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Abstract

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Keywords: price discrimination, information design, Consumer Scores, signaling, Ratchet Effect, Persistence, transparency

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Consumer Scores and Price Discrimination*

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June 18, 2018

Abstract

A long-lived consumer interacts with a sequence of firms in a stationary Gaussian setting. Each firm relies on the consumer's current *score*–an aggregate measure of past quantity signals discounted exponentially–to learn about her preferences and to set prices. In the unique stationary linear Markov equilibrium, the consumer reduces her demand to drive average prices below the no-information benchmark. The firms' learning is maximized by persistent scores, i.e., scores that overweigh past information relative to Bayes' rule when observing disaggregated data. Hidden scores–those only observed by firms–reduce demand sensitivity, increase expected prices, and reduce expected quantities.

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1 Introduction

The ability to tailor advertising, content, products, and prices to consumers' characteristics is a critical driver of firm profitability in online markets. To facilitate market segmentation, several data brokers (e.g., Acxiom, Equifax, and Experian) collect and distribute consumer-level behavioral and demographic information, a practice that has received significant attention by policymakers due to its potentially harmful and discriminatory consequences.[1] To correctly assess the impact of regulation, however, it is essential to understand how both technological and market forces affect consumers' incentives. In particular, if the final use of information impacts the distribution of surplus, the mechanisms by which consumer data are collected, aggregated and transmitted can affect the terms of the transactions in which the data are generated, and thus, the informational content of the data itself.

A prominent way in which data brokers operationalize individual-level data is by creating *consumer scores*–metrics that "describe an individual and predict a consumer's behavior, habit, or predilection" (Dixon and Gellman, 2014). Consumer scores combine information about individual customers' age, ethnicity, gender, household income, zip code, and purchase histories to quantify their profitability, health risk, job security, or credit worthiness. From a consumer welfare perspective, the effects of scores range from beneficial (personalized content and advertising) to ambiguous (tailored pricing of goods and services) to harmful (racial discrimination or identity theft). From a dynamic incentives perspective, however, any such score creates a link between a consumer's interaction with one firm and the terms of her future transactions with other firms and industries.[2] Understanding the welfare implications of the associated informational spillovers is thus of central importance, as these quantitative consumer-level metrics have become increasingly popular.[3]

To make progress towards understanding the economic impact of the collection and distribution of personal data, this paper examines the consequences of summarizing consumers' purchase histories into scores that are used for *price discrimination*. Our goal is to inform the policy debate on information aggregation and transparency. The former is related to the

[1]The European Union's General Data Protection Regulation seeks to improve consumers' control over the retention and distribution of their personal data (see Section 8). In the United States, the policy debate has largely focused on the transparency of data brokers' information (Federal Trade Commission, 2014).

[2]For example, information about a consumer's sporting goods purchases or eating habits can become a part of a predictive score for a health insurer. See "Very personal finance," *The Economist*, July 2nd 2012.

[3]Specific examples of consumer scores include: the Acxiom Consumer Prominence Indicator Score; the Equifax Discretionary Spending Limit Index; and the Experian Consumer View Profitability Score. All these scores fall outside the narrow definition of credit scores, and hence, they are not subject to the same regulation under the Fair Credit Reporting Act (Schmitz, 2014). Other examples of quantitative metrics include the Health Score under the Affordable Care Act, ratings for Uber riders, and "clickability" scores for Google advertisers.

composition of consumer scores: can these metrics aggregate information in a way that they benefit consumers in the absence of the usual "horizontal" arguments in favor of data collection and transmission (e.g., facilitating the matching of content to the consumer's taste)?[4] The latter is related to the observability of consumer scores: should consumers know how data brokers categorize them in real time?

Our approach leverages continuous-time methods to embed a model of the ratchet effect into an information design framework, which enables us to tractably examine how data aggregation and transparency impact a consumer's incentives. In the baseline model, a long-lived consumer faces a different monopolist at every instant of time. The consumer's preferences are quadratic in the quantity demanded, and linear in her privately observed willingness to pay, which is captured by a stationary mean reverting process. At any instant of time, an *intermediary* (an unmodeled data broker) observes a signal of the consumer's current purchase distorted by Brownian noise, and updates an *exponential score*: a one-dimensional linear aggregate of past signals discounted at an exponential rate.[5] Only the current value of the score is revealed to each monopolist, who then uses it to set prices, while the consumer knows the entire history of her score's values.

There are several reasons to focus on (third-degree) price discrimination based on purchase histories: (i) it is implicitly used in a number of markets, whether in the form of coupons, discounts, fees, return policies, order of search results, and reserve prices in advertising auctions;[6] (ii) because the quality of data has improved, the use of price discrimination is likely to expand in the future (Dubé and Misra, 2017); and (iii) it is a well-understood workhorse model through which we can uncover economic forces likely to be at play with other discriminatory uses of individual information.

Similarly, there are three reasons for focusing on exponential scores. First, many online scores (or "ratings") are arithmetic averages of historic data, and hence, examining linear aggregators is a natural exercise. Second, because the model is Gaussian, the optimal filter (i.e., Bayesian updating using the full history of past signals as opposed to an aggregate statistic) consists of a posterior belief that belongs to this class. Third, under a linear aggregation of signals, it is only with exponential kernels that the consumer's dynamic

[4] The tradeoff between surplus creation through horizontal matching and surplus distribution through customized pricing is the focus of recent work by Gomes and Pavan (2018) and Ichihashi (2018).

[5] Information collection is far from perfect, and attribution errors are more frequent than with credit scores. For example, a score may combine data from two different users with the same device ID, or fail to match a single user's activity across multiple devices (Rafieian and Yoganarasimhan, 2018).

[6] Google's bidder-specific dynamic price floor is described in Ad Exchange auction model at https://support.google.com/adxbuyer/answer/6077702. For a discussion of the scope of price discrimination see Council of Economic Advisers (2015) and Kehoe, Larsen, and Pastorino (2018). For more details on personalized reserve prices in online auctions, see Ostrovsky and Schwarz (2016), Syrgkanis, Kempe, and Tardos (2017), and the references therein.

optimization problem can be solved recursively with standard dynamic programming tools.

Overview of the results For any information structure, third-degree price discrimination is harmful to the consumer in a one-shot version of our model. Thus, the consumer benefits from the aggregation and distribution of her data solely to the extent that she can manipulate future prices through the impact of her purchases on future values of the score.

With exponential scores, the question of how to aggregate information reduces to how heavily to discount past quantity signals. Conversely, looking forward, any score discount rate is inversely related to the score's persistence: low discount rates imply that any signal can have important long-term consequences on the score. We show that, for any degree of persistence, there exists a unique *stationary linear Markov equilibrium*: the consumer's quantity demanded is a linear function of her type and, via the firms' linear pricing rule, of the firms' beliefs; the firms' beliefs are themselves a linear function of the score; and the outcome of the game is stationary.[7] In this equilibrium, the consumer reacts to the possibility of the firms ratcheting up prices by reducing her quantity demanded relative to the static optimum. In doing so, she optimally balances the benefit of lower future prices with the losses from lower current consumption and better-tailored future prices.

The ratchet effect drives average equilibrium prices down for all levels of the score's persistence. Not all consumers, however, have equal incentives to manipulate future prices. Due to the persistence of the type process, higher types expect to buy large quantities in the immediate future, and hence, have a stronger incentive to scale back their demand to manipulate prices downward. In equilibrium, this reduces the sensitivity of the consumer's actions to her type, relative to the static case. In turn, this reduces the informativeness of the quantity signals, and thus, the firms' ability to learn from the score.

How the sensitivity of the consumer's actions to her type varies across different levels of the score's persistence is critical to the firms' ability to price discriminate. A natural benchmark is the persistence of the beliefs that arise when firms have access to disaggregated data. To this end, we establish the existence and uniqueness of a *non-concealing* score that does not withhold any information relative to observing the full history of noisy signals.

The non-concealing score, however, never maximizes the amount of information conveyed to the firms in equilibrium: a relatively more persistent score increases the sensitivity of the consumer's actions to her type and facilitates the firms' learning. Indeed, by overweighing past information, a more persistent score also correlates less with the consumer's type, resulting in beliefs and prices that are less sensitive to changes in the score. Importantly, this result holds for all discount rates: even a very patient consumer signals her type more

[7] Uniqueness is up to a refinement motivated by discrete-time modeling that we discuss in Section 4.2.

aggressively in equilibrium, despite the consequences that a more persistent score has on long-term prices.

These results have important welfare implications. For very valuable market segments–i.e., markets with a high average willingness to pay–firms would prefer to operate under no information, so as to eliminate the ratchet effect. Conversely, less valuable segments buy little on average, so firms assign greater value to the ability to price discriminate on the basis of informative scores. Indeed, the ex ante variance of the firms' prices measures the value of the intermediary's information, which is often maximized by an even more persistent score than the learning-maximizing one.

The consumer-optimal score is uninformative if the mean of the consumer's type is sufficiently low. Instead, if the mean of the consumer's type is sufficiently high, so is the average quantity demanded. The presence of a score then leads firms to reduce prices, which, in turn, are applied to a large average number of units. Thus, market segments with high average willingness to pay prefer informative scores. When that is the case, consumer surplus can be maximized by a more or less persistent score than the non-concealing one, depending on the consumer's discount rate and on the persistence of her type.

We conclude the paper with an analysis of market transparency by examining the case in which the score is hidden from the consumer. The main difference relative to the baseline case is that firms' beliefs are now private, which leads to a *price-signaling effect*: by signaling the current level of firms' beliefs, today's price provides information about future prices.

In the unique stationary linear Markov equilibrium, the price perfectly reveals the current score. Thus, on the equilibrium path, the consumer has the same information as in the observable case. However, observing a high price realization also signals to the consumer that prices will be high in the future, i.e., that she will purchase relatively few units in the near term. Everything else equal, the signaling role of prices implies that the value to the consumer of manipulating her score by reducing her current quantity is diminished relative to the baseline model. Thus, with hidden scores, the consumer's demand is *less price sensitive*.

Firms take advantage of this reduced price sensitivity by making prices more responsive to the score. The ratchet effect is then stronger than in the case of observable scores, resulting in lower average quantities and higher average prices for all levels of the score's persistence. Numerical results show the point-wise ranking also extends to welfare, with average profits (consumer surplus) in the hidden case being uniformly above (below) the corresponding value in the observable case. Overall, these results make a strong case in favor of the transparency of scores, even in economies where consumers are already fully aware of their existence.[8] Furthermore, our analysis shows that the study of dynamic incentives should

[8]In practice, data brokers make few attempts at improving the transparency of their information. Two

be a central element of policy evaluation, as it uncovers a non-trivial mechanism of reduced price sensitivity by which score opacity can hurt consumers. In the conclusions (Section 8), we discuss our findings in the context of recent regulatory intervention.

Related Literature This paper builds on the literature on behavior-based price discrimination (Villas-Boas, 1999; Taylor, 2004), whose results are surveyed in great detail by Fudenberg and Villas-Boas (2006, 2015) and by Acquisti, Taylor, and Wagman (2016). Closest to our work is the two-period model of Taylor (2004) with observable actions and stochastic types. Taylor (2004) shows that ratcheting forces result in lower equilibrium prices when consumers are strategic (i.e., aware that they are being tracked). Qualitatively, our results differ in at least two respects. First, the market's average willingness to pay critically determines whether firms benefit from observing information about their consumers. Second, the presence of noisy signals in our model implies that the consumer's actions affect the information available to the firms. Therefore, the score's persistence and transparency levels affect the firms' ability to learn in a non-trivial way.[9]

The economic force driving the dynamics of our model is the ratchet effect (e.g., Freixas, Guesnerie, and Tirole (1985); Laffont and Tirole (1988)); and, more recently, Gerardi and Maestri (2016) for an infinite-horizon version with two types, perfectly observable actions, and a mixed-strategy equilibrium). The ratchet effect also underscores the analysis of privacy in settings with multiple principals. Calzolari and Pavan (2006) consider the case of two principals, and Dworczak (2017) that of a single transaction followed by an aftermarket. Relative to these papers, the presence of noise in our model and the restriction to linear pricing limit the ratcheting forces and allow firms to benefit from information transmission.[10]

Finally, our paper contributes to the literature on dynamic information design: our consumer can be seen as a privately informed sender whose preferences depend on the mean and variance of the receivers' beliefs. Our score process is an instance of a linear Gaussian *rating* introduced by Hörner and Lambert (2017). We maintain the assumptions of short-lived firms and additive signals, but we only consider scores with exponential weights. Our

exceptions are the Bluekai Registry http://www.bluekai.com/registry/ and Acxiom's *About the Data* initiative that reveal to consumers which interest groups they belong to. Recent regulatory efforts may change this outlook, as we discuss in Section 8.

[9] Cummings, Ligett, Pai, and Roth (2016) and Shen and Miguel Villas-Boas (2017) study two-period models in which advertisement messages are targeted on the basis of the information about consumers' purchases. These papers highlight trade-offs similar to ours, where the value of targeted advertising impacts the equilibrium price of the first-period good and the amount of information revealed by the consumer. In contrast, McAdams (2011) assumes that the buyer chooses whether the seller can observe her purchase history, and hence, the buyer's decision to disclose information can be used to facilitate price discrimination. Finally, Xu and Dukes (2017) consider the case of a seller with superior information.

[10] The ratchet effect appears, with a different interpretation or motivation, in relational contracts (Halac, 2012; Fong and Li, 2016) and in dynamic games with symmetric uncertainty (Cisternas, 2017b).

models differ along several further dimensions. First, our consumer is privately informed, which makes the informational content of the score endogenous. Second, the consumer's equilibrium payoff is nonlinear in the firms' posterior mean, which implies that optimal actions depend on the level of the firms' beliefs. Put together, dynamic programming is required to characterize incentives (as opposed to point-wise optimization with an exogenous variance, as in Holmström (1999)), thereby limiting the class of linear scores that can be studied. Third, the consumer's incentives depend critically on whether the agent knows her own score—the transparency question becomes critical in our setting.

2 Model

We develop a continuous-time analog of a repeated interaction between a long-run consumer and an infinite sequence of short-run firms. Our approach is characterized by two central features. First, the consumer faces a different monopolist in every period. Second, within each period, the consumer and the current firm play a sequential-move stage game in which the monopolist first posts a unit price for its product; having observed the price, the consumer then chooses a quantity to purchase.

Players, types, and payoffs. Directly in continuous time, consider a long-lived consumer who interacts with a continuum of firms over an infinite horizon. The consumer discounts the future at rate $r > 0$ and, at any instant in time $t \geq 0$, consuming $Q_t = q$ units of the good at price $P_t = p$ results in a flow utility

$$u(\theta, p, q) := (\theta - p)q - \frac{q^2}{2}, \tag{1}$$

where $\theta_t = \theta$ is the consumer's *type* at t, understood as a measure of her willingness to pay at that point in time. We assume throughout that the type process is stationary and mean reverting, with mean $\mu > 0$, speed of reversion $\kappa > 0$, and volatility $\sigma_\theta > 0$, i.e.,

$$d\theta_t = -\kappa(\theta_t - \mu)dt + \sigma_\theta dZ_t^\theta,\ t > 0, \tag{2}$$

where $(Z_t^\theta)_{t\geq 0}$ is a Brownian motion.[11] In particular, $(\theta_t)_{t\geq 0}$ is Gaussian and, by stationarity,

$$\mathbb{E}[\theta_t] = \mu \quad \text{and} \quad \text{Cov}[\theta_t, \theta_s] = \frac{\sigma_\theta^2}{2\kappa} e^{-\kappa|t-s|}, \quad \text{for all} \quad t, s \geq 0. \tag{3}$$

[11] Stationarity requires $\theta_0 \sim \mathcal{N}(\mu, \sigma_\theta^2/2\kappa)$ independent of $(Z_t^\theta)_{t\geq 0}$.

Each firm interacts with the consumer for only one instant, and only one firm operates at any time t; we refer to the monopolist operating at t simply as *firm* t. Production costs are normalized to zero, and hence, firm t's ex post profits are given by P_tQ_t, $t \geq 0$.

In the population interpretation of the model, each consumer is identified with a different path of $(\theta_t)_{t\geq 0}$. Consumers with high willingness to pay are those who exhibit large upward deviations from μ, which is average willingness to pay of the market segment under study; as (2) suggests, however, most consumers' types fluctuate around μ. Finally, if different goods are offered, (3) states that the consumer's associated willingness to pay is positively serially correlated, but such dependence weakens as the time between purchases increases (e.g., searching for a specific good leads to offers of similar products, but subsequent offers are less related to the original one as shopping progresses).

Score process and information. At any $t \geq 0$, firm t only observes the current value Y_t of a *score process* $(Y_t)_{t\geq 0}$ that is provided by an (unmodeled) intermediary. In contrast, the consumer observes the entire history of scores $Y^t := (Y_s : 0 \leq s \leq t)$ in addition to past prices and quantities and type realizations.[12]

Building a score process is a two-step procedure that involves data collection followed by data aggregation. We assume that the intermediary collects information about the consumer using a technology that records purchases with noise. Specifically, the intermediary observes

$$d\xi_t = Q_t dt + \sigma_\xi dZ_t^\xi,\ t > 0,$$

where $(Z_t^\xi)_{t\geq 0}$ is a Brownian motion independent of $(Z_t^\theta)_{t\geq 0}$, and Q_t is the realized purchase by the consumer at $t \geq 0$.

The intermediary then operationalizes the data by aggregating every history of the form $\xi^t := (\xi_s : 0 \leq s < t)$ into a real number Y_t that corresponds to the consumer's time-t score, $t \geq 0$. Building on Hörner and Lambert (2017), we restrict attention to the family of *exponential scores*, i.e., to Ito processes of the form

$$Y_t = Y_0 e^{-\phi t} + \int_0^t e^{-\phi(t-s)} d\xi_s,\ t \geq 0, \tag{4}$$

where $\phi \in (0, \infty)$. Under this specification, the consumer's current score is a linear function of the contemporaneous history of recorded purchases, and lower values of ϕ lead to scores processes that exhibit more *persistence*, as past information is discounted less heavily in

[12] In Section 5, we let firm t observe the entire history of scores Y^t, $t \geq 0$. On the other hand, in Section 7, firm t observes only Y_t, and the score is hidden to the consumer.

those cases. In differential form, the score process satisfies

$$dY_t = -\phi Y_t dt + d\xi_t = (Q_t - \phi Y_t)dt + \sigma_\xi dZ_t^\xi,\ t > 0.^{13} \tag{5}$$

Finally, the prior is that (θ_0, Y_0) is normally distributed; the exact distribution is determined in equilibrium so that the joint process $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian along the path of play.[14] In what follows, the expectation operator $\mathbb{E}[\cdot]$ is with respect to such prior, while $\mathbb{E}_0[\cdot]$ conditions on the realized value of (θ_0, Y_0). The former is the relevant operator for studying welfare, while the latter is used in the equilibrium analysis. The conditional expectations of the consumer and firm t are denoted by $\mathbb{E}_t[\cdot]$ and $\mathbb{E}[\cdot|Y_t]$, respectively.

Strategies and equilibrium concept. A strategy for the consumer specifies, for every $t \geq 0$, a quantity $Q_t \in \mathbb{R}$ to purchase as a function of the history of past prices, types, and score levels, $(\theta_s, P_s, Y_s : 0 \leq s \leq t)$. Instead, firm t must choose a price $P_t \in \mathbb{R}$ that is measurable with respect to Y_t only, $t \geq 0$. We say that a strategy for the consumer is *linear Markov* if, at any $t \geq 0$, $Q_t = Q(p, \theta_t, Y_t)$, where $Q : \mathbb{R}^3 \to \mathbb{R}$ is linear, and p denotes the current posted price (i.e., the consumer conditions her quantity demanded on the contemporaneous posted price). Similarly, for firm t, $P_t = P(Y_t)$ with $P : \mathbb{R} \to \mathbb{R}$, $t \geq 0$.

We focus on Nash equilibria in linear Markov strategies with the property that $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian. From this perspective, given a linear pricing rule $P(\cdot)$, an admissible strategy for the consumer is any process $(Q_t)_{t\geq 0}$ taking values in $\mathbb{R}$ and satisfying (i) progressive measurability with respect to the filtration generated by $(\theta_t, Y_t)_{t\geq 0}$, (ii) $\mathbb{E}_0\left[\int_0^T Q_s^2 ds\right] < \infty$ for all $T > 0$, and (iii) $\mathbb{E}_0\left[\int_0^\infty e^{-rt}(|\theta_t Q_t - Q_t^2/2| + |P_t(Y_t)Q_t|)dt\right] < \infty$. Requirement (i) states that, at histories where firms have chosen prices as prescribed by any candidate equilibrium, the history $(\theta_s, Y_s : 0 \leq s \leq t)$ captures all the information that is relevant for future decision-making; (ii) and (iii) are purely technical.[15]

[13]As mentioned in the introduction, only exponential kernels ensure that the consumer's dynamic optimization problem can be solved recursively with standard dynamic programming tools. Starting from a general kernel in (4), e.g., $Y_t = \int_0^t K(t,s)d\xi_s$, where $K(\cdot,\cdot)$ satisfies standard regularity conditions, it follows that $dY_t = K(t,t)d\xi_t + \int_0^t \frac{\partial K}{\partial t}(t,s)d\xi_s$. If the last integral is not affine in Y_t, either the increment dY_t depends on lagged quantities or scores (in which case the recursive structure that optimal control requires on the analog of (5) is lost), or it depends on a nonlinear function of Y_t (in which case linear aggregation is violated).

[14]See Proposition 1 in Section 4.1.

[15]Requirement (iii) is a mild strengthening of the condition $\mathbb{E}_0\left[\int_0^\infty e^{-rt}|u(\theta_t, P(Y_t), Q_t)|dt\right] < \infty$ that is usually imposed for verification theorems to hold (Sections 3.2 and 3.5 in Pham (2009)). In particular, it rules out strategies with the unappealing property of yielding high payoffs by making expenditures very negative (provided they exist). Finally, under (ii), the dynamic (5) admits a strong solution given any initial condition; therefore, the consumer's best-response problem is well-defined (Section 3.2 in Pham (2009)).

Definition 1. *A pair (Q, P) of linear Markov strategies is a Nash equilibrium if:*

(i) given any $(\vartheta, y) \in \mathbb{R}^2$, the process $(Q(P(Y_t^{Q\circ P}), \theta_t, Y_t^{Q\circ P}))_{t\geq 0}$, with $(Y_t^{Q\circ P})_{t\geq 0}$ the solution to $dY_t = (-\phi Y_t + Q(P(Y_t), \theta_t, Y_t))dt + \sigma_\xi dZ_t^\xi,\ t > 0$, $(\theta_0, Y_0) = (\vartheta, y)$, maximizes

$$\mathbb{E}_0\left[\int_0^\infty e^{-rt} u(\theta_t, P(Y_t), Q_t)dt\right]$$

subject to (2), (5) and $(\theta_0, Y_0) = (\vartheta, y)$, among all admissible strategies $(Q_t)_{t\geq 0}$; and

(ii) at any time-t history such that $Y_t = y$, $p = P(y)$ maximizes $p\mathbb{E}[Q(p, \theta_t, y)|Y_t = y]$.

We say that a linear pair (P, Q) is a stationary *linear Markov equilibrium if, in addition, the joint process $(\theta_t, Y_t^{Q\circ P})_{t\geq 0}$ is stationary Gaussian.*

In a Linear Markov (Nash) equilibrium, the optimality of the consumer's strategy is verified only when firms set prices according to $P_t = P(Y_t)$ for all $t \geq 0$, i.e., on the path of play. We elaborate more on this in Section 4.2, where we refine our solution concept to provide an analog of Markov Perfect equilibrium in our continuous time setting, a step that is required for determining the sensitivity if demand. Finally, the stationarity notion encompasses two ideas: the score must admit a proper long-run distribution, and such a distribution prevails from time zero. These properties allow us to perform a meaningful welfare analysis that is also time invariant.

Finally, we observe that the assumption of short-lived firms is useful to examine the role of scores but that, if the disaggregated histories of $(\xi_t)_{t\geq 0}$ are public, our Markovian equilibrium remains such in a model with a single firm.[16]

3 Price Discrimination Benchmark

To address our policy questions regarding score persistence and transparency, it is useful to begin with a review of (static and dynamic) price discrimination in our environment.

First, consider a consumer with preferences as in (1) who interacts with a firm only once. The firm's prior belief about the consumer's type has mean $\mu \in \mathbb{R}$ and variance $\text{Var}[\theta] > 0$. Before interacting with the consumer, a public signal Y about the consumer's type is realized. This signal can be general; i.e., it is not necessarily Gaussian.

[16] In a long-lived firm's best-response problem to a linear Markov strategy of the consumer, the only state variable is her belief about the consumer's type (which is linear in the score). However, due to the additivity of the price and the type in the consumer's strategy, a choice of price today does not affect future beliefs; i.e., experimentation effects are absent. This, in turn, makes myopic behavior optimal for the firm.

Let $M := \mathbb{E}[\theta|Y]$. Given a posted price p, maximizing the consumer's flow payoff (1) yields a demand with a unit slope $Q(p) = \theta - p$. The static equilibrium quantity and price are given by $Q = \theta - M/2$ and $P = M/2$, yielding the following ex ante surplus levels:

$$\Pi_Y^{\text{static}} = \frac{\mu^2}{4} + \mathrm{Var}[P] \quad \text{and} \quad CS_Y^{\text{static}} = \frac{1}{2}\mathrm{Var}[\theta] + \frac{\mu^2}{8} - \frac{3}{2}\mathrm{Var}[P].$$

By allowing the firm to better tailor its price to θ, better information increases profits: this is captured in the ex ante variance of the monopoly price $\mathrm{Var}[P]$, which measures the firm's ability to price discriminate. Conversely, more precise information results in a higher degree of correlation between the type and the price, which unequivocally reduces the consumer's expected surplus; this a consequence of linear demand, which implies that the average price and quantity levels are independent of the information structure (Schmalensee, 1981). In other words, we have deliberately chosen a setting where any benefit to consumers is derived from dynamic incentives.[17]

Now, suppose the consumer interacts with two firms. Firm 1 sets price P_1 using the common prior, while firm 2 observes a signal of the consumer's first-period quantity before choosing its price $P_2 = M/2$. Importantly, because the consumer realizes the impact of her first period quantity on the second period price, she attempts to manipulate firm 2's beliefs M downwards by adopting a lower first-period demand function than in the static benchmark. In equilibrium, however, firm 1 anticipates the consumer's manipulation incentives, and thus lowers its price. In other words, firm 1 must compensate the consumer for the information she reveals through her actions because firm 2 will use that information to price discriminate. Figure 1 illustrates this outcome.[18]

In a two-period interaction, the consumer therefore buys a lower quantity than she would in a static interaction without information ($Q' < \theta - \mu/2$), but also pays a lower price ($P' < \mu/2$). It is intuitive that a small amount of demand reduction is beneficial in the first period: the consumer gives up the marginal unit of consumption but receives an infra-marginal discount. Thus, the incentives to manipulate the firms' beliefs induce a tradeoff between lower prices today and tailored prices tomorrow.

The effects of strategic demand reduction on price *levels* are well-understood in the literature on behavior-based price discrimination (Villas-Boas, 1999; Taylor, 2004). In our setting where purchases are observed with noise, however, the consumer's incentive to manipulate the firms' beliefs is central to both the equilibrium price level and the *firms' information.*

[17]See Bergemann, Brooks, and Morris (2015) for a complete analysis of third-degree price discrimination in the static case.

[18]In Figure 1, the ratchet effect results in a parallel downward shift of the demand function. This is a property of equilibrium in our continuous-time setting but not in a two-period model.

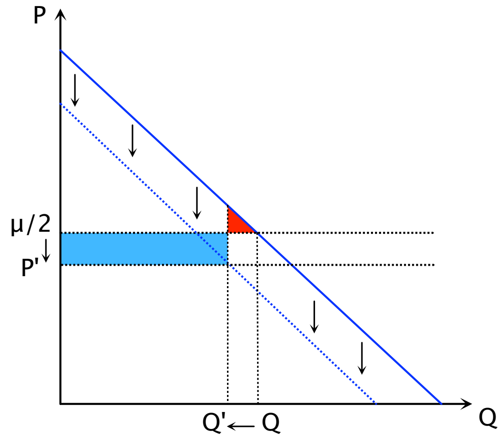

Figure 1: Strategic Demand Reduction

In particular, higher types θ (i.e., consumers who buy more units in period 2) stand to gain more by shading their period-1 demands. This effect reduces the responsiveness of actions to the consumer's unobserved willingness to pay, and hence, it lowers the informativeness of the first-period signal compared to a setting without information spillovers. In other words, the sensitivity of the value of manipulation to the consumer's type θ is crucial for determining the amount of information that is available to the firms in equilibrium.

Having developed some intuition for the drivers of strategic demand reduction, we now turn to the role of score persistence and transparency in shaping the equilibrium outcomes. The stationary setting that we analyze next has two key advantages in this regard: it eliminates end-game effects that can artificially influence policy implications; and it is considerably more tractable than its finite-horizon counterparts, while allowing for non-trivial information aggregation and endogenous learning.

4 Equilibrium Analysis

We begin our equilibrium analysis by studying how each firm learns about the consumer by observing the current level of her score (Section 4.1), and how it chooses a monopoly price (Section 4.2).

4.1 Learning under Linear Strategies

Along the path of play of a stationary linear Markov equilibrium, the consumer's quantity demanded at time t, Q_t, is a linear function of the contemporaneous pair (θ_t, Y_t), $t \geq 0$. As a result, the process $(\theta_t, Y_t)_{t\geq 0}$ evolves according to a linear stochastic differential equation

(SDE) with a normally distributed initial condition; therefore, it is a Gaussian process.[19]

Importantly, since firm t does not observe deviations by neither the consumer nor previous monopolists, (θ_t, Y_t) is also Gaussian from her perspective. Thus, her posterior belief $\theta_t|Y_t$ is normally distributed with mean and variance given by

$$\mathbb{E}[\theta_t|Y_t] = \mu + \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]}[Y_t - \mathbb{E}[Y_t]] \text{ and } \text{Var}[\theta_t|Y_t] = \text{Var}[\theta_t] - \frac{\text{Cov}^2[\theta_t, Y_t]}{\text{Var}[Y_t]}.$$

Furthermore, the stationarity requirement implies that the affine relationship between the posterior mean and the score, as well as the posterior variance, are time-independent. In what follows, we focus on the posterior mean to set up the consumer's best-response problem and defer the analysis of the posterior variance to Section 5.

Let $M_t := \mathbb{E}[\theta_t|Y_t]$, $t \geq 0$. By the previous discussion, we can characterize the outcome of any stationary linear Markov equilibria by using the firms' posterior belief process $(M_t)_{t\geq 0}$ instead of $(Y_t)_{t\geq 0}$. Specifically, we aim to find coefficients α, β and δ such that

$$Q_t = \alpha\theta_t + \beta M_t + \delta\mu, \ t \geq 0, \tag{6}$$

corresponds to the *quantity demanded along the path of play.* These coefficients will in turn depend on the degree of persistence of the score as captured by $\phi > 0$.

The next result characterizes learning in this stationary environment.

Proposition 1 (Stationarity and Beliefs). *A process $(\theta_t, Y_t)_{t\geq 0}$ with $(Q_t)_{t\geq 0}$ as in (6) and $M_t = \mathbb{E}[\theta_t|Y_t]$ for all $t \geq 0$ is stationary Gaussian if and only if:*

(i) $M_t = \mu + \lambda[Y_t - \bar{Y}]$, with $\bar{Y} = \mu(\alpha + \beta + \delta)/\phi$ and

$$\lambda = \frac{\alpha\sigma_\theta^2(\phi - \beta\lambda)}{\alpha^2\sigma_\theta^2 + \sigma_\xi^2\kappa(\phi - \beta\lambda + \kappa)}; \tag{7}$$

(ii) the score process (4) is mean reverting: $\phi - \beta\lambda > 0$;

(iii) $(\theta_0, Y_0) \sim \mathcal{N}([\mu, \bar{Y}]^\top, \Gamma)$ is independent of $(Z_t^\theta, Z_t^\xi)_{t\geq 0}$, where the long-run covariance matrix Γ is given in (A.4).

If $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian under (6), there exist $\bar{Y} \in \mathbb{R}$ given in part (i) and a covariance matrix Γ, such that $(\theta_t, Y_t) \sim \mathcal{N}([\mu, \bar{Y}]^\top, \Gamma)$ for all $t \geq 0$. In this sense, (7) is a constraint on the *sensitivity of beliefs to the score*: as a regression coefficient, $\lambda = \text{Cov}[\theta_t, Y_t]/\text{Var}[Y_t] = \Gamma_{12}/\Gamma_{22}$, and so the right-hand side in (7) accounts for how Γ depends

[19] The technical details are in the proof of Proposition 1, which is stated below.

on both (α, β, δ) and on the weight λ that past firms placed on the score to form their beliefs. Part (ii), in turn, indicates that the effective rate of decay of $(Y_t)_{t\geq 0}$ under (6) is $\phi - \beta\lambda$ due to the contribution of Y_t to Q_t via $M_t = \mu + \lambda[Y_t - \bar{Y}]$; non-positive values for this rate then imply that the scores process' long-run distribution is improper.[20] Finally, equipped with a proper long-run distribution, part (iii) states that for this distribution to obtain from time zero, (θ_0, Y_0) must be drawn from the same distribution.

An implication of Proposition 1 is that the consumer can control the future firms' beliefs by affecting the evolution of the score. Specifically, since $M_t = \mu + \lambda[Y_t - \bar{Y}]$ holds path-by-path of $(Y_t)_{t\geq 0}$, given any admissible strategy $(Q_t)_{t\geq 0}$, the law of motion of $(M_t)_{t\geq 0}$ is

$$dM_t = \left[-\phi\left(M_t - \mu + \lambda\bar{Y}\right) + \lambda Q_t\right] dt + \lambda\sigma_\xi dZ_t^\xi, \ t \geq 0. \tag{8}$$

Thus, the consumer's choice of quantity Q_t affects firm t's beliefs linearly with a slope of λ, but this effect decays at rate ϕ. We can now formalize the key tradeoff between the *persistence* and *sensitivity* of the firms' beliefs.

Lemma 1 (Persistence and Sensitivity)**.** *Suppose that $\alpha > 0$. Then the regression coefficient λ that solves (7) is strictly increasing in ϕ.*

Intuitively, the faster the score discounts old information, the easier it is to manipulate the firm's beliefs in the short run, and vice-versa. This tension underscores all the welfare and information properties of our equilibrium. We now connect the manipulation of beliefs to the manipulation of prices through the analysis of the time-t monopolist's problem.

4.2 Monopoly Pricing

To solve firm t's pricing problem, we must specify the slope of demand, i.e., the weight that a candidate equilibrium linear Markov strategy attaches to the contemporaneous price.

Crucially, because the score Y_t is publicly observed and the firms adopt a linear strategy $P_t = P(Y_t)$, the consumer is able to perfectly anticipate the candidate equilibrium price. The sensitivity of the consumer's demand is then equivalent to the (optimal) change in her quantity demanded in response to a price deviation $p \neq P_t$. This poses a challenge in continuous time because imposing optimality of the consumer's strategy at such off-path histories does not pin down her response to a deviation by a single (zero-measure) firm.

To overcome this challenge, we refine our stationary linear Markov equilibrium concept by requiring that prices be supported by the *limit sensitivity* of demand along a natural

[20]In particular, (ii) and (7) imply that $\lambda > 0$ when $\alpha > 0$.

sequence of discrete-time games indexed by their period length. Along such sequence, as the period length shrinks to zero, the limit demand sensitivity is equal to -1.

Heuristically, consider a discrete-time version of our model in which the period length given by $\Delta > 0$ is small. Given any posted price p, we write the consumer's continuation value V_t recursively with M_t as a state as follows:

$$\begin{aligned} V_t &= \max_q \left[(\theta_t - p)q - \frac{q^2}{2}\right]\Delta + \mathbb{E}_t[V_{t+\Delta}] \\ &= \max_q \left[(\theta_t - p)q - \frac{q^2}{2}\right]\Delta + V_t + \frac{\partial V_t}{\partial M_t}\underbrace{\left[-\phi\left(M_t - \mu + \lambda\bar{Y}\right) + \lambda q\right]\Delta t}_{=\mathbb{E}[\Delta M_t] \text{ from } (8)} + \text{other terms.} \end{aligned}$$

When Δ is sufficiently small, the missing terms that are affected by q on the right-hand side have only second-order effects on the consumer's payoff; therefore, the impact of quantities on the continuation value becomes asymptotically linear. Furthermore, because the noise in the score prevents price deviations from being observed by future firms, the continuation game is unaffected by the actual value of p. The consumer's best reply is then given by

$$Q_t = \theta_t - p + \lambda\frac{\partial V_t}{\partial M_t},$$

where $\partial V_t/\partial M_t$ is independent of the posted price, thus showing that the sensitivity of demand is -1.[21] The details of the formal argument can be found in the online Appendix; except for the case of hidden scores, a stationary linear Markov equilibrium is always understood to have such sensitivity.[22]

A key advantage of continuous time is that it allows us to pin down the sensitivity of demand independent of the remaining equilibrium coefficients, which simplifies the analysis. In particular, the next result uses our refinement to deliver a clean characterization of the monopoly price process along the path of play of any stationary linear Markov equilibrium.

Lemma 2 (Monopoly Price)**.** *Consider a stationary linear Markov equilibrium in which the quantity demanded follows (6). Then, prices are given by*

$$P_t = (\alpha + \beta)M_t + \delta\mu,\ t \geq 0. \tag{9}$$

[21] In particular, observe that the incentives to manipulate the firms' beliefs affect the intercept but not the slope of the demand function, as in Figure 1.

[22] The sequence of discrete-time games examined employ the traditional discrete-time analog of $(\theta_t, Y_t)_{t\geq 0}$, which involves noise scaled by $\sqrt{\Delta}$. Along this sequence, we can show that (i) linear best replies along the path of play are also optimal after having observed off-path prices, and (ii) the weight that linear best replies attach to the current price converges to -1 as the period length goes to zero.

The intuition is simple: because demand has unit slope, the monopoly price along the path of play of such an equilibrium satisfies $P_t = \mathbb{E}[Q_t|Y_t]$, $t \geq 0$.

Equipped with this result, we can formulate the consumer's best-response problem as

$$\max_{(Q_t)_{t\geq 0}} \mathbb{E}\left[\int_0^\infty e^{-rt}\left[(\theta_t - P_t)Q_t - \frac{Q_t^2}{2}\right]dt\right]$$

subject to

$$\begin{aligned} d\theta_t &= -\kappa(\theta_t - \mu)dt + \sigma_\theta dZ_t^\theta \\ dM_t &= (-\phi[M_t - \mu + \lambda\bar{Y}] + \lambda Q_t)dt + \lambda\sigma_\xi dZ_t^\xi \\ P_t &= (\alpha + \beta)M_t + \delta\mu, \end{aligned}$$

where λ satisfies (7). To tackle this best-response problem and also the characterization of stationary linear Markov equilibria, we use dynamic-programming tools.

4.3 Stationary Linear Markov Equilibria

Let $V(\theta, M)$ denote the value of the consumer's best-response problem when the current value of the state is $(\theta, M) \in \mathbb{R}^2$. The Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\begin{aligned} rV(\theta, M) = \sup_{q\in\mathbb{R}} \Bigg\{ &\big(\theta - \underbrace{[(\alpha+\beta)M + \delta\mu]}_{=P_t}\big)q - \frac{q^2}{2} - \kappa(\theta - \mu)\frac{\partial V}{\partial \theta} \\ &+ (\lambda q - \phi[M - \mu + \lambda\bar{Y}])\frac{\partial V}{\partial M} + \frac{\lambda^2\sigma_\xi^2}{2}\frac{\partial^2 V}{\partial M^2} + \frac{\sigma_\theta^2}{2}\frac{\partial^2 V}{\partial \theta^2}\Bigg\}. \end{aligned} \tag{10}$$

Moreover, if (α, β, δ) is such that the policy delivered by the HJB equation (10) (subject to standard transversality conditions) coincides with (6), and the tuple $(\alpha, \beta, \delta, \lambda)$ satisfies the stationarity condition $\phi - \beta\lambda > 0$ in (ii) of Proposition 1, then the coefficients (α, β, δ) fully characterize the outcome of a stationary linear Markov equilibrium.

The combination of (i) quadratic flow payoffs and (ii) Gaussian types and shocks makes the analysis quite tractable. In particular, the consumer's best-response problem is a linear-quadratic optimization problem. We then look for a quadratic value function

$$V(\theta, M) = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M \tag{11}$$

that solves (10), and hence, for a linear best response

$$Q(\theta, M) = \theta - [(\alpha + \beta)M + \delta\mu] + \lambda \frac{\partial V(\theta, M)}{\partial M}. \tag{12}$$

From the previous expression, the consumer's optimal quantity is a combination of the myopic purchase level, $\theta - P$, and the marginal value of manipulating the firms' beliefs via the score, $\lambda \cdot \partial V(\theta, M)/\partial M$. The latter term encodes the consumer's strategic demand reduction, i.e., it captures the strength of the *ratchet effect*.

By imposing the condition that the firms correctly anticipate the consumer's behavior and substituting into the value function, we obtain a sub-system of equations for the equilibrium coefficients (α, β, δ). This system is then coupled with equation (7) to pin down the equilibrium sensitivity of beliefs λ. This procedure allows us to establish the existence and uniqueness of an equilibrium. Furthermore, the equilibrium can be computed in closed form, up to the solution of a single algebraic equation for the coefficient α.

We use $\Lambda(\phi, \alpha, \beta) \in \mathbb{R}_+$ to denote the unique positive solution to (7) for λ when $(\phi, \alpha, \beta) \in (0, \infty) \times [0, 1] \times (-\infty, 0)$; the closed-form expression can be found in (A.12) in the Appendix.

Theorem 1 (Existence and uniqueness)**.** *For any $\phi > 0$, there exists a unique stationary linear Markov equilibrium. In this equilibrium, $0 < \alpha < 1$ is characterized as the unique solution to the equation*

$$x \;\; = \;\; 1 + \frac{\Lambda(\phi, x, B(\phi, x))xB(\phi, x)}{r + \kappa + \phi}, \; x \in [0, 1]. \tag{13}$$

Moreover, $\beta = B(\phi, \alpha) \in (-\alpha/2, 0)$ and $\delta = D(\phi, \alpha) \in \mathbb{R}$, where $B : (0, \infty) \times [0, 1] \to \mathbb{R}$ and $D : (0, \infty) \times [0, 1] \to \mathbb{R}$ are defined in (A.11) and (A.15). Finally, $\lambda = \Lambda(\phi, \alpha, B(\phi, \alpha)) > 0$.

Figure 2 illustrates the equilibrium coefficients $(\alpha, -\beta, \delta)$, the average equilibrium price

$$\mathbb{E}[P_t] = [\alpha + \beta + \delta]\mu, \tag{14}$$

as well as their static benchmark levels. We discuss their properties in the next subsection.

4.4 Strategic Demand Reduction

The consumer's incentives to reduce her demand—which ultimately shape the equilibrium coefficients and prices—depend on both her current type θ and the firms' contemporaneous beliefs, M. Having solved for the value function (11), it is possible to obtain the following

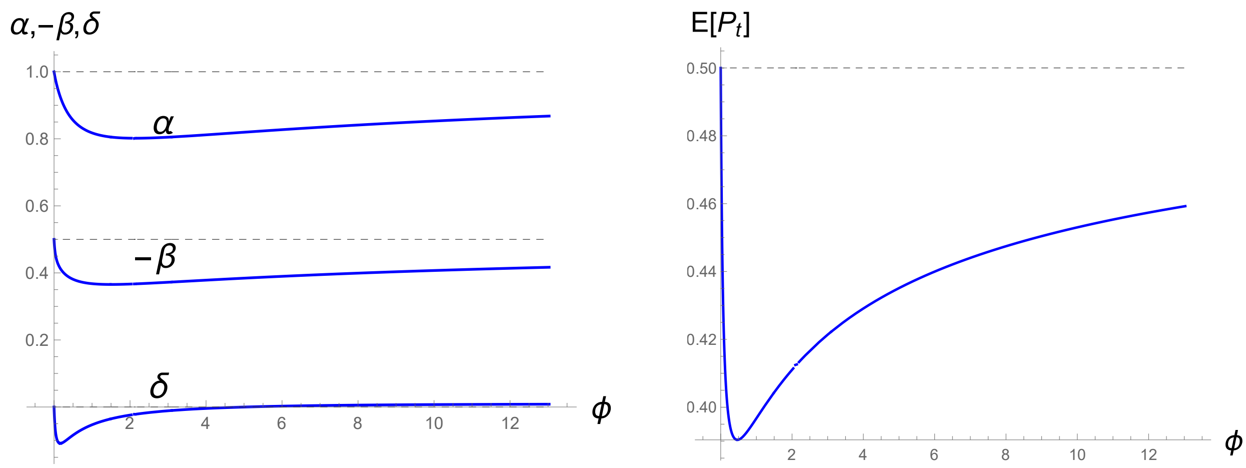

Figure 2: $(r, \sigma_\theta, \sigma_\xi, \kappa) = (1/10, 1, 1/3, 1)$.

linear expression for the *ratchet effect* term in the first-order condition (12):

$$\lambda \frac{\partial V(\theta, M)}{\partial M} = \frac{\lambda \alpha \beta}{r + \kappa + \phi} \theta - \frac{\lambda \beta(\alpha + \beta)}{r + 2\phi - \beta\lambda} M + 2\delta\mu. \tag{15}$$

To guide intuition, Proposition 2 introduces an alternate representation of the strength of the consumer's incentives to reduce her demand.

Proposition 2 (Value of Future Savings)**.** *Equilibrium prices and quantities satisfy*

$$Q_t = \theta_t - P_t - \lambda \mathbb{E}_t \left[\int_t^\infty e^{-(r+\phi)(s-t)} (\alpha + \beta) Q_s ds \right], \ t \geq 0. \tag{16}$$

The last term on the right-hand side of (16) is the (equilibrium) *value of future savings*. By the Envelope Theorem, the benefit of a marginal reduction in today's quantity along the "optimal trajectory" is equal to the net present value of the associated reduction in future prices, holding the future quantities constant. In particular, lowering Q_t by one unit reduces the posterior belief M_{t+dt} by λ and the contemporaneous price P_{t+dt} by $\lambda(\alpha + \beta)$; the impact on subsequent prices then vanishes at the rate ϕ at which beliefs discount the initial change.[23]

Comparing the first-order condition (12) with (16) allows us to conclude that the value of manipulating the firms' beliefs (15) is precisely given by such savings: the larger the savings, the stronger the incentive to engage in a downward deviation from the static Nash equilibrium. Moreover, taking expectations under the prior distribution of $(\theta_t)_{t \geq 0}$ in (16),

[23]While we prove the result only in equilibrium, (16) is an optimality condition and thus holds at a greater level of generality: if a best-response to a price process that is affine in the belief process exists, standard variational arguments show that an expression analogous to (16) must hold.

we find that the average strength of the ratchet effect is given by

$$\lambda \frac{\partial V(\mu,\mu)}{\partial M} = -\lambda \frac{(\alpha+\beta)(\alpha+\beta+\delta)\mu}{r+\phi}. \tag{17}$$

As we shall establish shortly, both $\mathbb{E}[P_t] = (\alpha+\beta+\delta)\mu$ and $\alpha+\beta$ are strictly positive. Therefore, the right-hand side of (17) is strictly negative, and hence, the average quantity demanded contracts. Coupled with the fact that $\mathbb{E}[P_t] = \mathbb{E}[Q_t]$, it is easy to conclude from (16) that the average price lies below the static level $\mu/2$, as in Figure 2 (right panel).

The incentives for demand reduction are not, however, uniform across consumer types. To see this, notice from Proposition 1 that the right-hand side of equation (13) which characterizes $\alpha(\phi)$ is simply the (static) unit weight attached to the type θ, plus the weight placed on θ in (15), i.e., $\lambda\alpha\beta/(r+\kappa+\phi)$. Proposition 2 then allows us to conclude that

$$\frac{\partial}{\partial\theta}\left(-\lambda\mathbb{E}_t\left[\int_t^\infty e^{-(r+\phi)(s-t)}(\alpha+\beta)(\alpha\theta_s+\beta M_s+\delta\mu)ds \mid \theta_t=\theta\right]\right) = \frac{\lambda\alpha\beta}{r+\kappa+\phi}, \tag{18}$$

i.e., that the weight on the consumer's type is diminished exactly by the *sensitivity of the value of future savings to* θ. From this perspective, therefore, the denominator of the right-hand side of (18) indicates that the impact of θ_t on future types θ_s, $s>t$, depreciates at rate κ, so the impact of a marginal change in today's type on future savings decays at the augmented rate of $r+\kappa+\phi$. The numerator, in turn, indicates that an increase in the current type θ_t positively affects not only future types θ_s but also future belief realizations M_s, $s>t$; in equilibrium, these two effects reduce to $\alpha\beta$ using expression (A.11).

We now use this intuition to derive properties of the equilibrium as a function of the score's persistence, making explicit the dependence of (α,β,δ) on ϕ when required.

Proposition 3 (Equilibrium Properties)**.**

(i) $\lim_{\phi\to 0,+\infty}(\alpha(\phi),\beta(\phi),\delta(\phi)) = (1,-1/2,0)$; $\lim_{\phi\to 0,+\infty}\mathbb{E}[(P_t-\mu/2)^2]=0$.

(ii) for all $\phi>0$,

$$1/2 < \frac{r+\kappa+\phi}{r+\kappa+2\phi} < \alpha(\phi) < 1;\ -\alpha(\phi)/2 < \beta(\phi) < 0;\ \textit{and}\ \mathbb{E}[P_t] \in (\mu/3,\mu/2). \tag{19}$$

(iii) $\alpha(\cdot)$ *is strictly quasiconvex.*

(iv) $\alpha(\phi)$ *and* $\mathbb{E}[P_t]$ *are increasing in* σ_ξ/σ_θ *for all* $\phi>0$.

Part (i) shows that the value of manipulation (15) vanishes when the score becomes uninformative: the equilibrium coefficients and price all converge to the static benchmark as $\phi \to 0$ and $+\infty$. If the score has no memory ($\phi \to \infty$), for instance, new information is forgotten instantaneously: in this case, the consumer's actions have no impact on future prices almost surely, and hence, behaving myopically is optimal. On the other hand, if the score is fully persistent ($\phi \to 0$), it places an arbitrarily large weight on arbitrarily old information that is uncorrelated with the current type: in this case, the firms' beliefs are simply not sensitive to new information, leading to the same conclusion.[24]

Part (ii) formalizes the intuition that the benefits of a downward belief manipulation are greater for higher types: because $\beta < 0$, the last term in (18) is strictly negative, and hence, $\alpha < 1$. In fact, because types are persistent, a high θ_t type is more likely to buy larger quantities in the future, and hence, is more willing to invest in strategic demand reduction. Furthermore, the equilibrium coefficient α and the expected price level are also bounded from below: if firms believed that $\alpha = 0$, then the beliefs and prices would not respond to changes in the score, and consumers would not attempt to manipulate them.

Part (iii) shows that the sensitivity of the value of manipulation to the type θ is strongest for an intermediate level of persistence ϕ. Consider (18) and recall the persistence vs. sensitivity tradeoff in Lemma 1: strategic demand reduction has a long-lasting effect for a low ϕ, i.e., if the score is persistent; however, a low ϕ is associated with a low value of λ, and hence, a lower sensitivity of the value of manipulation to the consumer's type, everything else equal. Conversely, because $\phi \mapsto \lambda(\phi)$ is bounded (due to the noise present in the score), the net present value of reducing prices vanishes for all types θ as ϕ grows without bounds.

Finally, part (iv) considers the effects of noise on the value of demand reduction and contrasts it with the effect of persistence. As the exogenous noise in the purchase signals increases, beliefs become less responsive to changes in the score; therefore, the consumer's incentives to manipulate decrease, and the equilibrium price increases in expectation. Furthermore, the incentives to manipulate decrease more quickly for higher types, which reduces the equilibrium α. Instead, both the expected price level and the coefficient α attain their minimum at some interior values of ϕ.

5 Information Revelation

The firms' ability to price discriminate critically depends on how much they learn about the consumer from the score. In this sense, there are two channels through which information aggregation can affect a score's informativeness. First, the persistence level ϕ directly determines how information is weighed relative to the optimal (statistical) *filter*, i.e., relative

[24]Formally, the proof of the proposition shows that $\lim_{\phi \to \infty} \text{Var}[Y_t] = 0$ and $\lim_{\phi \to 0} \lambda(\phi) = 0$.

to the posterior belief about $(\theta_t)_{t\geq 0}$ given the observations of $(\xi_t)_{t\geq 0}$ keeping behavior fixed. Second, the score indirectly affects the consumer incentives, most notably captured by α. In this section, we show that the two effects combined lead firms' learning to be maximized by relatively persistent scores.

To quantify the extent of the firms' learning, we consider the variance of the firms' beliefs

$$\text{Var}[\theta_t|Y_t] = \text{Var}[\theta_t]\left(1 - \frac{\text{Cov}^2[\theta_t, Y_t]}{\text{Var}[Y_t]\text{Var}[\theta_t]}\right).$$

The amount of the firms' learning is captured by strength of the correlation between the consumer's type and the score. Given a persistence level ϕ and coefficients $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_-$ (i.e., not necessarily the equilibrium ones), if $(\theta_t, Y_t)_{t\geq 0}$ is stationary, then

$$\frac{\text{Cov}^2[\theta_t, Y_t]}{\text{Var}[Y_t]\text{Var}[\theta_t]} = \frac{\alpha\Lambda(\phi, \alpha, \beta)}{\phi + \kappa - \beta\Lambda(\phi, \alpha, \beta)} := G(\phi, \alpha, \beta) \in [0, 1], \tag{20}$$

where the first equality comes from the stationary variance of $(\theta_t, Y_t)_{t\geq 0}$ (see (A.4) in the Appendix). We refer to G as the *gain* function: with Gaussian signals, the coefficient of determination (R^2) is a measure of how much information about the consumer's current type is gained by observing the contemporaneous score relative to the prior distribution.

A natural learning benchmark is given by the case in which each firm t observes the disaggregated history of purchases $\xi^t := (\xi_s : 0 \leq s < t)$, as opposed to a linear aggregate. Holding the consumer's behavior $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_-$ fixed, it is then intuitive that the corresponding optimal filter weighs past signals in such a way to maximize the gain function (20). Therefore, we define

$$\nu(\alpha, \beta) := \kappa + \frac{\gamma(\alpha)\alpha(\alpha + \beta)}{\sigma_\xi^2} = \arg\max_{\phi}\ G(\phi, \alpha, \beta), \tag{21}$$

where $\gamma(\alpha) > 0$ is the steady state variance of beliefs when the histories of $(\xi_t)_{t\geq 0}$, are observable.[25] We now establish an equivalence result between learning from the history of disaggregated signals and from the current level of a score with persistence $\nu(\alpha, \beta) > 0$.

Proposition 4 (Learning under Public Histories)**.** *Consider a quantity process $(Q_t)_{t\geq 0}$ as in (6) with $\alpha > 0$ and $\beta < 0$ such that $\nu(\alpha, \beta) > 0$.*

1. *If firms observe the histories of $(\xi_t)_{t\geq 0}$ and their beliefs are stationary, then the posterior mean process is affine in a stationary Gaussian score (4) with $\phi = \nu(\alpha, \beta)$.*

[25] $\gamma(\alpha)$ is the unique positive root of $x \mapsto \alpha^2 x^2/\sigma_\xi^2 + 2\kappa x - \sigma_\theta^2 = 0$. That $\nu(\alpha, \beta)$ maximizes $\phi \mapsto G(\phi, \alpha, \beta)$ is shown in Lemma 6 in the online Appendix, the statement of which can be also found in Appendix A.

2. *Conversely, if firms only observe the current value of a stationary Gaussian score with $\phi = \nu(\alpha, \beta)$, then their beliefs coincide with those of an observer who has access to the history of $(\xi_t)_{t\geq 0}$.*

By Bayes' rule, if firm t could observe the entire history ξ^t, $t \geq 0$, and her belief was stationary, then the posterior mean would linearly aggregate past signals with an exponential weight $\nu(\alpha, \beta)$; thus, the optimal filter is an exponential score with persistence $\phi = \nu(\alpha, \beta)$. Conversely, those beliefs can be induced by an exponential score with persistence $\phi = \nu(\alpha, \beta)$.

Because the consumer's optimal behavior is affected by the score's persistence, we now seek a score that induces the same outcome as the observation of the full history of signals *in equilibrium*, i.e., when the consumer's behavior is allowed to respond.

Definition 2 (Non-concealing score)**.** *A score with persistence $\phi > 0$ is* non-concealing *if*

$$\phi = \nu(\alpha(\phi), \beta(\phi)). \tag{22}$$

In other words, if a score with persistence ϕ generates equilibrium coefficients (α, β) such that $\nu(\alpha(\phi), \beta(\phi)) = \phi$, then the aggregation of signals into the score does not conceal any further information about the consumer's history. We now establish the existence and uniqueness of a solution to (22), with the corresponding implications on signaling.

Proposition 5 (Uniqueness of a Non-Concealing Score and Signaling)**.**

(i) There exists a unique $\phi^* \in \mathbb{R}_+$ solving $\phi = \nu(\alpha(\phi), \beta(\phi))$.

(ii) The fixed point ϕ^* satisfies $\kappa < \phi^* < \sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2}$.

(iii) The coefficient $\alpha(\cdot)$ is strictly decreasing at the fixed point $\phi = \phi^*$.

Part (i) states that there is a unique non-concealing score; combined with Proposition 4, moreover, it establishes the uniqueness of stationary linear Markov equilibrium when the history of $(\xi_t)_{t\geq 0}$ is observable.[26] Part (ii), in turn, shows that beliefs discount past signals more heavily than the type process discounts past shocks to taste, reflecting the identification problem faced by the firms while observing the histories of $(\xi_t)_{t\geq 0}$; the upper bound is a consequence of α (and hence, the informativeness of $(\xi_t)_{t\geq 0}$) being also bounded.

Part (iii) suggests that scores that are more persistent than the non-concealing one (i.e., values $\phi < \phi^*$) can increase firms' learning by inducing the consumer to be more responsive

[26]Intuitively, the firms' conjecture of α and the consumer's choice of α are strategic substitutes: given a low conjecture, firms believe that the quantity signals are uninformative, but the consumer then has no reason to manipulate them. The opposite holds if the firms assign a large weight to the purchase signals.

to her type. Indeed, by increasing the informativeness of the purchase signals, a more persistent score can augment the precision of the firms' beliefs despite concealing some of the information contained in the signals (i.e., despite the score being a suboptimal filter given the consumer's behavior). We formalize this intuition in the next result where, without fear of confusion, we let $G(\phi) := G(\phi, \alpha(\phi), \beta(\phi))$, $\phi \in [0, \infty)$, denote the *equilibrium gain* function.

Proposition 6 (Maximizing Learning). *$\phi \in [0, \infty) \mapsto G(\phi) \in [0, 1]$ is maximized in $(0, \phi^*)$.*

By definition of an optimal filter, $G_\phi(\phi^*, \alpha(\phi^*), \beta(\phi^*)) = 0$, and hence, changing the persistence of the score has only a second-order effect on learning, holding α and β constant. Marginally increasing β at $(\phi^*, \alpha(\phi^*), \beta(\phi^*))$, in turn, has no first-order effect on the amount of information transmitted either: β corresponds to the coefficient on the firm's beliefs in the consumer's strategy, and at ϕ^*, the score perfectly accounts for the contribution of the beliefs to the recorded purchases. Increasing α, however, has a first-order positive effect on learning, as the score is now more sensitive to the consumer's type. Put differently, the indirect effect of affecting the consumer's incentives determines the firms' learning locally around ϕ^*. Conversely, away from ϕ^*, the direct effect of ϕ on the score as a filter dominates, as further increasing or decreasing ϕ leads to a suboptimal aggregation of information that offsets any additional information coming from an increase in α. Figure 3 plots G as a function of $\phi - \nu(\phi)$: its maximum is located to the left of the vertical axis.

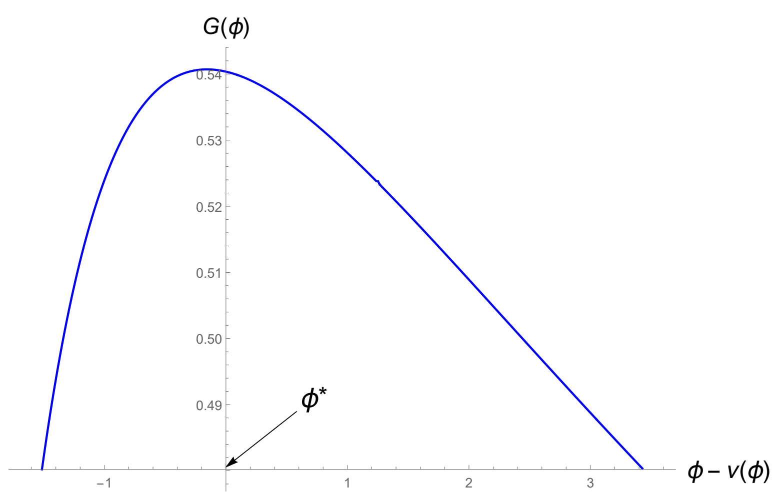

Figure 3: $(r, \sigma_\theta, \sigma_\xi, \kappa) = (3, 1, 2, 1)$.

That a reduction in ϕ from ϕ^* increases the equilibrium coefficient α is particularly interesting. Specifically, this result holds *for all discount rates*, meaning that even a very patient consumer finds it optimal to attach a higher weight to her type, despite the consequences that more persistent scores can have on long-term prices. Opposing this force is the fact that a score that attaches an excessive weight to past signals also correlates less with the consumer's current type, which results in a reduced sensitivity of beliefs (and hence, of prices) to changes in the score. This, in turn, makes the consumer less concerned about purchasing large quantities.[27] One might have expected the consumer's discount rate to non-trivially affect the relative strength of these effects, but the "score-sensitivity effect" turns out to be relatively stronger for all $r > 0$.

To see why this is the case, recall that the coefficient α reflects the *relative* value of future savings for a marginally higher type θ_t, as derived in (18). From Section 4.4, the sensitivity of the value of future savings to the consumer's type satisfies

$$\frac{\partial^2 V(\theta, M)}{\partial\theta\partial M} \propto \int_t^\infty \lambda e^{-(s-t)(r+\phi+\kappa)} ds = \frac{\lambda}{r+\phi+\kappa},$$

reflecting a combination of the direct impact of a shock to θ_t on future types (which decays at a rate κ) and its indirect impact on future prices (which depreciates at rate $r+\phi$) via the change in the quantity demanded.

We now contrast this expression with the gain function. With a linear relationship between $(Y_t)_{t\geq 0}$ and $(\theta_t)_{t\geq 0}$, total learning $G(\phi) = \text{Cov}^2[\theta_t, Y_t]/(\text{Var}[Y_t]\text{Var}[\theta_t])$ is akin to an impulse response. In fact, recalling that $\lambda = \text{Cov}[\theta_t, Y_t]/\text{Var}[Y_t]$, the linearity of the covariance operator yields (up to constant terms)[28]

$$G(\phi) \propto \lambda \cdot \text{Cov}[\theta_t, Y_t] = \lambda \int_{-\infty}^t e^{(s-t)(\phi-\beta\lambda+\kappa)} ds = \frac{\lambda}{\phi - \beta\lambda + \kappa}.$$

Intuitively, a shock to a past type θ_s, $s < t$, has an impact proportional to λ on the past score Y_s, which depreciates at rate $\phi - \beta\lambda$. But the type shock itself depreciates at rate κ, and hence, enters the covariance of θ_t and Y_t with weight $e^{-(s-t)(\phi-\beta\lambda+\kappa)}$. In particular, maximizing the gain function is akin to maximizing the *undiscounted* impulse response of the marginal value of future savings to shock to θ_t.

Consider now a marginal change in ϕ holding $(\alpha, \beta) \in [0, 1] \times \mathbb{R}_-$ constant: in this case, the only change in the previous expressions is that λ becomes $\Lambda(\phi, \alpha, \beta)$. As a result,

$$\nu(\alpha, \beta) = \arg\max \frac{\Lambda(\phi, \alpha, \beta)}{\kappa + \phi - \beta\Lambda(\phi, \alpha, \beta)} = \arg\max \frac{\Lambda(\phi, \alpha, \beta)}{\kappa + \phi} < \arg\max \frac{\Lambda(\phi, \alpha, \beta)}{r + \kappa + \phi},$$

which means that $\partial^2 V/\partial\theta\partial M$ falls whenever $G(\phi, \alpha, \beta)$ peaks (i.e., at $\phi = \nu(\alpha, \beta)$). Put

[27]The trade-off between persistence and sensitivity also arises in signal-jamming models with symmetric uncertainty. See, for example, Cisternas (2017a) in the context of career concerns.

[28]The expression below is written "as if" the game had an infinite past. An identical expression can be obtained by integrating on $[0, t]$ and using the covariance of the stationary distribution of (θ_0, Y_0).

differently, the problems of maximizing learning and of maximizing the value of future savings to shocks to θ_t (for a fixed consumer behavior) differ only in that discounting gives the immediate future more relevance in the latter problem. As a result, the sensitivity-persistence tradeoff is tilted in the favor of the sensitivity effect. This, in turn, leads to a lower α when the impact of quantities on future prices is *frontloaded* relative to the non-concealing score.

6 Welfare Analysis

Having examined the learning implications of score persistence, we turn to its welfare consequences. We begin by discussing the firms' profits. Omitting the dependence of P_t and M_t on ϕ, and using that $\mathbb{E}[Q_t|Y_t] = P_t$, firm t's ex ante profits are given by

$$\Pi(\phi) := \mathbb{E}[P_t Q_t] = \mathbb{E}[P_t^2] = \mathbb{E}[P_t]^2 + \text{Var}[P_t],\ t \geq 0. \tag{23}$$

Firms' profits thus depend on the expected price level, and on their ability to tailor prices based on the information contained in the score, as measured by the ex ante variability of the price. Moreover, by the projection theorem for Gaussian random variables, the ex ante variance of the firms' beliefs is proportional to the variance of the fundamental, scaled by the equilibrium gain factor. We then have

$$\begin{aligned}\mathbb{E}[P_t] &= (\alpha(\phi) + \beta(\phi) + \delta(\phi))\mu \quad \text{and} \\ \text{Var}[P_t] &= (\alpha(\phi) + \beta(\phi))^2 \text{Var}[M_t] = (\alpha(\phi) + \beta(\phi))^2 \text{Var}[\theta_t] G(\phi).\end{aligned}$$

The variance of the equilibrium price is not only a driver of firm profits; it also captures the equilibrium *value of information* that can be written as

$$\text{Var}[P_t] = \mathbb{E}[P_t Q_t] - \mathbb{E}[P_t] \cdot \mathbb{E}[Q_t].$$

This difference measures firms' supplemental profits relative to pricing under the consumer's equilibrium strategy with the knowledge of the prior distribution only. We now ask which score persistence level maximizes the value of information.[29]

Proposition 7 (Value of Information)**.**

(i) $\text{Var}[P_t]$ *is maximized to the left of* ϕ^**.*

[29] Because $(\alpha(\phi), \beta(\phi), \delta(\phi), G(\phi))$ converges to $(1, -1/2, 0, 0)$ as $\phi \to 0$, the equilibrium under study converges to the repetition of static Nash when firms use only the prior distribution to price. Thus, we can include $\phi = 0$ as part of the policy space, suitably understood as the case in which firms possess no information about the consumer.

(ii) If $r \geq \kappa$, then $\mathrm{Var}[P_t]$ *is maximized to the left of* $\phi^\dagger := \arg\max_{\phi \geq 0} G(\phi) < \phi^*$.

The variance of the equilibrium price attains its global maximum for $\phi < \phi^*$, i.e., for scores that are more persistent than the non-concealing one. Furthermore, if $r \geq \kappa$, the price variance-maximizing score is even more persistent than the learning-maximizing one because further increasing persistence induces prices that are more responsive to beliefs.[30]

Consider now the normalized ex ante consumer surplus:

$$CS(\phi) := r \cdot \mathbb{E}\left[\int_0^\infty e^{-rt}\{(\theta_t - P_t)Q_t - Q_t^2/2\}dt\right],$$

where $P_t = (\alpha+\beta)M_t + \delta\mu$ and $Q_t = \alpha\theta + \beta M_t + \delta\mu$, $t \geq 0$. Moreover, as we show in the online Appendix,

$$CS(\phi) = \mathbb{E}[P_t]\left(\mu - \frac{3}{2}\mathbb{E}[P_t]\right) + L(\phi)\mathrm{Var}[\theta_t]G(\phi) + \alpha(\phi)\left(1 - \frac{\alpha(\phi)}{2}\right)\mathrm{Var}[\theta_t], \tag{24}$$

where,

$$L(\phi) := \frac{\alpha(\phi)^2}{2} + \beta(\phi) - \frac{3}{2}(\alpha(\phi)+\beta(\phi))^2 < 0, \quad \text{for all } \phi > 0.$$

Thus, the welfare consequences of persistence critically depend on three equilibrium objects: the expected price level $\mathbb{E}[P_t]$; the sensitivity $\alpha(\phi)$ of the consumer's purchases to her type; and the firms' ability to learn the consumer's type in equilibrium, as measured by $G(\phi)$.

We know from the bounds in Proposition 3 that the first term in $CS(\phi)$ is positive and decreasing in the expected price. Thus, ratchet-type forces benefit the consumer through a lower average price. This effect (the first term in (24)) is proportional to μ^2: the consumer's benefit is higher when her average willingness to pay μ is high, and discounts are applied to a larger number of units.

Opposing this dynamic benefit are two forces. First, by shading down her demand ($\alpha(\phi) < 1$), the consumer moves away from her static optimum, which reduces her surplus (the third term in (24) is increasing in α). Second, the consumer transmits information about her willingness to pay to future firms. This makes the price positively covary with the consumer's type and reduces her surplus proportionally to the firms' information gain $G(\phi)$.

Unlike the expected price level, however, the costs of price discrimination for the consumer are independent of the average willingness to pay μ. It is then natural that consumers coming from high-μ segments benefit more from the availability of information than their low-μ counterparts, and that firms benefit from the scoring mechanism only if μ is low enough. The next result formalizes this; to this end, let $\phi^f := \arg\max_{\phi \geq 0} \Pi(\phi)$ and $\phi^c := \arg\max_{\phi \geq 0} CS(\phi)$.

[30] The condition $r \geq \kappa$ does not appear necessary from numerical examples.

Proposition 8 (Optimal Persistence)**.**

(i) Informative optima: there exists $\underline{\mu}_f > 0$ *such that* ϕ^f *is interior if* $\mu < \underline{\mu}_f$*; and there exists* $\bar{\mu}_c > 0$ *such that* ϕ^c *is interior if* $\mu > \bar{\mu}_c$*.*

(ii) Uninformative optima: there exists $\underline{\mu}_c > 0$ *such that* $\phi^c \in \{0, \infty\}$ *if* $\mu < \underline{\mu}_c$*; and there exists* $\bar{\mu}_f > 0$ *such that* $\phi^f \in \{0, \infty\}$ *if* $\mu > \bar{\mu}_f$*.*

In the limit case $\mu = 0$, the firm's optimal score is strictly positive and, under the conditions of Proposition 7, it is given by $\phi^f < \phi^\dagger$: for consumers with low average willingness to pay, the firms' ideal score is more persistent than the score that maximizes learning. Intuitively, these consumers buy few units on average, so the costs of price discrimination are low, and persistent scores maximize the value of information. Conversely, for consumers with high average willingness to pay, firms would rather commit to not observing any information, so that equilibrium prices attain the static benchmark.

For the same reason, when μ is sufficiently large, consumers prefer an informative score to anonymous purchases. Figure 4 illustrates consumer surplus as a function of ϕ for different levels of μ, and Figure 5 compares profit levels for the same parameter values.

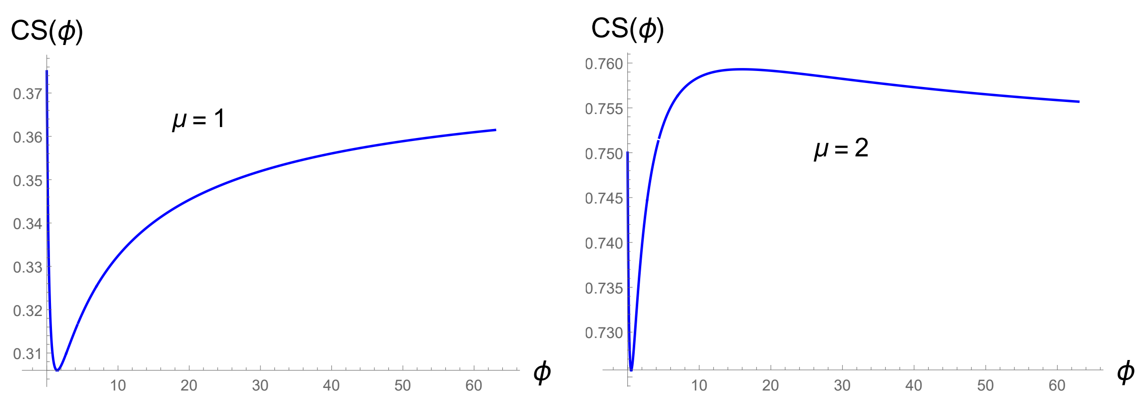

Figure 4: Consumer Surplus, $(\sigma_\theta, \sigma_\xi, \kappa, r) = (1, 2, 1, 1)$

When interior, the consumer-optimal persistence level can be higher or lower than the non-concealing ϕ^*. In the case of an arbitrarily large μ, the comparative statics of consumer surplus are essentially driven by the expected price level that, in turn, reflects the consumer's *average* value of future savings (16). Consistent with intuition, very patient consumers value having a long-term impact on the price, and hence, a very persistent score increases the benefit of reducing the quantity *today* to obtain a lower price. Conversely, impatient consumers prefer having an immediate effect on the price. Therefore, a score that forgets quickly, and this, that is very sensitive to new information, can further reduce the price relative to ϕ^*. Proposition (9) offers a summary of these claims.

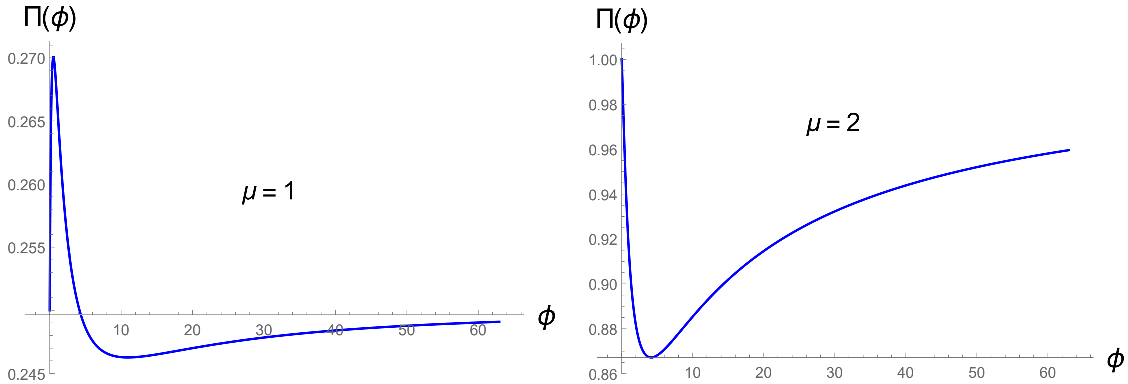

Figure 5: Producer Surplus, $(\sigma_\theta, \sigma_\xi, \kappa, r) = (1, 2, 1, 1)$

Proposition 9 (Expected Price Minimizer)**.** *Let $\alpha(\phi; r)$ and $\phi^*(r)$ denote the equilibrium weight on the type and non-concealing persistence as a function of $r \geq 0$, respectively. Then,*

(i) $\arg\min \mathbb{E}[P_t] \in (0, \arg\min \alpha(\phi; r))$*;*

(ii) There exists $0 < \underline{r} < \kappa$ such that for all $r < \underline{r}$, $\arg\min \mathbb{E}[P_t] < \phi^*(r) < \arg\min \alpha(\phi; r)$;

(iii) There exists $\bar{r} > \kappa$ such that for all $r > \bar{r}$, $\mathbb{E}[P_t]$ is strictly decreasing at $\phi^*(r)$.

Numerical simulations show that the optimal persistence level is increasing in the discount rate r, with the consumer-optimal persistence level falling even below the learning-maximizing $\phi^\dagger < \phi^*$.

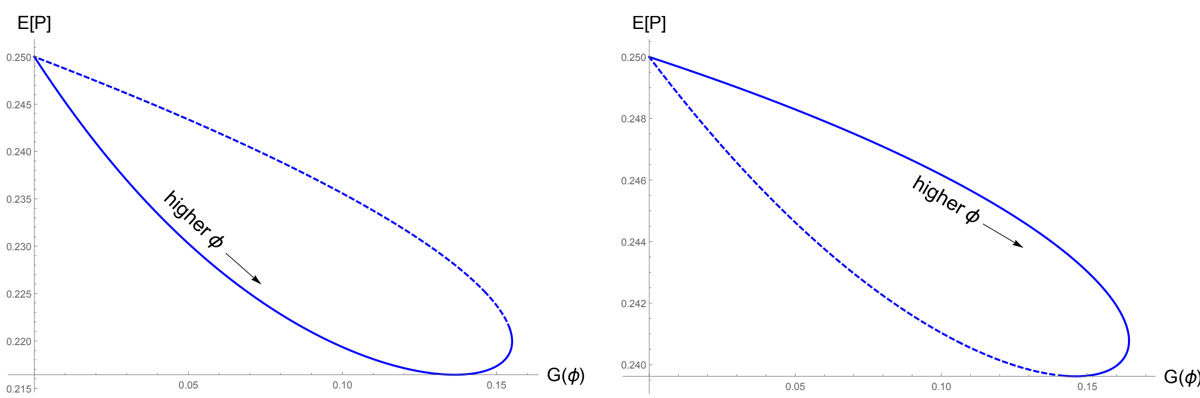

Figure 6: Average price minimizer vs. learning maximizer: $(\sigma_\theta, \sigma_\xi, \kappa) = (1, 2, 1)$; $r = 1/2$ (left); $r = 3$ (right). For a more patient consumer (left), the average price is minimized to the left of $\phi^\dagger$.

7 Hidden Scores

To study the role of transparency, we analyze the case in which the score Y_t is observed by firm t but hidden to the consumer for all $t \geq 0$. The critical difference with the baseline model is that the firms' beliefs are private, and hence, prices carry information that is relevant to the consumer. In particular, by potentially signaling the current level of the firms' beliefs, today's price can provide information about future prices.

When the consumer does not directly observe her score, we define a strategy for the consumer to be *linear Markov* if it is a linear function of (θ_t, p) only, where p is the contemporaneous price. The notion of a linear strategy for the firms is as in the baseline model. The objects of interest are

$$\begin{aligned} Q(\theta, p) &= \delta^h \mu + \alpha^h \theta + \zeta^h p \ \text{ and} \\ P(Y) &= \pi_0^h + \pi_1^h Y, \end{aligned}$$

where the superscript h stands for hidden. The corresponding concepts of admissible strategies, equilibrium, and stationarity are all straightforward modifications of those introduced in Section 2.[31] The focus is also on stationary linear Markov equilibrium.

Before turning to equilibrium analysis, we elaborate on two strategic implications of hidden scores. First, observe that ζ^h denotes the price sensitivity of demand. Critically, unlike in the observable-score case where off-path intra-temporal price variation is used to determine the slope of demand, we will now determine ζ^h along the equilibrium path. The justification comes again from discrete time: if a score process is hidden and has full-support noise, then (i) the consumer is not able to predict the next period's price using today's observation, and (ii) any price realization is possible. Thus, the (discrete time) price process induced by a linear strategy exhibits the required intra-temporal variation along the path of play to enable the consumer's best-response problem to pin down the sensitivity of demand.[32]

Second, when scores are hidden, both the firms and the consumer can signal their private information. In particular, if (as we will show) $\alpha^h > 0$, then $\pi_1 > 0$. This means the price is a fully separating signal, i.e., the consumer perfectly learns her score along the path of play. While, on the equilibrium path, the consumer has the same information as in the observable case, the signaling effect of prices deeply affects the consumer's incentives.

Turning to the equilibrium analysis, note that any equilibrium must entail $\zeta^h \neq 0$, and

[31] There are two changes only: Y_t is suppressed in the notion of a linear Markov strategy for the consumer, and admissible strategies are conditioned on $(\theta_t, P_t)_{t\geq 0}$ rather than on $(\theta_t, Y_t)_{t\geq 0}$. The latter change is innocuous in the consumer's best-response problem to a linear pricing strategy—we work with $(\theta_t, P_t)_{t\geq 0}$ as states for consistency only.

[32] In continuous time, the price process that results from a linear Markov pricing strategy will have continuous paths, so a deviation by a single firm can be detected. Because with full-support noise this issue arises only in continuous time, we refine our equilibrium in the continuous-time game by assuming that the firms conjecture that the consumer responds to the deviation with a sensitivity that coincides with the sensitivity of the quantity demanded along the path of play of a candidate Nash equilibrium. Thus, as it occurs in discrete time, the same candidate policy $Q(\theta, p)$ is used by the firms in their pricing problem.

hence, firm t sets a price $P(Y_t) = -[\delta^h\mu + \alpha^h M_t(Y_t)]/[2\zeta^h]$. We therefore seek to characterize an equilibrium in which the on-path purchases process is of the form

$$Q_t = \delta^h\mu + \alpha^h\theta_t + \zeta^h \underbrace{\left[-\frac{\delta^h\mu + \alpha^h M_t}{2\zeta^h}\right]}_{P_t=} = \frac{\delta^h}{2}\mu + \alpha^h\theta_t + \beta^h M_t, \tag{25}$$

where $\beta^h := -\alpha^h/2$, $M_t = \mu + \lambda^h[Y_t - \bar{Y}^h]$, and some λ^h and $\bar{Y}^h$, $t \geq 0$. In particular, realized prices and quantities satisfy $P_t = -\mathbb{E}[Q_t|Y_t]/\zeta^h$ along the path of play.

Since the quantity demanded (25) has the same structure as (6), the characterization of stationary beliefs in Proposition 1 applies to the hidden case with the additional restriction that $\beta = -\alpha^h/2$. Moreover, recall that when the score process is observed by the consumer, Theorem 1 reduces the quest for stationary linear Markov equilibria to a single equation (13) for α. With hidden scores, existence and uniqueness are reduced to an almost identical equation for the coefficient on the consumer's type.[33]

Proposition 10 (Existence and Uniqueness)**.** *There exists a unique stationary linear Markov equilibrium. In this equilibrium, $\alpha^h \in (0, 1)$ is the unique solution to*

$$x \;=\; 1 + \frac{\Lambda(\phi, x, -x/2)x(-x/2)}{r + \kappa + \phi},\ x \in [0, 1]. \tag{26}$$

Moreover, $\lambda^h = \Lambda(\phi, \alpha^h, -\alpha^h/2) > 0$, where Λ is defined in (A.12), and the price sensitivity of demand is given by

$$\zeta^h = -\frac{2(r + 2\phi)}{2(r + 2\phi) + \alpha^h\lambda^h} \in \left(-1, -\frac{r + 2\phi}{r + 3\phi}\right). \tag{27}$$

The first part of the proposition states that there is always a unique stationary linear Markov equilibrium, and the quantity demanded responds positively to changes in the consumer's type. Moreover, the coefficient α^h is determined by the same equation as α in the observable case replacing $B(\phi, \alpha) \in (-\alpha/2, 0)$ with $-\alpha/2$, $\alpha \in (0, 1)$.

The second part of the Proposition 10 states a key result: demand is less sensitive to price in the hidden case than in the baseline model. The reason lies in the informational content of prices when scores are hidden: by informing the consumer that her score is high, a high price today is a signal of high prices in the future and hence a signal of a low value of reducing today's quantity demanded. This effect is due to the convexity of the consumer's value as

[33] All formal proofs are in the Online Appendix. However, Appendix B outlines some of the arguments and derives additional equilibrium properties that are direct corollaries of the baseline model (e.g., $\alpha^h(\phi)$ is quasiconvex, uniqueness of a non-concealing rating $\phi^{*,h}$, gain factor G maximized to the left of $\phi^{*,h}$).

a function of the current price, whereby the advantage of reducing prices is greater when prices are low, and the consumer is likely to buy more units in the near future. Everything else equal, therefore, the price-signaling effect reduces the ratchet effect (at any given price) relative to the observable-scores case.

The next result summarizes the economic implications that hiding scores have on equilibrium outcomes, via a lower demand sensitivity. To unify notation, we let

$$\begin{aligned} Q_t &= \delta^o \mu + \alpha^o \theta_t + \zeta^o P_t \\ P_t &= \pi_0^o + \pi_1^o Y_t \end{aligned} \tag{28}$$

denote the realized demand and prices along the path of play of a stationary linear Markov equilibrium when the score is observed by the consumer.[34] In addition, let $(\mathrm{Q}^o, \mathrm{P}^o)$ and $(\mathrm{Q}^h, \mathrm{P}^h)$ denote the average price-quantity pairs in the observable and hidden cases, respectively. Recall that all the coefficients are functions of ϕ.

Proposition 11 (Role of Transparency). *In equilibrium, for all $\phi > 0$:*

(i) Sensitivity of demand to price: $0 > \zeta^h(\phi) > \zeta^o(\phi) > -1$.

(ii) Sensitivity of price to score: $\pi_1^h(\phi) > \pi_1^o(\phi) > 0$.

(iii) Sensitivity of demand to type: $1 > \alpha^o(\phi) > \alpha^h(\phi) > 0$

(iv) Average prices and quantities: $\mu/2 > \mathrm{P}^h(\phi) > \mathrm{P}^o(\phi) = \mathrm{Q}^o(\phi) > \mathrm{Q}^h(\phi) > \mu/4$.

To understand these results, let us begin with $0 > \zeta^h > -1$, i.e., demand is more inelastic when the scores are hidden. Facing a demand that is less sensitive to price, each firm charges a higher price that perfectly offsets the consumer's tendency to react less strongly to price increases (as in all linear-demand monopoly problems, the slope ζ^h cancels out in (25)); this makes future prices more sensitive to changes in the score relative to the observable case. In equilibrium, this effect is always strong enough to offset the reduction in signaling (parts (ii) and (iii)).

When prices are more sensitive to the score, the equilibrium amount of strategic demand reduction increases ($\mathrm{Q}^o > \mathrm{Q}^h$ in part (iv)). By the envelope theorem, an analog of the representation result (16) in Proposition 2 holds for the hidden case:

$$Q_t = \theta_t - P_t - \pi_1^h \mathbb{E}_t \left[\int_t^\infty e^{-(r+\phi)(s-t)} Q_s ds \right], \; t \geq 0.$$

[34] It is easy to see that, in the (θ, P) space, α^o is simple α solving (13). On the other hand, the coefficient of the quantity demanded on the price, $\zeta^o \neq -1$, combines the slope of demand (-1) and the change in the value of future savings associated with a change in the current price.

Because $\pi_1^h > \pi_1^o = \lambda(\alpha + \beta)$, the value of future savings increases (everything else equal), thereby inducing a lower average quantity demanded. Furthermore, since any change in the consumer's type translates into a stronger ratcheting of future prices, the sensitivity of value of future savings to the consumer's type decreases, which explains (iii). Conversely, since in the observable-score case prices are ratcheted up less strongly, the consumer has a relatively weak incentive to deviate from her myopic demand (which has unit sensitivity). As a result, the sensitivity of the value of future savings is relatively small in the observable case, which results in a corresponding sensitivity ζ^o of the *quantity demanded* that falls in between the two price sensitivities of demand, -1 and ζ^h (part (i)).

Finally, while average prices remain below the no-information (i.e., $\phi = 0$) case, they are higher than when scores are observed by the consumer ($\mu/2 > \mathrm{P}^h > \mathrm{P}^o$ in (iv)). In other words, the compensatory effect that emerges as a response to the consumer transmitting information to the firms diminishes on average.

These properties of the average prices and quantity, coupled with the lower price sensitivity of demand when scores are hidden, strongly suggest that firms are better off without transparency, while consumers are worse off. While establishing a comparison of ex ante surpluses across cases is challenging due to the different nonlinear components affecting each one, numerical results show that the ranking of prices and quantities holds, in fact, point-wise in ϕ. We illustrate this in figure 7.

8 Concluding Remarks

We have explored the allocation, informational, and welfare consequences of scoring consumers based on their purchase histories and using the information so-gained to price discriminate. Our analysis placed special emphasis on score persistence and transparency, two themes of critical importance for recent regulatory efforts aimed at protecting consumers.

Policy evaluation Our model clearly makes a number of simplifying assumptions, the strongest of which is perhaps the restriction to a continuous score with exponential weights. Nonetheless, our results can help shed light on the equilibrium effects of policies regulating the aggregation and the transmission of consumer data. For concreteness, we discuss our findings in the context of the General Data Protection Regulation (GDPR), which recently came into effect in the European Union. This comprehensive policy focuses on improving consumer awareness and control over their data, and on the transparency of media companies' and data brokers' information.[35]

[35]See https://www.eugdpr.org/key-changes.html for an overview of the GDPR.

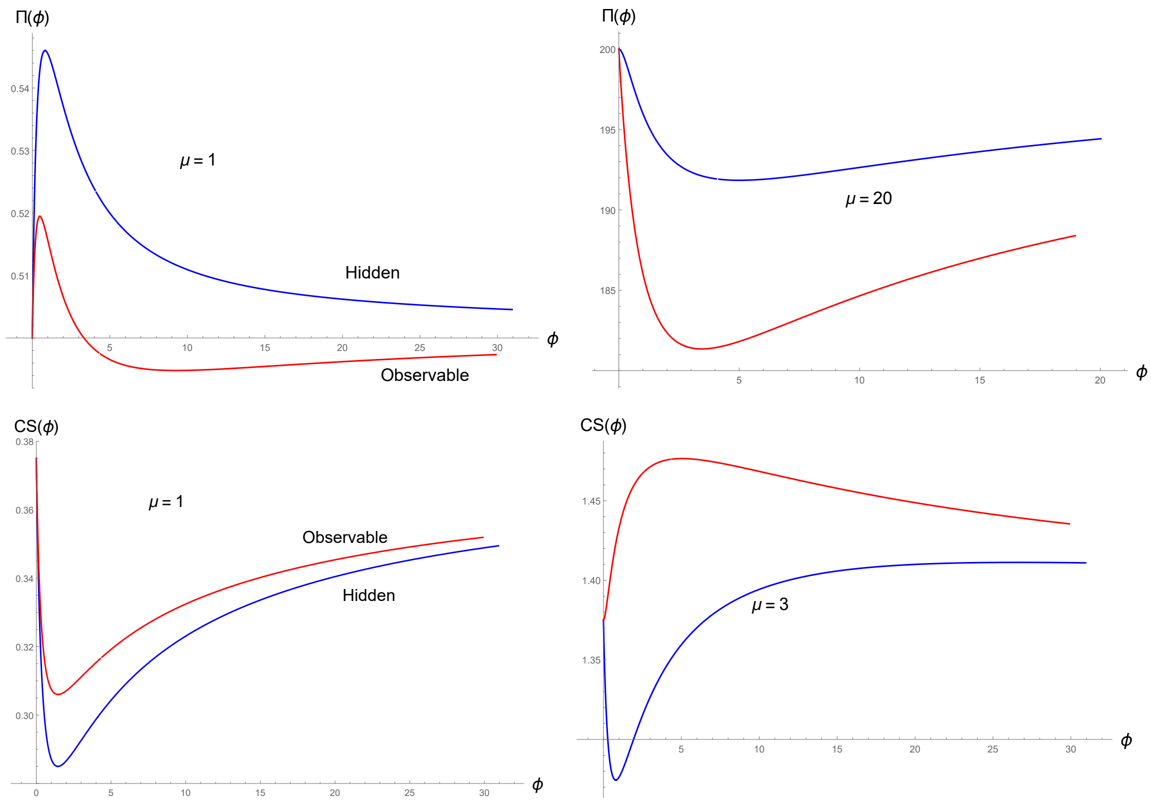

Figure 7: Producer and consumer surplus. Parameter values $\sigma_\xi = 1/3; \sigma_\theta = 1; r = 3; \kappa = 1$

One of the pillars of the GDPR is the Principle of Transparency (Art. 5) that grants consumers the right to be informed about the collection and use of their personal data. Viewed through the lenses of our model, it is straightforward to see that awareness of score-based market segmentation generates immediate benefits for consumers: an unaware (i.e., naive) consumer, who demands the static quantity as in Section 3, suffers the costs of tailored prices without reaping the benefits of lower average prices. Unawareness does not just hurt consumers directly through higher prices. Indeed, greater consumer awareness critically affects both the welfare implications of market segmentation and the consumers' preferences over different levels of persistence. In particular, with unaware consumers, firms would choose the learning-maximizing persistence level (given the consumer's myopic behavior), which would in turn minimize the consumers' welfare.

Conversely, with aware consumers, the GDPR recognizes the possibility of positive informational externalities across various transactions. In fact, the right to Data Portability allows consumers to move their data from one service provider (e.g. a social network) to another. In light of this provision, the contribution of our paper is twofold. First, to show how simple ways of aggregating data can benefit consumers even if the data is used adversely from their perspective. Second, to offer insights on the corresponding mechanisms underlying information aggregation that are otherwise hard to identify, such as why relatively persistent scores induce more information revelation from a consumer.

Equally important, our paper uncovers a non-trivial mechanism by which transparency can benefit consumers beyond the effect of increased awareness. Indeed, as argued earlier, transparent scores do not add to the consumer's information in equilibrium: the consumer has the same information in both the observable- and hidden-scores case along the corresponding paths of play. However, by allowing the consumer to disentangle the direct effect of prices from their signaling effect, transparency increases demand's price sensitivity, which results in lower prices. This finding provides support for the Right to Access provision of the GDPR, which enables consumers to learn in real time what firms know about them.

Finally, our model stresses that consumers may not necessarily oppose being tracked, to the extent to which the relevant data is collected as part of a monetary (or monetizable) transaction. Along these lines, our findings do not support blanket regulation that limits data collection and transmission, but instead suggest that any such policy must be motivated with distributional concerns, such as a focus on improving the welfare of low-μ segments that are unlikely to internalize the benefits of lower prices. Conversely, any information obtained by the firms from exogenous sources is bound to harm consumers if it is later used against them. Considerations such as these might motivate other GDPR provisions, such as the Right to Be Forgotten. Based on our findings, we would expect this right to be exercised by high-θ types. That said, noiseless signaling, such as a consumer requesting their data to be removed, would yield unraveling effects akin to disclosure problems that our model is not adequately equipped to capture.

Future directions. One advantage of our framework is its flexibility to easily accommodate different stage games, and thus shed light on other uses of consumer-level information. For example, one could examine a game where scores are used to tailor products of varying quality to the consumer's tastes.[36] Alternatively, our model can be amended to capture other forms of score-based discrimination, such as racial profiling.[37]

The most promising future direction perhaps involves formalizing a market for summary statistics, towards an understanding of the endogenous dissemination of consumer information. In particular: the mechanisms by which information is sold, and whom it is sold to, will likely influence the consumer's incentives to reveal it. We pursue this avenue of research in ongoing work.

[36] See Turow (2012) for a discussion of how firms use individual data to segment the market according to the quality of the products they offer.

[37] For example, Brayne (2017) describes the role of risk and merit scores in driving law enforcement's "stratified surveillance" practices–yet another form of segmentation.

Appendix A

Proofs for Section 4

Proof of Proposition 1. Suppose that $(\theta_t, Y_t)_{t\geq 0}$ is stationary Gaussian. By stationarity, $\mathbb{E}[Y_t]$ and $\mathrm{Cov}[\theta_t, Y_t]/\mathrm{Var}[Y_t]$ are independent of time; let $\bar{Y}$ and λ denote their respective values (to be determined). Moreover, by normality,

$$M_t := \mathbb{E}[\theta_t|Y_t] = \mu + \lambda[Y_t - \bar{Y}],\ t \geq 0.$$

Let $\hat{\delta} := \delta\mu + \beta(\mu - \lambda\bar{Y})$ and $\hat{\beta} = \beta\lambda$. We can then write the quantity demanded (6) as

$$Q_t = \delta\mu + \alpha\theta_t + \beta M_t = \hat{\delta} + \alpha\theta_t + \hat{\beta}Y_t,\ t \geq 0. \tag{A.1}$$

Using that $d\xi_t = Q_t dt + \sigma_\xi dZ_t^\xi$, we can conclude that $(\theta_t, Y_t)_{t\geq 0}$ evolves according to

$$\begin{aligned} d\theta_t &= -\kappa(\theta - \mu)dt + \sigma_\theta dZ_t^\theta, \\ dY_t &= [-(\phi - \hat{\beta})Y_t + \hat{\delta} + \alpha\theta_t]dt + \sigma_\xi dZ_t^\xi\ t > 0. \end{aligned} \tag{A.2}$$

The previous system is linear, and thus admits an analytic solution. Specifically, letting

$$X := \begin{bmatrix} \theta \\ Y \end{bmatrix},\ A_0 := \begin{bmatrix} \kappa\mu \\ \hat{\delta} \end{bmatrix},\ A_1 := \begin{bmatrix} \kappa & 0 \\ -\alpha & \phi - \hat{\beta} \end{bmatrix},\ B := \begin{bmatrix} \sigma_\theta & 0 \\ 0 & \sigma_\xi \end{bmatrix} \text{ and } Z := \begin{bmatrix} Z_t^\theta \\ Z_t^\xi \end{bmatrix},$$

we can write $dX_t = [A_0 - A_1 X_t]dt + BdZ_t$, $t > 0$, which has as unique (strong) solution

$$X_t = e^{-A_1 t}X_0 + \int_0^t e^{-A_1(t-s)}A_0 dt + \int_0^t e^{-A_1(t-s)}BdZ_s,\ t \geq 0, \tag{A.3}$$

where $e^{A_1 t}$ denotes the matrix exponential (Section 1.7 in Platen and Bruti-Liberati (2010)).

From the additive structure of (A.3), X_t is Gaussian for all $t \geq 0$ if and only if X_0 is Gaussian. But this implies that X_0 must be independent of $Z := (Z_t)_{t\geq 0}$ for Z to be a Brownian motion under the (null-sets augmented) filtration generated by Z and X_0.[38] Letting $\mathcal{N}(\vec{\mu}, \Gamma)$ denote the stationary distribution of X_t, $t \geq 0$, it follows that $\vec{\mu} \in \mathbb{R}^2$ and

[38] Denote such filtration by $(\mathcal{G}_t)_{t\geq 0}$. In the absence of independence, there must $t \geq 0$ such that Z_t is not independent of $\mathcal{G}_0$; but this violates the independent-increments requirement of a Brownian motion.

the 2×2 covariance matrix Γ must satisfy the equations

$$\mathbb{E}[X_t] = \hat{\mu} \quad \Leftrightarrow \quad e^{-A_1 t}\hat{\mu} + [A_1^{-1} - e^{-A_1 t}A_1^{-1}]A_0 = \hat{\mu} \;\text{ and}$$
$$\text{Var}[X_t] = \Gamma \quad \Leftrightarrow \quad e^{-A_t t}\Gamma e^{-A_1^T t} + e^{-A_1 t}\text{Var}\left[\int_0^t e^{A_1 s} B dZ_s\right] e^{-A_1^T t} = \Gamma,$$

where $\text{Var}[\cdot]$ and T denote the covariance matrix and transpose operators, respectively.

Observe that the first condition leads to $\vec{\mu} = A_1^{-1}A_0$ provided A_1 is invertible. This, in turn, happens when $\phi - \beta\lambda \neq 0$—we assume this in what follows. Regarding the second condition, differentiating it and using that $\text{Var}\left[\int_0^t e^{A_1 s} B dZ_s\right] = \int_0^t e^{A_1 s} B^2 e^{A_1^T s} ds$ yields

$$-A_1\Gamma - \Gamma A_1^T + B^2 = 0.$$

Using that $\vec{\mu} = (\mathbb{E}[\theta_t], \mathbb{E}[Y_t])^T = (\mu, \bar{Y})^T$, and that $\Gamma_{11} = \text{Var}[\theta_t]$, $\Gamma_{12} = \Gamma_{21} = \text{Cov}[\theta_t, Y_t]$ and $\Gamma_{22} = \text{Var}[Y_t]$, it is then easy to verify that the previous system has as solution

$$\vec{\mu} = \begin{bmatrix} \mu \\ \frac{\hat{\delta}+\alpha\mu}{\phi-\hat{\beta}} \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} \frac{\sigma_\theta^2}{2\kappa} & \frac{\alpha\sigma_\theta^2}{2\kappa(\phi-\beta\lambda+\kappa)} \\ \frac{\alpha\sigma_\theta^2}{2\kappa(\phi-\beta\lambda+\kappa)} & \frac{\alpha^2\sigma_\theta^2+\kappa\sigma_\xi^2(\phi-\beta\lambda+\kappa)}{2\kappa(\phi-\beta\lambda)(\phi-\beta\lambda+\kappa)} \end{bmatrix}. \tag{A.4}$$

To guarantee that the previous expressions indeed correspond to the first two moments of stationary Gaussian process, however, we must verify that Γ is both positive semi-definite and finite. Since $\sigma_\theta^2/2\kappa > 0$, positive semi-definiteness reduces to

$$\det(\Gamma) \geq 0 \Leftrightarrow \frac{\sigma_\xi^2\kappa(\phi - \hat{\beta} + \kappa)^2 + \alpha^2\sigma_\theta^2\kappa}{(\phi - \hat{\beta} + \kappa)^2(\phi - \hat{\beta})} > 0 \Leftrightarrow \phi - \underbrace{\hat{\beta}}_{=\beta\lambda} \geq 0.$$

Because $\phi - \beta\lambda \neq 0$, however, it follows that $\phi - \beta\lambda$ must be strictly positive. As a byproduct, $\Gamma_{22} > 0$ is finite. This proves (ii) and (iii).

To finish the proof, we find λ and $\bar{Y}$ that are consistent with Bayes' rule given a score process that is driven by (6). Using (A.4), and after some simplification,

$$\lambda \;=\; \frac{\text{Cov}[\theta_t, Y_t]}{\text{Var}[Y_t]} = \frac{\alpha\sigma_\theta^2(\phi - \beta\lambda)}{\alpha^2\sigma_\theta^2 + \kappa\sigma_\xi^2(\phi + \kappa - \beta\lambda)} \;\text{ and} \tag{A.5}$$

$$\bar{Y} \;=\; \frac{\mu[\alpha + \beta + \delta]}{\phi} \tag{A.6}$$

where the last equality follows from $\bar{Y} = [\hat{\delta} + \alpha\mu]/(\phi - \beta\lambda)$ and $\hat{\delta} = \delta\mu + \beta(\mu - \lambda\bar{Y})$; this proves (i). The converse part of the Proposition is true by the previous constructive argument. This concludes the proof. □

Proof of Lemma 1. The result follows from partially differentiating (7) with respect to ϕ.□

Proof of Lemma 2. Consider a linear Markov strategy $Q(p, \theta, Y)$ for the consumer with weight equal to -1 on the contemporaneous price. Because the time-t monopolist assumes that past purchases followed (6), we have that $M_t = \mu + \lambda[Y_t - \bar{Y}]$, $t \geq 0$, where $\bar{Y}$ and λ are given in (i) in Proposition 1. Thus, we can write $Q(p, \theta_t, M_t) = q_0 + \alpha\theta_t + q_2 M_t - p$ for some coefficients q_0, α and q_2. Importantly, the weight that the strategy attaches to the contemporaneous price does not change under this linear transformation.

The monopolist operating at time t therefore solves

$$\max_p \; p\mathbb{E}[q_0 + \alpha\theta_t + q_2 M_t - p | Y_t] \Leftrightarrow P(M_t) = \frac{q_0}{2} + \frac{\alpha + q_2}{2} M_t,$$

which leads to a realized purchase

$$Q_t \;\; = \;\; q_0 + \alpha\theta_t + q_2 M_t - P(M_t) = \frac{q_0}{2} + \alpha\theta_t + \frac{q_2 - \alpha}{2} M_t, \; t \geq 0.$$

We conclude that when demand is linear with unit sensitivity, if realized purchases are given by $Q_t = \delta\mu + \alpha\theta_t + \beta M_t$, contemporaneous prices satisfy $P_t = \delta\mu + (\alpha + \beta)M_t$, $t \geq 0$. Importantly, once the coefficients (α, β, δ) are determined, simple algebra shows that prices are supported by the linear Markov strategy

$$Q(p, \theta_t, Y_t) = 2\delta\mu + [\mu - \lambda\bar{Y}][\alpha + 2\beta] + \alpha\theta_t + \lambda[\alpha + 2\beta]Y_t - p,$$

where λ satisfies (7). This concludes the proof. □.

Proof of Theorem 1. Under the set of admissible strategies defined in Section 2, Verification Theorem 3.5.3 in Pham (2009) applies. Specifically, we look for a quadratic solution $V = v_0 + v_1\theta + v_2 M + v_3 M^2 + v_4\theta^2 + v_5\theta M$ to the HJB equation (10)

$$\begin{aligned} rV(\theta, M) = \sup_{q \in \mathbb{R}} \; & \Big\{ (\theta - [(\alpha + \beta)M + \delta\mu])q - q^2/2 - \kappa(\theta - \mu)V_\theta \\ & [\lambda q - \phi\left(M - \mu + \lambda\bar{Y}\right)] \frac{\partial V}{\partial M}(\theta, M) + \frac{\lambda^2\sigma_\xi^2}{2}\frac{\partial^2 V}{\partial M^2} + \frac{\sigma_\theta^2}{2}\frac{\partial^2 V}{\partial \theta^2} \Big\} \end{aligned}$$

subject to standard transversality conditions. To find a stationary linear Markov equilibrium, however, (i) we impose the fixed-point condition that the optimal policy is of the form $\delta\mu + \alpha\theta + \beta M$, and (ii) with the use of the equation for λ (equation (7)), find coefficients that satisfy the stationarity condition $\phi-\beta\lambda>0$ (part (ii) in Proposition 1).

To this end, observe that the first-order condition of the HJB equation reads

$$\begin{aligned} q &= \theta-[\delta\mu+(\alpha+\beta)M]+\lambda[v_2+2v_3M+v_5\theta] \\ &= -\delta\mu+\lambda v_2+[1+\lambda v_5]\theta+[2\lambda v_3-(\alpha+\beta)]M \end{aligned}$$

which leads to the following system matching-coefficient conditions:

$$\delta\mu=-\delta\mu+\lambda v_2,\ \alpha=1+\lambda v_5,\ \text{ and } \beta=2\lambda v_3-(\alpha+\beta). \tag{A.7}$$

By the Envelope Theorem, moreover,

$$\begin{aligned} (r+\phi)[v_2+2v_3M+v_5\theta] &= -(\alpha+\beta)[\delta\mu+\alpha\theta+\beta M]-\kappa(\theta-\mu)v_5 \\ &\quad +[\lambda(\delta\mu+\alpha\theta+\beta M)-\phi(M-\mu+\lambda\bar{Y})]2v_3, \end{aligned} \tag{A.8}$$

which yields the following system of equations

$$\begin{cases} (r+\phi)v_2=-(\alpha+\beta)\delta\mu+\kappa\mu v_5+[\lambda\delta\mu+\phi(\mu-\lambda\bar{Y})]2v_3 \\ (r+2\phi)2v_3=-(\alpha+\beta)\beta+2v_3\lambda\beta \\ (r+\kappa+\phi)v_5=-(\alpha+\beta)\alpha+2v_3\lambda\alpha. \end{cases} \tag{A.9}$$

Using that v_2, v_3 and v_5 can be written as a function of α,β and $\delta\mu$, this system becomes

$$\begin{cases} (r+\phi)\frac{2\delta\mu}{\lambda}=-(\alpha+\beta)\delta\mu+\kappa\mu\frac{\alpha-1}{\lambda}+[\lambda\delta\mu+\phi(\mu-\lambda\bar{Y})]\frac{\alpha+2\beta}{\lambda} \\ (r+2\phi)\frac{\alpha+2\beta}{\lambda}=\underbrace{-(\alpha+\beta)\beta+\beta(\alpha+2\beta)}_{=(\beta)^2} \\ (r+\kappa+\phi)\frac{\alpha-1}{\lambda}=\underbrace{-(\alpha+\beta)\alpha+\alpha(\alpha+2\beta)}_{=\alpha\beta}. \end{cases} \tag{A.10}$$

where we have assumed that $\lambda\neq 0$. In fact, since $\phi-\beta\lambda>0$ in any stationary linear Markov equilibrium, the equation for λ (i.e., (7)) implies that $\lambda\neq 0$ as long as $\alpha\neq 0$; but the latter is a corollary of the following lemma.

Lemma 3. *Any stationary linear Markov equilibrium must satisfy* $\alpha\in(0,1)$.

Proof. Consider a stationary linear Markov equilibrium with coefficients (α,β,δ). Straightforward integration shows that the consumer's equilibrium payoff is quadratic, and thus the system of equations (A.10) holds.

Suppose that $\alpha = 0$. From (7), $\lambda = 0$, and so $M_t = \mu$ for all $t \geq 0$; but this implies that prices do not respond to changes in the score, and hence, it is optimal for the consumer to behave myopically by choosing $Q_t = \theta_t - p$, a contradiction. If instead $\alpha < 0$, the last equation in (A.10) yields

$$\phi - \beta\lambda = (r + \kappa)\left(\frac{1}{\alpha} - 1\right) + \frac{\phi}{\alpha} < 0,$$

which is a contradiction with the equilibrium being stationary ((ii) in Proposition 1).

The case $\alpha = 1$ can be easily ruled out too: since $\lambda > 0$ in this case, the last equation in the system (A.10) yields that $\beta = 0$, but the second equation then implies that $\alpha = 0$, a contradiction. As a corollary, $\beta \neq 0$ in a stationary linear Markov equilibrium.

Suppose now that $\alpha > 1$. The last two equations of (A.10) can be used to solve for β and thus to find an expression for λ as a function of ϕ, α, and the parameters r and κ. In addition, from the last equation in (A.10),

$$L := \phi - \beta\lambda = \frac{\phi - \alpha(\kappa + r) + \kappa + r}{\alpha},$$

and hence, we can solve for $\phi = \phi(\alpha, L)$. We conclude that in the equation for λ, (7), ϕ can be replaced by expressions that depend on L and α. Specifically, the resulting equation is

$$\frac{\alpha L \sigma_\theta^2}{\kappa(\kappa + L)\sigma_\xi^2 + \alpha^2\sigma_\theta^2} + \frac{(\alpha - 1)(\kappa + L + r)(3\alpha(\kappa + L + r) - 3\kappa + L - r)}{\alpha(2\alpha(\kappa + L + r) - 2\kappa - r)} = 0.$$

By stationarity, $L > 0$. Since $\alpha > 1$, however, this implies that the left-hand side of this expression is strictly positive, which is a contradiction. Thus, $\alpha \in (0, 1)$. $\square$

We continue with the proof of the proposition. From the proof of the previous lemma, $\beta \neq 0$. In the system (A.10), we can multiply the second equation by $\alpha \neq 0$ and the third by $\beta \neq 0$ to obtain $(r + 2\phi)\alpha(\alpha + 2\beta) = (r + \kappa + \phi)\beta(\alpha - 1)$. From here, $\beta = B(\phi, \alpha)$ where

$$B(\phi, x) := \frac{-x^2(r + 2\phi)}{2(r + 2\phi)x - (r + \kappa + \phi)(x - 1)} \in \left(-\frac{x}{2}, 0\right) \text{ when } x \in (0, 1). \tag{A.11}$$

Moreover, since $\alpha \in (0, 1)$ and $\phi - \beta\lambda > 0$, it follows from (7) that $\lambda > 0$. However, when

$\alpha > 0$ and $\beta < 0$, the unique strictly positive root of (7) is given by

$$\Lambda(\phi,\alpha,\beta) := \frac{\sigma_\theta^2\alpha(\alpha+\beta)+\kappa\sigma_\xi^2(\kappa+\phi)-\sqrt{[\sigma_\theta^2\alpha(\alpha+\beta)+\kappa\sigma_\xi^2(\kappa+\phi)]^2-4\kappa(\sigma_\theta\sigma_\xi)^2\alpha\beta\phi}}{2\beta\kappa\sigma_\xi^2}. \tag{A.12}$$

In particular, since $\alpha^2+\alpha B(\phi,\alpha)=\alpha[\alpha+B(\phi,\alpha)]\geq \alpha^2/2>0$ when $\alpha\in[0,1]$, $\sigma_\theta^2\alpha(\alpha+B(\phi,\alpha))+\kappa\sigma_\xi^2(\kappa+\phi)>0$ over the same range.

We conclude that $\lambda=\Lambda(\phi,\alpha,B(\phi,\alpha))$ in equilibrium, and so, using the last equation of (A.10), we arrive to equation (13): namely, $\alpha\in(0,1)$ must satisfy $A(\phi,\alpha)=0$, where

$$A(\phi,x) := (r+\kappa+\phi)(x-1)-\lambda(\phi,x,B(\phi,x))xB(\phi,x),\ x\in[0,1]. \tag{A.13}$$

Lemma 4. *For every $\phi>0$, there exists a unique $\alpha\in(0,1)$ satisfying the previous equation. Moreover, the resulting function $\alpha:(0,\infty)\to(0,1)$ is of class of class C^2.*

Proof: Fix $\phi>0$. Observe that as $x\to 1$, $B(\phi,x)\to -1/2$ and $\lim_{x\to 1}\lambda(\phi,x,B(\phi,x))>0$. Hence, $\lim_{x\to1}A(\phi,x)>0$. Similarly, as $x\to 0$, $B(\phi,x)\to 0$ and $B(\phi,x)\lambda(\phi,x,B(\phi,x))\to 0$. Hence, $\lim_{x\to 0}A(\phi,x)<0$. The existence of $\alpha\in(0,1)$ satisfying $A(\phi,\alpha)=0$ follows from the continuity of $x\in[0,1]\mapsto g(\phi,x)$ and the Intermediate Value Theorem.

To show uniqueness, we prove that $x\mapsto -\Lambda(\phi,x,B(\phi,x))xB(\phi,x)$ is strictly increasing in $[0,1]$. To this end, notice first that since

$$H(\phi,x) := -\Lambda(\phi,x,B(\phi,x))B(\phi,x)>0,\ x \text{ in } (0,1),$$

it suffices to show that $x\mapsto H(\phi,x)$ is strictly increasing in the same region.

From the previous limits, $\lim_{x\to0}H(\phi,x)=0$ and $\lim_{x\to1}H(\phi,x)>0$; thus, there must exist a point at which $H_x(\phi,x)>0$. Towards a contradiction, suppose that there is $\hat{x}\in(0,1)$ s.t. $H_x(\phi,\hat{x})=0$, where H_x denotes the partial derivative of H with respect to x. Also, let $\ell(\phi,x):=\sigma_\theta^2x(x+B(\phi,x))+\kappa\sigma_\xi^2(\kappa+\phi)$. At any such $\hat{x}$,

$$\ell_x(\phi,\hat{x})\underbrace{\left[\ell(\phi,\hat{x})-\left(\ell^2(\phi,\hat{x})-4\kappa\sigma_\xi^2\sigma_\theta^2B(\phi,\hat{x})\hat{x}\phi\right)^{1/2}\right]}_{<0,\ \text{as}\ B<0}=2\kappa(\sigma_\theta\sigma_\xi)^2[B_x(\phi,\hat{x})\hat{x}+B(\phi,\hat{x})]\phi. \tag{A.14}$$

Moreover, straightforward algebra shows that

$$B_x(\phi,x)x=\underbrace{B(\phi,x)}_{<0}-\underbrace{\frac{x^2(r+2\phi)(r+\kappa+\phi)}{[2x(r+2\phi)-(r+\kappa+\phi)(x-1)]^2}}_{>0}<0 \ \text{ for } \ x\in[0,1],$$

so $B_x(\phi,x)x+B(\phi,x)<0$ for all $x\in[0,1]$. It then follows that $\ell_x(\phi,\hat{x})=\sigma_\theta^2[2x+B_x(\phi,\hat{x})\hat{x}+B(\phi,\hat{x})]>0$, otherwise the left-hand side of (A.14) is positive, while the right-hand side is negative.

Isolating the square root and squaring both sides in the first-order condition leads to the cancellation of $\ell^2\ell_x^2$ in (A.14). Dividing the resulting expression by $4\kappa(\sigma_\theta\sigma_\xi)^2\phi$ then yields

$$\begin{aligned} 0 &= \ell_x(\phi,\hat{x})\underbrace{\{\ell(\phi,\hat{x})[-B_x(\phi,\hat{x})\hat{x}-B(\phi,\hat{x})]+\ell_x(\phi,\hat{x})B(\phi,\hat{x})\hat{x}\}}_{K:=} \\ &\quad+\underbrace{\kappa(\sigma_\theta\sigma_\xi)^2[-B_x(\phi,\hat{x})\hat{x}-B(\phi,\hat{x})]^2\phi}_{>0}. \end{aligned}$$

But since $\ell_x(\phi,\hat{x})>0$, we must have that $K<0$. In particular, using that $\ell(\phi,x)=\sigma_\theta^2 x[x+B(\phi,x)]+\kappa\sigma_\xi^2(\phi+\kappa)$ and $\kappa\sigma_\xi^2(\phi+\kappa)[-B_x(\phi,\hat{x})\hat{x}-B(\phi,\hat{x})]>0$, it must be that

$$\begin{aligned} \sigma_\theta^2\{[\hat{x}^2+\hat{x}B(\phi,\hat{x})][-B_x(\phi,\hat{x})\hat{x}-B(\phi,\hat{x})]+[2\hat{x}+\hat{x}B_x(\phi,\hat{x})+B(\phi,\hat{x})]\hat{x}B(\phi,\hat{x})\} &< 0 \\ \Leftrightarrow \hat{x}^2[-\hat{x}B_x(\phi,\hat{x})+B(\phi,\hat{x})] &< 0. \end{aligned}$$

However, from the expression for $B_x(\phi,x)x$, we have that $-xB_x(\phi,x)+B(\phi,x)=x^2(r+2\phi)(r+\kappa+\phi)/[2x(r+2\phi)-(r+\kappa+\phi)(x-1)]^2\geq 0$, reaching a contradiction. The continuity of H_x implies that $x\mapsto H(\phi,x)$ is strictly increasing.

To conclude, since $(0,1)\times(0,\infty)\mapsto A(x,\phi)$ is of class C^1 and $\partial A/\partial x>0$, the Implicit Function Theorem guarantees that $\phi\in(0,1)\mapsto\alpha(\phi)\in(0,1)$, where $\alpha(\phi)$ satisfies $A(\phi,\alpha(\phi))=0$, $\phi>0$, is of class C^1. In fact, since such $\alpha(\cdot)$ is unique and defined over the whole domain $(0,\infty)$, the local property of continuous differentiability trivially extends globally. Moreover, using $H(\phi,x)$ defined above, it follows that

$$\alpha'(\phi)=\frac{1-\alpha(\phi)-\alpha(\phi)H_\phi(\phi,\alpha(\phi))}{r+\kappa+\phi+H(\phi,\alpha(\phi))+\alpha(\phi)H_\alpha(\phi,\alpha(\phi))}.$$

It is straightforward to verify that the right-hand side of the previous equality is of class C^1 as a function of $\phi\in(0,\infty)$. Thus, $\alpha(\cdot)$ is of class C^2. $\square$

It remains to characterize δ. Recall that the first equation in (A.10) reads

$$(r+\phi)\frac{2\delta\mu}{\lambda}=-(\alpha+\beta)\delta\mu+\kappa\mu\frac{\alpha-1}{\lambda}+[\lambda\delta\mu+\phi(\mu-\lambda\bar{Y})]\frac{\alpha+2\beta}{\lambda},$$

where $\bar{Y} = \mu[\delta + \alpha + \beta]/\phi$. Plugging this expression in the previous equation yields

$$\left[\frac{2(r+\phi)}{\lambda} + \alpha + \beta\right]\delta\mu \quad = \quad \mu\left[\frac{\kappa(\alpha-1)}{\lambda} + \frac{\alpha+2\beta}{\lambda}[\phi - (\alpha+\beta)\lambda]\right].$$

Observe that since $\alpha + \beta > 0$, the bracket on the left-hand side is strictly positive. If $\mu = 0$ this equation is trivially satisfied, i.e., the price and quantity demanded along the path of play have no deterministic intercept (and $v_2 = 0$, leaving the rest of the system unaffected). If $\mu \neq 0$, we have that $\delta = D(\phi, \alpha)$ where

$$D(\phi, x) := \frac{\kappa(\alpha-1) + [\alpha + 2B(\phi,\alpha)][\phi - (\alpha + B(\phi,\alpha))\Lambda(\phi,\alpha,B(\phi,\alpha))]}{2(r+\phi) + (\alpha + B(\phi,\alpha))\Lambda(\phi,\alpha,B(\phi,\alpha))}, \tag{A.15}$$

for $(\phi, x) \in (0,\infty) \times (0,1)$, which is well-defined for all values $\phi > 0$.

To conclude the proof of the theorem, there are two final steps:

1. Determination of the remaining coefficients. From the three matching coefficient conditions (A.7), v_2, v_3 and v_5 are determined using δ, α and β as follows:

$$v_2 = \frac{2\delta\mu}{\lambda},\ v_3 = \frac{\alpha+2\beta}{2\lambda} > 0, \quad \text{and} \quad v_5 = \frac{\alpha-1}{\lambda} < 0.$$

As for v_1 and v_4 (corresponding to θ and θ^2 in the value function) these can be obtained by differentiating the HJB equation with respect to θ. Specifically,

$$(r+\kappa)[v_1 + 2v_4\theta + v_5 M] \quad = \quad (\delta\mu + \alpha\theta + \beta M)[1 + v_5\lambda] - v_5\phi\left[M - \mu + \lambda\bar{Y}\right] - 2v_4\kappa(\theta-\mu)$$

leads to the additional equations

$$\begin{aligned} 2(r+\kappa)v_4 &= \alpha \cdot \underbrace{[1+\lambda v_5]}_{=\alpha;\text{ system } (A.7)} - 2v_4\kappa \Rightarrow v_4 = \frac{\alpha^2}{2(r+2\kappa)}, \quad \text{and}, \\ (r+\kappa)v_1 &= \delta\mu\alpha + v_5\phi(\mu - \lambda\bar{Y}) \Rightarrow v_1 = \frac{\delta\mu\alpha}{r+\kappa} + \frac{\phi(\mu-\lambda\bar{Y})\alpha\beta}{(r+\kappa+\phi)(r+\kappa)}. \end{aligned}$$

The coefficient v_0 can be found by equating the constant terms in the HJB equation—since the value function is quadratic, there is no constraint on this coefficient.

2. Transversality conditions and admissibility of the candidate equilibrium strategy (6). This is verified in the online Appendix.

This conclude the proof. □

Proofs for Section 4.4

Proof of Proposition 2. Consider the partial differential equation (PDE)

$$\begin{aligned} -(\alpha+\beta)[\delta\mu+\alpha\theta+\beta M]+\mathcal{L}F(\theta,M)-(r+\phi)F(\theta,M) &= 0 \\ \lim_{r\to\infty} e^{-rt}\mathbb{E}_0[F(\theta_t^{\vartheta},M_t^m)] &= 0, \end{aligned}$$

where $\mathcal{L}F := -\kappa(\theta-\mu)F_\theta+[-\phi(M-\mu+\lambda\bar{Y})+\lambda(\delta\mu+\alpha\theta+\beta M)]F_M+\frac{\sigma_\theta^2}{2}F_{\theta\theta}+\frac{(\lambda\sigma_\xi)^2}{2}F_{MM}$ and $(\theta_t^{\vartheta},M_t^m))_{t\geq 0}$ is the type-belief process starting from $(\theta_0,M_0)=(\vartheta,m)\in\mathbb{R}^2$.

From the proof of Proposition 1, the previous equation admits as solution the function

$$V_M(\theta,M)=v_2+2v_3M+v_5\theta$$

where v_2, v_3 and v_5 are the coefficients of the consumer's value function on M, M^2, and $M\theta$, respectively. In fact, display (A.8) shows that the previous function satisfies the PDE, while the transversality condition follows directly from $(\theta_t^{\vartheta})_{t\geq 0}$ and $(M_t^m)_{t\geq 0}$ being mean reverting and V_M being linear.

Importantly, $V_M(\cdot,\cdot)$ (i) is of class C^2 and (ii) exhibits quadratic growth. Thus, the Feynman-Kac Representation Theorem (Remark 3.5.6 in Pham 2009) applies: namely,

$$V_M(\vartheta,m)=-\mathbb{E}_0\left[\int_0^\infty e^{-(r+\phi)t}(\alpha+\beta)Q_t dt\right],\ \forall t\geq 0,$$

where we used that $Q_t=\delta\mu+\alpha\theta_t+\beta M_t$ in equilibrium. The result then follows from $V_M(\theta_t,M_t)=-\mathbb{E}_0\left[\int_0^\infty e^{-(r+\phi)t}(\alpha+\beta)Q_t dt\right]=-\mathbb{E}_t\left[\int_t^\infty e^{-(r+\phi)(s-t)}(\alpha+\beta)Q_s ds\right]$ if $(\theta_t,M_t)=$ $(\theta_0,M_0)=(\vartheta,m)\in\mathbb{R}^2$. This concludes the proof. $\square$

Proof of Proposition 3. (i) Limits. Let $\ell(\phi,\alpha):=\alpha\sigma_\theta^2[\alpha+B(\phi,\alpha)]+\kappa\sigma_\xi^2(\phi+\kappa)$ and

$$J(\phi):=\sqrt{[\ell(\phi,\alpha(\phi))]^2-4\kappa(\sigma_\xi\sigma_\theta)^2B(\phi,\alpha(\phi))\alpha(\phi)\phi}-\ell(\phi,\alpha(\phi)).$$

With this in hand, observe that (13) (or, equivalently, $A(\phi,\alpha(\phi))=0$, where $A(\phi,x)$ is defined in (A.13)), becomes $(r+\kappa+\phi)(\alpha(\phi)-1)+\alpha(\phi)J(\phi)/[2\kappa\sigma_\xi^2]=0$.

Since $\alpha(\phi)\in(0,1)$ for all $\phi>0$, and $0<|B(\phi,\alpha)|<1/2$ for all $\alpha\in(0,1)$ and $\phi>0$, we have that $0<-4\kappa(\sigma_\xi\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi\to 0$ as $\phi\to 0$. In addition, because $\alpha(\phi)+\beta(\phi)>0$, it follows that $\ell(\phi,\alpha) > \kappa^2\sigma_\xi^2$. Using that $\beta(\phi) = B(\phi,\alpha(\phi))$ then yields,

$$0 < J(\phi) = \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi}{\sqrt{[\ell(\phi,\alpha(\phi))]^2 - 4\kappa(\sigma_\xi\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi} + \ell(\phi,\alpha(\phi))} < \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2\beta(\phi)\alpha(\phi)\phi}{2\kappa^2\sigma_\xi^2}.$$

We conclude that $\lim_{\phi\to 0}\alpha(\phi)$ exists and takes value 1.

As for the limit to $+\infty$, notice that since $\ell(\phi,\alpha(\phi)) \geq \kappa\sigma_\xi^2\phi$ and $\alpha(\phi)B(\phi,\alpha(\phi)) < 0$,

$$0 < J(\phi) \;=\; \frac{-4\kappa(\sigma_\xi\sigma_\theta)^2 B(\phi,\alpha(\phi))\alpha(\phi)}{\sqrt{\left[\frac{\ell(\phi,\alpha(\phi))}{\phi}\right]^2 - \frac{4\kappa(\sigma_\xi\sigma_\theta)^2 B(\phi,\alpha(\phi))\alpha(\phi)}{\phi}} + \frac{\ell(\phi,\alpha(\phi))}{\phi}} \leq -\frac{4\kappa(\sigma_\xi\sigma_\theta)^2 B(\phi,\alpha(\phi))\alpha(\phi)}{2\sigma_\xi^2\kappa}.$$

But since $\alpha(\cdot)$ and $B(\cdot,\alpha(\cdot))$ are bounded, $J(\cdot)$ is bounded too. Thus, from $A(\phi,\alpha(\phi)) = 0$,

$$1 - \alpha(\phi) = \underbrace{\frac{\alpha(\phi)J(\phi)}{2\kappa\sigma_\xi^2}}_{\text{bounded}}\underbrace{\frac{1}{(r+\kappa+\phi)}}_{\to 0 \text{ as } \phi\to\infty} \to 0 \text{ as } \phi \to \infty.$$

Regarding the limit values for $\beta(\phi) = B(\phi,\alpha(\phi))$, these follow from the limit behavior of $\alpha(\phi)$ and the definition of B given by (A.11). As for $\delta(\phi)$, recall from (A.15) that

$$\delta(\phi) \;=\; \frac{\kappa(\alpha(\phi)-1) + [\alpha(\phi)+2\beta(\phi)][\phi - (\alpha(\phi)+\beta(\phi))\lambda(\phi)]}{2(r+\phi) + (\alpha(\phi)+\beta(\phi))\lambda(\phi)}.$$

From Lemma 5 next, however, $\lambda(\phi) \to 0$ as $\phi \to 0$. Using that $\alpha(\phi) \to 1$ and $\alpha(\phi)+2\beta(\phi) \to 0$ as $\phi \to 0$, and that $\alpha(\phi)+\beta(\phi) > 0$, it direct that $\delta(\phi) \to 0$ as $\phi \to 0$. Also from the same lemma, $\lambda(\phi) \to \sigma_\theta^2/\kappa\sigma_\xi^2$ as $\phi \to \infty$. Thus, $[\phi - (\alpha(\phi)+\beta(\phi))\lambda(\phi)]/[2(r+\phi) + (\alpha(\phi)+\beta(\phi))\lambda(\phi)] \to 1/2$ as $\phi \to \infty$. The limit $\delta(\phi) \to 0$ as $\phi \to \infty$ then follows from $\alpha(\phi) \to 1$ and $\alpha(\phi)+2\beta(\phi) \to 0$ as $\phi \to \infty$.

It remains to show the limit result on prices. To this end, we start with a preliminary

Lemma 5. $\lim_{\phi\to 0}\lambda(\phi) = 0$, $\lim_{\phi\to\infty}\lambda(\phi) = \sigma_\theta^2/\kappa\sigma_\xi^2$ *and* $\lim_{\phi\to 0}\lambda(\phi)/\phi = 2\sigma_\theta^2/[\sigma_\theta^2 + 2\sigma_\xi^2\kappa^2]$.

Proof. $\lim_{\phi\to 0}\lambda(\phi) = 0$ is direct consequence of the first bound in (A.18) which we establish shortly in the proof of part (ii) of the Proposition. Also, letting $\ell(\phi,\alpha) := \sigma_\theta^2[\alpha + B(\phi,\alpha)] + \kappa\sigma_\xi^2[\phi+\kappa]$, it is straightforward to verify that

$$\lambda(\phi) = \frac{4\kappa(\sigma_\xi\sigma_\theta)^2\alpha(\phi)}{2\kappa\sigma_\xi^2\left(\sqrt{\left[\frac{\ell(\phi,\alpha(\phi))}{\phi}\right]^2 - \frac{4\kappa(\sigma_\xi\sigma_\theta)^2 B(\phi,\alpha(\phi))\alpha(\phi)}{\phi}} + \frac{\ell(\phi,\alpha(\phi))}{\phi}\right)} \to \frac{4\kappa(\sigma_\xi\sigma_\theta)^2}{2\kappa\sigma_\xi^2[\kappa\sigma_\xi^2 + \kappa\sigma_\xi^2]} = \frac{\sigma_\theta^2}{\kappa\sigma_\xi^2}$$

as $\phi \to \infty$, and thus the second limit holds. The third limit follows directly from the first equality in the previous display. This ends the proof of the lemma. $\square$

Using the lemma, we first show that $\lim_{\phi\to\infty} \mathrm{Var}[\lambda(\phi)Y_t] = \lim_{\phi\to 0} \mathrm{Var}[\lambda(\phi)Y_t] = 0$. Recall that $(\delta(\phi), \alpha(\phi), \beta(\phi)) \to (0, 1, -1/2)$ as $\phi \to 0$ and $+\infty$. Also, from (A.4),

$$\mathrm{Var}[Y_t] = \frac{1}{2(\phi - \beta(\phi)\lambda(\phi))}\left[\sigma_\xi^2 + \frac{\alpha(\phi)\sigma_\theta^2}{\kappa[\phi - \beta(\phi)\lambda(\phi) + \kappa]}\right]. \tag{A.16}$$

By the previous lemma, therefore, $\lim_{\phi\to\infty} \mathrm{Var}[Y_t] = 0$, and so $\lim_{\phi\to\infty} \mathrm{Var}[\lambda(\phi)Y_t] = 0$. As for the other limit, we can write (A.16) as

$$\mathrm{Var}[\lambda(\phi)Y_t] \;=\; \underbrace{\frac{1}{2(\frac{\phi}{\lambda(\phi)} - \beta(\phi))}}_{\to\text{ constant}} \underbrace{\lambda(\phi)}_{\to 0} \underbrace{\left[\sigma_\xi^2 + \frac{\alpha(\phi)\sigma_\theta^2}{\kappa[\phi - \beta(\phi)\lambda(\phi) + \kappa]}\right]}_{\to \sigma_\xi^2 + \sigma_\theta^2/\kappa^2} \to 0 \text{ as } \; \phi \to 0.$$

The L^2-limits then follow directly from the following results: $(\delta(\phi), \alpha(\phi), \beta(\phi)) \to (0, 1, -1/2)$ as $\phi \to 0, \infty$; $P_t = \delta\mu + (\alpha+\beta)M_t$ and $M_t = \mu + \lambda[Y_t - \bar{Y}]$; $\mathbb{E}[P_t] = \mu[\alpha(\phi) + \beta(\phi) + \delta(\phi)] \to \mu/2$ as $\phi \to 0$ and $+\infty$; and the triangular inequality.

(ii) Bounds. Observe that the bounds for $\beta(\phi)$ were already determined from (A.11) and $\alpha(\phi) \in (0, 1)$. As for the lower bound for α, we will show the stronger result

$$\max\left\{\frac{r+\kappa+\phi}{r+\kappa+2\phi}, \frac{r+\kappa+\phi}{r+\kappa+\phi+\sigma_\theta^2/2\kappa\sigma_\xi^2}\right\} \le \alpha(\phi).$$

The bound is tight in the sense that it converges to 1 when $\phi \to 0$ and $+\infty$.

To obtain the bound, observe that from (A.12), $\lambda(\phi)$ satisfies

$$\lambda(\phi) = \frac{2\sigma_\theta^2\alpha(\phi)\phi}{\sqrt{\ell^2(\phi, \alpha(\phi)) - 4\kappa(\sigma_\theta\sigma_\xi)^2\alpha(\phi)B(\phi, \alpha(\phi))\phi} + \ell(\phi, \alpha(\phi))} < \frac{\sigma_\theta^2\alpha(\phi)\phi}{\ell(\phi, \alpha(\phi))}, \tag{A.17}$$

where $\ell(\phi, \alpha(\phi)) := \sigma_\theta^2\alpha(\phi)[\alpha(\phi) + B(\phi, \alpha(\phi))] + \kappa\sigma_\xi^2[\phi + \kappa]$. From here, we get two bounds:

$$\lambda < \frac{2\phi}{\alpha(\phi)} \quad \text{and} \quad \lambda < \frac{\sigma_\theta^2}{\kappa\sigma_\xi^2}. \tag{A.18}$$

In fact, since $B(\phi, \alpha(\phi)) \ge -\alpha/2$, then $\ell(\phi, \alpha(\phi)) > \sigma_\theta^2\alpha(\phi)^2/2$; but using this in (A.17) leads to the first inequality in (A.18). Similarly, the second upper bound follows from (A.17) using that $\alpha(\phi) < 1$ and that $\ell(\phi, \alpha(\phi)) > \kappa\sigma_\xi^2\phi$ due to $\alpha + B(\phi, \alpha) > 0$. In particular, $\lambda(\phi)$ is bounded over $\mathbb{R}_+$, and it converges to zero as $\phi \to 0$, as promised.

Consider now the locus $A(\phi, \alpha(\phi)) = 0$. Using the first bound in (A.18) yields

$$0 = (r+\kappa+\phi)(\alpha(\phi)-1) + \underbrace{\lambda(\phi)}_{\leq 2\phi/\alpha(\phi)} \alpha(\phi) \underbrace{[-B(\phi, \alpha(\phi))]}_{\in(0, \alpha(\phi)/2)} < (r+\kappa+\phi)(\alpha(\phi)-1) + \phi\alpha(\phi)$$

$$\Rightarrow \alpha(\phi) > \frac{r+\kappa+\phi}{r+\kappa+2\phi} > \frac{1}{2}, \quad \text{for all} \quad \phi > 0.$$

Similarly, using the second bound, $0 < (r+\kappa+\phi)(\alpha(\phi)-1) + [\sigma_\theta \alpha(\phi)]^2/[2\kappa\sigma_\xi^2]$; the desired second bound for alpha follows from imposing that $\alpha^2 < \alpha$ in the previous inequality.

We conclude this part by establishing the bounds for the expected price, and omit the dependence of $(\alpha, \beta, \delta, \lambda)$ on ϕ in the process. Observe that $\mathbb{E}[P_t] = \delta\mu + (\alpha+\beta)\mathbb{E}[M_t] = [\delta+\alpha+\beta]\mu$. Now, adding the second and third equation in the system (A.10) yields $(\alpha+2\beta)(\alpha+\beta)\lambda = (r+2\phi)(\alpha+2\beta) + (r+\kappa+\phi)(\alpha-1) + (\alpha+\beta)^2\lambda$. Thus, in equilibrium,

$$\begin{aligned}
\delta &= \frac{\kappa(\alpha-1) + [\alpha+2\beta][\phi - (\alpha+\beta)\lambda]}{2(r+\phi) + (\alpha+\beta)\lambda} \\
&= \frac{\kappa(\alpha-1) + (\alpha+2\beta)\phi - (r+2\phi)(\alpha+2\beta) - (r+\kappa+\phi)(\alpha-1) - (\alpha+\beta)^2\lambda}{2(r+\phi) + (\alpha+\beta)\lambda} \\
&= \frac{-(r+\phi)[2(\alpha+\beta)-1] - (\alpha+\beta)^2\lambda}{2(r+\phi) + (\alpha+\beta)\lambda},
\end{aligned}$$

from where it is easy to conclude that

$$\mathbb{E}[P_t] \;=\; \mu[\alpha+\beta+\delta] = \mu\frac{r+\phi}{2(r+\phi) + (\alpha+\beta)\lambda}. \tag{A.19}$$

In particular, $\mathbb{E}[P_t] < \mu/2$ when $\mu \neq 0$ follows directly from $\lambda(\alpha+\beta) > 0$.

On the other hand, from (A.17) and $\ell(\phi, \alpha) > \sigma_\theta^2 \alpha[\alpha + B(\phi, \alpha)] = \sigma_\theta^2\alpha[\alpha+\beta]$,

$$(\alpha+\beta)\lambda < (\alpha+\beta)\frac{\sigma_\theta^2 \alpha\phi}{\ell(\phi, \alpha)} < (\alpha+\beta)\frac{\sigma_\theta^2\alpha(\phi)\phi}{\sigma_\theta^2\alpha[\alpha+\beta]} = \phi.$$

Using this latter bound in (A.19) leads to $\mathbb{E}[P_t] > \mu/3$ whenever $\mu \neq 0$, as $r > 0$, .

(iii) Quasiconvexity of α. To prove this property, it is more useful to solve the last two equations in the system (A.10) for λ and β, namely,

$$\lambda(\phi, \alpha) = -\frac{(\alpha-1)(\kappa+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}{\alpha^3(r+2\phi)}, \tag{A.20}$$

$$\beta(\phi, \alpha) = -\frac{\alpha^2(r+2\phi)}{\kappa+\alpha(-\kappa+r+3\phi)+r+\phi}. \tag{A.21}$$

Substituting both expressions into (7) that defines λ, and recalling $s := \sigma_\xi^2/\sigma_\theta^2$, we obtain an alternate locus $(\phi, \alpha(\phi))$ that satisfies

$$\tilde{A}(\phi,\alpha) := \frac{(\alpha-1)(\kappa+r+\phi)(\kappa+\alpha(-\kappa+r+3\phi)+r+\phi)}{\alpha^3(r+2\phi)} + \frac{\alpha(\kappa-\alpha(\kappa+r)+r+\phi)}{\alpha^3+\kappa s(\kappa-\alpha r+r+\phi)} = 0. \tag{A.22}$$

Observe that $\tilde{A}$ is increasing in α whenever $\tilde{A}(\phi,\alpha) = 0$. In fact, since Proposition 1 establishes the uniqueness of an equilibrium, there is a unique $\alpha(\phi) \in [0,1]$ solving $\tilde{A}(\phi,\alpha) = 0$. In addition, $\tilde{A}(\phi,1) = \phi/[\kappa s(\kappa+\phi)+1] > 0$. Thus $\tilde{A}(\phi,\cdot)$ must cross zero from below.

Now, the second partial derivative

$$\frac{\partial^2 \tilde{A}(\phi,\alpha)}{(\partial\phi)^2} = -\frac{2(\alpha-1)^2(2\kappa+r)^2}{(r+2\phi)^3} - \frac{2\alpha^5\kappa s\left(\alpha^2+\kappa^2 s\right)}{\left(\alpha^3+\kappa s(\kappa-\alpha r+r+\phi)\right)^3}$$

is strictly negative because, by inspection, the first term is nonpositive and the second term is strictly negative. Furthermore, we know from Proposition 1 that $\alpha(\phi)$ is twice continuously differentiable. Combined with the fact that $\tilde{A}$ is increasing in its second argument whenever $\tilde{A} = 0$, the Implicit Function Theorem implies that $\alpha''(\phi) > 0$ at any critical point $\alpha'(\phi) = 0$.

(iv) Effect of noise terms σ_ξ/σ_θ. To show that $\phi \mapsto \alpha(\phi)$ is increasing in σ_ξ/σ_θ point-wise, consider again the locus $\tilde{A}(\phi,\alpha) = 0$ in (A.22), and differentiate with respect to $s := \sigma_\xi^2/\sigma_\theta^2$. We obtain

$$\frac{\partial \tilde{A}}{\partial s} = -\frac{\alpha^4\kappa((1-\alpha)(\kappa+r)+\phi)(\kappa+(1-\alpha)r+\phi)}{\left(\alpha^3+\kappa s(\kappa+(1-\alpha)r+\phi)\right)^2} < 0.$$

Because $\tilde{A}$ is increasing in α at $(\phi, \alpha(\phi))$, we conclude that α is increasing in s.

Finally, using the three equations (A.20)–(A.22), the derivative of the expected price $[\alpha+\beta+\delta]\mu$ with respect to α can be written as

$$\mu\frac{\alpha(r+\phi)(r+2\phi)(\kappa+r+\phi)(2(\kappa+r+\phi)-\alpha(2\kappa+r))}{\left[\alpha^2\left(\kappa^2+r(\kappa+2r)+5r\phi+3\phi^2\right)-\alpha(2\kappa+r)(\kappa+r+\phi)+(\kappa+r+\phi)^2\right]^2} > 0.$$

Furthermore, when using (A.20)–(A.22), the expected price does not depend on s directly. Using that $\phi \mapsto \alpha(\phi)$ is increasing in σ_ξ/σ_θ, therefore, the expected price is also increasing in s. This concludes the proof. □

Proofs for Section 5

Proof of Proposition 4. Refer to the Online Appendix. □

The following results are used in the subsequent analysis, and their proofs can be found in the online Appendix.

Lemma 6. *Suppose $\alpha > 0$ and $\beta < 0$ satisfy $\nu(\alpha, \beta) > 0$, where $\nu(\alpha, \beta)$ is defined in (21). Then, $\phi \mapsto G(\phi, \alpha, \beta)$ has a unique maximizer located at $\phi = \nu(\alpha, \beta)$. Moreover,*

(i) $\Lambda_\phi(\nu(\alpha, \beta), \alpha, \beta) = \Lambda(\nu(\alpha, \beta), \alpha, \beta)/(\nu+\kappa)$, where $\Lambda_\phi(\phi, \alpha, \beta)$ denotes the partial derivative of $\Lambda(\phi, \alpha, \beta)$ with respect to ϕ, and,

(ii) $\Lambda(\nu(\alpha, \beta), \alpha, \beta) = \alpha\gamma(\alpha)/\sigma_\xi^2$, where $\gamma(\alpha) = \sigma_\xi^2[(\kappa^2+\alpha^2\sigma_\theta^2/\sigma_\xi^2)^{1/2}-\kappa]/\alpha^2$ is the posterior belief's stationary variance when the histories ξ^t, $t \geq 0$, are public.

Lemma 7. *$\kappa < \arg\min \alpha < \infty$, and $[\alpha+\beta]'(\phi) < 0$, $\phi \in [\kappa, \arg\min \alpha]$; hence, $\alpha+\beta$ is strictly decreasing at any point satisfying (22). If $r > \kappa$, $[\alpha + \beta]'(\phi) < 0$ for all $\phi \in [0, \arg\min \alpha]$.*

Proof of Proposition 5. We first show that $\alpha'(\phi) < 0$ at any ϕ satisfying (22), i.e., $\phi = \nu(\alpha(\phi), \beta(\phi))$. To this end, recall that $\alpha(\phi)$ is the only value in $(0, 1)$ satisfying $(r+\kappa+\phi)(\alpha(\phi)-1) + \alpha(\phi)H(\phi, \alpha(\phi)) = 0$, where

$$H(\phi, \alpha) := -\Lambda(\phi, \alpha, B(\phi, \alpha))B(\phi, \alpha) = \frac{\sqrt{\ell^2(\phi, \alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha B(\phi, \alpha)\phi} - \ell(\phi, \alpha)}{2\kappa\sigma_\xi^2}$$

and $\ell(\phi, \alpha) := \sigma_\theta^2\alpha[\alpha + B(\phi, \alpha)] + \kappa\sigma_\xi^2[\phi + \kappa]$. Also, recall from the proof of Lemma 4 in the proof of proposition 1 that $\alpha \mapsto H(\phi, \alpha)$ is strictly increasing over $[0, 1]$.

Thus, denoting the partial derivatives with subindices,

$$\alpha'(\phi)\,[r + \kappa + \phi + H(\phi, \alpha(\phi)) + \alpha(\phi)H_\alpha(\phi, \alpha(\phi))] = 1 - \alpha(\phi) - \alpha(\phi)H_\phi(\phi, \alpha(\phi)).$$

Consequently, because $H > 0$, we conclude that the sign of α' is always determined by the sign of the right-hand side of the previous expression. We now show that the latter side is negative at any point ϕ s.t. $\phi = \nu(\alpha(\phi), \beta(\phi)) = \kappa + \alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi) + \beta(\phi)]/\sigma_\xi^2$.

To simplify notation, let $\Delta(\phi, \alpha) := \sqrt{\ell^2(\phi, \alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha B(\phi, \alpha)\phi}$. Omitting the dependence on $(\phi, \alpha(\phi))$ of H, Δ, ℓ, B, and of their respective partial derivatives,

$$H_\phi = \frac{1}{2\kappa\sigma_\xi^2}\left[\frac{\ell\ell_\phi - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha[\phi B_\phi + B]}{\Delta} - \ell_\phi\right].$$

Moreover, since $\ell_\phi = \sigma_\theta^2\alpha B_\phi + \kappa\sigma_\xi^2$ we can write

$$H_\phi \quad = \quad \frac{\kappa\sigma_\xi^2[\ell - \Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha B}{2\kappa\sigma_\xi^2\Delta} + \frac{\sigma_\theta^2\alpha B_\phi[\ell - \Delta] - 2\kappa(\sigma_\xi\sigma_\theta)^2\alpha\phi B_\phi}{2\kappa\sigma_\xi^2\Delta}.$$

Consider now the first term of the previous expression. In fact,

$$\frac{\kappa\sigma_\xi^2[\ell-\Delta]-2\kappa(\sigma_\xi\sigma_\theta)^2\alpha B}{2\kappa\sigma_\xi^2\Delta}=-B\frac{\partial\Lambda}{\partial\phi}(\phi,\alpha,B).$$

From Lemma 6, moreover, $\Lambda_\phi(\nu(\alpha,\beta),\alpha,\beta)=\Lambda(\nu(\alpha,\beta),\alpha,\beta)/[\nu(\alpha,\beta)+\kappa]$; therefore, this equality must holds at any ϕ such that $(\phi,\alpha(\phi),\beta(\phi))=(\nu(\alpha(\phi),\beta(\phi)),\alpha(\phi),\beta(\phi))$.

On the other hand, the second term of H_ϕ can be written as

$$\frac{\sigma_\theta^2B_\phi}{\Delta}\left[\alpha\frac{\ell-\Delta}{2\kappa\sigma_\xi^2}-\phi\alpha\right]=\frac{\sigma_\theta^2B_\phi}{\Delta}\left[(r+\kappa+\phi)(\alpha-1)-\phi\alpha\right],$$

where we used that $\alpha H=\alpha(\Delta-\ell)/2\kappa\sigma_\xi^2$. We deduce that, at the point of interest,

$$1-\alpha-\alpha H_\phi\;=\;\underbrace{1-\alpha+\frac{\lambda\alpha\beta}{\phi+\kappa}}_{K_1:=}-\underbrace{\frac{\sigma_\theta^2\alpha B_\phi}{\Delta}\left[(r+\kappa+\phi)(\alpha-1)-\phi\alpha\right]}_{K_2:=}\tag{A.23}$$

Straightforward differentiation shows that

$$B_\phi=\frac{\partial}{\partial\phi}\left(\frac{-\alpha^2(r+2\phi)}{2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)}\right)=\frac{\alpha^2(\alpha-1)(r+2\kappa)}{[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]^2}<0,$$

so $K_2>0$. As for the other term, $(r+\kappa+\phi)(\alpha-1)-\lambda\alpha\beta=0$ yields

$$K_1=\frac{(\phi+\kappa)(1-\alpha)+\lambda\alpha\beta}{\phi+\kappa}=\frac{r(\alpha-1)}{\phi+\kappa}<0.$$

We conclude that $\alpha'(\phi)<0$ at any point satisfying (22), provided any such point exists.

For existence, let $\eta(\phi):=\phi-\nu(\alpha(\phi),\beta(\phi))$, where $\nu(\alpha,\beta)=\kappa+\alpha\gamma(\alpha)[\alpha+\beta]/\sigma_\xi^2$, and

$$\gamma(\alpha):=\frac{\sigma_\xi^2}{\alpha^2}\left[\sqrt{\kappa^2+\alpha^2\frac{\sigma_\theta^2}{\sigma_\xi^2}}-\kappa\right],$$

(i.e., $\gamma(\alpha)$ is the unique positive solution of $0=\sigma_\theta^2-2\kappa\gamma-(\alpha\gamma/\sigma_\xi)^2$). Since $\alpha\in(1/2,1)$, γ is bounded, and so $\eta(\phi)>0$ for ϕ large. Also, using that $\lim\limits_{\phi\to0}(\alpha(\phi),\beta(\phi))=(1,-1/2)$, we have that $\lim\limits_{\phi\to0}\eta(\phi)<0$. The existence of ϕ s.t. $\eta(\phi)=0$ follows from the continuity of $\eta(\cdot)$.

Now, observe that $\alpha>0$ and $\alpha+\beta>\alpha/2>0$ imply that $\nu(\alpha(\phi),\beta(\phi))>\kappa$. On the other hand, since $\beta < 0$ and $\alpha < 1$,

$$\nu(\alpha(\phi), \beta(\phi)) < \kappa + \frac{\alpha(\phi)^2\gamma(\alpha(\phi))}{\sigma_\xi^2} = \sqrt{\kappa^2 + \alpha(\phi)^2\frac{\sigma_\theta^2}{\sigma_\xi^2}} < \sqrt{\kappa^2 + \frac{\sigma_\theta^2}{\sigma_\xi^2}}.$$

As a result, any such point must lie on $[\kappa, \sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2}]$, which establishes (ii).

We conclude by proving the uniqueness of a non-concealing point. Notice first that, because a point like that exists and α is decreasing at any such point, the previous lower bound and α's properties (bounds, limits and quasiconvexity) imply that $\arg\min\alpha > \kappa$; as a byproduct any such point must lie on $[\kappa, \arg\min\alpha]$.

With this in hand, observe that it suffices show that $[\nu(\alpha(\phi), \beta(\phi))]' < 0$ over $[\kappa, \arg\min\alpha]$. In fact, because the identity function is increasing, the existence of two such points would imply the existence of an intermediate third point at which $\eta(\cdot)$ vanishes and $[\nu(\alpha(\phi), \beta(\phi))]' > 0$, yielding a contradiction. To this end, write

$$[\nu(\alpha(\phi), \beta(\phi))]' = \frac{d}{d\phi}\left(\frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2}\right)(\alpha(\phi) + \beta(\phi)) + \left(\frac{\alpha(\phi)\gamma(\alpha(\phi))}{\sigma_\xi^2}\right)\frac{d(\alpha(\phi) + \beta(\phi))}{d\phi}.$$

From Lemma 7, $\alpha(\phi) + \beta(\phi)$ is strictly decreasing over $[\kappa, \arg\min\alpha]$. Since $\alpha + \beta > 0$ and $\alpha\gamma(\alpha(\phi)) > 0$ is suffices to show that $[\alpha(\phi)\gamma(\alpha(\phi))]' < 0$ over the same region. However,

$$\frac{\alpha\gamma(\alpha)}{\sigma_\xi^2} = \frac{1}{\alpha}\left[\sqrt{\kappa^2 + \alpha^2\frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa\right] = \frac{\sigma_\theta^2}{\sigma_\xi^2}\left(\sqrt{\frac{\kappa^2}{\alpha} + \frac{\sigma_\theta^2}{\sigma_\xi^2}} + \frac{\kappa}{\alpha}\right)^{-1}$$

which is strictly increasing in α. We conclude by using that $\alpha' < 0$ over $[\kappa, \arg\min\alpha]$. ☐

Proof of Proposition 6. Recall that $G(\phi) = \alpha(\phi)\lambda(\phi)/[\phi + \kappa - \beta(\phi)\lambda(\phi)] \geq 0$ for all $\phi \geq 0$. Since $\lambda(\phi)$ is bounded (second bound in A.18), $\lim_{\phi\to\infty} G(\phi) = 0$. Also $G(0) = 0$. By continuity, therefore, G has a global optimum that is interior.

From the definition of $\nu(\alpha, \beta)$,

$$G(\phi) := G(\phi, \alpha(\phi), \beta(\phi)) \leq G(\nu(\alpha(\phi), \beta(\phi)), \alpha(\phi), \beta(\phi)),$$

with equality only at ϕ^*. Also, from Lemma 6, $\Lambda(\nu(\alpha, \beta), \alpha, \beta) = \alpha\gamma(\alpha)/\sigma_\xi^2$. Thus, letting $\nu(\phi) := \nu(\alpha(\phi), \beta(\phi))$,

$$G(\nu(\phi), \alpha(\phi), \beta(\phi)) = \frac{\alpha(\phi)\Lambda(\nu(\phi), \alpha(\phi), \beta(\phi))}{\nu(\phi) + \kappa - \beta(\phi)\Lambda(\nu(\phi), \alpha(\phi), \beta(\phi))} = \frac{\alpha^2(\phi)\gamma(\alpha(\phi))}{\alpha^2(\phi)\gamma(\alpha(\phi)) + 2\kappa\sigma_\xi^2}, \quad \text{(A.24)}$$

where we used that $\nu(\phi) = \kappa + \alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi) + \beta(\phi)]/\sigma_\xi^2$.

However, by definition of $\gamma(\alpha)$, $\alpha^2\gamma(\alpha) = \sigma_\xi^2[(\kappa^2 + \alpha^2(\phi)\sigma_\theta^2/\sigma_\xi^2)^{1/2} - \kappa]$; thus, from (A.24), $G(\nu(\phi), \alpha(\phi), \beta(\phi))$ is decreasing when $\alpha(\phi)$ is decreasing. Since $G(\phi)$ is bounded from above by a decreasing function of ϕ on $[\phi^*, \arg\min \alpha]$, $G(\phi^*) > G(\phi)$ over the same interval.

To conclude the proof, we show that $G(\phi)$ is decreasing when $\alpha(\phi)$ is increasing, i.e., over $(\arg\min \alpha(\phi), \infty)$. Using that $G(\phi, \alpha, \beta) := \alpha\Lambda(\phi, \alpha, \beta)/[\phi + \kappa - \beta\Lambda(\phi, \alpha, \beta)]$, and equations (A.20) and (A.21) to substitute for λ and β, we obtain that $G(\phi) = \tilde{G}(\phi, \alpha(\phi))$ where

$$\tilde{G}(\alpha, \phi) := (1 - \alpha)\frac{(\kappa + r + \phi)}{(\kappa + (1 - \alpha)r + \phi)}\frac{(\kappa + \alpha(-\kappa + r + 3\phi) + r + \phi)}{\alpha(r + 2\phi)}. \tag{A.25}$$

Using that $\alpha \in (0, 1)$, it is easy to verify that the two fractions on the right-hand side are strictly positive and strictly decreasing in ϕ. Thus, $\frac{\partial \tilde{G}}{\partial \phi} < 0$. Also, up to a positive term,

$$\frac{\partial \tilde{G}}{\partial \alpha} = \alpha^2 \left(-\left(-\kappa^2 + r^2 + 2\phi(\kappa + r) + 2\kappa r + 3\phi^2\right)\right) + 2\alpha r(\kappa + r + \phi) - (\kappa + r + \phi)^2.$$

Observe that the right-hand side of the previous expression is a concave quadratic in α when $\phi > \kappa$. Moreover, over the same region, straightforward maximization yields that such quadratic is negative for all parameter values. Also, by Proposition 5 and α's quasiconvexity, $\kappa < \arg\min \alpha$. Thus, $\frac{d\tilde{G}}{d\phi} = \frac{\partial \tilde{G}}{\partial \alpha}\alpha'(\phi) + \frac{\partial \tilde{G}}{\partial \phi} < 0$, $\phi \in (\arg\min \alpha, \infty)$. □

Proofs for Section 6

Proof of Proposition 7. Observe that $\text{Var}[P_t] = [\alpha(\phi) + \beta(\phi)]^2\text{Var}[\theta_t]G(\phi)$. From the proof of Proposition 6, $\lim_{\phi \to 0,\infty} G(\phi) = 0$. Also, $\alpha + \beta$ is bounded. By continuity, we conclude that $\text{Var}[P_t]$ has a global optimum that is interior.

From Proposition 6, however, $G(\phi)$ is maximized to the left of ϕ^*. Also, from Lemma 7, $\alpha'(\phi) + \beta'(\phi) < 0$ over $[\kappa, \arg\min \alpha]$. Since $\kappa < \phi^* < \arg\min \alpha$, $\text{Var}[P_t]$ cannot attain a maximum in $[\phi^*, \arg\min \alpha]$. One can then verify that the total derivative of $\text{Var}[P_t]$ with respect to ϕ is negative over $[\arg\min \alpha, +\infty)$ for all parameter values $(r, \kappa, \sigma_\theta, \sigma_\xi)$. This is done in `scores.nb` posted in our websites.

As for part (ii), this one follows from the $\alpha + \beta$ being strictly decreasing in $[0, \arg\min \alpha]$ when $r \geq \kappa$ (Lemma 7) as the previous arguments over $[\kappa, \arg\min \alpha]$ and $[\arg\min \alpha, +\infty)$ are not modified. □

Proof of Proposition 8. We start with (i) for profits. From (A.19), $\alpha + \beta + \delta > 1/3$.

Thus, omitting the dependence of the equilibrium coefficients on ϕ,

$$\Pi(\phi) := \mu^2[\alpha+\beta+\delta]^2 + \frac{\sigma_\theta^2}{2\kappa}(\alpha+\beta)^2 G(\phi) \geq \frac{\mu^2}{9} + \frac{\sigma_\theta^2(\alpha+\beta)^2}{2\kappa} G(\phi), \quad \text{for all} \quad \phi > 0.$$

On the other hand, from the proof of Proposition 7, $\lim_{\phi \to 0,\infty} (\alpha+\beta)^2 G(\phi) = 0$, and so $\lim_{\phi \to 0,\infty} \Pi(\phi) = \mu^2/4$. Thus, if

$$\frac{\mu^2}{9} + \frac{\sigma_\theta^2(\alpha+\beta)^2}{2\kappa} G(\phi) \geq \frac{\mu^2}{4} \Leftrightarrow \mu^2 \leq \frac{18\sigma_\theta^2}{5\kappa}(\alpha+\beta)^2 G(\phi),$$

it follows that $\Pi(\phi) > \mu^2/4$. Since $\phi \mapsto [\alpha(\phi)+\beta(\phi)]^2 G(\phi)$ is continuous, strictly positive, and converges to 0 as $\phi \to 0$ and $+\infty$, it has a global maximum; denote it by $\phi^\dagger$. Letting

$$\underline{\mu}_f := \left[\frac{18\sigma_\theta^2}{5\kappa}(\alpha(\phi^\dagger)+\beta(\phi^\dagger))^2 G(\phi^\dagger)\right]^{1/2} > 0$$

the result follows.

As for the consumer, let $CS_\mu(\phi)$ denote her surplus for a given $\mu \geq 0$ and observe that

$$CS_\mu(\phi) = CS_0(\phi) + \mu^2 R(\phi),$$

where $R(\phi) := [\alpha(\phi)+\beta(\phi)+\delta(\phi)]\left(1 - \frac{3}{2}[\alpha(\phi)+\beta(\phi)+\delta(\phi)]\right)$ and

$$CS_0(\phi) = \frac{\sigma_\theta^2}{2\kappa} G(\phi) L(\phi) + \frac{\sigma_\theta^2}{2\kappa}\left[\alpha(\phi) - \frac{(\alpha(\phi))^2}{2}\right].$$

Importantly since $\alpha(\phi)+\beta(\phi)+\delta(\phi) \to 1/2$ as $\phi \to 0$ and $+\infty$, we have that $\lim_{\phi \to 0} R(\phi) = \lim_{\phi \to +\infty} R(\phi) = 1/8$. In addition we know that $1/3 < \alpha(\phi)+\beta(\phi)+\delta(\phi) < 1/2$ for all $\phi > 0$. Because the function $x \mapsto x - 3x^2/2$ is strictly decreasing in $[1/3, 1/2]$, we have that $R(\phi) > 1/8, \quad \text{for all} \quad \phi > 0$.

Fix any $\hat{\phi} > 0$. Then, using that $CS_\mu(0) = \mu^2/8$,

$$CS_\mu(\hat{\phi}) - CS_\mu(0) = \mu^2\left[R(\hat{\phi}) - \frac{1}{8}\right] + \frac{\sigma_\theta^2}{2\kappa}\underbrace{\left[G(\hat{\phi})L(\hat{\phi}) + \alpha(\hat{\phi}) - \frac{(\alpha(\hat{\phi}))^2}{2} - \frac{1}{2}\right]}_{=:K(\hat{\phi})}.$$

Observe that $K(\cdot)$ and $R(\cdot)$ are independent of μ, we can choose μ arbitrarily large such that the right-hand side is strictly positive. Since $CS_\mu(0) = \lim_{\phi \to \infty} CS_\mu(0)$, the consumer's global maximum ϕ^c must be interior.

We now turn to (ii), starting with the firms' case. Towards a contradiction, suppose that there are sequences $\mu_n \nearrow \infty$ and $\phi_n > 0$ for all $n \in \mathbb{N}$ such that $\Pi_{\mu_n}(\phi_n) \geq \Pi_{\mu_n}(0) = \Pi_{\mu_n}(+\infty)$. Then,

$$\Pi_{\mu_n}(\phi_n) \geq \Pi_{\mu_n}(0) \Leftrightarrow \frac{\sigma_\theta^2}{2\kappa}[\alpha(\phi_n)+\beta(\phi_n)]^2 G(\phi_n) \geq \mu_n^2 \left[\frac{1}{4} - [\alpha(\phi_n)+\beta(\phi_n)+\delta(\phi_n)]^2\right]$$

Observe first that $(\phi_n)_{n\in\mathbb{N}}$ cannot have a cluster point different from zero. Otherwise, along such subsequence, say $(\phi_{n_k})_{k\in\mathbb{N}}$, both $[\alpha(\phi_{n_k})+\beta(\phi_{n_k})]^2 G(\phi_{n_k})$ and $1/4-[\alpha(\phi_{n_k})+\beta(\phi_{n_k})+\delta(\phi_{n_k})]^2$ converge to strictly positive numbers; the inequality is then violated for large k.

Suppose now that there is a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ that diverges. Using that $\alpha+\beta+\delta = (r+\phi)/[2(r+\phi)+\lambda(\alpha+\beta)]$ and $G = \alpha\lambda/[\phi+\kappa-\beta\lambda]$, we obtain

$$\Pi_{\mu_{n_k}}(\phi_{n_k}) \geq \Pi_{\mu_n}(0) \Leftrightarrow \frac{\sigma_\theta^2}{2\kappa} \frac{4(\alpha+\beta)^2\alpha\lambda[2(r+\phi)+\lambda(\alpha+\beta)]^2}{[4(r+\phi)(\alpha+\beta)\lambda+(\alpha+\beta)^2\lambda^2](\phi+\kappa-\beta\lambda)}\Bigg|_{\phi=\phi_{n_k}} \geq \mu_{n_k}.$$

However, because α, β and λ are all bounded and $(\alpha,\beta,\lambda) \to (1,-1/2,\sigma_\theta^2/[\kappa\sigma_\xi^2])$ as $\phi \to +\infty$, both the numerator and denominator are $O(\phi^2)$ for large ϕ, so the limit of the left-hand side of the second inequality exists. The inequality is then violated for large k, a contradiction.

From the previous argument, the only remaining possibility is that $(\phi_n)_{n\in\mathbb{N}}$ converges to zero. However, from Proposition 3, $\lim_{\phi\to 0}(\alpha(\phi),\beta(\phi),\lambda(\phi)) = (1,-1/2,0)$, and so

$$\begin{aligned}\frac{4(\alpha+\beta)^2\alpha\lambda[2(r+\phi)+\lambda(\alpha+\beta)]^2}{[4(r+\phi)(\alpha+\beta)\lambda+(\alpha+\beta)^2\lambda^2](\phi+\kappa-\beta\lambda)} &= \frac{4(\alpha+\beta)^2\alpha[2(r+\phi)+\lambda(\alpha+\beta)]^2}{[4(r+\phi)(\alpha+\beta)+(\alpha+\beta)^2\lambda](\phi+\kappa-\beta\lambda)} \\ &\to \frac{2r}{\kappa}, \quad \text{as} \quad \phi \to \infty,\end{aligned}$$

and so the same inequality is again violated, a contradiction.

The case for the consumer is proved in an analogous fashion. Namely, towards a contradiction, assume that there are $(\mu_n)_{n\in\mathbb{N}}$ decreasing towards zero and $(\phi_n)_{n\in\mathbb{N}}$ strictly positive such that $CS_{\mu_n}(\phi_n) \geq CS_{\mu_n}(0)$. Straightforward algebraic manipulation shows that

$$CS_{\mu_n}(\phi_n) \geq CS_{\mu_n}(0) \Leftrightarrow \frac{1}{\frac{\text{Var}[\theta_t]}{2}\left(\frac{(\alpha(\phi_n)-1)^2}{R(\phi_n)-1/8} - 2L(\phi_n)\frac{G(\phi_n)}{R(\phi_n)-1/8}\right)} \geq \frac{1}{\mu_n},$$

with $R(\phi)$ defined in part (i) of the proof and $L(\phi) := \frac{\alpha(\phi)^2}{2} + \beta(\phi) - \frac{3}{2}(\alpha(\phi)+\beta(\phi))^2 < 0$. As in the firms' case, there can't be a subsequence of $(\phi_n)_{n\in\mathbb{N}}$ converging to a value different from zero; otherwise, the left-hand side of the inequality on the right converges, but the right-hand side converges. In the Online Appendix we show that

$$\lim_{\phi\to 0,+\infty}\frac{[\alpha(\phi)-1]^2}{R(\phi)}=0, \quad \text{and} \quad \lim_{\phi\to 0,+\infty}\frac{G(\phi)}{R(\phi)-1/8}>0. \tag{A.26}$$

But since $\lim_{\phi\to 0,+\infty} L(\phi)=-1/8$, the left-hand side of the same inequality is again violated, a contradiction. Thus, there must exist $\underline{\mu}^c>0$ such that for all $\mu<\underline{\mu}^c$, $CS_\mu(0)>CS_\mu(\phi)$ for all $\phi\in(0,\infty)$. This concludes the proof. $\square$

Proof of Proposition 9. Let $\phi_{\underline{\alpha}}(r):=\arg\min\{\alpha(\phi;r):\phi\geq 0\}$, where the dependence of α on $r>0$ is explicit. Similarly, we write $\phi^*(r)$ whenever needed. Also, recall from Proposition 5 that $\phi^*(r)\in[\kappa,\sqrt{\kappa^2+\sigma_\theta^2/\sigma_\xi^2}]$ for all $r>0$.

Let $J(\phi):=\lambda(\phi)[\alpha(\phi)+\beta(\phi)]$, and recall from (A.19) that $\mathbb{E}[P_t]=\mu(r+\phi)/[2(r+\phi)+J(\phi)]$. Thus, we need to study $J(\phi)/(r+\phi)$. However, using that, $(r+\kappa+\phi)(\alpha-1)-\lambda\beta\alpha=0$ and $\beta=-\alpha^2(r+2\phi)/[2(r+2\phi)\alpha-(r+\kappa+\phi)(\alpha-1)]$, it follows that

$$\lambda(\alpha+\beta)=\underbrace{\frac{(r+\kappa+\phi)(\alpha-1)}{\alpha\beta}}_{=\lambda}(\alpha+\beta)=\frac{(1-\alpha)(r+\kappa+\phi)[\alpha(r+2\phi)-(r+\kappa+\phi)(\alpha-1)]}{\alpha^2(r+2\phi)},$$

and so,

$$\frac{J(\phi)}{r+\phi}=\frac{(1-\alpha)(r+\kappa+\phi)}{\alpha(r+\phi)}\times\frac{\alpha(r+2\phi)-(r+\kappa+\phi)(\alpha-1)}{\alpha(r+2\phi)}=:h_1(\phi)h_2(\phi).$$

To prove (i), observe that since $h_i>0$, it suffices to show that $h_i'<0$ at $\phi>\phi_{\underline{\alpha}}(r)$ for all $r>0$, $i=1,2$. To this end, it is easy to see that

$$\begin{aligned}
h_1'<0 \quad &\Leftrightarrow \quad [-\alpha'[r+\kappa+\phi]+1-\alpha]\alpha[r+\phi]-(1-\alpha)[r+\kappa+\phi][\alpha'[r+\phi]+\alpha]<0\\
&\Leftrightarrow \quad \underbrace{-\alpha'[r+\kappa+\phi]\alpha[r+\phi]-(1-\alpha)[r+\kappa+\phi]\alpha'[r+\phi]}_{=-[r+\kappa+\phi]\alpha'[r+\phi]}-\kappa\alpha(1-\alpha)<0,
\end{aligned}\tag{A.27}$$

which clearly holds when $\alpha'>0$. On the other hand,

$$\begin{aligned}
h_2'(\phi)<0 \quad &\Leftrightarrow \quad [\{\alpha[r+2\phi]\}'+1-\alpha-\alpha'[r+\kappa+\phi]]\alpha[r+2\phi]\\
&\qquad -\{\alpha[r+2\phi]\}'[\alpha(r+2\phi)-(r+\kappa+\phi)(\alpha-1)]<0\\
&\Leftrightarrow \quad [1-\alpha-\alpha'[r+\kappa+\phi]]\alpha[r+2\phi]-(r+\kappa+\phi)(1-\alpha)[\alpha'[r+2\phi]+2\alpha]<0\\
&\Leftrightarrow \quad \underbrace{(1-\alpha)\alpha[r+2\phi]-(r+\kappa+\phi)(1-\alpha)2\alpha}_{=(1-\alpha)\alpha[-r-2\kappa]<0}-(r+\kappa+\phi)\alpha'[r+2\phi]<0,
\end{aligned}\tag{A.28}$$

which is also true when $\alpha' > 0$.

In order to prove (ii), we show that for small enough discount rates, the function

$$R(\phi) = \frac{\nu(\phi) - \kappa}{\nu(\phi) + r},$$

with $\nu(\phi) := \kappa + \alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi) + \beta(\phi)]/\sigma_\xi^2$, is a decreasing upper bound to $J(\phi)/(r+\phi)$ over $[\phi^*(r), \phi_{\underline{\alpha}}(r)]$ that coincides with it at $\phi^*(r)$; as a byproduct, $1/[2 + R(\phi)]$ is increasing lower bound for the expected price that guarantees the latter cannot attain its global minimum in that region. The proof relies on the following technical lemma, which is proved in the online Appendix:

Lemma 8. $(\alpha(\phi), \beta(\phi)) \in I := [1/2, 1] \times [-1/2, -1/8]$ *for all* $\phi > \kappa$. *Moreover, there exists* $0 < \underline{r} < \kappa$ *such that for all* $r < \underline{r}$

$$\phi \mapsto \frac{\Lambda(\phi, \alpha, \beta)}{r + \phi}$$

is decreasing over $[\kappa, \infty)$ *for all* $(\alpha, \beta) \in I$.

Consider now $r < \underline{r}$ as in the lemma, and observe that $\underline{r}$ is independent of the specific values of $(\alpha, \beta) \in I$ (but it depends on the latter set, of course). For notational simplicity, we omit the dependence of all variables on r. Proposition 5 shows that $\nu(\phi)$ crosses the identity (only once) from above. Thus, we have that $\phi \geq \nu(\phi)$ for all $\phi \in [\phi^*, \phi_{\underline{\alpha}}]$ with equality only at ϕ^*. Since $\phi^* > \kappa$, the lemma yields

$$\frac{J(\phi)}{r+\phi} := \frac{\Lambda(\phi, \alpha(\phi), \beta(\phi))[\alpha(\phi) + \beta(\phi)]}{r + \phi} \leq \frac{\Lambda(\nu(\phi), \alpha(\phi), \beta(\phi))[\alpha(\phi) + \beta(\phi)]}{r + \nu(\phi)} \tag{A.29}$$

for all $\phi \in [\phi^*, \phi_{\underline{\alpha}}]$, with equality only at ϕ^*. However, from Lemma 6, $\Lambda(\nu(\alpha, \beta), \alpha, \beta) = \alpha\gamma(\alpha)/\sigma_\xi^2$, where $\gamma(\alpha) := \sigma_\xi^2[(\kappa^2 + \sigma_\theta^2/\sigma_\xi^2)^{1/2} - \kappa]/\alpha^2$. Thus,

$$\frac{J(\phi)}{r+\phi} \leq \frac{1}{\nu(\phi) + r}\left[\frac{\alpha(\phi)\gamma(\alpha(\phi))[\alpha(\phi) + \beta(\phi)]}{\sigma_\xi^2}\right] = \frac{\nu(\phi) - \kappa}{\nu(\phi) + r} = R(\phi)$$

with equality only at ϕ^*. However, $R'(\phi) = \nu'(\phi)(r + \kappa)/[\nu(\phi) + r]^2$, and $\nu'(\phi) < 0$ over $[\kappa, \phi_{\underline{\alpha}}]$ was already established as part of the uniqueness step in the proof of Proposition 5.

Finally, to prove (iii), we show that $J(\phi)/(r + \phi)$ is increasing at $\phi^*(r) < \phi_{\underline{\alpha}}(r)$ for r large enough. From the last expressions in (A.27)–(A.28) in part (i) of the proof, it suffices to show that

$$
\begin{aligned}
\Xi_1(\phi) &:= -[r+\kappa+\phi]\alpha'[r+\phi] - \kappa\alpha(1-\alpha) > 0 \text{ and} \\
\Xi_2(\phi) &:= (1-\alpha)\alpha[-r-2\kappa] - (r+\kappa+\phi)\alpha'[r+2\phi] > 0
\end{aligned}
$$

at $\phi^*(r)$ for large r. We start with Ξ_1.

From the equation that defines α, i.e., (13), $\alpha(\phi; r) \nearrow 1$ as $r \to \infty$ for all $\phi > 0$. Since $\phi^* = \phi^*(r) \in K := [\kappa, \sqrt{\kappa^2 + \sigma_\theta^2/\sigma_\xi^2}]$ and $\{\alpha(\cdot; r) : K \to [0,1] \mid r > 0\}$ is a family of continuous functions, it follows that $\{\alpha(\cdot; r) : r > 0\}$ converges uniformly to 1 over K (Dini's Theorem). Thus, $\alpha(\phi^*(r); r) \to 1$ as $r \to \infty$ as well. Letting $\ell(\phi, \alpha) := \sigma_\theta^2\alpha[\alpha + B(\phi, \alpha)] + \kappa\sigma_\xi^2[\phi + \kappa]$, and omitting the dependence on r, therefore, it is direct to verify that:

$$
\begin{aligned}
\lim_{r\to\infty} \beta(\phi^*) &= -1/2 \\
\lim_{r\to\infty} \phi^* &= \kappa + \frac{\gamma(1)}{2\sigma_\xi^2} > \kappa, \\
\lim_{r\to\infty} \ell(\phi^*, \alpha(\phi^*)) &= \frac{\sigma_\theta^2}{2} + \kappa\sigma_\xi^2[\lim_{r\to\infty} \phi^* + \kappa] > 0 \\
\lim_{r\to\infty} -\lambda(\phi^*)\beta(\phi^*) &= \lim_{r\to\infty} \frac{\sqrt{\ell^2(\phi,\alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha B(\phi,\alpha)\phi} - \ell(\phi,\alpha)}{2\kappa\sigma_\xi^2}\Big|_{(\phi,\alpha)=(\phi^*,\alpha(\phi^*))} > 0 \\
\lim_{r\to\infty} r(1-\alpha(\phi^*)) &= \lim_{r\to\infty} -\lambda(\phi^*)\beta(\phi^*)\alpha(\phi^*) > 0,
\end{aligned}
$$

where the last equality follows from the equation that defines α. Moreover, from the proof of Proposition 5,

$$
\alpha'(\phi^*) = \left[\frac{1}{r+\kappa+\phi+H+\alpha H_\alpha} \times \left(\frac{r(\alpha-1)}{\phi^*+\kappa} - \underbrace{\frac{\sigma_\theta^2\alpha B_\phi}{\Delta(\phi,\alpha)}[(r+\kappa+\phi)(\alpha-1) - \phi\alpha]}_{K(\phi):=}\right)\right]_{\phi=\phi^*}
$$

where $H = H(\phi, \alpha) := -\Lambda(\phi, \alpha, B(\phi, \alpha)B(\phi, \alpha)$, $\Delta(\phi, \alpha) := \sqrt{\ell^2(\phi,\alpha) - 4\kappa(\sigma_\xi\sigma_\theta)^2\alpha B(\phi,\alpha)\phi}$, and

$$
B_\phi = \frac{\partial}{\partial\phi}\left(\frac{-\alpha^2(r+2\phi)}{2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)}\right) = \frac{\alpha^2(\alpha-1)(r+2\kappa)}{[2(r+2\phi)\alpha - (r+\kappa+\phi)(\alpha-1)]^2}.
$$

It is easy to see that $\lim_{r\to\infty} \Delta(\phi^*, \alpha(\phi^*)) > 0$ and $\lim_{r\to\infty} B_\phi(\phi^*, \alpha(\phi^*)) = 0$. Thus, $\lim_{r\to\infty} K(\phi^*) = 0$. Also, from the proof of Proposition 1, $H_\alpha > 0$, and it is easy to show that $\lim_{r\to\infty} H_\alpha(\phi^*, \alpha(\phi^*))$

exists and is finite. This yields

$$\lim_{r\to\infty}(r+\kappa+\phi^*)\alpha'(\phi^*) = \lim_{r\to\infty}(r+\phi^*)\alpha'(\phi^*) = \lim_{r\to\infty}\frac{r(\alpha(\phi^*)-1)}{\phi^*+\kappa} < 0,$$

and thus $\Xi_1(\phi^*) > 0$ for large $r > 0$ (the first term in Ξ_1 diverges, while the second vanishes). Regarding $\Xi_2 := (1-\alpha)\alpha[-r-2\kappa] - (r+\kappa+\phi)\alpha'[r+2\phi]$, observe that when $\phi > \kappa$,

$$\Xi_2 > (1-\alpha)\alpha[-r-\kappa-\phi] - (r+\kappa+\phi)\alpha'[r+2\phi] = -[r+\kappa+\phi][\alpha(1-\alpha)+\alpha'(r+2\phi)].$$

We conclude that $\Xi_2(\phi^*(r)) > 0$ for large r, as

$$\lim_{r\to\infty}(r+2\phi^*)\alpha'(\phi^*) = \lim_{r\to\infty}\frac{r(\alpha(\phi^*)-1)}{\phi^*+\kappa} < 0 \ \text{ and } \ \lim_{r\to\infty} -\alpha(\phi^*)[1-\alpha(\phi^*)] = 0.$$

This ends the proof of the proposition. $\square$

Appendix B: Hidden Scores

We briefly illustrate some key arguments used in the proofs of the observable-scores model that have a direct analog in the hidden case.

Existence and uniqueness of stationary linear Markov equilibria. A key step in the proof of Theorem 1 is Lemma 4, which establishes the existence and uniqueness of a solution to (13). From part 1. in the proof of Proposition 10 in the online Appendix (Section 1.4 in that file), replacing $B(\phi,\alpha) \in (-\alpha/2, 0)$ as defined in (A.11) by $-\alpha/2$ in (13) leads $\alpha \mapsto H^h(\phi,\alpha) := -\Lambda(\phi,\alpha,-\alpha/2)[-\alpha/2]$ to be strictly increasing in $(0,1)$, which was the main property driving existence and uniqueness in the observable counterpart.

α^h is decreasing at any non-concealing point In the hidden case, $\alpha^h(\phi)$ solves $(r+\kappa+\phi)(\alpha^h-1)+\alpha H^h(\phi,\alpha^h) = 0$. Moreover, from the previous paragraph $\alpha \mapsto H^h(\phi,\alpha)$ is strictly increasing in $(0,1)$. Thus, the sign of $[\alpha^h]'$ is given by the sign of $1-\alpha^h(\phi) - H^h_\phi(\phi,\alpha^h(\phi))$, as in the proof of Proposition 5. In particular, replacing B_ϕ by 0 in (A.23) in that proof yields that, at any $\phi^{*,h}$ satisfying $\phi^{*,h} = \nu(\alpha^h(\phi^{*,h}), -\alpha^h(\phi^{*,h})/2)$,

$$\begin{aligned} &\operatorname{sign}([\alpha^h]'(\phi^{*,h})) \\ = \ &\operatorname{sign}([1-\alpha^h(\phi) - \alpha^h(\phi)H_\phi(\phi,\alpha^h(\phi))]|_{\phi=\phi^{*,h}}) = \operatorname{sign}\left(\left[1-\alpha^h + \frac{\lambda\alpha^h[-\alpha^h/2]}{\phi+\kappa}\right]_{\phi=\phi^{*,h}}\right) < 0.\end{aligned}$$

Existence and uniqueness of a non-concealing score. Since $\alpha^h + \beta^h = \alpha^h/2$ in the hidden case,

$$\nu^h(\phi) := \nu(\alpha^h(\phi), -\alpha^h(\phi)/2) = \kappa + \frac{[\alpha^h(\phi)]^2\gamma(\alpha^h(\phi))}{2\sigma_\xi^2}.$$

Thus, the existence of $\phi^{*,h}$ and the corresponding bounds follow directly from the arguments in the proof of (ii) of Proposition 5. Moreover, using the definition of $\gamma(\alpha)$,

$$\nu(\phi) = \kappa + \frac{1}{2}\left[\sqrt{\kappa^2 + [\alpha^h(\phi)]^2\frac{\sigma_\theta^2}{\sigma_\xi^2}} - \kappa\right],$$

so $\text{sign}([\nu^h]'(\phi)) = \text{sign}(\alpha'(\phi))$. Since $[\alpha^h]'(\phi) < 0$ over $[\kappa, \arg\min \alpha^h]$, we conclude that there is only one such a point. Moreover, since $\alpha + \beta \geq \alpha/2$ in the observable case, $\nu^o(\phi) \geq \nu^h(\phi)$; therefore, $\phi^{*,o} \geq \phi^{*,h}$, as each ν crosses the identity from above.

Quasiconvexity of α^h and $G^h(\phi) := G(\phi, \alpha^h(\phi), -\alpha^h(\phi)/2)$ maximized to the left of $\phi^{*,h}$. They follow identical arguments as the ones used in the observable case (Propositions 3 and 6).

References

Acquisti, A., C. Taylor, and L. Wagman (2016): "The Economics of Privacy," *Journal of Economic Literature*, 54(2), 442–92.

Bergemann, D., B. Brooks, and S. Morris (2015): "The Limits of Price Discrimination," *American Economic Review*, 105(3), 921–957.

Brayne, S. (2017): "Big Data Surveillance: The Case of Policing," *American Sociological Review*, 82(5), 977–1008.

Calzolari, G., and A. Pavan (2006): "On the optimality of privacy in sequential contracting," *Journal of Economic Theory*, 130(1), 168–204.

Cisternas, G. (2017a): "Career Concerns and the Nature of Skills," *American Economic Journal: Microeconomics*, forthcoming.

——— (2017b): "Two-Sided Learning and the Ratchet Principle," *Review of Economic Studies*, forthcoming.

Council of Economic Advisers (2015): *Big Data and Differential Pricing.*

Cummings, R., K. Ligett, M. Pai, and A. Roth (2016): "The Strange Case of Privacy in Equilibrium Models," in *Proceedings of the 2016 ACM Conference on Economics and Computation*, pp. 659–659. ACM.

Dixon, P., and R. Gellman (2014): "The Scoring of America: How Secret Consumer Scores Threaten Your Privacy and Your Future," Discussion paper, World Privacy Forum.

Dubé, J.-P., and S. Misra (2017): "Scalable price targeting," Discussion paper, National Bureau of Economic Research.

Dworczak, P. (2017): "Mechanism Design with Aftermarkets: Cutoff Mechanisms," Discussion paper, Stanford University.

Federal Trade Commission (2014): *Data Brokers: a Call for Transparency and Accountability.* https://www.ftc.gov/reports/data-brokers-call-transparency-accountability-report-federal-trade-commission-may-2014.

Fong, Y.-F., and J. Li (2016): "Information Revelation in Relational Contracts," *Review of Economic Studies*, 84(1), 277–299.

Freixas, X., R. Guesnerie, and J. Tirole (1985): "Planning under Incomplete Information and the Ratchet Effect," *Review of Economic Studies*, 52(2), 173–191.

Fudenberg, D., and J. M. Villas-Boas (2006): "Behavior-Based Price Discrimination and Customer Recognition," in *Handbook on Economics and Information Systems*, ed. by T. Hendershott. Elsevier.

——— (2015): "Price Discrimination in the Digital Economy," in *The Oxford Handbook of the Digital Economy*, ed. by M. Peitz, and J. Waldfogel. Oxford University Press.

Gerardi, D., and L. Maestri (2016): "Dynamic Contracting with Limited Commitment and the Ratchet Effect," Discussion paper, Collegio Carlo Alberto.

Gomes, R., and A. Pavan (2018): "Price Customization and Targeting in Platform Markets," Discussion paper, Centre for Economic Policy Research.

Halac, M. (2012): "Relational Contracts and the Value of Relationships," *American Economic Review*, 102(2), 750–779.

Holmström, B. (1999): "Managerial Incentive Problems: A Dynamic Perspective," *Review of Economic Studies*, 66(1), 169–182.

Hörner, J., and N. Lambert (2017): "Motivational Ratings," Discussion paper, Yale University.

Ichihashi, S. (2018): "Online Privacy and Information Disclosure by Consumers," Discussion paper, Stanford University.

Kehoe, P. J., B. J. Larsen, and E. Pastorino (2018): "Dynamic Competition in the Era of Big Data," Discussion paper, Stanford University.

Laffont, J.-J., and J. Tirole (1988): "The Dynamics of Incentive Contracts," *Econometrica*, 56(5), 1153–1175.

McAdams, D. (2011): "Discounts For Qualified Buyers Only," Discussion paper, Fuqua School of Business.

Ostrovsky, M., and M. Schwarz (2016): "Reserve Prices in Internet Advertising Auctions: A Field Experiment," Discussion paper, Stanford University.

Pham, H. (2009): *Continuous-Time Stochastic Control and Optimization with Financial Applications*. Springer-Verlag, Berlin.

Platen, E., and N. Bruti-Liberati (2010): *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*. Springer.

Rafieian, O., and H. Yoganarasimhan (2018): "Targeting and Privacy in Mobile Advertising," Discussion paper, University of Washington.

Schmalensee, R. (1981): "Output and welfare implications of monopolistic third-degree price discrimination," *American Economic Review*, 71(1), 242–247.

Schmitz, A. J. (2014): "Secret Consumer Scores and Segmentations: Separating Haves from Have-Nots," *Michigan State Law Review*, pp. 1411–1473.

Shen, Q., and J. Miguel Villas-Boas (2017): "Behavior-Based Advertising," *Management Science*, forthcoming.

Syrgkanis, V., D. Kempe, and E. Tardos (2017): "Information Asymmetries in Common-Value Auctions with Discrete Signals," Discussion paper, Cornell University.

Taylor, C. R. (2004): "Consumer privacy and the market for customer information," *Rand Journal of Economics*, 35(4), 631–650.

Turow, J. (2012): *The daily you: How the new advertising industry is defining your identity and your worth*. Yale University Press.

Villas-Boas, J. (1999): "Dynamic competition with customer recognition," *Rand Journal of Economics*, 30(4), 604–631.

Xu, Z., and A. Dukes (2017): "Personalized Pricing with Superior Information on Consumer Preferences," Discussion paper, University of Southern California.