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MARKETS FOR FINANCIAL INNOVATION

Abstract

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JEL Classification: D47, D86, G23

Keywords: security design, market structure, price impact

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We propose a model where both security design and market structure are endogenously determined to explain why standardized securities are frequently traded in decentralized markets. We find that issuers offer debt contracts in thinner markets where investors have a higher price impact, and equity in deeper markets. In turn, investors accept to trade in thinner markets to elicit less variable securities from issuers if gains from trade are small. Otherwise, investors choose to trade in deeper markets where their price impact is minimized. We also show that there exist equilibrium market structures in which both debt and equity are traded.

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1 Introduction

Financial securities are traded in a wide variety of market structures. Over-the-counter markets facilitate the creation and trade of customized contracts, and exchanges are venues for many standardized securities. Frequently, however, standardized contracts, such as covenant-lite debt securities, are also traded in decentralized, fragmented market structures. While it is widely acknowledged that the structure of the market affects the price and trading efficiency of a security, little has been studied about the properties of a security and the market structure in which it is traded. Yet, regulators favor, for instance, shifting the trade of standardized securities from fragmented markets to more centralized venues. Ensuring that such proposals improve trade, or that they are at least innocuous, requires first answering a critical question: Why do standardized securities trade in decentralized markets to begin with?

To address this question, we propose a model in which financial intermediaries issue securities taking into account the markets in which the securities will be traded. Markets can be thinner and more fragmented, or deeper and more concentrated. Securities are offered to investors who strategically choose which market to participate in, understanding that their choice affects the design of the security they will be trading. Thus, the novelty in our model is that both the securities issued and the structure of the market are endogenously determined.

A central finding in our paper is that financial intermediaries have an incentive to issue debt when markets are thinner and equity when markets are deeper. As in Hébert (2018), a debt contract has the least variance among all limited liability securities with the same expected value. The key insight of our model is that financial intermediaries issue debt only if investors dislike variable payoffs more than they dislike impacting the price of a security when they trade it strategically. This is the case when investors anticipate they will need to trade little. In contrast, financial intermediaries issue equity when investors prioritize minimizing their price impact over trading a risky security. This is the case when investors anticipate they may need to take large positions. Even though investors are willing to trade risky securities, we show that their welfare is not necessarily higher in deeper and larger markets. Financial intermediaries, on the other hand, benefit when markets are deep. We also show that there exist equilibrium market structures in which both debt and equity are traded.

Our model has three dates and finite numbers of issuers and strategic investors. Financial intermediaries can issue asset-backed securities, therefore we refer to them as issuers. A security is a portion of an asset (or future cash flow) that is offered to investors. Issuers are risk neutral, while investors have mean-variance preferences. Investors are ex-ante homogeneous but have different ex-post valuations of the security they are offered. This allows investors to benefit from trading with each other.

We consider that issuers choose which securities to design taking as given the market structure in which securities will be trading, but not the idiosyncratic valuations of investors. Indeed, in practice bookrunners that manage security offerings initially prospect the market and probe the parties interested to determine demand. At the same time, the idiosyncratic valuations of investors are realized after issuers design securities, and an issuer cannot customize his security to address the specific requirements of any particular investor. The focus of our paper is thus on standardized security issuances.

Initially, investors choose a market in which to trade. Each investor understands that her choice affects the design of the security that will be offered in this market. Once markets open, investors can trade given their idiosyncratic valuation shocks. While there is an increasingly diverse set of trading protocols in practice, a key commonality is that market participants are strategic and, in particular, that they exercise market power relative to other participants. To capture this feature, we model investors' trading strategies as quantity-price schedules and consider that each investor understands the impact of her trade on the price of the security. Thus, investors act strategically both when markets form and when they trade.

In this environment, we consider two frictions. First, investors cannot directly invest in the same assets as issuers. Indeed, financial intermediaries frequently issue asset-backed securities which allow investors to gain exposure to markets that they could not otherwise invest in. Mortgage-backed securities are such an example. Second, issuers design securities bounded by limited liability. That is, a security's payoff cannot exceed the payoff of the asset that backs it in any given state of the world. In practice, most securities are implicitly designed to respect this constraint. In our set-up, limited liability is equivalent to the spanning constraint in the financial innovation literature (Duffie and Rahi (1995)) which requires that the securities a financial intermediary issues span the payoff of the asset that backs them.

We obtain two sets of results. The first set of results characterizes the security that an issuer finds optimal to offer taking as given the market structure. We show that this security depends monotonically on the depth of the issuer's market. In particular, we show (i) that the optimal security belongs to the family of debt contracts, paying the lesser of a flat payoff and the full value of the underlying asset in every state of the world, and (ii) that the state in which the security starts paying the flat payoff is higher in markets with more investors. This implies that, as the market deepens, the security approaches the payoff of the underlying asset in all states, and thus becomes equity.

The intuition for this first set of results is as follows. When choosing how to design a security, the issuer's main incentive is to obtain a high price for it. As usual, the equilibrium price at which the security is traded is increasing in its mean payoff and decreasing in the variance of its payoffs across states. The issuer thus faces a trade-off between the mean and the variance of the security he designs, making a debt contract the optimal one. Importantly, though, the equilibrium price decreases less with the variance of the security in deeper markets where investors have a lower price impact. Thus, the strength of the mean-variance trade-off faced by the issuer (and hence where on the spectrum of debt contracts the security sits) depends on the depth of the market. The deeper the market, the less pronounced the trade-off and the more equity-like the issuer makes his security.

The second set of results focuses on the equilibrium market structure. This is crucial to ensure that the securities issuers design in a given market structure can indeed be supported in equilibrium. If no investor benefits from trading in a particular market structure, then we should not expect the corresponding securities to arise in equilibrium. When choosing which market to trade in, an investor weighs the gains from trade with other investors against the ability to influence the security that the issuer designs. An investor who trades in a thinner market will have a larger price impact. On one hand, this amplifies the mean-variance trade-off in the issuer's security design problem and delivers a less risky security. On the other hand, it also amplifies the extent to which the investor will move the price of the security against herself when trading with other investors. When investors expect to be relatively homogeneous in their valuations of the same security, they anticipate limited benefits from trading with each other and are therefore willing to accept a larger price impact in order to elicit a less variable security from the issuer. In contrast, when investors expect to be relatively heterogeneous, they understand that they may want to engage in large trades with each other so they seek to limit their price impact by trading in a large market, albeit with a riskier security. Thus, an important outcome of our model is that debt is traded in thinner, more fragmented markets while equity is traded in deeper, more concentrated markets.

To gain further insights into our question, we analyze a simpler version of the model in which asset returns are uniformly distributed. The welfare implications are different for issuers and investors. If heterogeneity among investors is not too high, then the symmetric equilibrium that achieves the highest welfare for investors exists in the set of equilibria where debt is traded in thinner, more fragmented markets. In contrast, issuers are always better off designing a security for a large market than for a small market. Investors thus benefit at the expense of issuers in any equilibrium where debt is traded. In aggregate, however, the benefits to investors in an equilibrium where debt is traded are outweighed by the losses to issuers, such that total welfare is higher when markets are deeper, even though the security that emerges in these markets has more variable payoffs.

Related Literature

This paper relates to several strands of literature. The most relevant studies are those on security design and endogenous market structure.

The literature on security design has been very prolific over recent decades. The classic problem explored in these papers is that of a firm needing to raise funds from an investor to finance an investment project. In exchange, the firm proposes a security to the investor. A common result in this literature is that debt is the optimal security in the presence of asymmetric information or moral hazard. An incomplete list of papers includes DeMarzo and Duffie (1999), Biais and Mariotti (2005), Yang (2017), Asriyan and Vanasco (2018). Malenko and Tsoy (2018) have shown recently that a mixture of debt and equity can be optimal when the investor faces Knightian uncertainty about the underlying project's returns. We explore a variation of the typical set-up. In particular, financial intermediaries issue securities which allow investors to have exposure to assets in which they cannot invest directly. We find that debt securities are optimal, even in the absence of information asymmetries. In our model, this is because debt optimizes a mean-variance trade-off that investors, and consequently financial intermediaries, face. A similar channel explains why debt is the optimal security in Hébert (2018). Although Hébert (2018) analyzes a moral hazard problem, he shows that the mean-variance trade-off is key to the issuer's security design problem. Our main contribution, however, is to show that a financial intermediary chooses to issue debt only when investors trade in a thin market. As the market gets deeper, the optimal security becomes equity.

Parallel to the literature on security design, there is a body of work on financial innovation that studies the role of security issuances in completing markets. From the seminal paper of Allen and Gale (1991) to the more recent contribution of Carvajal, Rostek, and Weretka (2012), the main focus of this line of research is to analyze whether competition among asset-holders affects their incentives to introduce new securities. Complimentary to this literature, we study a model in which a financial intermediary's decision to issue securities is affected by the strategic competition between investors when trading the securities they are offered.

There is a young but growing literature on endogenous market structure. Babus and Parlatore (2018), Cespa and Vives (2018), Dugast, Uslu, and Weill (2018), Lee and Wang (2018), and Yoon (2018) provide models that seek to explain why trade takes places in a variety of venues, centralized or decentralized. However, in these papers, the asset traded is taken to be exogenous. In contrast, both the security design and the market participation decision are endogenous in our model.

A selected set of papers study the effect of market structure on security design. Axelson (2007) shows that debt is optimal if the degree of competition among investors is low. The driving force in his model is that investors are better informed about the prospects of the issuer than the issuer himself. Rostek and Yoon (2018) analyze the role of market structure for introducing non-redundant derivatives. In both of these papers, however, the market structure is taken to be exogenous. In our paper, the market structure is endogenously determined.

There is a lack of direct empirical work on the joint determination of security design and market structure. However, Biais and Green (2018) provide a thorough documentation of developments in the bond market in the 20th century. They find that when institutions became more important in bond markets, bond markets became thinner. While our model is inherently static (as it develops over three periods), it builds on ingredients that are relevant for investigating these issues. For instance, changes in the dispersion in investors' valuations or in the variance of the underlying asset over time could potentially account for market structure dynamics.

The rest of the paper proceeds as follows: Section 2 introduces the model environment; Section 3 defines and characterizes the equilibrium; Section 4 discusses the welfare implications of our model; Section 5 presents some extensions of our model; and Section 6 concludes. All proofs are collected in the Appendix.

2 The Model Set-Up

We consider an economy with three dates, $t = 0, 1, 2$, and two types of agents, issuers and investors. There are $M \geq 2$ risk neutral, impatient issuers indexed by $m = 1, ..., M$. Each issuer has access to an asset, Z, which yields a payoff $z(s) \geq 0$ if the aggregate state $s \in [0, S]$ is realized at date $t = 2$. The cumulative distribution function for states is $F(s)$, with $F(\cdot)$ continuous and differentiable, and the probability density function is $f(s)$. Without loss of generality, we assume $z'(\cdot) > 0$. A market m is associated with each issuer m. In each market m, the issuer can design a security W_m that pays $w_m(s)$ in state s at date $t = 2$. As in the literature on the spanning role of securities (Duffie and Rahi (1995)), we assume that the security payoff is subject to the feasibility constraint

$$w_m(s) \leq z(s), \forall s \in [0, S]. \tag{1}$$

There are $N \geq 3$ mean-variance, patient investors, indexed by $i = 1, ..., N$. Investor i is subject to a preference shock θ^i that shifts her marginal utility of consumption, as we describe in detail below. The heterogeneity that θ^i introduces across investors can be interpreted as differences in liquidity needs, in the use of the asset as collateral, in technologies to repackage and resell cash flows, or in risk-management constraints, for example. The shock θ^i is independently distributed across investors according to a distribution $G(\cdot)$ with mean μ_θ and standard deviation σ_θ. The realization of the shock θ^i is also independent of the realization of the state s. Investors do not have access to the asset Z, but they can acquire the securities that issuers design.

The timing of events is as follows. At date $t = 0$, investors choose a market m in which to trade. An investor can choose at most one market. However, multiple investors can choose the same market. Next, the issuer in market m designs the security W_m. We assume that the issuer supplies one unit per capita of the security W_m in his market.[1] At date $t = 1$, each

[1] The idea is that each unit of W_m is backed by one unit of the asset Z. For this, we assume that the issuer has access to a sufficiently large pool of the asset Z. The assumption that Z and W_m are scalable allows us to

investor i learns her preference shock θ^i. After this, all markets open and investors in each market m trade the security W_m. At date $t = 2$, the state s is realized. Investors receive payoffs according their final holdings of the security. Each issuer m pays $w_m(s)$ and receives $z(s)$ per capita. Consumption takes place.

There are two important aspects to the timing we use. First, the issuer designs the security before the preference shocks, θ^i, are realized. Thus, a security cannot be customized to address the specific requirements of any particular investor. This is our approach to model the innovation of standardized securities. Second, the issuer designs the security after the market structure is determined. While this timing does not allow investors to search and target particular securities, it is in line with security offering practices in which bookrunners initially prospect the market and track the parties interested to help determine demand.[2]

The investors' choices at date $t = 0$ determine a market structure $\mathcal{M}$. When an investor i chooses a market m, we say that $i \in m$. We denote by n_m the number of investors that choose market m. We consider a market m to be active if and only if $n_m > 2$. In this case, we say that $m \in \mathcal{M}$. A market structure $\mathcal{M}$ is characterized by the number of active markets, M', and by the number of investors in each market, $\{n_m\}_{m=1}^{M'}$. We define a market structure to be *symmetric* if each active market m has the same number of investors $n_m = n$.

We model investors' trading strategies as quantity-price schedules, as in Kyle (1989) and Vives (2011). In particular, the strategy of an investor is a map from her information set to the space of demand functions, as follows. The demand function of an investor $i \in m$ with preference shock θ^i is a continuous function $Q^i_m : \mathbb{R} \to \mathbb{R}$ which maps the price p_m of the security W_m in market m into a quantity q^i_m she wishes to trade

$$Q^i_m\left(p_m; \theta^i\right) = q^i_m.$$

An investor i who trades q^i_m units of security W_m in market m at date $t = 1$ consumes C^i_m at date $t = 2$, where

$$c^i_m(s) = q^i_m w_m(s), \tag{2}$$

abstract from any mechanical effects that would arise from having a fixed supply of Z in each market. Restricting the supply of Z reinforces mechanically our results. In Section 5.1, we allow the issuer to optimally choose how many units of Z back each unit of W_m, subject to a cost of procuring Z.

[2] In Section 5.2, we discuss an alternative timing in which we allow investors to choose markets after issuers design securities.

for each state s. Given a market structure $\mathcal{M}$ and a security W_m that issuer m designs at date $t = 0$, the expected payoff of an investor i in market m at date $t = 1$ as she engages in trade is

$$V_m^i = \theta^i E_1 \left(C_m^i\right) - \frac{\gamma}{2}\mathcal{V}_1 \left(C_m^i\right) - p_m q_m^i, \tag{3}$$

where $\mathcal{V}(\cdot)$ is the variance operator. We use $E_1(\cdot)$ and $\mathcal{V}_1(\cdot)$ to denote that expectations are being taken over the state s, which is the only unknown at date $t = 1$. The price p_m in Eq. (3) is the price at which local market m clears, given that issuer m supplies one unit of the security per capita. That is, p_m is such that

$$\sum_{i \in m} Q_m^i \left(p_m; \theta^i\right) = n_m.$$

Substituting Eq. (2) into Eq. (3), we obtain that investor i's objective function at date $t = 1$, before the uncertainty about the state of the world s has been resolved, is

$$V_m^i = \left[\theta^i E_1 (W_m) - p_m\right] q_m^i - \frac{\gamma}{2}\mathcal{V}_1 (W_m) \left(q_m^i\right)^2, \tag{4}$$

where $E_1(W_m) \equiv \int_0^S w_m(s)\,dF(s)$ and $\mathcal{V}_1(W_m) \equiv \int_0^S \left[w_m(s) - E_1(W_m)\right]^2 dF(s)$. In this reformulation, the preference shock θ^i captures investor i's valuation of the payoff she expects to obtain from one unit of the security W_m.

An issuer m supplies n_m units of the security W_m that he designs in market m, and he receives the price p_m per unit of the security. Aside from designing the security and supplying it to the market, the issuer is not directly involved in the trade between investors at date $t = 1$. Given a market structure $\mathcal{M}$ and a security W_m that the issuer designs in a market m with n_m investors at date $t = 0$, issuer m's expected payoff at date $t = 1$ is

$$V_m = \left[p_m + \beta E_1 (Z - W_m)\right] \times n_m,$$

where $\beta \in [0, 1]$ is a discount factor that captures the impatience of issuers relative to investors.

3 Equilibrium

In this section, we define and characterize the equilibrium. We start by solving for the trading equilibrium in each market m at date $t = 1$, given a market structure $\mathcal{M}$ and the securities W_m that issuers design at date $t = 0$. We then characterize the security that each issuer designs in equilibrium for his market m at date $t = 0$, given a market structure $\mathcal{M}$. Lastly, we analyze the market formation game which determines the equilibrium market structure $\mathcal{M}$ at $t = 0$.

Definition 1 *A subgame perfect equilibrium is a market structure $\mathcal{M}$, a set of securities $\{W_m\}_{m\in\mathcal{M}}$, and a set of demand functions $\left\{Q_m^i\right\}_{i\in m}$ for investors in each active market m such that:*

1. *Q_m^i solves each investor i's problem at date $t = 1$*

$$\max_{Q_m^i}\left\{\left[\theta^i E_1\left(W_m\right)-p_m\right]Q_m^i\left(p_m;\theta^i\right)-\frac{\gamma}{2}\left(Q_m^i\left(p_m;\theta^i\right)\right)^2\mathcal{V}_1\left(W_m\right)\right\}; \tag{5}$$

2. *W_m solves each issuer m's problem at date $t = 0$*

$$\max_{W_m}\left\{E_0\left(p_m\right)+\beta\left[E_1\left(Z\right)-E_1\left(W_m\right)\right]\right\}\times n_m, \tag{6}$$

   subject to the feasibility constraint (1);

3. *No investor i benefits from deviating and joining a different local market at date $t = 0$, i.e. the expected payoff an investor receives in market m is at least as large as the expected payoff from deviating to market m'*

$$E_0\left(V_m^i\right)\geq E_0\left(V_{m'}^i\right)\ \textit{for all } i\in m \textit{ and all } m'\neq m. \tag{7}$$

Our notion of equilibrium market structure, described in the third bullet of Definition 1, is related to the concept of pairwise stability introduced in Jackson and Wolinsky (1996), with the difference that we allow for deviations to be unilateral.

It is important to note that all agents act strategically. This implies that each investor $i \in m$ takes into account her price impact in market m when submitting her demand. Similarly, an issuer understands how the security he designs at date $t = 0$ affects the price at which investors trade it at date $t = 1$. At the market formation stage, each investor also takes into account how her market choice shapes the security that the issuers design, as well as the price at which trade takes place at date $t = 1$. For tractability, we restrict our attention to equilibria in which the market structure is symmetric, issuers design the same security, and agents have linear trading strategies.

The rest of this section characterizes the equilibrium. As mentioned earlier, we solve first for the trading equilibrium conditional on a market structure and a set of securities (Section 3.1), then for the equilibrium security conditional on a market structure (Section 3.2), and finally for the equilibrium market structure (Section 3.3).

3.1 The Trading Equilibrium

At date $t = 1$, after each investor i learns her preference shock θ^i, all active markets open and trade takes place. In each market m, an investor chooses her trading strategy in order to maximize her expected payoff, understanding that she has impact on the price p_m. As is standard in similar models, we simplify the optimization problem (5), which is defined over a function space, to finding the functions $Q_m^i\left(p_m; \theta^i\right)$ pointwise. For this, we fix a realization of the set of preference shocks, $\left\{\theta^i\right\}_{i=1}^N$. Then, we solve for the optimal quantity q_m^i that each investor $i \in m$ demands in market m when she takes as given the demand functions of the other investors in market m. Thus, we obtain investor i's best response quantity q_m^i in market m for each realization of the preference shocks of the other investors in market m. This gives us a map from prices to quantities, or the investor's optimal demand function point by point. We describe the procedure in detail below.

The first order condition for an investor i in market m is

$$\theta^i E_1\left(W_m\right) - p_m - \left(\frac{\partial p_{m,-i}}{\partial q_m^i} + \gamma \mathcal{V}_1\left(W_m\right)\right) q_m^i = 0, \tag{8}$$

where $p_{m,-i}$ is the residual inverse demand of investor i implied by

$$q_m^i + \sum_{j \in m, j \neq i} Q_m^j\left(p_m; \theta^j\right) = n_m. \tag{9}$$

An investor $i \in m$ chooses to trade a quantity q_m^i of the security W_m so that her marginal benefit equalizes her marginal cost of trading. The first term in the first order condition (8) is the marginal benefit of increasing the final holdings of the security W_m for investor i, which is given by the expected value of the security scaled by the investor's preference shock θ^i. The remaining terms in Eq. (8) represent investor i's marginal cost of increasing her demand. The second term represents the price that the investor pays to acquire one unit of the security W_m. Investors also incur indirect costs, captured in the last term in Eq. (8). First, since the investors trade strategically, increasing the quantity demanded has an impact on the market clearing price. Second, investors are risk averse, which maps into a holding cost of the security that increases proportionally to the variance of W_m as the quantity demanded increases. The following proposition characterizes the trading equilibrium in a market m.

Proposition 1 *Given a market structure $\mathcal{M}$ and a set of securities $\{W_m\}_{m\in\mathcal{M}}$, there exists a unique symmetric linear equilibrium that characterizes investors' trading strategies in each market m, as follows. The equilibrium demand function of an investor i in market m is*

$$Q_m^i\left(p_m;\theta^i\right) = \frac{1}{\left(1+\lambda_m\right)\gamma\mathcal{V}_1\left(W_m\right)}\left[\theta^i E_1\left(W_m\right) - p_m\right], \tag{10}$$

where $\lambda_m^{-1} \equiv (n_m - 2)$ is an index of market depth. The equilibrium price in market m is

$$p_m = \left(\frac{1}{n_m}\sum_{i\in m}\theta^i\right)E_1\left(W_m\right) - \left(1+\lambda_m\right)\gamma\mathcal{V}_1\left(W_m\right). \tag{11}$$

Proposition 1 shows that investor i buys or sells the security W_m depending on whether her valuation $\theta^i E_1\left(W_m\right)$ of the security's expected payoff is above or below the price p_m at which she can trade. Eq. (10) implies that the investor will restrict the size of her trade for two reasons. First, she is risk averse and the security is risky. Thus, the more risk averse the investor is, the less she will trade. Similarly, the more risky the security is (as reflected in a higher variance of payoffs across states), the less of it the investor trades, everything else constant. Second, the investor has a price impact, $\partial p_{m,-i}/\partial q_m^i = \lambda_m\gamma\mathcal{V}_1\left(W_m\right)$, that decreases with market depth. In other words, the larger the market is, the more the investor can trade without moving the price against herself.

The equilibrium price in market m, characterized by Eq. (11), is the expected payoff of the security W_m, scaled by the average valuation of the investors in market m, minus a risk premium. The risk premium exists because investors are risk averse and, in expectation, have to hold one unit of a risky security. Indeed, it is easy to check that the expected traded quantity is $E_0\left(q_m^i\right) = 1$ for any $i \in m$.

Given a realization of investors' preference shocks, $\left\{\theta^i\right\}_{i=1}^N$, it follows from Eq. (11) that the price of the security W_m is lower in a thinner market. The price of the security also decreases with the variance of the security, everything else constant. However, the price decreases less with the variance of the security as the market becomes deeper.[3]. These effects arise because investors are strategic and dislike risk. In a smaller market, changes in the demand of an individual investor have a larger impact on the price of the security. Furthermore, the riskier the security is, the less of it a risk averse investor will demand. If an investor demands less of the security, more will be available to other investors. The price will then have to fall so that, on average, other investors are content with holding more of the security. As the size of the market increases, the price impact of any one investor falls. An increase in riskiness is thus met with a smaller decrease in price compared to a smaller market where a strategic decrease in demand by one investor leads to a bigger price drop.

The effects of market depth and the variance of the security on the price are typical of models in which investors strategically trade risky assets in positive net supply by submitting demand functions. In contrast to standard models, however, in our model both the variance of the security and the market depth are endogenous. In particular, the security is the choice of the issuers, while the market structure, and implicitly the market depth, is the outcome of investors' choices.

3.2 The Equilibrium Security

At the end of date $t = 0$, after the market structure is determined, each active issuer m designs a security W_m that investors can trade in market m at date $t = 1$. The issuer chooses the payoff $w_m(s)$ of the security for each state s to maximize his expected profit in (6), subject to the feasibility constraint (1). The constraint (1) restricts the issuer to offer investors a security with a payoff that does not exceed what the issuer realizes on the asset Z in any state

[3] To verify this, consider the cross-partial derivative of the price p_m with respect to the variance of the security W_m and the number of investors in market m, holding everything else constant. This derivative is given by $\frac{\partial}{\partial n_m} \left.\frac{\partial p_m}{\partial \mathcal{V}_1(W_m)}\right|_{E_1(W_m)=cst} = -\frac{\partial \lambda_m}{\partial n_m} > 0$.

s. Alternatively, since the issuer is the residual claimant on the payoff of the asset Z, he is effectively designing two securities: one that he offers to investors and one that he keeps for himself. Thus, the constraint (1) simply requires that the two securities exhaust the returns to issuer m's asset, as is commonly assumed in the financial innovation spanning literature.

Taking the expectation at date $t = 0$ of the price p_m at which investors in market m trade the security W_m (i.e., the price in Eq. (11)) and substituting it into (6), we obtain that issuer m designs the security W_m to maximize the following objective function:

$$E_0 (V_m) = [\beta E_1 (Z) + (\mu_\theta - \beta) E_1 (W_m) - (1 + \lambda_m) \gamma \mathcal{V}_1 (W_m)] \times n_m. \tag{12}$$

It is transparent that the issuer benefits from offering a security that pays well in expectation, as the expected price at which investors trade is increasing in $E_1 (W_m)$.[4] At the same time, the issuer increases his expected profit if he offers a security with low variance, as the expected price at which investors trade is decreasing in $\mathcal{V}_1 (W_m)$. In fact, if he were unconstrained, the issuer would offer a security with infinite mean and zero variance. However, because the payoff of the security W_m cannot exceed the payoff of the asset Z, the issuer faces a trade-off between the mean and the variance of the security he designs. Since the weight on the variance in the issuer's expected profit in Eq. (12) depends on the depth λ_m^{-1} of the market in which the security is traded, how exactly this trade-off is resolved will depend on the market structure.

Proposition 2 *Suppose $\mu_\theta > \beta$ so that issuers find it profitable to design securities for investors. In any market m with n_m investors, issuer m designs a security W_m with payoffs*

$$w_m (s) = \begin{cases} z (s) & \text{if} \quad s < \overline{s}_m \\ E_1 (W_m) + \frac{\mu_\theta - \beta}{2\gamma} \frac{n_m - 2}{n_m - 1} & \text{if} \quad s \geq \overline{s}_m \end{cases} \tag{13}$$

where the threshold state $\overline{s}_m \in [0, S]$ is defined by

$$\overline{s}_m = \begin{cases} z^{-1} \left(E_1 (W_m) + \frac{\mu_\theta - \beta}{2\gamma} \frac{n_m - 2}{n_m - 1} \right), & \forall n_m < n_S \\ S, & \forall n_m \geq n_S \end{cases} \tag{14}$$

[4] By the law of iterated expectations, $E_1 (W_m) = E_0 (W_m)$.

with n_S finite if and only if the equation

$$\frac{n_S - 2}{n_S - 1} = \frac{2\gamma}{\mu_\theta - \beta} \left[z\left(S\right) - E_1\left(Z\right) \right] \tag{15}$$

has a solution $n_S \geq 3$.

Proposition 2 shows that issuer m finds it optimal to design a security that will pay the lesser of a flat payoff and the full value of the asset Z in every state of the world. The security payoff depends on the market structure, the distribution of the underlying asset Z, and the preferences of investors and issuers. We say that the security is debt if it pays the flat payoff in at least some states (i.e., the security is debt if $\bar{s}_m < S$). We say that the security is equity if it replicates the payoffs of the asset Z in all states. In our model, equity can be seen as a special case of debt, where the threshold state above which the security pays a flat payoff is $\bar{s}_m = S$.

We have the following cases from Proposition 2. If $\frac{2\gamma}{\mu_\theta - \beta} \left[z\left(S\right) - E_1\left(Z\right) \right] \leq \frac{1}{2}$, then the issuer finds it optimal to offer equity in any market structure. If $\frac{2\gamma}{\mu_\theta - \beta} \left[z\left(S\right) - E_1\left(Z\right) \right] \geq 1$, then the issuer finds it optimal to offer debt in any market structure, including in markets with infinitely many investors. These two cases represent corner solutions of the issuer's optimization problem. If instead $\frac{2\gamma}{\mu_\theta - \beta} \left[z\left(S\right) - E_1\left(Z\right) \right] \in \left(\frac{1}{2}, 1\right)$, then issuer m offers debt if the number of investors n_m in market m is below a threshold n_S, otherwise he offers equity.

We provide the intuition for why a debt contract is the security that the issuer chooses from the set of all possible security profiles. A debt contract has the following property: there are no two states, s' and s'', such that $w_m\left(s'\right) < z\left(s'\right)$ and $w_m\left(s'\right) < w_m\left(s''\right)$. In other words, if the constraint (1) does not bind in either state s' or state s'', then the security yields the same payoff in both states, and, if the constraint (1) binds only in one of the two states, the payoff in that state must be smaller than in the flat part of the debt contract. Suppose issuer m chooses a security that does not have this property. Then a deviation which increases the payoff of the security in state s' by $\varepsilon_{s'} > 0$ and decreases the payoff of the security in state s'' by $\varepsilon_{s''} = \frac{f(s')}{f(s'')}\varepsilon_{s'}$ decreases the variance of the security without changing its mean. Since the issuer's expected profit in Eq. (12) is decreasing in the variance of the security, it follows that such a deviation is profitable. Therefore, it cannot be optimal for the issuer to choose any security other than a debt contract. This argument is similar to the one Hébert (2018) uses to show that debt is the optimal contract in the presence of moral hazard. Novel to our framework, however, is how the equilibrium security depends on the market structure in which it is traded. The following proposition characterizes the relationship between the market structure and the debt contract that the issuer chooses.

Proposition 3 *Suppose that Eq. (15) has a finite solution $n_S \geq 3$. The threshold state $\bar{s}_m$ defined by (14) is increasing in the number of investors n_m in market m as long as $n_m \leq n_S$.*

Proposition 3 shows that when the issuer designs a debt security, he will adjust its payoff depending on the market in which the security is traded. In particular, the lowest state in which a security W_m pays the flat payoff increases with the number of investors in market m. Thus, the larger the market, the more state-contingent the security that the issuer designs. This property of the equilibrium security extends automatically to the case when Eq. (15) does not have a finite solution and the issuer offers debt in markets of any size.

To understand Proposition 3, we appeal to the intuition developed at the end of Section 3.1 about the forces that affect the price of a security W_m. To start, consider a state s where the security that issuer m designs pays $w_m(s) < z(s)$. If the issuer increases $w_m(s)$ slightly, holding constant the payoffs in all other states, then he increases both the mean and the variance of the security W_m. The increase in the mean of the security works in favor of the issuer because it increases the price he expects to receive, whereas the increase in the variance of the security decreases the issuer's expected profit. However, as we explained in Section 3.1, a higher variance has a greater impact on the expected price in a small market than in a large market. In contrast, as we can see from Eq. (12), the impact of a higher mean on the expected price does not depend on the size of the market. Therefore, the marginal benefit to the issuer of an increase in $w_m(s)$ is independent of n_m, while the marginal cost is decreasing in n_m. Since a profit-maximizing issuer sets $w_m(s)$ to equate marginal benefit and marginal cost, it follows that he will increase $w_m(s)$ by more in a large market than in a small market. Given that the issuer finds it optimal to issue a debt contract, he can accomplish this by increasing the threshold state above which the security pays a flat payoff. The next corollary formalizes this discussion and follows immediately from Proposition 3.

Corollary 1 *Suppose that Eq. (15) has a finite solution $n_S \geq 3$. The security W_m that the issuer designs in market m has the following properties:*

1. $\frac{\partial E_1(W_m)}{\partial n_m} > 0$ *for any* $n_m \leq n_S$*;*

2. $\frac{\partial \mathcal{V}_1(W_m)}{\partial n_m} > 0$ *for any* $n_m \leq n_S$*.*

Two polar securities can be of interest: riskless debt, which is a security that has a flat payoff in all states of the world, and equity, which replicates the payoff of the asset Z in every state. Proposition 2 allows us to understand whether these securities can be offered by issuers in equilibrium. The results are collected in the following corollary.

Corollary 2 *Fix a market structure* $\mathcal{M}$*.*

1. *In any market* $m \in \mathcal{M}$ *with* $n_m \geq n_S$ *investors, where* $n_S \in [3, \infty)$ *and satisfies Eq. (15), the issuer offers a security that pays the payoff of the asset* Z *in every state.*

2. *There is no market* $m \in \mathcal{M}$ *in which the issuer offers a security that pays a flat payoff in all states of the world.*

The first part of Corollary 2 is a direct implication of Proposition 2 and the discussion that follows it. Any issuer with at least n_S investors will find it optimal to offer equity. The second part of Corollary 2 says that issuers will never offer riskless debt. Suppose to the contrary that there is a market size $n_m \geq 3$ for which an issuer would find it optimal to offer riskless debt. The variance of riskless debt is zero so, from Eq. (12), it must be the case that the issuer finds it optimal to offer riskless debt for any market size, including in markets with at least n_S investors. This contradicts the first part of Corollary 2, hence the issuer never finds it optimal to offer riskless debt.

The results in this section characterize the security that an issuer chooses to design, taking as given the market structure. However, to show that a security can indeed be supported in equilibrium, we need to verify that the market structure in which it trades is also supported in equilibrium. We address this question in the next section.

3.3 The Equilibrium Market Structure

The goal in this section is to analyze whether there exist equilibrium market structures in which the securities that issuers design can be traded. We focus on symmetric market structures. In particular, we characterize market structures where each active market m has the same number of investors $n_m = n$ and no investor has an incentive to deviate to a different market at date $t = 0$. We discuss asymmetric equilibrium market structures in Section 5.3.

To understand the incentives of investor i at date $t = 0$ when she chooses a market in which to trade, we need to first evaluate her expected payoff $E_0\left(V_m^i\right)$ from being in market m, given a market structure $\mathcal{M}$. Substituting the equilibrium demand function $Q_m^i\left(p_m; \theta^i\right)$ from Eq. (10) and the equilibrium price p_m from Eq. (11) into the expression for V_m^i in Eq. (4) then taking expectations at date $t = 0$, before the realization of θ^i is known, we obtain

$$E_0\left(V_m^i\right) = \frac{\sigma_\theta^2}{2\gamma}\frac{n_m - 1}{n_m}\left(1 - \frac{1}{\left(1+\lambda_m^{-1}\right)^2}\right)\frac{\left[E_1\left(W_m\right)\right]^2}{\mathcal{V}_1\left(W_m\right)} + \frac{\gamma}{2}\left(1 + \frac{1}{\lambda_m^{-1}}\right)^2\left(1 - \frac{1}{\left(1+\lambda_m^{-1}\right)^2}\right)\mathcal{V}_1\left(W_m\right).$$

If we further substitute the market depth index $\lambda_m^{-1} = n_m - 2$, investor i's expected payoff becomes

$$E_0\left(V_m^i\right) = \frac{\sigma_\theta^2}{2\gamma}\frac{n_m - 2}{n_m - 1}\frac{\left[E_1\left(W_m\right)\right]^2}{\mathcal{V}_1\left(W_m\right)} + \frac{\gamma}{2}\frac{n_m}{n_m - 2}\mathcal{V}_1\left(W_m\right). \tag{16}$$

The expected payoff at date $t = 0$ of an investor who will trade the security W_m at date $t = 1$ in a market with n_m investors has two components. The first term in Eq. (16) is proportional to the variance of the investors' preference shocks, σ_θ^2, and captures the gains from trade with other investors. The larger σ_θ^2 is, the more heterogeneous investors are in how they value the mean payoff of the same security, and the more they benefit from trading with each other. In fact, when σ_θ^2 is small, investors are very similar in their valuation of the security and the equilibrium holdings of each investor approaches 1, which is the per capita supply offered by the issuer in market m. In this case, an investor's payoff is mainly driven by the risk premium that she commands as compensation for holding a risky security. The second term in Eq. (16) captures the part of the investor's expected payoff that comes from this compensation for risk.

Both the gains from trade and the compensation for risk depend on the depth of the market in which the investor trades. For a given security W_m, the gains from trade term in Eq. (16) increases with n_m, both because the fundamental gains from trade, $\frac{n_m-1}{n_m}\sigma_\theta^2$, are increasing in the number of market participants (even though the asset supply scales up linearly with the size of the market) and because the price impact of an investor is smaller in a larger market. In contrast, the compensation for risk term is decreasing in n_m, for a given security W_m, because the investor's price impact falls with the size of the market. The security that issuer m finds optimal to offer (see Proposition 2) also changes with n_m, affecting both terms in Eq. (16) through W_m. Investor i in market m weighs all of these effects at date $t = 0$ when deciding whether to deviate from market m, which has $(n-1)$ other investors, to a deeper market m', which has n other investors. The following proposition provides sufficient conditions for the existence of an equilibrium market structure.

Proposition 4 *Suppose that the asset Z satisfies $\frac{z(k)-E_1(Z|s\leq k)}{\sqrt{\mathcal{V}_1(Z|s\leq k)}} < \sqrt{2}, \forall k \in (0, S]$. Consider all $n \in [3, N]$ such that there exist integers $M_1 \in \mathbb{N}^+$ and $M_2 \in \mathbb{N}^0$ solving*

$$M_1 \times n + M_2 \times (n+1) = N, \tag{17}$$

with $M_1 + M_2 \leq M$. Then, there exists a scalar $\bar{\sigma} > 0$ such that:

1. *For any $\sigma_\theta^2 \leq \overline{\sigma}$, any market structure with M_1 issuers each getting n investors and M_2 issuers each getting $n+1$ investors is stable;*
2. *For any $\sigma_\theta^2 > \overline{\sigma}$, there is at least one stable market structure with one active issuer and all investors trading in the same market ($n = N$).*

As explained above, we focus on equilibria in which the market structure is symmetric. However, given a total number of investors N, a symmetric market structure in which each active market contains n investors may not exist for every value of n. To address the non-divisibility of investors, we extend our definition of a symmetric market structure to allow for a distribution of investors across markets such that there are n investors in some markets and $(n+1)$ investors in others. Condition (17) specifies when such generalized symmetric market structures exist.[5]

The main insight that follows from Proposition 4 is that a variety of market structures can be supported in equilibrium. This is important, as it informs us about which securities are indeed traded in equilibrium. Using Proposition 2, we can infer that when $\sigma_\theta^2 \leq \overline{\sigma}$, issuers offer debt if $n < n_S$ and equity if $n \geq n_S$. Similarly, when $\sigma_\theta^2 > \overline{\sigma}$, the issuer in the equilibrium with a single active market offers equity unless $N < n_S$. This implies the following properties of the

[5] Consider $N = 100$. A market structure in which there are $n = 8$ investors in each market does not exist, as it would require a fractional number of issuers. However, there exists a market structure in which there are $M_1 = 8$ markets each with $n = 8$ investors and $M_2 = 4$ markets each with $n = 9$ investors.

equilibrium. (i) Debt securities are traded in a larger number of smaller, less liquid markets. In this case, the market structure is more fragmented. (ii) Equity is traded in a smaller number of larger, more liquid markets. In this case, the market structure is more concentrated.[6]

Proposition 4 shows that the variance of investor preference shocks, σ_θ^2, helps determine which market structures can be supported in equilibrium. When σ_θ^2 is small, investors will not differ much in their valuations of the same security. The gains from trade are therefore low and investors anticipate that they will trade little with each other. Given this, investors are willing to trade in smaller markets, where they can use their larger price impact to obtain from issuers a less variable security whose remaining risk is well compensated. While the larger price impact also hurts the investor when she trades the security with other investors, this concern is muted because she anticipates trading little with other investors. In contrast, when σ_θ^2 is large, the gains from trade are also large. Investors understand that they may want to make large trades with each other in order to reap these gains, hence they seek to minimize their price impact by trading in a large market, albeit with a riskier security.

It is important to notice that investors' preferences shape the payoffs of the security traded in equilibrium both directly and indirectly. First, because the expected price at which a security W_m trades is increasing in the mean μ_θ of the investor preference shocks, μ_θ directly enters the optimization problem of issuer m and thus directly affects the payoffs of the security that he finds optimal to design. Second, although the variance of the investor preference shocks does not appear directly in the payoffs of the security derived in Proposition 2, σ_θ^2 plays an important role in determining which securities are traded in equilibrium. The payoffs of the equilibrium security in market m depend directly on the number of investors n_m, and σ_θ^2 affects an investor's decision about which market to trade in. Thus, as we discussed above, when σ_θ^2 is high, investors value trading in deeper markets, which induces the issuer to offer riskier securities, while, when σ_θ^2 is low, investors prefer trading in thinner markets, which induces the issuer to offer less variable securities.

[6] A market is liquid if the security can be traded with little impact on its price. We discuss this more formally in Section 4.

4 Welfare, Profits, and Liquidity

In this section, we explore several implications of our model. In particular, our goal is to gain insights into welfare, expected profits, and liquidity by exploring some simple examples.

We start by analyzing which equilibrium market structure yields the highest welfare for investors, which equilibrium yields the highest welfare for issuers, and whether any of the equilibria coincide with the solution to a social planning problem.

As in Proposition 4, we consider equilibrium market structures with M_1 issuers each getting n investors and M_2 issuers each getting $n+1$ investors, such that condition (17) is satisfied. In each active market, an investor obtains an expected profit $E_0\left(V_m^i\right)$ given by Eq. (16), while the issuer receives an expected profit $E_0\left(V_m\right)$ given by Eq. (12). Aggregate welfare can then be defined as

$$\begin{aligned}\mathcal{W} &= n \times M_1 \times E_0\left(V_m^i|_{n_m=n}\right) + (n+1) \times M_2 \times E_0\left(V_m^i|_{n_m=n+1}\right) \\ &\quad + M_1 \times E_0\left(V_m|_{n_m=n}\right) + M_2 \times E_0\left(V_m|_{n_m=n+1}\right).\end{aligned}$$

To understand which driving forces determine total welfare, it is useful to review the profits of the investors and issuers, paying special attention to how they depend on the depth of the market. Eq. (4) gives the profit of investor i after her preference shock θ^i is realized but before the state s is known. Evaluated at the equilibrium demand function $Q_m^i\left(p_m;\theta^i\right)$ derived in Proposition 1, Eq. (4) simplifies to

$$V_m^i = \frac{1+2\lambda_m}{2\gamma\left(1+\lambda_m\right)^2}\frac{\left[\theta^i E_1\left(W_m\right)-p_m\right]^2}{\mathcal{V}_1\left(W_m\right)}. \tag{18}$$

Given a market depth λ_m^{-1} (in essence, a market size n_m) and a security price p_m, Eq. (18) implies that investor i would prefer to trade a security with the least variance $\mathcal{V}_1\left(W_m\right)$ among all securities with the same mean payoff $E_1\left(W_m\right)$. In other words, investor i would prefer debt. Compare this to Eq. (16), which represents investor i's expected profit $E_0\left(V_m^i\right)$ when Eq. (4) is evaluated at both the equilibrium demand function $Q_m^i\left(p_m;\theta^i\right)$ and the equilibrium price p_m derived in Proposition 1. Given a market depth λ_m^{-1}, Eq. (16) implies that, when investor i takes into account her impact on the price, she would only prefer to trade the security with the least variance $\mathcal{V}_1(W_m)$ among all securities with the same mean payoff $E_1(W_m)$ if she expects investors to have very disperse valuations. In other words, investor i would prefer debt if σ_θ^2 is high but equity if σ_θ^2 is low.

However, a key feature of the equilibrium in our model is that investors take into account not only that they have a price impact when they trade but also that their market choice affects market depth and hence the payoffs of the securities that issuers design. Thus, an investor's expected profit as given by Eq. (16) depends on the depth of the market directly but also indirectly through the equilibrium security W_m derived in Proposition 2. In fact, the two terms in Eq. (16) – the gains from trade term and the compensation for risk term – can move in opposite directions as the market becomes deeper. For any underlying asset Z satisfying the sufficient conditions on $z(\cdot)$ in Proposition 4, the compensation for risk term is decreasing in n_m when evaluated at the equilibrium security.[7] In contrast, the gains from trade term is potentially increasing in n_m, as shown in the next proposition.

Proposition 5 *Consider an asset Z with payoffs $z(s) = z(0) + s^\alpha$, where $z(0) \geq 0$ and $\alpha > 0$. Suppose that the state s is uniformly distributed according to $f(\cdot) = \frac{1}{S}$. Evaluating an investor's expected profit in Eq. (16) at the equilibrium security $W_m^* \equiv W(n_m)$ derived in Proposition 2, an increase in n_m:*

1. *Increases the gains from trade term, $\frac{\sigma_\theta^2}{2\gamma}\frac{n_m-2}{n_m-1}\frac{[E_1(W(n_m))]^2}{\mathcal{V}_1(W(n_m))}$, if $z(0)$ is not too large;*
2. *Decreases the compensation for risk term, $\frac{\gamma}{2}\frac{n_m\mathcal{V}_1(W(n_m))}{n_m-2}$, if $\left(\frac{1+\alpha}{4\alpha}\frac{\mu_\theta-\beta}{\gamma S^\alpha}\right)^{\frac{1}{1+\alpha}} > \frac{2\alpha-1}{1+2\alpha}$.*

Proposition 5 implies that, when investor i takes into account the effect of her market choice on issuers' security design, she prefers equity if σ_θ^2 is high but debt if σ_θ^2 is low. Naturally, this result is consistent with Proposition 4 and the intuition we developed at the end of Section 3.3. It also underscores the role of investors' preferences when the security payoff is endogenous. In particular, comparing Proposition 5 to what one would conclude from Eq. (16) taking

[7] See the proof of Proposition 4. We emphasize that these conditions on $z(\cdot)$ are sufficient but not necessary. For example, in the class of functions $z(s) = z(0) + s^\alpha$ with a uniformly distributed aggregate state $f(s) = \frac{1}{S}$:

$$\frac{z(k) - E_1(Z|s \leq k)}{\sqrt{\mathcal{V}_1(Z|s \leq k)}} = \sqrt{2\alpha + 1}$$

and, therefore, $\frac{z(k)-E_1(Z|s\leq k)}{\sqrt{\mathcal{V}_1(Z|s\leq k)}} < \sqrt{2}$ if and only if $\alpha < \frac{1}{2}$. However, as shown in Proposition 5, the compensation for risk term is also decreasing in n_m if, for example, $\alpha = 1$ and $\frac{\mu_\theta-\beta}{\gamma S} > \frac{2}{9}$.

as exogenous the security W_m, we see a reversal in the type of security that investors prefer as a function of the expected dispersion in their valuations. Interestingly, then, increasing heterogeneity in investor preferences endogenously pushes towards trading equity in our model.

We can now proceed to characterize the welfare of investors and issuers.

Proposition 6 *Consider an asset Z with payoffs $z(s) = z(0) + s^{\alpha}$, where $z(0) \geq 0$ and $\alpha > 0$. Suppose that the state s is uniformly distributed according to $f(\cdot) = \frac{1}{S}$ and that $N > n_S$ with $n_S \in [3, \infty)$. It follows that:*

1. *An active issuer's expected profit $E_0(V_m)$ in Eq. (12) is increasing in n_m when W_m is the equilibrium security derived in Proposition 2.*

2. *If $z(0)$ is not too large, then, for any value of σ_θ^2, the equilibrium in which all investors trade in a single market and the issuer offers equity achieves the highest aggregate welfare.*

The first part of Proposition 6 says that an issuer is always better off designing a security for a large market than for a small market. Investors have less price impact in large markets, so the issuer is able to command a higher price for whatever security he designs. At the same time, Proposition 5 implies that an investor will be worse off in a large market than in a small market when σ_θ^2 is sufficiently low. Therefore, investors benefit at the expense of issuers in any equilibrium where debt is traded. Recall from Proposition 4 that there exist multiple symmetric equilibria when $\sigma_\theta^2 < \overline{\sigma}$. If the variance of investor preference shocks is low enough, the symmetric equilibrium that achieves the highest welfare for investors exists in the set of equilibria where debt securities are traded in many small markets.

The second part of Proposition 6 says that the benefits to investors of an equilibrium where debt is traded are outweighed by the losses to issuers, at least in environments where it is impossible to design a security that has high returns in all states of the world (i.e., environments where $z(0)$ is low). First, the expected, per-capita profit of an active issuer increases more quickly with n_m than the expected profit of an investor decreases with n_m. Second, the non-linear relationship between market size and the price impact of investors means that total welfare across issuers is maximized when there is only one active issuer.

The results in the second part of Proposition 6 also characterize the solution to a fully constrained social planning problem; that is, the problem of a social planner who chooses a market structure, a set of securities, and a set of demand functions to maximize aggregate welfare subject to the equilibrium conditions in Definition 1. However, as an alternative, we can consider a social planner who: (i) opens M_1 markets with n investors and M_2 markets with $n+1$ investors such that condition (17) holds; (ii) designs a security W_m subject only to the feasibility condition (1); (iii) allocates to investor i in market m a quantity q_m^i of the security W_m after the realization of investor preference shocks, where $\sum_{i \in m} q_m^i = n_m$ for each market m; and (iv) allocates to the issuer in each market m a quantity n_m of the security $(Z - W_m)$. The planner in this alternative planning problem still seeks to maximize the aggregate welfare of issuers and investors, but he is no longer constrained to choose among solutions that arise as a decentralized equilibrium.

In the alternative planning problem just described, it is straightforward to show that the social planner opens a single market in which all investors trade a zero-variance security (i.e., riskless debt). We omit the proof for brevity, but the intuition is as follows. A security with zero variance neutralizes the risk aversion of the investors. Maximum aggregate welfare is then achieved by allocating unboundedly positive positions q_m^i to investors whose realization of θ^i exceeds the market average and unboundedly negative positions to the rest to satisfy $\sum_{i \in N} q_m^i = N$. If the planner is restricted to design a positive-variance security, then he will open a single market in which investors take large but finite positions on the closest possible security to riskless debt.

The lesson from the alternative planning problem is that the planner can achieve higher welfare by decoupling the security design choice from the market structure choice. In particular, the planner would like to design a debt security for risk averse investors and he would like all investors to trade this security in the same market in order to maximize the gains from trade. The problem is that security design cannot be decoupled from market structure in equilibrium. Issuers respond to market-based incentives when designing a security that investors want to trade. These incentives come from the price of the security, which is endogenously less sensitive to investors' risk aversion in a large market because the price impact of an individual investor is decreasing in market size. Thus, when $n_S \in [3, \infty)$, the decentralized equilibrium supports equity in a large market or debt in small markets, but not debt in a large market.

We close this section by discussing liquidity. A natural measure of liquidity in our model is the price impact of an individual investor. Specifically, a market is liquid if the security can be traded with little impact on its price. Recall from Section 3.1 that the price impact of investor i in market m is $\partial p_{m,-i}/\partial q_m^i = \lambda_m \gamma \mathcal{V}_1(W_m)$, where $\lambda_m^{-1} \equiv (n_m - 2)$ and, in equilibrium, W_m depends on n_m as demonstrated in Proposition 2. Under the conditions stated in Proposition 4, the total derivative of $\partial p_{m,-i}/\partial q_m^i$ with respect to n_m is negative.[8] In other words, a larger market in our model is also a more liquid market. Thus, our model suggests that securities with less variable payoffs, controlling for the riskiness of the underlying asset Z and for investors' preferences, trade in less liquid markets.

5 Robustness

This section considers alternative formulations of our model. We demonstrate that the trading of debt in small markets and the trading of equity in large markets exist as equilibria in all of these formulations.

5.1 Costly Supply

Up to this point, we have assumed that each issuer m has access to a large pool of the original asset Z and backs each unit of the security W_m with one unit of Z. This assumption allowed us to abstract from mechanical effects that arise from having a fixed supply of Z in each market, which would only reinforce our results. We can relax this assumption and allow the issuer to choose how many units of Z back each unit of W_m, subject to a cost of procuring Z. In particular, issuer m incurs a cost $c(A_m)$ to acquire A_m units of Z which he then uses to back n_m units of W_m. The cost function satisfies the standard conditions $c(0) = 0$ and $c'(\cdot) > 0$. The issuer now chooses W_m and A_m subject to the feasibility constraint

$$n_m w_m(s) \leq A_m z(s), \forall s \in [0, S].$$

This constraint replaces (1). The rest of the model is as before.

Appendix B shows that the key insights of Propositions 2, 3, and 4 continue to hold. The equilibrium security W_m is a debt contract with threshold state $\overline{s}_m \in [0, S]$, where $\overline{s}_m$ is increasing in n_m so that W_m becomes more equity-like as the market size increases. All

[8] This follows immediately from the proof of Proposition 4.

investors trading equity in a single market is an equilibrium when the total number of investors N is large, but, for the same N, there also exist equilibria where debt is traded in many small markets if heterogeneity in investor preference shocks, σ_θ^2, is low.

5.2 Timing

Another assumption in our set-up relates to the timing of events. Specifically, we have assumed that issuers design securities after investors choose markets. An alternative is that issuers design securities before investors choose markets. While investors make their market choice before the realization of their preference shocks, issuers can now compete for investors through security design.

The trading equilibrium is still characterized by Proposition 1. In Propositions 2 to 4, investors chose markets taking into account how their choices would affect security design, while issuers designed securities taking as given the market structure. Now, it is issuers who choose securities taking into account how their choices affect market structure, while investors choose markets taking as given the securities.

Appendix C presents the details of this alternative formulation. We consider two issuers and study the existence of symmetric equilibria where each issuer attracts half of the total number of investors (i.e., each issuer attracts $\frac{N}{2}$ investors). For σ_θ^2 low, we find that (i) such an equilibrium exists, (ii) the security that prevails is a debt contract, and (iii) the threshold state above which the security delivers a flat payoff is increasing in $\frac{N}{2}$. In other words, a symmetric equilibrium with two large markets will involve the trading of a more equity-like security than a symmetric equilibrium with two small markets. The result that debt is traded in small markets while equity is traded in large markets thus continues to hold.

5.3 Asymmetric Equilibrium

We conclude this section by discussing the existence of asymmetric equilibria. The trading equilibrium in Section 3.1 and the issuer's security design in Section 3.2 were derived for an arbitrary market size, but attention was restricted to symmetric equilibria – that is, equilibria where all markets were equally sized – when deriving stable market structures in Section 3.3.

We now show that there are equilibria in the class of asymmetric market structures. The investor's expected profit, $E_0\left(V^i\left(n_m\right)\right)$, in a market of size n_m is still given by Eq. (16), with W_m evaluated at the equilibrium security derived in Proposition 2. We write $E_0\left(V^i\left(n_m\right)\right)$ rather than just $E_0\left(V_m^i\right)$ to make explicit the dependence of the investor's expected profit on n_m, both directly in Eq. (16) and indirectly through the dependence of W_m on n_m in Proposition 2.

Consider an asymmetric market structure with one market of size n_B and $(M'-1)$ markets of size n_m, where $n_B > n_m + 1$ and $n_B + (M'-1) \times n_m = N$. This market structure is stable if and only if

$$E_0\left(V^i\left(n_B\right)\right) > E_0\left(V^i\left(n_m+1\right)\right)$$

and

$$E_0\left(V^i\left(n_m\right)\right) > \max\left\{E_0\left(V^i\left(n_m+1\right)\right), E_0\left(V^i\left(n_B+1\right)\right)\right\}.$$

In words, no investor in the large market n_B wants to move to a smaller market (i.e., a market that has n_m other investors as opposed to $n_B - 1$ other investors). Similarly, no investor in a small market n_m wants to move to a slightly larger market (i.e., a market that has n_m other investors as opposed to $n_m - 1$ other investors) or to a much larger market (i.e., a market that has n_B other investors). Since there is only one market with n_B investors, it is not possible for one of them to move to an even larger market (i.e., a market that has n_B other investors as opposed to $n_B - 1$), hence we do not need $E_0\left(V^i\left(n_B\right)\right)$ to exceed $E_0\left(V^i\left(n_B+1\right)\right)$.

To fix ideas, consider the following parameterization: $z(s) = s$ and $f(s) = \frac{1}{S}$ for all $s \in [0, S]$, with $S = 1$, $\frac{\mu_\theta - \beta}{\gamma} = 1.25$, and $\frac{\sigma_\theta}{\gamma} = 0.275$. This implies $n_S = 6$ in Proposition 2, meaning that equity is traded in any market with six or more investors. It is straightforward to verify that one large market with $n_B = 75$ investors trading equity and any number of small markets each with $n_m = 4$ investors trading debt is a stable asymmetric equilibrium.[9] Our model therefore admits asymmetric equilibria and, in particular, asymmetric equilibria where debt and equity coexist.

[9] The relevant expected profits for an investor are: $E_0\left(V^i(4)\right) = 0.3151669\frac{\gamma}{2}$; $E_0\left(V^i(5)\right) = 0.3088872\frac{\gamma}{2}$; $E_0\left(V^i(75)\right) = 0.3094256\frac{\gamma}{2}$; and $E_0\left(V^i(76)\right) = 0.3094356\frac{\gamma}{2}$. Notice $E_0\left(V^i(75)\right) > E_0\left(V^i(5)\right)$ and $E_0\left(V^i(4)\right) > \max\left\{E_0\left(V^i(5)\right), E_0\left(V^i(76)\right)\right\}$, which are the stability conditions outlined above.

6 Conclusion

In this paper, we developed a model of financial innovation to address a critical question: Why do standardized securities trade in decentralized markets?

The novelty in our model is that both the securities issued and the structure of the market are endogenously determined. We first characterized the security that an issuer finds optimal to offer taking as given the market structure. We then characterized the set of stable market structures to determine which securities can indeed be supported in equilibrium. Our focus was primarily on symmetric market structures, but we explored asymmetric market structures in an extension. We also verified the robustness of our main insights to an extension where issuers are subject to a cost of procuring the assets that back their securities, and we discussed an alternative timing in which investors choose markets after issuers have posted securities.

The security that an issuer finds optimal to design belongs to the family of debt contracts, paying the lesser of a flat payoff and the full value of the underlying asset in every state of the world. This is a consequence of the mean-variance trade-off that investors face and of the fact that a debt contract has the least variance among all limited liability securities with the same expected value. We also showed that the state in which the security starts paying the flat payoff is higher in markets with more investors. Issuers respond to market-based incentives when designing a security that investors want to trade. These incentives come from the price of the security, which is endogenously less sensitive to investors' risk aversion in a large market because the price impact of an individual investor is decreasing in market size. On one hand, a lower price impact dulls the mean-variance trade-off that the issuer's security design problem inherits from investors' preferences, thus eliciting a riskier security. On the other hand, a lower price impact also dulls the extent to which an investor will move the price of the security against herself when trading with other investors.

As in Dugast, Uslu, and Weill (2018), investors' types play a key role in determining the market structure in which trade occurs. However, in our model, investors' preferences impact both directly an indirectly the security that will be traded. When investors expect to be relatively heterogeneous in their valuations of the same security, they understand that they may want to engage in large trades with each other so they seek to limit their price impact by trading in a large market, albeit with a riskier security. In contrast, when investors expect to be relatively homogeneous in their valuations, they anticipate trading little with each other and are thus willing to accept a larger price impact in smaller, more fragmented markets in order to elicit less variable securities from issuers.

The trading of simple debt in decentralized markets thus emerges as an equilibrium outcome in a model where both security design and market structure are endogenously determined.

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Appendix A – Proofs

Proof of Proposition 1

Rearrange the first order condition of investor i in Eq. (8) to isolate:

$$q_m^i = \frac{\theta^i E_1(W_m) - p_m}{\frac{\partial p_{m,-i}}{\partial q_m^i} + \gamma \mathcal{V}_1(W_m)} \tag{A.1}$$

for any $i \in m$. Use this expression to substitute out $Q_m^j(\cdot)$ from Eq. (9) for all investors $j \neq i$ in market m:

$$q_m^i + \sum_{j\in m, j\neq i} \frac{\theta^j E_1(W_m) - p_m}{\frac{\partial p_{m,-j}}{\partial q_m^j} + \gamma \mathcal{V}_1(W_m)} = n_m \tag{A.2}$$

We focus on symmetric linear equilibria in which the price impact $\frac{\partial p_{m,-j}}{\partial q_m^j}$ does not vary across investors within the same market. This permits rearranging Eq. (A.2) to isolate:

$$p_m = \frac{\sum_{j\in m, j\neq i} \theta^j}{n_m - 1} E_1(W_m) - \frac{n_m - q_m^i}{n_m - 1}\left(\frac{\partial p_{m,-j}}{\partial q_m^j} + \gamma \mathcal{V}_1(W_m)\right)$$

which then implies:

$$\frac{\partial p_{m,-i}}{\partial q_m^i} = \frac{1}{n_m - 1}\left(\frac{\partial p_{m,-j}}{\partial q_m^j} + \gamma \mathcal{V}_1(W_m)\right)$$

Invoking symmetry ($\frac{\partial p_{m,-i}}{\partial q_m^i} = \frac{\partial p_{m,-j}}{\partial q_m^j}$), we obtain:

$$\frac{\partial p_{m,-i}}{\partial q_m^i} = \lambda_m \gamma \mathcal{V}_1(W_m) \tag{A.3}$$

where $\lambda_m \equiv \frac{1}{n_m - 2}$. Substituting Eq. (A.3) into Eq. (A.1) delivers the equilibrium demand function $Q_m^i\left(p_m; \theta^i\right)$ in Eq. (10). Substituting Eq. (10) into the market clearing condition $\sum_{i\in m} Q_m^i\left(p_m; \theta^i\right) = n_m$ then delivers the equilibrium price p_m in Eq. (11). ■

Proof of Proposition 2

Issuer m designs a security W_m to maximize his expected payoff in Eq. (6), subject to the state-by-state feasibility constraint (1).

Letting $\upsilon(s) \geq 0$ denote the Lagrange multiplier on the feasibility constraint for state s, we can write the Lagrangian for issuer m's optimization problem as:

$$\mathcal{L}_m = E_0(V_m) + \int_0^S \upsilon(s)\left[z(s) - w_m(s)\right] dF(s)$$

or, equivalently:

$$\begin{aligned}\mathcal{L}_m \;=\;& \beta E_1(Z)\, n_m + (\mu_\theta - \beta)\, n_m \int_0^S w_m(s)\, dF(s) \\ & -\gamma \frac{n_m(n_m-1)}{n_m-2}\left[\int_0^S (w_m(s))^2\, dF(s) - \left(\int_0^S w_m(s)\, dF(s)\right)^2\right] \\ & + \int_0^S \upsilon(s)\left[z(s) - w_m(s)\right] dF(s)\end{aligned}$$

where the issuer is choosing $w_m(s)$ for each state $s \in [0, S]$ taking as given the market size n_m. We restrict attention to $n_m \geq 3$ so that the trading equilibrium in Proposition 1 involves a well-defined equilibrium price for market m.

The first order condition with respect to $w_m(s)$ delivers:

$$\upsilon(s) \overset{sign}{=} E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1} - w_m(s) \tag{A.4}$$

where $\upsilon(s) \geq 0$ and $w_m(s) \leq z(s)$ hold with complementary slackness.

If $\upsilon(s) > 0$, then:

$$w_m(s) = z(s)$$

and, invoking (A.4), we need:

$$z(s) < E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1}$$

to confirm $\upsilon(s) > 0$.

If $\upsilon(s) = 0$, then (A.4) pins down:

$$w_m(s) = E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1}$$

and we need:

$$z(s) \geq E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1}$$

to confirm $w_m(s) \leq z(s)$.

The payoffs of the equilibrium security are therefore:

$$w_m(s) = \begin{cases} z(s) & \text{if} \quad z(s) < E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m-2}{n_m-1} \\ E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m-2}{n_m-1} & \text{if} \quad z(s) \geq E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m-2}{n_m-1} \end{cases}$$

Suppose there exists an $\bar{s}_m \in (0, S)$ solving:

$$z(\bar{s}_m) \equiv E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1} \tag{A.5}$$

Then $z'(\cdot) > 0$ implies:

$$E_1(W_m) = \int_0^{\bar{s}_m} z(s)\, dF(s) + \int_{\bar{s}_m}^{S} z(\bar{s}_m)\, dF(s) \tag{A.6}$$

and we can rewrite Eq. (A.5) as:

$$\int_0^{\bar{s}_m} \left[z(\bar{s}_m) - z(s)\right] dF(s) \equiv \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1} \tag{A.7}$$

The left-hand side of Eq. (A.7) is increasing in $\bar{s}_m$ so there will be a unique solution $\bar{s}_m \in (0, S)$ if and only if:

$$z(S) - E_1(Z) > \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1} \tag{A.8}$$

The ratio $\frac{n_m-2}{n_m-1}$ is increasing in n_m and asymptotes to 1 as $n_m \to \infty$.

If the parameters satisfy $z(S) - E_1(Z) \in \left[\frac{\mu_\theta - \beta}{4\gamma}, \frac{\mu_\theta - \beta}{2\gamma}\right)$, then Eq. (15) has a unique solution $n_S \in [3, \infty)$. For any $n_m \in [3, n_S)$, condition (A.8) holds and the equilibrium security is given by Eq. (13) with $\bar{s}_m$ as defined in Eq. (A.5). For any $n_m \in [n_S, \infty)$, condition (A.8) does not hold, meaning that there is no $\bar{s}_m \in (0, S)$ solving Eq. (A.5). The equilibrium security is still given by Eq. (13) but with $\bar{s}_m = S$ instead of Eq. (A.5).

If the parameters satisfy $z(S) - E_1(Z) \geq \frac{\mu_\theta - \beta}{2\gamma}$, then condition (A.8) is true for any $n_m \in [3, \infty)$. The equilibrium security is thus given by Eq. (13) with $\bar{s}_m$ as defined in Eq. (A.5). Condition (A.8) being true for any $n_m \in [3, \infty)$ means that there is no solution $n_S \in [3, \infty)$ to Eq. (15). Assigning $n_S = \infty$ here recovers Eq. (A.5) from Eq. (14) for any $n_m \geq 3$.

If the parameters satisfy $z(S) - E_1(Z) < \frac{\mu_\theta - \beta}{4\gamma}$, then condition (A.8) is false for any

$n_m \in [3, \infty)$. The equilibrium security is thus given by Eq. (13) with $\overline{s}_m = S$ for all $n_m \in [3, \infty)$. Assigning $n_S = -\infty$ here recovers $\overline{s}_m = S$ from Eq. (14) for any $n_m \geq 3$.

We have now shown that the solution to the issuer's F.O.C.s belongs to the family of debt securities: W_m pays the entirety of the underlying asset Z up to some threshold state $\overline{s}_m$, after which it pays a flat amount that does not vary with the state. A perturbation argument similar to Hébert (2018) can be used to confirm the optimality of debt securities in our environment. We sketch this argument in the main text (see the second paragraph after the statement of Proposition 2) so do not reproduce it here. Instead, we confirm that $\overline{s}_m$ as defined by Eq. (A.7) satisfies the S.O.C. for a maximum in an auxiliary problem where the issuer chooses a threshold state $\widetilde{s}_m$ to maximize his expected profit within the family of debt securities.

The objective function for this auxiliary problem is:

$$\begin{aligned}\mathcal{L}_m^{(A)} \quad &= \quad (\mu_\theta - \beta)\left[z\left(\widetilde{s}_m\right) - \int_0^{\widetilde{s}_m}\left[z\left(\widetilde{s}_m\right) - z\left(s\right)\right]dF\left(s\right)\right] \\ &\quad -\gamma\frac{n_m - 1}{n_m - 2}\left[\int_0^{\widetilde{s}_m}\left[z\left(\widetilde{s}_m\right) - z\left(s\right)\right]^2 dF\left(s\right) - \left(\int_0^{\widetilde{s}_m}\left[z\left(\widetilde{s}_m\right) - z\left(s\right)\right]dF\left(s\right)\right)^2\right]\end{aligned}$$

The first derivative with respect to $\widetilde{s}_m$ is:

$$\frac{\partial \mathcal{L}_m^{(A)}}{\partial \widetilde{s}_m} = \left[\mu_\theta - \beta - 2\gamma\frac{n_m - 1}{n_m - 2}\int_0^{\widetilde{s}_m}\left[z\left(\widetilde{s}_m\right) - z\left(s\right)\right]dF\left(s\right)\right]\left[1 - F\left(\widetilde{s}_m\right)\right]z'\left(\widetilde{s}_m\right)$$

If $n_m < n_S$, then Eq. (A.7) has a unique interior solution $\overline{s}_m \in (0, S)$, which is also the unique interior solution to $\frac{\partial \mathcal{L}_m^{(A)}}{\partial \widetilde{s}_m} = 0$. The second derivative evaluated at this solution is:

$$\left.\frac{\partial^2 \mathcal{L}_m^{(A)}}{\partial \widetilde{s}_m^2}\right|_{\widetilde{s}_m = \overline{s}_m} = -2\gamma\frac{n_m - 1}{n_m - 2}\left(z'\left(\overline{s}_m\right)\right)^2 F\left(\overline{s}_m\right)\left[1 - F\left(\overline{s}_m\right)\right] < 0$$

where the inequality follows from $\overline{s}_m \in (0, S)$. Eq. (A.7) thus defines a local maximum and, since there are no local minima, the local maximum is also the global maximum.

If $n_m > n_S$, then there is no solution $\overline{s}_m < S$ to Eq. (A.7). The only solution to $\frac{\partial \mathcal{L}_m^{(A)}}{\partial \widetilde{s}_m} = 0$

is therefore $\widetilde{s}_m = S$, in which case the second derivative is:

$$\left.\frac{\partial^2 \mathcal{L}_m^{(A)}}{\partial \widetilde{s}_m^2}\right|_{\widetilde{s}_m = S} = -\left[\mu_\theta - \beta - 2\gamma \frac{n_m - 1}{n_m - 2}\left[z(S) - E_1(Z)\right]\right] f(S) z'(S)$$

This is negative if and only if $\frac{n_m-2}{n_m-1} > \frac{2\gamma}{\mu_\theta-\beta}\left[z(S) - E_1(Z)\right]$ or, equivalently, $n_m > n_S$.

Notice that Eq. (A.7) is only defined if $\mu_\theta > \beta$. We now demonstrate that $\mu_\theta > \beta$ is necessary and sufficient for the issuer's participation constraint to be satisfied. The participation constraint requires that the issuer's maximized expected profit, as given by $E_0(V_m)$ in Eq. (6) when evaluated at the equilibrium security, must be at least as large as $\beta E_1(Z) \times n_m$, which is what the issuer could get by consuming n_m units of Z at date $t = 2$ instead of using these units to design the security for market m.

If $n_m \geq n_S$, then the issuer's maximization problem yields $W_m = Z$ and the participation constraint simplifies to:

$$(\mu_\theta - \beta) E_1(Z) \geq \gamma \frac{n_m - 1}{n_m - 2} \mathcal{V}_1(Z) \tag{A.9}$$

Assume $\mu_\theta > \beta$ so that the left-hand side of (A.9) is positive. The right hand side of (A.9) is decreasing in n_m so (A.9) will hold for all $n_m \geq n_S$ if it holds for $n_m = n_S$. Evaluating (A.9) at the definition of n_S in Eq. (15), we get:

$$2z(S) E_1(Z) \geq E\left(Z^2\right) + \left(E_1(Z)\right)^2$$

which is true because Z has the property $z'(\cdot) > 0$.

If $n_m < n_S$, then $\bar{s}_m \in (0, S)$ is defined by Eq. (A.7). The participation constraint requires:

$$(\mu_\theta - \beta) E_1(W_m) \geq \gamma \frac{n_m - 1}{n_m - 2} \mathcal{V}_1(W_m) \tag{A.10}$$

where $E_1(W_m)$ is given by Eq. (A.6) and:

$$\mathcal{V}_1(W_m) = \int_0^{\bar{s}_m} \left[z(\bar{s}_m) - z(s)\right]^2 dF(s) - \left(\int_0^{\bar{s}_m} \left[z(\bar{s}_m) - z(s)\right] dF(s)\right)^2 \tag{A.11}$$

Use Eq. (A.7) to rewrite (A.10) as:

$$2E_1\left(W_m\right)\int_0^{\overline{s}_m}\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)\geq\mathcal{V}_1\left(W_m\right)$$

then substitute in for $E_1\left(W_m\right)$ and $\mathcal{V}_1\left(W_m\right)$ to get:

$$\begin{aligned}&2z\left(\overline{s}_m\right)\frac{1}{F\left(\overline{s}_m\right)}\int_0^{\overline{s}_m}z\left(s\right)dF\left(s\right)+\left(\frac{1}{F\left(\overline{s}_m\right)}-1\right)\left[\left(z\left(\overline{s}_m\right)\right)^2-\frac{1}{F\left(\overline{s}_m\right)}\int_0^{\overline{s}_m}\left(z\left(s\right)\right)^2dF\left(s\right)\right]\\ \geq\ &\frac{1}{F\left(\overline{s}_m\right)}\int_0^{\overline{s}_m}\left(z\left(s\right)\right)^2dF\left(s\right)+\left(\frac{1}{F\left(\overline{s}_m\right)}\int_0^{\overline{s}_m}z\left(s\right)dF\left(s\right)\right)^2\end{aligned}$$

which is again true because of $z'\left(\cdot\right)>0$. ■

Proof of Proposition 3

For any $n_m<n_S$, Eq. (14) simplifies to Eq. (A.7) from the proof of Proposition 2. Differentiating Eq. (A.7) yields:

$$\frac{d\overline{s}_m}{dn_m}=\frac{\mu_\theta-\beta}{2\gamma}\frac{1}{\left(n_m-1\right)^2}\frac{1}{z'\left(\overline{s}_m\right)F\left(\overline{s}_m\right)}>0$$

Therefore, $\frac{d\overline{s}_m}{dn_m}>0$ for any $n_m\in[3,n_S)$ and $\lim\limits_{n_m\to n_S^-}\frac{d\overline{s}_m}{dn_m}>0$.

A corollary is that the same properties hold for the mean and variance of the equilibrium security. To see why, differentiate Eq. (A.6) and (A.11) to get:

$$\frac{dE_1\left(W_m\right)}{d\overline{s}_m}=z'\left(\overline{s}_m\right)\left[1-F\left(\overline{s}_m\right)\right]$$

and:

$$\frac{d\mathcal{V}_1\left(W_m\right)}{d\overline{s}_m}=2z'\left(\overline{s}_m\right)\left[1-F\left(\overline{s}_m\right)\right]\int_0^{\overline{s}_m}\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)$$

Both of these derivatives are strictly positive because $n_m<n_S$ implies $\overline{s}_m\in(0,S)$. It then follows immediately that $E_1\left(W_m\right)$ and $\mathcal{V}_1\left(W_m\right)$ increase with n_m as $\overline{s}_m$ increases with n_m, up until the point where $n_m=n_S$. ■

Proof of Proposition 4

A market structure with one active issuer and all investors trading in the same market is always stable since there is no other active issuer to which an investor can deviate. The rest of this proof will therefore focus on symmetric market structures with two or more active issuers.

A market structure where each active issuer gets n investors is stable if and only if:

$$E_0\left(V^i(n)\right) > E_0\left(V^i(n+1)\right)$$

From Eq. (16), the expected profit of an investor in a market of size n is:

$$E_0\left(V^i(n)\right) = \frac{\sigma_\theta^2}{2\gamma}\frac{n-2}{n-1}\frac{\left[E_1(W(n))\right]^2}{\mathcal{V}_1(W(n))} + \frac{\gamma}{2}\frac{n}{n-2}\mathcal{V}_1(W(n)) \tag{A.12}$$

where we write $E_0\left(V^i(n)\right)$ to make explicit that we are evaluating the investor's expected profit at the equilibrium security derived in Proposition 2, denoted here by $W(n)$ to make explicit its dependence on the market size n.

We first show that the term $\frac{n-2}{n-1}\frac{[E_1(W(n))]^2}{\mathcal{V}_1(W(n))}$ in Eq. (A.12) is bounded. The ratio $\frac{n-2}{n-1}$ is increasing in n with $\lim_{n\to\infty}\frac{n-2}{n-1}=1$. Therefore, we only need to show that $\frac{[E_1(W(n))]^2}{\mathcal{V}_1(W(n))}$ is bounded. To do so, take the derivative with respect to n:

$$\frac{d}{dn}\left(\frac{\left[E_1(W(n))\right]^2}{\mathcal{V}_1(W(n))}\right) = \frac{E_1(W(n))}{\mathcal{V}_1(W(n))}\left[2\frac{dE_1(W(n))}{dn} - \frac{E_1(W(n))}{\mathcal{V}_1(W(n))}\frac{d\mathcal{V}_1(W(n))}{dn}\right]$$

If $n \geq n_S$, then $W(n) = Z$ and this derivative is zero. If instead $n < n_S$, then we can use the derivatives in the proof of Proposition 3 to write:

$$\frac{d}{dn}\left(\frac{\left[E_1(W(n))\right]^2}{\mathcal{V}_1(W(n))}\right) \overset{sign}{=} 1 - \frac{E_1(W(n))}{\mathcal{V}_1(W(n))}\int_0^{\bar{s}}\left[z(\bar{s}) - z(s)\right]dF(s)$$

and, with $E_1(W(n))$ as per Eq. (A.6) and $\mathcal{V}_1(W(n))$ as per Eq. (A.11), we get:

$$\frac{d}{dn}\left(\frac{\left[E_1(W(n))\right]^2}{\mathcal{V}_1(W(n))}\right) \overset{sign}{=} -\frac{\int_0^{\bar{s}} z(s)\left[z(\bar{s}) - z(s)\right]dF(s)}{\mathcal{V}_1(W(n))} < 0$$

We can now conclude:

$$\frac{\left[E_1\left(W\left(n\right)\right)\right]^2}{\mathcal{V}_1\left(W\left(n\right)\right)} \leq \frac{\left[E_1\left(W\left(3\right)\right)\right]^2}{\mathcal{V}_1\left(W\left(3\right)\right)}$$

where $n = 3$ is the smallest market size for which there can be a well-defined equilibrium price in Eq. (11). If $n_S > 3$, then Eq. (A.7) defines $\bar{s} \in (0, S)$ and hence $E_1\left(W\left(3\right)\right) \in (0, \infty)$ and $\mathcal{V}_1\left(W\left(3\right)\right) \in (0, \infty)$. In other words, $\frac{[E_1(W(3))]^2}{\mathcal{V}_1(W(3))}$ is bounded. If instead $n_S = 3$, then $\frac{[E_1(W(3))]^2}{\mathcal{V}_1(W(3))} = \frac{[E_1(Z)]^2}{\mathcal{V}_1(Z)}$ which is also bounded.

Next, we show that the term $\frac{n\mathcal{V}_1(W(n))}{n-2}$ in Eq. (A.12) is decreasing in n if the payoffs of the asset Z satisfy $\frac{z(k)-E_1(Z|s\leq k)}{\sqrt{\mathcal{V}_1(Z|s\leq k)}} < \sqrt{2}, \forall k \in (0, S]$. Taking derivatives:

$$\frac{d}{dn}\left(\frac{n\mathcal{V}_1\left(W\left(n\right)\right)}{n-2}\right) = -\frac{2\mathcal{V}_1\left(W\left(n\right)\right)}{\left(n-2\right)^2} + \frac{n}{n-2}\frac{d\mathcal{V}_1\left(W\left(n\right)\right)}{dn}$$

If $n \geq n_S$, then $W\left(n\right) = Z$ and this derivative is negative. If instead $n < n_S$, then we can use the derivatives in the proof of Proposition 3 to write:

$$\frac{d}{dn}\left(\frac{n\mathcal{V}_1\left(W\left(n\right)\right)}{n-2}\right) \stackrel{sign}{=} -\mathcal{V}_1\left(W\left(n\right)\right) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n\left(n-2\right)}{\left(n-1\right)^2}\frac{1-F\left(\bar{s}\right)}{F\left(\bar{s}\right)}\int_0^{\bar{s}}\left[z\left(\bar{s}\right) - z\left(s\right)\right]dF\left(s\right)$$

Using Eq. (A.7) and the expression for $\mathcal{V}_1\left(W\left(n\right)\right)$ in Eq. (A.11), we obtain the following necessary and sufficient condition for $\frac{d}{dn}\left(\frac{n\mathcal{V}_1(W(n))}{n-2}\right) < 0$ when $n < n_S$:

$$\int_0^{\bar{s}}\left[z\left(\bar{s}\right) - z\left(s\right)\right]^2 dF\left(s\right) > \frac{1}{F\left(\bar{s}\right)}\left(1 + \frac{1-F\left(\bar{s}\right)}{n-1}\right)\left(\int_0^{\bar{s}}\left[z\left(\bar{s}\right) - z\left(s\right)\right]dF\left(s\right)\right)^2$$

This rearranges to:

$$\frac{z\left(\bar{s}\right) - E_1\left(Z|s \leq \bar{s}\right)}{\sqrt{\mathcal{V}_1\left(Z|s \leq \bar{s}\right)}} < \sqrt{\frac{n-1}{1-F\left(\bar{s}\right)}} \tag{A.13}$$

where:

$$E_1\left(Z|s \leq \bar{s}\right) \equiv \frac{1}{F\left(\bar{s}\right)}\int_0^{\bar{s}} z\left(s\right)dF\left(s\right)$$

and:

$$\mathcal{V}_1\left(Z|s \leq \bar{s}\right) \equiv \frac{1}{F\left(\bar{s}\right)}\int_0^{\bar{s}}\left(z\left(s\right)\right)^2 dF\left(s\right) - \left(\frac{1}{F\left(\bar{s}\right)}\int_0^{\bar{s}} z\left(s\right)dF\left(s\right)\right)^2$$

Since $F(\overline{s}) \in (0,1)$ and $n \geq 3$, a sufficient condition for (A.13) is:

$$\frac{z(k) - E_1(Z|s \leq k)}{\sqrt{\mathcal{V}_1(Z|s \leq k)}} < \sqrt{2}, \forall k \in (0, S]$$

which is the condition in the statement of Proposition 4.

Invoking this condition, we can now conclude that there exists a bound $\overline{\sigma} > 0$ such that $\frac{dE_0(V^i(n))}{dn} < 0$ for all $n \geq 3$ if $\sigma_\theta^2 \leq \overline{\sigma}$.

In words, any symmetric market structure is stable when $\sigma_\theta^2 \leq \overline{\sigma}$. For any integer $n \geq 3$ such that $\frac{N}{n}$ is also an integer, the symmetric market structure involves $\frac{N}{n}$ active issuers each getting n investors. For any integer $n \geq 3$ such that $\frac{N}{n}$ is not an integer, we can only consider n if there exist positive integers, M_1 and M_2, such that $M_1 \times n + M_2 \times (n+1) = N$, in which case the symmetric market structure involves M_1 active issuers each getting n investors and M_2 active issuers each getting $n+1$ investors.

To see what values of n will be consistent with the existence of such integers, consider an arbitrary total number of active issuers M'. If each active issuer gets n investors, then there are $N - M' \times n$ investors left to be allocated to the M' active issuers. For a market structure where each of the M' active issuers gets either n or $n+1$ investors, we need $N - M' \times n \geq 0$ (so that no active issuer gets fewer than n investors) and $N - M' \times n \leq M'$ (so that no active issuer gets more than $n+1$ investors). In other words, we need $M' \in \left[\frac{N}{1+n}, \frac{N}{n}\right]$. We also need M' to be an integer and hence we need an integer to exist between $\frac{N}{1+n}$ and $\frac{N}{n}$. This implies that we can only consider n such that $\left\lfloor \frac{N}{n+1} \right\rfloor < \left\lfloor \frac{N}{n} \right\rfloor$, where the notation $\lfloor X \rfloor$ means X is rounded down to the nearest integer. As long as N is not too low, $\left\lfloor \frac{N}{4} \right\rfloor < \left\lfloor \frac{N}{3} \right\rfloor$ will be satisfied, meaning that there will exist a stable market structure where $M_1 \in \mathbb{N}^+$ active issuers get 3 investors each and $M_2 \in \mathbb{N}^0$ active issuers get 4 investors each. ■

Proof of Proposition 5

For ease of reference, define the gains from trade term in Eq. (16) as:

$$G(n_m) \equiv \frac{\sigma_\theta^2}{2\gamma} \frac{n_m - 2}{n_m - 1} \frac{[E_1(W(n_m))]^2}{\mathcal{V}_1(W(n_m))}$$

and the compensation for risk term as:

$$R\left(n_m\right) \equiv \frac{\gamma}{2}\frac{n_m}{n_m-2}\mathcal{V}_1\left(W\left(n_m\right)\right)$$

where we write $W\left(n_m\right)$ to make explicit the dependence of the equilibrium security W_m on the market size n_m in Proposition 2.

Taking derivatives, we get:

$$G'\left(n_m\right) \overset{sign}{=} \frac{E_1\left(W\left(n_m\right)\right)}{\left(n_m-1\right)\left(n_m-2\right)} + 2\frac{dE_1\left(W\left(n_m\right)\right)}{dn_m} - \frac{E_1\left(W\left(n_m\right)\right)}{\mathcal{V}_1\left(W\left(n_m\right)\right)}\frac{d\mathcal{V}_1\left(W\left(n_m\right)\right)}{dn_m}$$

and:

$$R'\left(n_m\right) \overset{sign}{=} -\left(\frac{2\mathcal{V}_1\left(W\left(n_m\right)\right)}{n_m\left(n_m-2\right)} - \frac{d\mathcal{V}_1\left(W\left(n_m\right)\right)}{dn_m}\right)$$

If $n_m > n_S$, then $W\left(n_m\right) = Z$ and thus $\frac{dE_1(W(n_m))}{dn_m} = \frac{d\mathcal{V}_1(W(n_m))}{dn_m} = 0$, which further implies $G'\left(n_m\right) > 0$ and $R'\left(n_m\right) < 0$. If instead $n_m \leq n_S$, then Proposition 2 defines:

$$z\left(\overline{s}_m\right) = E_1\left(W\left(n_m\right)\right) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m-2}{n_m-1}$$

With $z\left(s\right) = z\left(0\right) + s^\alpha$ and $f\left(s\right) = \frac{1}{S}$, the equilibrium security has:

$$E_1\left(W\left(n_m\right)\right) = z\left(0\right) + \overline{s}_m^\alpha\left(1 - \frac{\alpha}{1+\alpha}\frac{\overline{s}_m}{S}\right)$$

and:

$$\mathcal{V}_1\left(W\left(n_m\right)\right) = \frac{\alpha^2}{1+\alpha}\frac{\overline{s}_m^{1+2\alpha}}{S}\left(\frac{2}{1+2\alpha} - \frac{1}{1+\alpha}\frac{\overline{s}_m}{S}\right)$$

where:

$$\frac{\alpha}{1+\alpha}\frac{\overline{s}_m^{1+\alpha}}{S} = \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m-2}{n_m-1}$$

The total derivatives of $E_1\left(W\left(n_m\right)\right)$ and $\mathcal{V}_1\left(W\left(n_m\right)\right)$ with respect to n_m are therefore:

$$\frac{dE_1\left(W\left(n_m\right)\right)}{dn_m} = \left(1 - \frac{\overline{s}_m}{S}\right)\frac{\mu_\theta - \beta}{2\gamma\frac{\overline{s}_m}{S}}\frac{1}{\left(n_m-1\right)^2}$$

and:

$$\frac{d\mathcal{V}_1\left(W\left(n_m\right)\right)}{dn_m} = \frac{2\alpha}{1+\alpha}\frac{\overline{s}_m^{1+\alpha}}{S}\frac{dE_1\left(W_m\left(n_m\right)\right)}{dn_m}$$

Substituting into the expressions for $G'(n_m)$ and $R'(n_m)$, we get:

$$G'(n_m) \overset{sign}{=} \frac{\mu_\theta - \beta}{2\gamma \frac{\overline{s}_m}{S}} \frac{1}{(n_m - 1)^2} \frac{\overline{s}_m^\alpha \left[\frac{2\alpha}{1+2\alpha} + \frac{1-4\alpha}{1+2\alpha}\frac{\overline{s}_m}{S} + \frac{\alpha}{1+\alpha}\left(\frac{\overline{s}_m}{S}\right)^2\right] - z(0)\left[\frac{2\alpha}{1+2\alpha} - \frac{\overline{s}_m}{S}\right]}{\alpha \overline{s}_m^\alpha \left(\frac{2}{1+2\alpha} - \frac{1}{1+\alpha}\frac{\overline{s}_m}{S}\right)}$$

and:

$$R'(n_m) \overset{sign}{=} -\frac{2\alpha^2}{(1+\alpha)^2} \frac{\overline{s}_m^{1+2\alpha}}{S} \frac{\left[\left(\frac{1+\alpha}{\alpha}\frac{\mu_\theta - \beta}{2\gamma S^\alpha}\frac{n_m - 2}{n_m - 1}\right)^{\frac{1}{1+\alpha}} - \frac{2(1+\alpha) - n_m}{1+2\alpha}\right]}{n_m (n_m - 1)(n_m - 2)}$$

To help establish $R'(n_m) < 0$, notice that $R'(n_m) < 0$ for all $n_m \geq 3$ if and only if $R'(3) < 0$. Therefore, $\left(\frac{1+\alpha}{4\alpha}\frac{\mu_\theta - \beta}{\gamma S^\alpha}\right)^{\frac{1}{1+\alpha}} > \frac{2\alpha - 1}{1+2\alpha}$ is sufficient for $R'(n_m) < 0$. To help establish $G'(n_m) > 0$, define the function $h(x) \equiv \frac{2\alpha}{1+2\alpha} + \frac{1-4\alpha}{1+2\alpha}x + \frac{\alpha}{1+\alpha}x^2$, where $x \in [0, 1]$. Notice $h(0) > 0$. Also notice $h''(x) > 0$ so any solution to $h'(x) = 0$ is a minimum. If $\alpha \in \left(0, \frac{1}{4}\right]$, then there is no $x_0 \in [0, 1]$ solving $h'(x_0) = 0$, hence $h(x) > 0$ for all $x \in [0, 1]$. If instead $\alpha > \frac{1}{4}$, then $x_0 = \frac{(1+\alpha)(4\alpha - 1)}{2\alpha(1+2\alpha)}$ and $h(x_0) = \frac{7\alpha - 1}{4\alpha(1+2\alpha)^2} > 0$, where the inequality follows from $\alpha > \frac{1}{4}$. We again have $h(x) > 0$ for all $x \in [0, 1]$. Notice that $h\left(\frac{\overline{s}_m}{S}\right) > 0$ for all $\frac{\overline{s}_m}{S} \in (0, 1]$ implies $G'(n_m) > 0$ when $z(0) = 0$ so, by continuity, $G'(n_m) > 0$ for any $z(0)$ below some positive upperbound. We can exclude $\overline{s}_m = 0$ when discussing $h\left(\frac{\overline{s}_m}{S}\right)$ since $n_m \geq 3$ implies $\overline{s}_m > 0$. ■

Proof of Proposition 6

Start with the issuer's expected payoff, $E_0(V_m)$. Substituting $\lambda_m^{-1} \equiv (n_m - 2)$ into Eq. (12):

$$E_0(V_m) = \left[\beta E_1(Z) + (\mu_\theta - \beta) E_1(W(n_m)) - \frac{n_m - 1}{n_m - 2}\gamma \mathcal{V}_1(W(n_m))\right] \times n_m$$

This expression for $E_0(V_m)$ is increasing in n_m holding constant the security W_m, implying that $E_0(V_m)$ is increasing in n_m for $n_m > n_S$ since the equilibrium security for any $n_m > n_S$ is simply $W_m = Z$. It only remains to check that $E_0(V_m)$ is also increasing in n_m for $n_m \leq n_S$ when evaluated at the equilibrium security $W(n_m)$.

Using the expressions for $E_1(W(n_m))$, $\mathcal{V}_1(W(n_m))$, and $\overline{s}_m$ from the proof of Proposition 5, we can write:

$$E_0(V_m) = S^\alpha \left[\frac{\beta}{1+\alpha} + \frac{\mu_\theta z(0)}{S^\alpha} + (\mu_\theta - \beta)\left(\frac{\overline{s}_m}{S}\right)^\alpha \left(\frac{1+\alpha}{1+2\alpha} - \frac{\alpha}{2(1+\alpha)}\frac{\overline{s}_m}{S}\right)\right] n_m$$

for $n_m \leq n_S$. It is easy to show that $\left(\frac{\overline{s}_m}{S}\right)^{\alpha}\left(\frac{1+\alpha}{1+2\alpha} - \frac{\alpha}{2(1+\alpha)}\frac{\overline{s}_m}{S}\right)$ is increasing in $\frac{\overline{s}_m}{S}$ for any $\frac{\overline{s}_m}{S} \in [0,1]$. We also know from Proposition 3 that $\overline{s}_m$ is increasing in n_m. Therefore, $\frac{dE_0(V_m)}{dn_m} > 0$ for $n_m \leq n_S$.

Turn now to total welfare. Ignore the integered nature of investors for the moment. There are N investors, each getting the expected payoff $E_0\left(V_m^i\right)$ in Eq. (16). There are also $\frac{N}{n_m}$ active issuers, each getting the expected payoff $E_0\left(V_m\right)$ in Eq. (12). Inactive issuers receive a payoff of zero. Total (expected) welfare at date $t = 0$ is then:

$$\mathcal{W} = N \times E_0\left(V_m^i\right) + \frac{N}{n_m} \times E_0\left(V_m\right)$$

where $\frac{1}{n_m} \times E_0\left(V_m\right)$ is the expected, per-capita payoff of an active issuer.

Substituting in Eq. (12) and (16):

$$\mathcal{W} = N\left(\beta E_1\left(Z\right) + \left(\mu_\theta - \beta\right)E_1\left(W_m\right) + \frac{\sigma_\theta^2}{2\gamma}\frac{n_m - 2}{n_m - 1}\frac{\left[E_1\left(W_m\right)\right]^2}{\mathcal{V}_1\left(W_m\right)} - \frac{\gamma}{2}\mathcal{V}_1\left(W_m\right)\right) \tag{A.14}$$

Notice that the utility investors receive from the risk premium (i.e., compensation for risk term) is outweighed by the negative effect of variance on the price that the issuer receives.

The expression for $\mathcal{W}$ is increasing in n_m holding constant the security W_m. Therefore, $\mathcal{W}$ is increasing in n_m for $n_m > n_S$ and it only remains to check that $\mathcal{W}$ is also increasing in n_m for $n_m \leq n_S$ when evaluated at the equilibrium security. Using the expressions for $E_1\left(W\left(n_m\right)\right)$, $\mathcal{V}_1\left(W\left(n_m\right)\right)$, and $\overline{s}_m$ from the proof of Proposition 5, we can write:

$$\begin{aligned}\mathcal{W} \;\; = \;\; & \beta\left(z\left(0\right) + \frac{S^\alpha}{1+\alpha}\right)N + z\left(0\right)\left[\widehat{\mu}_\theta + \frac{1}{\alpha}\frac{\widehat{\sigma}_\theta^2}{\widehat{\mu}_\theta}\frac{2\left(1 - \frac{\alpha x}{1+\alpha}\right) + \frac{z(0)}{x^\alpha S^\alpha}}{\frac{2}{1+2\alpha} - \frac{x}{1+\alpha}}\right]\gamma S^\alpha N \\ & + \left[\widehat{\mu}_\theta x^\alpha\left(1 - \frac{\alpha x}{1+\alpha}\right) + \frac{1}{\alpha}\frac{\widehat{\sigma}_\theta^2}{\widehat{\mu}_\theta}\frac{x^\alpha\left(1 - \frac{\alpha x}{1+\alpha}\right)^2}{\frac{2}{1+2\alpha} - \frac{x}{1+\alpha}} - \frac{\alpha^2 x^{1+2\alpha}}{2\left(1+\alpha\right)}\left(\frac{2}{1+2\alpha} - \frac{x}{1+\alpha}\right)\right]\gamma S^{2\alpha}N\end{aligned}$$

for $n_m \leq n_S$, where $x \equiv \frac{\bar{s}_m}{S}$, $\widehat{\mu}_\theta \equiv \frac{\mu_\theta - \beta}{\gamma S^\alpha}$, and $\widehat{\sigma}_\theta \equiv \frac{\sigma_\theta}{\gamma S^\alpha}$. Taking derivatives:

$$\begin{aligned}
\frac{d\mathcal{W}}{dx} &= \frac{\widehat{\sigma}_\theta^2}{\widehat{\mu}_\theta} \frac{\frac{2}{1+\alpha}\frac{1}{1+2\alpha} - \frac{z(0)}{x^{1+\alpha}S^\alpha}\left(\frac{2\alpha}{1+2\alpha} - x\right)}{\left(\frac{2}{1+2\alpha} - \frac{x}{1+\alpha}\right)^2} \frac{\gamma S^\alpha N}{\alpha} z(0) \\
&\quad + \alpha x^{\alpha-1}\left[\frac{\widehat{\sigma}_\theta^2}{\widehat{\mu}_\theta} \frac{\left(1 - \frac{\alpha x}{1+\alpha}\right) h(x)}{\alpha^2\left(\frac{2}{1+2\alpha} - \frac{x}{1+\alpha}\right)^2} + \left(\widehat{\mu}_\theta - \frac{\alpha x^{1+\alpha}}{1+\alpha}\right)(1-x)\right]\gamma S^{2\alpha} N
\end{aligned}$$

with the function $h(x) > 0$ as defined in the proof of Proposition 5. The expression for $\bar{s}_m$ from the same proof implies:

$$\frac{2\alpha x^{1+\alpha}}{1+\alpha} = \widehat{\mu}_\theta \frac{n_m - 2}{n_m - 1}$$

and hence:

$$\widehat{\mu}_\theta - \frac{\alpha x^{1+\alpha}}{1+\alpha} = \frac{\widehat{\mu}_\theta n_m}{2(n_m - 1)} > 0$$

Therefore, the second line in the expression for $\frac{d\mathcal{W}}{dx}$ is positive. If $z(0) = 0$, then it follows immediately that $\frac{d\mathcal{W}}{dx} > 0$. If instead $z(0) > 0$, then the first line in the expression for $\frac{d\mathcal{W}}{dx}$ is positive if and only if:

$$\left[\frac{\alpha(1-x)}{x^{1+\alpha}S^\alpha} - \frac{1}{2x^\alpha S^\alpha}\right] z(0) < \frac{1}{1+\alpha}$$

A sufficient condition is $\frac{\alpha z(0)}{x^{1+\alpha}S^\alpha} < \frac{1}{1+\alpha}$ evaluated at $x = \left(\frac{(1+\alpha)\widehat{\mu}_\theta}{4\alpha}\right)^{\frac{1}{1+\alpha}}$, which is the lowest possible x, specifically the x associated with $n_m = 3$. In other words, $z(0) < \frac{\widehat{\mu}_\theta S^\alpha}{4\alpha^2}$ is sufficient for $\frac{d\mathcal{W}}{dx} > 0$. The fully constrained planner thus chooses $n_m = N$ and $W_m = Z$ when $z(0)$ is not too large.

Return now to the integered nature of investors. Denote by $\mathcal{W}(n_m)$ the right-hand side of Eq. (A.14), where $W_m \equiv W(n_m)$ is the equilibrium security. In a market structure satisfying condition (17), aggregate welfare is:

$$\mathcal{W} = n \times M_1 \times \frac{\mathcal{W}(n)}{N} + (N - n \times M_1) \times \frac{\mathcal{W}(n+1)}{N} \leq \mathcal{W}(n+1) \leq \mathcal{W}(N)$$

where the inequalities follow from the fact that $\mathcal{W}(n)$ is increasing in n when $z(0)$ is not too large. Recalling that $\mathcal{W}(N)$ is welfare when all investors trade in one market completes the proof. ■

Appendix B – Costly Supply

Given n_m, issuer m chooses a security W_m to supply in market m. He still supplies one unit of W_m per capita but now he chooses the number of units A_m of the asset Z that back the n_m units of W_m. Previously, we had assumed $A_m = n_m$. We now let the issuer choose A_m at a cost $c(A_m)$, where $c(0) = 0$ and $c'(\cdot) > 0$. To fix ideas, consider $c(A_m) = \frac{\delta}{2} A_m^2$.

Issuer m's expected payoff at date $t = 1$ is:

$$V_m = p_m n_m + \beta E_1 (A_m Z - n_m W_m) - \frac{\delta}{2} A_m^2$$

The equilibrium price p_m is still given by Eq. (11) so:

$$E_0 (V_m) = \left[(\mu_\theta - \beta) E_1 (W_m) - \frac{n_m - 1}{n_m - 2} \gamma \mathcal{V}_1 (W_m) \right] n_m + \beta E_1 (Z) A_m - \frac{\delta}{2} A_m^2$$

The Lagrangian for the issuer's problem can then be written as:

$$\begin{aligned} \mathcal{L} \; = \; & (\mu_\theta - \beta) n_m \int_0^S w_m(s) \, dF(s) \\ & - \frac{\gamma n_m (n_m - 1)}{n_m - 2} \left[\int_0^S (w_m(s))^2 \, dF(s) - \left(\int_0^S w_m(s) \, dF(s) \right)^2 \right] \\ & + \beta E_1 (Z) A_m - \frac{\delta}{2} A_m^2 + \int_0^S \upsilon(s) \left[A_m z(s) - n_m w_m(s) \right] dF(s) + \upsilon_A A_m \end{aligned} \tag{B.1}$$

where $\upsilon(s) \geq 0$ is the Lagrange multiplier on the feasibility constraint for state s, and $\upsilon_A \geq 0$ is the multiplier on $A_m \geq 0$.

The first order condition for $w_m(s)$ is:

$$\upsilon(s) = \mu_\theta - \beta - 2\gamma \frac{n_m - 1}{n_m - 2} \left[w_m(s) - E_1 (W_m) \right] \tag{B.2}$$

where $\upsilon(s) \geq 0$ and $A_m z(s) \geq n_m w_m(s)$ hold with complementary slackness. This implies that the equilibrium security, conditional on n_m, has payoffs:

$$w_m(s) = \begin{cases} \frac{A_m}{n_m} z(s) & if \quad s < \overline{s}_m \\ \frac{A_m}{n_m} z(\overline{s}_m) & if \quad s \geq \overline{s}_m \end{cases}$$

where:

$$\overline{s}_m = \arg\min_{k\in[0,S]} \left| z(k) - \frac{n_m}{A_m}\left(E_1(W_m) + \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m - 2}{n_m - 1}\right)\right| \tag{B.3}$$

and:

$$E_1(W_m) = \frac{A_m}{n_m}\left(z(\overline{s}_m) - \int_0^{\overline{s}_m} [z(\overline{s}_m) - z(s)]\, dF(s)\right) \tag{B.4}$$

The first order condition for A_m is:

$$\delta A_m = \beta E_1(Z) + \int_0^S \upsilon(s)\, z(s)\, dF(s) + \upsilon_A \tag{B.5}$$

Using Eq. (B.2) with $E_1(W_m)$ as defined in Eq. (B.4), we can rewrite Eq. (B.5) as:

$$\begin{aligned} \delta A_m &= \mu_\theta E_1(Z) + \upsilon_A \\ &\quad -2\gamma\frac{n_m - 1}{n_m - 2}\frac{A_m}{n_m}\left[E_1(Z)\int_0^{\overline{s}_m}[z(\overline{s}_m) - z(s)]\, dF(s) - \int_0^{\overline{s}_m} z(s)[z(\overline{s}_m) - z(s)]\, dF(s)\right] \end{aligned} \tag{B.6}$$

Consider $A_m > 0$ so that $\upsilon_A = 0$:

1. If $\overline{s}_m = S$, then Eq. (B.6) reduces to:

$$A_m = \frac{\mu_\theta E_1(Z)}{\delta + 2\gamma\frac{n_m - 1}{n_m(n_m - 2)}\mathcal{V}_1(Z)}$$

   which confirms $A_m > 0$. To confirm that Eq. (B.3) delivers $\overline{s}_m = S$, we need:

$$z(S) - E_1(Z) < \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m}{A_m}\frac{n_m - 2}{n_m - 1}$$

   Substituting in the solution for A_m, the condition for $\overline{s}_m = S$ simplifies to:

$$\frac{n_m(n_m - 2)}{n_m - 1} > \frac{2\gamma}{\delta}\frac{\mu_\theta\left[E_1(Z)\, z(S) - E_1\left(Z^2\right)\right] + \beta\mathcal{V}_1(Z)}{\mu_\theta - \beta}$$

   The left-hand side of this inequality is increasing in n_m and the right-hand side is positive. Therefore, $\overline{s}_m = S$ if n_m is above some threshold.

2. If the solution to Eq. (B.3) is interior, then $\overline{s}_m$ is defined by:

$$\int_0^{\overline{s}_m} [z(\overline{s}_m) - z(s)]\, dF(s) \equiv \frac{\mu_\theta - \beta}{2\gamma}\frac{n_m}{A_m}\frac{n_m - 2}{n_m - 1} \tag{B.7}$$

and we can use Eq. (B.7) to simplify Eq. (B.6) to:

$$A_m\left(\delta-2\gamma\frac{n_m-1}{n_m\left(n_m-2\right)}\int_0^{\overline{s}_m}z\left(s\right)\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)\right)=\beta E_1\left(Z\right)\tag{B.8}$$

Using Eq. (B.8) to substitute A_m out of Eq. (B.7), we can then rewrite Eq. (B.7) as:

$$\frac{\beta E_1\left(Z\right)}{\mu_\theta-\beta}\int_0^{\overline{s}_m}\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)+\int_0^{\overline{s}_m}z\left(s\right)\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)=\frac{\delta}{2\gamma}\frac{n_m\left(n_m-2\right)}{n_m-1}\tag{B.9}$$

which implies $\frac{\partial\overline{s}_m}{\partial n_m}>0$. Notice from Eq. (B.7) that $A_m>0$ and, to confirm $\overline{s}_m<S$, we need n_m below the threshold that delivered $\overline{s}_m=S$ in the previous bullet.

We have now shown that the key insights of Propositions 2 and 3 continue to hold. If small markets are stable, then debt will be traded in that market structure. If large markets are stable, then equity will be traded in that market structure.

A market structure where all investors trade in the same market is trivially stable, so, as long as N is large, there always exists an equilibrium where investors trade equity in large markets.

It remains to show that the key insights of Proposition 4 also continue to hold. In particular, we want to show that small markets are also stable if heterogeneity in investor preference shocks, σ_θ^2, is sufficiently low. The investor's expected profit is still given by Eq. (16) so we follow the steps in the proof of Proposition 4. Specifically, if we can show $\frac{d\mathcal{V}_1(W_m)}{dn_m}<\frac{2\mathcal{V}_1(W_m)}{n_m(n_m-2)}$ for any n_m where the equilibrium security is $W_m\neq Z$, then we can conclude that small markets are stable when σ_θ^2 is sufficiently low.

The variance of the equilibrium security derived above is:

$$\mathcal{V}_1\left(W_m\right)=\left(\frac{A_m}{n_m}\right)^2\left[\int_0^{\overline{s}_m}\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]^2dF\left(s\right)-\left(\int_0^{\overline{s}_m}\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)\right)^2\right]$$

where A_m depends on $\overline{s}_m$ and n_m as per Eq. (B.7) and $\overline{s}_m$ depends on n_m as per Eq. (B.9). Therefore:

$$\frac{d\mathcal{V}_1\left(W_m\right)}{dn_m}=\frac{2\mathcal{V}_1\left(W_m\right)}{A_m}\left(\frac{dA_m}{dn_m}-\frac{A_m}{n_m}\right)+\frac{2A_m^2}{n_m^2}\left[1-F\left(\overline{s}_m\right)\right]z'\left(\overline{s}_m\right)\frac{d\overline{s}_m}{dn_m}\int_0^{\overline{s}_m}\left[z\left(\overline{s}_m\right)-z\left(s\right)\right]dF\left(s\right)$$

where:

$$z'(\overline{s}_m)\frac{d\overline{s}_m}{dn_m} = \frac{\frac{\delta}{2\gamma}\frac{n_m^2-2n_m+2}{(n_m-1)^2}}{\frac{\beta E_1(Z)}{\mu_\theta-\beta}F(\overline{s}_m)+\int_0^{\overline{s}_m} z(s)\,dF(s)}$$

and:

$$\frac{dA_m}{dn_m} = \frac{A_m}{n_m}\frac{n_m^2-2n_m+2}{(n_m-1)(n_m-2)}\left(1-\frac{\delta\frac{A_m}{\mu_\theta-\beta}F(\overline{s}_m)}{\frac{\beta E_1(Z)}{\mu_\theta-\beta}F(\overline{s}_m)+\int_0^{\overline{s}_m} z(s)\,dF(s)}\right)$$

The condition we want to check, $\frac{d\mathcal{V}_1(W_m)}{dn_m} < \frac{2\mathcal{V}_1(W_m)}{n_m(n_m-2)}$, simplifies to:

$$\begin{aligned}
&[1-F(\overline{s}_m)]\,z'(\overline{s}_m)\frac{d\overline{s}_m}{dn_m}\int_0^{\overline{s}_m}[z(\overline{s}_m)-z(s)]\,dF(s)\\
<\;&\frac{1}{n_m}\left[\frac{n_m-1}{n_m-2}-\frac{n_m}{A_m}\frac{dA_m}{dn_m}\right]\left[\int_0^{\overline{s}_m}[z(\overline{s}_m)-z(s)]^2\,dF(s)-\left(\int_0^{\overline{s}_m}[z(\overline{s}_m)-z(s)]\,dF(s)\right)^2\right]
\end{aligned}$$

$\Leftrightarrow$

$$1+[1-F(\overline{s}_m)]\frac{[z(\overline{s}_m)-E_1(Z|s\leq\overline{s}_m)]^2}{\mathcal{V}_1(Z|s\leq\overline{s}_m)} < \frac{\frac{\delta}{2\gamma}\frac{n_m(n_m-2)}{n_m-1}\frac{1}{F(\overline{s}_m)}\left(n_m^2-2n_m+2\right)}{\left[\frac{\beta E_1(Z)}{\mu_\theta-\beta}+E_1(Z|s\leq\overline{s}_m)\right][z(\overline{s}_m)-E_1(Z|s\leq\overline{s}_m)]}$$

$\Leftrightarrow$

$$\frac{[1-F(\overline{s}_m)]}{(n_m-1)^2}\frac{[z(\overline{s}_m)-E_1(Z|s\leq\overline{s}_m)]^2}{\mathcal{V}_1(Z|s\leq\overline{s}_m)} < 1-\frac{\frac{n_m^2-2n_m+2}{(n_m-1)^2}\mathcal{V}_1(Z|s\leq\overline{s}_m)}{\left[\frac{\beta E_1(Z)}{\mu_\theta-\beta}+E_1(Z|s\leq\overline{s}_m)\right][z(\overline{s}_m)-E_1(Z|s\leq\overline{s}_m)]}$$

It follows from $z'(\cdot)>0$ that $E_1(Z)\geq E_1(Z|s\leq\overline{s}_m)$ and $[z(\overline{s}_m)-E_1(Z|s\leq\overline{s}_m)] > \frac{\mathcal{V}_1(Z|s\leq\overline{s}_m)}{E_1(Z|s\leq\overline{s}_m)}$, so a sufficient condition for $\frac{d\mathcal{V}_1(W_m)}{dn_m} < \frac{2\mathcal{V}_1(W_m)}{n_m(n_m-2)}$ is:

$$\frac{[z(\overline{s}_m)-E_1(Z|s\leq\overline{s}_m)]^2}{\mathcal{V}_1(Z|s\leq\overline{s}_m)} < \left(n_m^2-2n_m+2\right)\frac{\beta}{\mu_\theta}-1$$

The right-hand side is increasing in n_m so, with $n_m\geq 3$, it will be enough to have:

$$\frac{z(k)-E_1(Z|s\leq k)}{\sqrt{\mathcal{V}_1(Z|s\leq k)}} < \sqrt{\frac{5\beta}{\mu_\theta}-1}, \forall k\in(0,S]$$

with $\beta > \frac{\mu_\theta}{5}$.

Appendix C – Alternative Timing

Suppose the timing is such that issuers post securities first, then investors choose markets. Market choice is still made before the realization of investor preference shocks, but now issuers can compete for investors through security design. By posting securities first, we mean that the issuer commits to a particular payoff profile before investors choose their markets. The issuer is rational so his security design problem will take into account the best responses of investors. However, the issuer cannot post a security whose payoff profile is contingent on the number of investors who show up. That would constitute a customized contract, not a standardized contract. The focus of our paper is on standardized contracts.

Consider two issuers, 1 and 2. Issuer 1 offers a security W_1 and attracts n_1 investors. Issuer 2 offers a security W_2 and attracts $N - n_1$ investors.

The expected value to investor i of trading W_m in a market of size n_m is still given by $E_0\left(V_m^i\right)$ in Eq. (16). In the extreme case of $\sigma_\theta^2 = 0$:

$$E_0\left(V_m^i\right) = \frac{\gamma}{2}\frac{n_m}{n_m - 2}\mathcal{V}_1\left(W_m\right) \tag{C.1}$$

By a continuity argument, all results derived under $\sigma_\theta^2 = 0$ will extend to $\sigma_\theta^2 \in \left(0, \overline{\overline{\sigma}}\right)$, where $\overline{\overline{\sigma}}$ is some positive upperbound.

Notice that $\frac{n_m}{n_m - 2}$ in Eq. (C.1) is decreasing in n_m. Also recall that W_m is no longer responsive to n_m at the stage where investors choose their markets. Eq. (C.1) says that investors want a more variable security when σ_θ^2 is low. This is because the trading equilibrium delivers a low enough price (or, equivalently, a high enough risk premium) to compensate them for taking the risk. Investors also want to take this risk in very small markets, reflecting the fact that the risk premium increases with an individual investor's price impact.

Given the securities W_1 and W_2, investors will move around until they are indifferent between the two issuers. We abstract from the integered nature of investors here to avoid unnecessary algebra. The best response of investors then yields a market structure characterized by n_1^*, where n_1^* solves:

$$\frac{n_1^*}{n_1^* - 2}\mathcal{V}_1\left(W_1\right) = \frac{N - n_1^*}{N - n_1^* - 2}\mathcal{V}_1\left(W_2\right) \tag{C.2}$$

Eq. (C.2) defines n_1^* as a function of $\frac{\mathcal{V}_1(W_1)}{\mathcal{V}_1(W_2)}$. Differentiate Eq. (C.2) to get:

$$\frac{dn_1^*}{d\mathcal{V}_1(W_1)} = \frac{n_1^*}{2} \frac{1}{\frac{1}{n_1^*-2} + \frac{n_1^*}{(N-n_1^*-2)(N-n_1^*)}} \frac{1}{\mathcal{V}_1(W_1)}$$

This derivative is positive. If issuer 1 posts a more variable security than issuer 2, then issuer 1 will attract more investors.

Each issuer seeks to maximize his expected profit subject to a state-by-state feasibility constraint on the payoffs of the security he designs. He still offers one unit of the security to each investor in his market and, as in Appendix B, pays a cost to procure the assets that back this security. The Lagrangian for issuer 1's problem is thus given by Eq. (B.1) but with $n_1 = n_1^*$, where n_1^* depends on W_1 as per Eq. (C.2). The choice variables are the payoffs $w_1(s)$ for each state $s \in [0, S]$ and the number of units A_1 of Z that will back the n_1^* units of W_1.

The first order condition for $w_1(s)$ is:

$$\begin{aligned} \upsilon(s) &= \mu_\theta - \beta - \frac{\gamma}{n_1^*-2}\left(2(n_1^*-1) + \frac{(n_1^*)^2 - 4n_1^* + 2}{1 + \frac{n_1^*(n_1^*-2)}{(N-n_1^*-2)(N-n_1^*)}}\right)[w_1(s) - E_1(W_1)] \\ &\quad + \frac{1}{\frac{1}{n_1^*-2} + \frac{n_1^*}{(N-n_1^*-2)(N-n_1^*)}} \frac{w_1(s) - E_1(W_1)}{\mathcal{V}_1(W_1)}\left[(\mu_\theta - \beta) E_1(W_1) - \int_0^S \upsilon(s) w_1(s) dF(s)\right] \end{aligned}$$

Multiply both sides by $w_1(s)$ then integrate over $s \in [0, S]$ to isolate:

$$\int_0^S \upsilon(s) w_1(s) dF(s) = (\mu_\theta - \beta) E_1(W_1) - \frac{\gamma}{n_1^*-2}\left(\frac{2(n_1^*-1) + \frac{(n_1^*)^2-4n_1^*+2}{1+\frac{n_1^*(n_1^*-2)}{(N-n_1^*-2)(N-n_1^*)}}}{1 + \frac{1}{\frac{1}{n_1^*-2}+\frac{n_1^*}{(N-n_1^*-2)(N-n_1^*)}}}\right)\mathcal{V}_1(W_1)$$

We can now rewrite the first order condition for $w_1(s)$ as:

$$\upsilon(s) = \mu_\theta - \beta - \frac{\gamma n_1^*}{n_1^*-2} \frac{n_1^* - 2 + \frac{2(n_1^*-1)(n_1^*-2)}{(N-n_1^*-2)(N-n_1^*)}}{n_1^* - 1 + \frac{n_1^*(n_1^*-2)}{(N-n_1^*-2)(N-n_1^*)}}[w_1(s) - E_1(W_1)] \tag{C.3}$$

The first order condition for A_1 still takes the form of (B.5).

In a symmetric equilibrium, both issuers offer the same security W. Eq. (C.2) implies $n_1^* = \frac{N}{2}$ which, when substituted into Eq. (C.3), implies:

$$v(s) = \mu_\theta - \beta - \gamma \frac{N^2 - 8}{N(N-4)} [w(s) - E_1(W)]$$

for each $s \in [0, S]$. Therefore, the security that prevails in a symmetric equilibrium has payoffs:

$$w(s) = \begin{cases} \frac{2A}{N} z(s) & \text{if} \quad s < \overline{s} \\ \frac{2A}{N} z(\overline{s}) & \text{if} \quad s \geq \overline{s} \end{cases}$$

where the threshold $\overline{s} \in [0, S]$ is defined by:

$$\overline{s} = \arg \min_{k \in [0,S]} \left| z(k) - \frac{N}{2A} \left(E_1(W) + \frac{\mu_\theta - \beta}{\gamma} \frac{N(N-4)}{N^2 - 8} \right) \right| \tag{C.4}$$

and A solves:

$$A = \frac{\beta E_1(Z) + (\mu_\theta - \beta) \int_0^{\overline{s}} z(s)\, dF(s)}{\delta + \frac{2\gamma(N^2-8)}{N^2(N-4)} \left[\int_0^{\overline{s}} (z(s))^2\, dF(s) - \left(\int_0^{\overline{s}} z(s)\, dF(s) \right)^2 - \int_0^{\overline{s}} z(s)\, dF(s) \int_{\overline{s}}^{S} z(\overline{s})\, dF(s) \right]} \tag{C.5}$$

If the solution to Eq. (C.4) is interior, we can combine Eq. (C.4) and (C.5) to get:

$$\frac{\beta E_1(Z)}{\mu_\theta - \beta} \int_0^{\overline{s}} [z(\overline{s}) - z(s)]\, dF(s) + \int_0^{\overline{s}} z(s) [z(\overline{s}) - z(s)]\, dF(s) = \frac{\delta}{2\gamma} \frac{N^2(N-4)}{N^2 - 8}$$

We then need:

$$\delta < \frac{2\gamma}{\mu_\theta - \beta} \left[\mu_\theta \left[z(S) E_1(Z) - E_1\left(Z^2\right) \right] + \beta \mathcal{V}_1(Z) \right] \frac{N^2 - 8}{N^2(N-4)}$$

for the solution to indeed be interior, in which case:

$$\frac{d\overline{s}}{dN} = \frac{\delta}{2\gamma} \frac{N\left(N^3 - 24N + 64\right)}{\left(N^2 - 8\right)^2} \frac{1}{z'(\overline{s}) \int_0^{\overline{s}} \left[z(s) + \frac{\beta E_1(Z)}{\mu_\theta - \beta} \right] dF(s)} > 0$$

where the inequality follows from $N \geq 6$ to ensure $\frac{N}{2} \geq 3$. Therefore, the alternative timing considered here does not change the result that debt is traded in smaller markets than equity.