on each asset expected return.

2.2. The pricing kernel

In the absence of arbitrage, there exists a pricing kernel ξ_t that prices all the assets. We assume that it is solution to

$$\frac{d\xi_t}{\xi_t} = -r(\mathbf{y}_t)\, dt - \int_0^1 \phi(\mathbf{y}_t, i)\, dZ_t(i)\, di - \boldsymbol{\lambda}(\mathbf{y}_t)\, d\mathbf{W}_t, \tag{2}$$

where r is the instantaneous interest rate, $\boldsymbol{\lambda}$ is a vector valued function, including the unit prices of risk related to the fluctuations of the Brownian motion $\mathbf{W}_t$, and $\phi(\mathbf{y}, i)_{i\in(0,1)}$ is the collection of the unit prices of risk related to the fluctuations of the string $Z_t(i)_{i\in(0,1)}$. We assume that these prices of risk are continuous functions of the state vector $\mathbf{y}$ and i. From now on, we focus on the asset pricing implications of the pure string component. Appendix B contains extensions that allow for the existence of a priced Brownian risk. We now turn to the cross-sectional restrictions on each asset expected return.

2.3. Conditional CAPM and the correlation premium

In a frictionless market, the expected return on each asset-i satisfies the following standard restriction

$$\mathcal{E}(\mathbf{y}_t, i)\, dt \equiv E\left(\frac{dP_t(i) + D_t(i)\, dt}{P_t(i)}\right) = r(\mathbf{y}_t) - cov\left(\frac{dP_t(i)}{P_t(i)}, \frac{d\xi_t}{\xi_t}\right), \quad i \in (0,1). \tag{3}$$

We have

$$
\begin{aligned}
cov\left(\frac{dP_t(i)}{P_t(i)}, \frac{d\xi_t}{\xi_t}\right) &= -E\left(\sigma(\mathbf{y}_t, i)\, dZ_t(i) \int_0^1 \phi(\mathbf{y}_t, j)\, dZ_t(j)\, dj\right) \\
&= -\sigma(\mathbf{y}_t, i) \int_0^1 \phi(\mathbf{y}_t, j)\, E\left(dZ_t(i)\, dZ_t(j)\right) dj \\
&= -\sigma(\mathbf{y}_t, i) \left(\int_0^1 \phi(\mathbf{y}_t, j)\, \rho(\mathbf{y}_t, i, j)\, dj\right) dt. \qquad (4)
\end{aligned}
$$

Replacing these results into Eq. (3) leaves the following cross-equation restrictions on the cross-section of expected returns:

Proposition 1. (Correlation premium) *The expected return* $\mathcal{E}(\mathbf{y}_t, i)$ *on asset-*i, $i \in (0, 1)$, *satisfies*

$$\mathcal{E}(\mathbf{y}_t, i) - r(\mathbf{y}_t) = \mathcal{C}(\mathbf{y}_t, i), \qquad (5)$$

where

$$\mathcal{C}(\mathbf{y}_t, i) \equiv \sigma(\mathbf{y}_t, i) \int_0^1 \phi(\mathbf{y}_t, j)\, \rho(\mathbf{y}_t, i, j)\, dj. \qquad (6)$$

The term $\mathcal{C}(\mathbf{y}_t, i)$ in this proposition summarizes the evaluation of the asset-i granular exposure to the market, and we are referring to it as the *correlation premium* for asset-i. The proposition provides a novel theory of the cross-section of the expected returns, based on this correlation premium. Eq. (5) tells us that each asset expected excess return i is the premium required to compensate an investor for the exposure of the asset-i return to all remaining asset returns. The contribution of asset-j return to the premium for asset-i, when the state is $\mathbf{y}$, is $\sigma(\mathbf{y}, i)\, \phi(\mathbf{y}, j)\, \rho(\mathbf{y}, i, j)\, dj$. That is, $\rho(\cdot, i, j)$ is the correlation between asset-i and asset-j returns, correlation arising from the realization of the string; $\phi(\cdot, j)$ is the unit risk premium that compensates for any risk correlated with asset-j return; finally, $\sigma(\cdot, i)$ defines the size of the overall exposure of the asset-i return to the whole string, as explained in Section 2.1.

To illustrate Proposition 1, consider the following heuristic example based on a N-asset market. Consider, say, asset-i. Its returns are exposed to the risk of co-movements with returns on asset-1, a risk summarized by the correlation, $\rho(\mathbf{y}_t, 1, i)$; then, $\sigma(\mathbf{y}_t, i)\,\rho(\mathbf{y}_t, 1, i)$ is the risk of co-variation that returns on asset-i have with returns on asset-1. We term this co-variation "exposure," in analogy with standard asset pricing terminology. Now, there are obviously N such exposures arising from the realization of the string, including the variation of the very same asset-i returns. According to the model, each of these exposures receives a compensation. The correlation premium is the average premium, $\mathcal{C}$, as summarized by Table 1, i.e., the counterpart to Eq. (6) in this heuristic example.

	Exposure to asset-j	Compensation	Premium
$j=1$	$\sigma(\mathbf{y}_t, i)\,\rho(\mathbf{y}_t, i, 1)$	ϕ_1	$\sigma(\mathbf{y}_t, i)\,\rho(\mathbf{y}_t, i, 1)\,\phi_1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$j=N$	$\sigma(\mathbf{y}_t, i)\,\rho(\mathbf{y}_t, i, N)$	ϕ_N	$\sigma(\mathbf{y}_t, i)\,\rho(\mathbf{y}_t, i, N)\,\phi_N$
			$\mathcal{C} = \sigma(\mathbf{y}_t, i)\,\frac{1}{N}\sum_{j=1}^{N}\rho(\mathbf{y}_t, i, j)\,\phi_j$

Table 1: This table provides a heuristic construction of the expected return required to hold any asset i. The second column indicates how asset-i is exposed to fluctuations of any asset j. The second column is the unit risk premium required to bear a given exposure to any asset j. The premium is the average of the exposures weighted by the unit risk premiums.

This example illustrates that, in the model, exposures are the counterparts to the familiar "betas" in standard factor models. That is, betas are asset returns sensitivities to changes in common factors; instead, in our model, exposures result from the asset returns sensitivities to changes in all the asset returns that arise through the realization of the string. Similarly, compensations are the counterparts to "lambdas." But while lambdas are unit risk premiums relating to the fluctuations of common and exogenous factors, compensations are, in our model, unit premiums relating to how each asset return co-varies with all remaining asset returns: there exists, then, a compensation for each asset return in the assets universe. In Section 3, we formulate assumptions that help deal with these infinite dimensional problems, rendering our model tractable for empirical purposes.

3. A model with random correlations

This section provides model specifications that account for the salient empirical properties of (i) asset return correlations and (ii) the premiums required to bear time-variation in these correlations.

It is well-known that asset correlations do indeed vary over time (see, e.g., Figure 1 below). Initially, however, it is instructive to focus on our model implications in the simple case where correlations, variances and premiums are all constant. Assume, then, that for all $i, j \in (0,1)$,

$$\sigma(\mathbf{y}_t, i) = \sigma_i, \quad \rho(\mathbf{y}_t, i, j) = \rho(i,j), \quad \phi(\mathbf{y}_t, j) = \phi_o, \quad \boldsymbol{\lambda}(\mathbf{y}_t) = \boldsymbol{\lambda}_o,$$

for some constants σ_i, $\rho(i,j)$, ϕ_o and a vector of constants $\boldsymbol{\lambda}_o$. Given these assumptions, Proposition 1 predicts that the expected excess returns on each asset-i are

$$\mathcal{E}(\mathbf{y}_t, i) - r(\mathbf{y}_t) = \mathcal{C}(i), \quad \mathcal{C}(i) = \phi_o \sigma_i \underbrace{\left(\int_0^1 \rho(i,j)\, dj \right)}_{\equiv \rho_i \text{ (global correlation exposure)}}. \tag{7}$$

We call ρ_i *global correlation exposure* for asset-i, consistent with terminology in Section 2.3 (see Table 1): the risk premium on asset-i equals the product of a risk exposure, $\sigma_i \rho_i$, times the unit price of risk, ϕ_o. We refer to ρ_i as "global" because it is the average correlation of asset-i returns with all other asset returns. This decomposition of the expected returns is neat, but obtains due to the assumption that the unit prices of risk are constant in the cross-section. We now generalize the insights from this basic model and account for both time-variation in correlations and cross-sectional variations in the unit risk premiums.

3.1. A factor model of asset correlations

Figure 1 summarizes well-known evidence regarding time-variation in asset correlations. We construct 25 Size and Book-to-Market sorted portfolios and calculate realized correlations for each portfolio pair through one-month rolling windows estimates. Consider the empirical counterpart to the global correlation exposures ρ_i in Eq. (7), $\rho_t^{\$}(i) = \frac{1}{n} \sum_{j=1}^{n} \rho_t^{\$}(i,j)$,where $\rho_t^{\$}(i,j)$ denotes the realized correlation between portfolios j and i, and $n = 25$. We find that nearly 90% of the variability in these exposures is explained by the first principal component. Figure 1 plots the average correlation exposure, defined as $\rho_t^\$ = \frac{1}{n}\sum_{i=1}^{n}\rho_t^\$(i)$.

The fact that a large portion of the correlation exposures is driven by a single principal component suggests that a parsimonious model may help explain time-variation in these correlations. We now proceed with such a model while still assuming that correlation is priced in accordance with the string model in Section 2.

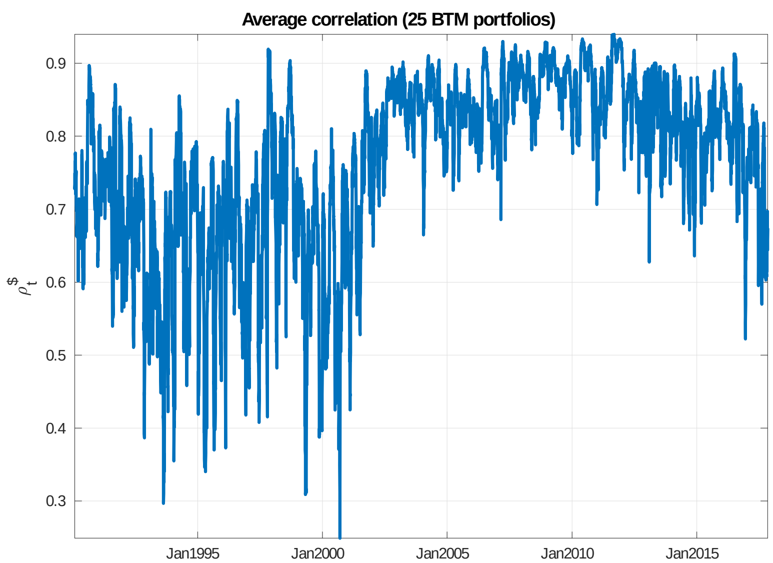

Figure 1. This picture depicts the average correlation exposure for 25 Size and Book-to-Market sorted portfolios, defined as $\rho_t^\$ = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_t^\$(i,j)$, where $\rho_t^\$(i,j)$ is the realized correlation between portfolios j and i, obtained through a rolling window equal to 22 days.

We assume that the asset correlations are driven by a diffusion process $\mathbf{y}_t \equiv y_t$, a scalar. To keep the model as simple as possible, we still assume that the exposures to strings are constant and independent of i, i.e., $\sigma(\mathbf{y}_t, i) = \sigma_i$; and we assume that the string correlation function is

$$\rho(y_t; i, j) = \varrho_0(i, j) + \varrho_1(i, j) e^{-y_t}, \tag{8}$$

where $\varrho_0(i, j)$ and $\varrho_1(i, j)$ are matrix coefficients independent of time, and y_t is solution to a square root process

$$dy_t = \kappa(m - y_t) dt + \eta\sqrt{y_t} dW_t, \tag{9}$$

for three positive constants κ, m and η. Under standard parameter restrictions, y_t stays strictly positive, hence, this specification for y_t bounds $\rho(y_t, i, j)$ to be inside the unit circle as soon as $|\varrho_0(i, j) + \varrho_1(i, j)| < 1$.

For the purpose of identifying the model, we need to fix the sign of $\varrho_1(i, j)$, and we work with $\varrho_1(i, j) > 0$. We, then, interpret y_t as a pro-cyclical variable: all correlations are down when y_t is up.

3.2. The correlation premium

The next corollary summarizes cross-section restrictions resulting from the assumptions formulated in Section 3.1.

Corollary 1. (One-factor correlation premiums) *Assume that the correlation function satisfies Eq. (8), where y_t is solution to Eq. (9), and that each asset return variance is constant and equal to σ_i^2 for asset-i. Then, the expected excess returns in Proposition 1 (Eqs. (5)-(6)) are*

$$\mathcal{E}(y_t, i) - r(y_t) = \mathcal{C}(y_t, i), \quad \mathcal{C}(y_t, i) = \sigma_i \int_0^1 \phi(y_t, j) \left(\varrho_0(i, j) + \varrho_1(i, j) e^{-y_t}\right) dj. \tag{10}$$

Thus, the cross-section of the expected excess returns is driven by a single, procyclical state variable, y_t. Moreover, under conditions on $\phi(y_t, j)$ discussed in a moment, expected excess returns are decreasing and convex in y_t, that is, they are countercyclical and react asymmetrically to y_t: they increase in bad times more than they lower in good, a property that is known to be empirically pervasive at the aggregate level since at least Mele (2007).

One simple condition leading to the previous predictions is that $\phi(y_t, j)$ is positive and constant, both in time and in the cross-section. More generally, expected returns are countercyclical if ϕ is. In all the specifications of ϕ provided below, expected excess returns are both countercyclical and convex in y_t, given the parameter estimates in Section 4.

We consider three specifications for the premiums that we use in our empirical work.

(I) Constant premiums. The correlation premium is constant both in time and in the cross-section, that is, $\phi(y_t, j) \equiv \bar{\phi}$. In this case, the correlation premium in Eq. (10) collapses to

$$\mathcal{C}(y_t, i) = \bar{\phi}\sigma_i \underbrace{\left(\varrho_0(i) + \varrho_1(i) e^{-y_t}\right)}_{\equiv \rho_i(y_t) \text{ (dynamic GCE)}}, \tag{11}$$

where $\varrho_q(i) = \int_0^1 \varrho_q(i, j)\, dj$, $q = 0, 1$. This model specification is a very minimal generalization of the constant correlation model in Eq. (7), whereby the global correlation exposure (GCE), ρ_i, is replaced by its dynamic counterpart, $\rho_i(y_t)$.

(II) Cross-sectional variation. The correlation premium for shocks on the asset return-j links to the dynamic GCE in (11) for the same asset, $\rho_j(t)$, according to $\phi(y_t, j) = \phi_0\varrho_0(j) + \phi_1\varrho_1(j)$, for two constants ϕ_0 and ϕ_1. The correlation premium for asset-i in Corollary 1 is

$$\mathcal{C}(y_t, i) = \sigma_i \int_0^1 \left(\phi_0\varrho_0(j) + \phi_1\varrho_1(j)\right)\left(\varrho_0(i, j) + \varrho_1(i, j) e^{-y_t}\right) dj. \tag{12}$$

(III) Time series and cross-sectional variation. The correlation premium for shocks on asset-j links to $\rho_j(t)$, according to $\phi(y_t, j) = \phi_{v0}\varrho_0(j) + \phi_{v1}\varrho_1(j) e^{-y_t}$, for two constants ϕ_{v0} and ϕ_{v1}, such that the correlation premium for asset-i is

$$\mathcal{C}(y_t, i) = \sigma_i \int_0^1 \left(\phi_{v0}\varrho_0(j) + \phi_{v1}\varrho_1(j) e^{-y_t}\right)\left(\varrho_0(i, j) + \varrho_1(i, j) e^{-y_t}\right) dj. \tag{13}$$

The rationale behind the specifications in (II) and (III) is the following. A parsimonious modeling assumption is that the premium $\phi_j(y_t, j)$ for exposure on asset returns-j reflects information on the dynamic GCE for the very same asset, $\rho_j(y_t)$. In these formulations, then, this premium reflects both the unconditional part of $\rho_j(y_t)$, i.e., $\varrho_0(j)$, and the exposure of $\rho_j(y_t)$ to movements in the state variable y_t, $\varrho_1(j)$. The difference between (II) and (III) is that the latter reflects both cross-sectional (i.e., $\varrho_0(j)$ and $\varrho_1(j)$) and time series (i.e., the state of y_t) information.

The following proposition gathers the expressions for the cross-section of the unconditional expected returns in the three specifications formulated above.

Proposition 2. (Unconditional correlation premiums) *The unconditional expected returns predicted by (I) the constant premiums model, (II) the cross-sectional variation model, and (III) the time series and cross-sectional variation model, are*

$$E\left[\mathcal{C}(y_t,i)\right] = \begin{cases} \bar{\phi}\sigma_i\left(\varrho_0(i)+\varrho_1(i)\bar{Y}_{(1)}\right) & \text{(I)} \\ \sigma_i\int_0^1\left(\phi_0\varrho_0(j)+\phi_1\varrho_1(j)\right)\left(\varrho_0(i,j)+\varrho_1(i,j)\bar{Y}_{(1)}\right)dj & \text{(II)} \\ \sigma_i\int_0^1\left[\phi_{v0}A_{0,0}(i,j)+\left(\phi_{v1}A_{1,0}(i,j)+\phi_{v0}A_{0,1}(i,j)\right)\bar{Y}_{(1)}+\phi_{v1}A_{1,1}(i,j)\bar{Y}_{(2)}\right]dj & \text{(III)} \end{cases}$$

where $A_{h,q}(i,j)\equiv\varrho_h(j)\,\varrho_q(i,j)$ *and*

$$\bar{Y}_{(\ell)}\equiv E\left(e^{-\ell y_t}\right)=\left(\frac{2\kappa}{2\kappa+\ell\eta^2}\right)^{\frac{2\kappa m}{\eta^2}},\quad \ell=1,2.$$

In Section 4, we test our string model while relying on its unconditional version predicted by Proposition 2, similarly as with standard methodology used with the Conditional CAPM (e.g., Jagannathan and Wang, 1996; Lettau and Ludvigson, 2001). We now develop additional cross-equation restrictions that we use while estimating the model. We address the question: is y_t a source of priced risk?

3.3. *The correlation-risk premium*

A key concept that has been extensively investigated in the empirical literature is the *correlation-risk premium*, defined as the difference between the expected integrated correlation under the risk-neutral probability and the physical probability, denoted hereafter as Q and P, respectively. If correlation was not a priced risk, this difference would always be zero. Figure 2 depicts the *realized* correlation-risk premium for S&P 500 stocks, defined as the difference between option implied integrated correlations (that is, correlations expected under Q) and realized correlations (proxies for expectations under P). Section 4 contains a detailed description of our input data and computations used in Figure 2.

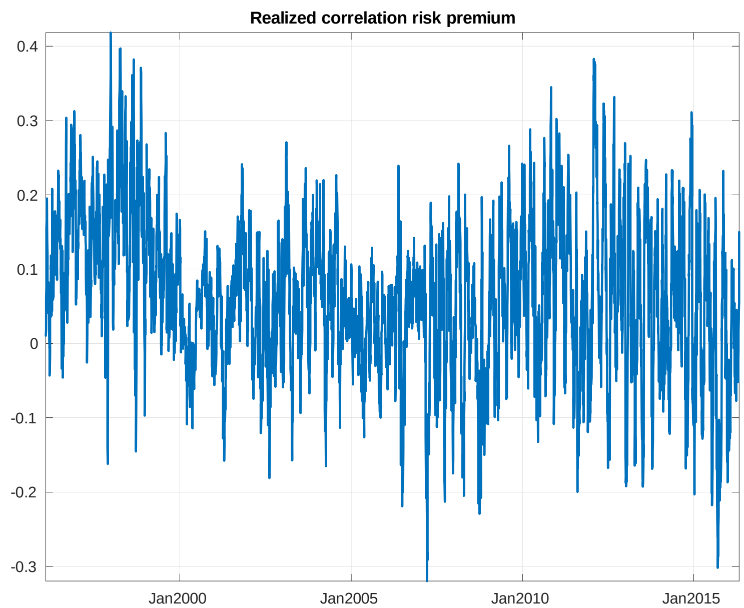

Figure 2. This picture plots the realized correlation-risk premium for S&P 500 stocks, defined as the difference between (i) risk-adjusted expectations of one-month average correlations, and implied by option prices, and (ii) realized correlations, calculated throughout a one-month window.

Consistent with the empirical evidence, we assume that time-variation in correlations is a priced risk. Our point of departure is the string correlation function $\rho(y_t; i, j)$ in Eq. (8). Let us integrate this function twice with respect to all asset pairs, obtaining the average correlation amongst all asset returns,

$$\rho(y_t; \boldsymbol{\varrho}) = \iint_{i,j\in[0,1]^2} \rho(y_t; i, j)\, didj = \varrho_0 + \varrho_1 e^{-y_t}, \tag{14}$$

where we have defined $\boldsymbol{\varrho} = [\varrho_0, \varrho_1]$ and $\varrho_q = \int_0^1 \varrho_q(i)\, di$, $q = 0, 1$. The model-implied correlation-risk premium is defined as the difference between the average expected integrated correlation $\rho(y_t; \boldsymbol{\varrho})$ in (14) under Q and that under P

$$\mathcal{P}_t \equiv \frac{1}{T-t}\left[\int_t^T E_t^Q\left(\rho\left(y_\tau; \boldsymbol{\varrho}\right)\right) d\tau - E_t\left(\rho\left(y_\tau; \boldsymbol{\varrho}\right)\right) d\tau\right], \tag{15}$$

where $E_t^Q(\cdot)$ denotes the expectation under Q given information at time-t, and $T-t$ is a given horizon.

In words, the *correlation-risk premium* compensates an investor for the fluctuations in the asset correlations. Note, also, that this definition is distinct from the *correlation premium*, i.e., $\mathcal{C}(\cdot, i)$ in Proposition 1. The correlation premium, $\mathcal{C}(\cdot, i)$, compensates for any asset return exposure to all remaining asset returns. The correlation-risk premium, $\mathcal{P}_t$, compensates for randomness in this exposure. Furthermore, note that y_t, the factor driving this random exposure, is not priced in the cross-section of the expected returns. Appendix B indicates how to proceed under the assumption that y_t is also priced in the cross-section of the expected returns. However, to keep the model as simple as possible, we do not consider this extension.

To render Eq. (15) operational, we specify the unit risk premium for y_t. We assume that $\lambda(y) = \nu\sqrt{y}$ for some constant ν, such that, under the risk neutral probability, Q,

$$dy_t = \tilde{\kappa}\left(\tilde{m} - y_t\right) dt + \eta\sqrt{y_t}\, d\tilde{W}_t, \tag{16}$$

where $\tilde{W}_t$ is a standard Brownian motion under Q, and

$$\tilde{\kappa} = \kappa + \nu\eta, \quad \tilde{m} = \frac{\kappa m}{\kappa + \nu\eta}.$$

Because y_t is interpreted as a pro-cyclical variable, we expect, empirically, that $\nu > 0$, meaning that y_t is more frequently in bad times under Q than under P (see Proposition A.1 in Appendix A).

Let $\boldsymbol{\vartheta} = [\boldsymbol{\theta}, \varrho_1, \nu]$, where $\boldsymbol{\theta} = [\kappa, m, \eta]$ denotes the parameter vector under the physical probability. Accordingly, denote with $\mathcal{P}_t = \mathcal{P}(y_t; \boldsymbol{\vartheta})$ the model-based correlation-risk premium in Eq. (15) for a given set of parameter values $\boldsymbol{\vartheta}$. The next proposition, proved in Appendix A, provides motivation for this notation as well as some properties of this correlation-risk premium.

Proposition 3. (Correlation-risk premium) *Assume that the premium related to Brownian fluctuations is* $\lambda(y) = \nu\sqrt{y}$. *Then, the correlation-risk premium is*

$$\mathcal{P}(y_t; \boldsymbol{\vartheta}) = \frac{\varrho_1}{T-t} \int_t^T \left(u(y_t, \tau - t; \boldsymbol{\theta}, \nu) - u(y_t, \tau - t; \boldsymbol{\theta}, 0) \right) d\tau, \tag{17}$$

where

$$u(y, x; \boldsymbol{\theta}, \nu) = a(x;\nu)\, e^{-b(x;\nu)y}, \quad a(x;\nu) = \left(\frac{2\tilde{\kappa}}{2\tilde{\kappa} + \eta^2 \left(1 - e^{-\tilde{\kappa}x}\right)} \right)^{\frac{2\kappa m}{\eta^2}}, \quad b(x;\nu) = \frac{2\tilde{\kappa} e^{-\tilde{\kappa}x}}{2\tilde{\kappa} + \eta^2 \left(1 - e^{-\tilde{\kappa}x}\right)}. \tag{18}$$

Moreover, for $\nu > 0$, *the correlation-risk premium is (i) strictly positive; and is (ii) increasing and concave in* y_t *for all* y_t *lower than some* y_1*; and (iii) decreasing and convex in* y_t *for all* y_t *higher than some* y_2.

Proposition 3 tells us that, provided correlation is positively priced, $\nu > 0$, the correlation-risk premium achieves a maximum. In good times, when the pro-cyclical variable y_t is high, the correlation-risk premium rises as y_t lowers. As times deteriorate further, additional drops in y_t lead to a fall in the correlation-risk premium. This fall reflects that fact that, in bad times, correlations under P and under Q are already very high; because they are obviously both bounded, then, as y_t lowers, their difference tends to vanish. These properties are illustrated by the left panel in Figure 3, which plots the correlation-risk premium $\mathcal{P}(y_t; \boldsymbol{\vartheta})$ in Eq. (17), and its unconditional expectation, based on the parameter estimates obtained in Section 4.

The right panel of Figure 3 depicts the correlation-risk premium against the instantaneous correlation predicted by the model, $\rho(y_t; \boldsymbol{\varrho})$ in Eq. (14), obtained while varying the state variable y_t driving them. The descending part of the curve in this right panel does then correspond to the ascending part of the curve in the left panel. The prediction is that, for most values of the instantaneous correlation, correlations and the correlation-risk premium are inversely related, with the premium achieving its maximum when correlation is at about as low as 20%. These predictions are useful because while y_t is not observed, we may estimate correlations and the correlation-risk premium based on observable quantities. Section 4 provides additional details on the testable implications of the model in this dimension, and evidence of a strong negative relation between correlations and the correlation-risk premium.

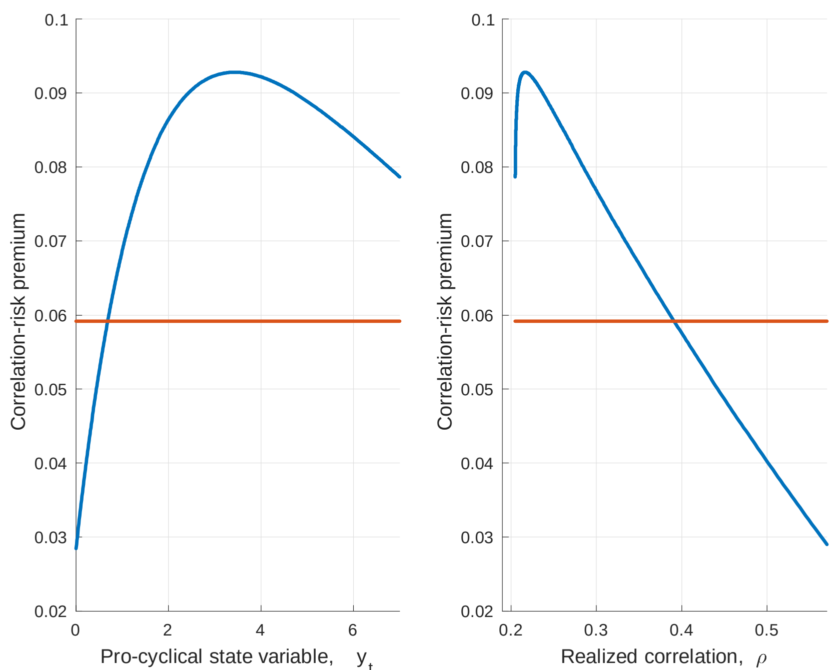

Figure 3. This picture plots the one-month correlation-risk premium $\mathcal{P}(y_t; \boldsymbol{\vartheta})$ in Eq. (17) against the state variable y_t (left panel) and the average correlation predicted by the model, $\rho(y_t; \boldsymbol{\varrho})$ in Eq. (14) (right panel). Parameter values are set equal to their estimates obtained in Section 4 (see Table 2). The red line is the unconditional expected value of the correlation-risk premium predicted by the model, i.e., $E(\mathcal{P}(y_t; \boldsymbol{\vartheta}))$ in Eq. (22).

4. Empirical analysis

4.1. Data and preparation of variables

4.1.1. Sources

For the model calibration, we require data on a wide panel of individual stocks belonging to a large index with traded options, and also data on a smaller panel of realized returns for a set of test assets. The first large panel is used to estimate the correlation state variable, and the smaller panels are then used to test our cross-sectional predictions. The data sample is daily and runs from January 1996 until April 2016.

For the smaller panels, we use returns on Fama-French portfolios and carry out estimation on 25 book-to-market and size sorted portfolios. We calculate second moments (volatilities, correlations, and factor betas) based on daily returns, and then proceed to estimate risk premiums relying on monthly portfolios returns.

As a broad sample of individual stocks we select all constituents of a market-wide index, namely, S&P500. The composition of S&P500 index is obtained from Compustat and merged with CRSP through the CCM Linking Table using GVKEY and IID to link to PERMNO, following the second best method from Dobelman, Kang, and Park (2014). The data on daily returns and market capitalization are obtained from CRSP, and as a proxy for index weights on each day, we use the relative market cap of each stock in an index from the previous day.

For the cross-sectional tests we use a number of standard portfolios, sorted one- or two-way by characteristics like market equity (ME) , book-to-market (BTM), investments (INV), operating profitability (OP), momentum (MOM), and reversal (REV). We obtain daily and monthly returns for these portfolios from Kenneth French data library. The cross-sectional pricing results are based on six sets of portfolios, each with 25 assets stemming from different two-way sorting procedures. We use the following data sets: 5x5 ME-BTM, 5x5 ME-INV, 5x5 ME-MOM, 5x5 ME-REV, 5x5 ME-OP, and 5x5 ME-BTM Global portfolios.

Because our model should not only deliver the cross-sectional pricing performance, but also be consistent with the price of correlation-risk, we use the options data on the S&P500 index and all its constituents to compute the time series of the implied correlations and the respective correlation-risk premiums as defined below. Implied correlations are estimated by comparing the index variance with the variance of the portfolio of index components. Matching the historical data with options happens through the historical CUSIP link provided by OptionMetrics. S&P500 index is directly used as underlying for options. PERMNO is used as the main identifier of individual stocks in the merged database. For computing the option-based variables we rely on the Surface File from OptionMetrics, selecting for each underlying the options with 30, 91 and 365 days to maturity and deltas in the out-the-money range (that is, absolute delta weakly less than 0.5). While the surface data is not suitable for testing trading rules due to extensive inter- and extrapolations of the market data, it proved to be a valuable source of information that can be used in asset pricing tests or in generating signals for trading.

4.1.2. Model inputs

The estimation of our model requires calibrating the string correlation function in (8) to its empirical counterparts. We calibrate the model in a way that the correlation state variable y_t reproduces model dynamics for the average correlation in (14) and its risk-neutral equivalent (defined in a moment) that match as closely as possible their empirical counterparts. As for these empirical counterparts, we rely on average correlations obtained through the equicorrelation amongst all S&P500 components. Equicorrelation is a useful measure of the average level of market-wide correlations and, hence, it may reasonably be based upon for the purpose of proxying the dynamics of our state-variable. Equicorrelations are computed assuming that, in each day, all pairwise correlations are equal.[4]

[4] Elton and Gruber (1973) are amongst the first to suggest this notion of correlation under the physical probability. Driessen, Maenhout, and Vilkov (2005) and Skinzi and Refenes (2005) extended this notion to the risk-neutral space to measure an average option-implied correlation representative of a universe of stocks.

Consider a basket of assets with a variance equal to σ_{It}^2 at time-t:

$$\sigma_{It}^2 = \sum_{i,j=1} w_i w_j \sigma_{it} \sigma_{jt} \rho_{ij,t},$$

where w_i are the asset portfolio weights. Given a time-series of variances of this basket, σ_{It}^2, of its components σ_{it}^2, and the index weights, w_i, equicorrelations are obtained as the single number $\rho_{ij,t} = \rho_t$ calculated in each day t as

$$\rho_t = \frac{\sigma_{It}^2 - \sum_{i=1} w_i^2 \sigma_{it}^2}{\sum_{i=1} \sum_{j \neq i} w_i w_j \sigma_{it} \sigma_{jt}}. \tag{19}$$

Note that the resulting correlation matrix of the assets in the basket is positive-definite, provided the equicorrelation is non-negative, which is the case in our empirical implementation of (19). In the sequel, we refer to "implied correlation" for the risk-neutral, and "realized correlation" for the realized equicorrelations.

Option-implied variances are computed as model-free implied variances (Dumas, 1995; Britten-Jones and Neuberger, 2000; Bakshi, Kapadia, and Madan, 2003). We compute realized variances using daily returns and a window length equal to one month. Thus, after plugging the implied or realized variances into Eq. (19), we end up with the monthly implied or realized correlations, respectively. The correlation-risk premium is constructed similarly as in Driessen, Maenhout, and Vilkov (2005) as an implied correlation at the end of day t minus 22-day moving averages of the realized correlations under P calculated through (19). We denote the estimate of this premium at time-t with $\mathcal{P}_t^{\$}$. Likewise, let $\rho_t^{\$}$ denote the realized correlation at time-t. As primary data series for calibrating the parameters governing the dynamics of y_t, we use one-month realized correlation, $\rho_t^{\$}$, and such is, then, the horizon of the corresponding correlation-risk premium, $\mathcal{P}_t^{\$}$. To calibrate the string correlation function (i.e., $\varrho_0(i,j)$ and $\varrho_1(i,j)$ in (8)), we need to fit Eq. (19) to the observed pairwise correlations between the test assets. Pairwise correlations are computed from daily returns by relying on standard formulas. Finally, the cross-sectional tests of our models are based on monthly realized excess returns of test portfolios. The excess returns are computed as realized monthly returns minus the one-month Treasury bill rate (from Ibbotson Associates) obtained from the Kenneth French data library.

4.2. Cross-equation restrictions and state variable estimates

We develop moment conditions that we use to estimate $\boldsymbol{\theta}$, the parameter vector related to the dynamics of the pro-cyclical state variable y_t under P (see Section 3), the correlation exposures $\varrho_0(i,j)$ and $\varrho_1(i,j)$, and the correlation-risk premium coefficient ν. Finally, we explain how we proceed to recover estimates of the pro-cyclical state variable for each date in our sample.

4.2.1. Matching correlations and correlation-risk premium

The next proposition provides moment conditions that we use to estimate $(\boldsymbol{\theta}, \varrho_1)$.

Proposition 4. (Correlation moment conditions) *For any integer n, the n-th uncentered unconditional moment of $\varrho(y_t)$ is*

$$E\left(\rho^n(y_t;\boldsymbol{\varrho})\right) = \sum_{i=0}^{n} \binom{n}{i} \varrho_0^i \varrho_1^{n-i} \left(\frac{2\kappa}{2\kappa + (n-i)\,\eta^2}\right)^{\frac{2\kappa m}{\eta^2}}. \tag{20}$$

For any fixed Δ, the unconditional covariance of $\rho(y_t;\boldsymbol{\varrho})$ with $\rho(y_{t+\Delta};\boldsymbol{\varrho})$ is

$$cov\left(\rho(y_t;\boldsymbol{\varrho}), \rho(y_{t+\Delta};\boldsymbol{\varrho})\right) = \varrho_1^2 \left[\left(\frac{4\kappa^2}{(2\kappa+\eta^2)^2 - \eta^4 e^{-\kappa\Delta}}\right)^{\frac{2\kappa m}{\eta^2}} - \left(\frac{4\kappa^2}{(2\kappa+\eta^2)^2}\right)^{\frac{2\kappa m}{\eta^2}}\right]. \tag{21}$$

Provided the state variable y_t is mean-reverting ($\kappa > 0$), the auto-covariance of the integrated correlation, $\rho(y_t;\boldsymbol{\varrho})$, is strictly positive and vanishes to zero, eventually. The higher κ, the higher the vanishing rate, just as for the original state variable y_t. Note, also, that m, the unconditional mean of y_t, can be identified with enough moment conditions. Intuitively, the variance of a square root process is level-dependent, such that the whole autocovariance function of y_t is level-dependent too.

Proposition 4 helps reconstructing the dynamics of y_t under the physical probability. Moreover, we rely on the model-implied correlation-risk premium in Proposition 3 and derive additional parameter restrictions. In Appendix A, we show that the unconditional mean of $\mathcal{P}(y_t; \boldsymbol{\vartheta})$ is

$$E(\mathcal{P}(y_t; \boldsymbol{\vartheta})) = \frac{\varrho_1}{T-t} \int_0^{T-t} (\bar{u}(x; \boldsymbol{\theta}, \nu) - \bar{u}(x; \boldsymbol{\theta}, 0))\, dx, \tag{22}$$

where

$$\bar{u}(x; \boldsymbol{\theta}, \nu) = \left(\frac{2\tilde{\kappa}\kappa}{2\tilde{\kappa}\kappa + (\kappa + \nu\eta e^{-\tilde{\kappa}x})\, \eta^2} \right)^{\frac{2\kappa m}{\eta^2}}.$$

We use a moment condition based on Eq. (22) as an additional cross-equation restriction for $[\boldsymbol{\theta}, \varrho_1]$, but also because it helps pinning down the level of the correlation-risk premium to its historical average, through ν. (The red line depicted in Figure 3 is the value of $E(\mathcal{P}(y_t; \boldsymbol{\vartheta}))$ implied by our parameter estimates.) Precisely, let $\boldsymbol{\zeta} = [\boldsymbol{\theta}, \varrho_0, \varrho_1, \nu]$ and let N denote the sample size. Define

$$h_N(\boldsymbol{\zeta}) \equiv \begin{bmatrix} E_N(\rho_t^{\$}) - E(\rho(y_t; \boldsymbol{\varrho})) \\ var_N(\rho_t^{\$}) - var(\rho(y_t; \boldsymbol{\varrho})) \\ E_N(\rho_t^{\$3}) - E(\rho^3(y_t; \boldsymbol{\varrho})) \\ \{cov_N(\rho_t^{\$}, \rho_{t+\Delta}^{\$}) - cov(\rho(y_t; \boldsymbol{\varrho}), \rho(y_{t+\Delta}; \boldsymbol{\varrho}))\}_{\Delta \in \mathbb{L}} \\ E_N(\mathcal{P}_t^{\$}) - E(\mathcal{P}(y_t; \boldsymbol{\vartheta})) \end{bmatrix},$$

where N subscripts indicate empirical moment estimates and, finally, $\mathbb{L}$ denotes the set of lags chosen while calibrating the model-implied autocovariance function to its data counterparts: two weeks, one month and two months. Our GMM estimator is obtained as

$$\hat{\boldsymbol{\zeta}}_N = \arg\min_{\boldsymbol{\zeta}} h_N(\boldsymbol{\zeta})^\intercal W_N h_N(\boldsymbol{\zeta}), \tag{23}$$

where W_N is a weighting matrix that minimizes the asymptotic variance of the estimator, which we estimate, recursively, as $\hat{W}_N^{-1} \equiv h_N(\hat{\boldsymbol{\zeta}}_N)^\intercal h_N(\hat{\boldsymbol{\zeta}}_N)$.

Therefore, we rely on 7 moment conditions to estimate 6 parameters. Table 2 contains parameter estimates and associated t-statistics. All parameter estimates are highly statistically significant.

	Estimate	t-stat
ϱ_0	0.2046	39.47
$\ln \varrho_1$	-0.9970	-17.36
κ	2.0151	3.71
m	5.7994	2.51
η	7.4182	5.43
ν	3.6167	31.65

Table 2: GMM estimates and t-stats obtained relying on the moment conditions (23).

4.2.2. Estimates of correlation exposures

To implement cross-sectional estimates of the model in (10), we need to estimate the asset return correlation exposures in Eq. (8), $\varrho_0(i,j)$ and $\varrho_1(i,j)$ and, thus, to build up estimates of the state. We rely on estimates of y_t obtained while minimizing a distance of the model predictions to the data proxies $\rho_t^{\$}$ and $\mathcal{P}_t^{\$}$,

$$\hat{y}_t = \arg\min_{y_t} \left(\frac{(\rho_t^{\$} - \rho(y_t; \hat{\boldsymbol{\varrho}}_N))^2}{var(\rho_t^{\$})} + \frac{(\mathcal{P}_t^{\$} - \bar{\mathcal{P}}(y_t; \hat{\boldsymbol{\vartheta}}_N))^2}{var(\mathcal{P}_t^{\$})} \right), \tag{24}$$

where $\bar{\mathcal{P}}(y_t; \hat{\boldsymbol{\vartheta}}_N)$ denotes the model counterpart to $\mathcal{P}_t^{\$}$. Estimates of the correlation exposures, say $\hat{\varrho}_0(i,j)$ and $\hat{\varrho}_1(i,j)$ are, then, obtained while regressing data proxies, $\rho_t^{\$}(i,j)$ say, onto a constant and $e^{-\hat{y}_t}$, under the restriction that the coefficient estimates sum up to the GMM estimates in (23), viz

$$\hat{\varrho}_q = \iint_{i,j\in[0,1]^2} \hat{\varrho}_q(i,j)\, didj, \quad q \in \{0,1\}.$$

Finally, we use $\hat{\varrho}_0(i,j)$ and $\hat{\varrho}_1(i,j)$ in Eq. (10) and implement cross-sectional estimates of the prices of risk $\phi(\cdot)$ while fitting the unconditional version of the model predicted by Proposition 2 in its three versions, as implied by (11)-(12)-(13).

Figure 4 depicts the estimates of the state obtained through (24) as well as a comparison of the average correlations predicted by the model with those in the data. The model tracks all the episodes of spikes in correlations that occurred in 1998, 2003, 2008 and 2011, albeit in a way less pronounced than in the data.

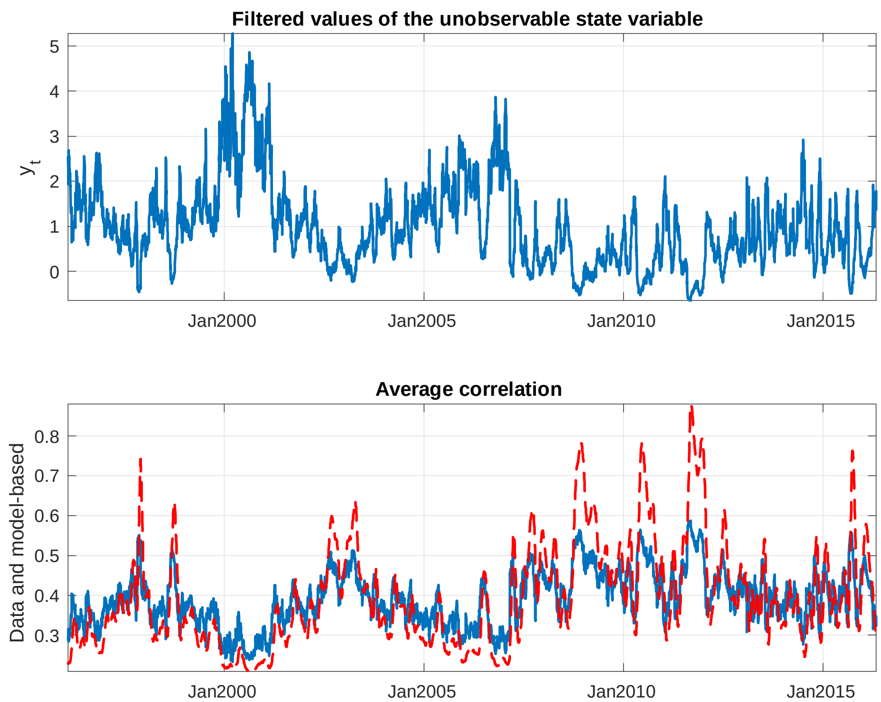

Figure 4. The top panel depicts estimates of the pro-cyclical state variable, y_t, obtained by matching the model predictions on realized correlations and correlation-risk premium, as in Eq. (24). The bottom panel depicts the average correlation in the data (dashed line) and the average correlation predicted by the model (solid line).

4.3. Cross-sectional pricing

We test the asset pricing model from (10) in its unconditional version implied by the three specifications of the correlation-risk premium predicted by Proposition 2: (I) constant premium $\phi(y_t, j) = \bar{\phi}$

both in time and cross-sections, (II) premium with cross-sectional variation $\phi(y_t, j) = \phi_0 \varrho_0(j) + \phi_1 \varrho_1(j)$, for two constants ϕ_0 and ϕ_1, and (III) premium with both time and cross-sectional variation $\phi(y_t, j) = \phi_{v0} \varrho_0(j) + \phi_{v1} \varrho_1(j) e^{-y_t}$, again for two constants ϕ_{v0} and ϕ_{v1}.

Portfolio set	$\bar{\phi}$	α	Slope	Data fit, R^2
5x5 ME-BTM	-0.032	0.008	1.999	0.082
	-0.164	2.555	1.770	–
5x5 ME-INV	-0.013	0.008	2.848	0.014
	-0.066	2.800	1.159	–
5x5 ME-MOM	-0.027	0.008	-0.927	0.027
	-0.114	2.476	-1.291	–
5x5 ME-REV	0.064	0.005	0.571	0.006
	0.299	1.794	1.070	–
5x5 ME-OP	-0.064	0.009	2.799	0.140
	-0.319	3.309	2.215	–
5x5 ME-BTM Global	-0.283	0.011	0.974	0.176
	-1.573	3.586	2.477	–

Table 3: This table provides parameter estimates of the constant risk premium $\bar{\phi}$ for Model I (with t-stats below), and the pricing performance expressed as the average pricing error (α per month) across a given set of portfolios, slope in the regression of average realized portfolio returns on the model-based unconditional returns (Slope), and the fit of the model (adjusted R^2) from this regression.

We follow a standard two-pass Fama-MacBeth (1973) procedure by first estimating the parametrized correlation risk premium $\phi(\cdot)$ each month, and then making a statistical inference about the significance of the risk premium parameters based on the estimated time series. The overall fit of the model is evaluated based on the comparison of the unconditional model-based average returns with the realized returns for the whole sample period.

For each asset i in a given set of test portfolios at the end of each month t, we compute the model-

Portfolio set	ϕ_0	ϕ_1	α	Slope	Data fit, R^2
5x5 ME-BTM	13.704	-10.775	0.016	1.106	0.495
	3.236	-3.148	4.440	4.949	–
5x5 ME-INV	17.635	-13.092	0.013	1.001	0.482
	2.746	-2.637	3.179	4.835	–
5x5 ME-MOM	16.458	-12.434	0.014	0.912	0.714
	2.596	-2.464	3.204	7.801	–
5x5 ME-REV	18.120	-13.544	0.013	0.848	0.459
	3.388	-3.228	3.592	4.619	–
5x5 ME-OP	28.101	-20.796	0.018	0.982	0.779
	4.124	-4.050	5.182	9.264	–
5x5 ME-BTM Global	23.567	-17.975	0.018	0.916	0.855
	4.356	-4.438	6.495	11.945	–

Table 4: This table provides parameter estimates of the coefficients ϕ_0 and ϕ_1 in the risk premium for Model II (with t-stats below) and the pricing performance expressed as the average pricing error (α per month) across a given set of portfolios, slope in the regression of average realized portfolio returns on the model-based unconditional returns (Slope), and the fit of the model (adjusted R^2) from this regression.

based unconditional expected return for the next month using as inputs 60-month historic-window volatility σ_i, the unconditional moments of the correlation level $\bar{Y}_{(\ell)}, \ell = 1, 2$, and the respective matrix correlation exposures $\varrho_0(i,j)$ and $\varrho_1(i,j)$. To obtain the correlation-risk premium parameters for models I through III, we minimize the sum of squared errors between model-based and observed returns for a given month. Tables 2 to 4 show the parameter estimates for the correlation premium and demonstrate the unconditional pricing performance of the three models for six sets of test assets.

The helicopter view at the models tells us that the constant risk premium in both cross-sectional and time-series dimensions does not do a good pricing job: the estimate of the string risk premium is not significant and comes with a counterintuitive negative sign; moreover, for half of the test portfolio

Portfolio set	ϕ_{v0}	ϕ_{v1}	α	Slope	Data fit, R^2
5x5 ME-BTM	5.091	-9.917	0.014	0.939	0.420
	3.241	-3.037	3.948	4.288	–
5x5 ME-INV	5.752	-9.996	0.011	0.822	0.255
	2.991	-2.576	2.799	3.035	–
5x5 ME-MOM	5.586	-10.108	0.013	0.838	0.553
	3.040	-2.687	3.308	5.541	–
5x5 ME-REV	6.526	-11.727	0.012	0.795	0.394
	4.090	-3.658	3.582	4.076	–
5x5 ME-OP	8.501	-15.362	0.016	0.939	0.537
	4.315	-4.087	4.749	5.369	–
5x5 ME-BTM Global	6.605	-13.996	0.018	0.849	0.766
	4.516	-4.877	6.396	8.929	–

Table 5: This table provides parameter estimates of the coefficients ϕ_{v0} and ϕ_{v1} in the risk premium for Model III (with t-stats below), and the pricing performance expressed as the average pricing error (α per month) across a given set of portfolios, slope in the regression of average realized portfolio returns on the model-based unconditional returns (Slope), and the fit of the model (adjusted R^2) from this regression.

sets, there is an insignificant or even negative relation between predicted and realized returns. The R^2 revealing the accuracy of the cross-sectional prediction is between 10% and 18% for the best three sets of portfolios.

Allowing for variation in the premium in the cross-sectional dimension turns out to be very important, and in most cases produces significant parameter estimates of ϕ_0 and ϕ_1. For all the test portfolios, the model has a reasonable pricing fit, with cross-sectional R^2 varying from 44% to 83%, with the best fit displaying at the level of the global ME-BTM portfolios. Extending to Model III by allowing time-variation in the string risk premium, does not seem to improve the unconditional pricing performance, producing results that are very similar to those of Model II. To see how well the unit risk premium tracks the risk premium on each portfolio, we also compute the unconditional expected returns predicted by the model and, for each portfolio, we plot it against the average realized returns in Figure 5. For space reasons, we only consider Model II in this exercise, as Model III delivers similar results.

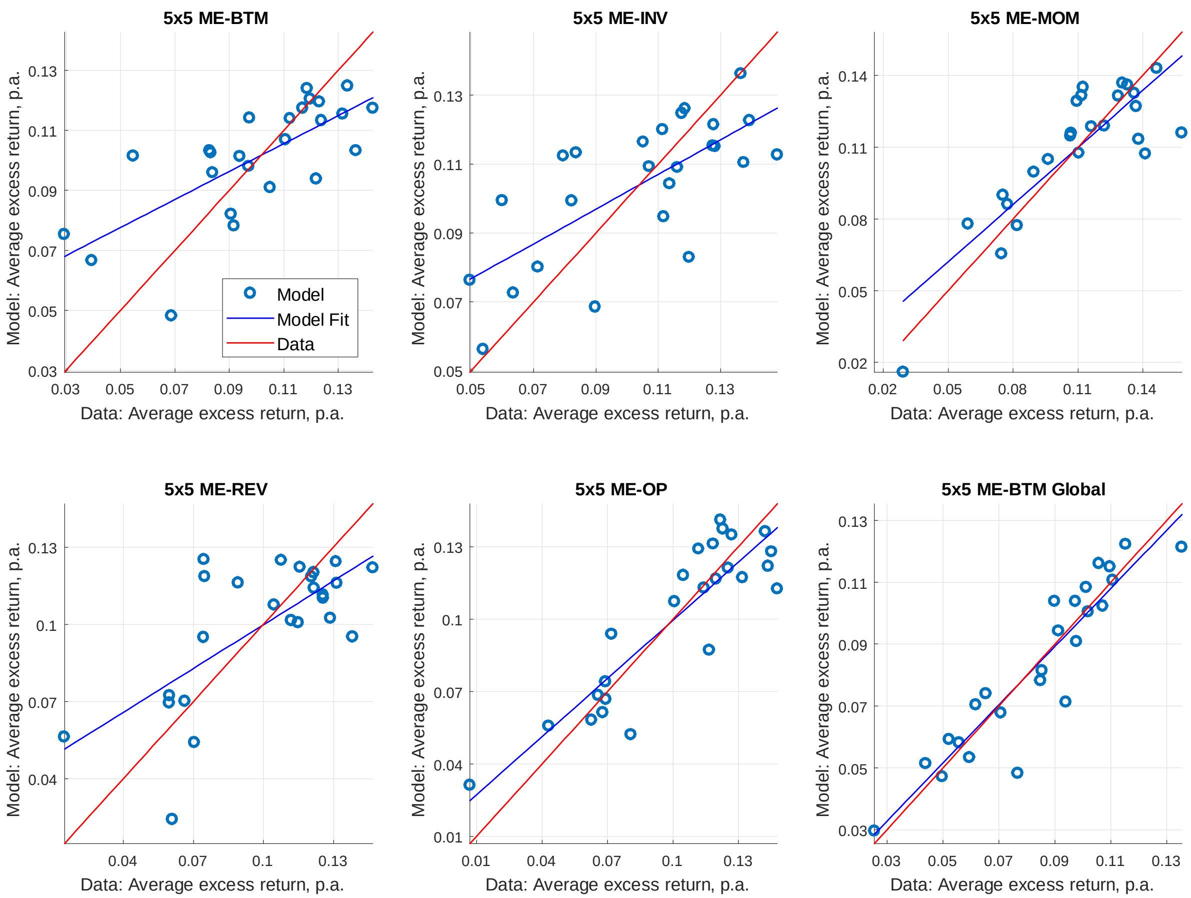

Figure 5. This picture depicts average excess returns and Model II predictions on the unconditional expected excess returns for 5x5 ME-BTM, 5x5 ME-INV, 5x5 ME-MOM, 5x5 ME-REV, 5x5 ME-OP, and 5x5 ME-BTM Global portfolios.

Finally, to disentangle exposure and pricing component in the correlation-risk premium, we isolate the pricing component $\phi(\cdot)$, which is plotted in Figure 6. While there is, on average, a positive relation between unit string risk premium and asset expected return, some unit risk premiums are negative. This feature of the string is in stark contrast to standard models with risk factors expressed as scalar random variables.

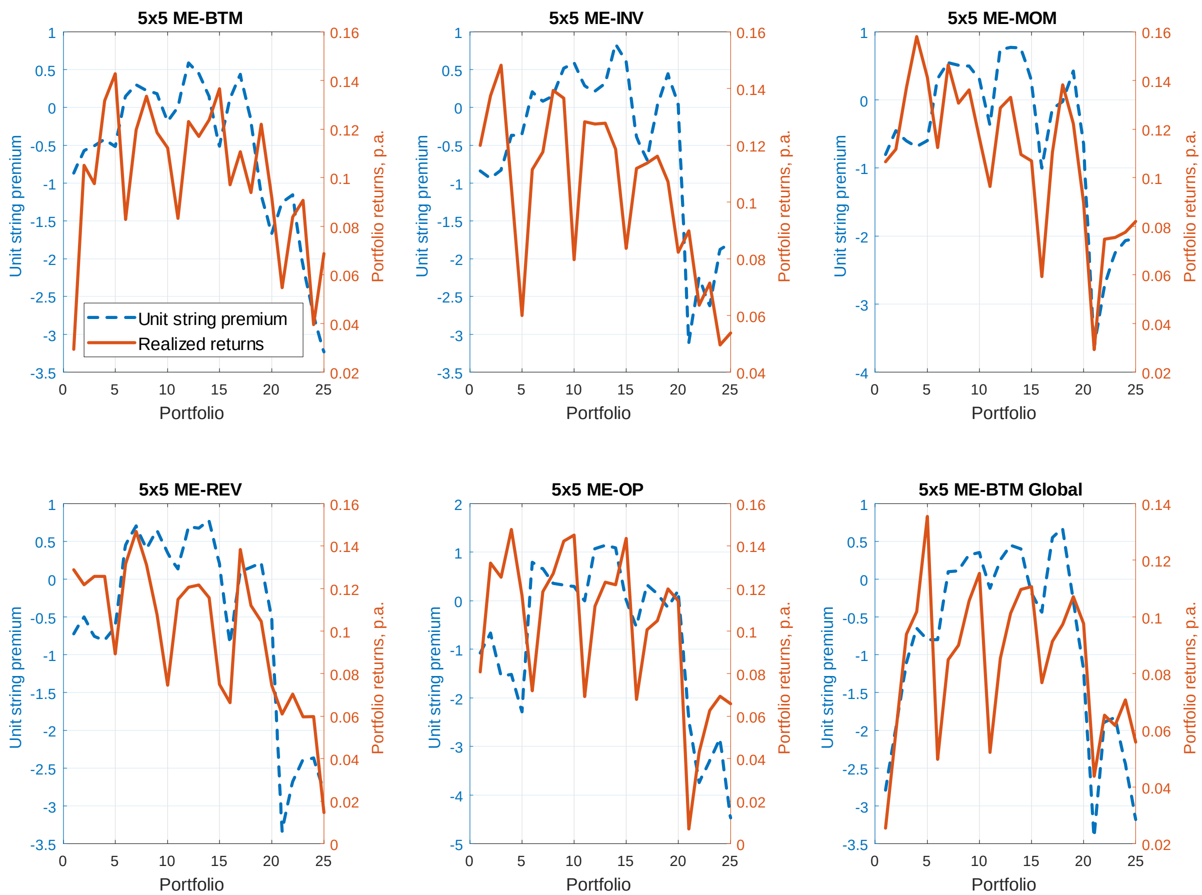

Figure 6. This picture depicts average excess returns and the unit risk premiums for each for each portfolio, with the latter estimated from Model II. The estimates are performed for 5x5 ME-BTM, 5x5 ME-INV, 5x5 ME-MOM, 5x5 ME-REV, 5x5 ME-OP, and 5x5 ME-BTM Global portfolios.

4.4. Correlation-risk premium

Next, we examine the model predictions on the correlation-risk premium. Proposition 3 (see Section 3) suggests a theoretical relation between realized correlation and correlation-risk premium. Given our parameter estimates, in Section 3 we explained that this relation is, statistically, roughly inverse for most of the time (see Figure 3). We calculate data counterparts to this relation. We approximate the correlation-risk premium with *realized* correlation-risk premium, defined as the difference between average correlations (implied and historical) over the last 22 days. We also compute the model-implied realized correlation-risk premium, estimating P-correlations through the average correlations $\rho(y_t; i, j)$ calculated over the last 22 days, and relying on the previously extracted y_t.

Figure 7 plots the results. The model predicts that the correlation-risk premium is statistically inversely related to realized correlations, as in the data. In terms of the explanations of Proposition 3 in Section 3, in bad times, when implied and realized correlations are both high, the correlation-risk premium decreases: implied correlations are obviously bounded and, then, a further increase in both correlations may translate into a decreasing difference between implied and realized correlations. Figure 7 shows that this effect is so strong to make the correlation-risk premium negatively related to realized correlations at any value for the realized correlations.

Because implied correlations are on average higher than realized, we might, then, also expect that implied correlations move less than one-to-one with realized correlations. It is indeed the case. Table 6 reports regression estimates that reveal this property both in the data and for the model. These properties seem to be in contrast with the empirical evidence in the equity volatility space, where volatility risk-premiums do actually increase in bad times (see Corradi, Distaso and Mele, 2013).[5]

	a	b	Adj-R^2
Data	0.1845	0.6209	52%
	(0.0030)	(0.0083)	
Model	0.3321	0.3015	72%
	(0.0011)	(0.0026)	

Table 6: This table provides estimates (with standard errors in parenthesis) for the coefficients a and b in the linear regression $\rho_Q = a + b\rho_P$, where ρ_Q is the one-month expected correlation for S&P 500 stocks under the risk-neutral probability, Q, and ρ_P is the one-month realized correlation. The figures in parenthesis are standard errors, and R^2 denotes the adjusted-R^2.

[5] Corradi, Distaso and Mele (2013) (Section 4.2.5) provide such evidence relying on ex-ante volatility risk-premiums, within a no-arbitrage model for equity volatility. In the interest rate volatility space, Mele, Obayashi and Shalen (2015) and Mele and Obayashi (2015) study some properties of the volatility risk-premium, without addressing the issue of premium sensitivity to market conditions.

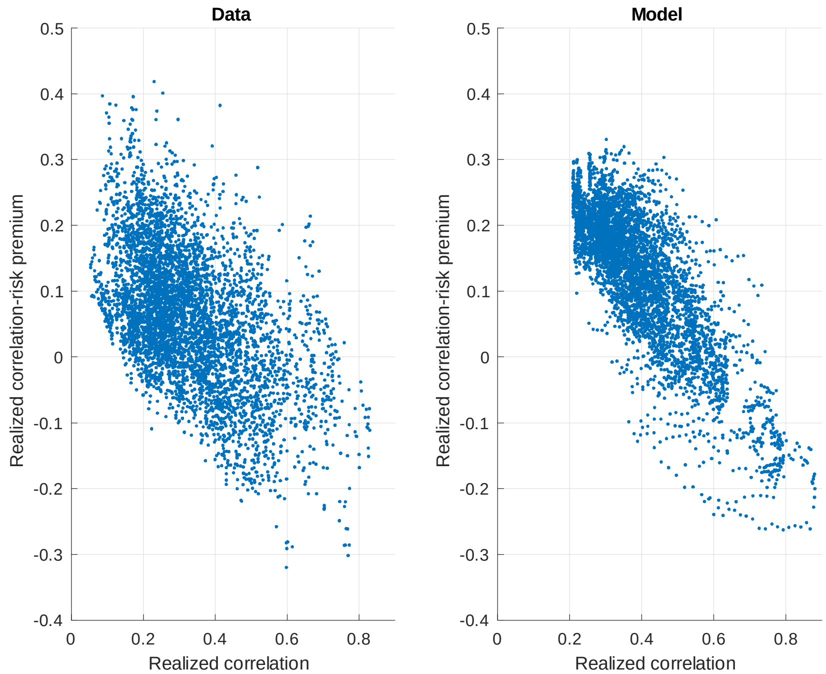

Figure 7. This picture depicts the realized correlation-risk premium for S&P 500 stocks against one-month realized correlations in the data (left panel) and predicted by the model (right panel).

Finally, we examine the model implications on the term structure of unconditional correlation-risk premiums. Our GMM in (23) contains a moment conditions that only regards the unconditional expectation of *one-month* correlation-risk premium. Yet our model allows us to consider any arbitrary horizon. Figure 8 plots the average correlation-risk premium estimated from data along with the expression for $E\left(\mathcal{P}\left(y_t;\boldsymbol{\vartheta}\right)\right)$ in (22), calculated with parameter values based on our GMM estimates. The model reproduces the upward sloping curve in the data and comes close to quantitatively match the unconditional correlation-risk premiums at all considered horizons.

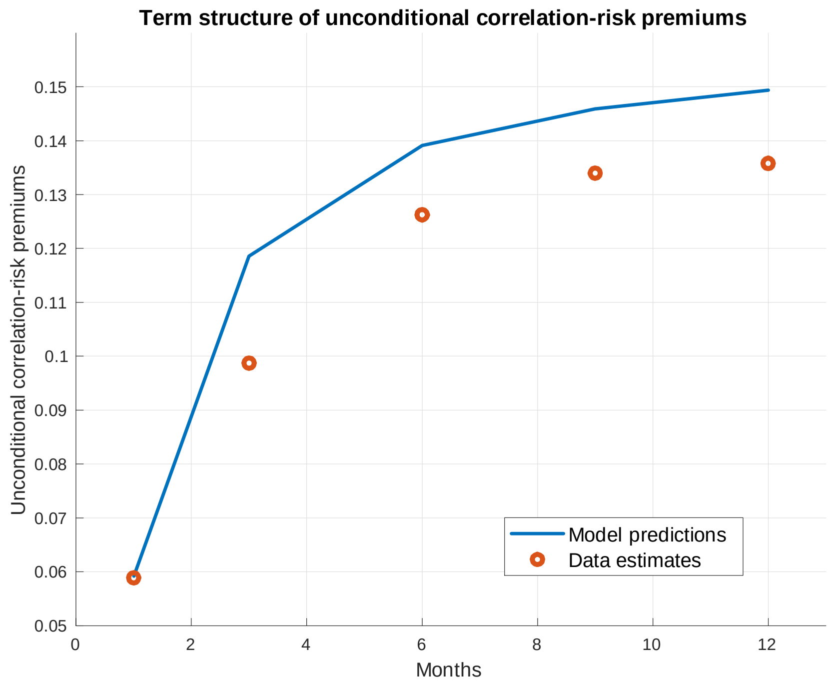

Figure 8. This picture depicts the unconditional correlation-risk premiums calculated for horizons equal to 1, 3, 6, 9, and 12 months. The circles are data estimates, computed as described in the main text. The solid curve depicts model predictions, obtained while fixing parameter values at the GMM estimates in (23), which rely on one moment condition based on one-month unconditional premium.

5. Conclusion

This paper introduces an arbitrage pricing model where asset returns are not driven by a pre-determined set of factors. Rather, in this model, asset returns are driven by the realization of a string, which, then, determines the asset returns co-movements and the whole set of correlations amongst asset returns. In this setup, "risk" is, thus, determined by the joint returns fluctuations in a given universe of securities, and the cross-section of equity returns reflect the exposures of any given asset price fluctuations to the fluctuations of the remaining asset prices. The cross-section of expected returns are simply these exposures, weighted through a common premium functional.

Within this theoretical framework, we specify a number of models that we may use in empirical work. We assume that the assets correlations in the string are random, and use the cross-section of options on individual S&P500 components, the S&P500 index, and the corresponding stock and index returns to extract information on the unobservable state underlying realized correlations at any given point in time. We develop method-of-moments conditions that we employ to estimate our model. With our estimates of the state, we reconstruct the dynamics of average correlations and correlation-risk premiums, and, naturally, the cross-section of expected returns that are predicted by the model. The model predicts the empirical patterns of correlation-risk premiums, but also explains cross-sectional pricing in a number of portfolios, both in the U.S. and in the international stock universe. Our framework offers opportunities for developing additional cross-sectional asset pricing while moving from a standard factor structure to a granular methodology, whereby the focus is to directly model and quantify risks that any individual portfolio may have in common with all others.

Appendices

A. Proofs

Proof of Proposition 2. Consider, first, the following preliminary result: for any given ℓ,

$$E_t\left(e^{-\ell y_T}\right)=\bar{a}_\ell\left(T-t\right)e^{-\bar{b}_\ell(T-t)\ell y_t}, \tag{A.1}$$

where

$$\bar{a}_\ell\left(x\right)=\left(\frac{2\kappa}{2\kappa+\ell\eta^2\left(1-e^{-\kappa x}\right)}\right)^{\frac{2\kappa m}{\eta^2}}, \quad \bar{b}_\ell\left(x\right)=\frac{2\kappa e^{-\kappa x}}{2\kappa+\ell\eta^2\left(1-e^{-\kappa x}\right)}.$$

Eq. (A.1) follows by a mere change in notation in a result to be stated below (see Eq. (A.5)). Taking the limits leaves

$$E\left(e^{-\ell y_t}\right)=\lim_{T\to\infty}E_t\left(e^{-\ell y_T}\right)=\bar{Y}_{(\ell)},$$

where $\bar{Y}_{(\ell)}$ is defined in the proposition. The expressions for the unconditional expected returns in Proposition 2 immediately follow.

Before providing the proof of Proposition 3, we prove a statement given in the main text regarding the dynamics of the state variable y_t under the risk-neutral probability.

Proposition A.1. (Dynamics of y under Q) *Consider two diffusion processes, x_{it}, $i=1,2$, solutions to Eq. (16), viz*

$$dx_{it}=\left(\kappa m-\left(\kappa+\nu_i\eta\right)x_{it}\right)dt+\eta\sqrt{x_{it}}d\tilde{W}_t,$$

where $\nu_1>\nu_2$. Then, $x_{1t}\leq x_{2t}$ a.s.

Proof. The drift of x_{1t} is strictly less than the drift of x_{2t}, and the proposition follows by a comparison theorem (e.g., Karatzas and Shreve (1991, p. 291-295)).

Proof of Proposition 3. We provide details regarding the function $w\left(y_t,T-t\right)\equiv u\left(y_t,\tau-t;\boldsymbol{\theta},0\right)=E_t\left(e^{-y_T}\right)$ in Eq. (18), as those regarding $E_t^Q\left(e^{-y_T}\right)$ follow through a change in notation. The function $w\left(y,T-\tau\right)$ satisfies the following partial differential equation

$$0=-w_2\left(y,T-\tau\right)+\kappa\left(m-y\right)w_1\left(y,T-\tau\right)+\frac{1}{2}\eta^2yw_{11}\left(y,T-\tau\right),\quad \text{for all } \tau\in[t,T),$$

where subscripts denote partial derivatives. The boundary condition is $w\left(y,0\right)=e^{-y}$. Conjecture that $w\left(y_t,T-t\right)=e^{\alpha(T-t)-b(T-t)y_t}$ and plug this suggested function into the previous partial differential equation. The result is that α and b satisfy the following ordinary differential equations: for all $x\in(0,T-t]$,

$$\begin{cases} 0=\dot{b}\left(x\right)+\kappa b\left(x\right)+\frac{1}{2}\eta^2b^2\left(x\right)\\ 0=\dot{\alpha}\left(x\right)+\kappa mb\left(x\right)\end{cases}$$

subject to the boundary conditions $\alpha(0) = 0$ and $b(0) = 1$. The solution for b and α follow by standard integration arguments and details are available upon request. Eq. (18) and, then, Eq. (17) follow by taking the exponential, $a = e^{\alpha}$, and noting that $\kappa m = \tilde{\kappa}\tilde{m}$.

Next, we show that, for $\nu > 0$, $\mathcal{P}$ is (i) strictly positive, (ii) increasing and concave in y for low y, and (iii) decreasing and convex in y for high y. (Note, also that the arguments below would equally go through if $\varrho_1 < 0$ and $\nu < 0$.)

The first property directly follows by Proposition A.1. However, we provide an alternative proof based on an argument that will be used to deal with the other proofs of the proposition. Note that the function $\Delta u(y, T-t) \equiv u(y, \tau - t; \boldsymbol{\theta}, \nu) - u(y, \tau - t; \boldsymbol{\theta}, 0)$ is solution to the following partial differential equation

$$0 = \mathcal{L}\Delta u(y, T-\tau) - \nu\eta y u_1(y, T-\tau), \quad \text{for all } \tau \in [t, T), \tag{A.2}$$

where $\mathcal{L}f(y, T-t) = \frac{\partial}{\partial t} f(y, T-t) + \kappa(m-y) \frac{\partial}{\partial y} f(y, T-t) + \frac{1}{2}\eta^2 y \frac{\partial^2}{\partial y^2} f(y, T-t)$, and subject to the boundary condition $\Delta u(y, 0) = 0$. Therefore, by the maximum principle for partial differential equations, we have that the sign of $\Delta u(y, T-t)$ is the same as the sign of $-\nu\eta y u_1(y, \tau - t)$. Since $u(y, \tau - t)$ is strictly decreasing in y for any finite T, it follows that $\Delta u(y, T-t)$ is strictly positive, and so is $\mathcal{P}$.

Regarding the second property (increasing and concave for low y) and the third (decreasing and convex for high y), differentiate Eq. (A.2) two times with respect to y, and denote with $\Delta u_1(y, T-t)$ and $\Delta u_{11}(y, T-t)$ the first and the second partial of $\Delta u(y, T-t)$ with respect to y. The result is that $\Delta u_1(y, T-t)$ and $\Delta u_{11}(y, T-t)$ are solutions to the following partial differential equations

$$0 = \mathcal{L}\Delta u_1(y, T-\tau) + \nu\eta b(T-\tau) u(y, T-\tau)(1 - b(T-\tau) y), \quad \text{for all } \tau \in [t, T), \tag{A.3}$$

and

$$0 = \mathcal{L}\Delta u_{11}(y, T-\tau) - \nu\eta b^2(T-\tau) u(y, T-\tau)(2 - b(T-\tau) y), \quad \text{for all } \tau \in [t, T), \tag{A.4}$$

subject to the boundary conditions $\Delta u_1(y, 0) = 0$ and $\Delta u_{11}(y, 0) = 0$.

Eq. (A.3) can be rearranged to yield

$$\begin{aligned}
\Delta u_1(y_t, T-t) &= \nu\eta \int_t^T b(T-\tau) E_t\left[u(y_\tau, T-\tau)(1 - b(T-\tau)) y_\tau\right] d\tau \\
&= \nu\eta E_t\left[u(y_\tau, T-\tau)\right] \int_t^T b(T-\tau) E_t^*\left[1 - b(T-\tau) y_\tau\right] d\tau \\
&= \nu\eta E_t\left[u(y_\tau, T-\tau)\right] \int_t^T b(T-\tau)\left[1 - b(T-\tau) E_t^*(y_\tau)\right] d\tau,
\end{aligned}$$

where $E_t^*(\cdot)$ denotes the expectation is taken under the probability P^*, defined as

$$\left.\frac{dP^*}{dP}\right|_{\mathbb{F}_\tau} = \frac{u(y_\tau, T-\tau)}{E_t\left[u(y_\tau, T-\tau)\right]}.$$

By the no-crossing property of a diffusion, the expectation $E_t^*(y_\tau)$ is increasing in the initial condition y_t and, thus, there exists a threshold y_A (resp., y_B) such that for all $y_t < y_A$ (resp., $y_t > y_B$), $\Delta u_1(y_t, T-t)$ is positive (resp., negative). Based on Eq. (A.4), we can make a similar argument and conclude that there exists a threshold y_C (resp., y_D) such that for all $y_t < y_C$ (resp., $y_t > y_D$), $\Delta u_{11}(y_t, T-t)$ is negative (resp., positive).

Proof of Proposition 4. The n-th conditional moment of $\rho(y_T;\boldsymbol{\varrho})$ is

$$\begin{aligned} E_t\left(\rho^n\left(y_T;\boldsymbol{\varrho}\right)\right) &= E_t\left(\varrho_0+\varrho_1 e^{-y_T}\right)^n \\ &= E_t\left(\sum_{i=0}^{n}\binom{n}{i}\varrho_0^i\varrho_1^{n-i}e^{-(n-i)y_T}\right) \\ &= \sum_{i=0}^{n}\binom{n}{i}\varrho_0^i\varrho_1^{n-i}E_t\left(e^{-(n-i)y_T}\right), \end{aligned}$$

where the second line follows by the binomial formula. Now, by Itô's lemma, $z_{i,t}\equiv(n-i)\,y_t$ is solution to

$$dz_{i,t}=\kappa\left(m_i-z_{i,t}\right)dt+\eta_i\sqrt{z_{i,t}}dW_t,$$

where $m_i=(n-i)\,m$ and $\eta_i=\sqrt{n-i}\eta$. Therefore, by the expression for the conditional expectation of e^{-y_T} in Proposition 4,

$$E_t\left(e^{-(n-i)y_T}\right)=a_i\left(T-t\right)e^{-b_i(T-t)(n-i)y_t}, \tag{A.5}$$

where, and using the fact that $m_i/\eta_i^2=m/\eta^2$,

$$a_i\left(x\right)=\left(\frac{2\kappa}{2\kappa+\eta_i^2\left(1-e^{-\kappa x}\right)}\right)^{\frac{2\kappa m}{\eta^2}},\quad b_i\left(x\right)=\frac{2\kappa e^{-\kappa x}}{2\kappa+\eta_i^2\left(1-e^{-\kappa x}\right)}.$$

Eq. (20) follows by taking the limit $E\left(\rho^n\left(y_t;\boldsymbol{\varrho}\right)\right)=\lim_{T\to\infty}E_t\left(\rho^n\left(y_T;\boldsymbol{\varrho}\right)\right)$.

Next, we determine the following unconditional uncentered covariance

$$c\rho_\infty^\Delta\equiv\lim_{T\to\infty}E_t\left(\rho\left(y_T;\boldsymbol{\varrho}\right)\rho\left(y_{T+\Delta};\boldsymbol{\varrho}\right)\right). \tag{A.6}$$

We have

$$E_t\left(\rho\left(y_T;\boldsymbol{\varrho}\right)\rho\left(y_{T+\Delta};\boldsymbol{\varrho}\right)\right)=\varrho_0^2+\varrho_0\varrho_1\left(E_t\left(e^{-y_T}\right)+E_t\left(e^{-y_{T+\Delta}}\right)\right)+\varrho_1^2E_t\left(e^{-(y_T+y_{T+\Delta})}\right).$$

By the Law of Iterated Expectations, and the expression for the conditional expectation of e^{-y_T} in Proposition 3,

$$E_t\left(e^{-(y_T+y_{T+\Delta})}\right)=E_t\left(e^{-y_T}E_T\left(e^{-y_{T+\Delta}}\right)\right)=a_\Delta E_t\left(e^{-(1+b_\Delta)y_T}\right),$$

where $a_\Delta=a\left(\Delta;0\right)$ and $b_\Delta=b\left(\Delta;0\right)$ and $a\left(x;\nu\right)$ and $a\left(x;\nu\right)$ are as in Eqs. (18) of Proposition 3. Applying again the expression for the conditional expectation of $e^{-y_{T+\Delta}}$ in Proposition 4 and relying on arguments nearly identical to those used to derive the conditional moment in Eq. (A.5),

$$E_t\left(e^{-(1+b_\Delta)y_T}\right)=a^\Delta\left(T-t\right)e^{-b^\Delta(T-t)(1+b_\Delta)y_t},$$

where

$$a^\Delta\left(x\right)=\left(\frac{2\kappa}{2\kappa+\left(1+b_\Delta\right)\eta^2\left(1-e^{-\kappa x}\right)}\right)^{\frac{2\kappa m}{\eta^2}},\quad b^\Delta\left(x\right)=\frac{2\kappa e^{-\kappa x}}{2\kappa+\left(1+b_\Delta\right)\eta^2\left(1-e^{-\kappa x}\right)}.$$

Hence,

$$E_t\left(e^{-(y_T+y_{T+\Delta})}\right)=a_\Delta a^\Delta(T-t)\,e^{-b^\Delta(T-t)(1+b_\Delta)y_t}.$$

Therefore, the limit in (A.6) is obtained as

$$\begin{aligned}c\rho_\infty^\Delta&=\varrho_0^2+2\varrho_0\varrho_1\lim_{x\to\infty}a(x;\nu)+\varrho_1^2a_\Delta\lim_{x\to\infty}a^\Delta(x)\\&=\varrho_0^2+2\varrho_0\varrho_1\left(\frac{2\kappa}{2\kappa+\eta^2}\right)^{\frac{2\kappa m}{\eta^2}}+\varrho_1^2\left(\frac{4\kappa^2}{4\kappa^2+4\kappa\eta^2+\eta^4\left(1-e^{-\kappa\Delta}\right)}\right)^{\frac{2\kappa m}{\eta^2}}.\end{aligned}$$

Eq. (21) follows by rearranging terms in

$$cov\left(\rho\left(y_t;\boldsymbol{\varrho}\right),\rho\left(y_{t+\Delta};\boldsymbol{\varrho}\right)\right)=c\rho_\infty^\Delta-E\left(\rho\left(y_t;\boldsymbol{\varrho}\right)\right)^2,$$

where the expression for $E\left(\rho\left(y_t;\boldsymbol{\varrho}\right)\right)$ is obtained through Eq. (20) of the proposition.

Proof of Eq. (22). We have, for $l>t$, and for fixed $\Delta t\equiv T-t$,

$$E\left(\mathcal{P}\left(y_t;\boldsymbol{\vartheta}\right)\right)=\lim_{l\to\infty}E_t\left(\mathcal{P}\left(y_l;\boldsymbol{\vartheta}\right)\right)=\frac{\varrho_1}{\Delta t}\int_0^{\Delta t}\lim_{l\to\infty}E_t\left(u\left(y_l,x;\boldsymbol{\theta},\nu\right)-u\left(y_l,x;\boldsymbol{\theta},0\right)\right)dx. \tag{A.7}$$

By Proposition 4, and arguments similar to those leading to Eq. (A.5),

$$E_t\left(u\left(y_l,x;\boldsymbol{\theta},\nu\right)\right)=a(x;\nu)\,E_t\left(e^{-b(x;\nu)y_l}\right)=a(x;\nu)\,a_B(l-t;\nu)\,e^{-b_B(l-t;\nu)b(x;\nu)y_t},$$

where

$$a_B(l-t;\nu)\equiv\left(\frac{2\kappa}{2\kappa+b(x;\nu)\,\eta^2\left(1-e^{-\kappa(l-t)}\right)}\right)^{\frac{2\kappa m}{\eta^2}},\quad b_B(l-t;\nu)\equiv\frac{2\kappa e^{-\kappa(l-t)}}{2\kappa+b(x;\nu)\,\eta^2\left(1-e^{-\kappa(l-t)}\right)}.$$

Eq. (22) follows by calculating the limits in (A.7), using the definition of $a(x;\nu)$ and $b(x;\nu)$ in Proposition 4, and rearranging terms.

B. Extensions

B.1. A string-and-factor model of asset returns

We extend the model in Section 2 to a market in which asset returns are strings, but they are also affected by systematic factors driven by Brownian motions, assuming that

$$\frac{dP_t(i)+D_t(i)\,dt}{P_t(i)}=\mathcal{E}\left(\mathbf{y}_t,i\right)dt+\sigma\left(\mathbf{y}_t,i\right)dZ_t(i)+\sigma_M\left(\mathbf{y}_t,i\right)d\mathbf{W}_t,\quad i\in(0,1), \tag{B.1}$$

where $\mathbf{W}_t$ is a standard multidimensional Brownian motion, $\sigma_M(\mathbf{y},i)$, is a continuous function, in $\mathbf{y}$ and i, and represents the asset returns exposures to the systematic factors, the “betas.” Remaining notation is as in Eq. (1).

The pricing kernel is still as in Eq. (2), such that repeating the arguments leading to Proposition 3, but relying on Eq. (B.1), leaves the following expression for the expected excess returns on any asset-$i \in (0,1)$

$$\mathcal{E}(\mathbf{y}_t, i) - r(\mathbf{y}_t) = \mathcal{C}(\mathbf{y}_t, i) + \sigma_M(\mathbf{y}_t, i)\,\boldsymbol{\lambda}(\mathbf{y}_t), \tag{B.2}$$

where $\mathcal{C}(\mathbf{y}, i)$ is as in Eq. (6). Compared to Proposition 1, this formulation adds a standard factor-risk premium to the explanation of the cross-section of asset returns, $\sigma_M(\mathbf{y}, i)\,\boldsymbol{\lambda}(\mathbf{y})$.

B.2. Compound strings

We consider the following extension to Eq. (1)

$$\frac{dP_t(i) + D_t(i)\,dt}{P_t(i)} = \mathcal{E}_t(\mathbf{y}_t, i)\,dt + \sigma(\mathbf{y}_t, i)\,dZ_t(i) + w(\mathbf{y}_t, i)\,dZ_t(\mathbf{Z}, \mathbf{y}_t), \tag{B.3}$$

where

$$dZ_t(\mathbf{Z}, \mathbf{y}_t) = \int_0^1 n(\mathbf{y}_t, j)\,dZ_t(j)\,dj,$$

for some functions $w(\mathbf{y}_t, i)$ and $n(\mathbf{y}_t, i)$. The additional term, $dZ_t(\mathbf{Z}, \mathbf{y}_t)$, is a linear functional of the whole string, and will be referred to as *compound string* in the sequel.

This extension accounts for economies in which each asset return reacts to shocks in the fundamentals pertaining to all remaining asset returns, that is, not only to "its own" string $dZ_t(i)$, but also to $dZ_t(j)$ for all j, directly. For example, in the illustrative model of Appendix B.3, each asset return is driven by a shock on its dividend and, due to market clearing, on those affecting all the dividend shares (i.e., the proportions of aggregate dividends paid by each asset), leading to price dynamics that are a special case of Eq. (B.3).

By arguments similar to those leading to Proposition 1, the expected excess returns on each asset are now given by

$$\begin{aligned}\mathcal{E}_t(\mathbf{y}_t, i) - r(\mathbf{y}_t) = \sigma(\mathbf{y}_t, i) &\int_0^1 \phi(\mathbf{y}_t, j)\,\rho(\mathbf{y}_t, i, j)\,dj \\ &+ w(\mathbf{y}_t, i) \iint_{u,v\in[0,1]^2} \phi(\mathbf{y}_t, u)\,n(\mathbf{y}_t, v)\,\rho(\mathbf{y}_t, u, v)\,du dv.\end{aligned} \tag{B.4}$$

The first term on the R.H.S. of Eq. (B.4) is the expected return predicted by Proposition 1. The second term captures the premium due to the compound string in Eq. (B.3). In our empirical work, we rely on the simple specification of the model that gives rise to Proposition 1. However, we now provide an example of a Consumption-based CAPM that leads to the assumptions underlying the predictions of both Proposition 1 and Eq. (B.4).

B.3. Example: a consumption-based CAPM

We consider an infinite horizon economy with a continuum of long-lived securities in $i \in (0,1)$. Each of these securities delivers an instantaneous dividend $D_t(i)$ at time-t, solution to

$$\frac{dD_t(i)}{D_t(i)} = g_t(i)\,dt + \sigma_{dt}(i)\,dZ_t(i), \tag{B.5}$$

where $dZ_t(i)$ is a string, and $g_t(i)$ and $\sigma_{dt}(i)$ are some functions described below. We assume that there is a single agent with instantaneous utility and constant relative risk aversion equal to γ, and subjective discount rate equal to δ. The model may well be extended throughout more general specifications of preferences, including habit formation.

We describe: (i) aggregate consumption and dividend shares, and, based on standard assumptions on the representative agent's preferences, the pricing kernel; (ii) volatilities, correlations and the cross-section of expected returns.

Aggregate consumption and pricing kernel. Denote the aggregate dividends with $D_t \equiv \int_0^1 D_t(i)\,di$, which satisfy

$$\frac{dD_t}{D_t} = \left(\int_0^1 g_t(i)\,s_t(i)\,di\right)dt + \int_0^1 \sigma_{dt}(i)\,s_t(i)\,dZ_t(i)\,di, \tag{B.6}$$

where $s_t(i) \equiv \frac{D_t(i)}{D_t}$ denotes the "dividend share" of asset-i. In equilibrium, aggregate dividends equal aggregate consumption, C_t say. Below, we show that

$$\frac{ds_t(i)}{s_t(i)} = \mu_t^s(i)\,dt + \sigma_{dt}(i)\,dZ_t(i) - \int_0^1 \sigma_{dt}(j)\,s_t(j)\,dZ_t(j)\,dj, \tag{B.7}$$

for some drift coefficient $\mu_t^s(i)$.

The stochastic discounting factor ξ_t is, thus, $\xi_t = e^{-\delta t}C_t^{-\gamma}$ and satisfies

$$\frac{d\xi_t}{\xi_t} = -r dt - \int_0^1 \phi(s_t(j))\,dZ_t(j)\,dj, \quad \phi(s_t(j)) = \gamma\sigma_{dt}(j)\,s_t(j)\,. \tag{B.8}$$

Note that $\phi(s_t(j))$ in Eq. (B.8) is the compensation received for holding any asset that is exposed to co-movements with the dividends paid by a given asset-j. Eq. (B.8) predicts that this compensation increases with the relative weight of asset-j in the economy, $s_t(j)$.

Asset returns. Asset prices do in general depend on the realization of the whole share process, a complication well understood since previous work on consumption based models (see, e.g., Menzly, Santos and Veronesi, 2004; MSV, in the sequel). We make a few simplifying assumptions to render the model analytically tractable. We assume that the representative agent has log-utility, $\gamma = 1$, and that the drift of each share process in (B.7) is linear in $s_t(i)$. These assumptions lead to an affine model for the price-dividend ratio, $p(s_t(i))$ say, similar as in MSV. Asset returns are, then, shown to equal

$$\frac{dP_t(i) + D_t(i)\,dt}{P_t(i)} = \mathcal{E}_t(i)\,dt + \underbrace{\sigma_{dt}(i)\left(1 + \frac{p'(s_t(i))}{p(s_t(i))}s_t(i)\right)}_{\equiv\sigma_t(i)}dZ_t(i) - \underbrace{\frac{p'(s_t(i))}{p(s_t(i))}s_t(i)}_{\equiv -w_t(i)}dZ_t(\mathbf{Z}, \mathbf{s}_t)\,, \tag{B.9}$$

where $dZ_t(\mathbf{Z}, \mathbf{s}_t) \equiv \int_0^1 \sigma_{dt}(j)\,s_t(j)\,dZ_t(j)\,dj$.

The volatility of asset-i returns has two components. The first is the volatility of the asset dividend growth, $\sigma_{dt}(i)$. The second stems from fluctuations in its price-dividend ratio, $p(s_t(i))$, which, in turn, originate from those in the dividend share, $s_t(i)$: the higher the semi-elasticity of $p(s_t(i))$, the more significant is this second source of volatility. The term $\sigma_t(i)$ reflects both dividend volatility and price-dividend ratio volatility. Instead, $w_t(i)$ only reflects price-dividend volatility.

Eq. (B.9) is then a special case of Eq. (B.3): the state vector is $\mathbf{s}_t = \mathbf{y}_t$, the compound string is $dZ_t(\mathbf{Z}, \mathbf{s}_t)$, and all other coefficients are independent of $\mathbf{s}_t$, with $n_t(i) = \sigma_{dt}(i) s_t(i)$. That is, while each asset return depends on its own share process, each share process is driven by the realization of the whole string (see Eq. (B.7)). Therefore, in equilibrium, asset returns are also driven by a compound string.

Expected returns on each asset, $\mathcal{E}_t(i)$, can now be determined through correlations and volatility, based on Eq. (B.4), which collapses to Eq. (B.9) in the case of the model in this appendix. These details are in Proposition B.1 below, and in its proof. First, we provide details regarding the price-dividend ratios in this economy given the assumptions formulated so far as well as additional ones.

Price-dividend ratios. The price-dividend ratio on each asset is

$$p(\mathbf{s}_t, i) \equiv \frac{P_t(i)}{D_t(i)} = E_t\left(\int_t^\infty \frac{\xi_u}{\xi_t}\frac{D_u}{D_t}\frac{s_u(i)}{s_t(i)}du\right),$$

where $\mathbf{s}_t = (s_t(i))_{i\in(0,1)}$ denotes the collection of all the share processes: the price-dividend ratio of any asset depends on the future paths of aggregate dividends, which, in turn, depend on all the shares process. This dimensionality problem simplifies when $\gamma = 1$, in which case, the price-dividend ratio on asset-i only depends on the asset relative share. Under the additional assumption that, in (B.7), $\mu_t^s(i) s_t(i) = \beta(\bar{s}_i - s_t(i))$, for some $(\bar{s}_i)_{i\in(0,1)}$ and β, we have that $p(s_t(i)) \equiv p(\mathbf{s}_t, i)$, where

$$p(s_t(i)) = \frac{1}{\delta+\beta} + \frac{\beta}{\delta(\delta+\beta)}\frac{\bar{s}_i}{s_t(i)}. \tag{B.10}$$

The constants $(\bar{s}_i)_{i\in(0,1)}$ satisfy $\int_0^1 \bar{s}_j dj = 1$, and β is constant in time and across assets, such that the shares sum up to one for all t. Note that the price-dividend ratio has the same functional form as in MSV. However, the model implications on the correlation of asset returns and the cross-section of expected returns are distinct, as we now explain.

We have:

Proposition B.1. (Correlation and expected returns) *We have*

$$\mathcal{E}_t(\mathbf{s}_t, i) - r(\mathbf{s}_t) = \sigma_{ct}^2 + \frac{1}{1+\frac{\beta}{\delta}\frac{\bar{s}_i}{s_t(i)}}\int_0^1 \sigma_{dt}(j) s_t(j) \rho(s_i, d_j) dj, \tag{B.11}$$

where σ_{ct}^2 denotes the instantaneous variance of aggregate consumption, which, in equilibrium, equals $\sigma_{dt}^2 \equiv var_t\left(\frac{dD_t}{D_t}\right)$, and

$$\rho(s_i, d_j) \equiv cov\left(\frac{ds_t(i)}{s_t(i)}, \frac{dD_t(j)}{D_t(j)}\right) = \sigma_{dt}(i)\rho(i,j) - \int_0^1 \rho(j,u)\sigma_{dt}(u) s_t(u) du.$$

The first term on the R.H.S. of Eq. (B.11), σ_{dt}^2, is the standard single Lucas' tree prediction. The second term can take either sign. For any asset-i such that the values of $\rho(s_i, d_j)$ across j make this second term positive, the expected excess returns are increasing in $s_t(i)$. Intuitively, asset-i is not a good hedge if its share is positively correlated with a sufficiently large set of the assets' dividends—for example, if $\rho(s_i, d_j)$ is positive for all dividends j. In this case, the expected return on asset-i is increasing in $s_t(i)$, as this asset pays a larger portion of consumption. This conclusion is reversed when $\rho(s_i, d_j)$ is such that the second term in the R.H.S. in (B.11) is negative.

These predictions are peculiar to this model, due to our granular account of the asset returns. In our model, returns and volatility are clearly disentangled: by Itô's lemma, the dynamics of the price-dividend ratio for any asset-i is

$$\frac{dp(s_t(i))}{p(s_t(i))} = (\cdots)\,dt + \frac{p'(s_t(i))}{p(s_t(i))} s_t(i) \left(\sigma_{dt}(i)\,dZ_t(i) - \int_0^1 \sigma_{dt}(j)\,s_t(j)\,dZ_t(j)\,dj\right),$$

such that the correlation of the price dividend ratios on any two assets i and j is

$$E\left(\frac{dp(s_t(i))}{p(s_t(i))}\frac{dp(s_t(j))}{p(s_t(j))}\right) = \frac{p'(s_t(i))}{p(s_t(i))}\frac{p'(s_t(j))}{p(s_t(j))} s_t(i)\,s_t(j)\,cov_{s_i s_j},$$

where

$$\begin{aligned} cov_{s_i s_j} &\equiv cov\left(\frac{ds_t(i)}{s_t(i)}, \frac{ds_t(j)}{s_t(j)}\right) \\ &= cov\left(\sigma_{dt}(i)\,dZ_t(i) - dZ_t(\mathbf{Z}, \mathbf{s}_t), \sigma_{dt}(j)\,dZ_t(j) - dZ_t(\mathbf{Z}, \mathbf{s}_t)\right) \\ &= \sigma_{dt}^2 - \sigma_{dt}(i)\,\sigma_{dt}(j)\,\rho(i,j) + \sigma_{dt}(j)\,\rho(s_i, d_j) + \sigma_{dt}(i)\,\rho(s_j, d_i). \end{aligned}$$

Moreover, expected returns in Eq. (B.11) are determined by how all shares correlate with aggregate consumption, but with all the asset dividends weighted by the relative shares.

We now provide proofs of two results stated in this appendix.

Proof of Eq. (B.7). By Itô's lemma, we have that $s_t(i) \equiv \frac{D_t(i)}{D_t}$ satisfies

$$\frac{ds_t(i)}{s_t(i)} = \frac{dD_t(i)}{D_t(i)} - \frac{dD_t}{D_t} + \left(\frac{dD_t}{D_t}\right)^2 - \frac{dD_t(i)}{D_t(i)}\frac{dD_t}{D_t}.$$

Using Eq. (B.5) and Eq. (B.6) leaves Eq. (B.7), with

$$\mu_t^s(i) = g_t(i) - \int_0^1 g_t(i)\,s_t(i)\,di + var_{dt} - cov_{d_i,d,t},$$

where

$$\sigma_{dt}^2 \equiv \iint_{i,j\in[0,1]^2} \sigma_{dt}(i)\,s_t(i)\,\rho(i,j)\,\sigma_{dt}(j)\,s_t(j)\,didj, \quad cov_{d_i d,t} \equiv \sigma_{dt}(i)\int_0^1 \sigma_{dt}(j)\,s_t(j)\,\rho(i,j)\,dj.$$

Proof of Proposition B.1. By Eq. (B.4), and the expression for the unit prices of risk, $\phi(s_t(j)) = \sigma_{dt}(j)\,s_t(j)$, the cross-section of expected excess returns is

$$\begin{aligned} \mathcal{E}_t(\mathbf{s}_t, i) - r(\mathbf{s}_t) &= \sigma_t(i)\int_0^1 \sigma_{dt}(j)\,s_t(j)\,\rho(i,j)\,dj \\ &\quad + w_t(i)\underbrace{\iint_{u,v\in[0,1]^2} \sigma_{dt}(u)\,s_t(u)\,n_t(v)\,\rho(u,v)\,dudv}_{=\sigma_{dt}^2}, \end{aligned} \tag{B.12}$$

where the term indicated in the brackets coincides with σ_{dt}^2 due to the expression of $n_t(v)$ given in the main text. Replacing the expressions for $\sigma_t(i)$ in the main text into (B.12), leaves

$$
\begin{aligned}
&\mathcal{E}_t(\mathbf{s}_t, i) - r(\mathbf{s}_t) \\
&= (1 - w_t(i))\,\sigma_{dt}(i) \int_0^1 \sigma_{dt}(j)\, s_t(j)\, \rho(i,j)\, dj + (1 + w_t(i) - 1)\,\sigma_{dt}^2 \\
&= \sigma_{dt}^2 + (1 - w_t(i)) \left(\sigma_{dt}(i) \int_0^1 \sigma_{dt}(j)\, s_t(j)\, \rho(i,j)\, dj - \sigma_{dt}^2 \right) \\
&= \sigma_{dt}^2 + (1 - w_t(i)) \left(\int_0^1 \sigma_{dt}(j)\, s_t(j) \left(\sigma_{dt}(i)\, \rho(i,j) - \int_0^1 \rho(j,u)\, \sigma_{dt}(u)\, s_t(u)\, du \right) dj \right),
\end{aligned}
$$

where the last line follows by the expression for σ_{dt}^2. Eq. (B.11) follows by the definition of $w_t(i)$ and by a direct calculation.

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