DISCUSSION PAPER SERIES

DP13181

ABOUT THE ORIGIN OF CITIES

André de Palma, Yorgos Y. Papageorgiou, Jacques-François Thisse and Philip Ushchev

INTERNATIONAL TRADE AND REGIONAL ECONOMICS

ISSN 0265-8003

ABOUT THE ORIGIN OF CITIES

André de Palma, Yorgos Y. Papageorgiou, Jacques-François Thisse and Philip Ushchev

Discussion Paper DP13181
Published 16 September 2018
Submitted 16 September 2018

Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

ABOUT THE ORIGIN OF CITIES

Abstract

We provide a bare--bones framework that uncovers the circumstances which lead either to the emergence of equally-spaced and equally-sized central places or to a hierarchy of central places. We show how these patterns reflect the preferences of agents and the efficiency of transportation and communication technologies. Under one class of agents, the economy is characterized by a uniform distribution or by a periodic distribution of central places having the same size. Under two asymmetric classes of agents, the interaction between agents may give rise to a hierarchy of settlements with one or several primate cities.

JEL Classification: N/A

Keywords: N/A

André de Palma - andre.depalma@ens-cachan.fr
ENS Cachan, CREST, University Paris-Saclay

Yorgos Y. Papageorgiou - yorgos@mcmaster.ca
McMaster University

Jacques-François Thisse - jacques.thisse@uclouvain.be
Higher School of Economics (St Petersburg) and CEPR

Philip Ushchev - ph.ushchev@gmail.com
National Research University Higher School of Economics

About the Origin of Cities*

André de Palma† Yorgos Y. Papageorgiou‡

Jacques-François Thisse§ Philip Ushchev¶

August 21, 2018

Abstract

We provide a bare–bones framework that uncovers the circumstances which lead either to the emergence of equally-spaced and equally-sized central places or to a hierarchy of central places. We show how these patterns reflect the preferences of agents and the efficiency of transportation and communication technologies. Under one class of agents, the economy is characterized by a uniform distribution or by a periodic distribution of central places having the same size. Under two asymmetric classes of agents, the interaction between agents may give rise to a hierarchy of settlements with one or several primate cities.

Keywords: central place, spatial externality, congestion, urban hierarchy

JEL Classification: R12; R14

*We wish to thank G. Duranton, two referees, H. Koster, L. Limonov, Y. Murata, the participants of the ITEA conference (Madrid 2017), and of the Interdisciplinary Leontief Centre seminar (Saint Petersburg 2018) for their comments. The first author would like to thank ANR (Elitisme) for financial support. The last two authors acknowledge the financial support of the Russian Science Foundation under the grant N°18-18-00253.

†ENS Cachan , CREST, University Paris-Saclay, France. andre.depalma@ens-cachan.fr

‡McMaster University, Hamilton, Ontario L8S 4K1, Canada. yorgos@mcmaster.ca

§CORE-UCLouvain, National Research University Higher School of Economics and CEPR. jacques.thisse@uclouvain.be

¶National Research University Higher School of Economics. ph.ushchev@gmail.com

1 Introduction

Despite the impact of variations in the natural landscape, cities can form a hierarchical system that shows some unexpected regularity (Marshall, 1989, chap. 5; Hohenberg and Lees 1985, chap. 2). This begs the following two questions. First, what are the conditions for systems of cities and urban hierarchies to emerge? Second, how can this process be derived within a microeconomic model from the interactions among agents pursuing their own interests? In response to these challenging questions, central place theory and new economic geography compete for adherents among economists and regional scientists.

Central place theory, based on the pioneering work of Christaller (1933) and Lösch (1940), is at both the core and periphery of spatial economics. On the one hand, it is at the core because it aims to explain why economic activities are distributed over a system of cities—or central places—in which the number of activities performed in a city rises with its size, and where cities having the same size are equally spaced (Mulligan *et al.*, 2012). On the other hand, it is at the periphery because the bulk of central place theory has been directed towards identifying geometric conditions under which a superposition of regular structures is possible. Here lies the Achilles Heel of central place theory. These considerations are only interesting if they are based on solid microeconomic foundations. Apart from a few contributions (see the literature review below), such geometric analyses have not been rationalized through the choices made by optimizing agents.

By contrast, new economic geography has the opposite problem. It relies on a full-fledged general equilibrium setup (Krugman, 1991; Fujita *et al.*, 1999a). However, since most models typically use a two-region setting, they do not permit to study the emergence of multiple settlements. For this reason, they are not able to deal with the formation of a hierarchy of settlements. Therefore, new economic geography has a limited applicability to modeling urban systems.

The term *central place* is used in this paper as shorthand for sites that accommodate a "high" number of agents. Our aim is to show how a pattern of equally-spaced central places having the same size and a hierarchy of central places, characterized by the existence of large and small cities, may emerge from a symmetry-breaking process (Matsuyama, 2008). To do this, we follow the main thread of spatial economics by assuming that the distribution of agents is the outcome of the interplay between an agglomeration force and a dispersion force. However, we differ from the gallery of existing models in one major respect. While these models emphasize a wide range of specific effects which are seldom combined within the same framework, we do not specify the details of a local congestion effect, which acts as a dispersion force, and of a global spatial externality, which generates the agglomeration force. In other words, both effects are captured by *reduced forms* that can account for different interpretations.[1]

In this paper, we build on Papageorgiou and Smith (1983) who maintained that the spatial distribution of identical individuals is determined by their attitudes regarding spatial interaction and by the properties of the spatial interaction field itself. Their analysis was limited to the necessary and sufficient conditions that render a uniform distribution of identical agents unstable. Our scope is much broader in that we determine the necessary and sufficient conditions for the emergence of uneven population patterns, which we are able to characterize. In order to achieve our goal, we consider a bare–bones framework in which agents' utility is negatively affected by a congestion effect that depends on the local population density, and positively affected by a spatial externality that depends on the whole distribution of agents over space. It is worth stressing here that our setting differs from most urban system models, which assume away spatial frictions across cities whose locations are therefore unspecified (see, e.g., Henderson, 1974, 1988; Behrens *et al.*, 2014; Davis and Dingel, 2019).

We consider a setting with continuous distributions of agents over the simplest possible, featureless, and unbounded space. In such a context, the spatial equilibria obtained are the solutions of Fredholm integral equations (Kolmogorov and Fomin, 2012), a mathematical concept which is seldom used in economics. Our main findings are as follows. In the case of one class of identical agents, we show that, when agents put a relatively high weight on spatial interaction as opposed to the local congestion effect, *the population is distributed unevenly over equally-spaced central places.* Or, to put it differently, the economic landscape is formed by the concatenation of identical human settlements. How packed are these places depends on the attitude of agents toward congestion and spatial interaction. The transition from the uniform distribution to an uneven distribution, which is reminiscent of the early stages of urbanization, is welfare-enhancing because agents benefit from a higher level of spatial interaction. However, a growing population leads to a lower welfare level when the equilibrium outcome involves a dispersed population because in this case the congestion effect is dominant. By contrast, the equilibrium utility level increases when central places emerge.

We also show that an improvement in spatial interaction possibilities has a non-monotone impact on the density of activities. Initially, as the interactions among agents become easier, the agglomeration of agents takes the form of a denser system of central places. However, beyond some

[1] For surveys of the various agglomeration and dispersion forces used in the literature, see Baldwin *et al.* (2003), Duranton and Puga (2004), Fujita and Thisse (2013), and Behrens and Robert-Nicoud (2015).

threshold, the additional gains associated with growing interactions are more than compensated by the costs generated by a rising congestion. In other words, technological devices that make interactions among agents easier lead, first, to a more compact system of central places and, then, to a sparse packing of activities on a more dispersed system. This is reminiscent of the bell-shaped curve of spatial development, which states that agents are dispersed when transport costs are either high or low and agglomerated when transport costs take on intermediate values (Krugman and Venables, 1995; Tabuchi, 1998; Puga, 1999; Fujita and Thisse, 2013). In those papers like in ours, *there is more agglomeration when spatial frictions are neither too strong nor too weak.*

The most enduring problem in spatial economics is probably the existence of an urban system involving large and medium-sized cities, towns, and villages. To deal with this problem, we must extend our baseline model in which central places have the same size. A natural extension is to introduce heterogeneity across agents by working with *two populations* such that agents are homogeneous within each population but heterogeneous between populations. This implies a complete spatial interdependence between the two populations. In this context, we show that *a hierarchy of tiered central places may emerge as an equilibrium outcome.* Furthermore, the share of each population in a central place varies with its location. Thus, we may conclude that the agglomerations are here the outcome of a richer pattern of interactions than in the case of a single population.

Depending on the type of interaction between the two populations and the parameters that characterize each population, the spatial economy may display the following patterns. In the first one, the spatial equilibrium involves two uniformly distributed populations. In this case, the interactions between the two populations are too weak for the blending of the two populations to generate a steady state that departs from what we observe with each population separately. The second pattern involves a periodic distribution. The third pattern leads to richer implications. Indeed, the superimposition of the two populations yields *an urban system in which one, two, or several central places are at the top of the urban system.* Although the distribution of the total population may exhibit one primate city and smaller cities whose size decreases with the distance to the primate city, as argued by Christaller and Lösch, the urban system may also involve several large cities, together with intermediate cities whose size does not decrease monotonically with the distance to the large cities. Thus, despite its great simplicity, our setting allows us to account for a rich set of urban patterns. In particular, it provides a background for two of the main features of central place theory: (i) a pattern of equidistant and identical settlements and (ii) a hierarchy of cities. However, our analysis also shows that there is no reason to expect the urban system to obey the rigid, pyramidal structure assumed by Christaller (1933) and Lösch (1940). Instead, the urban system may involve two or several very large urban agglomerations at the top of the urban hierarchy. To the best of our knowledge, no existing contribution has been able to generate similar results within a microeconomic framework as parsimonious as ours.

Related literature. Beckmann (1976) also shows that land use and social interactions give rise to a bell-shaped distribution of individuals over an interval whose limits are exogenous. In contrast to us, there is no transition from perfect dispersion to a regular pattern of cities in Beckmann. More importantly, his result seems to be driven by the assumption of a compact interval, as the peak tends to vanish when this interval becomes arbitrarily wide. Mossay and Picard (2011) revisit Beckmann's model when agents are distributed over a circle and show that the equilibrium involves multiple outcomes, which depend on the length of the circumference, having any odd number of identical and evenly spaced cities. Thus the choice of a mathematical space, characterized by specific boundary conditions, matters for the nature of the equilibrium outcome. Here, we work with the simplest possible seamless space, i.e., the real line which is unbounded. We determine the conditions that force a perfectly dispersed population to become a regular or a hierarchical pattern of settlements defined on the real line.

Eaton and Lipsey (1982) appealed to spatial competition theory and multipurpose shopping as an economic foundation for the existence of clusters in which spatially dispersed consumers buy different goods available in dispersed shops. That consumers combine their purchases to reduce travel costs creates demand externalities that firms can exploit by locating with firms selling other goods. Unfortunately, the spatial competition approach becomes very quickly intractable. Indeed, with several goods, trip-chaining implies a particular structure of substitution between outlets that makes it hard for a consumer to determine her optimal spatial structure of purchases. As a result, firms' demands are very complex and intricate, which renders the market game virtually impossible to solve. Hsu (2012) proposes to circumvent this difficulty by assuming that firms deliver their products to consumers. Firms now compete in price schedules, where a price schedule is a function which specifies the delivered price at which a firm is willing to supply a consumer at a particular location. In this case, a consumer buys each product from the cheapest firm regardless of where his other providers are located. When firms supply one good and when consumers have a perfectly inelastic demand for this good, this setting involves a continuum of free-entry equilibria with equidistant firms. Assuming a continuum of goods, Hsu (2012) shows that there exists an equilibrium that displays the hierarchical structure of regular central places, that is, all the goods available in a central place of a given order are also available in all central places of higher order.[2]

Models of economic geography developed in the wake of Krugman (1991) explain the emergence of agglomeration as a symmetry-breaking mechanism. This mechanism relies on the interplay between increasing returns in production, transport costs, and preference for variety in a two-region setting (see Baldwin *et al.*, 2003, for a synthesis of the main results). Models dealing with a large number of locations include Fujita and Mori (1997), Fujita *et al.* (1999b), and Tabuchi and Thisse (2011). Fujita and Mori (1997) use an economic geography setting to show that cities are created at more or less equal distances when the total population grows in an economy with one manufacturing sector. Fujita *et al.* (1999b) extend this setting to a multisector economy and show how a hierarchy of central places may emerge. However, Berliant and Kung (2006) show that those models have a continuum of equilibria in the sense that the distance between cities is indeterminate. Tabuchi and Thisse (2011) also use an economic geography setting to prove that decreasing transportation costs foster the emergence of big cities supplying two goods, which coexist with small and specialized cities in which only one good is produced.[3] This brief overview shows that our approach vastly differs from the existing literature.

In the next section, we develop our baseline model with homogeneous agents and discuss various interpretations of the agglomeration and dispersion forces. We study the equilibrium outcome and determine the necessary and sufficient conditions for a regular pattern of identical settlements to emerge in a world which is otherwise featureless. In Section 3, we consider the case of two classes of agents who interact both within and between classes. Superimposing the two individual distributions shows how a hierarchy of cities hosting agents belonging to both populations comes into being. Section 4 concludes.

2 Agglomeration in a homogeneous world

Our physical space is one-dimensional, featureless, and unbounded. Formally, we represent it as the real line $\mathbb{R} \equiv (-\infty, \infty)$, with locations $x, y \in \mathbb{R}$. The real line is the simplest possible homogeneous one-dimensional space on which we may expect to obtain a regular pattern of agglomerations. We start with a continuum of identical agents distributed over $\mathbb{R}$, with a population density denoted

[2] Quinzii and Thisse (1990) and Hsu *et al.* (2014) show, respectively, the optimality of a system of central places in each of these two settings.

[3] Henderson (1974, 1988) has developed a compelling approach to describe the formation of urban systems. However, shipping goods between cities is assumed to be costless, so that cities are not anchored in specific locations.

by $n(x) \geq 0$.

A *population density* is defined by a mapping from $\mathbb{R}$ to $\mathbb{R}_+$. We assume that $n(x)$ is twice continuously differentiable, bounded, and such that its mean $\overline{n}$ is given by

$$\overline{n} \equiv \lim_{b\to\infty} \frac{1}{2b} \int_{-b}^{b} n(x)\mathrm{d}x < \infty.$$

In other words, we focus on distributions such that the mean population density $\overline{n}$ across the entire space is finite. Clearly, the uniform density, $n(x) = \overline{n}$ for all $x \in \mathbb{R}$, satisfies these conditions.

Each agent stands at the center of a spatial interaction field with others. That humans are "social animals" is perhaps the most basic justification of the need for interaction among individuals. Likewise, an individual often learns more by being nested in a large pool of agents. In this case, the flow describes the amount of information and knowledge transferred between two agents. In line with the literature, we assume that the interaction flow between individuals located, respectively, at x and y decays by a factor of $\varphi(|x-y|)$, where $\varphi(\cdot)$ is a continuous and strictly decreasing function of the distance between x and y. Things work as if an individual were at the source of a flow which diffuses at a decreasing rate across space. At the same time, an individual at x receives an interaction flow emitted by those established at y, which is equal to $\varphi(|x-y|)n(y)$. Hence, the bilateral interactions between x and y, which are given by $n(x)n(y)\varphi(|x-y|)$, are gravitational in nature. If, in addition, the decay function φ is integrable over $\mathbb{R}_+$, the constant

$$G \equiv 2\int_0^{\infty} \varphi(z)\mathrm{d}z > 0, \tag{1}$$

reflects the global intensity of the spatial interaction field.[4]

In this paper, interactions across agents are expressed through a *spatial externality* generated by the sum of each agent's interactions with all the others in $\mathbb{R}$. More specifically, the spatial externality at x is finite and defined as follows:

$$E(x) = \int_{\mathbb{R}} \varphi(|x-y|)n(y)\mathrm{d}y > 0, \tag{2}$$

which is determined by the population density at that location. Since $n(\cdot)$ is bounded, $E(x)$ is finite for all x. We also assume that the impact of those in y to someone in x is given by an *exponential decay function* (see, e.g., Fujita and Ogawa, 1982; Lucas and Rossi-Hansberg, 2002; Desmet and Rossi-Hansberg, 2013):

$$\varphi(|x-y|) = \exp\left\{-\beta\,|x-y|\right\}, \tag{3}$$

[4]There exists a rich literature on social networks, which could be used to provide a wide range of justifications for what we call an interaction field (see, e.g., Jackson *et al.*, 2017).

where $\beta > 0$ is a spatial impedance parameter. Since $\exp\{-\beta |x-y|\} \approx 1-\beta |x-y|$ when y is close to x, β is an inverse measure of the spatial frictions that reduce the interactions among spatially dispersed agents. A high (low) value of β means that the impact of distance on the intensity of the externality is strong (weak), thus implying that the spatial externality is mainly local (global). Using (1) and (3), we obtain $G = 2/\beta$, which shows that the intensity of the interaction field decreases hyperbolically with β. Furthermore, (2) shows that the distribution of agents affects the value of $E(x)$.

Observe that $\exp\{-\beta |x-y|\}$ also has the nature of a distance-increasing iceberg transport cost (Fujita *et al.*, 1999a). Therefore, $E(x)$ may be re-interpreted as the *market potential* of location x. To put it differently, the total amount is the aggregate consumption of a continuum of varieties differentiated by the places where they are produced (Armington, 1969), while the consumption of a specific variety decreases with the distance between its place of origin and its destination (Head and Mayer, 2004).

The utility function of an agent at x, denoted by $v[n(x), E(x)]$, is the same across agents and depends on two variables: the population density $n(x)$ at the agent's location and the spatial externality $E(x)$ this agent enjoys from interacting with the other individuals. Intuitively, it is reasonable to assume that $v[n(x), E(x)]$ decreases with the local density $n(x)$, perhaps because agents compete locally for land or for buyers on the product market, so that agents' well-being decreases when the local population density rises, and increases with $E(x)$ that has by assumption the nature of a positive spatial externality. For example, if the population distribution is bell-shaped, individuals located near the maximizer of the distribution enjoy the benefits of a high potential of interactions because distances are shorter, but also bear the high costs associated with a denser local population.

To keep the analysis simple, let $v[n(x), E(x)]$ be linear in both variables:

$$v[n(x), E(x)] = E(x) - \alpha n(x), \tag{4}$$

where the parameter $\alpha > 0$ is the constant marginal rate of substitution between an increase in spatial externality and a decrease in local density. A high (low) value of α means that agents focus more (less) on what is going on at the local level than on the various benefits generated by the interactions with the rest of the economy.

We acknowledge that the general is preferable to the particular. However, it is impossible to pin down the spatial equilibrium without using a specific functional form for preferences. In this respect, a linear utility allows us to determine explicitly the equilibrium density for all admissible values of the parameters of the model. Despite its great simplicity, the expression (4) captures the trade-off between the benefit generated by spatial interaction with the entire population and the cost associated with local congestion, which corresponds to the fundamental trade-off of spatial economics.[5]

At the *spatial equilibrium* $n^*(x)$, all agents enjoy the same utility level v^*. Set

$$E^*(x) \equiv \int_{\mathbb{R}} \varphi(|x-y|)n^*(y)\mathrm{d}y.$$

Formally, we have

$$v[n^*(x), E^*(x)] = v^* \tag{5}$$

for any x such that $n^*(x) \geq 0$, while

$$v[n^*(x), E^*(x)] < v^*$$

implies $n^*(x) = 0$. Despite its simplicity, our setting captures a fundamental feature of spatial economics: the most preferred location of an agent depends on where the others are set up.

Our results are driven by the interplay between the parameters α and β. Our setting is thus very parsimonious. In addition, we often use the sole parameter:

$$\phi \equiv \frac{2}{\alpha\beta} > 0. \tag{6}$$

Roughly speaking, ϕ blends the two forces that affect agents' well-being; ϕ is low when the spatial decay factor β is high, and thus the spatial externality very localized, the marginal disutility of congestion α is high, or both. Intuitively, a low value of ϕ is associated with an economy in which individual agents "think and act locally," while a high value of ϕ is associated with agents who are globally oriented. The communication technology and the attitude of agents toward local and global interactions are critical for the nature of the spatial equilibrium.

Combining (5) with (2) and (4) shows that any spatial equilibrium is a non-negative solution to

$$n(x) = \frac{1}{\alpha}\left[\int_{\mathbb{R}} \varphi(|x-y|)n(y)\mathrm{d}y - v^*\right]. \tag{7}$$

Equation (7) is known as a Fredholm integral equation of the second kind because the unknown function $n(x)$ appears inside and outside the integral (Kolmogorov and Fomin, 2012, ch. 2, p. 74). The following proposition characterizes the solution to (7).

[5]Using linear utility functions is fairly standard in urban economics. An example of such preferences is given by the following indirect utility: $w(x) - s(x)R(x)$, where $w(x)$ is the wage, $s(x)$ the land consumption, and $R(x)$ the land rent at x.

The following result characterizes the spatial equilibrium in the case an exponential decay function.

Proposition 1. *Assume the utility function* (4) *and the spatial externality* (2)-(3).

(i) *The uniform distribution of agents is a spatial equilibrium. This equilibrium is unique and stable if and only if* $\phi < 1$.

(ii) *If* $\phi \geq 1$, *then the population density*

$$n^*(x) = \overline{n} + A \sin\left(\beta\sqrt{\phi - 1}\, x\right), \tag{8}$$

where A *is an arbitrary constant such that* $-\overline{n} \leq A \leq \overline{n}$, *is a spatial equilibrium.*

(iii) *At all spatial equilibria, the equilibrium utility level is given by*

$$v^* = \overline{n}\left(\frac{2}{\beta} - \alpha\right). \tag{9}$$

We give here a sketch of the proof; details are relegated to Appendix A.

(a) When $\phi < 1$, it can be shown that the integral operator in the right-hand side of (7) is a contraction mapping. Therefore, the Banach fixed-point theorem implies that the uniform density is the unique solution to (7) (Kolmogorov and Fomin, 2012, ch. 2, p. 66). This proves part (i).

(b) When $\phi > 1$, the equation (7) is equivalent to a special case of the Sturm-Liouville problem in which the differential operator is $-\mathrm{d}^2/\mathrm{d}x^2$ (Dunford and Schwartz, 1988, ch. XIII, p. 1291). The eigenfunctions of this operator are given (up to adding $\overline{n}$) by the right-hand side of (8). It remains to study the borderline case where $\phi = 1$. Taking pointwise the limit under $\phi \to 1$ on both sides of (8), we obtain $n^*(x) = \overline{n}$.

The population density is maximized at the *central places* given by

$$x^k_{\max} = \pm\left(\frac{1}{2} + 2k\right)\frac{\pi}{\beta}\sqrt{\frac{1}{\phi - 1}} \qquad k = 0, 1, ...,$$

It is minimized at

$$x^k_{\min} = \pm\left(\frac{3}{2} + 2k\right)\frac{\pi}{\beta}\sqrt{\frac{1}{\phi - 1}} \qquad k = 1, 2, ...,$$

and equal to the mean density at

$$\bar{x}^k = \pm\left(1 + 2k\right)\frac{\pi}{\beta}\sqrt{\frac{1}{\phi - 1}} \qquad k = 0, 1, ...$$

It follows from Proposition 1 that central places have the same size and are equally spaced. As the distance to a central place increases, the population density decreases, a pattern that concurs with what urban economics predicts. The limit of the urban area is reached at the distance $\left|x^k_{\max} - x^k_{\min}\right|$

from the central place. When the distance exceeds this threshold, the population density starts rising and takes again its highest value at the next central place. To be precise, the peaks and troughs of the spatial distribution of agents are distributed with a period equal to

$$T = \frac{2\pi}{\beta}\sqrt{\frac{1}{\phi - 1}}.$$

Therefore, when ϕ takes values larger than 1, the population density displays ups and downs that are periodically distributed. In other words, *the economic landscape is formed by a succession of central places whose area is equal to the period T.*

The value of T is an inverse measure of settlement density in the economy. Indeed, a decrease in T leads to a denser packing of central places over space. The distance between two central places increases when local congestion matters less to individuals (α decreases). In this case, agents are closer to each other to benefit more from the spatial externality. The impact of the spatial decay factor β is more involved. Differentiating T with respect to β shows that the distance between two neighboring central places first decreases and then increases when β increases from 0 to ∞, where the turning point is reached at $\beta = \alpha$. Indeed, when β is very large (or very small), individuals focus on their local environment (or benefit more or less equally from the others regardless of their locations). In both cases, each agent cares less about where the others are located. By contrast, when β takes on intermediate values, the intensity of the spatial externality depends strongly on the whole distribution of agents. In short, the packing of activities is denser for intermediate values of the decay parameter of the spatial externality function. Note also that the central places are pushed toward $+\infty$ or $-\infty$ when $\phi \to 1$. Thus, like in Krugman (1991), the process of agglomeration is the result of a symmetry-breaking mechanism that arises at $\phi = 1$. However, unlike in Krugman, since $\left|x^k_{\max}\right|$ is arbitrarily large when ϕ is arbitrarily close to 1, the transition from the uniform pattern to a landscape with central places is here continuous.

When the communication technology is inefficient (β is high), or agents are much affected by local congestion (α is high), or both ($\phi < 1$), *the world is flat.* Furthermore, (9) implies that the utility level decreases when the average population grows because the congestion effect is sufficiently strong for a denser population to make individuals worse-off. This finding agrees with historical evidence: before the onset of permanent settlements people remained dispersed in small hunter–gatherer bands because land could not support their way of life in large numbers, which implies that in early times $\phi < 1$ was satisfied.

With the passage of time, people gradually abandoned their nomadic life because food production generated higher returns per unit of land. Growth of social awareness, culture, kinship and religion in early humans increased the need for communication (extended the range of the spatial externality) and reduced the local effects of proximity until $\phi > 1$. Then, spatial uniformity was replaced by regular agglomerations representing the onset of early human settlements (the uniform distribution is no longer stable). Such settlements were forming by homogeneous populations well before 10,000 BC. In other words, when β became smaller than $2/\alpha$, from the historical point of view, *a seamless world organizes itself spontaneously as a system of identical central places*. Moreover, since $2/\beta - \alpha$ is now positive, such an agglomeration of the economy goes together with a rise in welfare. More specifically, *any shock that leads agents to agglomerate is welfare-enhancing*. In particular, a shock that makes individuals more prone to live together (α decreases) now leads to a higher welfare level. Similarly, a shock that facilitates spatial interaction (β decreases) also triggers a higher equilibrium utility level. Finally, an increase in population is now beneficial to all agents because the externality effect is dominant.

Part (ii) of Proposition 1 implies that there exists a continuum of spatial equilibria associated with the value of A. When $A = 0$, the spatial equilibrium is the uniform density. Since this equilibrium is unstable as shown by part (i), we may assume $A \neq 0$. Furthermore, as the choice of the origin is arbitrary, without loss of generality, we may assume that $A > 0$. Since the utility level is the same across spatial equilibria, we cannot use the principle of Pareto dominance as an equilibrium selection device. The period T , hence the location of central places, is independent of A while the overall shape of the equilibrium density remains the same for all $A \neq 0$. What changes with the value of A is the amplitude of the oscillations of the equilibrium density, that is, the size of settlements. Since the least populated areas in the real world have no or little activity ($n^*(x) \approx 0$), we find it intuitively appealing to focus on the case $A \approx \overline{n}$. This implies that the most populated areas have a density close to $2\overline{n}$ in the vicinity of the peaks of $n^*(x)$.

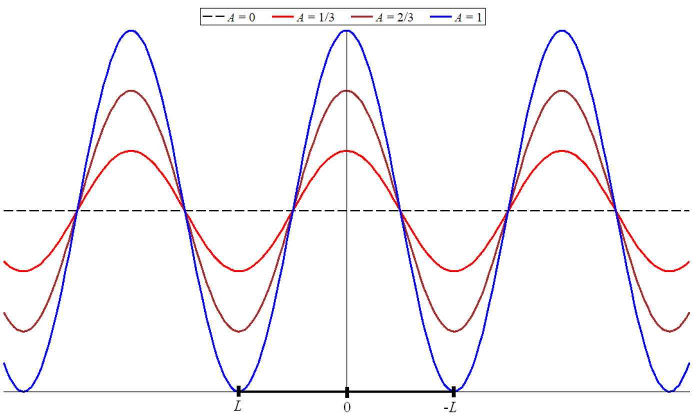

Fig. 1. Population densities in a homogeneous world

The results discussed above are illustrated by Figure 1, where have set $x^0_{\max} = 0$ without loss of generality since we work with a homogeneous space. The horizontal line represents the equilibrium density $n^*(x) = \bar{n}$ for $A = 0$, while $L = T/2$ is half the distance between two neighboring central places.

Food production in prehistoric times gradually accumulated food surpluses which, in turn, gradually freed some individuals from food production, created occupational specialties and accelerated social organization away from the roughly egalitarian norm of hunter–gatherer groups toward more complex schemes. Representing *specialization*, the mother of cities, requires at least two different populations. This is what we do in the next section, where we discuss how various urban hierarchies can emerge if we expand the current framework to account for two, rather than one, populations.

3 Hierarchy in a heterogeneous world

In this section, we show that hierarchy may emerge in the above setting when we consider two, rather than one, populations. In addition, we will see that alternative configurations may also arise at the equilibrium.

3.1 The model

Consider two populations specialized in different activities, which are distributed over $\mathbb{R}$ with densities $n_1(x)$ and $n_2(x)$, respectively. Both means $\overline{n}_1$ and $\overline{n}_2$ are finite. In this economy, the spatial distribution of both classes concerns everyone. This is reflected in the structure of the utility function of a j-type agent, $j = 1, 2$, which is given by:

$$v_j(x) = \gamma_{jj} E_{jj}(x) + \gamma_{jk} E_{jk}(x) - \alpha_{jj} n_j(x) - \alpha_{jk} n_k(x), \qquad j, k = 1, 2,\ j \neq k. \tag{10}$$

In (10), $E_{jk}(x)$ is the externality a j-type agent receives from k-type agents:

$$E_{jk}(x) \equiv \int_{-\infty}^{\infty} \exp\{-\beta_{jk}|x - y|\} n_k(y) \mathrm{d}y, \qquad j, k = 1, 2, \tag{11}$$

where $\beta_{jk} > 0$ are the spatial decay factors specific to j-type agents when they interact with k-type agents, where j may be equal to or different from k. In other words, the preferences (10) account for interactions within and between classes of agents.

One may think of very different types of interactions between the two populations. In (10), $\alpha_{jk} > 0$ is the marginal disutility a j-type agent suffers from the congestion associated with the fact of sharing the same local environment with the k-type agents, while $\gamma_{jk} > 0$ stands for the marginal utility a j-type agent gains from interacting with k-type agents. That α_{jk} is specific to each class of agents implies that the two classes of agents can generate various crowding effects. First, let us refer to the agents as *consumers* (type 1) and *firms* (type 2). In this case, consumers are attracted to places where the density of firms is high because there are more and better opportunities ($\alpha_{12} < 0$), while firms are attracted by places where consumers are numerous because there the expected volume of business is large ($\alpha_{21} < 0$); consumers are repulsed by places where the density of consumers is high because they dislike congestion ($\alpha_{11} > 0$), while firms dislike places when the density of firms is high because competition is localized and tough ($\alpha_{22} > 0$). Furthermore, we may also capture segregation behaviors by assuming $\alpha_{jj} < 0$ and $\alpha_{jk} > 0$, that is, j-type agents are attracted by their peers, but repelled by the members of the other population. Thus, depending on the sign of the parameters α_{jk}, (10) is able to account for a rich set of local interaction patterns.

Let $\mathbf{A}$ be the (2×2)-matrix

$$\mathbf{A} \equiv \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \tag{12}$$

of congestion disutilities. Without much loss of generality, we assume throughout the paper that this matrix is invertible, i.e., $\det(\mathbf{A}) \equiv \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$. On the other hand, it is reasonable to assume that each agent (weakly) benefits from interacting with the others, thus meaning $\gamma_{jk} \geq 0$ and $\gamma_{jj} \geq 0$ for $j, k = 1, 2$. If j-agents do not value interacting with k-agents, we have $\gamma_{jk} = 0$.

Very much like in the case of one type of agents, the equilibrium densities of agents must satisfy the following equilibrium conditions:

$$v_j[n_j^*(x), n_k^*(x), E_{jj}^*(x), E_{jk}^*] = v_j^*, \quad j = 1, 2, \tag{13}$$

where the equilibrium utility levels v_j^*, $j = 1, 2$, are two unknown constants.

We define the *mean* of the function $f(x) \in C(\mathbb{R})$, denoted by $\overline{f}$, as follows:

$$\overline{f} \equiv \lim_{b \to \infty} \frac{1}{2b} \int_{-b}^{b} f(x) \mathrm{d}x.$$

Combining the equilibrium conditions (13) with (10) and (11) and taking the deviations $\widetilde{n}_j(x) \equiv n_j(x) - \bar{n}_j$ from the mean on both sides, we obtain a system of two Fredholm integral equations:

$$\widetilde{n}_j(x) = \int_{\mathbb{R}} g_{jj}(x, y)\widetilde{n}_j(y)\mathrm{d}y + \int_{\mathbb{R}} g_{jk}(x, y)\widetilde{n}_k(y)\mathrm{d}y, \qquad j, k = 1, 2, \quad j \neq k, \tag{14}$$

where the own kernels $g_{jj}(x, y)$ and the cross kernels $g_{jk}(x, y)$ are defined—up to a coefficient equal to $1/\det(\mathbf{A})$—as follows:

$$g_{jj}(x, y) \equiv \alpha_{kk}\gamma_{jj}\exp\{-\beta_{jj}|x-y|\} - \alpha_{jk}\gamma_{kj}\exp\{-\beta_{kj}|x-y|\},$$

$$g_{jk}(x, y) \equiv \alpha_{kk}\gamma_{jk}\exp\{-\beta_{jk}|x-y|\} - \alpha_{jk}\gamma_{kk}\exp\{-\beta_{kk}|x-y|\}.$$

Note the difference between (14) and (7), as the latter only involves $n(x)$. By contrast, in (14), $n_j(x)$ also depends on $n_k(x)$. Hence, the distributions of the two populations are fully interdependent. The above expressions show that the spatial equilibrium depends on the nature and intensity of the *local* and *global* interactions *within* and *between* populations. We aim to determine what the economic landscape becomes when we superimpose the two distributions to form the distribution of the total population $n_1(x) + n_2(x)$. Note also there are two-way interactions between any two central places. Horizontal relations among central places are thus superimposed on the pyramidal structure of the urban system, like in Fujita *et al.* (1999b).

Repeating the argument of part (iii) in Appendix A, it is readily verified that

$$\overline{E}_{jk} = \frac{2}{\beta_{jk}}\overline{n}_k, \qquad j,k = 1,2,\ j \neq k. \tag{15}$$

Computing the mean of both sides of (10), we obtain:

$$v_j^* = \gamma_{jj}\overline{E}_{jj} + \gamma_{jk}\overline{E}_{jk} - \alpha_{jj}\overline{n}_j - \alpha_{jk}\overline{n}_k, \qquad j,k = 1,2,\ j \neq k. \tag{16}$$

Combining (15) and (16) yields *the equilibrium welfare level of j-type individuals*:

$$v_j^* = \left(2\frac{\gamma_{jj}}{\beta_{jj}} - \alpha_{jj}\right)\overline{n}_j + \left(2\frac{\gamma_{jk}}{\beta_{jk}} - \alpha_{jk}\right)\overline{n}_k, \qquad j,k = 1,2,\ j \neq k. \tag{17}$$

As in Section 2, the equilibrium welfare level v_j^* of a j-type agent is the same at all spatial equilibria. The expression (17), which is an extension of (9) to the case of two populations, has the following implication: depending on the values of the various parameters, the mean densities $\overline{n}_1$ and $\overline{n}_2$ have a positive or a negative impact on v_1^* and v_2^*. Indeed, (17) implies that the impact of an increase in $\overline{n}_j$ on v_k^* is positive if and only if the inequality

$$2\frac{\gamma_{jk}}{\beta_{jk}} > \alpha_{jk}$$

holds. This is so when the j-type individuals value much their interactions with the k-type individuals (γ_{jk} is large), when they have developed an efficient communication technology (β_{jk} is low), when the crossed congestion effect between the two populations is weak (α_{jk} is low), or when any combination of these conditions holds. On the other hand, if the opposite inequality holds, we have

$$\frac{\partial v_j^*}{\partial \overline{n}_k} < 0$$

because the congestion effect generated by the k-type individuals overcomes the benefit made from interacting with these individuals.

3.2 Equilibrium in a quasi-symmetric world

When the two populations are the mirror images of each other,[6] the distribution of the total population is uniform or periodic, as in Proposition 1. Therefore, for an urban hierarchy to arise, at least a few parameters characterizing the two populations must be different. However, the pattern of interactions associated with preferences (10) is too complex to allow for an intuitively appealing characterization such as the one obtained in Section 2. Indeed, in the asymmetric case, the number of parameters has increased from 2 to 10 (without loss of generality, one parameter may be normalized to 1 for each population). Despite this, we are able to characterize the different spatial configurations in Proposition 3. However, the results are difficult to interpret. This is why we first consider the special case of symmetric spatial impedances in Proposition 2. In what follows, we refer to this case as *quasi-symmetric.*

Consider the following simple case: (i) the preferences for local interactions depends on the agents of the same type and are population-specific, i.e., $\alpha_1 \equiv \alpha_{11} > 0$, $\alpha_2 \equiv \alpha_{22} > 0$, and $\alpha_{12} = \alpha_{21} = 0$; (ii) the intensity of the spatial externality both within and between populations is the same, i.e., $\beta \equiv \beta_{jk} > 0$, $j, k = 1, 2$; and (iii) the preferences for global interactions are the same within and between the two populations, i.e., $\gamma_{jk} = \gamma > 0$ for $j, k = 1, 2$. Thus, our quasi-symmetric setting involves 4 parameters, i.e., α_1, α_2, β, and γ. To insulate the effect of the selected parameters, we assume $\bar{n}_1 = \bar{n}_2 = 1$.

The following result is proven in Appendix B.

Proposition 2. *Assume that $\gamma < 1$, i.e., each agent values prefers to interact with members of her population than with members of the other population. Then, there exist a lower bound, $\underline{\beta}(\alpha_1, \alpha_2, \gamma) > 0$, and an upper bound, $\overline{\beta}(\alpha_1, \alpha_2, \gamma) > \underline{\beta}(\alpha_1, \alpha_2, \gamma)$, such that:*

(i) *under high spatial impedance, $\beta \geq \overline{\beta}(\alpha_1, \alpha_2, \gamma)$, there is a unique spatial equilibrium, and this equilibrium is such that both populations are uniformly distributed across space;*

[6] That is, $\alpha_{11} = \alpha_{22}$, $\alpha_{12} = \alpha_{21}$, while similar equalities hold for β and γ.

(ii) *under intermediate spatial impedance,* $\underline{\beta}(\alpha_1, \alpha_2, \gamma) \leq \beta < \overline{\beta}(\alpha_1, \alpha_2, \gamma)$*, the non-uniform equilibria involve central places having the same size;*

(iii) *under low spatial impedance,* $\beta < \underline{\beta}(\alpha_1, \alpha_2, \gamma)$*, the non-uniform equilibria involve central places of different sizes.*

In other words, we need two classes of agents to obtain a hierarchy of central places. More specifically, Proposition 2 shows that the spatial interdependence between the distributions of firms and consumers leads to the emergence of an urban hierarchy when the degree of spatial impedance is sufficiently low. Such a result is in the spirit of Proposition 1: weakening spatial frictions within and between populations favors the emergence of large urban agglomerations.

Figure 2 provides an illustration of Proposition 2 for the values $\alpha_1 = 0.25$, $\alpha_2 = 0.75$, $\beta = 1$, and $\gamma = 0.25$. Since $\underline{\beta}(\alpha_1, \alpha_2, \gamma) = 2.43$ and $\overline{\beta}(\alpha_1, \alpha_2, \gamma) = 8.24$, we are in case (iii) of Proposition 2 when $\beta < 2.43$. As shown by the black curve of Figure 2, which depicts the total population density $n_1^*(x) + n_2^*(x)$, the urban system involves *two top cities* (e.g., Beijing and Shanghai, Milan and Rome), as well as central places of lower orders (there are 4 levels corresponding to the local maxima). The densities $n_1^*(x)$, $n_2^*(x)$, and $n_1^*(x) + n_2^*(x)$ are symmetric about $x = 0$ over an interval (see $[-L, L]$ in Figure 2) beyond which the same urban pattern is repeated. That an infinity of primate cities exist over the real line should not as a surprise since our space is unbounded. Last, the size of lower order central places does not decrease monotonically with the distance to the primate cities: small cities may arise between two large cities.

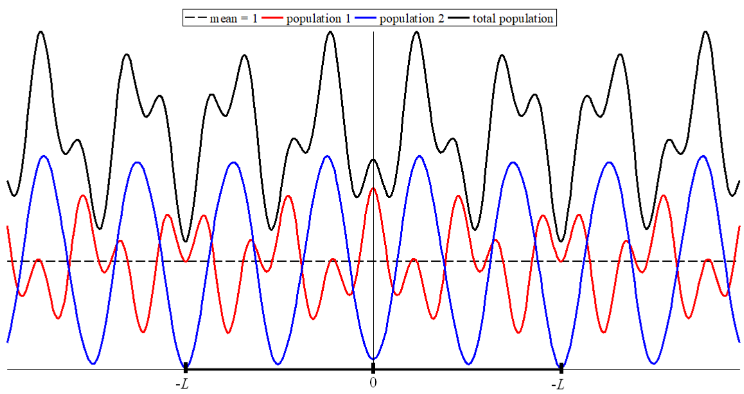

Fig. 2. Population densities in a quasi-symmetric world

To illustrate further, consider the case when 1-type agents are firms and 2-type agents are consumers. If consumers care more about local congestion than firms ($\alpha_1 < \alpha_2$), the red curve describes the equilibrium density of consumers, while the blue one is associated with firms. Since the former density involves more central places than the latter, consumers are more dispersed than firms across space, which seems plausible. In addition, each central place is *mixed* in that it hosts both firms and consumers, although the composition of a central place varies with its size.

3.3 Equilibrium in an asymmetric world

How far can we go beyond Proposition 2, or similar results? Solving the system (14) is a hard task because of technical complexity. Nevertheless, the following result provides a classification of possible spatial equilibria. For this, we need the following definitions. Let $\mathbf{D}$ be a (4×4)-matrix independent of x defined as follows:

$$\mathbf{D} \equiv \mathbf{Q} - 2\mathbf{B}\mathbf{A}^{-1}\mathbf{\Gamma},$$

where

$$\mathbf{Q} \equiv \begin{pmatrix} \beta_{11}^2 & 0 & 0 & 0 \\ 0 & \beta_{12}^2 & 0 & 0 \\ 0 & 0 & \beta_{21}^2 & 0 \\ 0 & 0 & 0 & \beta_{22}^2 \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} \beta_{11} & 0 \\ 0 & \beta_{12} \\ \beta_{21} & 0 \\ 0 & \beta_{22} \end{pmatrix}, \quad \mathbf{\Gamma} \equiv \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 & 0 \\ 0 & 0 & \gamma_{21} & \gamma_{22} \end{pmatrix},$$

while $\mathbf{A}$ is given by (12). The following result is proven in Appendix C.

Proposition 3. *Assume the utility functions* (10) *and the spatial externalities* (11).

(i) *If* $\mathbf{D}$ *has no strictly negative real eigenvalues, then the spatial equilibrium is unique and given by the uniform densities* $n_j(x) \equiv \overline{n}_j$, $j = 1, 2$;

(ii) *if* $\mathbf{D}$ *has one strictly negative real eigenvalue, then the non-uniform spatial equilibria are periodic;*

(iii) *if* $\mathbf{D}$ *has at least two strictly negative real eigenvalues, then the spatial equilibria involve different extrema.*

Though this proposition is difficult to interpret, it provides a useful guide to perform numerical exercises that allow one to gain more intuition about the nature of spatial equilibria. Figure 3 is drawn for a set of parameter values which describe totally asymmetric populations: $\alpha_{11} = 1.7$, $\alpha_{12} = 1.6$, $\alpha_{21} = 1.2$, $\alpha_{22} = 1.4$; $\beta_{11} = 0.25$, $\beta_{12} = 0.75$, $\beta_{21} = 0.4$, $\beta_{22} = 0.3$; $\gamma_{11} = 1$, $\gamma_{12} = 0.85$, $\gamma_{21} = 1.15$, and $\gamma_{22} = 1$.[7]

[7] Note that γ_{11} and γ_{22} have been normalized to 1 without loss of generality.

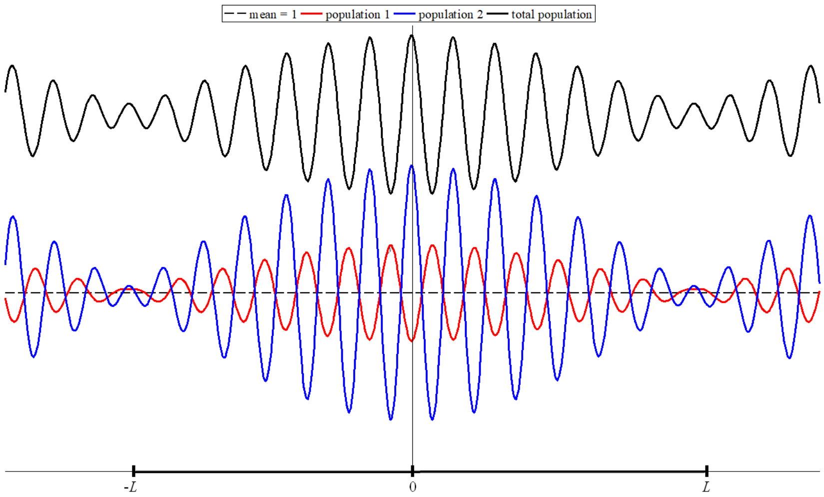

Fig. 3. Christaller-like population densities

The spatial equilibrium now involves *a primate city* located at $x = 0$ over the interval $[-L, L]$. As the distance to this city increases, the density $n_k^*(x)$ swings around the mean $\overline{n}_k = 1$, for $k = 1, 2$. More specifically, the fluctuations of $n_k^*(x)$ dampen and converge to 0 as $|x|$ increases. As for the least populated places, their size rises with the distance to $x = 0$. The pattern displayed in Figure 3 resembles the Christaller model of central place theory.

Another illustration is given by Figure 4 where we borrow the values used to build Figure 2. However, we now assume that $\beta_{11} = \beta_{22} = 0.5$ and $\beta_{12} = \beta_{21} = 1.5$. In other words, the spatial decay effect is now stronger between than within populations, which seems plausible. In this case, we obtain a pattern that differs from that of Figure 2 since five cities have almost the same largest size, e.g., the five most important German metropolitan areas.

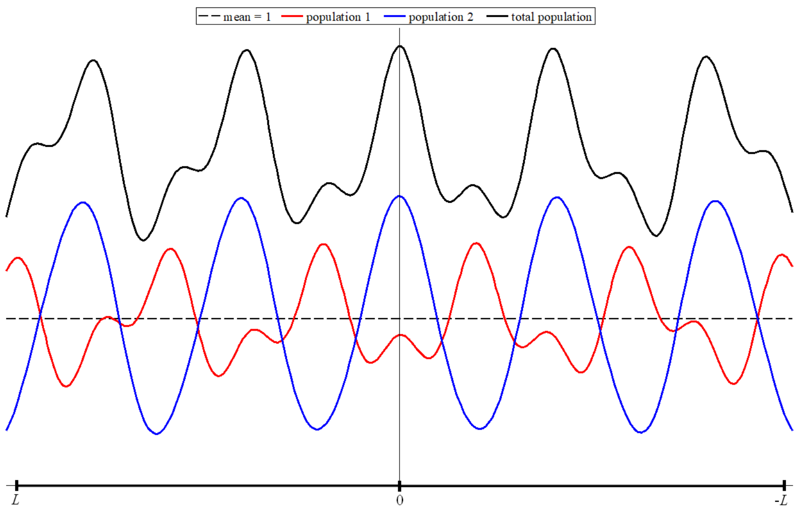

Fig. 4. The "Big Five" in a heterogeneous world

These examples are sufficient to show that our setting, despite its disarming simplicity, may generate a broad gallery of hierarchical urban systems. What matters for hierarchy across central places to arise is the presence of weak spatial impedance, that is, low transportation and/or communication costs. Thus, depending on the intensities of the forces at work, contrasted spatial patterns may emerge. This reflects the variety of urban systems we observe in the real world. In particular, our model shows that the possible equilibrium hierarchies of central places involve much richer structures than that hypothesized by Christaller (1933) and Lösch (1940), which arises only for specific values of the parameters.

4 Concluding remarks

Our paper may be viewed as a primer in central place theory. Indeed, we proposed a setting that captures local and global interactions across individuals. The main thrust of the paper was to study (i) when the interplay between agglomeration and dispersion forces leads to the emergence of central places and (ii) how the corresponding trade-off determines the sizes and locations of central places. Although we used simple explicit functional forms, we have seen that our model displays enough versatility to generate a wide range of equilibrium patterns, which we can characterize analytically. Admittedly, the flip side of our approach is the use of reduced forms, such as (4) and (10), to describe agents' preferences.

We acknowledge that a model with detailed micro-foundations is preferable to ours. Yet, we want to stress that the various attempts made to explain the urban hierarchy through the Zipf Law often rely on ad hoc assumptions or specific functional forms (see, e.g., Gabaix, 1999; Eeckhout, 2004; Hsu, 2012). More importantly, the Zipf Law ignores where cities are located. As a result, this law (if any!) is unable to predict distances between cities. The bulk of spatial economics stresses the importance of transportation costs. So, is it not weird to expect the size distribution of cities to be independent of their locations? Therefore, we find it fair to say that a full theory of urban systems is still lacking. This is why we believe that our approach, notwithstanding its drawbacks, may contribute to a better understanding of the reasons that lie behind the existence of striking regularities across urban systems.

A natural extension of our setting is to work with a featureless plane with the aim of getting closer to the pioneering, but highly speculative, work of Christaller and Lösch. However, as this task is likely to be technically hard, we leave it for future research.

References

Armington, P. S. (1969) A theory of demand for products distinguished by place of production. *International Monetary Fund Staff Papers* 16, 159-178.

Baldwin, R., Forslid, R., Martin, Ph., Ottaviano, G.I.P., Robert-Nicoud, F. 2003. *Economic Geography and Public Policy.* Princeton (NJ): Princeton University Press.

Beckmann, M.J. 1976. Spatial equilibrium in the dispersed city. In *Mathematical Land Use Theory*, edited by Yorgo Papageorgiou,117-25. Lexington (MA): Lexington Books.

Behrens, K., Duranton, G., Robert-Nicoud, F. 2014. Productive cities: Sorting, selection and agglomeration. *Journal of Political Economy* 122, 507-553.

Behrens, K., Robert-Nicoud, F. 2015. Agglomeration theory with heterogeneous agents. In: *Handbook of Regional and Urban Economics. Volume 5*, edited by G. Duranton, J.V. Henderson and W.C. Strange, 171-245. Amsterdam: Elsevier.

Berliant, M., Kung, F.-c. 2006. The indeterminacy of equilibrium city formation under monopolistic competition and increasing returns. *Journal of Economic Theory* 131, 101-133.

Christaller, W. 1933. *Die Zentralen Orte in Süddeutschland.* Jena: Gustav Fischer Verlag. English translation: *The Central Places of Southern Germany.* Englewood Cliffs, NJ: Prentice-Hall (1966).

Davis, D.R., Dingel, J. 2019. A spatial knowledge economy. *American Economic Review*, forthcoming.

Desmet, K., Rossi-Hansberg, E. 2014. Spatial development. *American Economic Review* 104, 1211-1243.

Dunford, N. and J.T. Schwartz. 1988. *Linear Operators. Part II.* Hoboken (NJ): Wiley.

Duranton, G. and D. Puga. 2004. Micro-foundations of urban increasing returns: Theory. In: *Handbook of Regional and Urban Economics. Volume 4*, edited by J.V. Henderson and J.-F. Thisse, 2063-2117. Amsterdam: Elsevier.

Eaton, B.C., Lipsey, R.G. 1982. An economic theory of central places. *Economic Journal* 92, 56-72.

Eeckhout, J. 2004. Gibrat's law for (all) cities. *American Economic Review* 94, 1429-1451.

Fujita, M., Mori, T. 1997. Structural stability and evolution of urban systems. *Regional Science and Urban Economics* 27, 399-442.

Fujita, M., Ogawa, H. 1982. Multiple equilibria and structural transition of non-monocentric urban configurations. *Regional Science and Urban Economics* 12, 161-196.

Fujita, M., Thisse J.-F. 2013. *Economics of Agglomeration. Cities, Industrial Location and Globalization.* Second edition. Cambridge: Cambridge University Press.

Fujita, M., Krugman, P., Venables, A.J. 1999a. *The Spatial Economy. Cities, Regions and International Trade.* Cambridge, MA: MIT Press.

Fujita, M., Krugman, P., Mori, T. 1999b. On the evolution of hierarchical urban systems. *European Economic Review* 43, 209-251.

Gabaix, X. 1999. Zipf's law for cities: An explanation. *Quarterly Journal of Economics* 114, 739-767.

Head, K. and T. Mayer. 2004. The empirics of agglomeration and trade. In: *Handbook of Regional and Urban Economics. Volume 4,* edited by J.V. Henderson, and J.-F. Thisse, 2609-2669. Amsterdam: Elsevier.

Henderson, J.V. 1974. The sizes and types of cities. *American Economic Review* 64, 640-656.

Henderson, J.V. 1988. *Urban Development. Theory, Fact and Illusion.* Oxford: Oxford University Press.

Hohenberg, P. and L.H. Lees. 1985. *The Making of Urban Europe (1000-1950).* Cambridge, MA: Harvard University Press.

Horn, R.A. and C.R. Johnson. 1985. *Matrix Analysis.* Cambridge: Cambridge University Press.

Hsu, W.-T. 2012. Central place theory and city size distribution. *Economic Journal* 122, 903-932.

Hsu, W.-T., T.J. Holmes and F. Morgan. 2014. Optimal city hierarchy: A dynamic programming approach to central place theory. *Journal of Economic Theory* 154, 245-273.

Jackson, M.O., B.W. Rogers and Y. Zenou. 2017. The economic consequences of social-network structure. *Journal of Economic Literature* 55, 49-95.

Kolmogorov, A.N. and S.V. Fomin. 2012. *Introductory Real Analysis.* New York: Dover Publications.

Krugman, P. 1991. Increasing returns and economic geography. *Journal of Political Economy* 99, 483-499.

Krugman, P., Venables, A.J. 1995. Globalization and the inequality of nations. *Quarterly Journal of Economics* 110, 857-880.

Lösch, A. 1940. *Die Räumliche Ordnung der Wirtschaft.* Jena: Gustav Fischer. English translation: *The Economics of Location.* New Haven, CN: Yale University Press (1954).

Lucas Jr., R.E., Rossi-Hansberg, E. 2002. On the internal structure of cities. *Econometrica* 70, 1445-1476.

Marshall, J.U. 1989. *The Structure of Urban Systems*. Toronto: University of Toronto Press.

Matsuyama, K. 2008. Symmetry-breaking. In *The New Palgrave Dictionary of Economics, 2nd Edition*, edited by L.E. Blume and S.Durlauf, 6484-6487. London: Palgrave Macmillan.

Mossay, P., Picard, P.M. 2011. On spatial equilibria in a social interaction model. *Journal of Economic Theory* 146, 2455-2477.

Mulligan, G.F., Partridge, M.D., Carruthers, J.I.. 2012. Central place theory and its reemergence in regional science. *Annals of Regional Science* 48, 405-431.

Papageorgiou, Y.Y., Smith, T.R. 1983. Agglomeration as local instability of spatially uniform steady–states. *Econometrica* 51, 1109-1119.

Pontryagin, L.S. 1962. *Ordinary Differential Equations*. Amsterdam: Elsevier.

Puga, D. 1999. The rise and fall of regional inequalities. *European Economic Review* 43, 303-334.

Quinzii, M., Thisse, J.-F. 1990. On the optimality of central places. *Econometrica* 58, 1101-1119.

Tabuchi, T. 1998. Urban agglomeration and dispersion: A synthesis of Alonso and Krugman. *Journal of Urban Economics* 44, 333-351.

Tabuchi T., Thisse, J.-F. 2011. A new economic geography model of central places. *Journal of Urban Economics* 69, 240-252.

Appendix

A. Proof of Proposition 1

We may rewrite the equilibrium condition (7) as follows:

$$\mathbf{n} = \mathbf{G}\mathbf{n} + v^* \mathbf{1}, \tag{A.1}$$

where $\mathbf{n}$ is the population density $n(x)$ viewed as an element of the Banach space $C(\mathbb{R})$ of bounded continuous real-valued functions, $\mathbf{1}$ is the function which is equal to 1 for all $x \in \mathbb{R}$, while $\mathbf{G} : C(\mathbb{R}) \to C(\mathbb{R})$ is the Fredholm-type integral operator associated with the negative-exponential convolution kernel:

$$(\mathbf{G}\mathbf{n})(x) \equiv \int_{\mathbb{R}} g(x,y)n(y)\mathrm{d}y, \qquad g(x,y) \equiv \frac{1}{\alpha} \exp\left\{-\beta|x-y|\right\}.$$

We first show the following intermediate result.

Lemma 1. *The dominant eigenvalue of* $\mathbf{G}$ *is given by* $\lambda_{\max}(\mathbf{G}) = \phi$.

Proof. It is readily verified that $\mathbf{1}$ is the eigenfunction of $\mathbf{G}$, while the corresponding eigenvalue is $1/\phi$. Furthermore, as the kernel $g(x,y)$ is symmetric, the operator $\mathbf{G}$ is self-adjoint. Hence, any two distinct eigenfunctions $\mathbf{n_1}$ and $\mathbf{n_2}$ must be pairwise orthogonal:

$$\int_{\mathbb{R}} n_1(x)n_2(x)\mathrm{d}x = 0.$$

This implies that $\mathbf{1}$ is the only eigenfunction of $\mathbf{G}$ (up to a scalar multiplier), which is strictly positive. Since the kernel $g(x,y)$ is strictly positive, the Frobenius-Perron theorem implies that both its dominant eigenvalue (i.e., the largest one in the absolute value) and the corresponding eigenfunction must be strictly positive. Since $\mathbf{1}$ is the only positive eigenfunction of $\mathbf{G}$, the corresponding eigenvalue must be the dominant one. Q.E.D.

We now proceed with the proof of Proposition 1.

(i-a) By Lemma 1, $\phi < 1$ implies $\lambda_{\max}(\mathbf{G}) < 1$, which means that the operator in the right-hand side of (A.1) is a contraction mapping. The unique solution to (A.1) is then given by the Neumann series:

$$\mathbf{n} = v^* \left(\mathbf{I_d} - \mathbf{G}\right)^{-1} \mathbf{1} = v^* \sum_{k=0}^{\infty} \mathbf{G}^k \mathbf{1}, \tag{A.2}$$

where $\mathbf{I_d}$ is the identity operator.

Furthermore, for each $k = 0, 1, 2, \ldots$, we have:

$$\mathbf{G}^k \mathbf{1} = \phi^k \mathbf{1}.$$

Plugging this expression into (A.2) and summing the resulting geometric series, we obtain:

$$\mathbf{n} = \frac{v^*}{1-\phi}\,\mathbf{1}.$$

Consequently, the equilibrium density can only be uniform: $n(x) = \overline{n}$.

(i-b) Since the equilibrium utility level v^* is the same for all equilibria (see (iii) below), we may consider the following continuous-time dynamic adjustment process:

$$\frac{\mathrm{d}n_t(x)}{\mathrm{d}t} = \gamma\left[v_t(x) - v^*\right], \tag{A.3}$$

where $\gamma > 0$ is a constant, $n_t(x)$ is the population density at location $x \in \mathbb{R}$ and time $t \in \mathbb{R}_+$, while $v_t(x)$ is the utility level of an agent located at x when the population density is $\mathbf{n}_t$. Clearly, a density $\mathbf{n} \in C(\mathbb{R})$ is a spatial equilibrium if and only if it is a steady state of the differential equation (A.3): for all $x \in \mathbb{R}$,

$$n_t(x) = n(x) \Longrightarrow \mathrm{d}n_t(x)/\mathrm{d}t = 0.$$

Using the operator $\mathbf{G} : C(\mathbb{R}) \to C(\mathbb{R})$

$$(\mathbf{G}\mathbf{n})(x) \equiv \frac{1}{\alpha}\int_{\mathbb{R}} g(x,y)n(y)\mathrm{d}y.$$

we may rewrite (A.3) as follows:

$$\frac{\mathrm{d}\mathbf{n}_t}{\mathrm{d}t} = \gamma(\mathbf{v}_t - v^*\mathbf{1}), \tag{A.4}$$

where $\mathbf{v}_t \in C(\mathbb{R})$ is defined by

$$\mathbf{v}_t \equiv \alpha(\mathbf{G} - \mathbf{I_d})\mathbf{n}_t.$$

By (9), we also have:

$$v^*\mathbf{1} = \alpha\,(\phi - 1)\,\overline{n}\mathbf{1} = \alpha\overline{n}(\mathbf{G} - \mathbf{I_d})\mathbf{1},$$

where the second equality holds because $\mathbf{1}$ is an eigenfunction of $\mathbf{G}$ corresponding to its principal eigenvalue, $\lambda_{\max}(\mathbf{G}) \equiv \phi$ (see Lemma 1).

Plugging the expressions for $\mathbf{v}_t$ and $v^*\mathbf{1}$ into (A.4), we obtain the following linear differential equation:

$$\frac{\mathrm{d}\widetilde{\mathbf{n}}_t}{\mathrm{d}t} = \alpha\gamma(\mathbf{G} - \mathbf{I_d})\widetilde{\mathbf{n}}_t, \tag{A.5}$$

where $\widetilde{n}(x) \equiv n(x) - \bar{n}$ is the deviation of the population density from the mean, while $\widetilde{\mathbf{n}} \equiv \mathbf{n} - \overline{n}\mathbf{1}$.

Studying the stability of the uniform density under (A.4) is equivalent to studying the stability of the zero steady state under (A.5). Since the linear operator $\mathbf{G} - \mathbf{I_d}$ in the right-hand side of (A.5) is a bounded operator, we may use the standard stability methods of linear differential equations to pin down the condition for all the eigenvalues of $\mathbf{G} - \mathbf{I_d}$ to be negative (Pontryagin, 1962). It follows from Lemma 1 that this condition is given by $\phi < 1$.

(ii) Assume now that $\phi \geq 1$. In this case, the solution to (7) need not be unique. Assume that a twice continuously differentiable and non-constant solution exists, and restate (7) as follows:

$$\widetilde{n}(x) = \frac{1}{\alpha}\int_{-\infty}^{x} \exp\left\{-\beta(x-y)\right\}\widetilde{n}(y)\mathrm{d}y + \frac{1}{\alpha}\int_{x}^{\infty} \exp\left\{\beta(x-y)\right\}\widetilde{n}(y)\mathrm{d}y, \tag{A.6}$$

where $\widetilde{n}(x) \equiv n(x) - \bar{n}$. Differentiating twice both sides of (A.6) with respect to x yields:

$$\widetilde{n}''(x) = -\beta^2(\phi - 1)\widetilde{n}(x), \tag{A.7}$$

We seek a non-trivial solution to (A.7) which satisfies the following condition: $\widetilde{n}(0) = \widetilde{n}(l) = 0$ for some $l > 0$. This is the simplest case of the Sturm-Liouville problem. It is well known that a non-trivial solution to (A.7) exists only if α and β satisfy the following condition:

$$\beta^2(\phi - 1) \in \left\{\lambda_k \middle|\ k = 1, 2, \ldots\right\}, \qquad \lambda_k \equiv \left(\frac{\pi k}{l}\right)^2. \tag{A.8}$$

The numbers λ_k in (A.8) are the eigenvalues of the Sturm-Liouville operator $-\mathrm{d}^2/\mathrm{d}x^2$. The corresponding non-trivial solutions to (A.7), that is, the eigenfunctions of $-\mathrm{d}^2/\mathrm{d}x^2$, are given by

$$\widetilde{n}_k(x) = A \sin\left(\sqrt{\lambda_k}\, x\right), \qquad k = 1, 2, \ldots \tag{A.9}$$

where A is an arbitrary non-zero constant. Combining (A.8) with (A.9) yields:

$$\widetilde{n}(x) = A \sin\left(\beta\sqrt{\phi - 1}\, x\right),$$

which is equal to (8) after adding $\overline{n}$ to both sides and taking (6) into account. Since a population density cannot be negative, it must be that $-\overline{n} \leq A \leq \overline{n}$.

(iii) The equilibrium condition is as follows:

$$E(x) - \alpha n(x) = v^*, \qquad \text{for all } x \in \mathbb{R},$$

where v^* is an unknown constant. To determine v^*, we first integrate both sides of this condition over $[-b, b]$ and multiply both sides by $1/(2b)$, where $b > 0$:

$$\frac{1}{2b}\int_{-b}^{b} E(x)\mathrm{d}x - \frac{\alpha}{2b}\int_{-b}^{b} n(x)\mathrm{d}x = v^*.$$

Since the mean population density is exogenous and given by

$$\overline{n} \equiv \lim_{b\to\infty} \frac{1}{2b}\int_{-b}^{b} n(x)\mathrm{d}x,$$

it is sufficient to show that

$$\lim_{b\to\infty} \frac{1}{2b}\int_{-b}^{b} E(x)\mathrm{d}x = \frac{2}{\beta}\overline{n}. \tag{A.10}$$

to obtain the desired result.

Let $M \equiv \sup_{x\in\mathbb{R}} n(x) < \infty$. Then, we have:

$$E(x) \equiv \int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}n(y)\mathrm{d}y \leq M\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}\mathrm{d}y = \frac{2}{\beta}M.$$

Hence, the limit in the left-hand side of (A.10) exists and is finite. This limit equals $2\overline{n}/\beta$ if the following equality

$$\lim_{b\to\infty} \frac{1}{2b}\int_{-b}^{b}\left[\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}n(y)\mathrm{d}y\right]\mathrm{d}x = \lim_{b\to\infty} \frac{1}{2b}\int_{-b}^{b}\left[\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}\mathrm{d}x\right]n(y)\mathrm{d}y \tag{A.11}$$

holds. The limit of the right-hand side of (A.11) is equal to $2\overline{n}/\beta$. As seen above, the limit of the right-hand side of (A.11) also exists. It remains to show that the two limits are equal. To do so, we set:

$$\mathcal{I}(b) \equiv \tfrac{1}{2b}\int_{-b}^{b}\int_{-b}^{b} \exp\{-\beta|x-y|\}n(y)\mathrm{d}x\mathrm{d}y,$$

$$\mathcal{R}_1(b) \equiv \tfrac{1}{2b}\int_{-b}^{b}\left[\int_{|y|>b} \exp\{-\beta|x-y|\}n(y)\mathrm{d}y\right]\mathrm{d}x,$$

$$\mathcal{R}_2(b) \equiv \tfrac{1}{2b}\int_{-b}^{b}\left[\int_{|x|>b} \exp\{-\beta|x-y|\}\mathrm{d}x\right]n(y)\mathrm{d}y.$$

It is readily verified that the following identities hold:

$$\frac{1}{2b}\int_{-b}^{b}\left[\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}n(y)\mathrm{d}y\right]\mathrm{d}x = \mathcal{I}(b) + \mathcal{R}_1(b), \tag{A.12}$$

$$\frac{1}{2b}\int_{-b}^{b}\left[\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}\mathrm{d}x\right]n(y)\mathrm{d}y = \mathcal{I}(b) + \mathcal{R}_2(b). \tag{A.13}$$

Since $n(x)$ is bounded above by M, we obtain:

$$0 < \mathcal{R}_k(b) < \frac{2M}{\beta^2}\cdot\frac{1}{b}, \qquad k = 1, 2,$$

which implies:

$$\lim_{b\to\infty}\mathcal{R}_1(b) = \lim_{b\to\infty}\mathcal{R}_2(b) = 0.$$

Taking the limit on both sides of (A.12) and (A.13) when $b \to \infty$ shows that (i) $\mathcal{I}(b)$ has a finite limit when $b \to \infty$, and (ii) the following equalities hold:

$$\lim_{b\to\infty}\frac{1}{2b}\int_{-b}^{b}\left[\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}n(y)\mathrm{d}y\right]\mathrm{d}x = \lim_{b\to\infty}\mathcal{I}(b) = \lim_{b\to\infty}\frac{1}{2b}\int_{-b}^{b}\left[\int_{-\infty}^{\infty} \exp\{-\beta|x-y|\}\mathrm{d}x\right]n(y)\mathrm{d}y,$$

which proves (A.11). Q.E.D.

B. Proof of Proposition 2

Define the following matrix:

$$\mathbf{M} \equiv \begin{pmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \\ \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{pmatrix}\begin{pmatrix} 1 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 1 \end{pmatrix}.$$

The $\mathbf{D}$-matrix can then be rewritten as follows:

$$\mathbf{D} = \beta^2 \mathbf{I} - 2\beta \mathbf{M}.$$

Let λ_i be the ith eigenvalue of $\mathbf{D}$, while μ_i is the ith eigenvalue of $\mathbf{M}$, with $i = 1, \ldots, 4$. Then, we have:

$$\lambda_i = \beta^2 - 2\beta\mu_i. \tag{B.1}$$

Since rank$(\mathbf{M}) = 2$, two eigenvalues are equal to zero, i.e., $\mu_3 = \mu_4 = 0$ (Horn and Johnson, 1985). Combining this with (B.1) implies that $\lambda_3 = \lambda_4 = \beta^2 > 0$. To find λ_1 and λ_2, we need to determine μ_1 and μ_2. To do so, we use the following result: if a matrix is represented as a product of two matrices, the set of non-zero eigenvalues is not sensitive to changing the order of multiplication (Horn and Johnson, 1985, p. 53, Theorem 1.3.20). Changing the order of multiplication in $\mathbf{M}$, we find that μ_1 and μ_2 are the eigenvalues of the following (2×2)-matrix:

$$\begin{pmatrix} \frac{1}{\alpha_1} & \frac{\gamma}{\alpha_2} \\ \frac{\gamma}{\alpha_1} & \frac{1}{\alpha_2} \end{pmatrix}.$$

The characteristic equation of this matrix is given by

$$\mu^2 - \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)\mu + \frac{1-\gamma^2}{\alpha_1\alpha_2} = 0.$$

Since $\gamma < 1$, the solution of this equation are both positive and given by:

$$\mu_{1,2} = \frac{1}{2}\left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \pm \sqrt{\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)^2 - \frac{4(1-\gamma^2)}{\alpha_1\alpha_2}}\right]. \tag{B.2}$$

Set

$$\underline{\beta}(\alpha_1, \alpha_2, \gamma) \equiv 2\mu_2, \qquad \overline{\beta}(\alpha_1, \alpha_2, \gamma) \equiv 2\mu_1, \quad \mu_1 > \mu_2.$$

Three cases may arise.

(i) If $\beta \geq \overline{\beta}(\alpha_1, \alpha_2, \gamma)$, using (B.1) and (B.2) shows that λ_1 and λ_2 are both non-negative, which means that $\mathbf{D}$ has no strictly negative real eigenvalues. By Proposition 3, the spatial equilibrium is unique and uniform, which proves part (i) of Proposition 2.

(ii) If $\underline{\beta}(\alpha_1, \alpha_2, \gamma) \leq \beta < \overline{\beta}(\alpha_1, \alpha_2, \gamma)$, we have: $\lambda_1 < 0 \leq \lambda_2$. Hence, Proposition 3 implies that all spatial equilibria are periodic, which proves part (ii) of Proposition 2.

(iii) If $\beta < \underline{\beta}(\alpha_1, \alpha_2, \gamma)$, we have: $\lambda_1 < \lambda_2 < 0$. Then, Proposition 3 implies part (iii) of Proposition 2. Q.E.D.

C. Proof of Proposition 3

The expression (11) may be rewritten as follows:

$$E_{jk}(x) = \int_{-\infty}^{x} \exp\{-\beta_{jk}(x-y)\}n_k(y)\mathrm{d}y + \int_{x}^{\infty} \exp\{\beta_{jk}(x-y)\}n_k(y)\mathrm{d}y.$$

Differentiating twice $E_{jk}(x)$ with respect to x yields:

$$E''_{jk}(x) = \beta_{jk}^2 E_{jk}(x) - 2\beta_{jk} n_k(x), \qquad j,k = 1,2.$$

or, in vector-matrix form,

$$\mathbf{E}''(x) = \mathbf{Q}\mathbf{E}(x) - 2\mathbf{B}\mathbf{n}(x),$$

where

$$\mathbf{n}(x) \equiv (n_1(x), n_2(x))^{\mathrm{T}}, \qquad \mathbf{E}(x) \equiv (E_{11}(x), E_{12}(x), E_{21}(x), E_{22}(x))^{\mathrm{T}}.$$

The equilibrium conditions in vector-matrix form are given by

$$\mathbf{v}^* = \boldsymbol{\Gamma}\mathbf{E}(x) - \mathbf{A}\mathbf{n}(x), \tag{C.1}$$

where $\mathbf{v}^* \equiv (v_1^*, v_2^*)^{\mathrm{T}}$. Computing the mean of (C.1) yields $\mathbf{v}^* = \boldsymbol{\Gamma}\overline{\mathbf{E}} - \mathbf{A}\overline{\mathbf{n}}$. Subtracting this expression from (C.1) and multiplying both sides by $\mathbf{A}^{-1}$, we obtain:[8]

$$\widetilde{\mathbf{n}}(x) = \mathbf{A}^{-1}\boldsymbol{\Gamma}\widetilde{\mathbf{E}}(x), \tag{C.2}$$

where $\widetilde{\mathbf{E}}(x) \equiv \mathbf{E}(x) - \overline{\mathbf{E}}$.

Plugging (C.2) into (C.1) and subtracting $\overline{\mathbf{E}}$ from both sides, we come to the following system of linear second-order differential equations:

$$\widetilde{\mathbf{E}}''(x) = \mathbf{D}\widetilde{\mathbf{E}}(x). \tag{C.3}$$

With almost no loss of generality, we may focus on the case where $\mathbf{D}$ has four linearly independent eigenvectors.[9] Let λ_j be the jth eigenvalue of $\mathbf{D}$ and let $\mathbf{S}_j = (s_{j1}, s_{j2}, s_{j3}, s_{j4})^{\mathrm{T}}$ be the corresponding eigenvector, for $j = 1, \ldots, 4$. Diagonalizing $\mathbf{D}$, we get:

$$\mathbf{D} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1},$$

[8] Since we assume that $\det(\mathbf{A}) \neq 0$, the inverse matrix $\mathbf{A}^{-1}$ is well defined.

[9] The case where this condition does not hold has a zero measure.

where

$$\mathbf{\Lambda} \equiv \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \qquad \mathbf{S} \equiv \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix}.$$

Consider now the following change of variables

$$\mathbf{Z}(x) \equiv \mathbf{S}^{-1}\widetilde{\mathbf{E}}(x). \tag{C.4}$$

and rewrite the system (C.3) in terms of $\mathbf{Z}(x)$:

$$\mathbf{Z}''(x) = \mathbf{\Lambda}\mathbf{Z}(x),$$

or, equivalently,

$$z_j''(x) = \lambda_j z_j(x), \qquad j = 1, \ldots, 4, \tag{C.5}$$

where $z_j(z)$ is the jth component of $\mathbf{Z}(x)$.

Since the system (C.5) is formed by four independent equations, we can solve each equation separately. Using (C.2) and (C.4), the equilibrium densities $\mathbf{n}^*(x)$ are related to the solutions $\mathbf{Z}^*(x)$ through the following relationship:

$$\mathbf{n}^*(x) = \mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{S}\mathbf{Z}^*(x) + \bar{\mathbf{n}}. \tag{C.6}$$

Since $\mathbf{n}^*(x)$ is bounded, the same must hold for $\mathbf{Z}^*(x)$.

The solution of (C.5) is given by

$$z_j^*(x) = A_j \exp\left\{\sqrt{\lambda_j}\, x\right\} + B_j \exp\left\{-\sqrt{\lambda_j}\, x\right\}, \qquad j = 1, \ldots, 4,$$

where A_j and B_j two arbitrary constants.

Three cases may arise.

(i) If $\mathbf{D}$ has no strictly negative real eigenvalues, $\mathbf{Z}^*(x)$ is bounded over $\mathbb{R}$ if and only if $A_j = B_j = 0$ for all j because $x \in \mathbb{R}$. In other words, $\mathbf{Z}^*(x) \equiv \mathbf{0}$ is the only bounded solution to (C.5). It then follows from (C.6) that $\mathbf{n}^*(x) \equiv \bar{\mathbf{n}}$, i.e., the spatial equilibrium is unique and uniform. This proves part (i) of Proposition 3.

(ii) If $\mathbf{D}$ has only one strictly negative real eigenvalue (λ_1, say), then, the equation (C.5) with $j = 1$ is such that the non-trivial solutions are the eigenfunctions of the Sturm-Liouville operator $-\mathrm{d}^2/\mathrm{d}x^2$. Following the same logic as in the proof of part (ii) of Proposition 1, we obtain:

$$z_1^*(x) = C_1 \sin(\sqrt{-\lambda_1}\, x), \tag{C.7}$$

where C_1 is an arbitrary constant. Clearly, $z_1^*(x)$ is bounded above over $\mathbb{R}$. Combining (C.6) with (C.7) yields:

$$\mathbf{n}^*(x) = \overline{\mathbf{n}} + C_1 \sin(\sqrt{-\lambda_1}\, x)\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{S}_1,$$

which proves part (ii).

(iii) If $\mathbf{D}$ has at least two distinct real and strictly negative eigenvalues (λ_1 and λ_2, say), then using the same reasoning as in the proof of part (ii) of Proposition 1 yields:

$$z_j^*(x) = C_j \sin(\sqrt{-\lambda_j}\, x), \qquad j = 1, 2.$$

As a result, any density of the form

$$\mathbf{n}^*(x) = \overline{\mathbf{n}} + C_1 \sin(\sqrt{-\lambda_1}\, x)\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{S}_1 + C_2 \sin(\sqrt{-\lambda_2}\, x)\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{S}_2$$

is an equilibrium. When $\lambda_1 \neq \lambda_2$, $\mathbf{n}^*(x)$ is not periodic, which implies that the corresponding spatial equilibria involve different extrema. This proves part (iii). Q.E.D.