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JEL Classification: C72, D74, D85

Keywords: network games, Contests, variational inequality

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Networks in Conflict: A Variational Inequality Approach*

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Abstract

We study a very general contest game in which players exert efforts in multiple battles. The conflict structure, which represents who participates in which battlefield, is arbitrary and can be represented by a hypergraph. We show, under mild conditions on the cost function and contest technology, that the set of pure strategy Nash equilibria is nonempty and convex, and provide equivalent characterizations using techniques from Variational Inequality (VI). We demonstrate that the strong monotonicity of the cost function always implies the uniqueness of Nash equilibrium regardless of the conflict structure. We also perform an extensive comparative statics analysis with respect to the parameters of the model and discuss several applications of our model. Our general model incorporates many existing models of single or multi-battle contests as special cases when the conflict network and/or the cost function take particular forms.

Keywords: network games, multi-battle contests, variational inequality, uniqueness, comparative statics

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1 Introduction

In many situations, different agents who are linked through a common "space" (e.g., a spatial or a social or a ethnic space) are involved in conflictual relationships. One could think of R&D, rent seeking, political campaigns, patent races, advertisement, ethnic conflicts, etc.[1] For example, in the case of geographical conflicts, different countries may face distinct border issues and fight with different neighboring countries to claim territory. In multi-party electoral campaigns, the same party may face heterogeneous populations of voters across districts, and must optimally allocate limited resources to maximize the total number of votes. In R&D races with multiple technologies and non-identical market values and participants, a firm may strategically choose to close the R&D office of one technology with low-market potential and fierce competition, in order to focus on more promising avenues. In ethnic conflicts, military alliances and enmities affect the intensity of a conflict. Each group may put all its resources in one conflict and neglect other conflicts.

In order to study these issues, we need a general framework in which the conflict structure is arbitrary and the conflict game involves multiple players competing in multiple battles. This is what we do in this paper by using an arbitrary network to model the conflict structure.

In our model, there is a finite number of players who compete against each other on different battles. Each battle has a certain value, and the probability of each participant winning a battle is described by a general contest success function. The conflict structure is modeled as a network in which each node corresponds to a player and each edge (or link) between two players means that these two players participate in a battle against each other. For example, in a complete network, all players compete simultaneously in all battles. Thus, each player can be involved in *multiple battles* and different battles may involve different subsets of players. Each player cares about the expected values of winning battles net of the cost of efforts. We study the Nash equilibrium of this game in which each player optimally decides how much effort she exerts in each battle she is involved in.

Observe that, since the equilibrium in a single-battle contest does not exhibit a closed form solution unless symmetry is assumed, our general conflict game does not have an explicit solution either. Given the high dimensionality of efforts, the heterogeneity of players and battles, and the arbitrariness of the conflict topology (network), providing the existence and uniqueness of a Nash equilibrium is very challenging. The standard fixed-point theorem for existence does not work here because the payoff function is not continuous due to a jump in the winning probabilities at the origin and because of the multi-dimensional and possibly unbounded strategy space. Since there is more than one battle, the conflict game is not an aggregate game. Moreover, our game is not supermodular so we cannot use the techniques from this literature.

Our first result is to provide a theorem that shows that there always exists a Nash equilibrium under mild assumptions on the general contest function and the cost function. To prove this existence theorem, we use results from the discontinuous game literature (see Reny (1999)). More specifically, we verify that our conflict game satisfies all the conditions imposed in this

[1] See Tullock (1980); Snyder (1989); Nti (1999); Konrad (2009).

literature, that is compactness, quasi-concaveness, reciprocally upper semi-continuity and payoff security, so that we can apply Reny (1999)'s main theorem.

Our second main result is to characterize the set of equilibria and to show under which conditions a Nash equilibrium is unique. As each player joins multiple battles with increasing marginal cost of efforts, it is possible for her to strategically abandon some battles. As a result, it is very likely that corner solutions (no effort in some battles) may exist in equilibrium, which implies that the first order conditions do not always hold with equality. This, clearly, complicates the equilibrium characterization. To address this technical challenge, we employ techniques from Variational Inequality (VI) to re-formulate the equilibrium condition so that the solution of the Nash equilibrium is equivalent to the solution to a VI problem. The VI, defined on the strategy space, is associated with an operator $\mathbf{F}$ that is linked to the gradients of the payoffs of the original conflict game. Since the solution to the VI problem is not limited to interior points, we do not need to artificially distinguish interior equilibria from corner ones. Interestingly, the operator defining this VI naturally satisfies some monotonicity properties, which limits the possible solution set. Given the equivalence between the VI solution and the Nash equilibrium, we are able to characterize the set of equilibria. In particular, we show that the operator $\mathbf{F}$ is weakly monotone such that the set of equilibria is always convex. The convexity of the set of equilibria implies that an equilibrium, if not unique, is never isolated. The multiplicity of equilibria arises due to the lack of strict monotonicity of the operator $\mathbf{F}$. However, if the cost is strongly monotone, the equilibrium always lies in the space where the operator $\mathbf{F}$ is strictly monotone; hence uniqueness is obtained. On the other hand, given any conflict structure, we can always pin down the cost function such that the resulting conflict game has a continuum of equilibria. As a result, to characterize the equilibrium set, we need to combine both techniques from VI and necessary conditions imposed on equilibrium from the cost function and the conflict topology.

Our third main result is to provide general comparative statics results. As different battles are linked to each other, any local shock in one battle or one player naturally propagates to the rest of the conflict network. For policy applications and welfare analysis, it is important to measure these direct and indirect network effects analytically. Without a closed form solution, the comparative statics analysis usually relies on the Implicit Function Theorem applied to the first order conditions. However, since the equilibrium is not always interior, this standard approach does not directly work. The VI formulation of equilibrium is not only useful for equilibrium characterization, but also for comparative statics results. We show that near any non-degeneracy equilibrium, the mapping from the parameter space to the equilibrium is continuously differentiable. Moreover, we provide exact formulas for the comparative statics analysis. These expressions enable us to conduct further comparative statics on aggregate effort and payoffs. As applications, we briefly discuss optimal battle subsidy problems and provide some examples.

1.1 Literature

This paper is closely related to two branches of literature: multi-battle contests, and games played on networks. Also, the paper is linked to the recent literature using VI techniques in economics.

In the literature on multi-battle contests, the structure of conflict is often very special. For example, it can have a complete structure so that every player join every battle. Moreover, the focus tends to be on the valuation linkages among battles, dynamics, and alternative contest success functions such as all-pay, instead of conflict topology (Konrad and Kovenock (2009), Fu, Lu, and Pan (2015), Roberson (2006); Kvasov (2007); Roberson and Kvasov (2012)).[2] Typically, in these contest papers, the equilibrium is usually solved in explicit forms under certain symmetry assumptions on players. Therefore, their proof techniques cannot be applied to heterogeneous players and general conflict structures. For the same reason, comparative statics exercises are carried out only in some special cases due to tractability. The VI approach we adopt in this paper does not rely on the symmetry of the players or on some specific restrictions on the conflict structure and, therefore, has broader applications. Moreover, the VI approach derives sharper predictions on the set of equilibria, as well as extensive comparative statics analysis.

In the literature on games played on networks,[3] the network summarizes social relations among players in a group, and thus can be represented by a graph. In our paper, the conflict topology, in general, is an hyper-graph, as each (hyper-)edge (battle in our model) could involve more than two players. Apart from this distinction, most papers on network games do not have contest components.[4] Exceptions include Goyal and Vigier (2014), Jackson and Nei (2015), Franke and Öztürk (2015), Hiller (2017), König, Rohner, Thoenig, and Zilibotti (2017), Rietzke and Matros (2018), Kovenock and Roberson (2018), all of which have a different focus and use specific forms.[5] For example, König, Rohner, Thoenig, and Zilibotti (2017) only consider a single Tullock lottery contest with positive (negative) spillovers by friends (enemies) so that they can derive closed-form solutions, which allow them to structural estimate the model for the great war of Africa. Goyal and Vigier (2014) also use the Tullock contest function and focus on optimal network design. Hiller (2017) develops a game-theoretic model of signed *network formation*. Rietzke and Matros (2018) focus on the Tullock and all-pay auction contest functions and study special families of networks such as biregular graph, star, etc, with linear cost functions, so that closed form solutions of equilibrium can be obtained. Our paper is more closely related to Franke and Öztürk (2015), which model conflicts on graphs using quadratic total cost functions and use examples to show that peacefully resolution of a battle may induce intensified conflicts in other battles. Our paper goes much further both on the generality of the model and on the completeness of the analysis on equilibrium characterization and comparative statics.

Early adopters of the VI approach in economics include Gabay and Moulin (1978) and Harker (1984, 1991).[6] The strict monotonicity of the operator in VI is closely related to Rosen (1965)'s *diagonal strict concave condition* for uniqueness in concave games. However, our contest model does not satisfy strict monotonicity in the entire strategy space, and thus uniqueness is not always guaranteed. In fact, we construct examples with a continuum of equilibria. Moreover, the VI approach enables us to establish the convexity of equilibrium set and to conduct exten-

[2] See Kovenock and Roberson (2012) for a recent survey.

[3] For an overview, see Jackson and Zenou (2015).

[4] See Jackson (2008), Dziubiński, Goyal, and Minarsch (2016), Dziubiński and Goyal (2017), Bimpikis, Ehsani, and Ilkılıç (2019), Bimpikis, Ozdaglar, and Yildiz (2016), Malamud and Rostek (2017).

[5] For an overview, see Dziubiński, Goyal, and Vigier (2016)

[6] For recent applications of VI in economics, see Ui (2016) for Bayesian games, Ewerhart (2014) and Ewerhart and Quartieri (2015) for Tullock contest models, and Nagurney (1999) for an overview.

sive comparative statics analysis, which are not considered in Rosen (1965). There is also a very recent literature that use VI techniques to analyze network games (see Melo (2019); Parise and Ozdaglar (2019) and Zenou and Zhou (2019)). The network structure in these papers specifies how the neighbors' actions affect a player's payoff, while in our model, the conflict topology is represented by a hypergraph, which captures who participates in which battle. As a result, the conflict game in our paper does not have the local aggregation properties, which are essential for proving existence and uniqueness of equilibrium in these papers.

The rest of the paper unfolds as follows. In the next section, we present the model and discuss our main assumptions. Section 3 studies the existence and uniqueness of the Nash equilibrium of our game and provides some examples to illustrate our results. Section 4 investigates the comparative statics properties of the Nash equilibrium of our conflict game. Section 5 discusses some of our results. Finally, Section 6 concludes. Appendix A provides some notations and some preliminary results. Appendix B gives all the proofs for the existence and uniqueness of equilibrium, and the comparative statics results. Appendix C provides additional examples and results.

2 Model

2.1 Setup and notations[7]

Players and battles Consider a set of players, $\mathcal{N}$, and a set of battles, $\mathcal{T}$. We use $i = 1, 2, 3, \ldots, n$ to denote players, with $N = |\mathcal{N}| \geq 2$ and $t = a, b, c, \ldots,$ to denote battles, with $T = |\mathcal{T}| \geq 1$.

Conflict structure Let the $N \times T$ matrix $\mathbf{\Gamma} = (\gamma_i^t)$ represent the conflict structure. Specifically, $\gamma_i^t = 1$ if player i is a participant of battle t; otherwise $\gamma_i^t = 0$. Each player can be involved in multiple battles and different battles may involve different subsets of players. Let

$$\mathcal{N}^t = \{i \in \mathcal{N} : \gamma_i^t = 1\} \subseteq \mathcal{N}$$

denote the set of players involved in battle t. Let $n^t = |\mathcal{N}^t| \geq 2$ denote its cardinality. Similarly, let

$$\mathcal{T}_i = \{t \in \mathcal{T} : \gamma_i^t = 1\} \subseteq \mathcal{T}$$

denote the set of battles that player i takes part in. Let $t_i = |\mathcal{T}_i| \geq 1$ denote the cardinality. Clearly, $i \in \mathcal{N}^t$ if and only if $t \in \mathcal{T}_i$.

Let v^t denote the value of battle t, which might be heterogeneous across battles. Taking the conflict structure $\mathbf{\Gamma}$ as given, player i's strategy is to choose a *battle-specific* nonnegative effort x_i^t for each battle $t \in \mathcal{T}_i$ she is involved in. Thus, player i's strategy is a vector $\mathbf{x}_i = \{x_i^t\}_{t\in\mathcal{T}_i} \in \mathbf{R}_+^{t_i}$.[8] Given player i's strategy $\mathbf{x}_i$, we denote $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in \mathbf{R}_+^{\bar{n}}$ as the whole strategy profile, and $\mathbf{x}^t = \{x_i^t\}_{i\in\mathcal{N}^t} \in \mathbf{R}_+^{n^t}$ as the effort vector in battle t. Let $\bar{n} = \sum_{t\in\mathcal{T}} n^t = \sum_{i\in\mathcal{N}} t_i = \sum_{i\in\mathcal{N}, t\in\mathcal{T}} \gamma_i^t$ denote the dimension of the strategy profile $\mathbf{x}$.

[7]For notations of special classes of matrices and vectors and some preliminary results, see Appendix A.

[8]This is different from Rietzke and Matros (2018), who study a contest model on networks in which each player chooses a *single action* instead of battle-specific actions.

Contest technology Given $\mathbf{x}^t$, the profile of efforts in battle t, the probability of winning battle t for player i, also known as the *Contest Success Function* (CSF), is given by:[9]

$$p_i^t(\mathbf{x}^t) = \begin{cases} \frac{f^t(x_i^t)}{\sum_{j\in\mathcal{N}^t} f^t(x_j^t)} & \text{when } \mathbf{x}^t \neq (0,0,\cdots,0) \\ \frac{1}{|\mathcal{N}^t|} = \frac{1}{n^t} & \text{otherwise} \end{cases} \tag{1}$$

for some increasing function $f^t(\cdot)$, which is called the *contest production function*.[10]

Payoffs The payoff function of player $i \in \mathcal{N}$ equals

$$\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{t\in\mathcal{T}_i} v^t p_i^t(\mathbf{x}^t) - C_i(\mathbf{x}_i), \tag{2}$$

where the first term is the sum of battle values multiplied by the corresponding probability of winning, and the second part is just the cost of efforts.[11] Here the cost $C_i(\mathbf{x}_i) : \mathbf{R}_+^{t_i} \mapsto [0, +\infty]$ depends on all the efforts player i exerts in each battle she is involved in. Note that we allow the possibility of $C_i(\hat{\mathbf{x}}_i) = +\infty$ for some $\hat{\mathbf{x}}_i$, which simply means that $\hat{\mathbf{x}}_i$ is infeasible for player i due to either resource constraints or technological constraints. We will specify the cost functions thoroughly in the next subsection.

Conflict Game We now formally define the conflict game and introduce the equilibrium concept.

Definition 1. *A conflict game is a tuple $CF = (\mathcal{N}, \mathcal{T}, \mathbf{\Gamma}, \{v^t, f^t(\cdot)\}_{t\in\mathcal{T}}, \{C_i(\cdot)\}_{i\in\mathcal{N}})$ in which $\mathcal{N}$ is the set of players, $\mathcal{T}$, the set of battles, $\mathbf{\Gamma}$, the structure (network) of the conflicts, v^t, the value of battle t, $f^t(\cdot)$, the production technology of battle $t \in \mathcal{T}$, and $C_i(\cdot)$, the cost function for player $i \in \mathcal{N}$.*

The timing of the game is straightforward: players simultaneously choose efforts, and their payoffs are given by (2). We are interested in the pure strategy Nash equilibrium of this conflict game. A strategy profile $\mathbf{x}^* = (\mathbf{x}_1^*, \cdots, \mathbf{x}_n^*)$ is an equilibrium of the conflict game CF if for every player $i \in \mathcal{N}$,

$$\Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \geq \Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}^*), \quad \forall \mathbf{x}_i. \tag{3}$$

Our main objective is to fully characterize the set of equilibria, and to perform extensive comparative statics exercises with respect to the primitives of the model, such as battle valuations, contest technology, cost functions, and conflict structure.

2.2 Assumptions and examples

In this subsection, we present a few examples to illustrate the generality of our model, and discuss some technical assumptions for our analysis.

[9]See Section A.3 of the Appendix A for an analysis of the properties of the Contest Success Function.

[10]This logit form of the CSF is widely used in modeling contests and conflicts. See e.g.,Tullock (1980); Dixit (1987); Hirshleifer (1989); Clark and Riis (1998); Konrad (2009); Franke and Öztürk (2015); König et al. (2017). See Skaperdas (1996) for the axiomatization of this logit form.

[11]Since the main focus of this paper is to study the impact of conflict topology, we assume, for simplicity, no valuation linkage between winning different battles. See Kovenock and Roberson (2012) for the case where the marginal value of winning a certain battle depends on the probabilities of winning other battles.

2.2.1 On the conflict structure $\mathbf{\Gamma}$

The conflict structure considered in this paper can be arbitrary, hence it, in particular, nests several structures studied in the existing literature as special cases.

Definition 2 (Complete conflict structure)**.** *A conflict structure* $\mathbf{\Gamma}$ *is called* **complete** *if* $\gamma_i^t = 1$ *for every* $i \in \mathcal{N}, t \in \mathcal{T}$.

For instance, $\mathbf{\Gamma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is complete since players 1 and 2 compete simultaneously in three battles a, b and c. Complete conflict structures, which are a particular case of our conflict structure $\mathbf{\Gamma}$, have been extensively studied in several contest models with multiple battlefields. See e.g. Friedman (1958); Roberson (2006); Kvasov (2007); Roberson and Kvasov (2012).

Definition 3 (Conflicts on graphs)**.** *For any undirected graph* $\mathcal{G} = (\mathcal{N}, \mathcal{E})$*, where* $\mathcal{N}$ *is the set of nodes, and* $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ *is the set of edges, we can define a conflict structure as follows:* $\mathcal{N}$ *is the set of nodes, and* $\mathcal{T} = \mathcal{E}$ *is the set of edges in graph* $\mathcal{G}$*. For each edge* $e = (i, j) \in \mathcal{E}$ *between* i *and* j*, there exists a battle between* i *and* j.

Figure 1 illustrates a circle graph with four nodes and four battles so that

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

where rows correspond to players (1, 2, 3, 4) and columns to battles (a, b, a', b'). We see, for example, that player 1 is involved in battles a and b while player 2 is involved in battles a and a'. We have: $\mathcal{N} = \{1, 2, 3, 4\}$, $\mathcal{T} = \{a, a', b, b'\}$, $\mathcal{N}^a = \{1, 2\}$, $\mathcal{N}^b = \{1, 4\}$, $\mathcal{N}^{a'} = \{2, 3\}$, $\mathcal{N}^{b'} = \{3, 4\}$, $\mathcal{T}_1 = \{a, b\}$, $\mathcal{T}_2 = \{a, a'\}$, $\mathcal{T}_3 = \{a', b'\}$, $\mathcal{T}_4 = \{b, b'\}$.

Adding a new edge in a graph implies that a new battle is introduced. The new conflict structure constructed from adding an edge from 2 to 4 in the network is shown in Figure 2 with new conflict structure $\mathbf{\Gamma}'$ given by

$$\mathbf{\Gamma}' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Example 1 (Geographic conflicts)**.** *A geographic conflict* $\mathbf{\Gamma}$ *is not directly given, but constructed from the geographic relations among countries involved in battles. Figure 3 illustrates a geographic conflict for three countries:* $1, 2, 3$ *in which each pair of countries shares a border while all of them have a common border* d*. See Figure 4 for a representation of the conflict structure* $\mathbf{\Gamma}$.

Notice that there exist alternative ways of representing the same conflict structure. For example, in Figure 4, we display three equivalent representations of conflict structure from left to

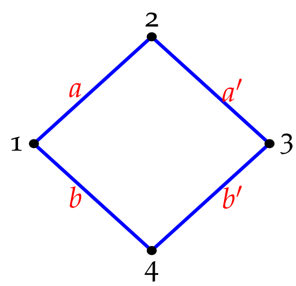

Figure 1: A ring graph

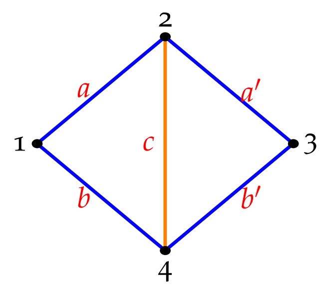

Figure 2: Adding one edge

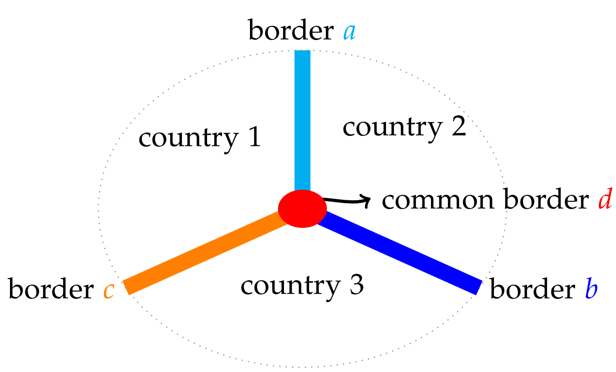

Figure 3: An example of geographic conflicts among countries 1, 2, and 3.

right: bipartite graph,[12] hypergraph,[13] and matrix representation. The former two graphical presentations are easier to visualize, while the matrix version is mathematically simpler. Depending on the specific context and concrete applications, we will adopt the most convenient form.

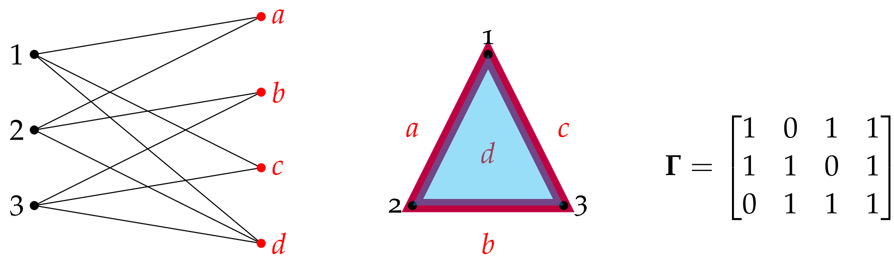

Figure 4: Equivalent representations

2.2.2 On the contest production function

A commonly used specification of contest production function is the following:

Example 2. *(Tullock, 1980) $f(z) = z^r$ for some $r > 0$.*

In this example, f is concave (convex) if $r \leq (\geq) 1$. Larger r implies more precise technology of mapping efforts into wining probabilities. It is well documented in contest models that pure strategy equilibrium may fail to exist for large enough r. In applications, r is often restricted to be in $(0, 1]$. In general, we make the following assumption on the contest production function f^t.

Assumption 1. *For every battle t, $f^t(\cdot)$ satisfies:*

$$f^t(0) = 0, \; df^t(z)/dz > 0, \; d^2 f^t(z)/dz^2 \leq 0 \text{ for all } z > 0. \tag{4}$$

In particular, $f^t(\cdot)$ is strictly increasing and concave (including linear case). Assumption 1 is often adopted to guarantee the existence of a pure strategy equilibrium.[14] Clearly, $f(z) = z^r$ in Example 2 satisfies Assumption 1 if and only if $0 < r \leq 1$. Our results about equilibrium existence and uniqueness do not critically rely on this Tullock form, although in many examples, we will adopt this form with $r = 1$ for convenience.

[12]Formally, the set of nodes in the constructed bipartite graph is $\mathcal{N} \cup \mathcal{T}$ and the adjacency matrix is $\begin{bmatrix} \mathbf{0} & \mathbf{\Gamma} \\ \mathbf{\Gamma}^T & \mathbf{0} \end{bmatrix}$, where $\mathbf{\Gamma}^T$ is the transpose of Γ.

[13]Different battles may involve the same subset of contestants (like the complete case), so the hypergraph in this paper allows multiple "hyper-edges" with the same subset of nodes.

[14]See Szidarovszky and Okuguchi (1997); Nti (1999); Cornes and Hartley (2005), among others.

2.2.3 On the cost function

Since our specification of the cost function allows for infeasible effort profile, it is natural to focus on the following cost function:

$$C_i(\mathbf{x_i}) = \begin{cases} c_i(\mathbf{x_i}) & \text{if } \mathbf{x}_i \in \Delta_i, \\ +\infty & \text{if otherwise} \end{cases} \tag{5}$$

for a finite-valued function $c_i(\cdot) : \mathbf{R}_+^{t_i} \mapsto [0, +\infty)$. $\Delta_i = \{\mathbf{x}_i = \{x_i^t\} \in \mathbf{R}_+^{t_i} : \sum_{t \in \mathcal{T}_i} x_i^t \leq k_i\}$ denotes the strategy that is feasible for player i,[15] where $k_i \in (0, +\infty]$. Obviously any $\mathbf{x}_i \notin \Delta_i$ will never be chosen in equilibrium as the cost is infinite. The specification in (5) includes the following three cases, which are commonly adopted in the literature.

(i) *Pure-budget case*

There is a finite cap on i's total effort but there is no additional cost, i.e., $k_i < +\infty$ and $c_i(\cdot) \equiv 0$. This just means that each agent i chooses $\mathbf{x}_i = \{x_i^t\}_{t \in \mathcal{T}_i} \in \mathbf{R}_+^{t_i}$ that maximizes $\sum_{t \in \mathcal{T}_i} v^t p_i^t(\mathbf{x}^t)$ under the budget constraint: $\sum_{t \in \mathcal{T}_i} x_i^t \leq k_i$.

(ii) *Pure-cost case*

There is no budget constraint, $k_i = +\infty$, i.e., any strategy $\mathbf{x}_i$ is feasible for player i, with potentially high but finite cost $c_i(\mathbf{x}_i)$. This implies that each agent i chooses $\mathbf{x}_i = \{x_i^t\}_{t \in \mathcal{T}_i} \in \mathbf{R}_+^{t_i}$ that maximizes $\sum_{t \in \mathcal{T}_i} v^t p_i^t(\mathbf{x}^t) - c_i(\mathbf{x}_i)$.

(iii) *Mixed case*

There is a budget constraint, i.e., $k_i < +\infty$ and $c_i(\cdot)$ is not identically zero. In that case, each agent i chooses $\mathbf{x}_i = \{x_i^t\}_{t \in \mathcal{T}_i} \in \mathbf{R}_+^{t_i}$ that maximizes $\sum_{t \in \mathcal{T}_i} v^t p_i^t(\mathbf{x}^t) - c_i(\mathbf{x}_i)$ under the budget constraint: $\sum_{t \in \mathcal{T}_i} x_i^t \leq k_i$.

Next, we impose a mild assumption on the c_i.

Assumption 2. *The cost function $c_i(\cdot)$ is twice continuously differentiable and is monotone and convex. In addition, if $k_i = +\infty$, we assume that $c_i(\mathbf{x}_i)$ is strongly monotone.*[16]

The conditions in Assumption 2 are fairly weak. In some applications, we need a slightly strengthened version.

Assumption 3. *The cost function $c_i(\cdot)$ is twice continuously differentiable, convex and strongly monotone.*

Assumption 3 is stronger than Assumption 2 because it imposes strong monotonicity of the cost function, even if $k_i < +\infty$.

[15] In computing the total effort in (5), we normalize the weight on the effort exerted in each battle x_i^t to be one, mainly for the ease of the presentation. The analysis and results can be easily extended to heterogeneous weights.

[16] Specifically, $c_i(\cdot)$ satisfies strong monotonicity if $c_i(\mathbf{x}_i') > c_i(\mathbf{x}_i'')$ whenever $\mathbf{x}_i' \geq \mathbf{x}_i''$ but $\mathbf{x}_i' \neq \mathbf{x}_i''$. And $c_i(\cdot)$ satisfies monotonicity if $c_i(\mathbf{x}_i') \geq c_i(\mathbf{x}_i'')$ whenever $\mathbf{x}_i' \geq \mathbf{x}_i''$. For example, $c_i(\cdot) \equiv 0$ is monotone but not strongly monotone.

We now illustrate the cost function given by (5) using some examples considered in the literature.

Example 3. *Kovenock and Roberson (2012) consider the following cost structure:*

$$C_i(\mathbf{x}_i) = \begin{cases} c\sum_{t\in\mathcal{T}_i} x_i^t & \text{if } \sum_{t\in\mathcal{T}_i} x_i^t \le k_i \\ +\infty & \text{if } \sum_{t\in\mathcal{T}_i} x_i^t > k_i \end{cases}, \tag{6}$$

This is referred to as the budget-constrained linear costs, for finite $k_i > 0$ and nonnegative constant marginal opportunity cost c. This cost function clearly satisfies Assumption 2. When $c = 0$, we are in the pure-budget case (i), also known as the budget-constrained use-it-or-lose-it costs *as each unit of effort up to the budget constraint has a zero opportunity cost. For a strictly positive c, this cost function belongs to the mixed case (iii).*[17]

Example 4. *Suppose that the cost function only depends on total effort, i.e.,*

$$C_i(\mathbf{x}_i) = g_i\left(\sum_{a\in\mathcal{T}_i} x_i^a\right) \tag{7}$$

for a single-variable function $g_i(\cdot)$. This cost function belongs to the pure-cost case (ii) and satisfies Assumption 2 if and only if $g_i(\cdot)$ is convex and strictly increasing.[18]

3 Equilibrium analysis

In this section, we conduct an analysis of the equilibrium of this multiple-battle multi-player conflict game. In Section 3.1, we show the existence of equilibrium. In Section 3.2, we study the issue of equilibrium uniqueness using techniques from Variational Inequality.

3.1 Existence of equilibrium

Let $\mathcal{NE}$ denote the set of equilibria for this conflict game CF. Given the generality of our conflict model, existence of equilibrium is obviously the first major issue to investigate. Our first Theorem shows that

$$\mathcal{NE} \neq \emptyset.$$

Theorem 1 (Existence)**.** *Suppose that Assumptions 1 and 2 hold. The conflict game CF has at least one equilibrium.*

Assumptions 1 (on the contest technology) and 2 (on the cost function) are fairly easy to satisfy, but they are sufficient for existence by Theorem 1. Note that no restriction is imposed on the conflict structure $\mathbf{\Gamma}$.

[17]See Friedman (1958); Roberson (2006); Kvasov (2007); Kovenock and Roberson (2012); Roberson and Kvasov (2012).

[18]For example, Franke and Öztürk (2015) consider a quadratic form with $g_i(z) = \frac{1}{2}z^2$.

Before presenting our proof technique for existence, we discuss a few challenging issues in showing the existence of an equilibrium. First (*i*), the winning probability function $p_i^t(\cdot)$ in battle t has a discrete jump when contestants exert zero effort in that battle. Therefore, the payoffs in the conflict game CF are *not* continuous.[19] Furthermore, the dimensionality of the discontinuity can be very high as each player simultaneously participates in multiple battles. Also, the strategy space can be unbounded. Second (*ii*), due to multiple battles and multi-dimensional efforts, the conflict game CF is not an *aggregate game*. Third (*iii*), the conflict game CF is not *supermodular*.[20] The first challenge (*i*) implies that standard existence theorems based on Kakutani's fixed-point theorem cannot be used here (see Glicksberg, 1952). The second one (*ii*) implies that techniques from aggregate games are not applicable (see Jensen, 2010).[21] The third challenge (*iii*) implies that we cannot use existence theorems based on lattice approach and Tarski's fixed-point theorem (see Milgrom and Roberts, 1990; Vives, 1990, 2001).

To deal with these technical challenges, we utilize results from discontinuous games; see Reny (1999). More specifically, we verify that the conflict game satisfies all the conditions such as compactness, quasi-concaveness, reciprocally upper semi-continuity and payoff security, so that Reny (1999)'s result is applicable. Here, we briefly provide some intuition behind the proof and highlight the roles played by Assumptions 1 and 2. First, it is without loss of generality that we restrict players' efforts to belong to a bounded strategy space as the winning probability is bounded above by one, while larger effort is associated with a higher cost. Second, winning probabilities always add up to unity, so that the sum of payoffs over all players does not exhibit any discontinuity, which implies reciprocally upper semi-continuity. Third, Assumption 1 on the contest production function guarantees that the winning probability $p_i^a(\cdot)$ in battle $a \in \mathcal{T}_i$ is concave in x_i^a, hence the payoff of each player i is concave as the benefit of efforts is linear in these winning probabilities and the cost function is convex. Since the payoff has points of discontinuity, the proof of payoff security is non-trivial and rather technical, and utilizes special properties of the contest success function. The formal definitions of these properties and verifications of these conditions are given in Section B.1 of the Appendix B.

Remark 1. *To show existence, there are alternative proofs based on approximation either on the strategy space or the contest success functions.*

1. *The first approach is based on an approximation and truncation of the strategy space. To be more precise, for each $\epsilon \in (0, 1)$, let us consider a modified conflict game CF^ϵ by imposing a uniform low bound ϵ on the efforts. We first show that the modified game CF^ϵ is* well-behaved, *and thus has a pure strategy Nash equilibrium $\mathbf{x}(\epsilon)$.*[22] *Next, we can show that there exists a subsequence $\{\mathbf{x}(\epsilon)\}$ that converges to some limiting profile $\mathbf{x}^*$. Finally, we can prove that $\mathbf{x}^*$ is an equilibrium of the original game CF.*[23]

[19] This type of discontinuity is well documented in the contest literature.

[20] The payoff (2) is not (log)-supermodular. Moreover when k_i is finite, the feasible set Δ_i is not even a lattice.

[21] When there is only a single battle, the contest game is an aggregate game using the sum of efforts as the aggregator. Several proofs in the contest literature on equilibrium existence utilize this aggregation property; see, for example, Szidarovszky and Okuguchi (1997); Cornes and Hartley (2005).

[22] For uniqueness in CF^ϵ, see Proposition C2 in Appendix C.

[23] At the end of Section B.1 of the Appendix B, we give a sketch of the proof of existence using this modified conflict game CF^ϵ.

2. *The second approach uses approximation of the CSF. More precisely, for each $\delta > 0$, we can modify the CSF as follows:*[24]

$$\tilde{p}_i^a(\mathbf{x}^a) = \frac{f^a(x_i^a)}{\delta + \sum_{j \in \mathcal{N}^a} f^a(x_j^a)}, \ \forall \mathbf{x}^a \in \mathbf{R}_+^{n^a}. \tag{8}$$

Denote by CF_δ the resulting conflict game with the modified winning function defined in (8) and by $\mathcal{NE}_\delta$ the set of equilibria. Thus δ can be interpreted as the exogenous probability of a tie. This modification removes the discontinuity in CF, and thus we can show that under Assumptions 1 and 2, for every $\delta > 0$, CF_δ has at least one equilibrium, i.e., $\mathcal{NE}_\delta \neq \emptyset$. Moreover, following similar steps of the proof of the existence of equilibrium using CF^ϵ, we can identify an equilibrium of CF by taking δ to zero. However, the equilibrium correspondence $\mathcal{NE}_\delta$ may not be continuous at $\delta = 0$, as we will see in Example 5 later.[25]

3.2 Uniqueness versus multiplicity of equilibria

Before stating our results about the uniqueness of equilibrium, we would like to discuss the properties of the equilibrium, provide a simple model that highlights our results and explains the Variational Inequality techniques.

3.2.1 Preliminary results on equilibrium strategy profile

Given the existence of equilibrium, what can we say about the properties of equilibria? Is the equilibrium locally unique? Do multiple equilibria exist? What can we say about the geometry of the $\mathcal{NE}$?

Recall that $\Delta_i = \{\mathbf{x}_i \in \mathbf{R}_+^{t_i} | \sum_{t \in \mathcal{T}_i} x_i^t \leq k_i\}$ is the strategy space of player i. Let $\mathcal{S} = \prod_i \Delta_i$ denote the whole strategy space. We highlight the distinction between different types of strategy profiles. The importance of such distinction will soon be clear.

Definition 4.

1. *A player i is active in battle t if $x_i^t > 0$, i.e, her effort in battle t is strictly positive. A strategy profile $\mathbf{x}$ is interior if every player is active in every battle she is involved in, i.e., $x_i^t > 0, \forall i, t$ with $\gamma_i^t = 1$.*

2. *A strategy profile $\mathbf{x}$ is of type S^1 if, for every battle t, there exists at least one active player under $\mathbf{x}$. A strategy profile $\mathbf{x}$ is of type S^2 if, for every battle t, there are at least two active players under $\mathbf{x}$.*

In our general conflict game with heterogeneous battle valuations and asymmetric players, we should not always expect an interior equilibrium (see, for instance, Example 9 below). Facing multiple battles and competitors, a player i must allocate her efforts by balancing marginal benefits and marginal costs. If the marginal benefit from exerting effort in battle a is low relative to the marginal cost or the shadow price of the effort (for the budget case of cost functions), a

[24] Similar results hold if we assume battle-specific tie parameter $\delta^a > 0, a \in \mathcal{T}$.

[25] Furthermore, we can prove that the equilibrium in CF_δ is unique for every $\delta > 0$ using the Variational Inequality approach; see Proposition C3 in Appendix C.

player might strategically choose to become inactive in battle a. However, the marginal benefit of winning battle a depends on the battle value v^a, and the efforts of other active participants in the same battle, as well as the marginal cost, which depends on her effort choices in other battles. It is thus possible that a player might strategically abandon certain battles to better concentrate on battles in which she holds an advantageous position. Note that for any interior strategy profile which is of type S^2, the reverse holds only when each battle has only two participants (for instance, in conflicts on graphs, Definition 3).

Let $\mathcal{S}^1$ and $\mathcal{S}^2$ denote the set of strategy profiles satisfying types S^1 and S^2, respectively. It is easily verified that both $\mathcal{S}^1$ and $\mathcal{S}^2$ are convex and relatively dense and open subsets of $\mathcal{S}$.[26] If a strategy profile $\mathbf{x}$ is not in $\mathcal{S}^1$, then there exists at least one battle with no active participant. Given the discrete jump of the winning probability at the origin, having a battle with no active players certainly will not occur at any equilibrium. In other words, in equilibrium, each battle has at least one active player, formally shown in Lemma 1.

Lemma 1. *Under Assumptions 1 and 2, any equilibrium of the conflict game is of type S^1. Formally, $\mathcal{NE} \subseteq \mathcal{S}^1$.*

Next, should we expect any equilibrium to be of type S^2? Suppose that $\mathbf{x}$ is of type S^1, but not S^2. Then there exists a battle, say a, and a player, say i, such that i is the only active contestant in battle a. Since player i faces no competitor in battle a, clearly she could reduce her effort in a slightly and still win battle a with probability one. This deviation reduces her cost at least weakly, without affecting her expected winning values. When i's cost function is strongly monotone, this deviation is strictly profitable for i. Therefore, such $\mathbf{x}$ cannot be an equilibrium. Formally, we have:

Lemma 2. *Under Assumptions 1 and 3, every equilibrium is of type S^2. Formally, $\mathcal{NE} \subseteq \mathcal{S}^2$.*

However, when the cost is monotone, but not strongly monotone, the equilibrium strategy may be of type S^1. An equilibrium in $\mathcal{S}^1$ but not in $\mathcal{S}^2$ places several restrictions on the equilibrium. For example, if player i is the only active contestant in battle a, and her cost is a pure-budget case, then she must be the single active contestant in any other battle $b \in \mathcal{T}_i$. Otherwise, she could reduce x_i^a and increase x_i^b by the same amount accordingly to meet the budget constraint while strictly improving her winning probability in battle b. We will see equilibrium of type S^1 in Example 5 below.

The distinction between these two types of equilibria has major consequence on the geometry of the equilibrium, as we demonstrate in Theorem 2 and Figure 5 below.

3.2.2 A simple toy model illustrating the uniqueness of equilibrium

Before discussing the technical aspects associated with the uniqueness of the equilibrium, let us highlight several subtle issues and motivate our approach using Variational Inequality with

[26]The properties of both sets are summarized in Lemma A3.

a simple toy model. Although such a simple model is well studied in contest literature, the approach we take below is different from existing ones.

Consider the setting with one battle with value v and two players, 1 and 2. Let $p(x,y) = \frac{f(x)}{f(x)+f(y)}$ be the winning probability of player 1, and x, y be the efforts of players 1 and 2, respectively. An interior equilibrium (x^*, y^*) must satisfy the following system of equations

$$\begin{aligned} F_1(x,y) &=: c_1'(x) - vp_x = 0, \\ F_2(x,y) &=: c_2'(y) + vp_y = 0 \end{aligned} \tag{9}$$

where we use the fact that the winning probability of player 2 is simply $1 - p(x,y)$. In general, the above system $\mathbf{F}(x,y) = (F_1, F_2)$ is non-linear even with a simple specification of $f(.)$ and the cost functions. Suppose that $f(.)$ satisfies Assumption 1 and costs are convex. Then, we claim that $\mathbf{F}(x,y)$ is injective on $\mathbf{R}^2_{++}$, which immediately implies the uniqueness of the solution to (9) and, therefore, the uniqueness of interior equilibrium in this model with one battle and two players. To verify the claim, we note that the Jacobian of the mapping $\mathbf{F}$ is given by:

$$\mathbf{M}(x,y) := \frac{\partial(F_1, F_2)}{\partial(x,y)} = \begin{bmatrix} c_1''(x) & 0 \\ 0 & c_2''(y) \end{bmatrix} + v \begin{bmatrix} -p_{xx} & -p_{xy} \\ p_{xy} & p_{yy} \end{bmatrix}$$

The convexity of the cost function implies that $c_1''(.) \geq 0, c_2''(.) \geq 0$. Assumption 1 on $f(.)$ implies that $p_{xx} < 0, p_{yy} > 0$ (See Lemma A2 in Section A.3 of the Appendix A for details). Since the off diagonal entries p_{xy} and $-p_{xy}$ have opposite sign, the symmetric part of this Jacobian matrix $\mathbf{M}$, $(\mathbf{M} + \mathbf{M}^T)/2$, is a diagonal matrix with positive entries:

$$(\mathbf{M} + \mathbf{M}^T)/2 = \begin{bmatrix} c_1''(x) & 0 \\ 0 & c_2''(y) \end{bmatrix} + v \begin{bmatrix} -p_{xx} & 0 \\ 0 & p_{yy} \end{bmatrix}$$

Thus, matrix $\mathbf{M}$ is positive definite. And it implies that $\mathbf{F}$ is a monotone operator in the following sense

$$(x' - x)(F_1(x', y') - F_1(x,y)) + (y' - y)(F_2(x', y') - F_2(x,y)) > 0, \quad \forall (x', y') \neq (x,y). \tag{10}$$

The above inequality implies the injectiveness of the mapping $\mathbf{F}$, i.e., if $F_1(x', y') = F_1(x,y)$, and $F_2(x', y') = F_2(x,y)$, then it must be the case that $x' = x$ and $y' = y$.[27]

Of course, this simple toy model is just an example to highlight the main ideas. Many other issues in the general conflict model are not illustrated here. For example, each battle could have more than three participants, instead of two, so that the positive definiteness of matrix $\mathbf{M}$ is more difficult to show. Moreover, equilibrium may not be interior, and hence it will not always satisfy $\mathbf{F} = \mathbf{0}$. How to deal with these issues in the general framework is the goal of the next subsection using the Variational Inequality approach.

[27] We could have used the global univalence theorem of Gale and Nikaido (1965) to prove injectiveness of $\mathbf{F}$, as the Jacobian of $\mathbf{F}$ is positive definite, hence a P-matrix.

3.2.3 Preliminary results of the proof of uniqueness of equilibrium

As can be seen from the simple toy model above, a key step of the proof of uniqueness of equilibrium is to show certain monotonicity properties of the mapping $\mathbf{F}$, which is closely related to the first-order conditions of the original game. The conditions on the cost and the CSF make sure that such monotonicity condition indeed holds. In the general conflict game with multiple battles and players, we would like to use a similar idea, except that we have to deal with the case when efforts are zero in equilibrium. This is why we will use techniques from Variational Inequalities.[28]

Indeed, to characterize the equilibrium, we pin down the equilibrium conditions. Suppose $\mathbf{x}^* \in \mathcal{NE}$. From (3), for each i, it holds that $\mathbf{x}_i^* \in \arg\max_{\mathbf{x}_i \in \Delta_i} \Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}^*)$. Moreover as each battle has at least one active player under $\mathbf{x}^*$ by Lemma 1, the payoff function Π_i is continuously differentiable at $\mathbf{x}^*$. Hence there exists a scalar λ_i such that the following Karush–Kuhn–Tucker (KKT) first-order conditions (FOCs) must hold:

$$\begin{cases} \frac{\partial \Pi_i(\mathbf{x}^*)}{\partial x_i^a} - \lambda_i \leq 0 \quad \text{(with equality if } x_i^{*a} > 0), \quad \forall a \in \mathcal{T}_i \\ (\sum_{a \in \mathcal{T}_i} x_i^{*a} - k_i)\lambda_i = 0, \lambda_i \geq 0, (\sum_{a \in \mathcal{T}_i} x_i^{*a} - k_i) \leq 0. \end{cases} \tag{11}$$

Here, λ_i is the Lagrange multiplier for the budget constraint of player i.

It is very difficult, if not impossible, to use the system above to characterize the set of Nash equilibria $\mathcal{NE}$. As stated above, to move forward, we apply some techniques from Variational Inequalities.

Definition 5 (Variational Inequality). *A vector $\mathbf{z}^* \in \mathbf{R}^m$ solves the Variational Inequality $VI(\mathbf{\Psi}, \mathcal{K})$ with set $\mathcal{K} \subset \mathbf{R}^m$, and operator $\mathbf{\Psi} : \mathcal{K} \to \mathbf{R}^m$ if and only if*

$$\langle \mathbf{\Psi}(\mathbf{z}^*), \mathbf{z} - \mathbf{z}^* \rangle \geq 0, \quad \forall \mathbf{z} \in \mathcal{K} \tag{12}$$

The solution set of this VI problem is denoted as $Sol(\mathbf{\Psi}, \mathcal{K})$.

For our conflict game, we consider the following operator $\mathbf{F}$:

$$\mathbf{F}(\mathbf{x}) = - \begin{bmatrix} \nabla_{\mathbf{x}_1} \Pi_1(\mathbf{x}) \\ \nabla_{\mathbf{x}_2} \Pi_2(\mathbf{x}) \\ \vdots \\ \nabla_{\mathbf{x}_n} \Pi_n(\mathbf{x}) \end{bmatrix} : \mathcal{S}^1 \subseteq \mathbf{R}^{\bar{n}} \to \mathbf{R}^{\bar{n}}. \tag{13}$$

[29]

Due to the discontinuity of CSFs, $\mathbf{F}(\mathbf{x})$ is only defined when all the payoffs are continuously differentiable, i.e., $\mathbf{x} \in \mathcal{S}^1$. As each $\mathbf{x}_i \in \Delta_i \subset \mathbf{R}^{t_i}$, $\nabla_{\mathbf{x}_i} \Pi_i$ is a column vector of dimension t_i. The

[28] In our simple model with one battle and two players, any equilibrium was necessarily interior by Lemma 2.

[29] The negative sign here is used to make sure that the operator $\mathbf{F}$ is monotone, instead of anti-monotone. $\mathbf{F}$ is sometimes called game Jacobian, see, for instance, Facchinei and Pang (2007); Melo (2019); Parise and Ozdaglar (2019).

operator $\mathbf{F}$ defines a mapping from set $\mathcal{S}^1 \subseteq \mathbf{R}^{\bar{n}}$ to $\mathbf{R}^{\bar{n}}$ as $\bar{n} = \sum_{i \in \mathcal{N}} t_i$.[30] This operator plays a key role in our equilibrium analysis due to the result below:

Proposition 1. *Under Assumptions 1 and 2, the following statements are equivalent:*

(i) A strategy profile $\mathbf{x}^*$ *is a Nash equilibrium of the conflict game CF;*

(ii) $\mathbf{x}^*$ *solves* $VI(\mathbf{F}, \mathcal{S}^1)$*, i.e.,*

$$\mathbf{x}^* \in \mathcal{S}^1, \textit{and } \langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0, \forall \mathbf{x} \in \mathcal{S}^1. \tag{14}$$

The equivalent characterization of a Nash equilibrium of a generic smooth concave game (not necessarily in a conflict game) using Variational Inequality is known in the literature.[31] In the standard definition of Variational Inequality, the domain $\mathcal{K}$ is usually assumed to be closed (see Facchinei and Pang (2007)). However, in our model $\mathcal{S}^1$ is *not* closed. Thus many existing results on Variational Inequality cannot be directly applied here.[32] Our proposition extends the VI characterization of equilibrium accommodating non-closeness of domain $\mathcal{S}^1$ and discontinuity in payoffs. Since the closure of $\mathcal{S}^1$ is $\mathcal{S}$, by continuity of inner product, equation (14) in item (ii) can be replaced by the following equivalent one:

$$\mathbf{x}^* \in \mathcal{S}^1, \text{and } \langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0, \forall \mathbf{x} \in \mathcal{S}. \tag{15}$$

Formally, $\mathcal{NE} = Sol(\mathbf{F}, \mathcal{S}^1)$. With slight abuse of notation, we might write $\mathcal{NE} = \mathcal{S}^1 \cap Sol(\mathbf{F}, \mathcal{S})$. The benefit of this equivalent formulation hinges on the monotonicity of the operator $\mathbf{F}$, which is *naturally* satisfied in this conflict game.[33]

Definition 6. *An operator* $\mathbf{\Psi}$*, from* $\mathcal{K}$ *to* $\mathbf{R}^m$*, is called monotone on set* $\hat{\mathcal{K}} \subseteq \mathcal{K}$ *if*

$$\langle \mathbf{\Psi}(\mathbf{z}') - \mathbf{\Psi}(\mathbf{z}''), \mathbf{z}' - \mathbf{z}'' \rangle \geq 0, \quad \forall \mathbf{z}', \mathbf{z}'' \in \hat{\mathcal{K}}, \mathbf{z}' \neq \mathbf{z}''.$$

It is called strictly monotone on set $\hat{\mathcal{K}}$ *if*

$$\langle \mathbf{\Psi}(\mathbf{z}') - \mathbf{\Psi}(\mathbf{z}''), \mathbf{z}' - \mathbf{z}'' \rangle > 0, \quad \forall \mathbf{z}', \mathbf{z}'' \in \hat{\mathcal{K}}, \mathbf{z}' \neq \mathbf{z}''.$$

[30] Observe that, for each strategy profile $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in \mathbf{R}^{\bar{n}}_+$, with $\mathbf{x}_i = \{x_i^t\}_{t \in \mathcal{T}_i} \in \mathbf{R}^{t_i}_+$, we present it as a column vector of dimension $\bar{n}$. Here we specify a concrete way to do so. By fixing a complete order on the set of battles, $\mathbf{x}$ can be displayed using a lexicographic order by first listing player 1's efforts sequentially according to the order of the battles that she joins, then listing player 2's efforts, and so on. Alternatively, we could use the lexicographic order with first priority on battles, instead of on players.

[31] Proposition 1.4.2 in Facchinei and Pang (2007) presents a formal statement of the characterization of a Nash equilibrium in smooth concave games using VI, where a smooth concave game is a game such that each player's strategy space $\mathcal{X}_i$ is closed and convex, each payoff is at least twice continuously differentiable in $\prod \mathcal{X}_i$, and concave in own strategy $\mathbf{x}_i \in \mathcal{X}_i$, fixing any $\mathbf{x}_{-i} \in \mathcal{X}_{-i}$.

[32] To give an example, the conditions for the existence of solution to VI in Facchinei and Pang (2007) do not hold here, so Theorem 1 (non-emptiness of $\mathcal{NE}$) is not a direct corollary of Proposition 1.

[33] In general, further assumptions on concave games are required in order to obtain monotonicity of this operator. For example, see Melo (2019); Parise and Ozdaglar (2019) and Zenou and Zhou (2019) for applications of VI in games played on networks by imposing certain restrictions on the spectral property of network matrix and the curvature of payoffs.

The monotonicity of the operator enables us to characterize the solution set $Sol(\mathbf{\Upsilon}, \mathcal{K})$ neatly. Given our equivalent characterization of $\mathcal{NE}$ in Proposition 1, it is useful to check the monotonicity properties of the operator $\mathbf{F}$ on its domain $\mathcal{S}^1$ and on a slightly smaller subdomain $\mathcal{S}^2$.

Proposition 2. *Under Assumptions 1 and 2, $\mathbf{F}$ is monotone on $\mathcal{S}^1$, and strictly monotone on $\mathcal{S}^2$. Furthermore, for any $\mathbf{x}', \mathbf{x}'' \in \mathcal{S}^1$ with $\mathbf{x}' \neq \mathbf{x}''$,*

$$\langle \mathbf{F}(\mathbf{x}') - \mathbf{F}(\mathbf{x}''), \mathbf{x}' - \mathbf{x}'' \rangle > 0, \tag{16}$$

when at least one of $\mathbf{x}'$ and $\mathbf{x}''$ is in $\mathcal{S}^2$.

Just as the monotonicity of a single-variable function is determined by the sign of its derivative, the monotonicity of $\mathbf{F}$ is reflected by the properties of its Jacobian matrix. As illustrated in the toy model above, let us define

$$\mathbf{M}(\mathbf{x}) := \nabla_{\mathbf{x}} \mathbf{F}(\mathbf{x}), \mathbf{x} \in \mathcal{S}^1 \tag{17}$$

as the Jacobian of the mapping $\mathbf{F}$. Note that $\mathbf{M}(\mathbf{x})$ is a square matrix of dimension $\bar{n}$, and it is defined only when $\mathbf{x}$ is in $\mathcal{S}^1$. In general, $\mathbf{M}$ is not symmetric.[34] The (strict) monotonicity of $\mathbf{F}$ is closely related to the positive (semi-)definiteness of $\mathbf{M}$. In fact, we can show the following result, which directly implies Proposition 2.

Proposition 3. *Under Assumptions 1 and 2, $\mathbf{M}(\mathbf{x})$ is positive semi-definite for any $\mathbf{x} \in \mathcal{S}^1$, and is positive definite for any $\mathbf{x} \in \mathcal{S}^2$.*

In Section 3.2.5, we provide several examples of $\mathbf{M}$s that illustrate this Proposition.

3.2.4 General results: Uniqueness of equilibrium

Let us now present our general result about the uniqueness of equilibrium.[35]

Theorem 2. *Under Assumptions 1 and 2, one and only one of the following mutually exclusive statements about $\mathcal{NE}$ is true:*

(i) $\mathcal{NE}$ contains an equilibrium $\mathbf{x}^*$ which is of type S^2. In this case, $\mathcal{NE}$ is a singleton, i.e., this $\mathbf{x}^*$ is the unique equilibrium;

(ii) all the equilibria are in the set $\mathcal{S}^1 \backslash \mathcal{S}^2$, i.e., any equilibrium is of type S^1, but not S^2.

Recall that $\mathcal{NE}$ is non-empty, so an immediate implication of Theorem 2 and Lemma 2 is:

Theorem 3 (Uniqueness)**.** *Suppose that Assumptions 1 and 3 hold. Then the equilibrium of a conflict game CF is unique.*

[34] See Example 7 below.

[35] All the proofs about the uniqueness of equilibrium can be found in Section B.2 of the Appendix B.

Let us show the implication of this result in terms of characterizing $\mathcal{NE}$, and more importantly in identifying conditions for uniqueness. Suppose the conflict game has two equilibria $\mathbf{x}^*, \mathbf{x}^{**}$ in $\mathcal{NE}$, then by (15)

$$\langle \mathbf{F}(\mathbf{x}^*), \mathbf{y}' - \mathbf{x}^* \rangle \geq 0, \quad \forall \mathbf{y}' \in \mathcal{S}$$
$$\langle \mathbf{F}(\mathbf{x}^{**}), \mathbf{y}'' - \mathbf{x}^{**} \rangle \geq 0, \quad \forall \mathbf{y}'' \in \mathcal{S}.$$

Substituting $\mathbf{y}' = \mathbf{x}^{**}$ into the first equation, and $\mathbf{y}'' = \mathbf{x}^*$ into the second equation, and summing both equations, we get the following key inequality:

$$\langle \mathbf{F}(\mathbf{x}^*) - \mathbf{F}(\mathbf{x}^{**}), \mathbf{x}^* - \mathbf{x}^{**} \rangle \leq 0. \tag{18}$$

However, by Proposition 2, equation (18) cannot hold whenever $\mathbf{x}^* \neq \mathbf{x}^{**}$, and at least one of $\mathbf{x}^*, \mathbf{x}^{**}$ is in $\mathcal{S}^2$. An immediate implication is that there is at most one equilibrium in $\mathcal{S}^2$. The characterization of $\mathcal{NE}$ in Theorem 2 immediately follows.

More generally, Theorem 3 proves uniqueness of equilibrium under fairly weak conditions, which is only imposed on the cost structure, but not on the conflict structure $\mathbf{\Gamma}$ or on the valuations. For example, the cost structure considered in Example 4 above is strongly monotone, and hence a unique equilibrium is obtained. Therefore, Theorem 3 immediately implies the existence and uniqueness theorems in Franke and Öztürk (2015). Note that Assumption 3 is slightly stronger than Assumption 2. We point out that, without strongly monotone cost (Assumption 3), multiple equilibria may arise even under very "regular" cases (see Example 5 below).

Moreover, a strong geometry restriction is imposed on $\mathcal{NE}$:

Proposition 4. *Under Assumptions 1 and 2, the set of equilibria $\mathcal{NE}$ is convex.*

Based on the characterizations above, only two scenarios can possibly occur; see Figure 5. Therefore, it is impossible to have one equilibrium in $\mathcal{S}^2$ and another equilibrium in $\mathcal{S}^1 \backslash \mathcal{S}^2$. If either we prove that any equilibrium must be of type S^2 (as in Lemma 2 and Corollary 2), or a particular equilibrium of type S^2 is identified (either through inspection or symmetry of the underlying game), then case (*i*) of Theorem 2 immediately implies that there is no other equilibrium to look for. Moreover, if there exist multiple equilibria, then any equilibrium cannot be locally unique.

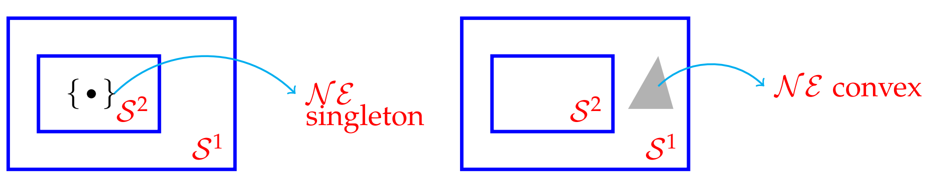

Figure 5: Two scenarios for $\mathcal{NE}$

3.2.5 Examples

In order to gain some intuition of our results, let us provide some specific examples.

Example 5 (Uniqueness versus continuum of equilibria: pure budget case)**.** *Consider a conflict game on a circle with three players:* 1, 2, 3 *and three battles:* a, b, c *as described in Figure 6.*

For each battle $t = a, b, c$, *assume the value* $v^t = 1$. *We also assume a Tullock Contest Success Function with* $r = 1$, *i.e.,*

$$p_i^t(\mathbf{x}^t) = \begin{cases} \frac{x_i^t}{\sum_{j \in \mathcal{N}^t} x_j^t} & \text{when } \mathbf{x}^t \neq (0, 0, \cdots, 0) \\ \frac{1}{|\mathcal{N}^t|} = \frac{1}{n^t} & \text{otherwise} \end{cases} \tag{19}$$

The cost of every player is of the pure budget case, in which $k_1 = k_3 = 1$ *and* $k_2 = \bar{K} > 8$. *Define* $\mathbf{x}^* = (x_1^a, x_1^c, x_2^a, x_2^b, x_3^b, x_3^c)$. *It is straightforward to show that this conflict game CF has a continuum of equilibria (of type* S^1*)*[36] *characterized by:*

$$\mathcal{NE} = \{\mathbf{x}^* = (0, 1, x_2^a, x_2^b, 0, 1) | x_2^a \geq 4, x_2^b \geq 4, x_2^a + x_2^b \leq \bar{K}\}. \tag{20}$$

In any of the equilibria, player 2 *is sure to win both battles* a *and* b. *Therefore, her best response is not unique, as long as both* x_2^a *and* x_2^b *are large enough to deter players* 1 *and* 3 *from exerting strictly positive efforts in battles* a *and* b*, respectively. Therefore, the multiplicity of equilibria arises here because* $\bar{K}$ *is relatively large and the budget difference between player 2 and the other players is too large. The equilibrium is thus not unique and of type* S^1. *Observe that* $\mathcal{NE}$ *is a convex set as showed in Proposition 4.*

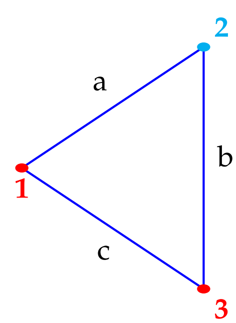

Figure 6: Conflict game on a circle

In Remark 1, we consider a modified game CF^ϵ *by imposing that the effort of every player on each battle has a minimal lower bound* $\epsilon > 0$*, i.e., no player can provide zero effort. In this example, we can show that, for each* $\epsilon \in (0, 1/2)$*, the modified game* CF^ϵ *has a unique equilibrium given by:*

$$\mathcal{NE}^\epsilon = \left\{ \mathbf{x}^\epsilon = \left(\epsilon, 1 - \epsilon, \frac{\bar{K}}{2}, \frac{\bar{K}}{2}, \epsilon, 1 - \epsilon \right) \right\}.$$

Indeed, in that case, players 1 and 3 will provide the minimum effort ϵ *in their battle with player 2 and maximal effort* $1 - \epsilon$ *in the other battles. Because of the concavity of the Contest Success Function given in* (1) *(Assumption 1), player 2 will exactly devotes half of budget,* $\bar{K}/2$*, in each battle she is involves in.*[37]

[36] If the equilibrium is of type S^2, then it is unique by Theorem 2.

[37] Note that ϵ cannot be greater than 1/2, as player 1 (or 3) participates in two battles with minimal effort ϵ in one of them with the constraint that $k_1 = 1$.

The fact that the modified game CF^ϵ has a unique Nash equilibrium is actually true for any network as shown in Proposition C2 in Appendix C.

As also shown in Remark 1, we can prove the existence of equilibrium by modifying the CSF as in (8), where δ can be interpreted as the probability of a tie. In this example, we can show that, for each $\delta > 0$, the modified game CF_δ has a unique equilibrium given by:

$$\mathcal{NE}_\delta = \left\{ \mathbf{x}_\delta = \left(0, 1, \frac{\bar{K}}{2}, \frac{\bar{K}}{2}, 0, 1\right) \right\}.$$

When $\delta > 0$, the uniqueness of equilibrium is obtained as player 2 wants to minimize the probability of ties in both battles a and b, even though her competitors are inactive in a and b.[38] *This result that the modified conflict game CF_δ with $\delta > 0$ has a unique equilibrium can be shown to be true for any network (See Proposition C3 in Appendix C for a general statement).*

Furthermore, as ϵ or δ approaches zero, we have:

$$\lim_{\epsilon \to 0} \mathbf{x}^\epsilon = \lim_{\delta \to 0} \mathbf{x}_\delta = \left(0, 1, \frac{\bar{K}}{2}, \frac{\bar{K}}{2}, 0, 1\right) \in \mathcal{NE},$$

i.e., the unique equilibrium converges to a particular equilibrium in $\mathcal{NE}$ of CF. Since there are many other equilibria in $\mathcal{NE}$, the equilibrium correspondence $\mathcal{NE}^\epsilon(\mathcal{NE}_\delta)$ is not continuous at $\epsilon = 0$ ($\delta = 0$). Consistent with Theorem 2, every equilibrium of CF is of type S^1 in $\mathcal{NE}$.

In Example 5, we show that multiple equilibria arise due to the existence of a powerful player 2, who has excessive resource to deter her competitors from being active in any of 2's battles. We consider a variant of Example 5 where the resource of player 2 is moderate.

Example 6 (Variation of Example 5)**.** *Consider exactly the same conflict game as in Example 5 (see Figure 6) but let us modify the value of k_2. Assume, now, that $k_2 = \bar{K} < 8$.* [39] *Then, the conflict game has a Nash equilibrium given by:*

$$\mathcal{NE} = \left\{ \mathbf{x}^* = \left(x_1^a, 1 - x_1^a, \frac{\bar{K}}{2}, \frac{\bar{K}}{2}, x_3^b, 1 - x_3^b\right) \right\}, \tag{21}$$

where

$$x_1^a = x_3^b = -\frac{3\bar{K}}{2} + \sqrt{2\bar{K}(1 + \bar{K})}.$$

Given $\bar{K} \in (0, 8)$, it is easy to show that this $\mathbf{x}^*$ is interior, hence of type S^2. As a consequence, such $\mathbf{x}^*$ is the unique *equilibrium by Theorem 2. Note that the cost function is not strongly monotone, so we can not directly apply Theorem 3 to obtain uniqueness.*

[38] Player 2's unique best response is to split her budget equally: when $\delta > 0$, $z/(z + \delta)$ is strictly concave in z, so $\frac{\bar{K}}{2} = \arg\max_{x_2^a \in [0, \bar{K}]} \left\{ \frac{x_2^a}{x_2^a + \delta} + \frac{\bar{K} - x_2^a}{\bar{K} - x_2^a + \delta} \right\}$.

[39] When $k_2 = \bar{K} = 8$, the conflict game has a unique equilibrium given by: $\mathcal{NE} = \{\mathbf{x}^* = (0, 1, 4, 4, 0, 1)\}$, which is of type S^1 since player 2 will be the only active player in battles a and b. This example shows that a unique equilibrium is not necessary of type S^2.

Example 6 shows that when the budget difference between player 2 and the other players is not too large, then a unique equilibrium emerges. In that case, no player is able to win any battle for sure, and the equilibrium will thus be of type S^2.

In Section C.1 of the Appendix C, we provide two other examples with different networks that illustrate the issue of multiple equilibria in our conflict game.

3.3 Discussions

We discuss several issues regarding the uniqueness or multiplicity of equilibria.

3.3.1 Cost function and conflict structure

First, does the existence of multiple equilibria in Example 5 hinges on a specific conflict structure $\mathbf{\Gamma}$? The answer is no, if we do not impose any restriction on the cost function.

Corollary 1. *Given any conflict structure, contest technology and values, there exists a cost function $C_i(\mathbf{x}_i)$, for each i, that satisfies Assumption 2, such that the resulting conflict game has a continuum of equilibria.*

The constructive proof of Corollary 1 follows from a similar idea to Example 5. Pick any player, say i. We assume that her cost is of the pure budget case with a very large k_i. For any player $j \neq i$, we assume that the cost is linear, i.e., $c_j(\mathbf{x}_j) = \sum_{t\in\mathcal{T}_j} x_j^t$. Given the linearity and separability of the cost function, we can rewrite player j's payoff Π_j as $\sum_{t\in\mathcal{T}_j}(v^j p_j^t(\mathbf{x}^t) - x_j^t)$, which is separable across battles. So each j's decision in different battles in $\mathcal{T}_j$ is fully independent.[40] Thus we can only focus on the battles that i joins. As long as i's effort in each battle $t \in \mathcal{T}_i$ is large enough, other players in this battle $t \in \mathcal{T}_i$ will optimally choose to become inactive. Therefore multiple equilibria arise for the same reason as in Example 5.

Second, can we still obtain uniqueness without strongly monotonic cost functions? The answer is yes, if we impose some restrictions on the conflict structure.

Corollary 2. *Suppose $\mathbf{\Gamma}$ is complete (see Example 2), and each player's cost is of the pure budget case. Any equilibrium must be of type S^2, therefore the equilibrium must be unique.*

The intuition of Corollary 2 is straightforward. Under the given assumptions, it is impossible to have any player who wins all the battles for sure. So, by the completeness of the conflict structure, any equilibrium profile must be of type S^2, hence uniqueness directly follows from Theorem 2 part (*i*). Thus, the multiplicity issue highlighted in Example 5 does not occur for any complete conflict structure. Moreover, the pure budget case of cost is also necessary, see Corollary 1.

Uniqueness in this Corollary crucially relies on the completeness of $\mathbf{\Gamma}$, which prevents the emergence of a "super powerful" node such as player 2 in Example 5. Note that Corollary 2 does

[40]In addition, if player i's cost happens to be separable across battles in $\mathcal{T}_i$, the whole conflict game can be decomposed into T independent battles, and there will be no linkage between different battles.

not contradict Corollary 1 as the cost structure considered in Corollary 1 is more general, and not limited to the pure-budget case.

3.3.2 Technical discussion on the monotonicity property of F

As we have seen above, the characterization results on $\mathcal{NE}$ are all direct consequences of the (strict) monotonicity of the operator **F** and the VI formulation of the equilibrium conditions. This observation leads us to ask why monotonicity holds naturally in such a setting. Why is the distinction between type S^1 and S^2 strategy profiles important? Recall that the conflict structure is arbitrary and we only impose Assumptions 1 and 2.

Let us start with an example with only one battle.

Example 7 (Monotonicity of **F**: single battle case)**.** *Consider a conflict game with three players:* 1, 2, 3 *and one battle, say* $t = a$, *so* $\mathbf{\Gamma}$ *is complete and equal to:*

$$\mathbf{\Gamma} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For the unique battle a, we assume that $v^a = v$. *Since there is only one battle,* $x_i^a := x_i$, *for each i. We also assume a Tullock Contest Success Function with* $r = 1$ *given by* (19) *and linear costs equal to:* $c_i(x_i) = c_i x_i, i = 1, 2, 3$[41] *so that the cost function is strongly monotone. The payoff function for each agent i is thus given by:* $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = v\frac{x_i}{\sum_{j=1}^3 x_j} - c_i x_i$. *Then,*

$$\mathbf{F}(x_1, x_2, x_3) = -\begin{bmatrix} \partial\Pi_1/\partial x_1 \\ \partial\Pi_2/\partial x_2 \\ \partial\Pi_3/\partial x_3 \end{bmatrix} = -\begin{bmatrix} v\frac{x_2+x_3}{(x_1+x_2+x_3)^2} - c_1 \\ v\frac{x_1+x_3}{(x_1+x_2+x_3)^2} - c_2 \\ v\frac{x_1+x_2}{(x_1+x_2+x_3)^2} - c_3 \end{bmatrix}.$$

The Jacobian of **F** *is*

$$\mathbf{M} = \frac{-v}{(x_1+x_2+x_3)^3} \begin{bmatrix} -2(x_2+x_3) & x_1-(x_2+x_3) & x_1-(x_2+x_3) \\ x_2-(x_1+x_3) & -2(x_1+x_3) & x_2-(x_1+x_3) \\ x_3-(x_1+x_2) & x_3-(x_1+x_2) & -2(x_1+x_2) \end{bmatrix}.$$

First, we note that, unless $x_1 = x_2 = x_3$, **M** *is* not *symmetric. As a result, to show that* **M** *is positive semi-definite, we need to show that the symmetric part of* **M** *is positive semi-definite (see Definition A1 in Appendix A). The latter is given by:*[42]

$$\frac{\mathbf{M}+\mathbf{M}^T}{2} = \frac{v}{(x_1+x_2+x_3)^3} \left\{ \begin{bmatrix} (x_2+x_3) & 0 & 0 \\ 0 & (x_1+x_3) & 0 \\ 0 & 0 & (x_1+x_2) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_1 & x_1 \\ 0 & x_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & 0 & x_2 \\ 0 & 0 & 0 \\ x_2 & 0 & x_2 \end{bmatrix} + \begin{bmatrix} x_3 & x_3 & 0 \\ x_3 & x_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

[41] The case with convex c_i is similar, except that there is an additional diagonal matrix in **M** due to the curvature of the cost. See, for instance, the toy example in Section 3.2.2.

[42] Such decomposition is inspired by Goodman (1980).

The last three matrices in the curly bracket are clearly positive semi-definite, while the first diagonal matrix is positive definite when at least two of x_1, x_2, x_3 are positive,[43] *in which case $\frac{\mathbf{M}+\mathbf{M}^T}{2}$, hence $\mathbf{M}$, is positive definite.*

However, when $x_1 > 0, x_2 = x_3 = 0$, then $\mathbf{x}$ is of type S^1, and $\mathbf{M}$ is positive semi-definite, but not positive definite as the first row/column of $\frac{\mathbf{M}+\mathbf{M}^T}{2}$ is zero.

From this example, we show that the distinction between type S^1 and type S^2 is critical for the positive definiteness versus the semi-definiteness of the matrix $\mathbf{M}$. For a battle with more than three players, we have a similar decomposition of the symmetric part of $\mathbf{M}$ into the sum of a few simpler matrices, where each of them is shown to be either positive semi-definite, or positive definite.

Next, we consider an example with multiple battles.

Example 8. *Consider a conflict game on the following start network (see Figure 7) with three players, 1, 2, 3, and two battles, a and b.*

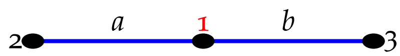

Figure 7: A star

Assume a Tullock Contest Success Function with $r = 1$, and the following quadratic cost function: $c_1(\mathbf{x}_1) = \frac{s_1}{2}(x_1^a + c_1^b)^2$, $c_2(\mathbf{x}_2) = \frac{s_2}{2}(x_2^a)^2$, $c_3(\mathbf{x}_3) = \frac{s_3}{2}(x_3^b)^2$. Thus, each player's payoff has the following form:

$$\begin{aligned} \Pi_1(\mathbf{x}_1, \mathbf{x}_{-1}) =& v^a p_1^a(x_1^a, x_2^a) + v^b p_1^b(x_1^b, x_3^b) - c_1(\mathbf{x}_1), \\ \Pi_2(\mathbf{x}_2, \mathbf{x}_{-2}) =& v^a p_2^a(x_2^a, x_1^a) - c_2(\mathbf{x}_2), \\ \Pi_3(\mathbf{x}_3, \mathbf{x}_{-3}) =& v^b p_3^b(x_3^b, x_1^b) - c_3(\mathbf{x}_3), \end{aligned} \tag{22}$$

where $\mathbf{x}_1 = (x_1^a, x_1^b)$, $\mathbf{x}_2 = (x_2^a)$ and $\mathbf{x}_3 = (x_3^b)$, and $\mathbf{x} = (x_1^a, x_1^b, x_2^a, x_3^b)$. We can show that

$$\frac{\mathbf{M}+\mathbf{M}^T}{2} = \mathbf{J}^b + \mathbf{J}^c \tag{23}$$

where

$$\mathbf{J}^b = v^a \begin{bmatrix} \frac{2x_2^a}{(x_1^a+x_2^a)^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2x_1^a}{(x_1^a+x_2^a)^3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + v^b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2x_3^b}{(x_1^b+x_3^b)^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2x_1^b}{(x_1^b+x_3^b)^3} \end{bmatrix}$$

and

$$\mathbf{J}^c = \begin{bmatrix} s_1 & s_1 & & \\ s_1 & s_1 & & \\ & & s_2 & \\ & & & s_3 \end{bmatrix}.$$

[43] This is equivalent to $\mathbf{x} = (x_1, x_2, x_3)$ being of type S^2.

Clearly, each diagonal block matrix of $\mathbf{J}^c$ is the Hessian matrix of the cost of player i, which is positive semi-definite by the convexity of the cost function. Together, we show that $\mathbf{J}^c$ is positive semi-definite. Similarly, $\mathbf{J}^b$ can be decomposed as the sum of two matrices which corresponds to battle a and battle b respectively. Each matrix is positive semi-definite following the same logic as in Example 7. Focusing on type S^2 strategy profile, $\mathbf{J}^b$ is positive definite, hence $\frac{\mathbf{M}+\mathbf{M}^T}{2}$ is positive definite as well.

The proof of (strict) monotonicity of $\mathbf{F}$ in Proposition 3 with general conflict structure and cost functions is just a generalization of Examples 7 and 8.

4 Comparative statics

Given the equilibrium characterization in the previous section, let us now investigate the comparative statics properties of the Nash equilibrium of our conflict game. Since an equilibrium may not be interior, the standard tools for deriving comparative statics results using the *Implicit Function Theorem* do not directly apply here. Moreover, since our conflict game is not supermodular, we cannot use the lattice approach from this strand of the literature. In this section, by using the equivalence of the solution between VI and Nash equilibrium, we show how to use the VI approach to conduct comparative statics of the equilibrium of the conflict game, even if the latter is not interior.

Formally, let $\boldsymbol{\theta}$ summarize all the exogenous parameters of the conflict game that can be adjusted smoothly in a set Θ. For example, $\boldsymbol{\theta}$ could include the valuations v^t of the battles, the precision of the contest technology f^t, the parameters in the functional form of the cost structure, etc. As a result, we now define the payoff function, the vector of the gradients of the payoff function and the Jacobian matrix as $\Pi_i(\mathbf{x};\boldsymbol{\theta}), \mathbf{F}(\mathbf{x};\boldsymbol{\theta}), \mathbf{M}(\mathbf{x};\boldsymbol{\theta})$, respectively. Finally, let $\mathcal{NE}(\boldsymbol{\theta})$ be the set of equilibria for the game $CF(\boldsymbol{\theta})$. Our task is to examine the properties of the mapping

$$\mathcal{NE}(\boldsymbol{\theta}) : \Theta \to \mathbf{R}_+^{\bar{n}}. \tag{24}$$

Let us first consider the pure-cost case and then, in Section 4.2, the general case.

4.1 Pure-cost case

Throughout this subsection, we assume that Assumptions 1 and 3 hold so that, by Theorem 3, a unique equilibrium exists. So for any $\boldsymbol{\theta}$, $\mathcal{NE}(\boldsymbol{\theta}) = \{\mathbf{x}^*(\boldsymbol{\theta})\}$ contains a single element $\mathbf{x}^*(\boldsymbol{\theta})$. Also $\mathbf{x}^*(\boldsymbol{\theta})$ is in $\mathcal{S}^2$ by Lemma 2. Moreover, since we are in a pure-cost case, we have for each i, $k_i = +\infty$.

Recall that, from (15), $\mathbf{x}^*(\boldsymbol{\theta})$ is an equilibrium if and only if

$$\mathbf{x} \in \mathcal{S}^1, \text{ and } \mathbf{x} \text{ solves VI}(\mathbf{F}(\cdot;\boldsymbol{\theta}), \mathbf{R}_+^{\bar{n}}),$$

as $\mathcal{S} = \mathbf{R}_+^{\bar{n}}$ in this case. For this special domain $\mathbf{R}_+^{\bar{n}}$, the solution to $\text{VI}(\mathbf{F}(\cdot;\boldsymbol{\theta}), \mathbf{R}_+^{\bar{n}})$ can be equivalently characterized as

$$x_i^{t*} \geq 0, \frac{\partial \Pi_i(\mathbf{x}^*;\boldsymbol{\theta})}{\partial x_i^t} \leq 0, \text{ and } x_i^{t*}\frac{\partial \Pi_i(\mathbf{x}^*;\boldsymbol{\theta})}{\partial x_i^t} = 0, \tag{25}$$

for all i, t such that $\gamma_i^t = 1$. The system above is obtained from the complementarity slackness conditions for each player's payoff maximization in equilibrium. This can be written as:[44]

$$\mathbf{F}(\mathbf{x}^*;\boldsymbol{\theta}) \geq \mathbf{0}, \ \ \mathbf{x}^* \geq \mathbf{0}, \ \text{ and } \langle \mathbf{F}(\mathbf{x}^*;\boldsymbol{\theta}), \mathbf{x}^* \rangle = 0. \tag{26}$$

Definition 7. *An equilibrium point $\mathbf{x}^*(\boldsymbol{\theta})$ satisfying (26) is called non-degenerate if $x_\ell^* + F_\ell(\mathbf{x}^*;\boldsymbol{\theta}) > 0, \forall \ell \in \{1, 2, \cdots, \bar{n}\}$.*

Since both x_ℓ^* and $F_\ell(\mathbf{x}^*;\boldsymbol{\theta})$ are non-negative, this condition is equivalent to the set $\{\ell \in \{1, 2, \cdots, \bar{n}\} : x_\ell^* = 0, F_\ell(\mathbf{x}^*;\boldsymbol{\theta}) = 0\}$ being empty. In other words, we assume that the complementarity slackness condition holds strictly. We will now only focus on non-degenerate equilibria for which, as in (25), either there is an interior solution, which is an extremum, or a corner solution, which is not an extremum. For any non-degenerate equilibrium point $\mathbf{x}^*$, it is useful to partition efforts into two disjoint sets:

$$\alpha := \{\ell \in \{1, 2, \cdots, \bar{n}\} : x_\ell^* > 0, F_\ell(\mathbf{x}^*;\boldsymbol{\theta}) = 0\}, \tag{27}$$

$$\breve{\alpha} := \{\ell \in \{1, 2, \cdots, \bar{n}\} : x_\ell^* = 0, F_\ell(\mathbf{x}^*;\boldsymbol{\theta}) > 0\}, \tag{28}$$

with $\alpha \cap \breve{\alpha} = \emptyset$, $\alpha \cup \breve{\alpha} = \{1, 2, \cdots, \bar{n}\}$ (due to non-degeneracy). Clearly, α is always non-empty since, in equilibrium, there exists at least one active player in every battle by Lemma 1. Note that $\breve{\alpha}$ is empty only when $\mathbf{x}^*$ is an interior equilibrium.[45]

Theorem 4. *Suppose that Assumptions 1 and 3 hold and that each cost function is of the pure-cost case. If the equilibrium $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is non-degenerate at $\boldsymbol{\theta}^*$, then there exists an open neighborhood $\mathcal{O}$ of $\boldsymbol{\theta}^*$ such that, for any $\boldsymbol{\theta} \in \mathcal{O}$, the unique equilibrium $\mathbf{x}^*(\boldsymbol{\theta})$ is continuously differentiable with*

$$\frac{\partial \mathbf{x}_\alpha^*(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = -(\mathbf{M}_{\alpha\alpha}(\mathbf{x};\boldsymbol{\theta}))^{-1} \frac{\partial \mathbf{F}_\alpha(\mathbf{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\Big|_{\mathbf{x}=\mathbf{x}(\boldsymbol{\theta})}, \quad \textit{and} \tag{29}$$

$$\frac{\partial \mathbf{x}_{\breve{\alpha}}^*(\boldsymbol{\theta})}{\partial \theta} = 0, \ \textit{as } \mathbf{x}_{\breve{\alpha}}(\boldsymbol{\theta}) = \mathbf{0}.$$

First, as stated above, the standard Implicit Function Theorem only works for interior equilibria when all the FOCs are satisfied with equalities. In our case, due to the possibility of corner solutions, we need to distinguish the slack variables in α from the binding variables in $\breve{\alpha}$. By forcing the variables in $\breve{\alpha}$ to remain binding, we can use the Implicit Function Theorem partially on the set of slack variables to compute their trajectory as $\boldsymbol{\theta}$ varies. The non-degeneracy condition guarantees that the binding variables indeed remain binding for $\boldsymbol{\theta}$ near $\boldsymbol{\theta}^*$. A critical prerequisite to apply this *"Partial Implicit Function Theorem approach"* is that the submatrix $\mathbf{M}_{\alpha\alpha}$ is non-singular.

[44]This problem in (26) is called a nonlinear complementarity problem $NLP(\mathbf{F}, \mathbf{R}_+^{\bar{n}})$, which is a special case of VI when $\mathcal{K}$ is the cone $\mathbf{R}_+^{\bar{n}}$; see Facchinei and Pang (2007).

[45]All the proofs about the comparative statics results can be found in Section B.3 of the Appendix B.

Note that α is endogenously determined in equilibrium; it is critical that $\mathbf{M}_{\alpha\alpha}$ is nonsingular for any index α, which is indeed the case by the positive definiteness of $\mathbf{M}$. This observation points out another implication of the positive definiteness of $\mathbf{M}$.

Second, the VI formulation of the equilibrium is the key step that enables us to apply this *"Partial Implicit Function Theorem approach"*. We will see this connection further in the general case that may include budget constraints. Finally, when $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is non-degenerate, by our construction, the index set α is locally constant. Therefore, if player i is inactive (active) in battle a at $\boldsymbol{\theta}^*$, she remains inactive (active) in a neighborhood of $\boldsymbol{\theta}^*$. This shows the local robustness of active and inactive players near a non-degenerate equilibrium. However, this is not the case when $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is degenerate. Indeed, a player can switch from inactive to active in a certain battle, at a point $\boldsymbol{\theta}$ arbitrarily close to $\boldsymbol{\theta}^*$ (see Example 9 below).

To illustrate these points, let us provide a brief sketch of the proof of Theorem 4. Consider the following system:

$$\begin{cases} F_\ell(\mathbf{x};\boldsymbol{\theta}) = 0, & \ell \in \alpha; \\ x_\jmath = 0, & \jmath \in \check{\alpha}, \end{cases} \text{ or equivalently } \begin{bmatrix} F_\alpha(\mathbf{x};\boldsymbol{\theta}) \\ \mathbf{x}_{\check{\alpha}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{30}$$

Clearly $\mathbf{x}^*(\boldsymbol{\theta}^*)$ satisfies this system of $\bar{n}$ equations with $\bar{n}$ unknowns at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ from equilibrium conditions. Moreover, the Jacobian of the mapping with respect to $\mathbf{x}$ is just $Diag\{\mathbf{M}_{\alpha\alpha}, \mathbf{I}_{\check{\alpha}}\}$, which can be shown to be nonzero at $(\mathbf{x}, \boldsymbol{\theta}) = (\mathbf{x}^*(\boldsymbol{\theta}^*), \boldsymbol{\theta}^*)$. So by the Implicit Function Theorem, the system in (30) implicitly defines a smooth function $\mathbf{x}^*(\boldsymbol{\theta})$ near $(\mathbf{x}^*(\boldsymbol{\theta}^*), \boldsymbol{\theta}^*)$. Next we can show that the solution $\mathbf{x}^*(\boldsymbol{\theta})$ to (30) indeed satisfies (26) under non-degeneracy, hence it is an equilibrium of $CF(\boldsymbol{\theta})$ for $\boldsymbol{\theta}$ close to $\boldsymbol{\theta}^*$.

An interior equilibrium is obviously non-degenerate, so we can state the following result:

Proposition 5. *Suppose that Assumptions 1 and 3 hold and that each cost function is of the pure-cost case. Assume also that $\mathbf{x}^*(\theta^*)$ is an interior equilibrium at $\boldsymbol{\theta}^*$. Then, for θ in a small open neighborhood of $\boldsymbol{\theta}^*$, $\mathbf{x}^*(\theta)$ remains interior with*

$$\frac{\partial \mathbf{x}^*(\boldsymbol{\theta})}{\partial \theta} := -(\mathbf{M}(\mathbf{x};\boldsymbol{\theta}))^{-1} \frac{\partial \mathbf{F}(\mathbf{x};\boldsymbol{\theta})}{\partial \theta}|_{\mathbf{x}=\mathbf{x}^*(\boldsymbol{\theta})}.$$

For the pure-cost case considered in this subsection, we can always prove that the unique equilibrium $\mathbf{x}^*(\boldsymbol{\theta})$ is continuous in $\boldsymbol{\theta}$, but may not always differentiable. Therefore, the non-degeneracy assumption is not redundant, and, in fact, critical for the differentiability of $\mathbf{x}^*(\boldsymbol{\theta})$ in $\boldsymbol{\theta}$. Since degeneracy only occurs non-generically, $\mathbf{x}^*(\boldsymbol{\theta})$ is continuously differentiable in $\boldsymbol{\theta}$ almost everywhere. These observations are illustrated in the following example.

Example 9 (Impact of an increase in the cost function in the pure-cost case for a complete network)**.** *Consider a one-battle-three-player game and the Tullock lottery contest as in Example 7 but with $v = 4$, $c_1(x_1) = x_1, c_2(x_2) = x_2$, and $c_3(x_3) = \theta x_3$, where $\theta > 0$ is the parameter of interest. It is easily verified that there is a unique equilibrium given by:*

$$\mathbf{x}^*(\theta) := (x_1(\theta), x_2(\theta), x_3(\theta)) = \begin{cases} \left(\frac{8\theta}{(2+\theta)^2}, \frac{8\theta}{(2+\theta)^2}, \frac{8(2-\theta)}{(2+\theta)^2}\right) & \textit{if } \theta \in (0, 2]; \\ (1, 1, 0) & \textit{if } \theta \geq 2. \end{cases} \tag{31}$$

Indeed, if θ is not too large, player 3 is active and we have a unique interior equilibrium. However, as θ increases, player 3 becomes weaker, and thus reduces her effort $x_3(\theta)$. When $\theta \geq 2$, player 3 becomes inactive. We show that $\mathbf{x}^*(\theta)$ is non-degenerate if and only if $\theta \neq 2$.[46] *Indeed, from (31), $\mathbf{x}^*(\theta)$ is continuously differentiable in θ except at $\theta = 2$, which is consistent with Theorem 4. At $\theta = 2$, $\mathbf{x}^*(\theta)$ is continuous, but not differentiable.*[47]

4.2 General case

In the general case with budget caps, multiple equilibria may arise. Moreover, whenever multiple equilibria occur, due to the convexity of $\mathcal{NE}$, an equilibrium is never locally unique. To facilitate our comparative statics exercises , we make the following assumption throughout this subsection.

Assumption 4. *Assume $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is of type S^2.*

In particular, this assumption implies that $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is the unique equilibrium of $CF(\boldsymbol{\theta})$ at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ by Theorem 2. For ease of notation, we define the following

$$\mathbf{B} = \begin{bmatrix} \mathbf{1}_{t_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{t_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{t_n} \end{bmatrix}_{\bar{n}\times n}, \text{ and } \mathbf{k} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}_{n\times 1},$$

where t_i is the dimension of the strategy space of player i and $\mathbf{1}_{t_i}$ is the vector of 1 of dimension t_i. So the budget constraints can collectively be expressed as $\mathbf{k} - \mathbf{B}^T\mathbf{x} \geq \mathbf{0}$. Since the budget constraints are imposed on each player independently, the columns of $\mathbf{B}$ are linear independent, and so the rank of $\mathbf{B}$ is equal to n. For the same reason, we can reformulate the Nash equilibrium as the solution to a VI problem.

Proposition 6. *$\mathbf{x}^*$ is an equilibrium of the game if and only if $\mathbf{x}^*$ is of type S^1, and there exists $\boldsymbol{\lambda}^* = (\lambda_1^*, \cdots, \lambda_n^*)^T$ such that*

$$(\mathbf{x}^*, \boldsymbol{\lambda}^*) \text{ solves } VI(\tilde{\mathbf{F}}, \mathbf{R}_+^{\bar{n}} \times \mathbf{R}_+^n)$$

where

$$\tilde{\mathbf{F}}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{bmatrix} \lambda_1\mathbf{1}_{t_1} - \nabla_{\mathbf{x}_1}\Pi_1 \\ \vdots \\ \lambda_n\mathbf{1}_{t_n} - \nabla_{\mathbf{x}_n}\Pi_n \\ \mathbf{k} - \mathbf{B}^T\mathbf{x} \end{bmatrix}, \quad \mathbf{R}_+^{\bar{n}} \times \mathbf{R}_+^n \to \mathbf{R}^{\bar{n}+n}. \tag{32}$$

Recall that in the original VI($\mathbf{F}, \mathcal{S}$), the domain of the characterization of equilibrium in Proposition 1 was $\mathcal{S} = \prod \Delta_i$, which potentially explicitly depends on $\boldsymbol{\theta}$, in particular $k_i, i \in \mathcal{N}$. The advantage of this augmented VI($\tilde{\mathbf{F}}, \mathbf{R}_+^{\bar{n}+n}$) is that the domain $\mathbf{R}_+^{\bar{n}+n}$ is independent of $\boldsymbol{\theta}$.

[46] Indeed, when $\theta < 2$, the equilibrium is interior, hence non-degenerate. For $\theta \geq 2$, $x_1(\theta) = x_2(\theta) = 1 > 0$, and $x_3(\theta) = 0$ with $F_3(\mathbf{x}^*(\theta)) = -\frac{\partial\{v\frac{x_3}{x_1+x_2+x_3} - \theta x_3\}}{\partial x_3}|_{\mathbf{x}=\mathbf{x}^*(\theta)} = \theta - 2$, which is zero only at $\theta = 2$.

[47] More precisely, $x_3(\theta)$ is not differentiable at $\theta = 2$ as the left derivative is $-\frac{1}{2}$, while the right derivative is 0.

Moreover, this reformulation enables us to exploit a similar method as in the previous subsection to conduct a sensitivity analysis. Also, the new VI formulation explicitly takes into account the λ_is, the shadow prices of the budget constraints, which are useful for the comparative statics exercises. Finally, the Jacobian matrix of the mapping $\tilde{\mathbf{F}}$, that is $\tilde{\mathbf{M}}$, is a bordered matrix of $\mathbf{M}$, defined as:

$$\tilde{\mathbf{M}} := \begin{bmatrix} \mathbf{M} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0}_{n\times n} \end{bmatrix}$$

which only depends on $\mathbf{x}$, but not on $\boldsymbol{\lambda}$.[48]

Given $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, let α be the index set with positive efforts, and β be the index set with binding budget constraints.

$$\begin{aligned} \alpha &:= \{\ell \in \{1,2,\cdots,\bar{n}\} : x^*_\ell > 0, \tilde{F}_\ell(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0\}; \\ \breve{\alpha} &:= \{\ell \in \{1,2,\cdots,\bar{n}\} : x^*_\ell = 0, \tilde{F}_\ell(\mathbf{x}^*, \boldsymbol{\lambda}^*) > 0\}; \\ \beta &:= \{i \in \{1,2,\cdots,n\} : k_i - \sum_{a\in\mathcal{T}_i} x_i^{*a} = 0, \lambda^*_i > 0\}; \\ \breve{\beta} &:= \{i \in \{1,2,\cdots,n\} : k_i - \sum_{a\in\mathcal{T}_i} x_i^{*a} > 0, \lambda^*_i = 0\}. \end{aligned}$$

We modify the condition for non-degeneracy accordingly, so that the following set:

$$\{\ell \in \{1,2,\cdots,\bar{n}\} : x^*_\ell = 0, \tilde{F}_\ell(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0\} \cup \{i \in \{1,2,\cdots,n\} : k_i - \sum_{a\in\mathcal{T}_i} x_i^{*a} = 0, \lambda^*_i = 0\}$$

is empty.

Theorem 5. *Suppose that Assumptions 1, 2 and 4 hold and that $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is a non-degenerate equilibrium with multiplier vector $\boldsymbol{\lambda}^*$ at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$. Then, there exists an open neighborhood $\mathcal{O}$ of $\boldsymbol{\theta}^*$ such that, for any $\boldsymbol{\theta} \in \mathcal{O}$, the conflict game $CF(\boldsymbol{\theta})$ has a unique equilibrium $\mathbf{x}(\boldsymbol{\theta})$ with associated multiplier $\boldsymbol{\lambda}(\boldsymbol{\theta})$. Both $\mathbf{x}(\boldsymbol{\theta})$ and $\boldsymbol{\lambda}(\boldsymbol{\theta})$ are continuously differentiable in $\boldsymbol{\theta} \in \mathcal{O}$ with*

$$\begin{bmatrix} \frac{\partial \mathbf{x}_\alpha(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \\ \frac{\partial \boldsymbol{\lambda}_\beta(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \end{bmatrix} = - \begin{bmatrix} \mathbf{M}_{\alpha\alpha}(\mathbf{x}) & \mathbf{B}_{\alpha\beta} \\ \mathbf{B}^T_{\alpha\beta} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \tilde{\mathbf{F}}_\alpha(\mathbf{x},\boldsymbol{\lambda};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \\ \frac{\partial \tilde{\mathbf{F}}_\beta(\mathbf{x},\boldsymbol{\lambda};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \end{bmatrix} \Big|_{\mathbf{x}=\mathbf{x}(\boldsymbol{\theta}),\boldsymbol{\lambda}=\boldsymbol{\lambda}(\boldsymbol{\theta})}. \tag{33}$$

$$\frac{\partial \mathbf{x}_{\breve{\alpha}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0, \text{ as } \mathbf{x}_{\breve{\alpha}} = \mathbf{0}.$$

$$\frac{\partial \boldsymbol{\lambda}_{\breve{\beta}}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0, \text{ as } \boldsymbol{\lambda}_{\breve{\beta}} = \mathbf{0}.$$

Theorem 5 provides the comparative statics result in the general setting with arbitrary conflict structure and general cost and contest technologies.[49] Given a non-degenerate type S^2 equilibrium $\mathbf{x}^*(\boldsymbol{\theta}^*)$ at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, then for any $\boldsymbol{\theta}$ near $\boldsymbol{\theta}^*$, the conflict game $CF(\boldsymbol{\theta})$ has a unique equilibrium that changes smoothly with the parameter $\boldsymbol{\theta}$. Since some players' budget constraints may be

[48] $\tilde{\mathbf{M}}$ is obviously not positive definite due to zeros in the lower right block.

[49] See Nti (1997) and Jensen (2016) for comparative statics results for a single battle contest model.

binding, the shadow price of budget is also changing smoothly as players are optimally adjusting efforts across different battles, given the movement of the components of the game.[50]

In Theorems 4 (pure-cost case) and 5 (general case), we provide the theoretical results of the comparative static exercises when equilibria are not degenerate. These results are useful for further analysis of equilibrium payoffs and for the effectiveness of several policy interventions.

4.3 Equilibrium payoff and aggregate effort

So far we have studied the impact of an increase in a given parameter on individual effort. We would now like to analyze the impact on equilibrium payoff and aggregate efforts. Define

$$\Pi_i^*(\boldsymbol{\theta}) := \Pi_i(\mathbf{x}^*(\boldsymbol{\theta}); \boldsymbol{\theta}) \tag{34}$$

as the equilibrium payoff for player i, and

$$X(\boldsymbol{\theta}) = \langle \mathbf{1}_{\bar{n}}, \mathbf{x}^*(\boldsymbol{\theta}) \rangle \tag{35}$$

as the aggregate efforts across all battles.[51] Define by X the aggregate effort, i.e., the sum of efforts of all players.

Proposition 7. *Under the same assumptions as in Theorem 5, both equilibrium payoffs and aggregate efforts are continuously differentiable in $\boldsymbol{\theta} \in \mathcal{O}$ with*

$$\frac{\partial \Pi_i^*}{\partial \boldsymbol{\theta}} = \frac{\partial \Pi_i}{\partial \mathbf{x}_\alpha} \frac{\partial \mathbf{x}_\alpha}{\partial \boldsymbol{\theta}} + \frac{\partial \Pi_i}{\partial \boldsymbol{\theta}} \tag{36}$$

$$\frac{\partial X}{\partial \boldsymbol{\theta}} = \langle \mathbf{1}_{\bar{n}}, \frac{\partial \mathbf{x}}{\partial \boldsymbol{\theta}} \rangle = \langle \mathbf{1}_{|\alpha|}, \frac{\partial \mathbf{x}_\alpha}{\partial \boldsymbol{\theta}} \rangle \tag{37}$$

evaluated at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta})$, where $\frac{\partial \mathbf{x}_\alpha}{\partial \boldsymbol{\theta}}$ is given by (33) in Theorem 5.

We can also study the comparative statics for aggregate effort for each specific battle t and/or each player i. For that, we decompose the impacts of $\boldsymbol{\theta}$ into direct effect and strategic (indirect) effect:

$$\frac{\partial \Pi_i^*}{\partial \boldsymbol{\theta}} = \underbrace{\frac{\partial \Pi_i}{\partial \mathbf{x}_\alpha} \frac{\partial \mathbf{x}_\alpha}{\partial \boldsymbol{\theta}} + \frac{\partial \Pi_i}{\partial \mathbf{x}_{\breve{\alpha}}} \overbrace{\frac{\partial \mathbf{x}_{\breve{\alpha}}}{\partial \boldsymbol{\theta}}}^{=\mathbf{0}}}_{\text{strategic effect}} + \underbrace{\frac{\partial \Pi_i}{\partial \boldsymbol{\theta}}}_{\text{direct effect}} .$$

Indeed, when a parameter θ increases, it directly affects Π_i^*, the equilibrium payoff of i, but also indirectly affects Π_i^* through the change in efforts of active players ($\mathbf{x}_\alpha$) and inactive players ($\mathbf{x}_{\breve{\alpha}}$). The latter $\mathbf{x}_{\breve{\alpha}}$ vanishes as $\frac{\partial \mathbf{x}_{\breve{\alpha}}}{\partial \boldsymbol{\theta}} = \mathbf{0}$ by Theorem 4. In some situations, the direct effect may also be zero. Everything depends on which parameter θ we consider. For example, consider a change in

[50] Observe that, when $\beta = \emptyset$ (all the budget constraints are strictly slack), we are back to the case discussed in Theorem 4.

[51] The proof of Proposition 7 is straightforward as the results follow by direct differentiation and chain rule.

the budget cap for player i, i.e., $\theta = k_i$. Clearly, the parameter k_i does not directly enter the payoff function of any player but the strategic effect still exists and may be significant. If we, instead, consider a change in the valuation of battle t, i.e., $\theta = v^t$, clearly both effects co-exist. Indeed, the direct effect is zero for players not competing in battle t, but positive for the other players. Although the multipliers $\lambda(\boldsymbol{\theta})$ do not enter the payoff directly, they affect the strategic effect as shown in (33).

We would now like to answer the following question: Suppose that the planner has one additional dollar to add to a battle valuation. Which battle should she choose? The answer depends on the objective function of the planner. We consider two cases. First, the planner chooses the subsidy that maximizes *aggregate efforts*, i.e.,

$$OPT^X = \arg\max_{t\in\mathcal{T}} \frac{\partial X}{\partial v^t}. \tag{38}$$

Second, the planner chooses the subsidy that maximizes *aggregate payoffs*, i.e.,[52]

$$OPT^\Pi = \arg\max_{t\in\mathcal{T}} \sum_{i\in\mathcal{N}} \frac{\partial \Pi_i^*}{\partial v^t}. \tag{39}$$

In a general conflict game, we can compute the solutions OPT^X and OPT^Π to both programs, as those terms $\frac{\partial X}{\partial v^t}, \frac{\partial \Pi_i^*}{\partial v^t}$ are explicitly given by Proposition 7. To gain some intuition, let us illustrate this issue with the following example.

Example 10 (Optimal subsidies in a pure-cost case in a bipartite network)**.** *Consider the network in Figure 4 with 3 players ($\mathcal{N} = \{1,2,3\}$) and 4 battles ($\mathcal{T} = \{a,b,c,d\}$). The cost function is assumed to be quadratic and given by: $c_i = \frac{1}{2}(X_i)^2$, for $i = 1,2,3$, where X_i is the aggregate effort of player i. For example, for player 1, $X_1 = x_1^a + x_1^c + x_1^d$, as she participates in three battles a, c, and d. X_2 and X_3 are defined in a similar way. The values of the battles are equal to: $\boldsymbol{\theta}^* = (v^a, v^b, v^c, v^d) = (v,v,v,V)$ and we consider changes in $\boldsymbol{\theta}^*$, the values of the battles.*

First, we show that there is a unique (symmetric) equilibrium given by:[53]

$$x_1^a = x_1^c = x_2^a = x_2^b = x_3^b = x_3^c = \frac{3v}{2\sqrt{2(9v+4V)}}, \quad x_i^d = \frac{2\sqrt{2}V}{3\sqrt{9v+4V}}, \quad for\ i = 1,2,3.$$

We obtain the following comparative statics results, summarized in Table 1:

We see that a change in $v^a = v$ affects in a same way players 1 and 2. Intuitively, as v^a increases, x_1^a increases, but x_1^b and x_1^d are reduced by substitution of efforts. However, player 3 reduces her efforts in both battles $x_3^b = x_3^c$, but interestingly x_3^d increases. However, as $v^d = V$ increases, all players behave in the same way: only efforts in battle d increase, while efforts in battles a,b,c decrease.

From Table 1, it is straightforward to see that the solutions to OPT^X and OPT^Π are given by:

$$OPT^X = \{d\}, \quad OPT^\Pi = \{a,b,c\}.$$

[52]We can also consider weighted payoffs $\sum w_i \Pi_i^*$, for any positive weighting vector $\mathbf{w} = (w_1, \cdots, w_N)$.

[53]Unless the conflict game is highly symmetric in both values and cost structures, we do not have a closed-form solution for the equilibrium. The comparative statics results are obtained using Theorem 5. Details are available upon request.

∂/∂	X	$\sum \Pi_i^*$	Π_1^*	Π_2^*	Π_3^*
$v^a = v$	$\frac{3}{2\sqrt{2(9v+4V)}}$	$\frac{9}{12}$	$\frac{(135v+88V)}{144(3v+2V)}$	$\frac{(135v+88V)}{144(3v+2V)}$	$\frac{(27v+20V)}{72(3v+2V)}$
$v^d = V$	$\frac{4}{2\sqrt{2(9v+4V)}}$	$\frac{8}{12}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$

Table 1: Comparative statics results in Example 10

Indeed, since $\partial X/\partial v^a < \partial X/\partial v^d$, the planner should target the battle with the larger size (battle d) in order to maximize aggregate effort. However, since $\partial(\sum \Pi_i^*)/\partial v^a > \partial(\sum \Pi_i^*)/\partial v^d$, the planner should target the battle with the smaller size (either a, b or c) in order to maximize aggregate payoffs.

As shown in the above Example, the optimal policy generally depends on the objective function and also on the underlying conflict game. The richness of our model enables us to explore these interesting questions in different combinations of conflict topology and specifications of values and cost structures.[54]

5 Discussions

5.1 Stability of equilibrium

Let us investigate the issue of stability for the Nash equilibrium in the conflict game. Under the strong monotonicity of the cost function (Assumption 3), the equilibrium $\mathbf{x}^*$ is unique by Theorem 3. What can we say about the stability of the equilibrium? We follow Dixit (1986) and consider the adjustment or tatonnement process defined by

$$\frac{d\mathbf{x}_i(t)}{dt} = \sigma_i(BR_i(\mathbf{x}_{-i}) - \mathbf{x}_i), \quad i \in \mathcal{N} \tag{40}$$

where $\sigma_1, \cdots, \sigma_n > 0$ are the adjustment speeds, and $BR_i(\cdot)$ is the best-reply function of player i.[55] Clearly $\mathbf{x} = \mathbf{x}^*$ is a stationary point of the system. We call the unique equilibrium $\mathbf{x}^*$ *locally asymptotically stable* if there exists a neighborhood of $\mathbf{x}^*$ such that if the system above starts at any point inside this neighborhood, the solution to the system converges to $\mathbf{x}^*$. The following Proposition shows the stability result for this general conflict game with multi-dimensional actions and non-linear best-reply functions.

Proposition 8. *Assume that Assumptions 1 and 3 hold. Then, the unique equilibrium $\mathbf{x}^*$ is locally asymptotically stable under the adjustment process (40).*

In fact, this Proposition is a direct consequence of the positive definiteness of the matrix $\mathbf{M}$ at $\mathbf{x}^*$, which plays a key role in this paper.

[54] More examples are available upon request.

[55] Here we consider only local stability. It can be shown that in an open neighbourhood of $\mathbf{x}^*$, for each i, the best-reply $BR_i(\cdot)$ exists and is a singleton, not a correspondence.

5.2 Related VIs and implications

To apply Variational Inequality to characterize an equilibrium, we utilize the (strict) monotonicity of $\mathbf{F}$, which does not explicitly depend on the feasibility set Δ_i. Therefore, we can adopt similar techniques to analyze the equilibrium under a general feasibility set $\mathcal{X}_i \subset \mathbf{R}_+^{t_i}$. In Section C.2 of Appendix C, we provide some results of uniqueness of equilibrium of our conflict game under a more general feasibility set.

6 Conclusion

In this paper, we present a theory of conflicts among a group of players competing in multiple heterogeneous battlefields with an arbitrary conflict topology. We utilize tools from Variational Inequality (VI), and provide equivalent characterizations of the equilibrium. We show that the set of equilibria is always non-empty and convex. Furthermore, we demonstrate the power of VI in this particular class of games, and exploit VI techniques to conduct intensive comparative statics for equilibrium efforts and payoffs.

We have mainly been focusing on the theoretical issues in conflict games, such as existence, uniqueness of equilibrium and comparative statics exercises. The results obtained are very general and we believe can be useful in many applications. In particular, our comparative statics results lead to many interesting applications that can be empirically investigated. For instance, we can easily answer the following question: how a shock on the valuation of a battle affects the intensity of the conflict in that battle but also how it *propagates* to other battles in the network?

From a theoretical viewpoint, other aspects on conflicts could be studied in our game. For example, what are the optimal battle values for a given contest architecture? What is the optimal network design of conflict architecture? Another natural extension would be to introduce incomplete information in the conflict game either on the battle value or on the cost, and address the impact of information disclosure on efforts and welfare.[56] Furthermore, since the strategic alliance is commonly observed in geographic conflicts, it would be interesting to explore the formation and stability of coalition outcomes.[57] Finally, studying the dynamics of conflicts in a network setting could also be another research avenue.[58] We leave these exciting topics for future research.

[56] See Ewerhart and Quartieri (2015); Zhang and Zhou (2016); Ui (2016); Serena (2016).

[57] See Tan and Wang (2010); Jackson and Nei (2015); Huremovic (2019); Dziubiński et al. (2016).

[58] See Dziubinski, Goyal, and Minarsch (2019).

Appendix

A Notations and preliminary results

In this section, we provide some notations and present a few lemmas, which will be used for the proofs of our main results.

A.1 Notations

Let $\mathbf{I}_n$ denote the n-dimensional identity matrix, and $\mathbf{1}_n$ be the the column vector of 1s. The inner product of two column vectors $\mathbf{x}, \mathbf{y}$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ or $\mathbf{x}^T\mathbf{y}$. For an $m \times n$ matrix $\mathbf{B}$, the transpose is denoted as $\mathbf{B}^T$. Given index sets α, β, let $\mathbf{B}_{\alpha\beta}$ denote the submatrix of $\mathbf{B}$ with elements b_{ij}, in which $i \in \alpha, j \in \beta$. Also $\mathbf{x}_\alpha$ will denote the subvector of $\mathbf{x}$ with elements $x_i, i \in \alpha$ and $|\alpha|$ will denote the number of elements in the index set α, such that $\mathbf{x}_\alpha \in \mathbf{R}_+^{|\alpha|}$.

For a multi-variable function $f(\mathbf{x}, \mathbf{y}) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$, let $\nabla f = (\nabla_\mathbf{x} f^T, \nabla_\mathbf{y} f^T)^T$ denote the gradient, where $\nabla_\mathbf{x} f = (f_{x_1}, \cdots, f_{x_n})^T$, and $\nabla_\mathbf{y} f = (f_{y_1}, \cdots, f_{y_m})^T$. Similarly, the Hessian matrix ∇_f^2 can be represented by $\begin{bmatrix} \nabla^2_{\mathbf{xx}} f & \nabla^2_{\mathbf{xy}} f \\ \nabla^2_{\mathbf{yx}} f & \nabla^2_{\mathbf{yy}} f \end{bmatrix}$ where $\nabla^2_{\mathbf{xx}} f, \nabla^2_{\mathbf{yy}} f$ denote the sub-Hessian matrices, and $\nabla^2_{\mathbf{xy}} f = (\nabla^2_{\mathbf{yx}} f)^T$ denotes the matrix of cross partials in $x_i, y_j, i \in \{1, 2, \cdots, n\}, j \in \{1, 2, \cdots, m\}$.

Given an $n \times n$ matrix $\mathbf{A}$, not necessarily symmetric, $\mathbf{A}$ is called positive definite (semi-definite), if $\mathbf{x}^T\mathbf{A}\mathbf{x} > (\geq) 0$ for any nonzero vector $\mathbf{x} \in \mathbf{R}^n$. In the next section, we give the properties of these matrices.

A.2 Positive (semi)definite matrix

Definition A1. *Given an $n \times n$ real matrix $\mathbf{A}$, not necessarily symmetric, $\mathbf{A}$ is called positive definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0, \forall \mathbf{0} \neq \mathbf{x} \in \mathbf{R}^n$. Similarly, $\mathbf{A}$ is called positive semi-definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0, \forall \mathbf{0} \neq \mathbf{x} \in \mathbf{R}^n$.*

Clearly from this definition $\mathbf{A}$ is positive definite if and only if the symmetric part $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is positive definite. Since $\frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric, its eigenvalues are real number. It is obvious that $\mathbf{A}$ is positive semi-definite if and only if $\mathbf{A} + \epsilon \mathbf{I}_n$ is positive definite for any $\epsilon > 0$.

Lemma A1. *Suppose $\mathbf{A}$ is positive definite. The following results hold:*

(1) $det[\mathbf{A}] > 0$.

(2) The transpose $\mathbf{A}^T$ and the inverse matrix $\mathbf{A}^{-1}$ are both positive definite.

(3) For any nonempty index set $\alpha \subset \{1, 2, \cdots, n\}$, the submatrix $\mathbf{A}_{\alpha\alpha}$ is positive definite, hence $det[\mathbf{A}_{\alpha\alpha}] > 0$.

(4) $\mathbf{A}$ is a P-matrix.[1]

(5) All the eigenvalues of $\mathbf{A}$ have positive real parts.[2]

(6) For any $n \times k$ matrix $\mathbf{B}$ with rank k, $\mathbf{B}^T\mathbf{A}\mathbf{B}$ is positive definite.

Remark: If $\mathbf{A}$ is a symmetric and positive definite matrix, then all the results listed are well-known (see Meyer (2000)). The proofs are more involved if $\mathbf{A}$ is not necessarily symmetric. Our main application of this concept is to matrix $\mathbf{M}$ in (17), which is indeed asymmetric. The results stated in Lemma A1 are well-known, and the proofs are omitted.

A.3 Properties of CSF and payoffs

Lemmas A2, A3, A4 and A5 provide some preliminary results on the properties of contest success functions and payoffs in the conflict game. We only give the proof of Lemma A5 as those of Lemmas A2, A3 and A4 are immediate.

Lemma A2. *Suppose $f(\cdot)$ is strictly increasing and weakly concave with $f(0) = 0$.*

[I] For $\delta > 0$, define the following functions

$$p_i^\delta(x_1, \cdots, x_n) := \frac{f(x_i)}{\sum_{k=1}^n f(x_k) + \delta}, i = 1, 2, \cdots, n.$$

1. *$p_i^\delta(x_1, \cdots, x_n; \delta)$ is continuously differentiable on $\mathbf{R}_+^n$ with*

$$\frac{\partial p_i^\delta}{\partial x_i} = [1 - p_i^\delta]\frac{f'(x_i)}{\sum_{k=1}^n f(x_k) + \delta} > 0, \quad (A.1)$$

$$\frac{\partial^2 p_i^\delta}{\partial x_i \partial x_i} = [1 - p_i^\delta]\frac{f''(x_i)}{\sum_{k=1}^n f(x_k) + \delta} - 2[1 - p_i^\delta]\left[\frac{f'(x_i)}{\sum_{k=1}^n f(x_k) + \delta}\right]^2 < 0. \quad (A.2)$$

2. *p_i^δ is strictly increasing and strictly concave in x_i.*
3. *p_i^δ is convex in $x_{-i} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$.*
4. *$\sum_{k=1}^n p_k^\delta(\mathbf{x})$ is concave in $(x_1, \cdots, x_n)$.*

[II] Consider the following functions, defined on $\mathbf{R}_+^n$,

$$p_i(x_1, \cdots, x_n) := \begin{cases} \frac{f(x_i)}{\sum_{k=1}^n f(x_k)}, & \text{if } \max(x_1, \cdots, x_n) > 0 \\ \frac{1}{n} & \text{if } x_1 = \cdots = x_n = 0 \end{cases}, i = 1, 2, \cdots, n.$$

[1] See Gale and Nikaido (1965) for definition and properties of P-matrix.

[2] Note that the eigenvalue of $\mathbf{A}$ can be complex numbers, for example $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is positive definite, and the eigenvalues are $1 \pm \sqrt{-1}$.

On the set $\mathbf{R}_+^n \backslash \{(0,0,\cdots,0)\}$,

1. $p_i(x_1,\cdots,x_n)$ *is continuously differentiable with*

$$\frac{\partial p_i}{\partial x_i} = [1-p_i]\frac{f'(x_i)}{\sum_{k=1}^n f(x_k)}, \tag{A.3}$$

$$\frac{\partial^2 p_i}{\partial x_i \partial x_i} = [1-p_i]\frac{f''(x_i)}{\sum_{k=1}^n f(x_k)} - 2[1-p_i]\left[\frac{f'(x_i)}{\sum_{k=1}^n f(x_k)}\right]^2. \tag{A.4}$$

2. *When one of the* $x_k, k \neq i$ *is strictly positive,* p_i *is strictly increasing and strictly concave in* x_i *with* $\frac{\partial p_i}{\partial x_i} > 0$ *and* $\frac{\partial^2 p_i}{\partial x_i \partial x_i} < 0$. *When* $x_k = 0, \forall k \neq i$, $p_i = 1$ *for every* $x_i > 0$, *hence* p_i *is only weakly increasing and weakly concave in* $x_i > 0$ *with* $\frac{\partial p_i}{\partial x_i} = 0$ *and* $\frac{\partial^2 p_i}{\partial x_i \partial x_i} = 0$.

3. *When* $x_i = 0$, p_i *is trivially convex in* $(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n) \in \mathbf{R}_+^{n-1}\backslash\{(0,0,\cdots,0)\}$ *as* $p_i \equiv 0$ *in this case. When* $x_i > 0$, p_i *is convex in* $(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n) \in \mathbf{R}_+^{n-1}$.

4. $\sum_{k=1}^n p_k(\mathbf{x}) = 1$ *for any* $(x_1,\cdots,x_n)$.

Lemma A3. *Under Assumptions 1 and 2, the following statements hold:*

(1) $\mathcal{S}^2 \subseteq \mathcal{S}^1 \subseteq \mathcal{S}$. *Moreover, the closure of* $\mathcal{S}^1$ *equals the closure of* $\mathcal{S}^2$, *which is the same as* $\mathcal{S}$.

(2) *For any* $\mathbf{y} \in \mathcal{S}$, *for any* $\mathbf{x}' \in \mathcal{S}^1, \mathbf{x}'' \in \mathcal{S}^2$, *and for any* $t \in (0,1)$, *we have* $t\mathbf{y} + (1-t)\mathbf{x}' \in \mathcal{S}^1$ *and* $t\mathbf{y} + (1-t)\mathbf{x}'' \in \mathcal{S}^2$. *In particular,* $t\mathbf{x}' + (1-t)\mathbf{x}'' \in \mathcal{S}^2$. *Moreover, both* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *are convex.*

(3) *Both* $\mathcal{S}^1$ *and* $\mathcal{S}^2$ *are relatively open subsets of* $\mathcal{S}$. *More precisely, for any* $\mathbf{x}' \in \mathcal{S}^1$ *there exists* $\epsilon_1 > 0$ *such that whenever* $||\mathbf{y} - \mathbf{x}'|| < \epsilon_1$ *and* $\mathbf{y} \in \mathcal{S}$, *we have* $\mathbf{y} \in \mathcal{S}^1$. *Similarly, for any* $\mathbf{x}'' \in \mathcal{S}^2$ *there exists* $\epsilon_2 > 0$ *such that whenever* $||\mathbf{y} - \mathbf{x}''|| < \epsilon_2$ *and* $\mathbf{y} \in \mathcal{S}$, *we have* $\mathbf{y} \in \mathcal{S}^2$.

(4) *If* $\mathbf{x}$ *is of type* S^1, *then* $\Pi_i(\mathbf{x})$ *is continuously differentiable at* $\mathbf{x}$. *Moreover at any point* $\tilde{\mathbf{x}} \in \mathcal{S}\backslash\mathcal{S}^1$, *at least one player's payoff is not continuous at* $\tilde{\mathbf{x}}$.

(5) *If* $\mathbf{x}$ *is of type* S^2, *for any* i *and any* $\mathbf{y}_i \in \Delta_i$, $\Pi_i(\mathbf{y}_i, \mathbf{x}_{-i})$ *is continuously differentiable at* $(\mathbf{y}_i, \mathbf{x}_{-i})$.

Lemma A4. *Suppose the effort profile* $\mathbf{x}$ *is of type* S^1. *Then* $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i})$ *is continuously differentiable at* $\mathbf{x}$. *Moreover, for any* $a \in \mathcal{T}_i$, *the partial derivative of* Π_i *with respect to* x_i^a *equals*

$$\begin{aligned}\frac{\partial \Pi_i(\mathbf{x})}{\partial x_i^a} &= v^a \frac{\partial p_i^a(\mathbf{x}^a)}{\partial x_i^a} - \frac{\partial c_i(\mathbf{x}_i)}{\partial x_i^a} \\ &= v^a[1-p_i^a(\mathbf{x}^a)]\frac{f^{a\prime}(x_i^a)}{\sum_{j\in\mathcal{N}^a} f^a(x_j^a)} - \frac{\partial c_i(\mathbf{x}_i)}{\partial x_i^a}.\end{aligned} \tag{A.5}$$

Lemma A5. *Under Assumptions 1 and 2, for each fixed* $\mathbf{x}_{-i} \in \Delta_{-i}$, $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i})$ *is concave in* $\mathbf{x}_i$ *on* Δ_i, *i.e.,*

$$t\Pi_i(\mathbf{x}_i', \mathbf{x}_{-i}) + (1-t)\Pi_i(\mathbf{x}_i'', \mathbf{x}_{-i}) \geq \Pi_i(t\mathbf{x}_i' + (1-t)\mathbf{x}_i'', \mathbf{x}_{-i}) \tag{A.6}$$

for all $t \in [0,1]$ *and* $\mathbf{x}_i', \mathbf{x}_i'' \in \Delta_i$.

Although $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is always concave in $\mathbf{x}_i$, it might not be continuous at every $\mathbf{x}_i$ due to jump in the winning probability.

Proof of Lemma A5: Recall that $\Pi_i(\mathbf{x}) = \sum_{a\in\mathcal{T}_i} v^a p_i^a(\mathbf{x}^a) - c_i(\mathbf{x}_i)$. Since the cost term c_i is continuously differentiable and convex, it suffices to show that CSF $p_i^a(\mathbf{x}^a)$ in (1) is concave in $\mathbf{x}_i$ for each $a \in \mathcal{T}_i$. Since CSF $p_i^a(\mathbf{x}^a)$ depends on $\mathbf{x}$ only through $\mathbf{x}^a$, it suffices to show that p_i^a is concave in x_i^a. When there exists at least another player $j \in \mathcal{N}^a\backslash\{i\}$ with x_j^a positive, then p_i^a is continuously differentiable and concave in x_i^a by Part [II] of Lemma A2. However, when $x_j^a = 0$ for all $j \in \mathcal{N}^a\backslash\{i\}$, the winning probability p_i^a is either $1/n^a$ if $x_i^a = 0$, or 1 if x_i^a is positive. This step function, although not continuous at $x_i^a = 0$, is also concave in x_i^a. Combining these results together, we prove that $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is concave in $\mathbf{x}_i$ on Δ_i. □

B Proofs

B.1 Proofs of the existence results

Each player $i = 1, \cdots, n$ has a pure strategy set, Δ_i, and a payoff function: $\Pi_i : \Delta \mapsto \mathbf{R}_+$, where $\Delta = \times_{i=1}^n \Delta_i$. Then there is a fixed normal form game

$$G(\Delta_1, \cdots, \Delta_n, \Pi_1, \cdots, \Pi_n) = G(\Delta_i, \Pi_i)_{i=1}^n.$$

$G(\Delta_i, \Pi_i)_{i=1}^n$ is called a compact game if each Δ_i is a nonempty compact set, and each Π_i is a bounded payoff function. If for each i and every $\mathbf{x}_{-i}$, $\Pi_i(\cdot, \mathbf{x}_{-i})$ is quasi-concave on Δ_i, $G(\Delta_i, \Pi_i)_{i=1}^n$ is quasi-concave.

Since the conflict game is discontinuous, the proof of existence theorem follows from Reny (1999). To better understand Reny's results, we first recall some key notions.

Definition B2. *Player i can secure a payoff of $\alpha \in \mathbf{R}$ at $\mathbf{x} \in \Delta$ if there exists $\bar{\mathbf{x}}_i \in \Delta_i$, such that $\Pi_i(\bar{\mathbf{x}}_i, \mathbf{x}'_{-i}) \geq \alpha$ for all $\mathbf{x}'_{-i}$ in some open neighborhood of $\mathbf{x}_{-i}$.*

Definition B3. *A game $G = (\Delta_i, \Pi_i)_{i=1}^n$ is payoff secure if for every $\mathbf{x} \in \Delta$ and every $\epsilon > 0$, each player i can secure a payoff of $\Pi_i(\mathbf{x}) - \epsilon$ at $\mathbf{x}$.*

Definition B4. *A game $G = (\Delta_i, \Pi_i)_{i=1}^n$ is reciprocally upper semi-continuous if, whenever $(\mathbf{x}_i, \Pi_i)_{i=1}^n$ is in the closure of the graph of its vector payoff function and $\Pi_i(\mathbf{x}) \leq \Pi_i$ for every player i, then $\Pi_i(\mathbf{x}) = \Pi_i$ for every player i.*

Definition B5. *For any two vectors $\mathbf{u} = (u_1, \cdots, u_T)$ and $\mathbf{v} = (v_1, \cdots, v_T)$, the L-infinite norm is defined by*

$$||\mathbf{u} - \mathbf{v}||_\infty = \max_{i\in\{1,\cdots,T\}} |u_i - v_i|.$$

Theorem B1. *(Reny, 1999) If $G = (\Delta_i, \Pi_i)_{i=1}^n$ is compact, quasi-concave, reciprocally upper semi-continuous and payoff secure, then it possesses a pure strategy Nash equilibrium.*

We now show our existence result as follows.

Proof of Theorem 1: The existence theorem follows from verifying that the conflict game satisfies the conditions of Theorem B1 for existence of equilibria in discontinuous games.

(i) Under Assumptions 1 and 2, the conflict game is compact.

First we consider the case that every k_i is finite. Recall that the strategy space of player i is $\Delta_i = \{\mathbf{x}_i = \{x_i^a\} \in \mathbf{R}_+^{t_i} : \sum_{a\in\mathcal{T}_i} x_i^a \le k_i\}$, which is clearly compact and convex, and non-empty.

When $k_i = +\infty$ for some i, then by Assumption 2, the cost function c_i must be strongly monotone for this player. Note that $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{a\in\mathcal{T}_i} v^a p_i^a - c_i(\mathbf{x}_i)$. Since the winning probability is bounded above by 1, then there exists a large enough $M_i > 0$ such that in equilibrium a player would never choose $x_i^a > M_i$ in any battle $a \in \mathcal{T}_i$. As a result we can , without loss of generality, restrict player i's strategy space to $\hat{\Delta}_i := \Delta_i \cap [0, M_i]^{t_i}$, which is clearly compact and convex.

(ii) Under Assumptions 1 and 2, by Lemma A5, the conflict game is concave , hence quasi-concave. In other words, for each fixed $\mathbf{x}_{-i} \in \Delta_{-i}$, $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is concave in $\mathbf{x}_i$ on Δ_i, i.e.,

$$t\Pi_i(\mathbf{x}_i', \mathbf{x}_{-i}) + (1-t)\Pi_i(\mathbf{x}_i'', \mathbf{x}_{-i}) \ge \Pi_i(t\mathbf{x}_i' + (1-t)\mathbf{x}_i'', \mathbf{x}_{-i}) \tag{B.1}$$

for all $t \in [0,1]$ and $\mathbf{x}_i', \mathbf{x}_i'' \in \Delta_i$.

(iii) The conflict game is reciprocally upper semi-continuous.

Note that the sum of payoffs equals

$$\sum_{i\in\mathcal{N}} \Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{a\in\mathcal{T}} v^a - \sum_{i\in\mathcal{N}} c_i(\mathbf{x}_i)$$

as the winning probabilities for every battle a add up to 1. Since each $c_i(\mathbf{x}_i)$ is assumed to be continuous on Δ_i, the sum of the payoffs is also continuous. As a consequence, by Reny (1999), the conflict game is reciprocally upper semi-continuous.

(iv) The conflict game CF is payoff secure at any $\mathbf{x}$ for all the players.

Without loss of generality, we focus on player i. For each fixed $\mathbf{x}$, define $\mathcal{T}_i^+ = \{a \in \mathcal{T}_i : x_j^a = 0, \forall j \in N^a\}$. Let $\mathcal{T}_i^- = \mathcal{T}_i \backslash \mathcal{T}_i^+$ be the complement. Note that for any battle $a \in \mathcal{T}_i^+$, $\mathbf{x}^a = \mathbf{0}$, hence the winning probability $p_i^a(\mathbf{x}^a)$ has a jump at $\mathbf{x}^a = \mathbf{0}$; If $b \in \mathcal{T}_i^-$, $p_i^b(\cdot)$ is continuous at $\mathbf{x}^b$.

For any $\epsilon > 0$, since the cost function $c_i(\cdot)$ is continuous, we can find $\delta_1 > 0$ such that

$$|c_i(\mathbf{x}_i) - c_i(\mathbf{x'}_i)| \le \epsilon, \tag{B.2}$$

whenever $||\mathbf{x}_i - \mathbf{x'}_i||_\infty \le \delta_1$.

For any battle $b \in \mathcal{T}_i^-$, since $p_i^b(\cdot)$ is continuous at $\mathbf{x}^b$, so we can find $\delta_b > 0$ such that:

$$|p_i^b(\mathbf{x}^b) - p_i^b(\mathbf{x'}^b)| \le \epsilon \tag{B.3}$$

whenever $||\mathbf{x}^b - \mathbf{x'}^b||_\infty \le \delta_b$. Moreover, denote $\delta_2 = \min_{b\in\mathcal{T}_i^-} \delta_b$.

There are two cases to consider:

(iv-a) Player i's budget constraint is slack.

We define $\bar{\mathbf{x}}_i$ as follows:

$$\bar{x}_i^a = \begin{cases} \bar{\delta} & \text{if } a \in \mathcal{T}_i^+; \\ x_i^a & \text{if } a \in \mathcal{T}_i^-. \end{cases}$$

where $\bar{\delta} = \min(\delta_1, \min_{b \in \mathcal{T}_i^-} \delta_b) > 0$.[3] Now we claim that by choosing $\bar{\mathbf{x}}_i$, player i can secure a payoff $\Pi_i(\mathbf{x}) - \beta\epsilon$ at $\mathbf{x}$, i.e.,

$$\Pi_i(\bar{\mathbf{x}}_i, \mathbf{x}'_{-i}) \geq \Pi_i(\mathbf{x}) - \beta\epsilon, \tag{B.4}$$

whenever

$$||\mathbf{x}'_{-i} - \mathbf{x}_{-i}||_\infty \leq \bar{\delta}, \tag{B.5}$$

where $\beta = 1 + \sum_{a \in \mathcal{T}_i^-} v^a$.

To see that, we note that

$$\Pi_i(\bar{\mathbf{x}}_i, \mathbf{x}'_{-i}) - \Pi_i(\mathbf{x}) = \sum_{a \in \mathcal{T}_i} v^a (p_i^a(\bar{x}_i^a, \mathbf{x'}_{-i}^a) - p_i^a(\mathbf{x}^a)) - c(\bar{\mathbf{x}}_i) + c(\mathbf{x}_i). \tag{B.6}$$

For any $a \in \mathcal{T}_i^+$, $p_i^a(\bar{x}_i^a, \mathbf{x'}_{-i}^a) \geq \frac{1}{n^a}$, as player i's effort is $\bar{\delta}$ while the other player's effort in this battle is at most $\bar{\delta}$ by (B.5). In addition, $p_i^a(\mathbf{x}^a) = \frac{1}{n^a}$ by Eq.(1). Therefore, $(p_i^a(\bar{x}_i^a, \mathbf{x'}_{-i}^a) - p_i^a(\mathbf{x}^a)) \geq 0$.

For any $a \in \mathcal{T}_i^-$, by (B.3), we have $(p_i^a(\bar{x}_i^a, \mathbf{x'}_{-i}^a) - p_i^a(\mathbf{x}^a)) \geq -\epsilon$.

On the other hand, since $||\mathbf{x}_i - \bar{\mathbf{x}}_i||_\infty \leq \bar{\delta} \leq \delta_1$, we have $-c(\bar{\mathbf{x}}_i) + c(\mathbf{x}_i) \geq -\epsilon$ by (B.2).

Combining these results, we have:

$$\Pi_i(\bar{\mathbf{x}}_i, \mathbf{x}'_{-i}) - \Pi_i(\mathbf{x}) \geq -(1 + \sum_{a \in \mathcal{T}_i^-} v^a)\epsilon = -\beta\epsilon, \tag{B.7}$$

Since β is a constant, payoff security is proved.

(iv-b) Now we consider the case where the budget constraint of i is binding, i.e.,

$$\sum_{a \in \mathcal{T}_i} x_i^a = k_i > 0. \tag{B.8}$$

The proof is similar, except that we need to reconstruct $\bar{\mathbf{x}}_i$ to satisfy budget constraint.

Let f is a battle in the set $\mathcal{T}_i^-$ such that $x_i^f > 0$, such f must exist by (B.8). We define $\bar{\bar{\mathbf{x}}}_i$ as follows:

$$\bar{\bar{x}}_i^a = \begin{cases} \bar{\bar{\delta}} & \text{if } a \in \mathcal{T}_i^+; \\ x_i^a & \text{if } a \in \mathcal{T}_i^-, a \neq f; \\ x_i^f - \bar{\bar{\delta}}|\mathcal{T}_i^+| & \text{if } a = f. \end{cases}$$

[3]We could shrink $\bar{\delta}$ appropriately so that i's budget constraint is still satisfied.

We can redistribute δ_f form battle f to each battle $\mathcal{T}_i^+$ at most, where $\delta_f < x_i^f$. Thus, define $\bar{\bar{\delta}}$ in the similar way:

$$\bar{\bar{\delta}} = \min\left(\frac{\delta_1}{|\mathcal{T}_i^+|}, \frac{\delta_f}{|\mathcal{T}_i^+|}, \frac{x_i^f}{|\mathcal{T}_i^+|} \min_{b \in \mathcal{T}_i^- \setminus \{f\}} \delta_b\right) > 0. \tag{B.9}$$

Essentially, from $\mathbf{x}_i$, we redistribute $\bar{\bar{\delta}}$ from battle f to each battle in $\mathcal{T}_i^+$ to obtain $\bar{\bar{\mathbf{x}}}_i$. By construction of $\bar{\bar{\delta}}$, we have $\bar{\bar{x}}_i^a \geq 0$ for any battle $a \in \mathcal{T}_i$. Then, we claim that for such $\bar{\bar{\mathbf{x}}}_i$,

$$\Pi_i(\bar{\bar{\mathbf{x}}}_i, \mathbf{x}'_{-i}) \geq \Pi_i(\mathbf{x}) - \beta\epsilon, \tag{B.10}$$

whenever $||\mathbf{x}'_{-i} - \mathbf{x}_{-i}||_\infty \leq \bar{\bar{\delta}}$. Here $\beta = 1 + \sum_{a \in \mathcal{T}_i^-} v^a$.

Similar to the proof without budget constraint, for any battle $a \in \mathcal{T}_i^-$, we need to prove that

$$(p_i^a(\bar{\bar{x}}_i^a, \mathbf{x}'^a_{-i}) - p_i^a(\mathbf{x}^a)) \geq -\epsilon. \tag{B.11}$$

By (B.3), it is sufficient to prove the following

$$||(\bar{\bar{x}}_i^a, \mathbf{x}'^a_{-i}) - \underbrace{(x_i^a, \mathbf{x}^a_{-i})}_{=\mathbf{x}^a}||_\infty \leq \delta_a. \tag{B.12}$$

Except for battle f, the above is obviously as $\bar{\bar{\delta}} \leq \delta_a$.

For $a = f$, note that $|\bar{\bar{x}}_i^f - x_i^f| = |\mathcal{T}_i^+|\bar{\bar{\delta}} \leq \delta_f$ by (B.9). While

$$||\mathbf{x}'^f_{-i} - \mathbf{x}^f_{-i}||_\infty \leq \bar{\bar{\delta}} \leq \delta_f / |\mathcal{T}_i^+| \leq \delta_f, \tag{B.13}$$

so we get

$$||(\bar{\bar{x}}_i^f, \mathbf{x}'^f_{-i}) - (x_i^f, \mathbf{x}^f_{-i})||_\infty \leq \delta_f. \tag{B.14}$$

Moreover, for battle $a \in \mathcal{T}_i^+$, similar to the case without budget constraint, we have

$$(p_i^a(\bar{\bar{x}}_i^a, \mathbf{x}'^a_{-i}) - p_i^a(\mathbf{x}^a)) \geq 0.$$

In addition, since $||\mathbf{x}_i - \bar{\bar{\mathbf{x}}}_i||_\infty \leq |\mathcal{T}_i^+|\bar{\bar{\delta}} \leq \delta_1$, by (B.2), the following holds

$$-c(\bar{\bar{\mathbf{x}}}_i) + c(\mathbf{x}_i) \geq -\epsilon \tag{B.15}$$

Combined these results, payoff security is proved.

Since the conflict game is compact, quasi-concave, reciprocally upper semi-continuous and payoff secure, by Theorem B1 of Reny (1999), a pure strategy Nash equilibrium of the conflict game must exist. □

Sketch of the proof of of existence of equilibrium using the modified conflict game CF^{ϵ}[4]

[4]The details of this proof are available from the authors upon request.

First, we approximate the original game by truncating the strategy space so that the game CF^{ϵ} is defined by requiring that the effort of every player on each battle has a minimal lower bound $\epsilon > 0$ while the other elements of the payoff function such as the cost, the values of the battles and the conflict topology remain the same. The major part of the proof is to show that: (*i*) the truncated game CF^{ϵ} has an equilibrium for every ϵ; (*ii*) the limiting strategy $\mathbf{x}^*$ must avoid points of discontinuity in the original game, i.e., for every battle, there exists at least one participant exerting strictly positive effort; and (*iii*) there is no profit deviation under the limiting strategy $\mathbf{x}^*$ in the original conflict game CF.

Truncation of efforts removes the discontinuity in the winning probabilities. Then, under Assumptions 1 and 2, we can show that CF^{ϵ} satisfies the usual conditions (such as continuity and concavity of payoffs, convexity and compactness of strategy spaces) for equilibrium existence in a continuous game, which implies (*i*).

Proving (*ii*) is the most difficult step, as the payoff might not be continuous at the limiting strategy $\mathbf{x}^*$, and thus we cannot directly go to the limit. Instead, we prove that a subsequence of $\lambda_i(\epsilon)$, the Lagrange multiplier associated with player i's budget constraint (or the shadow price of the budget), has a uniform finite upper bound as ϵ goes to zero. The finiteness of this bound implies that, in the limiting strategy $\mathbf{x}^*$, for every battle, there is at least one player exerting positive effort as the winning probability has a discrete jump at the origin, but the shadow price of effort is uniformly bounded from above.

Finally, we prove (*iii*) by using (*ii*) and the properties of the payoff function $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i})$. □

B.2 Proofs of the uniqueness results

Proof of Lemma 1: Suppose there exists a battle a such that in equilibrium all the participants in battle a are inactive, i.e., $\mathbf{x}^a = \mathbf{0}$. Take any player i participating in battle a. We claim that i has a profitable deviation. There are two cases to consider: (1) Suppose $k_i = +\infty$. If player i increases her effort in battle a by $\eta > 0$, her probability of winning jumps from $1/n^a$ to 1, but the increment in cost is continuous in η. For a sufficiently small η, this deviation is profitable. (2) Suppose $k_i < +\infty$. If her budget constraint is slack, then we can apply the same logic as in case (1). On the other hand, if the budget constraint is binding for player i, then there exists at least one battle b in which i is exerting strictly positive effort. For $\eta > 0$ but sufficiently small, shifting η effort from battle b to a is a profitable deviation for player i since her cost function is continuous and the winning probability in battle b is continuous in η, but the winning probability in battle a has a discrete jump. □

Proof of Lemma 2: From Lemma 1, we know that for every battle a, at least one player i is exerting positive effort. But i cannot be the only active contestant in battle a, otherwise she can profit by lowering her effort in battle a slightly to still ensure a win in a, while strictly lowering her total cost by the strong monotonicity assumption on the cost function. □

Proof of Proposition 1: From (i) to (ii). Suppose $\mathbf{x}^*$ is a Nash equilibrium of the conflict game CF. Then $\mathbf{x}^* \in \mathcal{S}^1$ by Lemma 1. Moreover, since each Π_i is continuous and differentiable at $\mathbf{x}^*$ by Lemma A3, $\mathbf{F}(\mathbf{x}^*)$ is well-defined. Take any player i; for any $\mathbf{y}_i \in \Delta_i$ and any $t \in [0,1)$, the following must hold:

$$\Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \geq \Pi_i((1-t)\mathbf{x}_i^* + t\mathbf{y}_i, \mathbf{x}_{-i}^*)$$

where $((1-t)\mathbf{x}_i^* + t\mathbf{y}_i, \mathbf{x}_{-i}^*)$ lies in the set $\mathcal{S}^1$ by item 2 of Lemma A3. Therefore,

$$\lim_{t\to 0^+} \frac{\Pi_i((1-t)\mathbf{x}_i^* + t\mathbf{y}_i, \mathbf{x}_{-i}^*) - \Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)}{t} \leq 0, \text{equivalently } \langle -\nabla_{\mathbf{x}_i}\Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*),\ \ \mathbf{y}_i - \mathbf{x}_i^*\rangle \geq 0$$

Adding up these inequalities for all i yields the following

$$\langle \mathbf{F}(\mathbf{x}^*),\ \mathbf{y} - \mathbf{x}^*\rangle \geq 0 \text{ for any } \mathbf{y} = (\mathbf{y}_1, \cdots, \mathbf{y}_n) \in \Delta_1 \times \cdots \times \Delta_n. \tag{B.16}$$

From (ii) to (i) . Suppose $\mathbf{x}^* \in \mathcal{S}^1$, and $\langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^*\rangle \geq 0, \forall \mathbf{x} \in \mathcal{S}^1$. Since the closure of $\mathcal{S}^1$ is $\mathcal{S}$ by item 1 of Lemma A3, it must hold that $\langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^*\rangle \geq 0$ for all $\mathbf{x} \in \mathcal{S}$. Substituting $\mathbf{x} = (\mathbf{y}_i, \mathbf{x}_{-i}^*)$ yields

$$\langle -\nabla_{\mathbf{x}_i}\Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*),\ \ \mathbf{y}_i - \mathbf{x}_i^*\rangle \geq 0$$

for any $\mathbf{y}_i \in \Delta_i$. Next, consider the following single-variable function, $\sigma(t) = \Pi_i((1-t)\mathbf{x}_i^* + t\mathbf{y}_i, \mathbf{x}_{-i}^*), t \in [0,1]$. Then $\sigma(t)$ is concave in t in $[0,1]$ by Lemma A5. Moreover since $\mathbf{x}^*$ is of type S^1, the strategy $((1-t)\mathbf{x}_i^* + t\mathbf{y}_i, \mathbf{x}_{-i}^*) = (1-t)(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) + t(\mathbf{y}_i, \mathbf{x}_{-i}^*)$ is also of type S^1 when t is very close to 0. So $\sigma(t)$ is continuously differentiable in t in a small neighborhood of 0 (note that $\sigma(t)$ may not be continuous on the whole interval $[0,1]$, especially at $t = 1$). In particular, we have $\sigma'(0) = \langle \nabla_{\mathbf{x}_i}\Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*),\ \ \mathbf{y}_i - \mathbf{x}_i^*\rangle \leq 0$. The concavity of σ implies that $\frac{\sigma(t)-\sigma(0)}{t-0} \leq \frac{\sigma(t')-\sigma(0)}{t'-0}$ for any $0 < t' < t \leq 1$. Taking t' to zero yields $\frac{\sigma(t)-\sigma(0)}{t-0} \leq \sigma'(0) \leq 0$. So $\sigma(t) \leq \sigma(0)$ for any $t \in [0,1]$. In particular, we have $\sigma(1) \leq \sigma(0)$, or equivalently $\Pi_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \geq \Pi_i(\mathbf{y}_i, \mathbf{x}_{-i}^*)$. Since this inequality holds for any $\mathbf{y}_i \in \Delta_i$ and any player i, $\mathbf{x}^*$ is a Nash equilibrium. □

Proof of Proposition 2: We use Proposition 3 in the proof. Suppose $\mathbf{x}' \neq \mathbf{x}''$ in $\mathcal{S}^1$. Consider the following single-variable function

$$\omega(t) := \langle \mathbf{F}(t\mathbf{x}' + (1-t)\mathbf{x}''), (\mathbf{x}' - \mathbf{x}'')\rangle, t \in [0,1].$$

Note that $t\mathbf{x}' + (1-t)\mathbf{x}''$ is in $\mathcal{S}^1$ for any $t \in [0,1]$ by Lemma A3. Therefore, $\mathbf{F}(t\mathbf{x}' + (1-t)\mathbf{x}'')$ is continuously differentiable in $t \in [0,1]$. So $\omega(t)$ is differentiable in $t \in [0,1]$. Moreover,

$$\omega(1) - \omega(0) = \langle \mathbf{F}(\mathbf{x}') - \mathbf{F}(\mathbf{x}''),\ \ \mathbf{x}' - \mathbf{x}''\rangle, \textit{and}$$

$$\omega'(t) = (\mathbf{x}' - \mathbf{x}'')^T \cdot \mathbf{M}(t\mathbf{x}' + (1-t)\mathbf{x}'') \cdot (\mathbf{x}' - \mathbf{x}'').$$

By the Mean Value Theorem, there exists a $\hat{t} \in (0,1)$ such that

$$\omega(1) - \omega(0) = \omega'(\hat{t}) = (\mathbf{x}' - \mathbf{x}'')^T \cdot \mathbf{M}(\hat{t}\mathbf{x}' + (1-\hat{t})\mathbf{x}'') \cdot (\mathbf{x}' - \mathbf{x}'').$$

Since the matrix $\mathbf{M}(\hat{t}\mathbf{x}' + (1-\hat{t})\mathbf{x}'')$ is positive semi-definite by Proposition 3, $\omega'(\hat{t}) \geq 0$. Therefore, $\omega(1) - \omega(0) \geq 0$. Moreover, if in addition one of $\mathbf{x}', \mathbf{x}''$ is in $\mathcal{S}^2$, then $\hat{t}\mathbf{x}' + (1-\hat{t})\mathbf{x}''$ is in

$\mathcal{S}^2$ as well by item (2) of Lemma A3. (It is important here that $\hat{t}$ is in the open interval $(0,1)$, not on the boundary points) Then $\mathbf{M}(\hat{t}\mathbf{x}' + (1-\hat{t})\mathbf{x}'')$ is positive definite by Proposition 3, and so $\omega'(\hat{t}) > 0$. The rest just follows. □

Proof of Proposition 3: We first show the case when $\mathbf{x}$ is in $\mathcal{S}^2$. The proof is based on the observations in Goodman (1980): $\mathbf{M} + \mathbf{M}^T$ is positive definite, if the following three conditions are satisfied: (1) $\Pi_i(\cdot, \mathbf{x}_{-i})$ is strictly concave in $\mathbf{x}_i$ with a Hessian matrix $\nabla^2_{\mathbf{x}_i}\Pi_i$ that is negative definite at every point $\mathbf{x}$.[5] (2) Each $\Pi_i(\cdots, \mathbf{x}_i, \cdots)$ is convex in $\mathbf{x}_{-i} = (\mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_n)$, and (3) the sum of payoffs $\sum_{i\in\mathcal{N}}\Pi_i$ is concave in $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$.

Next, we verify these three conditions.

1. Note that $\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{a\in\mathcal{T}_i} v^a p_i^a(\mathbf{x}^a) - c_i(\mathbf{x}_i)$. Cost c_i of player i is convex in $\mathbf{x}_i$, and the first summation term is separable in $\mathbf{x}_i$. Moreover, $p_i^a(\mathbf{x}^a)$ is strictly concave in x_i^a with $\frac{\partial^2 p_i^a}{\partial(x_i^a)^2} < 0$ by item 2 of Lemma A2, and there are at least two active contestants in battle a (recall that $\mathbf{x}$ is in $\mathcal{S}^2$). Note that Hessian matrix $\nabla^2_{\mathbf{x}_i}\Pi_i = D_i - \nabla^2_{\mathbf{x}_i}c_i(\mathbf{x}_i)$, where D_i is a diagonal matrix with strictly negative diagonal entries $\frac{\partial^2 p_i^a}{\partial(x_i^a)^2}, a \in \mathcal{T}_i$, and $\nabla^2_{\mathbf{x}_i}c_i(\mathbf{x}_i)$ is positive semi-definite by convexity of c_i. Therefore, Hessian matrix $\nabla^2_{\mathbf{x}_i}\Pi_i$ is negative definite, which verifies condition (1).
2. For condition (2), note that by fixing x_i^a, each $p_i^a(\mathbf{x}^a)$ depends only on $\mathbf{x}$ through $\mathbf{x}^a$ (the components of efforts exerted in battle a). Moreover $p_i^a(\mathbf{x}^a)$ is convex in $\{\mathbf{x}_j^a, j \in \mathcal{N}^a \backslash \{i\}\}$ by item 3 of Lemma A2, and hence is convex in $(\mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_n) = \mathbf{x}_{-i}$. Therefore, $\sum v^a p_i^a(\mathbf{x}) - c_i(\mathbf{x}_i)$ is convex in $\mathbf{x}_{-i}$ as the cost of player i does not depend on $\mathbf{x}_{-i}$ and a positive linear combination of convex functions is convex.
3. For condition (3), we note that the sum of payoffs equals
$$\sum_{i\in\mathcal{N}}\Pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{a\in\mathcal{T}} v^a - \sum_i c_i(\mathbf{x}_i)$$
as the winning probabilities for every battle a add up to 1. Since each $c_i(\mathbf{x}_i)$ is convex in $\mathbf{x}_i$, $\sum_i c_i(\mathbf{x}_i)$ is convex in $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$. As a consequence, $\sum \Pi_i$ is concave in $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$.

Note that every point $\mathbf{x} \in \mathcal{S}^1$ is the limit of a sequence of points in $\mathcal{S}^2$.[6] Since the limit of convergent positive definite matrices is positive semi-definite, we prove the first part of the Proposition 3 for the case where $\mathbf{x} \in \mathcal{S}^1$. □

Proof of Theorem 2: Just as the monotonicity of a single-variable function is determined by the sign of its derivative, the monotonicity of $\mathbf{F}$ is reflected by the properties of its Jacobian matrix.

[5] The second claim is not directly implied by the first part, as a strictly concave function may have a negative semi-definite Hessian matrix at some point.

[6] For example, consider $\mathbf{x}[n] = \frac{n}{1+n}\mathbf{x} + \frac{1}{n+1}\mathbf{x}'', n = 1, 2, \cdots, \infty$, where $\mathbf{x}''$ is an arbitrary point in $\mathcal{S}^2$. Then each $\mathbf{x}[n]$ is in $\mathcal{S}^2$ by item 2 of Lemma A3. Moreover, $x[n] \to \mathbf{x}$ as $n \to \infty$.

From (16), we know that

$$\langle \mathbf{F}(\mathbf{x}^*) - \mathbf{F}(\mathbf{x}^{**}), \mathbf{x}^* - \mathbf{x}^{**} \rangle \leq 0 \tag{B.17}$$

for any two Nash equilibria $\mathbf{x}^*, \mathbf{x}^{**}$. If $\mathcal{NE}$ contains an equilibrium $\mathbf{x}^*$ of type S^2, then $\mathbf{x}^*$ must be the unique equilibrium. Otherwise there exists another equilibrium $\mathbf{x}^{**} \neq \mathbf{x}^*$, and

$$\langle \mathbf{F}(\mathbf{x}^*) - \mathbf{F}(\mathbf{x}^{**}), \mathbf{x}^* - \mathbf{x}^{**} \rangle > 0$$

by Proposition 2, which contradicts (B.17). On the other hand, since $\mathcal{NE} \subseteq \mathcal{S}^1$ by Lemma 1, if there is no equilibrium in $\mathcal{S}^2$, then all the equilibria must belong to the set $\mathcal{S}^1 \backslash \mathcal{S}^2$. (Note that Theorem 2 does not rule out the case that an equilibrium is unique, and the unique equilibrium is in $\mathcal{S}^1 \backslash \mathcal{S}^2$.) □

Proof of Theorem 3: Under Assumptions 1 and 3, any equilibrium must be of type S^2 by Lemma 2, so the case (2) in Theorem 2 never occurs. The result just follows. □

Proof of Proposition 4: Suppose $\mathbf{x}'$ and $\mathbf{x}''$ are two equilibria. Fix $t \in [0,1]$, and let $\mathbf{x}(t) = t\mathbf{x}' + (1-t)\mathbf{x}''$. We want to show that $\mathbf{x}(t)$ is also an equilibrium. First note that $\mathbf{x}(t) \in \mathcal{S}^1$ as both $\mathbf{x}'$ and $\mathbf{x}''$ are in $\mathcal{S}^1$, which is convex (see Lemma A3). Pick an arbitrary $\mathbf{z} \in \mathcal{S}$, and define $\mathbf{y} = \mu\mathbf{x}(t) + (1-\mu)\mathbf{z}$, for $\mu \in (0,1]$. Then $\mathbf{y} \in \mathcal{S}^1$ as well by item (2) of Lemma A3. We have

$$\begin{aligned} \langle \mathbf{F}(\mathbf{x}'), \mathbf{y} - \mathbf{x}' \rangle &\geq 0, && (\text{as } \mathbf{x}' \in \mathcal{NE}) \\ \langle \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{x}' \rangle \geq \langle \mathbf{F}(\mathbf{x}'), \mathbf{y} - \mathbf{x}' \rangle &\geq 0. && (\text{by monotonicity of } \mathbf{F}) \end{aligned}$$

Similarly, we have $\langle \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{x}'' \rangle \geq 0$. Therefore, $\langle \mathbf{F}(\mathbf{y}), \mathbf{y} - \mathbf{x}(t) \rangle \geq 0$, i.e.,

$$\langle \mathbf{F}(\mu\mathbf{x}(t) + (1-\mu)\mathbf{z}), (\mu\mathbf{x}(t) + (1-\mu)\mathbf{z} - \mathbf{x}(t)) \rangle \geq 0,$$

which leads to

$$\langle \mathbf{F}(\mu\mathbf{x}(t) + (1-\mu)\mathbf{z}), \mathbf{z} - \mathbf{x}(t) \rangle \geq 0$$

for any $\mu \in (0,1)$. Taking $\mu \to 1$ yields

$$\langle \mathbf{F}(\mathbf{x}(t)), \mathbf{z} - \mathbf{x}(t) \rangle \geq 0.$$

Since $\mathbf{z}$ is arbitrary, $\mathbf{x}(t)$ is an equilibrium by Proposition 1. □

Proof of Corollary 1: The main intuition just follows from the discussion after Corollary 1 in the main paper. For each $a \in \mathcal{T}_i$, we denote k_i^a as the unique solution to

$$v^a \frac{d\left\{\frac{f^a(z)}{f^a(k_i^a)+f^a(z)}\right\}}{dz}\Big|_{z=0} = 1, \text{ or } v^a \frac{f^{a'}(0)}{f^a(k_i^a)} = 1.$$

Multiple equilibria arise when $k_i > \sum_{a \in \mathcal{T}_i} k_i^a$. □

Proof of Corollary 2: First, we show that all the equilibria must be of type S^2 under the given assumptions. Suppose, $\mathbf{x}^*$ is an equilibrium and there exists a battle, say a, such that player i is the only active contestant in that battle. Since every player participates in every battle by completeness of $\mathbf{\Gamma}$, i must win any other battle $b \neq a$ with probability 1 as well. Otherwise, he can shift effort from battle a to b to strictly improve his payoff. Consequently, i is the only active player in every battle. Hence all the other players obtain zero profit, which is clearly not an equilibrium, as any player $j \neq i$ could secure positive profit by allocating his budget equally among all battles. We apply Theorem 2 directly to obtain uniqueness. □

B.3 Proofs of the comparative statics results

Proof of Theorem 4: First, we prove that the Jacobian matrix $Diag\{\mathbf{M}_{\alpha\alpha}(\mathbf{x},\boldsymbol{\theta}), \mathbf{I}_{\breve{\alpha}}\}$ is non-singular at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}^*), \boldsymbol{\theta} = \boldsymbol{\theta}^*$. This is equivalent to non-singularity of $\mathbf{M}_{\alpha\alpha}(\mathbf{x};\boldsymbol{\theta})$. Invoking Proposition 3, at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}^*), \boldsymbol{\theta} = \boldsymbol{\theta}^*$, we establish that $\mathbf{M}(\mathbf{x};\boldsymbol{\theta})$ is positive definite as $\mathbf{x}^*(\boldsymbol{\theta}^*)$ is of type S^2 by Lemma 2. Therefore the submatrix $\mathbf{M}_{\alpha\alpha}$ is also positive definite, and hence has a positive determinant by Lemma A1 , at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}^*), \boldsymbol{\theta} = \boldsymbol{\theta}^*$. By the Implicit Function Theorem, the solution to system (30) implicitly defines a smooth function $\mathbf{x}^*(\boldsymbol{\theta})$ near $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}^*), \boldsymbol{\theta} = \boldsymbol{\theta}^*$.

Next, we prove that $\mathbf{x}^*(\boldsymbol{\theta})$ is a solution to the linear complementarity problem:

$$\mathbf{F}(\mathbf{x}^*;\boldsymbol{\theta}) \geq \mathbf{0}, \;\; \mathbf{x}^* \geq \mathbf{0}, \;\; \text{and } \langle \mathbf{F}(\mathbf{x}^*;\boldsymbol{\theta}), \mathbf{x}^* \rangle = 0 \tag{B.18}$$

for $\boldsymbol{\theta}$ near $\boldsymbol{\theta}^*$. Clearly $\langle \mathbf{F}(\mathbf{x}^*(\boldsymbol{\theta});\boldsymbol{\theta}), \mathbf{x}^* \rangle = \langle \mathbf{F}_\alpha(\mathbf{x}^*(\boldsymbol{\theta});\boldsymbol{\theta}), \mathbf{x}^*_\alpha \rangle + \langle \mathbf{F}_{\breve{\alpha}}(\mathbf{x}^*(\boldsymbol{\theta});\boldsymbol{\theta}), \mathbf{x}^*_{\breve{\alpha}} \rangle = 0 + 0 = 0$ by (30). Next, $\mathbf{x}^*_\alpha(\boldsymbol{\theta}^*) > \mathbf{0}$ (from definition of α), so by continuity, $\mathbf{x}^*_\alpha(\boldsymbol{\theta}) > \mathbf{0}$ for $\boldsymbol{\theta}$ near $\boldsymbol{\theta}^*$. Furthermore, $\mathbf{x}^*_{\breve{\alpha}}(\boldsymbol{\theta}) = \mathbf{0}$ by construction, so $\mathbf{x}^*(\boldsymbol{\theta}) \geq \mathbf{0}$ in an open neighborhood of $\boldsymbol{\theta}^*$. Similarly we can show that $\mathbf{F}(\mathbf{x}^*(\boldsymbol{\theta});\boldsymbol{\theta}) \geq \mathbf{0}$ near $\boldsymbol{\theta}^*$. Combining these results, we prove that the solution to system (30), $\mathbf{x}^*(\boldsymbol{\theta})$ must be a solution to VI$(\mathbf{F}(\cdot;\boldsymbol{\theta}), \mathbf{R}^{\tilde{n}}_+)$. Moreover $\mathbf{x}^*(\boldsymbol{\theta})$ clearly has the same type as $\mathbf{x}^*(\boldsymbol{\theta}^*) \in \mathcal{S}^2$. Consequently, $\mathbf{x}^*(\boldsymbol{\theta})$ is also in $\mathcal{S}^2$, and must be the unique equilibrium of the game $CF(\boldsymbol{\theta})$ by Proposition 1 and Corollary 3.

For the comparative results, first we have $\frac{\partial \mathbf{x}^*_{\breve{\alpha}}(\boldsymbol{\theta})}{\partial \theta} = 0$ as $\mathbf{x}_{\breve{\alpha}}(\boldsymbol{\theta}) = \mathbf{0}$ by construction. Second, $\frac{\partial \mathbf{x}^*_\alpha(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ can be obtained by differentiating $F_\alpha(\mathbf{x};\boldsymbol{\theta}) = \mathbf{0}$. □

Proof of Proposition 6: First, note that the solution to VI$(\tilde{\mathbf{F}}, \mathbf{R}^{\tilde{n}}_+ \times \mathbf{R}^n_+)$ is equivalent to a nonlinear complementarity problem

$$\tilde{\mathbf{F}}(\mathbf{x},\boldsymbol{\lambda}) \geq \mathbf{0}, (\mathbf{x},\boldsymbol{\lambda}) \geq \mathbf{0}, \text{ and } \langle \tilde{\mathbf{F}}(\mathbf{x},\boldsymbol{\lambda}), (\mathbf{x},\boldsymbol{\lambda}) \rangle = 0$$

which reduces to

$$\begin{aligned} \lambda_i \mathbf{1}_{t_i} - \nabla_{\mathbf{x}_i}\Pi_i \geq \mathbf{0}, \mathbf{x}_i \geq \mathbf{0}, \langle (\lambda_i \mathbf{1}_{t_i} - \nabla_{\mathbf{x}_i}\Pi_i), \mathbf{x}_i \rangle = 0, \forall i \\ \mathbf{k} - \mathbf{B}^T\mathbf{x} \geq \mathbf{0}, \boldsymbol{\lambda} \geq \mathbf{0}, \langle \mathbf{k} - \mathbf{B}^T\mathbf{x}, \boldsymbol{\lambda} \rangle = 0. \end{aligned}$$

This is just the KKT system for players' payoff maximization conditions subject to constraints. The result just follows from Proposition 1. □

Proof of Theorem 5: Given Proposition 6, the proof is similar to the proof of Theorem 4 with minor modifications. We consider the following system:

$$\begin{bmatrix} \tilde{F}_\alpha(\mathbf{x}, \boldsymbol{\lambda}; \boldsymbol{\theta}) \\ \mathbf{x}_{\breve{\alpha}} \\ \tilde{F}_\beta(\mathbf{x}, \boldsymbol{\lambda}; \boldsymbol{\theta}) \\ \boldsymbol{\lambda}_{\breve{\beta}} \end{bmatrix} = \mathbf{0}.$$

(1) To apply the Implicit Function Theorem, we need $\begin{bmatrix} \mathbf{M}_{\alpha\alpha}(\mathbf{x}) & \mathbf{B}_{\alpha\beta} \\ \mathbf{B}_{\alpha\beta}^T & \mathbf{0} \end{bmatrix}$ to be non-singular at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}^*)$. This is indeed true as

$$Det \begin{bmatrix} \mathbf{M}_{\alpha\alpha}(\mathbf{x}) & \mathbf{B}_{\alpha\beta} \\ \mathbf{B}_{\alpha\beta}^T & \mathbf{0} \end{bmatrix} = -Det[\mathbf{M}_{\alpha\alpha}(\mathbf{x})] Det[\mathbf{B}_{\alpha\beta}^T \mathbf{M}_{\alpha\alpha}(\mathbf{x})^{-1} \mathbf{B}_{\alpha\beta}].$$

At $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta}^*)$, which is of type S^2 by Assumption 4, $\mathbf{M}$ is positive definite by Proposition 3. So $\mathbf{M}_{\alpha\alpha}(\mathbf{x})$ is positive definite, and hence non-singular (Lemma A1). Moreover since $\mathbf{B}_{\alpha\beta}^T$ is always of full column rank, $\mathbf{B}_{\alpha\beta}^T \mathbf{M}_{\alpha\alpha}(\mathbf{x})^{-1} \mathbf{B}_{\alpha\beta}$ is also positive definite, and hence non-singular (Lemma A1).

(2) By the Implicit Function Theorem, the system above has a solution $(\mathbf{x}(\boldsymbol{\theta}), \boldsymbol{\lambda}(\boldsymbol{\theta}))$ near $\boldsymbol{\theta}^*$, which is also a solution to the VI($\tilde{\mathbf{F}}, \mathbf{R}_+^{\bar{n}} \times \mathbf{R}_+^{n}$), and hence an equilibrium of $CF(\boldsymbol{\theta})$, given the non-degeneracy of $\mathbf{x}^*(\boldsymbol{\theta}^*)$.

(3) By construction, $\mathbf{x}^*(\boldsymbol{\theta})$ has the same sign as $\mathbf{x}^*(\boldsymbol{\theta}^*)$, so it must be of type S^2 as well. By Theorem 2 , $\mathbf{x}^*(\boldsymbol{\theta})$ must be the unique equilibrium of $CF(\boldsymbol{\theta})$.

(4) The derivatives of $\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\lambda}(\boldsymbol{\theta})$ follow similarly by implicit differentiation. □

C Additional examples and results

C.1 Additional examples

Example C1 (Multiple equilibria)**.** *Consider the network depicted in Figure A1 with seven agents in which player 1 is in the center. Let* $k_2 = \cdots = k_7 = 1$ *and* $k_1 = \bar{K} > 24$*. Multiple equilibria occur for the same reason as in Example 5. Indeed, player 1 has a total budget of* $\bar{K}$*, but only needs to allocate at least* 4 *to each of the six battles. The dimension of* $\mathcal{NE}$ *is* 6 *whereas, in Example 5, it was equal to* 2*.*

Consider now the network depicted in Figure A2 with four agents. Let $k_2 = k_3 = 1$ *and* $k_1 = k_4 = \bar{K} > 8$*. In equilibrium, as in Example 5, both players* 2 *and* 3 *allocate their entire budget to battle c while players* 1*'s and* 4*'s best responses are not unique. So,* $\mathcal{NE}$*, which is of dimension* 4*, is isomorphic to a product of two simplexes, each with dimension* 2*.*

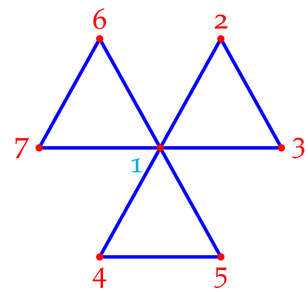

Figure A1: Seven agents

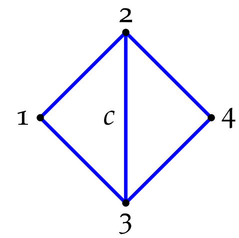

Figure A2: Four agents

C.2 Additional results

We here analyze the equilibrium under a general feasibility set $\mathcal{X}_i \subset \mathbf{R}_+^{t_i}$. The following result is parallel to Theorem 2.

Proposition C1. *Assume that $\mathcal{X}_i$ is nonempty and convex for each i. If $\mathbf{x}^*$ and $\mathbf{x}^{**}$ are two equilibria, and both are of type S^2, then $\mathbf{x}^* = \mathbf{x}^{**}$.*

An immediate application is for the truncated conflict game CF^ϵ with $\epsilon > 0$. Let $\mathcal{X}_i = \Delta_i \cap \{\mathbf{x}_i | x_i^a \geq \epsilon, \forall a \in \mathcal{T}_i\}$ and $\mathcal{S}(\epsilon) = \prod \mathcal{X}_i$.

Proposition C2. *For the conflict game CF^ϵ with $\epsilon > 0$,*

(i) $\mathbf{x}^\epsilon$ is an equilibrium of CF^ϵ if and only if $\mathbf{x}^\epsilon$ solves $VI(\mathbf{F}, \mathcal{S}(\epsilon))$.

(ii) $\mathbf{F}$ is strictly monotone on $\mathcal{S}(\epsilon)$, and $\mathcal{NE}^\epsilon$ is a singleton.

Thus, uniqueness is automatically obtained for CF^ϵ. Another closely related game is the conflict game CF_δ with $\delta > 0$ (see Remark 1). Let $\Pi_i^\delta(\mathbf{x}), i \in \mathcal{N}$ denote the modified payoff and $\mathbf{F}_\delta$ denote the operator constructed in (13) using Π_i^δ, instead of Π_i.

Proposition C3. *For the conflict game CF_δ with $\delta > 0$*

(i) $\mathbf{x}_\delta$ is an equilibrium of CF_δ if and only if $\mathbf{x}_\delta$ solves $VI(\mathbf{F}_\delta, \mathcal{S})$.

(ii) $\mathbf{F}_\delta$ is strictly monotone on $\mathcal{S}$, and $\mathcal{NE}_\delta$ is a singleton.

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