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SOCIAL MEDIA AND POLARIZATION

Abstract

Because of its impacts on democracy, there is an important debate on whether the recent trends towards greater use of social media increases or decreases (political) polarization. One challenge for understanding this issue is how social media affects the equilibrium prevalence of different types of media content. We address this issue by developing a model of a social media network where there are two types of news content: mass-market (mainstream news) and niche-market (biased or more "extreme" news) and two different types of individuals who have a preference for recommending one or other type of content. We find that social media will amplify the prevalence of mass-market content and may result in it being the only type of content consumed. Further, we find that greater connectivity and homophily in the social media network will concurrently increase the prevalence of the niche market content and polarization. We then study an extension where there are two lobbying agents that can and wish to influence the prevalence of each type of content. We find that the lobbying agent in favor of the niche content will invest more in lobbying activities. We also show that lobbying activity will tend to increase polarization, and that this effect is greatest in settings where polarization would be small absent of lobbying activity. Finally, we allow individuals to choose the degree of homophily amongst their connections and demonstrate that niche-market individuals exhibit greater homophily than the mass-market ones, and contribute more to polarization.

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Social Media and Polarization*

Arthur Campbell† C. Matthew Leister‡ Yves Zenou§

July 13, 2019

Abstract

Because of its impacts on democracy, there is an important debate on whether the recent trends towards greater use of social media increases or decreases (political) polarization. One challenge for understanding this issue is how social media affects the equilibrium prevalence of different types of media content. We address this issue by developing a model of a social media network where there are two types of news content: mass-market (mainstream news) and niche-market (biased or more "extreme" news) and two different types of individuals who have a preference for recommending one or other type of content. We find that social media will amplify the prevalence of mass-market content and may result in it being the only type of content consumed. Further, we find that greater connectivity and homophily in the social media network will concurrently increase the prevalence of the niche market content and polarization. We then study an extension where there are two lobbying agents that can and wish to influence the prevalence of each type of content. We find that the lobbying agent in favor of the niche content will invest more in lobbying activities. We also show that lobbying activity will tend to increase polarization, and that this effect is greatest in settings where polarization would be small absent of lobbying activity. Finally, we allow individuals to choose the degree of homophily amongst their connections and demonstrate that niche-market individuals exhibit greater homophily than the mass-market ones, and contribute more to polarization.

Keywords: Social Networks, polarization, homophily, information transmission, lobbying.

JEL Classification: D83, D85, L82.

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1 Introduction

The recent rise of populism and political polarization has raised concerns about how political opinions are formed and their impact on democracy. This has led to many research papers investigating the role of social media[1] on the formation of political beliefs. Many authors attribute political polarization to the rise of social media (see e.g., Mutz, 2006; Hindman, 2008; Pariser, 2011; Sunstein, 2009, 2018) because it creates "echo chambers" whereby an individual and their friends only consume media that conforms to a particular perspective, and argue that social media habits re-enforce this phenomenon (see e.g., El-Bermawy 2016; Sambrook, 2016; Allcott and Gentzkow, 2017; Allcott et al., 2019). There is a more recent literature that argues that social media does not necessarily increase polarization (Boxell et al., 2017) but, on the contrary, can reduce it (Barberá, 2015; Dubois and Blank, 2018; Algan et al., 2019) because it exposes users to different opinions and thus leads to less narrow political views. This is because social media (such as Twitter or Facebook) encompasses a much larger network than just direct friends, so that individuals are exposed to alternative political views through their weak ties and are thus less likely to hold extreme political opinions. Both views seem reasonable and have empirical support.

A challenge for the empirical literature is understanding how social media usage affects the equilibrium prevalence of different types of media content. The empirical literature has typically compared outcomes between groups of individuals that experience different levels of social media usage/exposure. The difficulty is that a population's usage of social media determines the equilibrium prevalence of various types of media content, and this is itself an important determinant of polarization. The aim of this paper is to provide a simple model that examines the impact of social media networks on the consumption of news content that accounts for these equilibrium effects. In particular, the model examines how social media affects individual level consumption patterns and how it may amplify or mitigate differences amongst groups in a population. With our results, we provide an exact mechanism explaining how social media increases or decreases polarization in news content consumption.

We develop a dynamic model in which individuals can be of two types and must choose what news to forward to their friends. They can either be of a mass-market type (type M) or of a niche-market type (type N). Mass-market individuals have a preference for recommending mainstream news while niche-market individuals prefer recommending more specific and/or biased news. For example, mainstream news can correspond to views that are in accordance with the majority of the population (e.g., moderate republican or

[1]One of the most striking consumer trends over the last two decades has been the changing nature of how individuals access and consume media. Individuals are shifting their consumption habits away from traditional media (television, newspapers, radio) towards internet sources accessed through personal computing devices (smart phones and personal computers). For example, in recent surveys by the Pew Research Center (see Bialik and Eva Matsa 2017), in 2017, 43 % of U.S. adults report often getting news online (up from 38% in 2016) compared to 50 % on television (down from 57 % in 2016), 25 % on radio (25 % in 2016) and 18 % from print newspapers (down from 20 % in 2016). Furthermore, two thirds get at least some news on social media and social media is as common a pathway to get news as a news organizations app or website. Finally, when news is sent directly from a family or friend via email or text message it is the most likely to lead to a follow up action (Shearer and Gottfried, 2017). These changes in the way people access media has led to an increased number of providers of media and to new technologies for how individuals search and/or find out about media, in particular, the advent of social media via online platforms (e.g. Facebook, Twitter, WhatsApp, YouTube) whereby content is recommended by individuals to their friends.

democratic views) while a niche-market news could be conspiracy-theory or ideologically extreme views.

In our model, social media is captured by the social network[2] of each individual. Indeed, at each period of time, each individual receives news (which could be either of type M or N) from k friends and then recommends news that is of the same type as herself in the following period if she has received at least one recommendation of her type of news from one of her k friends. If all her k friends have recommended news from the other type, then she will recommend this type of news.[3] There is homophily captured by the parameter α, which is the probability that, for each of her k friends, the individual draws someone from the same type. The way the network is modeled here is similar to a platform such as Twitter. Every day, each individual receives tweets from the people she follows and then decides which ones she wants to retweet to her followers.[4]

We establish how the characteristics of a social media network determines the equilibrium prevalence of each type of news content when the process of recommendations reaches a steady state. When the network is relatively disconnected (i.e., the expected number of friendship in the population $\mathbb{E}[k]$ is below a threshold), then mass-market content is the only content that is recommended in steady state. Above this threshold, niche content survives but the mass-market content is always amplified by social media (i.e., the fraction of mass-market (niche) recommendations is greater (less) than the fraction of mass-market (niche) individuals). In this second case, we find that increasing the connectivity or homophily of the social network reduces the prevalence of the mass-market content.

Our second set of results focus on individual consumption patterns and, in particular, polarization. We measure polarization as the difference between the types of content consumed across niche-market and mass-market individuals. Polarization is determined by two forces (i) systematic differences from whom each type of individual receives recommendations, and (ii) differences in the type of content that each type of individual recommends in steady state. In our model, homophily affects the first of these: greater levels of homophily mean that each type is more likely to receive recommendations from someone of the same type. Hence, the source of recommendations becomes increasingly different for higher levels of homophily. The second force, the steady state recommendation behavior of each type, is determined by the average connectivity and homophily of the social media network, and the fraction of mass-market individuals. When we are in a steady state where only mass-market content survives, then all types recommend identical content. However, as the network becomes more connected, if there is either greater homophily or the fraction of mass-market individuals becomes smaller, then, the prevalence of the niche-market content grows, and the difference in the recommendation

[2]For overviews of the network literature, see Vega-Redondo (2007), Newman (2010), Jackson (2008) and Jackson et al. (2017).

[3]This means that, in our model, discovering information through news on social media may happen with the purposeful application of information searching skills and strategies. However, users may also be exposed to news that they would otherwise not actively seek. Ahmadi and Wohn (2018) refer to this type of exposure as *incidental news exposure*. In fact, 59% of Facebook users have reported that they incidentally get exposed to news on this platform every day or almost every day (Purcell et al., 2010).

[4]The way we model social media is clearly much simpler than the way Twitter works since there are only two types of news contents and individuals only transmit one type of information. But, it is also similar since the network is dynamic (at each period of time, each individual draws a different set of people), there is a large number of active users and users prefer to follow actors whose position in the ideological space is similar to their own position.

behaviors of the two types becomes larger. Therefore, these forces increase polarization through this second channel. In fact, we show that more homophily always leads to more polarization. This result is not obvious, as we also show that more homophily can lead to more variety in consumers' news content.

We extend our framework by studying how two lobbying agents can influence the prevalence of each type of content and thus polarization between agents. In our model, a lobbying agent of one type directly spreads their content among consumers. We find that niche lobbying (promoting, for example, "extreme" views) is shown to be relatively more effective at changing the behavior of niche individuals than mass-market lobbying is at changing the behavior of mass-market individuals. Since the returns are higher for the niche lobbyist (and we assume that neither agent has a cost advantage), in equilibrium, the niche agent invests more. In an example, we show that lobbying will increase polarization and this effect is most significant when the network is sparse and polarization would otherwise be minimal.

In a second extension, we allow consumers to make a costly investment to increase the degree of homophily α amongst their own connections. We demonstrate that, when the society is not sufficiently connected, then there exists a steady-state equilibrium with zero homophily, which implies no polarization. However, if the cost of homophily effort is low enough, there also exists another equilibrium with strictly positive homophily. Moreover, if the society is sufficiently connected, then, in an equilibrium with positive homophily, the niche-type consumers exhibit greater homophily than mass-type consumers. This shows that when agents choose their degree of homophily, different equilibria may emerge, where some equilblira amplify polarization and some mitigate it.

2 Related Literature

Our paper is related to the literature on the economics of media markets. This literature has mainly focused on traditional forms of media (see Anderson et al., 2016 for an overview of media economics and Athey et al., 2018, for a recent example of a model of media competition). As such, these models typically assume there are a relatively small number of providers (e.g. a small number of TV channels) about which all consumers are perfectly informed (everyone knows what is shown on the different TV channels).[5] We believe that our work contributes to this literature by introducing the first model that captures the important distinguishing characteristic of social media, relative to traditional media, whereby consumers become aware of media content through receiving recommendations and influence the prevalence of that content to others through making recommendations. We also contribute to this literature by developing a *dynamic* model with an explicit *network* that influences the information transmission process, and allows us to calculate comparative statics (with respect to network characteristics) on the equilibrium.

Our paper is also related to the literature on learning in networks (see Jackson, 2008; Goyal, 2012; Mobius and Rosenblat, 2014; and Golub and Sadler, 2016, for overviews of this literature) and, more generally, to the literature on sequential learning, in particular, the papers by Banerjee (1993), Ellison and Fudenberg (1995) and Banerjee and Fudenberg (2004), and more recently, Wolitzky (2018). In this literature, successive generations of agents use information from either the experiences or observations of the choices of earlier

[5]See also Mullainathan and Shleifer (2005) who show why people tend to buy and consume news that they know will fit with their biased views.

generations to guide their own decisions and form their own beliefs. By looking at the convergence of this dynamic process, different authors have shown under which conditions there is a reinforcement of opinions over time, which may lead to polarization in steady state. For example, Golub and Jackson (2010), using the DeGroot model in which agents correct their heterogeneous initial opinions by averaging the opinions of their neighbors, show that, if the network is strongly connected, then all agents in the network will converge to the same norm so that there will be no polarization of beliefs and opinions in the long run. More recently, Bolleta and Pin (2019) extend this model by allowing the agents to form a network endogenously and show that, if the initial diameter of the network is at most two, the process will converge to a single opinion for the whole society while, with an initial diameter of 5, an initially connected society will disagree in the long run, which implies that polarization will prevail in the steady state.[6]

Compared to this literature, our model is quite different. First, each individual does not form an opinion by averaging the opinion of her neighbors as it is usually the case in the learning-in-networks literature. Here, the process of making a recommendation (the equivalent of opinion formation in the aforementioned literature) is based on homophily and on a strong preference for one's own type of content: if at least one of her friends transmits the same news content as the type of the individual, then the individual recommends this news content. Second, we study market rather than individual effects and thus can examine under which conditions niche-market news emerges. We can also study how polarization and variety of news, defined at the market level, are affected by the network structure. Finally, because of the tractability of our model, we can also study the impact of lobbyists on steady-state equilibrium and how individuals choose their degree of homophily.

3 Model, equilibrium and comparative statics results

3.1 Model

There is a unit mass of individuals who are either of a mass-market type or niche-market type $\theta \in \{M, N\}$. Mass-market type individuals have a preference for mainstream news while niche-market type individuals have a preference for more specialized content. A fraction $\rho > 1/2$ of consumers are of the mass-market type M while $1 - \rho$ are of the niche-market type N. There is a mass 1 of news content that is either mass-market or niche-market $\phi \in \{M, N\}$. The distinguishing characteristic of mass-market content is that there is a greater share of the population which is of type M.

Time is discrete $t = 1, 2, \ldots$. In each period, an individual receives a recommendation of news content from k friends. They consume all of these items and then, in the next period, recommend news that is the closest match to their type. In our two-type setting, each individual of type $\theta \in \{M, N\}$ will recommend news that is of the same type as herself provided that she received at least one recommendation of this type of news content from one of her k friends. Otherwise, in the event she only received information

[6]There is also a theoretical literature on "echo chambers" and their effects on outcomes, recently surveyed by Levy and Razin (2019). This literature focuses on the cognitive biases of information processing that lead to extremism and polarization. It also shows the link to segregation. This is very different to what we do since we have perfect information and the way people transmit news contents is through word-of-mouth communication. We believe that our model provides a different but complementary way of explaning "echo chambers".

from the other type of content, she will recommend the other type of news content.

In each period, an individual draws at random the number of friends k that provides her with recommendations from a distribution $\{p_k\}$.[7] Then, the consumer draws each friend uniformly at random from the measure of similar types with probability α, and draws a friend uniformly at random from the population with probability $1-\alpha$. The parameter α measures the extent of *homophily* in friendships since higher α means that the individual is more likely to have a friend of the same type.[8] When $\alpha = 0$, all friendships are drawn uniformly at random from the population (no homophily).

Our setup is a simple model of a platform such as Twitter. In each period, an individual receives recommendations (tweets of news content) from people she follows. Then, in the next period, she chooses one piece of content that best aligns with her type to recommend (retweet) to the set of people who follow her. In this setting, the distribution of friendships p_k captures the differences across people in terms of how many people they follow on Twitter and the degree of homophily α captures the tendency for people to follow people who are similar to themselves.

Let us now formally described this information transmission process. Denote by m_t, the probability that a type-M individual recommends news content M in period t, and by n_t, the probability that a type-N individual recommends news content N. The probabilities m_t and n_t evolve according to:

$$n_t = 1 - \sum_k p_k \left[(1-\alpha)\left((1-\rho)(1-n_{t-1}) + \rho m_{t-1}\right) + \alpha(1-n_{t-1})\right]^k \qquad (1)$$

$$m_t = 1 - \sum_k p_k \left[(1-\alpha)(\rho(1-m_{t-1}) + (1-\rho)n_{t-1}) + \alpha(1-m_{t-1})\right]^k \qquad (2)$$

To understand these equations, consider n_t (expression 1). The summation on the right-hand-side evaluates the likelihood that the niche-market consumer recommends news M in period t; 1 minus this probability gives the likelihood she recommends news content N in period t. Indeed, with probability p_k, the consumer finds $k \geq 1$ friends. Each friend is drawn uniformly at random from the population with probability $1-\alpha$ and uniformly at random from amongst niche-market individuals with probability α. When a friend is drawn uniformly at random from the population, with probability $1-\rho$ the friend is a niche-market type, and with probability ρ the friend is a mass-market type. In these respective instances, the likelihoods the friend recommends the mass-market news content are given by $1-n_{t-1}$ and m_{t-1}. Alternatively, with probability α, due to homophily, the friend is a niche-market type. In this instance, the likelihood the friend recommends the mass-market content is $1-n_{t-1}$. The period t niche-market consumer recommends the mass-market news only if every friend recommends the mass-market news, which is given by the individual likelihood each friend recommends the mass-market news content raised to the power k. A similar explanation can be constructed for m_t (expression 2).

[7]We assume $p_0 = 0$, so that each consumer always receives some news content. We also assume $p_1 < 1$, so that each consumer has the opportunity to compare news with positive probability.

[8]One of the most prominent ideas in the public discussion of modern trends in social media is the tendency for individuals to predominantly interact with people who are similar to themselves. This particular tendency –known as *homophily* or *assortativity*– has a rich intellectual history in sociology (see McPherson et al., 2001, for a review of the literature documenting this tendency).

3.2 Equilibrium

A steady-state equilibrium (n^*, m^*) satisfies $n_{t-1} = n_t = n^*$ and $m_{t-1} = m_t = m^*$. Writing n^* and m^*, the implicit functions defined by the steady state conditions (by dropping time indexes in the above equations) are given by:

$$n^* = 1 - \sum_k p_k \left[(1-\alpha)\left((1-\rho)(1-n^*) + \rho m^*\right) + \alpha(1-n^*)\right]^k \tag{3}$$

$$m^* = 1 - \sum_k p_k \left[(1-\alpha)(\rho(1-m^*) + (1-\rho)n^*) + \alpha(1-m^*)\right]^k \tag{4}$$

We assume that the starting point of the system is somewhere on the interior $0 < m_0, n_0 < 1$. Our analysis focuses on the steady state that occurs from any such point. As we will show this point will be unique and identical, given $\{p_k\}, \alpha, \rho$, for any starting point on the interior.

Define the prevalence of mass-market content by

$$h^* = (1-\rho)(1-n^*) + \rho m^*, \tag{5}$$

the steady-state probability that a randomly drawn individual in the population recommends the mass-market news content. Then, $1 - h^* = \rho(1-m^*) + (1-\rho)n^*$ gives the probability that a randomly drawn individual recommends the niche-market news content. The quantity h^* measures the relative prevalence of each type of news content. That is, higher (lower) h^* corresponds to the mass- (niche-) market constituting a higher fraction of recommendations, and hence, consumption of news content.

Define the positive value:

$$B := \left[\alpha + (1-\alpha)(1-\rho)\frac{(1-p_1\alpha)}{1-p_1(\alpha+(1-\alpha)\rho)}\right]^{-1}.$$

Let $\mathbb{E}[\cdot]$ denote the expectation operator over p_k. We can now characterize our model's steady state.

Proposition 1. *There is a unique stable steady-state equilibrium (n^*, m^*) for any starting point $0 < m_0, n_0 < 1$, which is characterized as follows:*

1. *if $\mathbb{E}[k] \leq B$, then $m^* = 1$, $n^* = 0$ and $h^* = 1$,*
2. *if $\mathbb{E}[k] > B$, then $\rho < m^* < 1$, $0 < n^* < 1$ and $\rho < h^* < 1$.*

Proposition 1 characterizes the unique steady state of our model. For low $\mathbb{E}[k]$ below the threshold B, on average, individuals have few friends and receive relatively few new recommendations in each period. In this case, the mass-market news content swamps that of the niche-market and we end up with a steady-state equilibrium for which $m^* = 1$ and $n^* = 0$, so that only mass-market news content prevails in the market. If, instead, individuals receive a sufficiently large number of recommendations in expectation so that $\mathbb{E}[k]$ lies above B, then the niche-market news content survives in equilibrium. However, the steady-state fraction of recommendations of mass-market news content exceeds the fraction of these types in the population, i.e., $h^* > \rho$.

We can gain further intuition for the social media mechanism by considering how the threshold B is related to the primitives of the model. First, B increases with the prominence of the mass-market, as captured by ρ. Indeed, when ρ increases, the fraction of $M-$type consumers increases and an equilibrium with $m^* = 1$ and $n^* = 0$ is more likely to emerge. An increase in homophily α reduces B, hence, homophily decreases the burden placed on the niche consumers to sufficiently socialize and perpetuate niche-market news. Combining these insights, for niche content that is more-niche (corresponding to larger ρ) to survive, it requires that homophily be greater in the population.

The main insights of the proposition are that (i) the social media mechanism amplifies content, and (ii) this amplification confers a disproportionate advantage to mass-market news content. In all cases, the steady-state pattern of recommendations in the population over-represents the mass-market news content relative to the fraction of the population of this type. Furthermore, when individuals are receiving sufficiently few recommendations in expectation in each period, the niche-market news content may fail to survive at all.

3.3 Comparative statics

We can also study the comparative statics properties of the interior steady-state equilibrium.

Proposition 2. *Assume $\mathbb{E}[k] > B$. Then:*

1. *an increase in α increases the probability a niche individuals recommends niche-market content n^* and decreases the prevalence of mass-market content h^*, with $\lim_{\alpha\to 1} m^* = \lim_{\alpha\to 1} n^* = 1$, and $\lim_{\alpha\to 1} h^* = \rho$,*

2. *a First-Order Stochastic Dominance (FOSD) change to the distribution of friendships p_k decreases h^* and increases n^*,*[9]

3. *an increase in ρ increases m^*, decreases n^* and h^*.*

First, we find increasing homophily α improves (reduces) the prevalence of niche-(mass-)market content. Interacting with similar individuals helps to propagate niche-market content and in-part offset the disadvantage social media confers on it vis-a-vis mass-market content. As homophily becomes extreme, α converges to 1, each type of individual recommends their own type of news content, and the relative share of recommendations are reflective of the share of types in the population. In the case of mass-market content, an increase in homophily α has an ambiguous effect on the probability m^* that mass-market news content is recommended by mass-market indiviudals. Indeed, on the one hand, mass-market individuals receive more news content from like-minded individuals but, on the other hand, when ρ is not too large, the increase in the prevalence of niche content reduces m^*. When α goes to 1, the first effect dominates the other and, thus, α has a positive impact on m^* while, when α is small, an increase in α reduces m^*.

Second, a FOSD change to the distribution of friendships, i.e., each individual has more friends in expectation, improves both the probability n^* a niche individual recommends niche content and the prevalence of the niche-market $1 - h^*$. Similarly to the case of increasing homophily, the effect on the mass-market is ambiguous because of the two aforementioned competing forces.

[9]Consider the distributions $\{p_{k,1}\}$ and $\{p_{k,2}\}$. The concept of *first-order stochastic dominance* captures the idea that $\{p_{k,1}\}$ is obtained by shifting the mass from $\{p_{k,2}\}$ to place it on higher values. Thus, $\{p_{k,1}\}$ first-order stochastically dominates $\{p_{k,2}\}$ if $\sum_k f(k)p_{k,1} \geq \sum_k f(k)p_{k,2}$ for all nondecreasing functions $f(k)$. So it requires a higher expectation of all nondecreasing functions.

Finally, when ρ, the fraction of consumers who are of the mass-market type M, increases, quite naturally, m^*, and decreases, n^*. The net effect is an increase in the prevalence of the mass-market content h^*.

So far, we have focused on the steady state recommendation behavior of each type of individual within the population. Also important for understanding the impact of social media are the steady state consumption patterns of individuals (i.e. the recommendations individuals receive). Moreover, two quantities of particular interest are the differences between what each type of individual sees (what we define as polarization in the next section) and the variety of content that individuals see (exposure to both types of content).[10] In the next two sections we develop measures of polarization and variety, and analyze how the social media mechanism influences each.

3.4 Polarization

In our model, we define polarization as the difference in the consumption patterns of news between the two types of consumers. Our measure of polarization is the difference in the probability that each type is recommended type M content by a friend in the steady state. We may calculate it by first defining x_n^* and x_m^*, the probability for an individual of type N and type M, respectively, of receiving news of their own type from a randomly chosen friend in the steady state:

$$x_n^* = \alpha n^* + (1-\alpha)(1-h^*) \tag{6}$$
$$x_m^* = \alpha m^* + (1-\alpha)h^*, \tag{7}$$

Our measure of polarization is then:[11]

$$P^*(m^*, n^*) = |x_m^* - (1 - x_n^*)|$$

where $1-x_n^*$ is the probability that a type$-N$ individual receives news of type M in steady state. Clearly, the larger is $P^*(m^*, n^*)$, the greater is the difference in consumption of news content between the two types. By direct calculation, we find that:[12]

$$P^*(m^*, n^*) = \alpha\,(m^* + n^* - 1)\,. \tag{8}$$

Polarization is zero $P^*(m^*, n^*) = 0$ when either there is no homophily $\alpha = 0$ or only the mass market steady state exists $\mathbb{E}[k] \leq B$. In this section we study the comparative statics of polarization and so restrict the analysis to the cases where polarization is non-zero. We provide the following result.

Proposition 3. *Assume $\alpha > 0$ and $\mathbb{E}[k] > B$. Then, an increase in homophily (increase in α), an improvement in connectivity (through a First-Order Stochastic Dominance (FOSD) change to the distribution of friendships p_k) or a larger niche market (smaller ρ), result in an increase in polarization $P^*(m^*, n^*)$.*

[10] It is straightforward to observe that an individual's recommendation behavior is a poor indication of an individual's consumption of media content and differences between groups in recommendations may or may not be reflective of differences in consumption. Moreover, measures of the variety of content being recommended in the population may also be a poor indication of the variety of content any individual consumes.

[11] We could have a similar definition in terms of type N with $P^*(m^*, n^*) = |x_n^* - (1 - x_m^*)|$. This will lead to the same definition of polarization given in (8).

[12] In the proof of Proposition 1, we show that $m^* + n^* > 1$ so we do not require the absolute value notation in $P^*(m^*, n^*)$.

The first comparative statics result of this proposition shows that, as each individual pays more attention to like-minded individuals (individuals are more homophilous, i.e., higher α), polarization in the population increases. This captures both the direct effect of homophily on polarization and the indirect effect through the changes to the steady state recommendation probabilities m^* and n^*. One can immediately observe from the relationship in equation (8) that the direct effect is positive. The indirect effect is somewhat less obvious, as homophily has a positive effect on n^* but an ambiguous effect on m^* (Proposition 2). Nonetheless, the net effect is positive because the positive effect on n^* offests any negative effects of homophily on polarization through m^*.

This result shows that greater homophily in the social media network creates larger differences in the consumption patterns of different types of people. Each type of individual consumes more content that aligns with their own type at the expense of the other type of content. This is consistent with the observation in the popular press about the impact of social media through the creation of so-called "echo-chambers". Society becomes increasingly polarized because individuals only interact with people who are similar to themselves. One of the contributions of this paper is that it provides an equilibrium theory of social media consumption patterns. Here, we see that the equilibrium level of polarization is increasing in the degree of homophily. This is distinct from individual level comparisons that empirical work has focused on where individuals with different degrees of homophily within their friendship groups are compared. In our model this would be equivalent to comparing two individuals with different levels of α holding m^*, n^* fixed. Hence, our model provides theoretical support for the "echo-chamber" effect on equilibrium levels of polarization.

The second part of the proposition considers the effect of improved connectivity of the network through a FOSD change to p_k. Our measure of polarization is a combination of the differences in who sends each type of recommendation (as captured by homophily α) and differences in the composition of content that each type recommends (as captured by $m^* + n^* - 1$). The effect of connectivity on polarization acts entirely through its effect on the composition of content that each type recommends (through m^* and n^*). In Proposition 1, we have seen that, when the network is not very connected, i.e., $\mathbb{E}[k] \leq B$, then every individual recommends mass-market content, this drives polarization to zero i.e., $P^*(m^*, n^*) = 0$. When the network becomes sufficiently connected, i.e., $\mathbb{E}[k] > B$, then each type exhibits different recommendation behavior ($m^* \neq 1 - n^*$) and as the network becomes better connected (by having a FOSD change to p_k) these differences become greater thereby increasing polarization. To summarize, conditional on the network being sufficiently connected, denser networks increase polarization.

Both homophily α and network density can influence polarization. Moreover, how influential each is on polarization depends on the level of the other (i.e. the cross effect $\frac{\partial^2 P}{\partial \alpha \partial k}$). For instance, in the case of no homophily $\alpha = 0$ or as the network becomes disconnected $\mathbb{E}[k] < B$, then polarization becomes negligible and the cross effect is small. To assess the interaction of parameters α and k in driving polarization across their full parameter range we develop an example. Figure 1 provides contour plots of $P(m^*, n^*)$ varying α and k for $\rho = 2/3$.[13] Lighter isoquants (i.e. toward the upper-right) correspond with greater polarization values; values $0.1, 0.2, \ldots, 0.9$ are plotted. Individuals are assumed to have a fixed number of friends k, which takes values between 1 and 5.[14]

First, parameters α and k display diminishing marginal effects on polarization, both

[13]Steady state values n^* and m^* were calculated by iterating (3) and (4) to convergence.

[14]Precisely, $p_k = 1$ for some $k > 1$; k is varied continuously in Figure 1 to yield smooth contours.

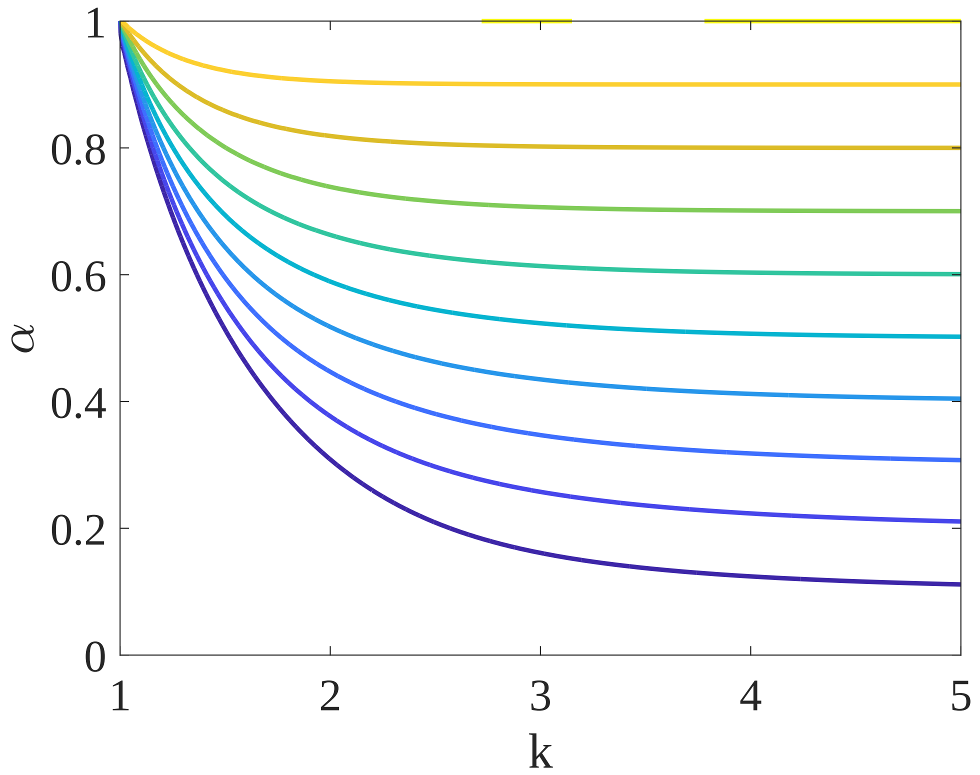

Figure 1: Polarization contour plot. Lighter isoquants give greater values.

within and across these parameters. Holding fixed $\alpha = 1/3$, an increase of k from 2 to 4 friends results in an increase in polarization from roughly 0.14 to 0.32. Holding fixed $\alpha = 2/3$, an increase of k from 2 to 4 friends results in an increase in polarization from roughly 0.60 to 0.66. We see that while increasing homophily carries a monotonic effect on polarization, greater α crowds the marginal effect of k on polarization.

Finally, in Proposition 3, we show that a decrease in ρ, the fraction of type$-M$ individuals in the society, leads to an increase in polarization. In this case, the more balanced (closer to 50-50 mass-market - niche-market) the population is, then polarization will tend to be higher.

If one compares the comparative statics in Proposition 2 with those in Proposition 3 we see that the effects on the recommendation behavior of niche individuals n^* are the same as those on polarization. Whereas, the effects on mainstream individual's behavior m^* are non-monotonic. This suggests that the effects on niche individual's behaviour are more important for driving polarization. We can formalize this idea by decomposing polarization into two components, each deriving from the respective consumption patterns of the mass and niche market consumers. We define the following measures:

$$P_m(m^*, n^*) = \alpha(m^* - h^*),$$

$$P_n(m^*, n^*) = \alpha(n^* - (1 - h^*)).$$

where $P(m^*, n^*) = P_m(m^*, n^*) + P_n(m^*, n^*)$ and P_i is the difference in the probabilities of receiving a recommendation of own type of content from a friend (given a level of homophily α) versus receiving a recommendation of own type of content from a person drawn uniformly at random from the population. We can define $S_m(m^*, n^*) =$

$P_m(m^*, n^*)/P(m^*, n^*)$ and $S_n(m^*, n^*) = 1 - S_m(m^*, n^*)$ the share of polarization deriving from each type of consumer.

Proposition 4. *Suppose* $\mathbb{E}[k] > B$. *Then,* $S_n(m^*, n^*) = \rho$.

We see that under this decomposition the niche market drives the majority share of polarization, equal to the constant $\rho > 1/2$. While increases to α and k have a clear impact on polarization that changes across the parameter range, the contribution of each consumer type to polarization remains constant.

3.5 Variety of the news content consumption

In this subsection, we introduce a metric of the *variety* or *diversity* of the news content that an individual consumes. As we will see, this is different to polarization. Indeed, for an individual that receives a share x of her own type content, we define a variety function $V(x)$, for $x \in [0, 1]$, where V is continuous, $V(0) = V(1) = 0$, single peaked at $x = \frac{1}{2}$ and symmetric so that $V(x) = V(1-x)$. We say that an individual i consumes a greater variety of news content than an individual j when $V(x_i) > V(x_j)$, where x_i and x_j are the respective share of own type content that each individual receives and are defined by (6) and (7).

Proposition 5. *Niche-market individuals consume a greater variety of news content than mass-market individuals, i.e.,* $V(x_n^*) \geq V(x_m^*)$ *where the inequality is strict if* $\alpha > 0$ *and* $\mathbb{E}[k] > B$.

This proposition shows that when there is some amount of homophily, i.e., $\alpha > 0$, and there is an interior steady-state equilibrium, i.e., $\mathbb{E}[k] > B$, then niche-market individuals will always consume a mix of news content with greater variety (closer to 50-50) than mass-market individuals. This result comes from the fact that niche-market individuals are naturally more exposed to mass-market news than the reverse and thus end up consuming more diverse news content. This precisely holds when some degree of homophily is allowed, as otherwise if $\alpha = 0$ or $\mathbb{E}[k] \leq B$, then $n^* = 0$ and both types only consume mass-market content.

The next result considers how the amount of homophily affects the variety of news content consumed by each type.

Proposition 6. *For* $\alpha \in [0, 1)$,

1. $\exists \widehat{\alpha}_n$ *such that the variety consumed by the niche-market individual* $V(x_n^*(\alpha))$ *is increasing in* α *for* $\alpha < \widehat{\alpha}_n$, *decreasing in* α *for* $\alpha > \widehat{\alpha}_n$ *and maximized for* $\alpha = \widehat{\alpha}_n$.

2. *Suppose* $\mathbb{E}[k] \leq \frac{1}{1-\rho}$, *then,* $\exists \widehat{\alpha}_{m1}, \widehat{\alpha}_{m2}$ *such that the variety consumed by the mass-market individual* $V(x_m^*(\alpha))$ *is initially increasing for* $\alpha < \widehat{\alpha}_{m1}$ *and eventually decreasing for* $\alpha > \widehat{\alpha}_{m2}$.

In a network with no homophily, the social media mechanism generates consumption patterns in which more mass-market content is consumed than niche-market content. In this case, an individual's variety is increasing in the amount of niche-market content she consumes. As homophily α is increased, niche individuals are exposed to more niche-market news content, thereby increasing their variety. Conversely, mass-market individuals are exposed to less niche-market news content and so their variety decreases.

This shows that the comparative statics results in terms of *variety* are different from that of *polarization*. Indeed, more homophily always leads to more polarization (Proposition 3) but not necessarily to more variety (Proposition 6). When becoming more homophilous, each individual pays more attention to like-minded individuals, which leads to more polarization but, as explained above, this may yield a more or a less diversified mix of news content for an individual.

Proposition 7. *For $\rho \in (\frac{1}{2}, 1)$,*

1. *The variety consumed by mass-market individuals is decreasing in the prominence of the mass-market ρ.*
2. *The variety consumed by niche-market individuals is:*
   (a) decreasing in ρ if $\lim_{\rho \to \frac{1}{2}^+} x_n^*(\rho) \leq \frac{1}{2}$,
   (b) increasing and then decreasing in ρ if $\lim_{\rho \to \frac{1}{2}^+} x_n^*(\rho) > \frac{1}{2}$.

We see that increasing the prominence of the mass-market always decreases the variety of news content consumed by mass-market individuals. Somewhat more surprisingly, it may improve the variety consumed by niche-market individuals when the variety they consume in the limit as $\rho \to \frac{1}{2}$ is biased towards niche-market content $x_m^*(\rho) > \frac{1}{2}$.

4 Lobbying

In this section, we analyze how strategic actors may influence the prevalence of each type of content. In a number of contexts, there may be parties that wish to promote some kinds of news content over others. For instance, when the content is related to public policy (climate change, openness to trade, etc.), parties that have an interest in promoting one outcome may seek to influence public opinion by facilitating the spread of content that is favorable to their preferred outcome. We address three questions: (i) Does the social media mechanism lead one or other actor (niche-market versus mass-market) to invest more in influence activities? (ii) Will banning influence activities lead to more niche-market or mass-market content being shared? (iii) Does lobbying activity affect the degree of polarization?

We assume there are two strategic players (lobbyists), a mass-market player and a niche-market player, where each makes up-front investment, e_m and e_n, respectively, at cost $C(\cdot)$, where $C'(0) = 0$, $C'(1-\rho) > 1$, and $C''(\cdot) > \overline{C}$, to increase the prevalence of their own respective content. Each player seeks to maximize the fraction of the population forwarding their content in the steady-state subject to the costs of investment. This implicitly assumes that these two strategic players are perfectly patient in the sense that they place zero weight on the forwarding behavior on the path to the steady-state. This assumption preserves a good deal of tractability for the analysis while, we believe, still capturing the elements of the trade-offs facing sufficiently patient players.

Under these assumptions, the benefits for each player may be captured by h^* and $1 - h^*$ for the mass-market and niche-market player, respectively, where $h^* = \rho m^* + (1-\rho)(1-n^*)$. We assume that the costly investments are to spread additional mass or niche-market content to their respective populations during each period. The players choose e_m and e_n, respectively, where e_i represents the fraction of the population that the player reaches with its content. The dynamic equations that describe the evolution of n_t and m_t are now given by:

$$n_t = 1-(1-e_n)\, f\left(z_{n,t-1}\right) \tag{9}$$
$$m_t = 1-(1-e_m)\, f\left(z_{m,t-1}\right) \tag{10}$$

where $f(x)=\sum_k p_k x^k$, $z_{n,t-1}=\alpha(1-n_{t-1})+(1-\alpha)h_{t-1}$ and $z_{m,t-1}=\alpha(1-m_{t-1})+(1-\alpha)(1-h_{t-1})$. When $e_n=e_m=0$, we are back to equations (1) and (2). In steady state, these equations become:

$$n^* = 1-(1-e_n)\, f\left(z_n^*\right) \tag{11}$$
$$m^* = 1-(1-e_m)\, f\left(z_m^*\right) \tag{12}$$

We see that the investments e_m and e_n only appear in their own respective equations for m^* and n^*. Indeed, the direct effect of an investment e_i only changes a consumer's behavior by reaching an individual of the same type who has exclusively received content from the other type.[15] Clearly, if an individual of type θ has received information from at least one friend about the news content of the same type, then the strategic actor of the other type has no impact on this individual. We assume that the cost function is in terms of the total mass of players that each investment affects. In particular, this implies that $C\left(\rho e_m\right)=C\left((1-\rho)\, e_n\right)$ and means that neither player has a cost advantage in reaching a mass of players of their own type. The mass-market lobbyist solves the following program:

$$\max_{e_m}\left\{h^*(e_m,e_n)-C(\rho e_m)\right\} \tag{13}$$

while, for the niche lobbyist, we have:

$$\max_{e_n}\left\{1-h^*(e_m,e_n)-C((1-\rho)e_n)\right\} \tag{14}$$

The first-order condition for each player are:[16]

$$C'\left(\rho e_m^*\right) = \frac{f\left(z_m^*\right)}{\Delta}\left[1-\alpha(1-e_n)f'(z_n^*)\right] \tag{15}$$
$$C'\left((1-\rho)\, e_n^*\right) = \frac{f\left(z_n^*\right)}{\Delta}\left[1-\alpha(1-e_m)f'(z_m^*)\right] \tag{16}$$

where

$$\Delta=\begin{pmatrix} \alpha(1-(1-e_n^*)f'(z_n^*))(1-(1-e_m^*)f'(z_m^*)) \\ -(1-\alpha)(1-((1-\rho)(1-e_n^*)f'(z_n^*)+\rho(1-e_m^*)f'(z_m^*))) \end{pmatrix}>0.$$

Our first result establishes that there is a unique Nash equilibrium in effort choices by the two players and that when there is no homophily the niche lobbyist invests more in influence activities but the mass-market content is nonetheless more prevalent.

[15]Note that an investment affects both steady state values m^* and n^* through the combination of direct and indirect effects of the investments.

[16]We provide a derivation of these conditions in Appendix B.

Proposition 8. *There is a unique interior Nash equlibrium in investments* $(e_m^*, e_n^*) \in (0,1)^2$. *Moreover, when* $\alpha = 0$ *the niche lobbyist invests more in lobbying than the mass-market lobbyist* $(1-\rho)\, e_n^* > \rho e_m^*$ *and the fraction of the population forwarding mass-market content is greater than a half, i.e.,* $h^* > \frac{1}{2}$.

A player's influence investments are effective when it changes the type of content that an individual forwards from the other type to its own type. This occurs when the player's investment reaches an individual of the same type who otherwise is only receiving content of the other type. The social media mechanism confers a benefit to mass-market content and so it is more likely that a niche-market individual only receives mass-market content than a mass-market individual only receives niche content. Hence, niche investments are relatively more likely to change the behavior of niche individuals than mass-market investments are to change the behavior of mass-market individuals. The returns are higher for the niche lobbyist from influence activities and there are no cost advantages to either lobbyist, so in equilibrium the niche lobbyist invests more.

We now turn to the question of whether banning lobbying activities benefits the niche-market or mass-market content.

Proposition 9. *When* $\alpha = 0$, *the fraction of individuals forwarding niche content is higher under lobbying than no lobbying,* $h\,(0,0) > h\,(e_m^*, e_n^*)$.

Proposition 9 shows that, in the absence of homophily, banning lobbying activities increases (decreases) the steady state prevalence of mass (niche) market content. The niche-market lobbyist invests more in equilibrium and so when lobbying activities are banned the shift in the steady state is towards more mass-market content. This unambiguously benefits the mass-market player because it results in a better steady state and reduces its investment costs. On the other hand, the impact on the niche-market player is ambiguous as it may also benefit from the change as although the steady state worsens under the ban it does not incur the costs of influence investments.

In a setting with no homophily, the steady state prevalence of mass-market content h^* is a proxy for the variety of content being consumed by each consumer. When it is closer to 1/2, each consumer is consuming a greater variety of content. Hence, a second conclusion from Proposition 9 is that the ban on lobbying activities reduces the variety of media content being consumed.

Our final set of results considers the impact of lobbying on polarization. A necessary condition for non-zero polarization is a positive level of homophily, so all the result are under this assumption.

Proposition 10. *When* $\alpha > 0$, *then, Polarization* $P(e_m^*, e_n^*) \to \alpha$ *as* $e_m^* \to 1$ *and* $e_n^* \to 1$. *Suppose,* $p_k = 1$ *for some* $k > 1$, *then* $e_m^* \to 0$, $e_n^* \to 0$ *and* $P(e_m^*, e_n^*) \to \alpha$ *as* $k \to \infty$.

Proposition 10 establishes that polarization approaches its upper bound α as lobbying investments increase to their maximum values. Hence, it also establishes that polarization will be increasing as the costs of investment are varied from prohibitively expensive where it is banned to inexpensive where the investments approach their upper bound $e_m^* = e_n^* = 1$. The second part of the proposition establishes that as social media becomes highly connected then it concurrently crowds out lobbying activities and increases polarization.

We investigate these issues further in an example illustrated in Figure 2. The Figure shows the equilibrium investments by each player and the level of polarization at the equilibrium and in the case without lobbying where $e_m = e_n = 0$. The network is

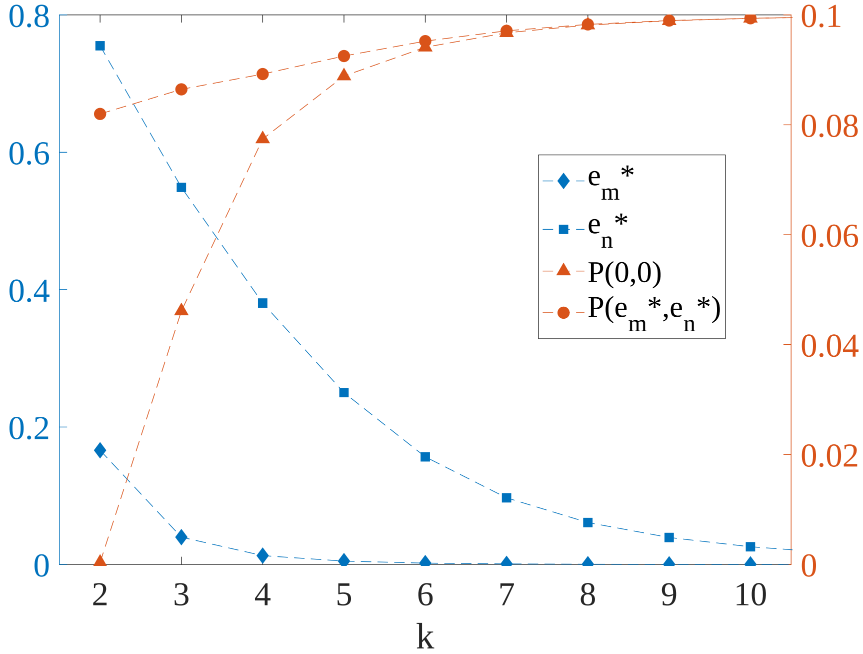

Figure 2: Lobbying and Polarization.

one where $p_k = 1$ for $k \geq 2$; costs are parametrized by a cost function $C(x) = 2x^2$, homophily is positive $\alpha = 0.1$ and the mass/niche market parameter is $\rho = 2/3$. First, we observe that the niche-market player invests more in equilibrium (as per our result without homophily) and that the equilibrium investments by both players are decreasing in the connectivity of the social media network. Second, polarization is greater under lobbying than without lobbying ($P(e_m^*, e_n^*) > P(0,0)$) and polarization in each case is increasing in the connectivity of the social media network. Finally, the impact of lobbying on polarization ($P(e_m^*, e_n^*) - P(0,0)$) is decreasing in the connectivity of the social network. Moreover, it is greatest in the case where the mass-market steady state exists in the absence of lobbying (where $k = 2$). As k decreases, the niche market player vigorously invests in lobby activities as the steady state is otherwise tending towards the mass-market steady state. We see that lobbying investments are an influential channel for market polarization in addition to connectivity and homophily established in the baseline model. Moreover, it is precisely in the settings where connectivity is low (and so polarization would otherwise be small) that lobbying activities are the most vigorously pursued and the impact is the greatest. In our example, the endogenous response of lobbying largely offsets the change in polarization that would otherwise have occurred at lower levels of connectivity of the social media network.

5 Endogenous Homophily

In online environments, individuals have a great deal of discretion over who they interact with and from/to who they share content. One factor that may affect the choice of who to be connected with in these online environments is the type of content that is shared by the other individual. In particular, a common observation of online environments is that people tend to interact with like-minded others and a common concern in recent times has been that this tendancy has resulted in "echo-chambers" where people only encounter opinions and content that is aligned with their view of the world. In this section, we consider how the endogenous choice of homophily by individuals and the social media mechanism affects the content that is being shared.

We allow agents to make a costly investment to increase the degree of homophily amongst their connections. We will assume it is costly to increase homophily and the costs are given by $kD(\alpha)$ where $D'(\alpha) > 0$, $D'(0) = 0$ and $D''(\alpha) > F$ and k is the number of friends. Also, for tractability, we simplify the network by only considering *regular* networks where everyone has the same number of friends $k \geq 2$.

We assume that agents maximize the amount of content that is their own type that they receive subject to the costs of homophily. For a mass-market individual, we have:

$$\max_{\alpha_m} \{k [\alpha_m m^* + (1-\alpha_m)h^* - D(\alpha_m)]\} \tag{17}$$

while, for a Niche individual,

$$\max_{\alpha_n} \{k [\alpha_n n^* + (1-\alpha_n)(1-h^*) - D(\alpha_n)]\} \tag{18}$$

We note that the steady state quantities m^*, n^*, h^* are all a function of the equilibrium levels of homophily chosen but each individual takes these quantities as given as she has no influence on them. Our steady-state equilibrium is determined by the solution to:

$$\begin{aligned} n^* &= 1 - [\alpha_n(1-n^*) + (1-\alpha_n)h^*]^k && (19) \\ m^* &= 1 - [\alpha_m(1-m^*) + (1-\alpha_m)(1-h^*)]^k && (20) \end{aligned}$$

where $h^* = \rho m^* + (1-\rho)(1-n^*)$.

Proposition 11. *For a given pair of homophily levels α_m and α_n, there is a unique stable steady-state equilibrium (m^*, n^*).*

We denote the steady-state equilibrium as a function of the actions of niche and mass-market individuals by $m^*(\alpha_m, \alpha_n), n^*(\alpha_m, \alpha_n), h^*(\alpha_m, \alpha_n)$. An equilibrium is a pair (α_m^*, α_n^*) that satisfy:

$$\alpha_m^* = \arg\max_{\alpha_m} \{k [\alpha_m m^*(\alpha_m^*, \alpha_n^*) + (1-\alpha_m)h^*(\alpha_m^*, \alpha_n^*) - D(\alpha_m)]\} \tag{21}$$

and

$$\alpha_n^* = \arg\max_{\alpha_n} \{k [\alpha_n n^*(\alpha_m^*, \alpha_n^*) + (1-\alpha_n)(1-h^*(\alpha_m^*, \alpha_n^*)) - D(\alpha_n)]\} \tag{22}$$

The first order conditions are given by:

$$\begin{aligned} D'(\alpha_m^*) &= (1-\rho)\,[m^*(\alpha_m^*, \alpha_n^*) + n^*(\alpha_m^*, \alpha_n^*) - 1)] && (23) \\ D'(\alpha_n^*) &= \rho\,[m^*(\alpha_m^*, \alpha_n^*) + n^*(\alpha_m^*, \alpha_n^*) - 1] && (24) \end{aligned}$$

Our first result shows that, when the underlying network is not well connected or the niche is particularly small, then there may be no homophily in the choices of individuals.

Proposition 12. *There exists an equilibrium with zero homophily if and only if $k \leq \frac{1}{1-\rho}$.*

The zero homophily equilibrium corresponds to a situation where the mass-market content is the only content shared in the market. When this is the case, there are no returns for an individual of either type from increasing their level of homophily because everyone is sharing the same type of content. In this case, polarization $P^*(m^*, n^*)$ is minimal and equal to zero. The condition $k \leq \frac{1}{1-\rho}$ demonstrates that it is a lack of connectivity in the underlying network that allows this equilibrium to exist. The empirical prediction of the model is that a lack of a homophily is associated with a lack of connectivity (or number of friends) and small niche (high ρ). It is important to note that these results are driven by the equilibrium prevalence of each type of content and not the difficulty of finding a particular type of individual.

The following proposition characterizes the equilibrium level of homophily when the underlying network is sufficiently well connected.

Proposition 13. *Suppose $k > \frac{1}{1-\rho}$. Then, there exists an equilibrium with positive levels of homophily for both types of individuals. Moreover, the niche-type players exhibit greater homophily than the mass-type agents, i.e., $\alpha_n^* > \alpha_m^*$.*

In sufficiently well connected networks, the mass-market content will not flood the market and so there is always some amount of benefit to connect with individuals that are similar to one's self. Moreover, the niche individuals benefit the most from this because it is relatively more difficult for these individuals to find their own type of content. Hence, in equilibrium, the niche individuals exhibit a greater degree of homophily than mass-market individuals.

We now assess the decomposition of polarization when homophily is endogenous. As in Section 3.4, we can decompose polarization into its respective market shares by defining:

$$P_m(m^*, n^*, \alpha_m^*, \alpha_n^*) = \alpha_m^*(m^* - h^*),$$

$$P_n(m^*, n^*, \alpha_m^*, \alpha_n^*) = \alpha_n^*(n^* - (1 - h^*)),$$

$$P(m^*, n^*, \alpha_m^*, \alpha_n^*) = P_m(m^*, n^*, \alpha_m^*, \alpha_n^*) + P_n(m^*, n^*, \alpha_m^*, \alpha_n^*),$$

and market shares:

$$S_m(m^*, n^*, \alpha_m^*, \alpha_n^*) = P_m(m^*, n^*, \alpha_m^*, \alpha_n^*)/P(m^*, n^*, \alpha_m^*, \alpha_n^*),$$

$$S_n(m^*, n^*, \alpha_m^*, \alpha_n^*) = P_n(m^*, n^*, \alpha_m^*, \alpha_n^*)/P(m^*, n^*, \alpha_m^*, \alpha_n^*).$$

Proposition 14. *Suppose $k > \frac{1}{1-\rho}$ and $\alpha_m^* < \alpha_n^*$. Then $S_n(m^*, n^*, \alpha_m^*, \alpha_n^*) > \rho$. Moreover, if $D(\alpha) = \beta\alpha^\gamma$ and $0 < \alpha_m^* < \alpha_n^* < 1$ then:*

$$S_n(m^*, n^*, \alpha_m^*, \alpha_n^*) = \frac{1}{1 + \left(\frac{1-\rho}{\rho}\right)^{\frac{\gamma}{\gamma-1}}}.$$

Focusing on the equilibrium described in Proposition 13 where $\alpha_m^* < \alpha_n^*$, endogenous homophily increases the niche market's contribution to market polarization. When costs to homophily take a power form $D(\alpha) = \beta\alpha^\gamma$, then market shares become independent of k and β, with the niche's market share strictly decreasing in γ with limit 1 as $\gamma \to 1^+$. That is, as the convexity in homophily costs decreases, and the equilibrium ratio

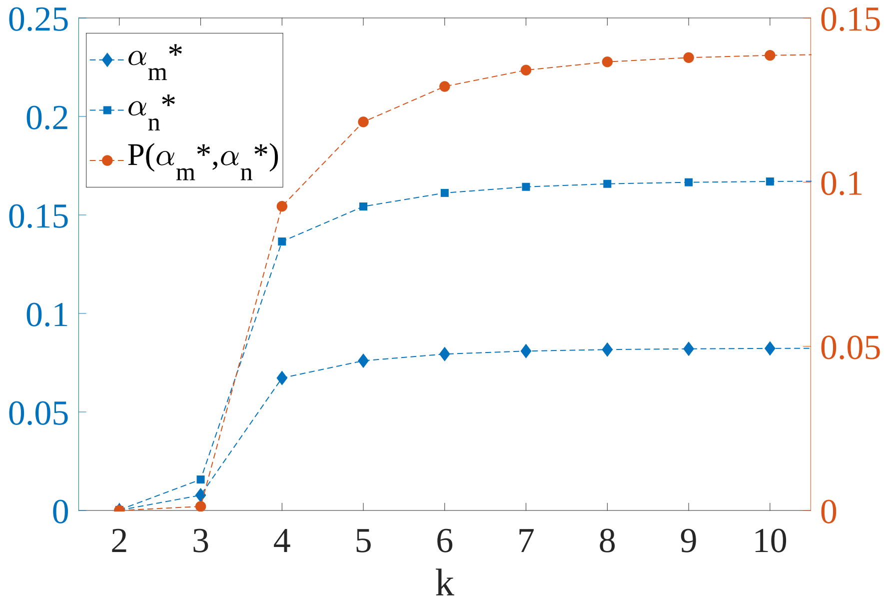

Figure 3: Exogenous homophily ($\alpha = 0.1$).

α_m^*/α_n^* decreases, the proportion of polarization accredited to the niche market grows unbounded.[17]

Figure 3 provides numerical solutions to homophily for each type (left axis), and the resulting polarization (right axis). We set the parameters to $\rho = 2/3$ and $\beta = \gamma = 2$. We again consider $p_k = 1$ for some $k > 1$. The figure illustrates that denser networks, higher in k, has a positive effect on the level of homophily each type chooses and the amount of polarization. The niche-market invests twice as much in homophily than the mass-market. Interestingly, per Proposition 14, $S_n(m^*, n^*, \alpha_m^*, \alpha_n^*) = 0.8$, and so the niche-market accounts for a polarization share that is four times that of the mass-market.

6 Conclusion

We have developed a model of social media to understand its role in promoting or suppressing particular types of news content. We find that social media promotes mass-market content at the expense of niche content. We show that either greater connectivity or greater homophily increase the prevalence of niche-market content. We also investigate how social media affects polarization and the variety of the news content consumed by individuals. The same forces that promote the prevalence of niche-market content tend to also increase polarization, whereas, the impact on the variety of content consumed by individuals is in general ambiguous.

In an extension to lobbying groups, we find that this may be an additional channel for increasing polarization and are particularly important in driving polarization when it would otherwise be small. Finally, we allow consumers to choose the degree of homophily

[17]The conditions $0 < \alpha_m^*, \alpha_n^* < 1$ of Proposition 14 implies an interior solution, and thus f.o.c.'s (23) and (24) hold. Under the power cost function $D(\alpha) = \beta\alpha^\gamma$, a sufficient condition for an interior equilibrium is that β is sufficient large, as at zero homophily marginal values to homophily are positive while marginal costs are zero, implying the two markets always invest in positive homophily.

amongst their connections and demonstrate that the niche-market individuals exhibit greater homophily than the mass-market consumers.

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Appendix

A Proofs of the results in the main text

Proof of Proposition 1. We prove the result under heterogeneous homophily (α_n, α_m) and lobbying efforts (e_n, e_m). Denote $a_n \equiv 1-e_n$ and $a_m \equiv 1-e_m$. We first establish the necessary and sufficient condition, $\mathbb{E}[k] > B$, for there to exists a unique interior steady state; no interior steady states exists when the condition is violated. We then show that all steady states are globally stable.

Define variables $x_t := 1-n_t$ and $y_t := 1-m_t$, which yields the equivalent system:

$$x_t = a_n \sum_k p_k \left[(1-\alpha_n)((1-\rho)x_{t-1} + \rho(1-y_{t-1})) + \alpha_n x_{t-1}\right]^k, \tag{A.1}$$

$$y_t = a_m \sum_k p_k \left[(1-\alpha_m)(\rho y_{t-1} + (1-\rho)(1-x_{t-1})) + \alpha_m y_{t-1}\right]^k. \tag{A.2}$$

Dropping time subscripts to focus on the steady state, the system can be written:

$$x = a_n \sum_k p_k \left[(1-\alpha_n)h + \alpha_n x\right]^k, \tag{A.3}$$

$$y = a_m \sum_k p_k \left[(1-\alpha_m)(1-h) + \alpha_m y\right]^k. \tag{A.4}$$

where, at the steady-state, $h^* = H(x^*, y^*) := (1-\rho)x^* + \rho(1-y^*)$. (A.3) and (A.4) define implicit functions $x(h)$ and $y(h)$. The functions are continuous and monotone increasing and decreasing over $h \in [0,1]$, respectively, with $x(0) = 0$, $x(1) = 1$, $y(0) = 1$ and $y(1) = 0$. We verify these properties below.

For uniqueness of an interior steady state, it suffices to show existence of at most one h^* solving (A.3), (A.4) and $h^* = H(x(h^*), y(h^*))$. For this, we show that (i) $H(x(h), y(h))$ is continuous, (ii) $\lim_{h\to 0} H(x(h), y(h)) \geq 0$, (iii) $\lim_{h\to 1} H(x(h), y(h)) \leq 1$, and (iv) $\frac{d^3}{dh^3} H(x(h), y(h)) \geq 0$, which imply $H(x(h), y(h))$ crosses the 45-degree line at at-most one unique point $h^* \in (0,1)$. (i) follows immediately from $x(h)$ and $y(h)$ continuous. (ii) and (iii) clearly hold for $x, 1-y, \rho \in [0,1]$.

To show (iv), we first show $\frac{\partial^3 x}{\partial h^3} > 0$. Define $f(z) := \sum_k p_k z^k$, and note that $\frac{\partial^n}{\partial z^n} f^z \geq 0$ for all $n \geq 0$. Then define $z_x := (1-\alpha_n)h + \alpha_n x$, giving $\frac{\partial z_x}{\partial h} = (1-\alpha_n) + \alpha_n \frac{\partial x}{\partial h}$, and $\frac{\partial z_x}{\partial h} = \alpha_n \frac{\partial^2 x}{\partial h^2}$. Repeated implicit differentiation of (A.3) gives:

$$\frac{\partial x}{\partial h} - \frac{\partial z_x}{\partial h} a_n f'(z_x) = 0, \tag{A.5}$$

$$\frac{\partial^2 x}{\partial h^2} - \alpha_n a_n \frac{\partial^2 x}{\partial h^2} f'(z_x) - a_n \left(\frac{\partial z_x}{\partial h}\right)^2 f''(z_x) = 0, \tag{A.6}$$

$$\frac{\partial^3 x}{\partial h^3} - \alpha_n a_n \frac{\partial^3 x}{\partial h^3} f'(z_x) - \alpha_n a_n \frac{\partial^2 x}{\partial h^2}\frac{\partial z_x}{\partial h} f''(z_x) - 2\frac{\partial z_x}{\partial h}\alpha_n a_n \frac{\partial^2 x}{\partial h^2} f''(z_x) - a_n \left(\frac{\partial z_x}{\partial h}\right)^3 f'''(z_x) = 0. \tag{A.7}$$

Now, (A.5) gives:

$$\frac{\partial x}{\partial h} = \frac{(1-\alpha_n)a_n f'(z_x)}{1-\alpha_n a_n f'(z_x)} > 0,$$

the inequality holding by $(1-\alpha_n)a_n f'(z_x) > 0$, and by $1-\alpha_n a_n f'(z_x) > 0$ holding where equality (A.3) holds, for each $h \in (0,1)$, because $a_n f((1-\alpha_n)h) > 0$, $a_n f((1-\alpha_n)h+\alpha_n) < 1$ and f is convex. This verifies that $x(h)$ is increasing and continuous, and establishes that $\frac{\partial z_x}{\partial h} > 0$. (A.6) gives:

$$\frac{\partial^2 x}{\partial h^2} = \frac{\left(\frac{\partial z_x}{\partial h}\right)^2 a_n f''(z_x)}{1-\alpha_n a_n f'(z_x)} > 0,$$

the inequality holding by $a_n f''(z_x) > 0$, and the above. (A.7) gives:

$$\frac{\partial^3 x}{\partial h^3} = \frac{3\alpha_n \frac{\partial z_x}{\partial h}\frac{\partial^2 x}{\partial h^2} a_n f''(z_x) + \left(\frac{\partial z_x}{\partial h}\right)^3 a_n f'''(z_x)}{1-\alpha_n a_n f'(z_x)} > 0,$$

the inequality holding by $\frac{\partial z_x}{\partial h} > 0$, $f''(z_x), f'''(z_x) > 0$, and by the above.

Still for (iv), we next show $\frac{\partial^3 y}{\partial h^3} < 0$. Now define $z_y := (1-\alpha_m)(1-h) + \alpha y$, giving $\frac{\partial z_y}{\partial h} = -(1-\alpha_m) + \alpha_m \frac{\partial y}{\partial h}$, and $\frac{\partial z_y}{\partial h} = \alpha_m \frac{\partial^2 y}{\partial h^2}$. Repeated implicit differentiation of (A.4) gives analogous expressions to (A.5), (A.6) and (A.7) but with y's in place of "x" and z_y's in place of "z_x". However, (A.5) now gives:

$$\frac{\partial y}{\partial h} = \frac{-(1-\alpha_m)a_m f'(z_y)}{1-\alpha_m a_m f'(z_y)} < 0,$$

the inequality holding by $-(1-\alpha_m)a_m f'(z_y) < 0$, and by $1-\alpha_m a_m f'(z_y) > 0$ holding where equality (A.4) holds, for each $h \in (0,1)$, because $a_m f((1-\alpha_m)(1-h)) > 0$, $a_m f((1-\alpha_m)(1-h)+\alpha_m) < 1$ and f is convex. This verifies that $y(h)$ is decreasing and continuous, and establishes that $\frac{\partial z_y}{\partial h} < 0$. (A.6) gives:

$$\frac{\partial^2 y}{\partial h^2} = \frac{\left(\frac{\partial z_y}{\partial h}\right)^2 a_m f''(z_y)}{1-\alpha_m a_m f'(z_y)} > 0,$$

the inequality holding by $f''(z_y) > 0$, and the above. Finally, (A.7) can be written:

$$\frac{\partial^3 y}{\partial h^3} = \frac{\partial z_y}{\partial h}\frac{3\alpha_m \frac{\partial^2 y}{\partial h^2} a_m f''(z_y) + \left(\frac{\partial z_y}{\partial h}\right)^2 a_m f'''(z_y)}{1-\alpha_m a_m f'(z_y)} < 0,$$

the inequality holding by the above.

It follows that:

$$\frac{d^3}{dh^3}H(x(h),y(h)) = (1-\rho)\frac{\partial^3 x}{\partial h^3} - \rho\frac{\partial^3 y}{\partial h^3} > 0.$$

We have shown that $H(x(h),y(h))$ crosses the 45-degree line at at-most one unique point $h^* \in (0,1)$.

Before establishing the condition characterizing an interior steady state (where $h^* \in (0,1)$), first note that, setting $y = 1-\rho$ on the right-hand-side of (A.4), and by $x^* \leq 1-y^*$

in the unique steady state, we have $H(x^*, 1-\rho) < H(\rho, 1-\rho) = \rho$, while setting $y = 1-\rho$ on the left-hand-side of (A.4) gives:

$$\begin{aligned} y &= 1-\rho = a_m \sum_k p_k \left[(1-\alpha_m)(1-H(x,1-\rho)) + \alpha_m(1-\rho)\right]^k \\ &> a_m \sum_k p_k \left[(1-\alpha_m)(1-H(\rho,1-\rho)) + \alpha_m(1-\rho)\right]^k = 1-\rho. \end{aligned}$$

This implies that $y^* < 1-\rho$ for condition (A.4) to hold with equality, or equivalently $m^* > \rho$. $n^* > 0$ when $h^* \in (0,1)$ then follows directly from condition (A.3), or equivalently $m^* > \rho$.

We now derive the condition characterizing when the steady state is interior, that is, $h^* \in (0,1)$, which implies $x^*(h^*) < 1$ and equivalently $n^* > 0$ (i.e. news of the niche market persists in steady state). Remember:

$$\begin{aligned} \frac{\partial}{\partial x} h &= \frac{(1-\alpha_n)a_n \sum_k k p_k \left((1-\alpha_n)h + \alpha_n x\right)^{k-1}}{1-\alpha_n a_n \sum_k k p_k \left((1-\alpha_n)h + \alpha_n x\right)^{k-1}}; \\ \frac{\partial}{\partial y} h &= -\frac{(1-\alpha_m)a_m \sum_k k p_k \left((1-\alpha_m)(1-h) + \alpha_m y\right)^{k-1}}{1-\alpha_m a_m \sum_k k p_k \left((1-\alpha_m)(1-h) + \alpha_m y\right)^{k-1}}. \end{aligned}$$

Note that:

$$\lim_{h\to 1} \frac{\partial}{\partial x} h = \frac{(1-\alpha_n)a_n \mathbb{E}[k]}{1-\alpha_n a_n \mathbb{E}[k]}; \qquad \lim_{h\to 1} \frac{\partial}{\partial y} h = -\frac{(1-\alpha_m)a_m p_1}{1-\alpha_m a_m p_1},$$

These give:

$$\begin{aligned} \lim_{h\to 1} \frac{d}{dh} H(x(h), y(h)) &= \lim_{h\to 1} \left[(1-\rho)\frac{\partial}{\partial h} x - \rho \frac{\partial}{\partial h} y\right] \\ &= \left((1-\rho)\frac{(1-\alpha_n)a_n \mathbb{E}[k]}{1-\alpha_n a_n \mathbb{E}[k]} + \rho \frac{(1-\alpha_m)a_m p_1}{1-\alpha_m a_m p_1}\right) \end{aligned}$$

When $\lim_{h\to 1} \frac{d}{dh} H(x(h), y(h)) < 1$ then the unique steady state satisfies $h^* < 1$ and is therefore interior. Rearranging gives:

$$\mathbb{E}[k] > B(\alpha_n, \alpha_m) := \frac{1}{a_n} \frac{1 - a_m p_1(\alpha_m + (1-\alpha_m)\rho)}{1 - a_m p_1(\alpha_m(1-\rho) + \alpha_n \rho) - (1-\alpha_n)\rho}.$$

When $\alpha_n = \alpha_m$ and $a_n = a_m = 1$ we obtain

$$\mathbb{E}[k] > B := B(\alpha, \alpha) = \left[\alpha + (1-\alpha)(1-\rho)\frac{1-p_1\alpha}{1-p_1(\alpha + (1-\alpha)\rho)}\right]^{-1}.$$

Toward establishing global stability, we next show the unique steady state is locally stable, a necessary condition for global stability. Again allow $\alpha_n \neq \alpha_m$. For this, writing $x^*(y)$ and $y^*(x)$ the implicit steady-state solutions to (A.1) and (A.2), respectively, and $x^{*-1}(x)$ the inverse of the former. Then:

$$\begin{aligned} & \left.\frac{d}{dm} n^*(m) \cdot \frac{d}{dn} m^*(n)\right|_{(n^*, m^*)} < 1 \\ \Leftrightarrow \quad & \left.\frac{d}{dy} x^*(y) \cdot \frac{d}{dx} y^*(x)\right|_{(x^*, y^*)} < 1 \\ \Leftrightarrow \quad & \left.\frac{d}{dx} y^*(x)\right|_{x^*} > \left.\frac{d}{dx} x^{*-1}(x))\right|_{x^*}. \end{aligned}$$

Implicit differentiation gives:

$$\lim_{x\to 1}\frac{dy^*}{dx} = \frac{-(1-\alpha_m)(1-\rho)a_m p_1}{1-((1-\alpha_m)\rho+\alpha_m)a_m p_1},$$
$$\lim_{x\to 1}\frac{dy}{dx^*} = \left(\lim_{y\to 0}\frac{dx^*}{dy}\right)^{-1} = \frac{1-((1-\alpha_n)(1-\rho)+\alpha_n)a_n\mathbb{E}[k]}{-(1-\alpha_n)\rho a_n\mathbb{E}[k]},$$

the second equality following from the inverse function theorem. Then, it suffice for the steady state (x^*, y^*) to be stable for:

$$\lim_{x\to 1}\tfrac{dy^*}{dx} > \lim_{x\to 1}\tfrac{dy}{dx^*} \quad if\ \mathbb{E}[k] > B,$$

$$\lim_{x\to 1}\tfrac{dy^*}{dx} \leq \lim_{x\to 1}\tfrac{dy}{dx^*} \quad if\ \mathbb{E}[k] \leq B,$$

as then $x^*(y)$ and $y^*(x)$ must intersect at (x^*, y^*) such that $\frac{d}{dx}y^*(x^*) > \frac{d}{dx}x^{*-1}(x^*)$, both when (x^*, y^*) and when $(x^*, y^*) = (1, 0)$. With algebra, the second strict inequality at limits $x \to 1$ and $y \to 0$ becomes:

$$\mathbb{E}[k] > \frac{1-a_m p_1(\alpha_m+(1-\alpha_m)\rho)}{1-a_m p_1(\alpha_m(1-\rho)+\alpha_n\rho)-(1-\alpha_n)\rho} = B(\alpha_n, \alpha_m).$$

We now show global stability, provided the starting point (x_0, y_0) is interior. Define the dynamic system $(x_t, y_t) = g(x_{t-1}, y_{t-1})$ by equations (A.1) and (A.2) where it is straightforward to verify that g is continuous in $(x, y) \in [0, 1] \times [0, 1]$. The quantities $x^*(y_{t-1})$ and $y^*(x_{t-1})$ give the unique interior fixed point of $g_z : [0, 1] \to [0, 1]$ for $z = x, y$ where $x = g_x(x, y_{t-1})$ and $y = g_y(x_{t-1}, y)$; when there is more than one fixed point (this may only occur when the argument is 0) then it is defined as the minimum. Define $\underline{x}(x_{t-1}, y_{t-1}) = \min\{x_{t-1}, x^*(y_{t-1})\}$ and $\bar{x}(x_{t-1}, y_{t-1}) = \max\{x_{t-1}, x^*(y_{t-1})\}$ and similarly define $\underline{y}(x_{t-1}, y_{t-1}), \bar{y}(x_{t-1}, y_{t-1})$. We now show the following property of the dynamic system.

Lemma A.1. *Suppose $x_{t-1} \neq x^*(y_{t-1})$ then $g_x(x_{t-1}, y_{t-1}) \in (\underline{x}(x_{t-1}, y_{t-1}), \bar{x}(x_{t-1}, y_{t-1}))$ and, similarly, suppose $y_{t-1} \neq y^*(x_{t-1})$ then $g_y(x_{t-1}, y_{t-1}) \in (\underline{y}(x_{t-1}, y_{t-1}), \bar{y}(x_{t-1}, y_{t-1}))$.*

Proof. We observe that $g : [0, 1]^2 \to [0, 1]^2$, $g_x(x_{t-1}, y_{t-1})$ and $g_y(x_{t-1}, y_{t-1})$ are increasing and convex in x_{t-1} and y_{t-1} respectively. Hence, if $x_{t-1} < x^*(y_{t-1})$ then $x_{t-1} < g_x(x_{t-1}, y_{t-1}) < x^*(y_{t-1})$ and if $x_{t-1} > x^*(y_{t-1})$ then $x^*(y_{t-1}) < g_x(x_{t-1}, y_{t-1}) < x_{t-1}$. The same argument applies for the y coordinate completing the proof. □

Take the case where the steady state is interior: $0 < m^*, n^*, x^*, y^* < 1$. Define a line segment $\tilde{y}_1(\tilde{x}_1)$ between the steady state (x^*, y^*) and point $(1, 1)$ by $\tilde{y}_1 = \frac{1-y^*}{1-x^*}\tilde{x}_1 + \frac{y^*-x^*}{1-x^*}$ for $\tilde{x}_1 \in [x^*, 1]$. Now, define a second line $\tilde{y}_2(\tilde{x}_2)$ for $\tilde{x} \in [0, x^*]$ in the following way. For each $x \in [x^*, 1]$ find the point $(x^*(\tilde{y}_1(x)), \max\{x^{*-1}(x), 0\})$; this point corresponds with the south-west corner of the dashed box in Figure A1. First, $x^*(\tilde{y}_1(x))$ is continuous and strictly decreasing in x for all $x^* \leq x \leq 1$, $x^*(\tilde{y}_1(1)) = 0$ and $x^*(\tilde{y}_1(x^*)) = x^*$. Second, note that we have defined the y-coordinate equal to 0 when the inverse of equation (A.3) $x^{*-1}(x)$ gives a negative solution. Finally, there exists $\bar{x} \leq 1$ such that $x^{*-1}(x) > 0$ for $x < \bar{x}$ and $x^{*-1}(x)$ is strictly decreasing in x. The line segments $\tilde{y}_1, \tilde{y}_2, x^*(y)$ segment the interior of (x, y) into four regions about the steady state as in the diagram below where the regions are labeled A, B, C, D:

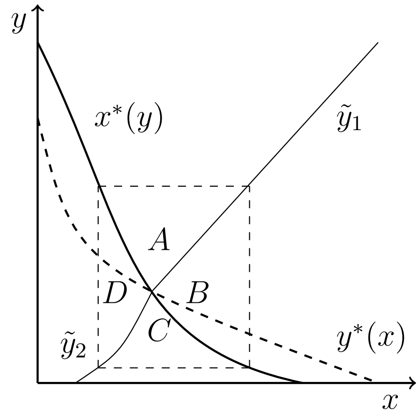

Figure A1: Four Segments

Now define a function $L(x, y) : [0, 1]^2 \to \Re$ as follows:

$$L(x,y) = \begin{cases} y^* - y & \text{if } (x,y) \in A \\ y^* - \tilde{y}_1(x) & \text{if } (x,y) \in B \\ y^* - \tilde{y}_1(x^*(y)) & \text{if } (x,y) \in C \\ y^* - x^{*-1}(x) & \text{if } (x,y) \in D \end{cases} \tag{A.8}$$

The dashed box in Figure A1 defines one isoquant of L.

For any interior starting point (x_0, y_0) we can define $X = \{(x, y) : L(x, y) \geq L(x_0, y_0)\}$ where X is a compact subset $X \subseteq \Re^2$. Moreover, the function $L(x, y) : X \to \Re$ is continuous in x, y and $L(x, y) \leq 0$ with equality if and only if $(x, y) = (x^*, y^*)$. We now show the following two lemmas

Lemma A.2. *Suppose* $(x_{t-1}, y_{t-1}) \in (0, 1)\times[0, 1]$ *and* $x_{t-1} \neq x^*(y_{t-1})$ *then* $L(g(x_{t-1}, y_{t-1})) > L(x_{t-1}, y_{t-1})$.

Proof. We construct the following upper x'', y'' and lower x', y' bounds from the value $L(x_{t-1}, y_{t-1})$:

$$\begin{aligned} y'' &= y^* - L(x_{t-1}, y_{t-1}) \\ x'' &= \tilde{y}_1^{-1}(y'') \\ y' &= \max\{x^{*-1}(x''), 0\} \\ x' &= x^*(y'') \end{aligned}$$

We note that from the definition of L that (x_{t-1}, y_{t-1}) lies on the boundary of $[x', x''] \times [y', y'']$ and $L(x, y) > L(x_{t-1}, y_{t-1})$ for any point on the interior $(x, y) \in (x', x'') \times (y', y'')$. We also note that $x^*(y)$ and $y^*(x)$ cross at most once in $(0, 1) \times (0, 1)$ where $\left|\frac{\partial x^*}{\partial y}\right| < (\left|\frac{\partial y^*}{\partial x}\right|)^{-1}$ so $y' < y^*(x) < y''$ for all $x \in [x', x'']$. Now, by lemma A.1 we have that $g_y(x_{t-1}, y_{t-1}) \in (\underline{y}, \bar{y}) \subset (y', y'')$ and for $x_{t-1} \neq x^*(y_{t-1})$ then $g_x(x_{t-1}, y_{t-1}) \in (\underline{x}, \bar{x}) \subset (x', x'')$ so $g(x_{t-1}, y_{t-1}) \in (x', x'') \times (y', y'')$ and hence $L(g(x_{t-1}, y_{t-1})) > L(x_{t-1}, y_{t-1})$. □

Lemma A.3. *Suppose* $(x_{t-1}, y_{t-1}) \in (0, 1) \times [0, 1]$, $x_{t-1} = x^*(y_{t-1})$ *and* $y_{t-1} \neq y^*(x_{t-1})$ *then* $L(g(g(x_{t-1}, y_{t-1}))) > L(x_{t-1}, y_{t-1})$.

Proof. In this case $g_x(x_{t-1}, y_{t-1}) = x_{t-1}$ and using lemma A.1 $g_y(x_{t-1}, y_{t-1}) \in (\underline{y}, \bar{y})$, hence, $L(g(x_{t-1}, y_{t-1})) = L(x_{t-1}, y_{t-1})$ and $g_x(x_{t-1}, y_{t-1}) \neq x^*(g_y(x_{t-1}, y_{t-1}))$. We now apply lemma A.2 to conclude that $L(g(g(x_{t-1}, y_{t-1}))) > L(g(x_{t-1}, y_{t-1})) = L(x_{t-1}, y_{t-1})$ □

To establish our result for an interior steady state and interior starting point (x_0, y_0) we define $X = \{(x, y) : L(x, y) \geq L(x_0, y_0)\}$ and $h(x, y) = g(g(x, y))$. X is a compact subset $X \subseteq \Re^2$. Moreover, the function $L(x, y) : X \to \Re$ is continuous in x, y, $L(x, y) \leq 0$ with equality if and only if $(x, y) = (x^*, y^*)$, and $h(x, y)$ is a continuous mapping $X \to X$ by virtue of g having the same properties. By lemmas A.2 and A.3 $L(h(x, y)) > L(x, y)$ for all $(x, y) \in X/(x^*, y^*)$ and $L(h(x^*, y^*)) = L(x^*, y^*)$. The conditions of Lemma 6.2 of Stokey and Lucas (1989) are satisfied for X and L as defined here, $g = h$ and $\bar{x} = (x^*, y^*)$. Therefore, (x^*, y^*) is the globally stable steady state of h and hence g in X.

For the case where the steady state is $m^* = 1 = 1 - x^*, n^* = 0 = 1 - y^*$ we define:

$$\tilde{L}(x, y) = -\max\{y, x^{*-1}(x)\} \tag{A.9}$$

Lemma A.4. *Suppose* $(x_{t-1}, y_{t-1}) \in (0, 1) \times [0, 1]$ *and* $x_{t-1} \neq x^*(y_{t-1})$ *then* $\tilde{L}(g(x_{t-1}, y_{t-1})) > \tilde{L}(x_{t-1}, y_{t-1})$.

Proof. We note that from the definition of $\tilde{L}$ that (x_{t-1}, y_{t-1}) lies on the boundary of the set $\{(x, y) : \tilde{L}(x, y) \geq \tilde{L}(x_{t-1}, y_{t-1})\}$. Moreover, $\tilde{L}(x, y) > \tilde{L}(x_{t-1}, y_{t-1})$ for any (x, y) such that $x > x^*(\tilde{L}(x, y))$ and $y < \tilde{L}(x, y)$. In the case that $m^* = 1 = 1 - x^*, n^* = 0 = 1 - y^*$ then $y^*(x) < x^{*-1}(x)\ \forall x \in (0, 1)$, hence, $y^*(x) < \tilde{L}(x_{t-1}, y_{t-1})\ \forall x \in (0, 1)$. Now, by lemma A.1 $g_y(x_{t-1}, y_{t-1}) \in (0, \tilde{L}(x_{t-1}, y_{t-1})$ and $g_x(x_{t-1}, y_{t-1}) \in (x^*(\tilde{L}(x_{t-1}, y_{t-1})), 1)$. □

Lemma A.5. *Suppose* $(x_{t-1}, y_{t-1}) \in (0, 1) \times [0, 1]$, $x_{t-1} = x^*(y_{t-1})$ *then* $\tilde{L}(g(g(x_{t-1}, y_{t-1}))) > \tilde{L}(x_{t-1}, y_{t-1})$.

Proof. In this case, $g_x(x_{t-1}, y_{t-1}) = x_{t-1}$ and observing that $y_{t-1} \neq y^*(x_{t-1}$ we can use lemma A.1 to conclude that $g_y(x_{t-1}, y_{t-1}) \in (0, y_{t-1})$. Hence, $\tilde{L}(g(x_{t-1}, y_{t-1})) = \tilde{L}(x_{t-1}, y_{t-1})$ and $g_x(x_{t-1}, y_{t-1}) \neq x^*(g_y(x_{t-1}, y_{t-1}))$. We can now apply lemma A.4 to conclude that $\tilde{L}(g(g(x_{t-1}, y_{t-1}))) > \tilde{L}(g(x_{t-1}, y_{t-1})) = \tilde{L}(x_{t-1}, y_{t-1})$ completing the proof. □

To establish our result for the mass market steady state and interior starting point (x_0, y_0) we define $X = \{(x, y) : \tilde{L}(x, y) \geq \tilde{L}(x_0, y_0)\}$ and $h(x, y) = g(g(x, y))$. X is a compact subset $X \subseteq \Re^2$. Moreover, the function $\tilde{L}(x, y) : X \to \Re$ is continuous in x, y, $\tilde{L}(x, y) \leq 0$ with equality if and only if $(x, y) = (1, 0)$, and $h(x, y)$ is a continuous mapping $X \to X$ by virtue of g having the same properties. By lemmas A.4 and A.5 $\tilde{L}(h(x, y)) > \tilde{L}(x, y)$ for all $(x, y) \in X/(x^*, y^*)$ and $\tilde{L}(h(1, 0)) = \tilde{L}(1, 0)$. The conditions of Lemma 6.2 of Stokey and Lucas (1989) are satisfied for X and $\tilde{L}$ as defined here, $g = h$ and $\bar{x} = (1, 0)$. Therefore, $(1, 0)$ is the globally stable steady state of h and hence g in X. □

Proof of Proposition 2. We start with a lemma.

Lemma A.6. $h^* \geq \rho$

Proof of Lemma A.6. We prove the lemma by showing that $H(x(\rho), y(\rho)) \geq \rho$. Then, given $H(x(h), y(h))$ crosses the 45-degree line at most once from above, the lemma follows. The quantities $x(\rho), y(\rho)$ are defined as:

$$\begin{aligned} f^{-1}(x) - \alpha x &= (1-\alpha)\rho \\ f^{-1}(y) - \alpha y &= (1-\alpha)(1-\rho) \end{aligned}$$

where the left-hand side is an increasing concave function of its argument and is equal to 0 at 0. Hence, using the line defined by the points $(y, (1-\alpha)(1-\rho))$ and $(x, (1-\alpha)\rho)$, the following relationship holds

$$(1-\alpha)(1-\rho) \geq \frac{(1-\alpha)[\rho - (1-\rho)]}{x(\rho) - y(\rho)} y(\rho)$$

rearranging

$$\frac{x(\rho)}{y(\rho)} \frac{1-\rho}{\rho} \geq 1$$

which implies that $H(x(\rho), y(\rho)) \geq \rho$ thereby establishing the result. □

We use the multivariate implicit function theorem. Define z_x and z_y as above; $h = (1-\rho)x + \rho(1-y)$. Write the system (A.3) *and* (A.4)*:*

$$\begin{bmatrix} g_x(x,y) \\ g_y(x,y) \end{bmatrix} = \begin{bmatrix} x - \sum_k p_k \left[(1-\alpha)h + \alpha x\right]^k \\ y - \sum_k p_k \left[(1-\alpha)(1-h) + \alpha y\right]^k \end{bmatrix}.$$

$g_x(x^*, y^*) = 0$ and $g_y(x^*, y^*) = 0$ then defines the steady state. The Jacobian of the system is:

$$J = \begin{bmatrix} 1 - ((1-\alpha)(1-\rho) + \alpha) f'(z_x) & (1-\alpha)\rho f'(z_x) \\ (1-\alpha)(1-\rho) f'(z_y) & 1 - ((1-\alpha)\rho + \alpha) f'(z_y) \end{bmatrix}$$

which has inverse:

$$J^{-1} = \frac{1}{|J|} \begin{bmatrix} 1 - ((1-\alpha)\rho + \alpha) f'(z_y) & -(1-\alpha)\rho f'(z_x) \\ -(1-\alpha)(1-\rho) f'(z_y) & 1 - ((1-\alpha)(1-\rho) + \alpha) f'(z_x) \end{bmatrix}.$$

We know that $|J| > 0$ by stability of the steady state.

The comparative statics with respect to α is then given by:

$$\begin{aligned} \begin{bmatrix} \frac{\partial x^*}{\partial \alpha} \\ \frac{\partial y^*}{\partial \alpha} \end{bmatrix} &= -J^{-1} \begin{bmatrix} \frac{\partial g_x(x^*, y^*)}{\partial \alpha} \\ \frac{\partial g_y(x^*, y^*)}{\partial \alpha} \end{bmatrix} = -J^{-1} \begin{bmatrix} -\rho(1 - (x^* + y^*)) f'(z_x^*) \\ (1-\rho)(1 - (x^* + y^*)) f'(z_y^*) \end{bmatrix} \\ &= \frac{-1}{|J|}(1 - (x^* + y^*)) \begin{bmatrix} \rho f'(z_x^*)(1 - f'(z_y^*)) \\ (1-\rho) f'(z_y^*)(1 - f'(z_x^*)) \end{bmatrix}. \end{aligned}$$

Now, $\frac{\partial}{\partial h} H(x(h^*), y(h^*)) < 1$ from the proof of Theorem 1. Also from the proof of Theorem 1, we can write:

$$\frac{\partial}{\partial h} H(x(h^*), y(h^*)) = (1-\rho) \frac{(1-\alpha) f'(z_x)}{1 - \alpha f'(z_x)} + \rho \frac{(1-\alpha) f'(z_y)}{1 - \alpha f'(z_y)}.$$

Therefore, either $f'(z_x) < 1$ or $f'(z_y) < 1$ must hold for $\frac{\partial}{\partial h} H(x(h^*), y(h^*)) < 1$ to obtain. By $y^* \leq x^*$ and $y^* < 1/2$, $h^* - (1-h^*) = (1-2y^*) + 2(1-\rho)(x^* - y^*) > 0$, and therefore

$z_x^* > z_y^*$. *It must therefore be that* $f'(z_y^*) < 1$. *With* $|J| > 0$, $1 > x^* + y^*$ *and* $f'(z_x^*) > 0$, *this gives* $\frac{\partial x^*}{\partial \alpha} < 0$, *or equivalently* $\frac{\partial n^*}{\partial \alpha} > 0$. *And, to show* $\frac{\partial h^*}{\partial \alpha} < 0$:

$$\begin{aligned}
\frac{\partial h^*}{\partial \alpha} &= (1-\rho)\frac{\partial x^*}{\partial \alpha} - \rho\frac{\partial y^*}{\partial \alpha} \\
&= \frac{-(1-\rho)\rho}{|J|}(1-(x^*+y^*))\left(f'(z_x^*)(1-f'(z_y^*)) - f'(z_y^*)(1-f'(z_x^*)\right) \\
&= \frac{-(1-\rho)\rho}{|J|}(1-(x^*+y^*))\left(f'(z_x^*) - f'(z_y^*)\right) \\
&< 0,
\end{aligned}$$

the inequality following from $f'(z_x^*) - f'(z_y^*) > 0 \Leftrightarrow z_x^* > z_y^* \Leftrightarrow$:

$$\begin{aligned}
(1-\alpha)h^* + \alpha x^* &> (1-\alpha)(1-h^*) + \alpha y^* \\
\alpha(x^* - y^*) &> (1-\alpha)(1-2h^*),
\end{aligned}$$

which holds by $x^* > y^* \Leftrightarrow n^* < m^*$ *and by* $h^* > \rho > 1/2$.

To show $\lim_{\alpha\to 1} m^* = \lim_{\alpha\to 1} n^* = 1$, *note that* $\lim_{\alpha\to 1} f(z_x) = f(x)$ *and* $\lim_{\alpha\to 1} f(z_y) = f(y)$. $x - f(x) = 0$ *and* $y - f(y) = 0$ *both having solutions* 1 *and* 0. *By* $\frac{\partial m^*}{\partial \alpha} > 0$, $\lim_{\alpha\to 1} m^* = 0$ *is excluded. For* $\lim_{\alpha\to 1} n^*$, $\lim_{\alpha\to 1} y^* = 0$ *implies that* $f'(z_y) = p_1 < 1$, *and therefore,* $\lim_{\alpha\to 1} \frac{\partial x^*}{\partial \alpha} < 0$, *equivalently* $\lim_{\alpha\to 1} \frac{\partial n^*}{\partial \alpha} > 0$, *which excludes* $\lim_{\alpha\to 1} n^* = 0$.

We now show the claim on FOSD shifts to $\{p_k\}$. *Any FOSD shift to* $\{p_k\}$ *can be decomposed into multiple shifts in probability densities* $\epsilon > 0$ *from* $p_{k'}$ *to* $p_{k''}$ *for some* $k'' > k'$. *Then, given this* ϵ, *we can write the system:*

$$\begin{aligned}
x &= \left(\begin{array}{c} ((p_{k'} - \epsilon)\left[(1-\alpha)h + \alpha x\right]^{k'}) + ((p_{k''} + \epsilon)\left[(1-\alpha)h + \alpha x\right]^{k''}) \\ + \sum_{k\neq k',k''} p_k \left[(1-\alpha)h + \alpha x\right]^k \end{array} \right), \\
y &= \left(\begin{array}{c} ((p_{k'} - \epsilon)\left[(1-\alpha)(1-h) + \alpha y\right]^{k'}) + ((p_{k''} + \epsilon)\left[(1-\alpha)(1-h) + \alpha y\right]^{k''}) \\ + \sum_{k\neq k',k''} p_k \left[(1-\alpha)h + \alpha y\right]^k \end{array} \right).
\end{aligned}$$

The comparative statics with respect to ϵ *is then given by:*

$$\begin{aligned}
\begin{bmatrix} \frac{\partial x^*}{\partial \epsilon} \\ \frac{\partial y^*}{\partial \epsilon} \end{bmatrix} &= -J^{-1} \begin{bmatrix} \frac{\partial g_x(z_x^*, z_y^*)}{\partial \epsilon} \\ \frac{\partial g_y(z_x^*, z_y^*)}{\partial \epsilon} \end{bmatrix} = -J^{-1} \begin{bmatrix} z_x^{*k'} - z_x^{*k''} \\ z_y^{*k'} - z_y^{*k''} \end{bmatrix} \\
&= \frac{-1}{|J|} \begin{bmatrix} (1-((1-\alpha)\rho + \alpha)f'(z_y^*))\delta_x - (1-\alpha)\rho f'(z_x^*)\delta_y^* \\ -(1-\alpha)(1-\rho)f'(z_y)\delta_x^* + (1-((1-\alpha)(1-\rho)+\alpha)f'(z_x^*))\delta_y^* \end{bmatrix}.
\end{aligned}$$

where $\delta_x := z_x^{k'} - z_x^{k''} > 0$ *and* $\delta_y := z_y^{k'} - z_y^{k''} > 0$.

$$\begin{aligned}
\frac{\partial h}{\partial \epsilon} &= (1-\rho)\frac{\partial x}{\partial \epsilon} - \rho\frac{\partial y}{\partial \epsilon} \\
&= \frac{-1}{|J|}\left(\delta_x(1-\rho)(1-\alpha f'(z_y)) - \delta_y\rho(1-\alpha f'(z_x))\right) > 0 \\
&\Leftrightarrow \frac{\delta_x}{\delta_y}\frac{1-\rho}{\rho} > \frac{1-\alpha f'(z_x)}{1-\alpha f'(z_y)}. \qquad \text{(A.10)}
\end{aligned}$$

The right-hand-side of (A.10) *is less than one at* (z_x^*, z_y^*)*, which is shown in the proof of Proposition 1. Now:*

$$\begin{aligned}
\frac{\delta_x^*}{\delta_y^*} \geq \frac{z_x^*}{z_y^*} &= \frac{(1-\alpha)h^* + \alpha x^*}{(1-\alpha)(1-h^*) + \alpha y^*} \\
&\geq \frac{(1-\alpha)\rho + \alpha x^*}{(1-\alpha)(1-\rho) + \alpha y^*} = \frac{\rho}{1-\rho}\left[\frac{(1-\alpha)\frac{1-\rho}{y^*} + \alpha\frac{x^*}{y^*}\frac{1-\rho}{\rho}}{(1-\alpha)\frac{1-\rho}{y^*} + \alpha}\right] \\
&\geq \frac{\rho}{1-\rho}
\end{aligned}$$

where the first inequality follows from $\frac{\delta_x}{\delta_y}$ *decreasing in* k'' *and increasing in* k' *upon* $k'' \to \infty$*, the second inequality follows from Lemma A.6, and the third inequality follows from* $\frac{x^*}{y^*}\frac{1-\rho}{\rho} > 1$ *which is shown in the proof of Lemma A.6. Therefor the left-hand-side of* (A.10) *is above one.*

Finally, to show n^* *increases with FOSD shifts to* $\{p_k\}$*, a decrease in* h^* *decreases each term in the right-hand-side of* (A.3) *(i.e. for each* k*). Moreover, an FOSD shift in* $\{p_k\}$ *shifts probabilities to larger* k*, also decreasing the right-hand-side of* (A.3)*. Therefore,* x^* *unambiguously decreases, or equivalently* n^* *increases.*

For the last claim, the comparative statics with respect to ρ *is then given by:*

$$\begin{aligned}
\begin{bmatrix} \frac{\partial x^*}{\partial \rho} \\ \frac{\partial y^*}{\partial \rho} \end{bmatrix} &= -J^{-1}\begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial \rho} \\ \frac{\partial g_y(x^*,y^*)}{\partial \rho} \end{bmatrix} = -J^{-1}\begin{bmatrix} -(1-\alpha)(1-(x^*+y^*))f'(z_x^*) \\ (1-\alpha)(1-(x^*+y^*))f'(z_y^*) \end{bmatrix} \\
&= \frac{-1}{|J|}(1-\alpha)(1-(x^*+y^*))\begin{bmatrix} -f'(z_x^*)(1-\alpha f'(z_y^*)) \\ f'(z_y^*)(1-\alpha f'(z_x^*)) \end{bmatrix}.
\end{aligned}$$

$x^* \geq y^*$ *implies* $x^* \leq 1 - y^*$*, equivalently* $x^* + y^* \leq 1$*.* $1 - \alpha f'(z_x^*)$ *where equality* (A.3) *holds, and* $1 - \alpha f'(z_y^*)$ *where equality* (A.4) *holds; see proof of Proposition 1. Therefore,* $\frac{\partial x^*}{\partial \rho} > 0$ *and* $\frac{\partial y^*}{\partial \rho} < 0$*, equivalently* $\frac{\partial n^*}{\partial \rho} < 0$ *and* $\frac{\partial m^*}{\partial \rho} > 0$*.*

□

Proof of Proposition 3. We construct the comparative statics of P^* with respect to α, ϵ and ρ using the proof of Proposition 2, and establish their signs to be negative, positive, and negative, respectively.

First:

$$\begin{aligned}
\frac{dP^*}{d\alpha} &= (m^* + n^* - 1) + \alpha\left(\frac{\partial x^*}{\partial \alpha} + \frac{\partial y^*}{\partial \alpha}\right) \\
&= (m^* + n^* - 1)\left(1 + \frac{\alpha}{|J|}(\rho f'(z_x^*) + (1-\rho)f'(z_y^*) - f'(z_x^*)f'(z_y^*))\right),
\end{aligned}$$

so, $\frac{dP^*}{d\rho} > 0$ if and only if:

$$|J| > -\alpha(\rho f'(z_x^*) + (1-\rho)f'(z_y^*) - f'(z_x^*)f'(z_z^*)). \tag{A.11}$$

From the proof of Proposition 2 we may calculate:

$$|J| = 1 - ((1-\alpha)(1-\rho) + \alpha)\, f'(z_x^*) - ((1-\alpha)\rho + \alpha)\, f'(z_y^*) + \alpha f'(z_x^*)f'(z_z^*),$$

and thus (A.11) is equivalent to:

$$1 > (1-\rho)f'(z_x^*) + \rho f'(z_y^*).$$

To show this inequality, from the proof of Proposition 2 we have:

$$\frac{\partial}{\partial h}H(x(h^*), y(h^*)) = (1-\rho)\frac{(1-\alpha)f'(z_x^*)}{1-\alpha f'(z_x^*)} + \rho\frac{(1-\alpha)f'(z_y^*)}{1-\alpha f'(z_y^*)} < 1.$$

If we define $g(z) \equiv \frac{(1-\alpha)z}{1-\alpha z}$, a convex function which is increasing for $z \geq 0$ and satisfies $g(0) = 0$ and $g(1) = 1$, this can be written:

$$(1-\rho)g(f'(z_x^*)) + \rho g(f'(z_y^*)) < 1.$$

By Jensen's inequality, we have:

$$1 > (1-\rho)g(f'(z_x^*)) + \rho g(f'(z_y^*)) > g((1-\rho)f'(z_x^*) + \rho f'(z_y^*)),$$

which implies $1 > (1-\rho)f'(z_x^*) + \rho f'(z_y^*)$.

Second:

$$\begin{aligned}\frac{dP^*}{d\epsilon} &= -\alpha\left(\frac{\partial x^*}{\partial \epsilon} + \frac{\partial y^*}{\partial \epsilon}\right)\\ &= \frac{1}{|J|}\left((1-f'(z_y^*))\delta_x + (1-f'(z_x^*))\delta_y\right)\\ &= \frac{(\delta_x+\delta_y)}{|J|}\left(1-\left(\frac{\delta_y}{\delta_x+\delta_y}f'(z_x^*) + \frac{\delta_x}{\delta_x+\delta_y}f'(z_y^*)\right)\right).\end{aligned}$$

From the Proposition 2, $\delta_x/\delta_y \geq \rho/(1-\rho)$, or equivalently $\rho \leq \frac{\delta_x}{\delta_x+\delta_y}$. With $1 > ((1-\rho)f'(z_x^*)) + \rho f'(z_y^*)$ from above and $f'(z_y^*) < f'(z_x^*)$, these give $1 > \frac{\delta_y}{\delta_x+\delta_y}f'(z_x^*) + \frac{\delta_x}{\delta_x+\delta_y}f'(z_y^*)$, and thus $\frac{dP^*}{d\epsilon} \geq 0$.

Third:

$$\begin{aligned}\frac{dP^*}{d\rho} &= -\alpha\left(\frac{\partial x^*}{\partial \rho} + \frac{\partial y^*}{\partial \rho}\right)\\ &= \frac{(1-\alpha)(1-(x^*+y^*))}{|J|}\left(-(f'(z_x^*) - f'(z_y^*))\right) < 0,\end{aligned}$$

as $f'(z_y^*) < f'(z_x^*)$ shown in the proof of Proposition 2.

□

Proof of Proposition 4. We can write polarization as:

$$P(m^*, n^*) = \alpha(n-(1-h^*)) + \alpha(m^*-h^*),$$

and thus the niche's share to polarization:

$$S_n^*(m^*, n^*) = \frac{1}{\frac{\alpha(m^*-h^*)}{\alpha(n-(1-h^*))} + 1}.$$

Substitute $h^* = (1-\rho)(1-n^*) + \rho m^*$, the fraction in the denominator becomes:

$$\frac{(m^*-h^*)}{(n-(1-h^*)} = \frac{(1-\rho)(m^*+n^*-1)}{\rho(m^*+n^*-1)} = \frac{1-\rho}{\rho},$$

which gives $S_n^*(m^*, n^*) = \rho$. □

Proof of Proposition 5. When $\alpha = 0$ or $\mathbb{E}[k] \leq B$ then both types consume the same mix of news content so the variety is the same for both. Now in the case where $\alpha > 0$ and $\mathbb{E}[k] > B$ observe first that:

$$x_m^* - x_n^* \quad = \quad \alpha(m^* - n^*) + (1-\alpha)(2h^* - 1) > 0 \tag{A.12}$$

where the inequality follows from $m^* > n^*$ and $h^* > \rho > \frac{1}{2}$ and second that

$$x_m^* + x_n^* - 1 \quad = \quad \alpha(m^* + n^* - 1) > 0 \tag{A.13}$$

where the inequality follows from $m^* + n^* > 1$, which has been shown in the proof of Proposition 1. These two inequalities imply $x_m^* > x_n^*$ and $x_m^* > 1 - x_n^*$, these and the properties of V imply the result. □

Proof of Proposition 6.

1. The comparative statics results in Proposition 2 imply that $x_n^* = (1-\alpha)(1-h^*) + \alpha n^*$ is increasing in α by virtue of h^* and n^* being respectively decreasing and increasing in α. To establish the result we note that for $x_n^*(\alpha = 0) = 1 - h^* < 1 - \rho < \frac{1}{2}$ and $x_n^*(\alpha = 1) = 1$ so variety is initially increasing in x_n^* until a level of homophily $\hat{\alpha}$ where $x_n^*(\hat{\alpha}) = \frac{1}{2}$ and is decreasing for $\alpha > \hat{\alpha}$.

2. When $\mathbb{E}[k] \leq \frac{1}{1-\rho}$ then for $\alpha = 0$ we are in the mass-market steady state where $m^* = 1, n^* = 0$ and variety is minimized. We noted that $B(\alpha)$ is decreasing in α and $\lim_{\alpha\ to 1} B(\alpha) = 1$ so there exists $bar \alpha$ such that $\mathbb{E}[k] = B(\bar{\alpha}))$ and for $1 > \alpha > \bar{\alpha}$ we are in the mixed equilibrium where $0 < m^*, n^* < 1$. First, this implies that there exists $\hat{\alpha}_m 1$ such that for $\bar{\alpha} \geq \alpha < \hat{\alpha}_m 1$ variety is increasing. Second, we observe that $\lim_{\alpha \to 1} x_m^*(\alpha) = 1$ and so there exists another threshold $\hat{\alpha}_m 2$ such that for $\alpha > \hat{\alpha}_m 2$ that variety is decreasing. □

Proof of Proposition 7. The comparative statics results in Proposition 2 imply that $x_m^*(\rho)$ is increasing in ρ and $x_n^*(\rho)$ is decreasing in ρ. Both results in the proposition then follow from the properties of V and noting that $x_m^*(\rho) > \frac{1}{2} \forall \rho > \frac{1}{2}$. □

Proof of Proposition 8. The proof of Propositon 1 shows that there is a unique locally stable steady state $m^*(e_n, e_m), n^*(e_n, e_m)$ for any pair $(e_n, e_m) \in [0,1) \times [0,1)$. We now proceed to establish that $\exists \overline{C}$ such that $C'' > \overline{C}$ each player's objective is concave, the first order condition is sufficient for the optimal effort choice and the best responses functions are continuous. The second derivative of each player's objective are:

$$\rho \frac{\partial^2 m^*}{\partial e_m^2} - (1-\rho)\frac{\partial^2 n^*}{\partial e_m^2} - \frac{\partial^2 C(\rho e_m)}{\partial e_m^2} \tag{A.14}$$

$$(1-\rho)\frac{\partial^2 n^*}{\partial e_n^2} - \rho \frac{\partial^2 m^*}{\partial e_n^2} - \frac{\partial^2 C((1-\rho)e_n)}{\partial e_n^2} \tag{A.15}$$

Hence, a sufficient condition for concavity under the condition $C'' > \overline{C}$ is that the magnitude of $\frac{\partial^2 m^*}{\partial e_m^2}, \frac{\partial^2 n^*}{\partial e_m^2}, \frac{\partial^2 m^*}{\partial e_n^2}, \frac{\partial^2 n^*}{\partial e_n^2}$ are bounded. In each case, the second derivative is a bounded term (given by $\frac{1}{|J|^2}$ where $|J|$ is the determinant of the Jacobian of the system of equations given in equation B) multiplied by a second term that is a sequence of product, addition or subtraction operations involving $\alpha, \rho, e_m, e_n, f(z_n), f(z_m), f'(z_n), f'(z_m), f''(z_n), f''(z_m)$ and the first derivatives $\frac{\partial m^*}{\partial e_m}, \frac{\partial n^*}{\partial e_m}, \frac{\partial m^*}{\partial e_n}, \frac{\partial n^*}{\partial e_n}$. All these terms are themselves bounded and so the entire term comprises a finite sequence of product, addition and subtraction operations will also be bounded. The best response of each player may be written as

$$\rho \frac{\partial m^*}{\partial e_m} - (1-\rho)\frac{\partial n^*}{\partial e_m} - \rho C'(\rho e_m) = 0 \tag{A.16}$$

$$(1-\rho)\frac{\partial n^*}{\partial e_n} - \rho \frac{\partial m^*}{\partial e_n} - (1-\rho)C'((1-\rho)e_n) = 0 \tag{A.17}$$

and via the implicit function theorem we find that:

$$\begin{aligned} \frac{\partial BR_m(e_n)}{\partial e_n} &= -\frac{\rho \frac{\partial^2 m^*}{\partial e_m \partial e_n} - (1-\rho)\frac{\partial^2 n^*}{\partial e_m \partial e_n}}{\rho \frac{\partial^2 m^*}{\partial e_m^2} - (1-\rho)\frac{\partial^2 n^*}{\partial e_m^2} - \rho C''(\rho e_m^*)} \\ \frac{\partial BR_n(e_m)}{\partial e_m} &= -\frac{(1-\rho)\frac{\partial^2 n^*}{\partial e_m \partial e_n} - \rho \frac{\partial^2 m^*}{\partial e_m \partial e_n}}{(1-\rho)\frac{\partial^2 n^*}{\partial e_n^2} - \rho \frac{\partial^2 m^*}{\partial e_n^2} - (1-\rho)C''((1-\rho)e_n^*)} \end{aligned}$$

The same argument that bounded the value of the second derivatives also bounds $\frac{\partial^2 m^*}{\partial e_m \partial e_n}$ and $\frac{\partial^2 n^*}{\partial e_m \partial e_n}$. It is then straightforward to observe that for $\delta < 1$ $\exists \overline{C}$ such that

$$\left|\frac{\partial BR_m(e_n)}{\partial e_n}\right|, \left|\frac{\partial BR_n(e_m)}{\partial e_m}\right| < \delta$$

and hence the best response functions are continuous and there exists a unique Nash equilibrium (e_m^*, e_n^*) where they coincide. We can verify that the equilibrium will be interior by first observing that for any $e_m \in [0,1]$ the right-hand side of equation 16 is positive and the marginal costs of investment go to zero for $e = 0$ so $BR_n(0) > 0$. Second, we observe that when $e_n > 0$, the right-hand side of equation 15 is strictly positive and so $BR_m(e_n) > 0$ for $e_n > 0$. These two observations rule out any equilibria where either player invests 0. Finally, our assumption on the convexity of the cost function guarantees that $e_n = 1$ or $e_m = 1$ is not part of an equilibrium.

First, observe in equation (A.20) that a steady state is increasing in the mass-market investment e_m and decreasing in the niche investment e_n.

We now proceed to establish that the niche-market player invests more than the mass-market player when $\alpha = 0$. In this case, the steady state relationships for m, n and h may be written as:

$$n^* = 1 - (1 - e_n) f(h^*) \tag{A.18}$$

$$m^* = 1 - (1 - e_m) f(1 - h^*) \tag{A.19}$$

$$h^* = \rho - \rho(1 - e_m) f(1 - h^*) + (1 - \rho)(1 - e_n) f(h^*) \tag{A.20}$$

First, for $\rho e_m = (1-\rho)e_n = \mu$ and $h^* = \frac{1}{2}$ the right-hand side of equation A.20 is greater than $\frac{1}{2}$. Then, using the properties of the right-hand side of equation A.20 (mapping $[0,1] \to [0,1]$, continuous, positive first and third derivatives) shown in the proof of Proposition 1 we may conclude that $h^*(\frac{\mu}{\rho}, \frac{\mu}{1-\rho}) > \rho$. Second, observe in equations A.19, A.20 and (A.20) that a steady state is increasing in the mass-market investment e_m and decreasing in the niche investment e_n. Now, to establish the results by way of contradiction, suppose $(1-\rho)\, e_n^* < \rho e_m^*$. The two points above then imply that the steady state $h^*(e_m^*, e_n^*) > \frac{1}{2}$. However, this is a contradiction of the first order conditions for each player (equations (15) and (16)), where, $h^*(\rho e_m^*, (1-\rho)e_n^*) > \frac{1}{2} \Rightarrow (1-\rho)\, e_n^* > \rho e_m^*$. Therefore, we conclude that $(1-\rho)\, e_n^* > \rho e_m^*$.

To show that $h^* > \frac{1}{2}$, by way of contradiction suppose this was not the case $h^* < \frac{1}{2}$. The first order conditions of the two players equations for the case $\alpha = 0$ simply to:

$$C'\left(\rho e_m^*\right) = \frac{f\left(1-h^*\right)}{\Delta} \tag{A.21}$$

$$C'\left(\left(1-\rho\right)e_n^*\right) = \frac{f\left(h^*\right)}{\Delta} \tag{A.22}$$

These immediately implies that $(1-\rho)\, e_n^* < \rho e_m^*$ if $h^* < \frac{1}{2}$ which contradicts our earlier result $(1-\rho)\, e_n^* > \rho e_m^*$. Thus establishing that $h^* > \frac{1}{2}$.. □

Proof of Proposition 9. Choose any A such that $\rho e_m^* < A < (1-\rho)\, e_n$. Now write $\widetilde{n}\,(h, A) = 1 - \left(1 - \frac{A}{1-\rho}\right) f(h)$ and $\widetilde{m}\,(h, A) = 1 - \left(1 - \frac{A}{\rho}\right) f\,(1-h)$. We can find $\frac{\partial \widetilde{n}}{\partial A}$ and $\frac{\partial \widetilde{m}}{\partial A}$ as $\frac{1}{1-\rho} f(h)$ and $\frac{1}{\rho} f\,(1-h)$ respectively and can conclude that the function $H = \rho\widetilde{m}\,(h, A) + (1-\rho)\,\widetilde{n}\,(h, A)$ is decreasing in A because $\frac{\partial H}{\partial A} = f\,(1-h) - f(h) < 0$ for $h > \frac{1}{2}$. Now, we can conclude (using the properties of H shown in Proposition 1) that the value $h^*\,(A, A)$ that satisfies $\rho\widetilde{m}\,(h^*, A) + (1-\rho)\,(1 - \widetilde{n}\,(h^*, A)) = h^*$ is also decreasing in A. Finally, to establish the result, we note that $\rho e_m^* < A < (1-\rho)\, e_n^*$ and so increasing e_n from $\frac{A}{1-\rho}$ to e_n^* and decreasing e_m from $\frac{A}{\rho}$ to e_m^* will reduce h so $h^*\,(e_m^*, e_n^*) < h^*\,(A, A) < h^*\,(0, 0)$. □

Proof of Proposition 10. To establish the first result, it is straightforward to observe that as $e_m, e_n \to 1$ then the solution to the system of equations A.19 and A.20 goes to $m^* = n^* = 1$ and polarization $P(e_m^*, e_n^*) = \alpha(m^* + n^* - 1) \to \alpha$. To establish the second result, we observe that when $f(x) = x^k$ then $lim_{k\to\infty} f(x) \to 0$ for all $x < 1$ so again the solution to the system of equations A.19 and A.20 goes to $m^* = n^* = 1$ and polarization $P(e_m^*, e_n^*) = \alpha(m^* + n^* - 1) \to \alpha$ for $k \to \infty$. Finally, we note that this also implies that $lim_{k\to\infty} h^* = \rho$ and so the right hand side of equations A.21 and A.22 goes to zero. This establishes that the equilibrium investments e_m^*, e_n^* also go to zero in this limit. □

Proof of Proposition 11. The proof of Proposition 1 includes the case of different homophily parameters α_m, α_n. □

Proof of Proposition 12.

When $E[k] \leq \frac{1}{1-\rho}$ then $m^*(0,0) = h^*(0,0) = 1$ and $n^*(0,0) = 0$. Moreover, this satisifies the first order conditions given in equations (23) and (24). To show the converse, suppose there is an equilibrium with zero effort. This implies that the right-hand side of equations (23) and (24) are 0. The only steady state where this is possible is $m^* = 1, n^* = 0$ which from Proposition 1 implies our result.

□

Proof of Proposition 13. Define $\hat{\alpha}_m(\alpha_n)$ as the implicit solution for α_m in equation 23 and $\hat{\alpha}_n(\alpha_m)$ as the implicit solution for α_n in equation 24. For sufficient convexity of D both are continuous functions $[0,1] \to [0,1]$. Hence there exists a point on $[0,1] \times [0,1]$ where equations 23 and 24 are satisfied. Moreover, when $E[k] > \frac{1}{1-\rho}$ we have $\hat{\alpha}_m(0), \hat{\alpha}_n(0) > 0$ and $\hat{\alpha}_m(1), \hat{\alpha}_n(1) < 1$ such that the equilibrium is interior. Finally, the returns to homophily for a niche individual shown on the right-hand side of equation 24 is greater than the returns to homophily for mass-market individuals shown in equation 23 since $\rho > \frac{1}{2}$. This immediately implies that the niche individuals will exhibit greater homophily $\alpha_n^* > \alpha_m^*$ in equilibrium. □

Proof of Proposition 13. Analogous to the proof of Proposition 4, it is straight forward to derive:

$$S_n^*(m^*, n^*, \alpha_m^*, \alpha_n^*) = \frac{1}{\frac{\alpha_m^*(m^*-h^*)}{\alpha_n^*(n-(1-h^*))} + 1} = \frac{1}{\frac{\alpha_m^*}{\alpha_n^*}\frac{1-\rho}{\rho} + 1}.$$

The condition $\alpha_m^* < \alpha_n^*$ then implies that $S_n^*(m^*, n^*, \alpha_m^*, \alpha_n^*) > \rho$.

Now, further impose $0 < \alpha_m^* < \alpha_n^* < 1$, which implies that f.o.c's (23) and (24) must hold. Taking the ratio of these f.o.c's gives:

$$\frac{D'(\alpha_m^*)}{D'(\alpha_m^*)} = \frac{m^* - h^*}{n^* - (1-h^*)} = \frac{1-\rho}{\rho},$$

the second equality shown in the proof of Proposition 4. If $D(\alpha) = \beta\alpha^\gamma$, we obtain:

$$\frac{D'(\alpha_m^*)}{D'(\alpha_m^*)} = \left(\frac{\alpha_m^*}{\alpha_m^*}\right)^{\gamma-1},$$

and therefore:

$$\frac{\alpha_m^*}{\alpha_n^*}\frac{(1-\rho)}{\rho} = \left(\frac{1-\rho}{\rho}\right)^{\frac{\gamma}{\gamma-1}},$$

which gives:

$$S_n(m^*, n^*, \alpha_m^*, \alpha_n^*) = \frac{1}{1 + \left(\frac{1-\rho}{\rho}\right)^{\frac{\gamma}{\gamma-1}}}.$$

□

B Derivation of the first-order conditions with lobbyists (Section 4)

Taking the derivative of equations (13) and (14) with respect to e_m and e_n respectively produces the following first order conditions for each player:

$$\rho\frac{\partial m^*}{\partial e_m} - (1-\rho)\frac{\partial n^*}{\partial e_m} - \rho C'(\rho e_m) = 0 \tag{B.1}$$

$$(1-\rho)\frac{\partial n^*}{\partial e_n} - \rho\frac{\partial m^*}{\partial e_n} - (1-\rho)C'((1-\rho)e_n) = 0 \tag{B.2}$$

We use the multivariate implicit function theorem to find the partial derivatives of the steady-state quantities with respect to the investments. Define z_x and z_y as before; $h = (1-\rho)x + \rho(1-y)$. The system of equations (A.19) and (A.20) may be written as:

$$\begin{bmatrix} g_x(x,y) \\ g_y(x,y) \end{bmatrix} = \begin{bmatrix} x - a_n \sum_k p_k \left[(1-\alpha)h + \alpha x\right]^k \\ y - a_m \sum_k p_k \left[(1-\alpha)(1-h) + \alpha y\right]^k \end{bmatrix}.$$

where $g_x(x^*, y^*) = 0$ and $g_y(x^*, y^*) = 0$ defines the steady state. The Jacobian of the system is:

$$J = \begin{bmatrix} 1 - ((1-\alpha)(1-\rho) + \alpha)a_n f'(z_x) & (1-\alpha)\rho a_n f'(z_x) \\ (1-\alpha)(1-\rho)a_m f'(z_y) & 1 - ((1-\alpha)\rho + \alpha)a_m f'(z_y) \end{bmatrix}$$

which has inverse:

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} 1 - ((1-\alpha)\rho + \alpha)a_m f'(z_y) & -(1-\alpha)\rho a_n f'(z_x) \\ -(1-\alpha)(1-\rho)a_m f'(z_y) & 1 - ((1-\alpha)(1-\rho) + \alpha)a_n f'(z_x) \end{bmatrix}.$$

The determinant $|J| > 0$ by stability of the steady state. The comparative statics with respect to a_m and a_n are then given by:

$$\begin{bmatrix} \frac{\partial x^*}{\partial a_m} \\ \frac{\partial y^*}{\partial a_m} \end{bmatrix} = -J^{-1}\begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial a_m} \\ \frac{\partial g_y(x^*,y^*)}{\partial a_m} \end{bmatrix} = -J^{-1}\begin{bmatrix} 0 \\ -f(z_y) \end{bmatrix} \tag{B.3}$$

$$= \frac{f(z_y)}{|J|}\begin{bmatrix} -(1-\alpha)\rho a_n f'(z_x) \\ 1 - ((1-\alpha)(1-\rho) + \alpha)a_n f'(z_x) \end{bmatrix}; \tag{B.4}$$

$$\begin{bmatrix} \frac{\partial x^*}{\partial a_n} \\ \frac{\partial y^*}{\partial a_n} \end{bmatrix} = -J^{-1}\begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial a_n} \\ \frac{\partial g_y(x^*,y^*)}{\partial a_n} \end{bmatrix} = -J^{-1}\begin{bmatrix} -f(z_x) \\ 0 \end{bmatrix} \tag{B.5}$$

$$= \frac{f(z_x)}{|J|}\begin{bmatrix} 1 - ((1-\alpha)\rho + \alpha)a_m f'(z_y) \\ -(1-\alpha)(1-\rho)a_m f'(z_y) \end{bmatrix}. \tag{B.6}$$

It is straightforward to observe that $\frac{\partial x^*}{\partial a_n} = \frac{\partial n^*}{\partial e_n}, \frac{\partial x^*}{\partial a_m} = \frac{\partial n^*}{\partial e_m}, \frac{\partial y^*}{\partial a_n} = \frac{\partial m^*}{\partial e_n}$ and $\frac{\partial y^*}{\partial a_m} = \frac{\partial m^*}{\partial e_m}$ and so we may substitute the relationships in equations B.6 and B.4 into the first order conditions in equations B.1 and B.2 for the respective partial derivatives. Re-arranging and defining $\Delta \equiv |J|$ establishes equations (15) and (16) in the main text.