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Abstract

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JEL Classification: E62, H20, H60

Keywords: Optimal Government Debt, incomplete markets, Capital taxation, Dynamically Optimal Taxation

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The Optimum Quantity of Capital and Debt*

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August 9, 2021

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1 Introduction

What are the optimal levels of capital and government debt? Should capital be taxed and if yes, how much? What is the optimal extent of redistribution? We study these classic questions in a heterogenous agents, incomplete markets, Aiyagari (1995) economy. In this economy households are exposed to idiosyncratic income shocks but no aggregate risk. They face exogenous credit constraints and the only assets are physical capital and government debt. The Ramsey planner commits itself ex-ante to a path of linear labor and capital taxes and government debt to maximize agents' discounted present value of lifetime utility.

We prove three main theoretical findings on optimal policies. First, we show that it is optimal to equalize the pre-tax return on capital and the rate of time preference in the long-run, i.e. the capital stock satisfies the modified golden rule. Our second theoretical result shows that the long-run steady-state allocations and policies are independent of initial conditions. In particular, the long-run level of government debt is uniquely determined and does not depend on the initial value of debt or capital. Similarly, steady-state tax rates on capital and labor are unique and independent of initial conditions. Our third contribution is a characterization of optimal steady-state policies which allows us to numerically compute the optimal long-run policies. Our characterization result will also enable researchers to compute optimal policy responses to aggregate shocks since a steady state with optimal policies is a necessary starting point for this endeavor. Without our characterization one would have to start from a non-optimal steady state which will unavoidably lead to an uninterpretable bias since the optimal policy would not only respond to aggregate shocks but would also tend to push the economy towards the optimal steady state.

Our result that the long-run steady state is independent of initial conditions renders tractable a quantitative analysis of the dynamic optimal taxation problem. Whereas the literature has focused mainly on characterizing the steady state which maximizes welfare, we develop a new computational algorithm that allows us to maximize welfare in the initial period by choosing the optimal path of taxes and debt.

A comparison with the optimal Ramsey taxation results in representative agents, complete markets economies without aggregate risk as in Lucas (1990), and Chari and Kehoe (1999) helps to elucidate our theoretical findings. As is well known the steady-state Ramsey planner solution depends on initial conditions, such as the initial government debt level, in this complete markets environment. The intuition for this result is straightforward. As in Barro (1979) the planner aims to smooth distortions over time using government debt. In the absence of any exogenous fluctuations it is optimal (perhaps after a few early periods) to keep government debt and labor taxes constant over time. This policy provides higher welfare than a deviating policy where, for example, labor taxes and distortions are lowered initially, more debt is issued to finance this tax cut, and then eventually labor taxes and distortions are increased to cover the higher interest rate burden on government debt. This alternative policy would reduce welfare since the gain of lower distortions in the beginning is outweighed by the loss of higher distortions later on, since distortions are "convex" as in Barro (1979).

If markets are incomplete though, this reasoning is only one part of the story (Heathcote, 2005). Lowering taxes today still means higher debt (as in the complete markets case) but now more debt has a welfare-enhancing element, as it improves households' ability to smooth consumption in response to income shocks. The costs of having higher debt - higher future taxes - remain if markets are incomplete, but there is now an additional benefit: better consumption smoothing. As a result the planner lowers taxes initially in light of two benefits - lower distortions today and higher debt (more liquidity) - and still just one cost (higher distortions tomorrow). Of course, there are limits to how high debt can become; eventually, future distortions get too big and outweigh the initial lower distortions and the benefits of higher liquidity. The optimal level of government debt is determined as equalizing the benefits and costs at the margin.

A conclusion common to both complete and incomplete markets is that the long-run capital stock satisfies the modified golden rule (see also Aiyagari, 1995).[1] In a representative agent economy distributional concerns are absent and investment efficiently transfers resources across time. If markets are incomplete distributional concerns are present, but we show that these do not interfere with efficiency in investment, reminiscent of the production efficiency result in Diamond and Mirrlees (1971). Aiyagari (1995) shows that this finding implies a positive capital income tax rate. One interpretation here is that this tax corrects households' overaccumulation of capital due to a precautionary savings motive. However, we show this interpretation to be inaccurate and that instead, the planner issues as much debt as is necessary to enhance consumption smoothing such that capital demand satisfies the modified golden rule. The capital income tax rate is positive, as Aiyagari's (1995) arguments are valid, and is such that the private sector is willing to absorb the optimal levels of both capital and government debt. Thus there is no need to implement a higher than efficient capital stock in order to achieve better consumption smoothing, simply because debt can be used instead to prevent the overaccumulation of capital. A higher capital stock could also be used to increase wages, which would benefit those depending primarily on labor income but we show that the planner could use labor taxes to increase the after-tax wage instead.

[1] Chen et al. (2018) claim to disprove Aiyagari (1995), that a Ramsey steady state does not exist and that the modified golden rule does not hold. We show in Appendix IV that their proof is flawed and incomplete. Park (2014), however, shows (correctly) that the modified-golden rule result does not extend to her limited commitment economy.

On the other hand, if either of the instruments, issuing debt or taxing labor, is unavailable to the planner, then the capital stock will not satisfy the modified golden rule.[2]

Our result showing independence of initial conditions allows for a quantitative analysis of the optimal dynamic taxation problem. Whereas the literature has focused on maximizing steady-state welfare, our task is to characterize the optimal policy along the full transition path. In particular our characterization has to take into account that the optimal policy, at each point in time during the transition, depends on the full transition path of capital, debt and tax rates.

Computing the path of tax rates and government debt that maximizes welfare in the initial period is a huge computational challenge. Several hundreds or thousands of variables must be chosen in a highly nonlinear optimization problem. However, our result that the optimal long-run policy is independent of initial conditions turns this unwieldly optimization problem into a manageable one. From a computational point of view, independence of initial conditions means that we know the optimal long-run policies and allocations without having to compute the transition. We know the initial conditions (economy calibrated to the US economy) and we know the terminal condition, the optimal long-run steady state characterized above. The (still daunting) computational problem is then to find the policy path that satisfies all necessary first-order conditions along the transition and at the same time the initial and terminal conditions. This is a challenge as it involves solving hundreds or thousands of nonlinear equations but it is significantly easier, and tractable as opposed to the original problem, which was to find the optimal transition and the optimal terminal point at the same time. Further, given the large number of variables involved in the original problem, there is no way to check whether a candidate solution is a global maximum. This is not a concern in our approach.

In the optimal steady state we find that capital taxes are always significantly positive in contrast to complete markets (see the seminal contributions of Chamley, 1986; Judd, 1985) although, for all calibrations, relatively low compared to most developed economies. In our benchmark calibration, aimed at resembling the high income inequality in the U.S. and with a Frisch elasticity of labor supply equal to one, the long-run taxes on capital and labor are around 21 and 50 percent respectively. The optimal long-run level of government debt equals 1.1 times GDP.

Our finding that government debt is high, capital is taxed at a low rate, and labor income is taxed at a high rate when compared to current U.S. values, is robust across various different alternative calibrations, although the precise numbers do depend on the details of

[2] It is important to remember that Dávila et al. (2012) consider a different policy problem - they characterize the constrained efficient outcome - and therefore unsurprisingly obtain different results for the optimal capital stock.

the calibration. Indeed, we reach the same conclusion for a low and a high Frisch elasticity of labor supply, for a low and a high income elasticity of labor supply, for low and high income inequalities and in a model with permanent income differences.

The high debt levels we find follow also from our assumption that the government always honors its debt, so that elements such as a default premium, not present in our model, do not restrict how much debt can be issued. Instead distortionary taxes is the only element keeping debt from becoming infinitely large and thus maximizing the liquidity services. The debt level is, however, not unrealistically high as some countries (Japan) have a debt to GDP ratio as high as 2.[3] The outcome that high debt and high labor taxes is optimal also follows from the fact that the standard Aiyagari model misses elements, such as endogenous human capital accumulation (Wu, 2021), which render high labor taxes more distortionary. The lesson here is that within the Aiyagari model distortionary taxes do not severely limit the level of government debt.

Knowing the optimal path of policies allows us to compute the welfare gains of switching to the optimal policy and to better understand the properties of the optimal steady-state policies, as these are linked tightly to the policies chosen during the transition. The optimal transition is characterized by an initial period of high capital income taxation and low labor taxation. While the high initial capital tax rates are well known from complete markets and are a result of initially inelastically supplied capital, the low initial taxation (close to 0) of labor income is new to the incomplete markets environment. As a result labor market distortions are low initially and government debt accumulates. Eventually labor taxes are increased to pay the interest rates on debt which converges to its high steady-state level. The transition path also explains why the planner chooses a policy such that welfare in the terminal optimal steady state is lower than in the initial steady state. Since it is evaluated in the initial period welfare is an accumulation of the welfare gains and losses along the full transition path to the new steady state. Welfare is enhanced as the welfare gains of low labor taxation in the beginning of the transition outweigh the (highly discounted) welfare losses in the terminal steady state.

Although most of the literature either maximizes steady-state welfare or, when considering transitions, assumes fixed tax rates throughout the transition,[4] a few papers deviate from these restrictive assumptions. For example, Dyrda and Pedroni (2020) also compute the

[3]Holter et al. (2019) and Kindermann and Krueger (2021) find that even higher debt levels can be sustained in OLG economies.

[4]Domeij and Heathcote (2004) are among the the first to look at the welfare impact during the transition, but their anlysis is limited to a one time tax change. Kindermann and Krueger (2021) focus on the taxation of top income households and solve for the one time change in the top marginal tax rate. They find that the optimal top marginal tax rate should be 90%. Boar and Midrigan (2021) consider non-linear income and wealth taxes and find a linear income tax (and lump-sum transfers) to approximate the optimum very well.

optimal transition path in an incomplete markets economy but do not characterize the optimal steady-state policies first.[5] Le Grand and Ragot (2021) is complementary to ours as they solve for the optimal unemployment insurance benefits in a heterogeneous-agent model over the business cycle.

Aiyagari and McGrattan (1998) study the optimal level of debt in an incomplete markets model but under the alternative assumption that the planner maximizes the utility at the steady state instead of ex-ante welfare. They find that the optimal level of debt is two-thirds of GDP, in line with the current US level. Much of the follow-up work in this literature also maximizes the steady-state welfare. For example, Röhrs and Winter (2014) find that if inequality is large, the optimal level of debt that maximizes the steady-state welfare is even lower and should be negative, -0.8. One reason why the optimal level of debt is low or even negative when steady-state welfare is maximized is that this optimality criterion ignores the welfare loss of reducing debt along the transition path to a low-debt steady state.[6]

In a series of papers Bhandari et al. (2017a,b, 2021) also consider optimal taxation in incomplete market models, building on the work of Aiyagari et al. (2002) who were the first to investigate the Ramsey policy in a Lucas and Stokey (1983) economy with incomplete markets (and aggregate risk). A key difference is that we follow Aiyagari (1995) and impose tight exogenous credit constraints, which is necessary for matching the joint distribution of earnings, consumption and wealth observed in the data and for generating a realistic distribution of marginal propensities to consume.[7] These credit constraints make the computational problem significantly more complicated, since a fraction of households is not operating on their consumption Euler equation, preventing us from using an easy backward shooting approach iterated on the Euler equation.

It is also the presence of credit constraints that generates a large demand for precautionary savings and thus potentially a positive capital income tax rate. The reason why nevertheless we do not find high capital income tax rates is the large amount of debt, which allows households to smooth consumption quite well but requires an after-tax interest rate close to the rate of time preference. For a higher capital income tax rate and thus a lower pre-tax interest rate, the private sector would not be willing to absorb the capital stock and the large

[5] Dyrda and Pedroni (2020) directly search for the optimal path of policies that maximizes welfare in a huge space which includes infinitely many possible steady states. They show that their policies converge to a steady state that is consistent with the steady state found by our algorithm, using the same calibration. Therefore, using our algorithm to find the steady state first can greatly reduce the complexity and facilitate their algorithm.

[6] Conesa et al. (2009) also maximizes steady-state welfare to solve for the optimal taxes on capital and labor (but not debt) in an OLG economy with incomplete markets and idiosyncratic income risk. They find a relatively high capital tax of 36%.

[7] Tight credit constraints are also an important difference between this paper and Gottardi et al. (2015) as the latter paper assumes credit constraints to be non-binding.

stock of debt. The planner finds it welfare-maximizing to reduce inequality through more debt and low capital income tax rates, instead of through low debt and high capital income taxes. Both a high level of debt and high capital tax rates are not possible since the asset market would not clear.

The paper is organized as follows. Section 2 presents our incomplete markets model and the Ramsey taxation problem. We provide our theoretical results in Section 3 before we move to the quantitative analysis. Section 4 shows optimal policy in the steady state and the optimal transition path is presented in Section 5. Section 6 concludes.

2 The Model

In this section, we present the incomplete markets model with heterogenous agents and uninsurable idiosyncratic labor productivity shocks. The setup is similar to Aiyagari (1995), except that our utility function takes the Greenwood–Hercowitz–Huffman (GHH) form and government spending is exogenous. In particular, the same tax instruments are used as this allows us to address several important questions left answered in Aiyagari (1995). What are the steady-state levels of government debt, labor and capital income taxes and what are the properties of the transition path?

2.1 The Environment

Time is discrete and infinite, denoted by $t \in \{0, 1, 2, ...\}$. There is a measure one of ex ante identical households, a representative firm and a government.

Endowment and Technology A household supplies labor $n_t \in [0, 1]$ in period t. She faces an idiosyncratic labor productivity shock $e_t \in E$, which follows a Markov process and is i.i.d. across households. She has access to an incomplete market and can only hold a non-state contingent one-period bond $a_t \in A$, subject to a constraint $a_t \geq -\underline{a}$.

A representative competitive firm produces final goods using capital K_t and labor N_t using the neoclassical constant-returns-to scale production function $F(K, N)$ which satisfies the standard conditions.[8] Capital depreciates at rate δ.

The government is a Ramsey planner with full commitment. It collects linear capital income tax at the rate τ_{kt} and linear labor income tax at the rate τ_{nt}. It issues government debt B_t to finance lump-sum transfer T_t and government expenditure G_t.

[8] The production function is assumed to be twice continuously differentiable, strictly increasing and concave in each argument and satisfying the standard Inada conditions: $\lim_{K\to 0} F_K = \infty, \lim_{K\to\infty} F_K = 0$ and $\lim_{N\to 0} F_N = \infty$.

Preferences The instantaneous utility of a household takes the Greenwood–Hercowitz–Huffman (GHH) form:

$$u(c,n) = \frac{\left(c - \chi \frac{n^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}\right)^{1-\sigma}}{1-\sigma}.$$

A household's lifetime utility is the expected discounted sum of utilities $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)\right]$. The GHH utility function implies that a household's labor supply decision depends on the wage, but not on the level of consumption or assets. This utility function is also used by Heathcote (2005) and Bayer et al. (2020), who show that GHH preferences approximate labor supply well since the wealth effect on labor supply is very small and is estimated to be close to 0 in DSGE models with heterogeneous agents, echoing the findings in Jaimovich and Rebelo (2009) who find that wealth effects on labor supply must be weak for models to match business cycle facts. In terms of result, we will discuss below that allowing for significant wealth effects would assign an unrealistically strong insurance role to labor supply and thus undermine the importance of government debt as a consumption smoothing device. A quantitative analysis of the amount of optimal government debt therefore requires low wealth effect preferences such as GHH.

Markets There are competitive markets for labor, capital, final goods, and bonds.

2.2 Competitive Equilibrium

Firm The optimality conditions for the firm imply that in each period, the interest rate and the wage are equal to the marginal return of capital and the marginal return of labor respectively,

$$\begin{aligned} r_t &= F_K(K_t, N_t) - \delta, \\ w_t &= F_N(K_t, N_t). \end{aligned}$$

Government The government collects linear taxes on capital income and labor income. Denote the after-tax capital return and wage as $\bar{r}_t$ and $\bar{w}_t$, so that $\bar{r}_t = (1-\tau_{kt}) r_t$ and $\bar{w}_t = (1-\tau_{nt}) w_t$. The government's budget constraint is

$$G_t + (1+\bar{r}_t) B_t + T_t \leq \tau_{kt} r_t A_t + \tau_{nt} w_t N_t + B_{t+1}, \tag{1}$$

where $A_t = K_t + B_t$ is the total amount of assets, the sum of physical capital and government debt. Standard arguments using the constant-return-to-scale assumption lead to the following equivalent resource constraint:

$$G_t + (1 + \bar{r}_t) B_t + \bar{r}_t K_t + \bar{w}_t N_t + T_t \leq F(K_t, N_t) - \delta K_t + B_{t+1}. \tag{2}$$

Households Starting from period 0 with asset a_0 and productivity e_0, a household solves the following problem

$$V_0(a_0, e_0) = \max_{\{a_{t+1}, c_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, n_t),$$

subject to

$$c_t + a_{t+1} \leq a_t (1 + \bar{r}_t) + \bar{w}_t e_t n_t + T_t, \tag{3}$$

$$a_{t+1} \geq -\underline{a}, \tag{4}$$

where $V_0(a_0, e_0)$ represents the lifetime utility of a household with initial state (a_0, e_0). The measure one of households allows us to use T_t to denote transfers both in the government and the household budget constraints. The optimality condition of n_t is

$$u_c(c_t, n_t) e_t \bar{w}_t + u_n(c_t, n_t) = 0,$$

which implies that labor supply

$$n_t = \left(\chi^{-1} e_t \bar{w}_t\right)^{\phi}, \tag{5}$$

and after tax labor income

$$y_t = e_t n_t \bar{w}_t = \chi^{-\phi} (e_t \bar{w}_t)^{1+\phi}. \tag{6}$$

In the rest of the paper, we can therefore treat n_t and y_t as known functions of $\bar{w}_t$, reducing the number of choice variables. The optimality condition for a_{t+1} and the borrowing constraint imply the necessary conditions:

$$u_c(c_t, n_t) \geq \beta (1 + \bar{r}_{t+1}) \mathbb{E}_t u_c(c_{t+1}, n_{t+1}), \tag{7}$$

$$(a_{t+1} + \underline{a}) (u_c(c_t, n_t) - \beta (1 + \bar{r}_{t+1}) \mathbb{E}_t u_c(c_{t+1}, n_{t+1})) = 0. \tag{8}$$

Equation (7) is the standard Euler equation, and equation (8) is the Kuhn-Tucker condition for the borrowing constraint.

Equilibrium The distribution of households with productivity e_t and asset a_t in period t is denoted by μ_t, a measure on $S = E \times A$. The asset market clearing conditions for assets, labor and capital are,

$$A_t = \int_S a_t d\mu_t, \tag{9}$$

$$N_t = \int_S e_t n_t d\mu_t, \tag{10}$$

$$K_t = A_t - B_t. \tag{11}$$

A sequence of prices and allocations and policies $\{\bar{r}_t, \bar{w}_t, T_t, B_{t+1}, K_{t+1}, a_{t+1}, c_t\}_{t=0}^{\infty}$ is a competitive equilibrium given initial conditions (B_0, K_0, μ_0) if

1. Households maximize utility (taking prices and policies as given).
2. Firms maximize profits (taking prices and policies as given).
3. Market clearing conditions (9), (10) and (11) hold.
4. The government budget constraints (1) are satisfied.
5. The resource constraints (2) are satisfied.

This competitive equilibrium exists, as originally proven in Aiyagari (1995) in a simpler environment and recently proven by Acikgoz (2018) with unbounded utility and Zhu (2020) with endogenous labor supply.

2.3 The Optimal Taxation Problem

The Ramsey planner maximizes the sum of lifetime utilities of all households, by choosing time paths for $\bar{r}_t, \bar{w}_t$ and B_t, consistent with the equilibrium conditions described above. These are the instruments considered in Aiyagari (1995) and in the representative agent literature, which excludes lump-sum taxation since otherwise the planner does not need to use distortionary taxes on labor or capital. In heterogeneous agents incomplete markets models with natural borrowing limits transfers are indetermined and this indeterminacy carries over to some degree to models with exogenous borrowing constraints (Bhandari et al., 2017b). While it is not known whether the optimal Ramsey solution is indetermined, the planner always has an incentive to front-load transfers, which might lead to non-binding borrowing constraints for all households and thus approximate the natural debt limit indeterminacy.[9]

[9] We conducted numerical experiments with transfers and confirmed that this is a concern.

We therefore study the Ramsey problem with standard tax instruments:

$$\max_{\{\bar{r}_t,\bar{w}_t,B_{t+1},a_{t+1},c_t\}} \int V_0(a_0,e_0)d\mu_0$$

subject to the resource constraint (2), households' budget constraints (3), households consumption Euler equation (7), and the credit constraint (8). The other unknowns, including $n_t, r_t, w_t, K_t, A_t, N_t$ can all be expressed as functions of the choice variables in the Ramsey problem, using the equations described in subsection 2.2.

One way to solve this problem would be to extend the primal approach used in complete markets models and to use first-order conditions to substitute for prices. We take a different route and we are the first to apply a Lagrangian maximization approach to study optimal taxation in an Aiyagari incomplete markets economy.[10] We assign present value Lagrangian multipliers γ_t, θ_{t+1} and η_{t+1} to constraints (2), (7) and (8), respectively. The Lagrangian can be written (see Appendix I) as

$$\begin{aligned}\mathcal{L} \;=\; & \int \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \\ & \left(u\left(c_t,n_t\right)+u_c\left(c_t,n_t\right)\left(\left(\eta_t\left(a_t-\underline{a}\right)-\theta_t\right)\left(1+\bar{r}_t\right)-\left(\eta_{t+1}\left(a_{t+1}-\underline{a}\right)-\theta_{t+1}\right)\right)\right) \\ & +\gamma_t\Big(F\left(K_t,N_t\right)-\delta K_t+B_{t+1}-G_t-T_t-\left(1+\bar{r}_t\right)B_t-\bar{r}_tK_t-\bar{w}_tN_t\Big)\Big\}d\mu_0. \quad (12)\end{aligned}$$

To simplify the notation, we define $\lambda_{t+1}=\eta_{t+1}\left(a_{t+1}+\underline{a}\right)-\theta_{t+1}$. We derive FOCs from the Lagrangian in Appendix I and show that the interior solution of the Ramsey problem satisfies the following conditions:

$$\begin{aligned}\lambda_{t+1}: \quad & u_c\left(c_t,n_t\right) \geq \beta\left(1+\bar{r}_{t+1}\right)\mathbb{E}_t\left[u_c\left(c_{t+1},n_{t+1}\right)\right] \\ & \text{with equality if } a_{t+1} > -\underline{a}, \quad (13) \\ a_{t+1}: \quad & u_c\left(c_t,n_t\right)+u_{cc}\left(c_t,n_t\right)\left(\lambda_t\left(1+\bar{r}_t\right)-\lambda_{t+1}\right) \\ & =\beta\left(1+\bar{r}_{t+1}\right)\mathbb{E}_t\left[u_c\left(c_t,n_t\right)+u_{cc}\left(c_{t+1},n_{t+1}\right)\left(\lambda_{t+1}\left(1+\bar{r}_{t+1}\right)-\lambda_{t+2}\right)\right] \\ & +\beta\gamma_{t+1}\left(F_K\left(K_{t+1},N_{t+1}\right)-\delta-\bar{r}_{t+1}\right) \\ & \text{if } a_{t+1} > -\underline{a}; \text{ otherwise } \lambda_{t+1}=0, \quad (14) \\ B_{t+1}: \quad & \gamma_t=\beta\left(1+F_K\left(K_{t+1},N_{t+1}\right)-\delta\right)\gamma_{t+1}, \quad (15) \\ \bar{r}_t: \quad & \gamma_tA_t=\mathbb{E}_t\left[u_c\left(c_t,n_t\right)\lambda_t\right. \\ & \left.+a_t\left(u_c\left(c_t,n_t\right)+u_{cc}\left(c_t,n_t\right)\left(\lambda_t\left(1+\bar{r}_t\right)-\lambda_{t+1}\right)\right)\right], \quad (16)\end{aligned}$$

[10]Note that the first draft of Açikgöz (2013) is from 2013.

$$\bar{w}_t: \quad \gamma_t N_t = \gamma_t \left(F_N\left(K_t, N_t\right) - \bar{w}_t\right) \frac{\partial N_t}{\partial \bar{w}_t}$$
$$+\mathbb{E}_t\left[e_t n_t u_c\left(c_t, n_t\right) + \left(\frac{\partial c_t}{\partial \bar{w}_t} u_{cc}\left(c_t, n_t\right) + \frac{\partial n_t}{\partial \bar{w}_t} u_{cn}\left(c_t, n_t\right)\right)\left(\lambda_t\left(1+\bar{r}_t\right) - \lambda_{t+1}\right)\right]. \quad (17)$$

$\partial c_t / \partial \bar{w}_t, \partial n_t / \partial \bar{w}_t$, etc. are known functions of the control variables. The explicit expressions of these functions are shown in the Appendix. Note that as in Marcet and Marimon (2019), we expand the state space of the problem to include Lagrange multipliers of the dynamic implementability constraints. Since there is a continuum of heterogeneous households, the relevant state variable for the Ramsey planner is the joint distribution of these multipliers.[11]

3 Analytical Results

A key first step in the quantitative analysis is to compute the optimal policy in the long run. The second step is to use the optimal long-run policy as a terminal condition when computing the optimal policy during the transition path. We therefore make the standard assumption that the optimal long-run policy is stationary, an assumption we maintain throughout the paper:

Assumption 1. *For each set of initial conditions (B_0, K_0, μ_0), the economy (including policy and all other variables) converges to a steady state.*

Note that the above does not assume our main result on the independence of initial conditions. Instead it assumes that for each set of initial conditions (B_0, K_0, μ_0), there is a solution to the maximization problem of the Ramsey planner. Note that this assumption also holds in representative agent economies, where given the initial level of debt B_0 and capital K_0 the steady state is unique. But at the same time, the steady state depends on the initial debt level that is, different steady states can be reached for different initial conditions. In contrast, we show the independence of initial conditions in our incomplete markets economy. The same steady state is reached independently of where the economy started.

Whereas uniqueness and existence are generic properties of maximization problems (it just rules out more than one global maximum),[12] the second assumption that the optimal solution converges to a steady state is standard and essential for tractability in incomplete market models. Aiyagari (1995) therefore assumes that government expenditures are endogenous and constant in a steady state, implying finite values for the associated steady-state multipliers. It

[11] Marcet and Marimon (2019) construct the recursive Lagrangian by "dualizing" the dynamic incentive constraints period by period, assuming that the solution to the primal problem is a saddle-point of the corresponding Lagrangian. An earlier draft of their paper (1994) features the Ramsey problem under complete markets as an example whose formulation looks very similar to this model.

[12] See Aiyagari (1994) for a proof that a solution to the optimal taxation problem exists.

would be straightforward to follow Aiyagari (1995) and to incorporate endogenous government expenditures into our analysis without changing our conclusions.

However, Straub and Werning (2020) show that in a different model, the capitalist-worker model of Judd (1985) without government debt, the optimal solution does not converge to a steady state if the intertemporal elasticity of substitution is below one (and the welfare weight on capitalists is zero). For these parameter values, Proposition 2 in Straub and Werning (2020) shows that no interior steady state exists, implying that the assumption of convergence to an interior steady state is invalid. Specifically, they show that the multiplier on the first-order savings decision cannot converge to an interior steady-state value. In contrast, we prove the existence of steady-state Lagrange multipliers on households savings decisions in our incomplete markets taxation environment.

Chen et al. (2018) argue that in incomplete markets models, a relaxed Ramsey planner who drops households' Euler equations from its constraints, chooses policy sequences that do not converge. This relaxed Ramsey problem is simpler but quite different to the original standard Ramsey problem in Aiyagari (1995) and our paper, so it is not directly linked to the discussion on the existence of steady states in incomplete and complete markets models.[13] Furthermore, issues relating to non-existence of steady-states do not arise in our numerical applications because we are always able to find a solution to the FOCs characterizing the steady state.[14] Whereas these findings show that the existence of a steady state is not an issue in our incomplete markets model, little is known about whether the optimal solution converges to this steady state for arbitrary initial conditions (see footnote 14 in Aiyagari (1995)). That is why we impose the standard Assumption 1.

3.1 Steady State

This assumption on the stationarity of the optimal long-run policy means that we can replace all variables in the above FOC with their steady-state values: We drop time subscripts and if necessary, add superscripts $'$ and $''$ to denote future variables next period and two periods later. Then the optimal stationary policy is a solution to:

$$u_c\left(c,n\right) \geq \beta\left(1+\bar{r}'\right)\mathbb{E}\left[u_c\left(c',n'\right)|e\right]$$

[13] Appendix IV discusses the details of Chen et al. (2018) and shows why their arguments are incorrect.

[14] Straub and Werning (2020) also consider the representative agent Ramsey taxation problem in Chamley (1986) and find that an exogenous upper bound on capital taxes can be binding forever if the initial level of government debt is close enough to the peak of a "Laffer curve". Again these issues seem not to arise in our incomplete markets model. We also impose an upper bound on capital taxation but find it to be binding only for the first period. Instead the planner finds it optimal to lower labor taxes and issue more bonds, which requires a sufficiently high after-tax return on assets if households are to be willing to absorb the additional debt.

$$\text{with equality if } a' > -\underline{a}, \tag{18}$$

$$u_{cc}(c,n)\left[\lambda(1+\bar{r})-\lambda'\right] = \beta\mathbb{E}\left[(1+r')u_{cc}(c',n')\left(\lambda'(1+\bar{r})-\lambda''\right)\right]$$
$$+\beta\gamma\left(F_K(K,N)-\delta-\bar{r}\right)$$
$$\text{if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0, \tag{19}$$

$$1 = \beta\left(1+F_K(K,N)-\delta\right), \tag{20}$$

$$\gamma A = \mathbb{E}\left[u_c(c,n)\lambda + a(u_c(c,n)+u_{cc}(c,n)(\lambda(1+\bar{r})-\lambda'))\right], \tag{21}$$

$$\gamma N = \gamma\left(F_N(K,N)-\bar{w}\right)\frac{\partial N}{\partial \bar{w}}$$
$$+\mathbb{E}\left[enu_c(c,n)+\left(\frac{\partial c}{\partial \bar{w}}u_{cc}(c,n)+\frac{\partial n}{\partial \bar{w}}u_{cn}(c,n)\right)\left(\lambda(1+\bar{r})-\lambda'\right)\right]. \tag{22}$$

3.2 Optimal Long-run Level of Capital

While most of our results are naturally based on numerical simulations, we can still analytically derive the optimal level of capital in the long run. A key property of the steady state is that the capital level satisfies the modified golden rule (see also Aiyagari (1995)). Equation (20) implies:

Theorem 1. *The steady-state capital stock satisfies the modified golden rule,*

$$\beta(1+F_K(K,N)-\delta) = 1. \tag{23}$$

The modified golden rule states that it is optimal to equalize the return on capital and the rate of time preference, that is resources are efficiently allocated across time. This result is familiar from representative agent economies where distributional concerns are by assumption absent. Theorem 1 shows that we obtain the same efficiency result in our incomplete market economy where redistribution might induce a deviation from production efficiency, reminiscent of the production efficiency result in Diamond and Mirrlees (1971).

As is well known agents engage in precautionary savings to smooth consumption in response to idiosyncratic income fluctuations, and smoothing is more effective the more assets are available. The planner does not issue more capital to increase the availability of assets, but instead issues more government debt. This measure has the advantage that debt can be used as well as capital for consumption smoothing, but does not interfere with production efficiency. This reasoning is reflected in the absence of a "precautionary savings" term in the FOC determining the optimal level of capital.

A higher than efficient capital stock could also be used to increase wages, benefiting those whose consumption is financed primarily from labor and not asset income, as is the case in Dávila et al. (2012). In our Ramsey taxation problem the planner can increase the capital stock too but only by lowering capital income taxes. The planner can also use labor taxes to change the after-tax labor income.[15] We show that the planner uses labor taxes to modify the after-tax wage and not a higher capital stock, as is again reflected in the absence of a "wage" term in the FOC determining the optimal level of capital.[16]

These arguments establish, moreover, that the availability of government debt and labor taxes is necessary for theorem 1 to be valid. Without these instruments the modified golden rule does not hold. If labor taxes are unavailable, the planner needs to take into account that a higher capital stock leads to higher wages; and if government debt is unavailable, the planner needs to take into account that a higher capital stock improves consumption smoothing.

3.3 Optimal Long-run Level of Debt

As is well known, the steady state Ramsey planner solution depends on initial conditions, i.e. the initial government debt level, when markets are complete (see e.g. Lucas (1990) and Chari and Kehoe (1999)). The next theorem shows that this result is overturned if markets are incomplete.

Theorem 2. *The long-run values of government debt, of the labor income tax rate, and of the capital income tax rate are generically independent of the initial level of government bonds (and the initial capital stock).*

To better understand this result, it is important to recognize that the key difference between complete and our incomplete markets model is that households face credit constraints in the incomplete markets world, and do not in the complete markets world. If markets are complete and thus lack credit constraints, the optimal steady state is linked to the initial steady state through the optimality conditions along the transition path. The optimality conditions enable computing the solution backwards starting at the optimal steady state. One can infer all period t variables from knowing all variables at period $t+1$. For example, from the capital stock in period $t+1$ one infers the interest rate, which using the consumption Euler equation, yields consumption in period t. This in turn allows to infer the level of investment and capital in period t. Credit constraints break this link. Knowing the interest

[15] Dávila et al. (2012) study a different problem, the constrained efficient allocation in a model with exogenous labor, where the planner also maximizes the discounted present value of lifetime utility but decides how much each individual has to save without the need to implement those decisions through a properly designed tax scheme. They find, in the economy calibrated to the U.S., that the optimal level of capital is much higher than the current U.S. level as the rich have to save more such that aggregate capital and thus wages increase.

[16] Lowering debt while keeping the total amount of households' assets constant increases capital but lowers the marginal product of capital (MPK). For a fixed after-tax interest rate $\bar{r}$ (which is necessary to keep total assets $K+B$ constant), a lower MPK is equivalent to lower capital income taxes.

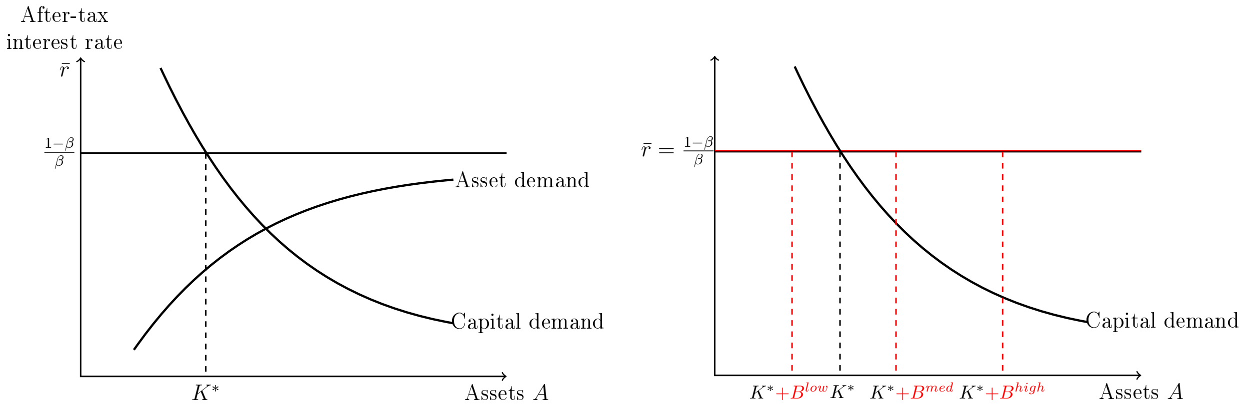
rate and period $t+1$ consumption of households who are credit constrained in period t is not sufficient to infer their period t consumption level; a binding credit constraint prevents us from using the consumption Euler equation as is possible in the complete markets case. As a result there is no deterministic link between the optimal and the initial steady state. Note that, from a computational perspective, this missing link prevents us also from using a simple "backward shooting" algorithm. But, as we explain in Section 5, it is Theorem 2 that renders the computational algorithm tractable, since we can first compute the steady state independently from the transition path and in a second step, solve for the transition path knowing both the initial and terminal conditions.[17]

The intuition for why there is a unique optimal level of government debt is straightforward. As in Barro (1979) and as in complete markets models the planner aims to smooth distortions over time using government debt. But with incomplete markets there is an additional benefit of providing more bonds: better consumption smoothing. The planner therefore deviates from full distortion smoothing and instead faces a trade-off between consumption and distortion smoothing. As a result the planner lowers labor taxes initially as there are two benefits - lower distortions today, and higher debt and thus better consumption smoothing - but just one cost (higher distortions tomorrow). Of course there are limits to how high debt can become as eventually future distortions become too big and outweigh the initial lower distortions and the benefits of higher liquidity. The optimal level of government debt is determined as equalizing the benefits and costs at the margin. As a result, in the long run both labor taxes and government bonds are high, which has the additional advantage that risky labor income is replaced with safe asset income.

A more formal intuition, and one that is moving us closer to how the proof works, is to note that there are not enough independent optimality conditions to determine the long-run steady state if markets are complete. Government bonds have no net worth since Ricardian equivalence holds in complete markets models, and therefore agents are willing to hold any amount of bonds in steady state. As a result bonds, B, appear only in the government budget constraint (the household budget constraint is dropped by Walras' Law) but this is not sufficient to pin down the long-run level of bonds. The steady-state government budget constraint determines only pairs of B and labor taxes τ_n which satisfy this constraint but does not determine each separately. In other words, an equation is missing. Thus, the long-run level of government debt (and also of labor taxes) is not determined from the steady state

[17] The credit constraints also explain why the optimal steady state wealth distribution is independent from initial conditions. One property of the Aiyagari model is that the credit constraint will be eventually binding for everyone. At the point in time when the credit constraint is binding a household's life is reset and the individual history until this point is wiped out. Eventually everyone's history was eliminated at some point such that the current situation is independent from the initial one, implying that each individual's initial income level will be irrelevant for the long-run income position.

Figure 1: Asset Markets in (In)complete Markets



(a) Asset Market: Aiyagari Model

(b) Asset Market: Complete Markets

FOCs, but only when initial conditions are taken into account.

We now argue that incomplete market models provide an additional equation - the asset demand equation - which serves to determine the long-run debt level since bonds have net worth in this class of models.[18]

As is well known, aggregate households' steady-state asset demand in the Aiyagari economy is described through a mapping between the after-tax interest rate $\bar{r}$ and assets A as illustrated in Figure 1a.[19] Since the capital stock is at its modified golden rule level K^* where the marginal product of capital equals $1/\beta$ (Theorem 1), total assets $A = K^* + B$ are one-to-one related to the number of bonds. Figure 2a shows that picking a specific capital income tax rate and therefore an after-tax interest rate $\bar{r}$, automatically also chooses a specific amount of bonds B and vice versa. The planner therefore faces a trade-off, illustrated

[18]Some intuition can also be gained from a simple reduced-form model where bonds by assumption have value, where the representative agent's utility equals

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) + \chi(B_{t+1}))$$

and the household budget constraint is (inelastic labor $n = 1$)

$$B_{t+1} = (1 + \bar{r}_{t+1})B_t - c_t + w_t.$$

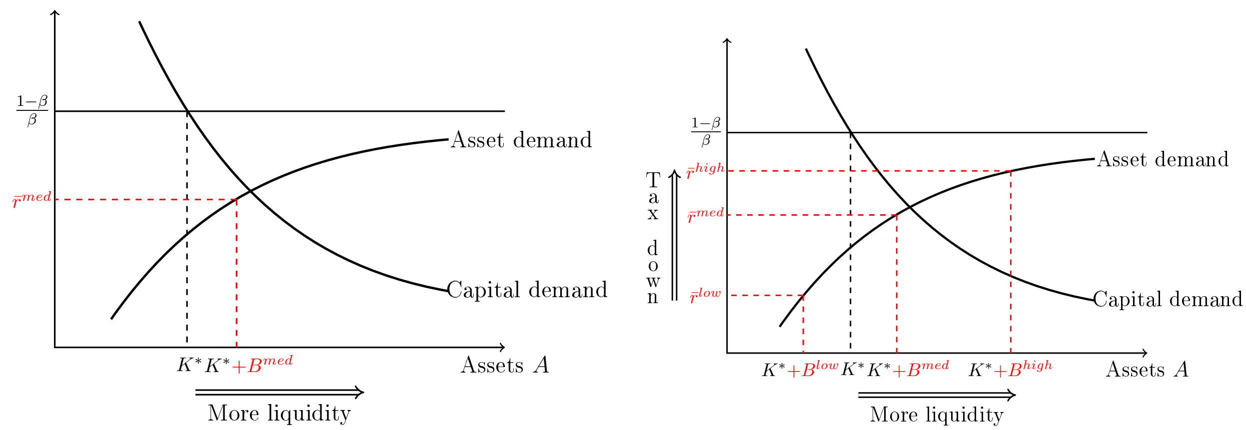
In steady state the planner has to respect households' demand for bonds function,

$$1 - \frac{\chi'(B)}{u'(c)} = \beta(1 + \bar{r}),$$

which is the additional equation that determines the long-run level of bonds in the Ramsey planner problem. The intuition in our incomplete markets model is the same with the important difference that bonds have a real value not by assumption but endogenously.

[19]For a textbook treatment of incomplete market models see Ljungqvist and Sargent (2012).

Figure 2: Additional Equation: B-$\bar{r}$ trade-off



(a) Additional Equation (b) B-$\bar{r}$ Trade-off

in Figure 2b, between supplying more bonds/liquidity and lower capital income tax rates. Choosing a low level of bonds, B^{low}, allows for a low after-tax interest rate $\bar{r}^{low}$, that is a high tax on capital income. Choosing higher levels of bonds, B^{med} or B^{high}, provides more liquidity and thus enhances consumption smoothing, but the capital income tax rates have to fall as households require higher after-tax interest rates, $\bar{r}^{med}$ or $\bar{r}^{high}$, to be willing to absorb the higher amount of assets $K^* + B$. This $B - \bar{r}$ trade-off provides the additional equation allowing us to determine the long-run level of debt using just the steady-state FOCs. This trade-off is absent in complete markets models and therefore the long-run level of bonds is not determined as illustrated in Figure 1b. In a steady state $1 + \bar{r} = 1/\beta$ and Ricardian equivalence implies that the representative agent is willing to hold any amount of bonds, $A^{low}, A^{med}, A^{high}$.

The formal proof uses ideas and concepts developed by Debreu (1970) to show the generic local uniqueness of competitive equilibria. We use the same approach since both in Debreu (1970) and here, one has to show that a set of equations is locally invertible and thus has a unique local solution. In Debreu (1970) this set of equations is given by the excess demand function and here it is the set of equations characterizing the optimal steady state. Local uniqueness is guaranteed generically, meaning it holds for a set of parameters of measure one (here, the distribution of idiosyncratic productivity; in Debreu (1970), initial endowments).[20]

The key step in the proof is however not related to Debreu's concepts. We prove the uniqueness of the steady-state distribution of the Lagrange multipliers of households' Euler equations in our incomplete markets economy and the indeterminacy in complete markets economies. As it turns out, this difference - uniqueness here and indeterminacy there - is

[20]The same proof to show local uniqueness can be used to show that the constraint qualification is generically satisfied such that the Karush-Kuhn-Tucker optimality conditions are necessary.

one-to-one related to the uniqueness/indeterminacy of the steady state.

As in Debreu (1970) local uniqueness implies that there are at most a finite number of solutions to the necessary FOCs of the optimal steady state. Figures 2 and 3 illustrate this reasoning. The two panels in Figure 2 show the simple case where the asset demand curve is monotonically increasing and therefore each level of assets is associated with a different after-tax capital income tax rate $\bar{r}$. That is, we obtain only one solution. Figure 3a illustrates that a finite number of solutions is possible, that is multiple levels of interest rates, $\bar{r}_1, \bar{r}_2, \bar{r}_3$, are associated with the same asset level A. What both figures have in common is that all solutions are locally unique; can be separated by open sets.[21] Adopting the arguments in Debreu (1970) and using our result that the distribution of Lagrange multipliers is unique shows that this is the generic case. Figure 3b shows a non-generic case where a continuum of interest rates $\bar{r}$ is associated with the same A and thus an infinite number of solutions would be possible. Following the arguments in Debreu (1970) and again using the uniqueness of the Lagrange multiplier distribution we show that this is a pathological case and not robust to small perturbations of fundamentals (distribution of productivity shocks).

While our proof considers very general cases including the one in Figure 3a and relies on generic local inconvertibility used in Debreu (1970), in many cases, including the quantitative exercise discussed later in this paper, the excess demand function is like the one in Figure 1a: smooth and monotonic. If so, the uniqueness can be intuitively understood as the result of a smoothly decreasing excess demand function instead of relying on the more general but also more technical property: generic local invertibility.[22]

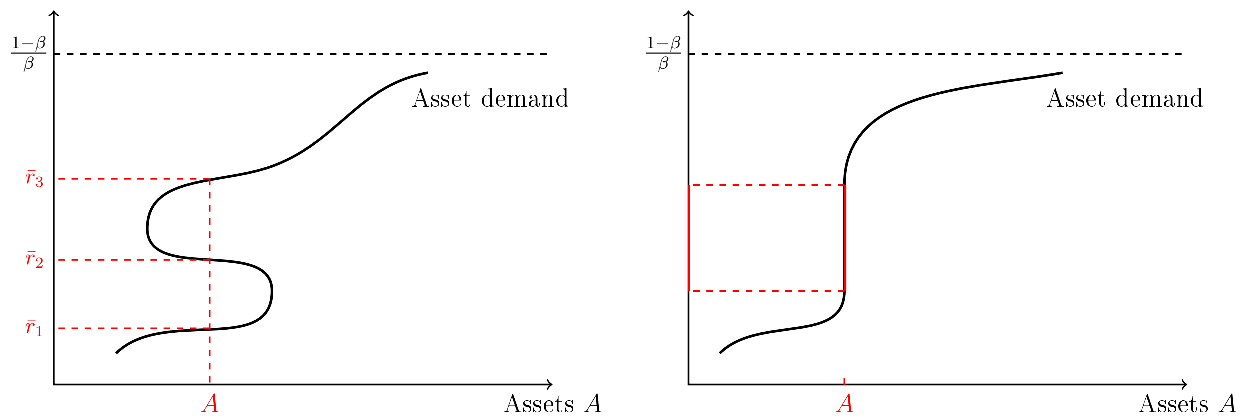
The inspection of the proof of Theorem 2 offers a simple way to check the uniqueness of the optimal steady state debt level through numerical methods. The steady state is characterized through hundreds of nonlinear equations and it is close to impossible to check whether this system of equations has a unique solution in a numerical analysis. Since we prove the uniqueness of the Lagrange multiplier distribution, our proof offers an easy solution as it just requires checking that the excess demand functions have no horizontal parts. If they do not, the solution is globally unique, including the transition. Note that we obtain uniqueness even for backward bending curves while a researcher without knowledge of our theorem could be inclined to conclude that multiple optimal steady states exist.

Corollary 1. *The Ramsey solution is globally unique if the excess demand functions feature no vertical parts. This property can be easily checked numerically.*

[21] For each solution e there is an open set U_e such that $e \in U_e$ and no other solution is in U_e.

[22] Similarly, while Debreu (1970) considers very general cases, many competitive equilibrium models used in economics admit smooth and monotonically decreasing excess demand functions and guarantee the uniqueness.

Figure 3: Generic Local Uniqueness



(a) Finite Number of SS

(b) Non-generic Indeterminacy

We apply this Corollary in our benchmark quantitative analysis below and show the results in Appendix II.

The final step of the proof first shows that the number of steady states is finite. Then, since the steady state depends continuously on initial conditions - such as the initial debt level - the finiteness of the number of steady states implies that the steady state does not depend on initial conditions. This concludes the proof.

4 Quantitative Analysis: Steady State

The quantitative analysis has two main parts. First, we compute the optimal policy in the long-run in this Section. Second, we use the optimal long-run policy as a terminal condition to compute the optimal policy along the transition path in Section 5.

We start by calibrating the model to the U.S. economy, and then compute the optimal values for the capital and labor tax rates, the capital stock and the level of debt in the steady state.[23] We also compute the optimal policy for a different Frisch elasticity, for a different elasticity of intertemporal substitution, for the income process as used in Aiyagari (1995) (with much lower income inequality than in our calibrated benchmark model) and for a specification of the income process allowing for permanent productivity differences.

4.1 Calibration

To calibrate the initial steady state of the benchmark economy to the U.S. economy, we first set the initial values of the following variables according to common practice in the

[23]These are the same policy instruments as used in Aiyagari (1995).

Table 1: Benchmark Calibration

Parameters	Value	Description	Source/Target
		Exogenous Parameters	
σ	2	Coefficient of Risk Aversion	
ϕ	1	Frisch Elasticity	
α	0.36	Capital Share	
δ	0.08	Depreciation Rate	
τ_l	28%	Labor Income Tax Rate	Trabandt and Uhlig (2011)
τ_k	36%	Capital Income Tax Rate	
B/Y	62%	Debt to GDP Ratio	Holter et al. (2019)
G/Y	7.3%	Gov. Expenditure to GDP Ratio	Prescott (2004)
		Calibrated Parameters	
ρ	0.94	Persistence of Labor Productivity	$a_{90}/a_{50} = 7.55$
σ_u	0.18	Std. Dev. of Labor Productivity Shock	$var\,(\log y) = 1.29$
β	0.96	Discount Rate	$K/Y = 3$
χ	2.56	Disutility from Labor	$mean(n) = 0.33$

literature. Following Trabandt and Uhlig (2011), the initial labor income tax rate is set to 28% and the capital income tax rate to 36%, as shown in table 1. The debt-to-GDP ratio is 61.85% as in Holter et al. (2019), and government expenditure is 7.3% of GDP, as in Prescott (2004). Then, we set some parameters in the utility function and production function as follows: $\sigma = 2, \phi = 1, \alpha = 0.36$ and $\delta = 0.08$. The values for σ, α and δ are those most commonly used in the literature. The value of the Frisch elasticity ϕ is set higher than what is considered a typical choice in the empirical labor literature, but lower than the choice among many macroeconomists. As this parameter is important for the size of labor taxes in standard models, we provide several robustness checks. Anticipating our result of high labor income taxation in the long run, the relatively high choice of $\phi = 1$ shows that this finding is not due to an inelastic household labor supply.

The remaining parameters are set to match related targets in the U.S. economy. Idiosyncratic labor productivity evolves according to the AR(1) process $\log e_t = \rho \log e_{t-1} + u_t$, $u_t \sim N(0, \sigma_u)$, where $\rho = 0.938$ and $\sigma_u = 0.184$ are calibrated to two targets in the U.S. economy, following Diaz-Gimenez et al. (2011): first, the variance of log labor income — 1.29, and second, the ratio of asset holdings at the 90 percentile over asset holdings at the 50 percentile — 7.55. In the benchmark this persistent stochastic process is the only source of individual heterogeneity while we add permanent differences in productivity in the robustness analysis below. The time preference is set as $\beta = 0.958$ to match a capital-output ratio of 3. The disutility from labor is $\chi = 2.56$ such that the labor supply on average is 0.33.

4.2 Results

We solve numerically the set of equations characterizing the steady state of the optimal policy problem – equation (18) to (22). Appendix III.1 describes the computational algorithm. Appendix II applies Corollary 1 to show that the optimal steady state (and the transition path) are unique.

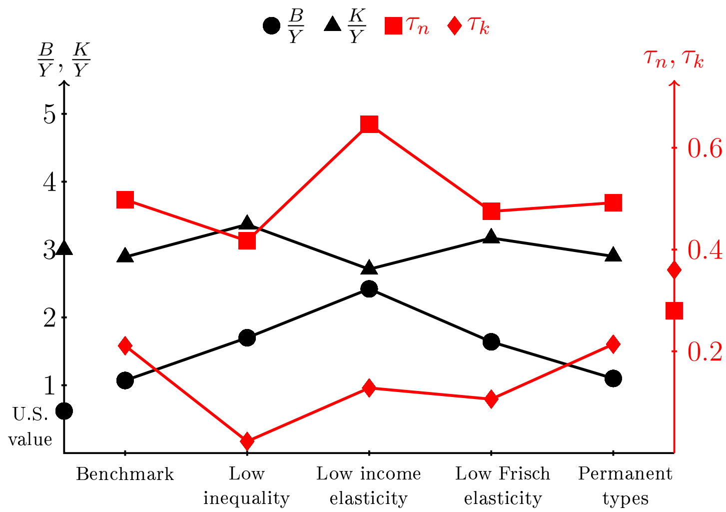
We conduct this experiment for the benchmark calibration shown in section 4.1, and several other parametrizations where we change one parameter at a time. We consider a low Frisch labor supply elasticity of $\phi = 0.5$ instead of 1, a small income effect of $\sigma = 1$ instead of 2 and a low inequality calibration as in Aiyagari (1995) where we set $\rho = 0.6$ and $\sigma_u = 0.2$ instead of 0.938 and 0.184 (we also recalibrate the model, including changing β and χ to match the benchmark capital output ratio and labor supply). Finally we allow for permanent productivity differences in addition to an stochastic element, implying that not all income states can be reached from any other state, e.g. the most productive worker today can never fall to the lowest productivity. While we discuss the results in detail in the following sections, Figure 4 provides an overview of optimal policies.

While the precise numbers and magnitudes of optimal policy vary across these parameterizations, we obtain several robust substantive conclusions. In the long run, the levels of government debt and labor income taxes are high and the capital tax is low relative to the current U.S. level. While a complete explanation requires computing the full transition path to the steady state as we show below, our steady-state results already provide several intuitive arguments. One line of reasoning is that higher labor taxes reduce the risky income stream and replace it with risk-free asset income from holding bonds. The high level of debt along with the modified golden rule for capital imply that households require a higher after-tax interest rate and thus the tax on capital income is low across parameterizations. The high level of debt also implies large interest rate payments requiring a quite high tax on labor income, again robustly across all calibrations. The results therefore show that in the long run, the planner does not use high capital income taxes for redistribution, but instead decides to tax the risky labor income at a high rate, and provides safe interest rate income from holding a large amount of debt which serves to smooth consumption very well.[24]

The transition analysis complements these findings and shows that the high steady-state debt level is a consequence of low labor taxes during the transition. Initially, it is optimal to have low labor distortions to redistribute toward households which hold low assets and primarily rely on their labor income. This policy is financed through issuing debt, and labor

[24] Note that GHH preferences ensure that the accumulation of assets and the associated wealth effect have no impact on labor supply. Strong wealth effects would imply a strong labor supply response and would bias our findings on the level of optimal government debt, establishing that using GHH preferences is important for our quantitative analysis.

Figure 4: Results Overview



taxes have to be raised eventually to pay for the interest payments on this debt. The transition analysis also explains why moving to a high debt, high labor taxes steady state is welfare-enhancing although the welfare in the final steady state is lower than for the calibrated initial steady state. The welfare gains during the transition when debt is accumulated and labor taxes are low outweighs the later welfare losses when labor taxes have to be raised.

The detailed descriptions and explanations of results for the benchmark calibration are considered in Section 4.2.1, for a low inequality economy in Section 4.2.2, for a low Frisch elasticity in Section 4.2.3, for a low income elasticity in section 4.2.4 and for permanent productivity differences in section 4.2.5. The details of the numerical approach are delegated to Appendix III.

4.2.1 Results: Benchmark Calibration

The findings for the optimal Ramsey policies are summarized in column (1) of Table 2 while the corresponding numbers — calibrated to the U.S. economy — are in column (2), as a comparison. In the long run, the optimal labor tax rate is as high as 49.8%, while the capital tax rate is 21.5% — higher than the optimal tax rate of 0 in a complete markets model, but lower than the current capital tax rate in the U.S. The quite large tax revenue is spent on redistribution through lump-sum transfer — 21.1% of GDP (larger than in the calibrated benchmark because of the lower GDP) — and more importantly, on interest payment on government debt (the debt level is high — 1.07 times GDP). The capital satisfies the modified golden rule, so the capital output ratio is 2.89, slightly lower than the current ratio in the U.S. The high labor tax rate reduces labor supply from 0.33 to 0.23. This policy leads to

Table 2: Steady-state Ramsey Solutions: Benchmark Economy

	Optimal Steady State (1)	U.S. Calibration (2)
τ_l	49.8%	28.0%
τ_k	21.1%	36.0%
T/Y	21.1%	14.1%
B/Y	1.07	0.62
K/Y	2.89	3.00
$\int n$	0.23	0.33
$N = \int en$	0.31	0.46
$var\,(\log(y+T))$	0.30	0.51
$var\,(\log(y+T+\bar{r}a))$	0.35	0.52
coeff. var. a	1.37	1.52
coeff. var. c	0.64	0.80

Note - The table contains the optimal Ramsey steady-state policies. (1): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (2): U.S. economy (calibration target)

lower inequality of disposable income and reduces the inequality of wealth.

This steady state features a high tax rate on labor and a large amount of redistribution. First, the social planner shrinks income inequality by setting a high labor tax rate, even though given the high Frisch elasticity the distortion of labor supply is sizeable, 30% lower compared to the level in the calibrated U.S. economy. The lump-sum transfer stays the same as the initial level, but it becomes larger relative to the lower labor income, the inequality of after tax labor income and transfer income $\log(y+T)$ decreases to 0.30 from 0.51. The planner lowers income risk further through paying large interest on high government debt and reducing wealth inequality. The coefficient of variation of household assets drops from 1.52 to 1.37, which manifests in lower inequality of disposable income $\log(y+T+\bar{r}a)$ of 0.35 relative to the benchmark level of 0.52. Eventually, the consumption inequality reduces to 0.64, 20% lower than the initial level.

Our findings show how a welfare-maximizing planner should redistribute and make use of government debt in an incomplete markets setting, as the asset-income of households can be increased. This means of redistribution involves issuing government debt while keeping the steady-state capital fixed at the level that satisfies the modified golden rule.

4.2.2 Results: A Low Inequality Economy

The income process in our benchmark calibration implies a large amount of inequality. If instead we target an economy with lower income inequality, as in Aiyagari (1995), the motive

Table 3: Steady-state Ramsey Solutions: Low Inequality Economy, $\rho = 0.6, \sigma_u = 0.2$

	Optimal Steady State (1)	U.S. Calibration (2)
τ_l	41.7%	28%
τ_k	2.3%	36%
T/Y	14.5%	13.4%
B/Y	1.70	0.62
K/Y	3.37	3.00
$\int n$	0.29	0.33
$N = \int en$	0.31	0.36
$var\,(\log(y+T))$	0.158	0.163
$var\,(\log(y+T+\bar{r}a))$	0.160	0.165
coeff. var. a	0.72	0.75
coeff. var. c	0.229	0.230

Note - The table contains the optimal Ramsey steady-state policies for the low inequality economy, $\log e_t = \rho \log e_{t-1} + u_t$ with $\rho = 0.6$ and $\sigma_u = 0.2$. (1): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (2): U.S. economy (calibration target)

for redistribution is smaller: while the main results stay qualitatively the same — high labor tax rate, low capital tax rate and high government debt, some of the optimal policy choices are different quantitatively, which illustrate how polices are determined according to income inequality. Based on the benchmark economy, we change the parameters of the income process to $\rho = 0.6$ and $\sigma_u = 0.2$. Then we recalibrate the model by changing β to 0.974 and χ to 2.56 to match a capital output ratio of 3 and an average labor supply of 0.33.

The optimal long-run tax rates in this low inequality economy are lower, compared to the benchmark economy. As shown in Table 3, the labor income tax rate drops to 41.7% though it is still quite high, compared to the level in the current U.S. economy. The capital income tax rate is close to 0, only 2.3%. The transfer is now 14.5% of GDP and the level of government debt is even larger than in the benchmark, at 1.7 times GDP, mainly due to the recalibration of parameters.

In this low inequality economy, the planner has, compared to the high inequality benchmark economy, fewer incentives to reduce the income risks and redistribute to low-income households, resulting in lower labor and capital tax rates. The initial income inequality is already quite low: the variance of log disposible income $var\,(\log(y+T+\bar{r}a))$ is only 0.165, much lower than the level in the benchmark 0.52. The Ramsey planner increases the labor tax rate to 41.7%, which reduces the labor supply to 0.29 and the risky labor income, but the change — both the increase of labor tax and the reduction of labor supply are smaller than in the benchmark calibration. Consequently, the reductions in the inequality of income, assets and consumption are much smaller than in the benchmark: the variance of log disposible income $var\left(\log(y+T+\bar{r}a)\right)$ slightly reduces to 0.160, from the already low initial level 0.163; consumption inequality, measured by the coefficient of variation of consumption, reduces to 0.229 from the initial level of 0.230.

4.2.3 Results: Low Labor Supply Elasticity

Now we consider a lower labor supply elasticity and set the Frisch elasticity to $\phi = 0.5$. Both labor supply and labor income become more volatile due to the drop in the Frisch elasticity, requiring an adjustment to the stochastic process for productivity. As a result of this recalibration to match the same targets as in the benchmark, we now use $\beta = 0.950$, $\rho = 0.925$, $\sigma_u = 0.27$ and $\chi = 6.68$. The modified golden rule capital stock computed using the recalibrated parameter values is 2.71. The steady-state results are reported in Table 4.

A lower elasticity of labor supply implies that labor supply is less sensitive to an increase in labor taxes, rendering a labor tax of 64.6% optimal. At the same time, although the steady-state labor tax rate is higher than in the benchmark economy, the labor supply is similar: 0.23, due to the lower distortion of the labor tax. The government takes advantage of the lower labor distortion to set a higher labor tax rate, which can sustain an even higher government debt level, 2.42 times GDP, with a lower capital tax rate of 12.8%. As in the benchmark, a high labor tax rate, a low capital tax rate and a high level of government debt reduce inequality. In this economy the reduction in our measures of inequality are larger than in the benchmark due to the higher labor tax rate: the variance of log disposable income decreases to 0.31, the coefficient of variation in assets to 1.13, and the coefficient of variation in consumption to 0.55. Comparing Tables 2 and 4 shows that the (percentage) reduction is larger than in the benchmark for each inequality measure.

4.2.4 Results: Low Risk Aversion

To explore the role of risk aversion in shaping our results, we consider a lower risk aversion by setting the coefficient of relative risk aversion $\sigma = 1$. As a result of a recalibration we use now $\beta = 0.968$, $\rho = 0.922$, $\sigma_u = 0.21$ and $\chi = 2.56$. Notice that the recalibrated β needs to be higher than in the benchmark, to match the same capital-output ratio. The high β here implies that the modified golden rule capital stock is now larger than in the benchmark. Results are reported in Table 5.

Like in the benchmark, the optimal labor tax rate and the debt level are high, and the capital tax rate and inequality are low. The quantitative difference is that tax rates are smaller in this calibration with low risk aversion, for two reasons. First, low risk aversion implies that households are less averse to income risks, and the planner is less averse to inequality, implying

Table 4: Steady-state Ramsey Solution: Low Labor Supply Elasticity, $\phi = 0.5$

	Optimal Steady State (1)	U.S. Calibration (2)
τ_l	64.6%	28%
τ_k	12.8%	36%
T/Y	21.9%	14.1%
B/Y	2.42	0.62
K/Y	2.71	3.00
$\int n$	0.23	0.33
$N = \int en$	0.30	0.44
$var\,(\log(y + T))$	0.22	0.51
$var\,(\log(y + T + \bar{r}a))$	0.31	0.53
coeff. var. a	1.13	1.55
coeff. var. c	0.55	0.75

Note - The table contains the optimal Ramsey steady-state policies for the low labor supply elasticity, $\phi = 0.5$.
(1): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level.
(2): U.S. economy (calibration target)

a reduced incentive to use a high labor tax to lower risk and to redistribute. Second, the calibration with low risk aversion implies a higher β than in the benchmark, and the modified golden rule implies a higher capital level in the optimal steady state, 3.17, higher than in the initial steady state and also higher than in the benchmark optimal steady state. A higher capital stock and output level allow the government to tax labor and capital at lower rates to finance government expenditures, transfers and interest rate payments on debt. Consequently, the labor tax rate is set at 47.5%, still high but lower than in the benchmark. The capital tax rate is set at 10.6%, even lower than the 21.1% level in the benchmark. The lower capital tax makes the households willing to absorb both the higher capital stock, 3.17 times GDP, and the high level of government debt, 1.64 times GDP. Although government debt is higher than in the benchmark it is still sustainable, even with lower tax rates, because the capital stock and output are higher in this calibration with low risk aversion. Given these policies, the inequality of disposable income, assets and consumption decreases while the reduction in inequality is similar to the benchmark, tax distortions are smaller and the capital stock and output are higher.

4.2.5 Results: Permanent Income Differences

In this experiment, in addition to the labor productivity shocks described in the benchmark, we introduce permanent differences in labor productivities. There are two types of households and we assume that the more productive type has a 20% higher permanent productivity than

Table 5: Steady-state Ramsey Solution: Low Risk Aversion, $\sigma = 1$

	Optimal Steady State (1)	U.S. Calibration (2)
τ_l	47.5%	28%
τ_k	10.6%	36%
T/Y	18.2%	14.1%
B/Y	1.64	0.62
K/Y	3.17	3.00
$\int n$	0.25	0.33
$N = \int en$	0.34	0.46
$var\,(\log(y+T))$	0.35	0.51
$var\,(\log(y+T+\bar{r}a))$	0.37	0.51
coeff. var. a	1.27	1.51
coeff. var. c	0.63	0.77

Note - The table contains the optimal Ramsey steady-state policies for low risk aversion, $\sigma = 1$. (1): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (2): U.S. economy (calibration target)

the other type. More specifically, there are both time-varying and time invariant components in the labor productivity of a household. In the more productive type's log labor productivities, the time-invariant fixed effect is 0.2, while it is 0 for the other type. The time-varying labor productivity of a household follows an AR(1) process like in the experiments above. In the recalibrated model, the parameters are very close to the values in the benchmark: $\beta = 0.958$, $\rho = 0.936$, $\sigma_u = 0.18$, $\chi = 2.56$.

The optimal policies also are very similar to the benchmark values. The optimal labor and capital taxes are 49.2% and 20.3% — very close to the solution in the benchmark. Other model moments - government debt, labor supply, income and asset inequalities - also are very close to those in the optimal steady state in the benchmark calibration. This experiment shows that introducing permanent income differences has a small impact on optimal policies. The reason is that the social planner wants to reduce income inequalities generated by both permanent income differences in labor productivities and risks, so similar to the benchmark with only income risks but no permanent differences, the planner still increases labor tax rate to reduce the unequal income source, increases government debt to provide liquidity to households, and incentivizes households to rely more on transfers and interest payments.

Table 6: Steady-state Ramsey Solutions: Permanent Income Differences

	Optimal Steady State (1)	U.S. Calibration (2)
τ_l	49.2%	28%
τ_k	21.4%	36%
T/Y	20.3%	13.8%
B/Y	1.10	0.62
K/Y	2.90	3.00
$\int n$	0.23	0.33
$N = \int en$	0.32	0.46
$var\,(\log(y+T))$	0.50	0.61
$var\,(\log(y+T+\bar{r}a))$	0.51	0.62
coeff. var. a	1.40	1.55
coeff. var. c	0.66	0.81

Note - The table contains the optimal Ramsey steady-state policies for permanent differences in productivity. (1): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (2): U.S. economy (calibration target)

5 Quantitative Analysis: Transition

Let us reiterate a main objective, and huge computational challenge, in this paper: to compute the paths of tax rates and government debt which maximize welfare at date 0. To achieve this, several hundred or thousands of variables must be chosen in a highly nonlinear optimization problem. However, our previous result on the optimal steady state turn this non-manageable optimization problem into a manageable one. We have shown that the optimal steady state is independent of initial conditions. From a computational standpoint this means that we know the optimal long-run policies and allocations without having to compute the transition. We know the initial conditions (economy calibrated to U.S. economy) and we know the terminal condition, the optimal steady state characterized above. The computational problem is then to find the optimal policy path that satisfies all necessary first-order conditions along the transition, and at the same time satisfies the initial and terminal conditions. This problem is still very daunting as it involves solving hundreds or thousands of nonlinear equations, but it is significantly easier (and therefore tractable) than the original problem of trying to find the optimal transition and the optimal terminal point at the same time. Given the large number of variables involved in the original problem there is no way to check the global validity of a candidate solution. However, with our approach this check is not necessary.

Knowing the optimal path of policies allows us to compute the welfare gains of switching to the optimal policy and helps us understanding the properties of the optimal steady-state policies better, since those obviously depend on the transition.

5.1 Computational Algorithm

Appendix III.2 outlines the details of computing an optimal transition, starting from the model calibrated to the U.S. economy and going to the optimal long-run steady state.

5.2 Calibration of Initial Steady State

We choose the simulation period to be 800 years since the distribution of assets and productivity converges only slowly. In order to facilitate the computation of the optimal transition path we recalibrate the model with 10-year periods. We have to recalibrate the model, and choose targets in our 10-year period economy to be consistent with those in the 1-year economy in the benchmark economy in Section 4. To choose the parameters governing the productivity process, we run a Monte Carlo simulation to obtain the variance of the log of 10-yearly earnings in the 1-year calibration exercise we performed in Section 4.1. We then adjust σ_u, the standard deviation of the innovations to labor productivity, and the persistence of labor productivity, ρ, to match the variance of this log of 10-yearly earnings, $var(log(y^{(10)})) = 1.06$, and the ratio between the asset holdings of a household at the 90th percentile and a household at the 50th percentile, $a_{90}/a_{50} = 7.55$. With a 10-year period we set $K^{(10)}/Y^{(10)} = 0.3$ consistent with a capital output ratio of three when Y is annual and not the 10 times higher, 10-year output $Y^{(10)}$. Then to get the same annual interest rate of 4% as in section 4.1, $r^{(10)} = 1.04^{10} - 1$, we adjust the 10-year depreciation rate to $\delta^{(10)} = 0.720$. Notice here we use the superscript $^{(10)}$ to denote a 10-year variable and when necessary, distinguish it from its annual counterpart without the superscript.

5.3 Results: Transition

We now compute the transition path with optimal choices of debt levels, of capital tax and labor income tax rates.[25] Transfers are kept at the calibrated benchmark level. Figure 5 plots the optimal transition path of capital taxes, τ_k, and labor income taxes, τ_l, and Figure 6 plots the optimal path of the capital stock, K, and government debt, B, normalized by annual output, so that they are comparable to the numbers in the previous section. It is optimal to reduce the labor tax rate to almost zero in the first few periods and to finance this by increasing debt and taxing capital highly. Low initial labor taxes reduce distortions

[25] Due to the re-calibration of the model with 10-year periods, the long-run optimal policies in this section are quantitatively similar but not identical to the benchmark results in Section 4.2.1 with 1-year periods.

Table 7: Benchmark Calibration with 10-year Periods

Parameters	Value	Description	Source/Target
		Exogenous Parameters	
σ	2	Coefficient of Risk Aversion	
ϕ	1	Frisch Elasticity	
α	0.36	Capital Share	
$\delta^{(10)}$	0.72	10-year Depreciation Rate	
τ_l	28%	Labor Income Tax Rate	Trabandt and Uhlig (2011)
τ_k	36%	Capital Income Tax Rate	
B/Y	62%	Debt to GDP Ratio	Holter et al. (2019)
G/Y	7.3%	Gov. Expenditure to GDP Ratio	Prescott (2004)
		Calibrated Parameters	
ρ	0.62	Persistence of Labor Productivity	$a_{90}/a_{50} = 7.55$
σ_u	0.38	Std. Dev. of Labor Productivity Shock	$var\left(\log y^{10}\right) = 1.06$
β	0.67	Discount Rate	$K/Y = 3$
χ	7.02	Disutility from Labor	$mean(n) = 0.33$

and increase the welfare of wealth-poor households. Through high initial taxes on capital the planner achieves redistribution, whereas debt issuance relaxes households' credit constraints. The labor income tax, τ_l, starts out at 0.2% and gradually increases to a level of about 50% after 80 years (the long-run steady-state level of τ_l is 50% with the 10-year calibration). The low initial taxation of labor income leads to an accumulation of debt to its new high steady-state level of 93% of annual GDP (9.3% 10-year GDP). This high debt level requires high labor taxes, implying that the long-run steady-state resulting from the path of optimal policy has lower welfare than the initial one. The transition explains why this policy change nevertheless is welfare-enhancing. The welfare gains realized during the early period of the transition outweigh the welfare losses later on.

As is well known in the literature, the planner typically wants to impose a very high tax on perfectly inelastic capital in the first period. With transfer as an instrument, the planner would like to confiscate the entire capital stock in period one and redistribute it. In our case, with labor tax, capital tax and debt as the instruments, this is not necessarily the case. However, following the literature, we still impose an upper bound of $\tau_k = 100\%$. The capital tax thus starts out at 100% in period one and gradually decreases to a level of about 19% after 100 years (the long-run steady-state level of τ_k is 19% with the 10-year calibration). In this model, the planner sets the initial capital tax rate high, for two reasons: First, as emphasized in the Ramsey taxation literature with a representative agent, the initial distortion of capital taxation is low; Second, new in the model with heterogenous agents, the high capital tax rate reduces wealth inequality, which is high in the initial steady state calibrated to the U.S.

economy. In the long run, the capital tax rate is low, again for two reasons: First, it is optimal to keep long-run inter-temporal distortions low, similar to the findings with a representative agent; Second and new in the Aiyagari world, the low capital tax rate renders household savings high enough to absorb both the high level of government debt and the capital stock implied by the modified golden rule.

Debt is rapidly increasing in the first few periods and reaches a peak after 50 years (5 periods) before it starts decreasing towards the steady-state level. At the peak, debt is about 1.97 times annual GDP (0.197 relative to 10-year GDP), whereas the long-run level is about 93% of annual GDP (0.093 in 10-year terms). The evolution of the optimal debt level reflects the optimal choices of labor and capital taxes. The sharp initial increase of government debt in the first 40 years is due to the very low labor tax rate, which starts at 0.2%, stays very low (below 2%) for 30 years and then increases to 12% and 34% in the following 2 periods (20 years), which are still low relative, compared with the long-run level. The initially low labor tax rates help the wealth-poor households, who are at or close to the borrowing constraint and rely mainly on labor income, to receive higher incomes, become richer, save more and smooth consumption better. The initial capital tax rate is high, but it cannot overturn the deficit created by the very low labor tax rate, so government debt increases to a high level, which is desirable as it provides more liquidity to constrained households in the beginning periods. The labor tax rates monotonically and smoothly increases to its steady-state level that we computed as described above. The optimal paths of labor and capital income taxes then determine the size of the government deficit and the accumulation of debt, including its hump-shaped evolution over time. The planner could have picked a monotonically increasing debt path but this would have implied higher initial labor taxes since the steady state labor tax rate is pinned down at 50%. This different labor tax rate path would hurt the poor and thus would have lowered welfare.

Furthermore the increases in labor taxes and the accumulation of debt interact in a welfare enhancing way. As the wealth-poor households gradually accumulate more and more assets, the planner can increase the labor tax rate to reduce income risks, without driving many households to the borrowing constraint. Over time, the labor tax rate increases to a high enough level ($>$ 40% after 50 years), which implies a positive primary surplus and thus a declining government debt level. The government debt declines rapidly between year 50 and 100, because while the labor tax rate is already close to the high steady state level in year 50, the capital tax rate is 50%, still much higher than its steady state level 19%. The government surplus is high and then gradually declines as the capital tax rate declines to its low steady state level in year 100. Then government debt becomes stable and gradually converges to its steady-state level — 93% of annual GDP, which is still much higher than the initial level.

Figure 5: Optimal Transition Path for labor tax τ_l (left) and capital tax τ_k (right)

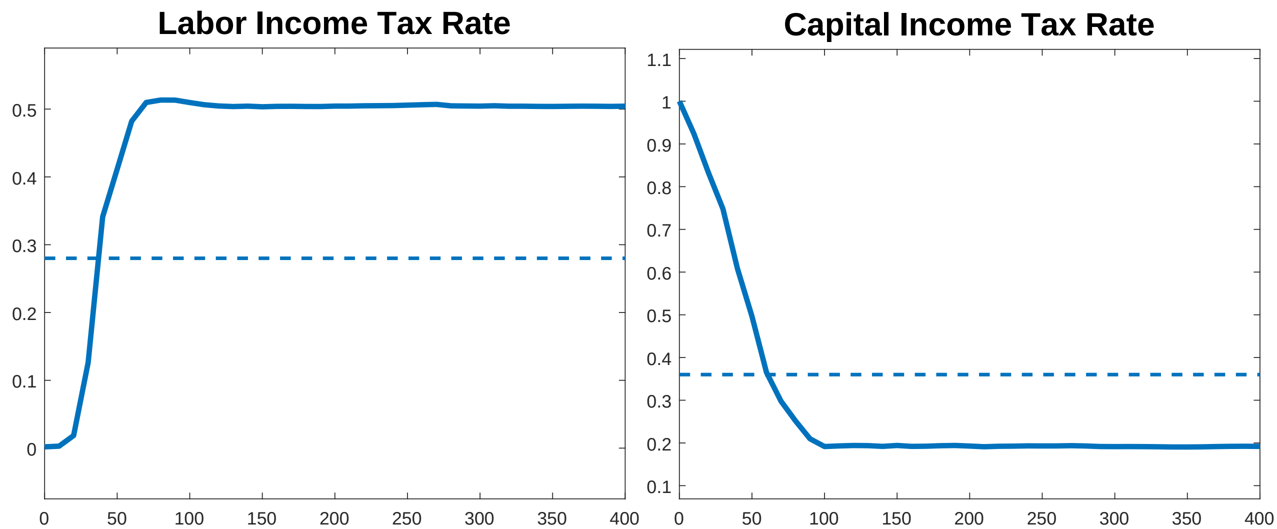
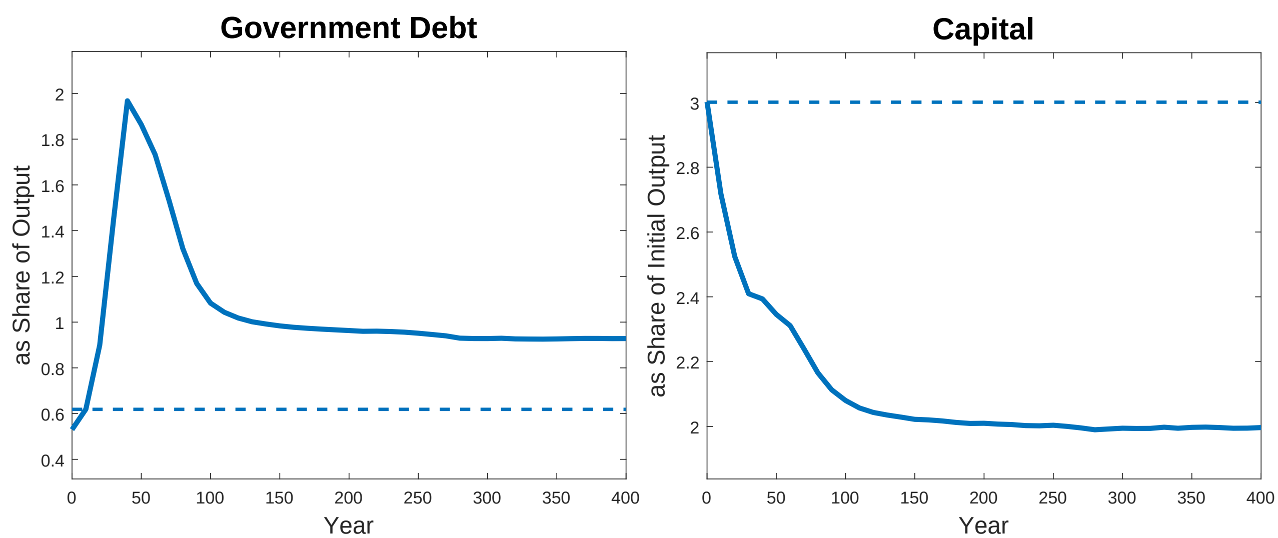


Figure 6: Optimal Transition Path for B (left) and K (right).



The inequality measures along the transition are illustrated in Figure 7 and 8. In the long-run, inequalities of incomes, assets and consumption are low. In the long-run steady state, the labor tax rate is high so labor income risk is low; moreover, the low capital tax rate and high government debt imply large household assets and asset incomes as a safe income source. The two features generate low consumption and asset inequality. Along the transition, tax rates are different from the long-run values, so inequalities of incomes, consumption and assets are different from the long-run counterparts, especially in the beginning. The initial labor tax rates are low, and labor income is the main source of the disposable income as asset income is low given the high capital tax rate. Consequently, the inequality of disposable income, measured by the variance of log disposable income, is as large as 0.57, even larger than the level inequality at the initial steady state — 0.53, as shown by the blue solid line at the left panel of Figure 7. Over time, as the labor tax rate increases, labor income inequality can be affected; More importantly, the level of labor income decreases, implying that risky labor income becomes a smaller fraction of disposable income, and the composition effect reduces inequality of disposable income. As shown by the blue solid line, inequality of disposable income decreases gradually to its low steady state level 0.43. The red dashed line plots the variance of log labor income plus transfers, which is very close to the variance of log disposable income. This similarity shows that the decline of disposable income inequality mainly comes from the composition effect: risky labor income becomes smaller and smaller relative to the constant transfers. Remember that transfers are fixed and the variance of log labor income is constant over time and not affected by the labor tax rate, given the GHH utility, so the change of the variance of log labor income plus transfers is purely due to a composition effect. There are small differences between the two inequalities shown by the blue solid and red dashed lines, and they are driven by the asset income inequality, whose impact is not large. The right panel of Figure 7 shows that asset income starts from 0, given the 100% capital income tax rate; then gradually increases until year 100, as capital income tax rate decreases during that period; and eventually, it converges to its steady state level 0.1. Throughout the transition, asset income inequality is low, relative to its level at the initial steady state. While asset income is an important part of disposable income and important for household consumption smoothing, its inequality level is low, so its contribution to the dynamics of disposable income inequality is small. After the 100% high capital tax rate in the first period, the asset inequality, measured by the variance of assets, slightly decreases to 1.44 from its initial level of 1.59, as shown by the left panel of Figure 8. While the initially high capital tax rate tends to decrease asset accumulation and inequality, the initially low labor tax rate tends to increase inequality in income and savings. After the first period, the asset inequality increases for 4 periods, but not much, suggesting that the impact of large labor income inequality is larger than the impact of high capital tax rate, but the difference is not larger. After period 5, the labor tax rate reaches and stays at the high level and income inequality declines, consequently, asset income inequality also declines towards the low long-run level 0.64. Finally, consumption inequality starts high and declines towards its low steady state level over time, similar to the trend of income inequality. In the first two periods, given the very low labor tax rates ($< 1\%$), consumption inequality levels are as high as 0.43, even slightly higher than the level at the initial steady state — 0.41. Notice that the large consumption inequality is not due to a low level of consumption of the poor. The planner tries to increase the consumption of the poor by setting a low labor tax rate, which also increases the consumption of households with high labor productivities, such that the large consumption inequality does not necessarily imply low welfare. Afterwards, as labor tax rates increase, consumption inequality follows the trend of income inequality: declining fast until about year 100 and then gradually converging to the low steady-state level 0.28. Finally, the economy reaches the steady state with high labor taxes and government debt, and low inequalities in disposable income and consumption.

Figure 7: Inequality of incomes: disposable income (left) and asset incomes (right)

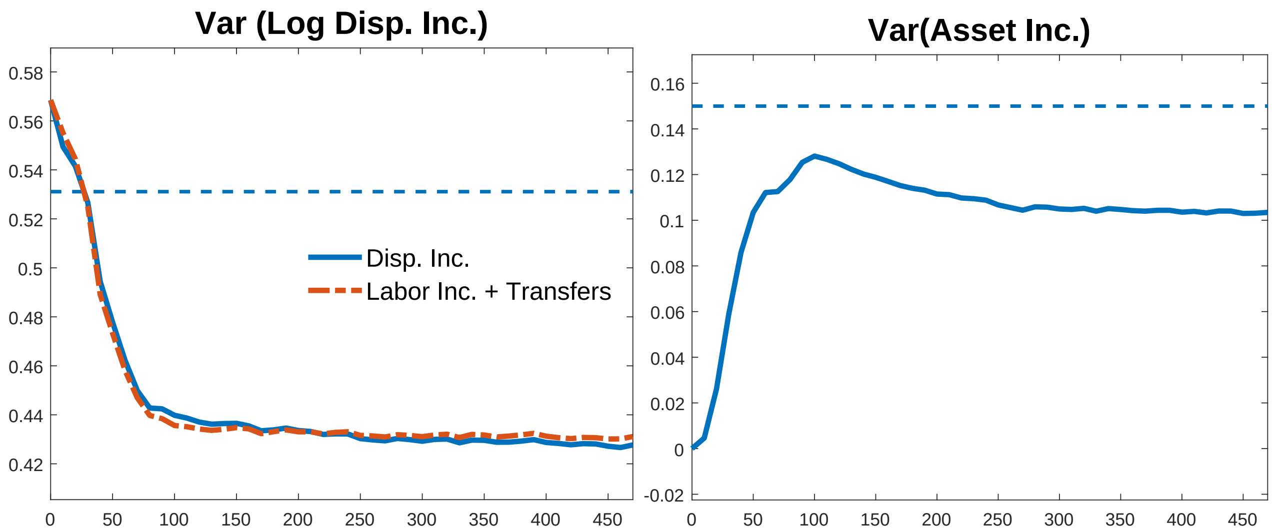
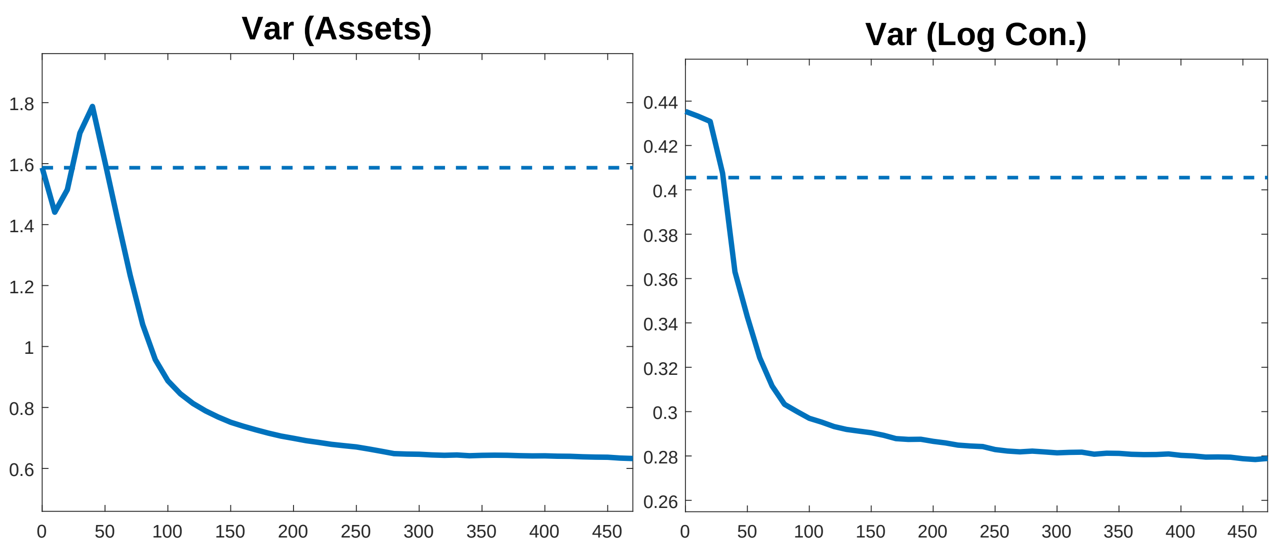


Figure 8: Inequality of assets (left) and consumption (right)



5.4 Welfare Analysis

To quantify the welfare gain of the optimal transition relative to the initial steady state, we compute the consumption equivalent gain, i.e., by what percentage we need to increase the consumption of all households in all periods and all states given the initial steady-state policies such that their aggregate expected lifetime utilities at period 0 equal the aggregate expected lifetime utilities of households at period 0 given the optimal transition. More specifically, we denote the consumption equivalent gain as φ, and then the expected lifetime utility of a household with asset a_0 and labor productivity e_0, given the consumption equivalence gain and initial steady-state policies, becomes

$$V_0^{ce}(a_0, e_0) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(\varphi c_t^{ini} - \chi \frac{\left(n_t^{ini}\right)^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}} \right)^{1-\sigma}}{1-\sigma},$$

where c_t^{ini} and n_t^{ini} are consumption and labor supply in different periods and states of a specific households, given the initial policies. Then we solve for the consumption equivalence gain φ, such that the period 0 aggregate welfare of all households, given the consumption equivalence gain and the initial policies, equals the corresponding aggregate welfare given the optimal transition:

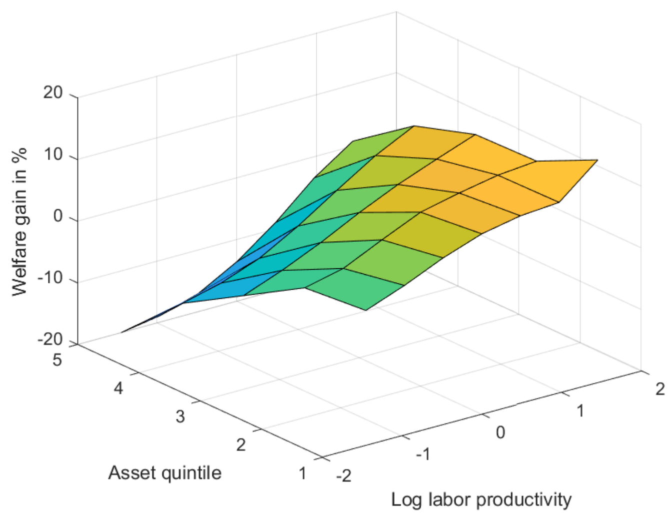
$$\int V_0^{ce}(a_0, e_0) d\mu_0 = \int V_0^{opt}(a_0, e_0) d\mu_0, \tag{24}$$

where $V_0^{opt}(a_0, e_0)$ is the expected lifetime utility from period 0 of the household given the optimal transition we computed in this section; recall that μ_0 represents the initial distribution of households.

The lifetime welfare gain achieved from moving to the optimal policy taking into account the full transition period is equivalent to increasing the consumption in all states of the initial steady state by 2.59%. This is a substantial welfare gain, especially because we here have 10-year time periods, and the welfare gain must be viewed as an average welfare gain over 10 years.

To understand who gains and who loses from the optimal tax reform, we can also solve Equation (24) for different parts of the state space. In Figure 9 we plot the welfare gain in consumption equivalents by log labor productivity, $\log e$, and asset quintile. As can be seen from the figure the welfare gain is decreasing in the asset level and increasing in labor productivity. This is as we would expect. The initial capital tax of 100% will hurt everyone, but wealthy households the most. The high capital tax thus achieves some redistribution but at the cost of reduced insurance (households have lower asset levels). The initial subsidies to labor are on the other hand welfare-improving for everyone. The gains are largest for more productive households. What the social planner in this economy is lacking is, of course, a progressive labor tax. With only flat labor taxes, the planner cannot help the least productive

Figure 9: Welfare Gain in Consumption Equivalents by Productivity Level and Wealth Decile



households without also helping the most productive ones.[26]

6 Conclusion

In incomplete markets models of the Bewley-Imrohoroglu-Huggett-Aiyagari type, inequality is, to a large degree, purely happenstance and this calls for considerable redistribution in an optimal welfare-maximizing policy. Several classic instruments are available: labor income taxation, capital income taxation, and government debt. However, massive redistribution could also inflict big efficiency losses, curbing some of its desirability. While all of these instruments can reduce inequality, what is then unclear is how much redistribution should come from which instrument. Each comes with efficiency losses in terms of distorting labor supply and/or capital accumulation. Furthermore, if markets are incomplete, the planner can also reduce wealth inequality through issuing more debt such that low-labor income households can also rely on their asset income for consumption purchases.

This paper offers the following conclusions on how to redistribute in a welfare maximizing way. The optimal policy to provide insurance is to tax labor heavily in the long run, and redistribute through government bonds. In particular redistribution works through high labor taxation with capital income taxed only at a low rate, a conclusion that holds in high- and low-inequality economies and is robust to changing parameters such as labor supply elasticity.

The results during the transition to the long-run optimum are, however, quite different.

[26]Optimal progressive labor income taxes as in Heathcote et al. (2017) and Boar and Midrigan (2021) is an interesting extension left for future work.

During the transition debt is accumulated, and the increase in government revenue is used to bring labor taxes down to below their current U.S. level. Only when the long-run steady state is approached, and the amount of debt and associated interest rate payments are high, does it become necessary to increase labor taxes to balance the budget. At that time, capital taxes will have converged also to a low level after initial periods of high taxation, a well known result as capital is supplied quite inelastically in the short-run.

We prove two theoretical results which enable this quantitative analysis. We show theoretically that the optimal capital stock is at the modified golden rule and that the long-run optimal steady state is independent of initial conditions. In particular, there is a unique long-run optimal level of government debt that is independent of the initial level of debt in our incomplete markets model, a result not valid in complete markets models.

The independence of initial conditions result renders tractable a quantitative analysis of the dynamic optimal taxation problem. This is the first paper to apply a Lagrangian maximization approach to study optimal taxation in an Aiyagari economy. Since we can compute the terminal point first, we are able to design a feasible computational algorithm for finding the entire sequence of optimal policies.

References

Acemoglu, D. and M. K. Jensen (2015): "Robust Comparative Statics in Large Dynamic Economies," *Journal of Political Economy*, 123, 587–640.

Acikgoz, O. T. (2018): "On the Existence and Uniqueness of Stationary Equilibrium in Bewley Economies with Production," *Journal of Economic Theory*, 173, 18–55.

Aiyagari, R., A. Marcet, T. Sargent, and J. Seppälä (2002): "Optimal Taxation without State-Contingent Debt," *Journal of Political Economy*, 110, 1220–1254.

Aiyagari, S. R. (1994): "Optimal Capital Income Taxation With Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Working Paper 508, Federal Reserve Bank of Minneapolis.*

——— (1995): "Optimal capital income taxation with incomplete markets, borrowing constraints, and constant discounting," *Journal of Political Economy.*

Aiyagari, S. R. and E. R. McGrattan (1998): "The Optimum Quantity of Debt," *Journal of Monetary Economics*, 42, 447–469.

ALIPRANTIS, C. D. AND K. BORDER (2006): *Infinite Dimensional Analysis. A Hitchhiker's Guide*, Springer-Verlag Berlin Heidelberg, 3 ed.

BARRO, R. J. (1979): "On the Determination of the Public Debt," *Journal of Political Economy*, 87, 940–971.

BAYER, C., B. BORN, AND R. LUETTICKE (2020): "The Liquidity Channel of Fiscal Policy," *CESifo Working Paper*.

BHANDARI, A., D. EVANS, M. GOLOSOV, AND T. J. SARGENT (2017a): "Fiscal Policy and Debt Management with Incomplete Markets," *The Quarterly Journal of Economics*, 132, 617–663.

——— (2017b): "Public Debt in Economies with Heterogeneous Agents," *Journal of Monetary Economics*, 91, 39–51.

——— (2021): "Inequality, Business Cycles, and Monetary-Fiscal Policy," *Econometrica*.

BOAR, C. AND V. MIDRIGAN (2021): "Efficient Redistribution," Working paper.

CARROL, C. D. (2006): "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems," *Economic Letters*, 91, 312–320.

CHAMLEY, C. (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54, 607–622.

CHARI, V. V. V. AND P. J. KEHOE (1999): "Optimal fiscal and monetary policy," in *Handbook of Macroeconomics*, ed. by J. Taylor and M. Woodford, Amsterdam: North-Holland, vol. 1, chap. 26, 1671–1745.

CHEN, Y., Y. CHIEN, AND C. C. YANG (2017): "Aiyagari Meets Ramsey: Optimal Capital Taxation with Incomplete Markets," .

——— (2018): "Implementing the Modified Golden Rule? Optimal Ramsey Capital Taxation with Incomplete Markets Revisited," .

CONESA, J. C., S. KITAO, AND D. KRUEGER (2009): "Taxing Capital? Not a Bad Idea After All!" *American Economic Review*, 99, 25–48.

DÁVILA, J., J. H. HONG, P. KRUSELL, AND J.-V. RÍOS-RULL (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80, 2431–2467.

Debreu, G. (1970): "Economies with a Finite Set of Equilibria," *Econometrica*, 38, 387–392.

Diamond, P. A. and J. A. Mirrlees (1971): "Optimal Taxation and Public Production I: Production Efficiency," *The American Economic Review*, 61, 8–27.

Diaz-Gimenez, J., A. Glover, and J.-V. Rios-Rull (2011): "Facts on the Distributions of Earnings, Income, and Wealth in the United States: 2007 Update," *Federal Reserve Bank of Minneapolis Quarterly Review*, 34, 2–31.

Dierker, E. and H. Dierker (1972): "The Local Uniqueness of Equilibria," *Econometrica*, 40, 867–881.

Domeij, D. and J. Heathcote (2004): "On The Distributional Effects Of Reducing Capital Taxes," *International Economic Review*, 45, 523–554.

Dyrda, S. and M. Pedroni (2020): "Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Shocks," *SSRN Electronic Journal*, 1–54.

Gottardi, P., A. Kajii, and T. Nakajima (2015): "Optimal Taxation and Debt with Uninsurable Risks to Human Capital Accumulation," *American Economic Review*, 105, 3443–3470.

Heathcote, J. (2005): "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *The Review of Economic Studies*, 72, 161–188.

Heathcote, J., K. Storesletten, and G. L. Violante (2017): "Optimal Tax Progressivity: An Analytical Framework*," *The Quarterly Journal of Economics*, 132, 1693–1754.

Holter, H. A., D. Krueger, and S. Stepanchuk (2019): "How Do Tax Progressivity and Household Heterogeneity Affect Laffer Curves?" *Quantitative Economics*, 10, 1317–1356.

Jaimovich, N. and S. Rebelo (2009): "Can News about the Future Drive the Business Cycle?" *American Economic Review*, 99, 1097–1118.

Judd, K. L. (1985): "Redistributive taxation in a simple perfect foresight model," *Journal of Public Economics*, 28, 59–83.

Kindermann, F. and D. Krueger (2021): "High Marginal Tax Rates on the Top 1Income Risk," *American Economic Journal: Macroeconomics, forthcoming.*

Le Grand, F. and X. Ragot (2021): "Managing Inequality over the Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks," Tech. rep.

Ljungqvist, L. and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, MIT Press.

Lucas, R. E. (1990): "Supply-Side Economics: An Analytical Review," *Oxford Economic Papers*, 42, 293–316.

Lucas, R. E. and N. L. Stokey (1983): "Optimal fiscal and monetary policy in an economy without capital," *Journal of Monetary Economics*, 12, 55–93.

Marcet, A. and R. Marimon (2019): "Recursive Contracts," *Econometrica*, 87, 1589–1631.

Park, Y. (2014): "Optimal Taxation in a Limited Commitment Economy," *The Review of Economic Studies*, 81, 884–918.

Prescott, E. C. (2004): "Why Do Americans Work So Much More Than Europeans ?" *Federal Reserve Bank of Minneapolis Quarterly Review*, 28, 2–13.

Röhrs, S. and C. Winter (2014): "Reducing Government Debt in the Presence of Inequality," *Journal of Economic Dynamics and Control*, 82, 1–20.

Shannon, C. (2006): "A Prevalent Transversality Theorem for Lipschitz Functions," *Proceedings of the American Mathematical Society*, 134.

Straub, L. and I. I. Werning (2020): "Positive Long Run Capital Taxation: Chamley-Judd Revisited," *American Economic Review*, 110, 86–119.

Tauchen, G. (1986): "Finite State Markov-chain Approximations to Univariate and Vector Autoregressions," *Economic Letters*, 20, 177–181.

Trabandt, M. and H. Uhlig (2011): "The Laffer Curve Revisited," *Journal of Monetary Economics*, 58, 305–327.

Wu, C. (2021): "More Unequal Income but Less Progressive Taxation: Economics or Politics?" *Journal of Monetary Economics*, 117, 949–968.

Zhu, S. (2020): "Existence of Stationary Equilibrium in an Incomplete-Market Model With Endogenous Labor Supply," *International Economic Review*, 61, 1115–1138.

ONLINE APPENDICES

I Derivations

In this section, we provide the derivations of key equations for the household problem, the Ramsey planner's problem and the steady state.

I.1 Households' Problem

A household's labor supply can be expressed as a function of effective wage and consumption, using the first-order condition (FOC) of n_t:

$$\begin{aligned} u_c(c_t, n_t) e_t \bar{w}_t + u_n(c_t, n_t) = 0 \Rightarrow \\ \frac{-u_n(c_t, n_t)}{u_c(c_t, n_t)} = e_t \bar{w}_t \Rightarrow \\ \chi n_t^{\frac{1}{\phi}} = e_t \bar{w}_t \Rightarrow \\ n_t = \left(\chi^{-1} e_t \bar{w}_t\right)^{\phi}, \end{aligned}$$

and labor income can be also expressed as a function of wage and consumption, as follows:

$$y_t = \chi^{-\phi} e_t^{1+\phi} \bar{w}_t^{1+\phi}.$$

Moreover,

$$e_t w_t u_{ct} + u_{nt} = 0$$

will be a useful expression to simplify expressions later. Given the expressions of n_t and y_t, using the FOC w.r.t. a_{t+1} and Kuhn-Tucker condition for the borrowing constraint, a household's policy functions solve the following system of necessary conditions:

$$\begin{aligned} u_c(c_t, n_t) &\geq \beta(1+\bar{r}_{t+1}) \mathbb{E}[u_c(c_{t+1}, n_{t+1})], \\ 0 &= (a_{t+1} + \underline{a})(u_c(c_t, n_t) - \beta(1+\bar{r}_{t+1}) \mathbb{E} u_c(c_{t+1}, n_{t+1})), \\ c_t + a_{t+1} &\leq a_t(1+\bar{r}_t) + y_t + T_t \\ a_{t+1} + \underline{a} &\geq 0. \end{aligned}$$

I.2 Planner's Problem

Given the planner's problem described in the main text, here we derive the Lagrangian equation (12). First, denote the history of a household's labor productivity from period 0 to t as $h^t = \{h^{t-1}, e_t\}$ where $h^0 = \{e_0\}$. Let θ_{t+1}, η_{t+1} and γ_t represent the present value Lagrangian multipliers for equation (7), (8) and (2) respectively. Then the Lagrangian can be expressed as

$$
\begin{aligned}
L =& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Bigg(u\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) \\
&+\theta_{t+1}\left(h^t\right)\Bigg(u_c\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) - \beta\left(1+\bar{r}_{t+1}\right) \sum_{h^{t+1}} \Pi\left(h^{t+1}|h^t\right) u_c\left(c_{t+1}\left(h^{t+1}\right), n_{t+1}\left(h^{t+1}\right)\right)\Bigg) \\
&-\eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right)+\underline{a}\right)\Bigg(u_c\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) \\
&-\beta\left(1+\bar{r}_{t+1}\right) \sum_{h^{t+1}} \Pi\left(h^{t+1}|h^t\right) u_c\left(c_{t+1}\left(h^{t+1}\right), n_{t+1}\left(h^{t+1}\right)\right)\Bigg)\Bigg) \\
&+\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) - \delta K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right) \\
=& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Bigg(u\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) + u_c\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) \\
&\Bigg(\theta_{t+1}\left(h^t\right) - \theta_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right)+\underline{a}\right) + \eta_t\left(h^{t-1}\right)\left(a_t\left(h^{t-1}\right)+\underline{a}\right)\left(1+\bar{r}_t\right)\Bigg)\Bigg) \\
&+\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) + (1-\delta) K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right).
\end{aligned}
$$

Define $\lambda_{t+1} \equiv \eta_{t+1}\left(a_{t+1}+\underline{a}\right) - \theta_{t+1}$, and the Lagrangian can be further simplified as

$$
\begin{aligned}
L =& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \left(u\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) + u_c\left(c_t\left(h^t\right), n_t\left(h^t\right)\right)\left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right)\right)\right) \\
&+\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) - \delta K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right),
\end{aligned}
$$

subject to equation (3), (4), (9), (10) and (11), starting from initial conditions $a_0\left(h^{-1}\right) = a_0, B_0$ and $\lambda_0\left(h^{-1}\right) = 0$.

The first order conditions can be obtained from the Lagrangian, by taking derivatives w.r.t. to the unknowns $\lambda_{t+1}, a_{t+1}, B_{t+1}, T_t, \bar{r}_t, \bar{w}_t$. Then we obtain the set of FOCs in the main text, i.e., equation (13) to (17). The FOCs, together with the constraints, i.e., equation

(3), (4), (11) and (10), characterize the necessary conditions for the interior solution of the planner's problem.

In these FOCs, partial derivatives including $\frac{\partial N_t}{\partial w_t}$ and so on can be expressed as

$$\frac{\partial N_t}{\partial \bar{w}_t} = \int \frac{\partial n_t}{\partial \bar{w}_t} d\mu_t.$$

Moreover, expressions for $\frac{\partial c_t}{\partial \bar{w}_t}$, and similar partial derivatives are easy to obtain given equation (5), $n_t = (\chi^{-1} e_t \bar{w}_t)^{\phi}$, and (6), $y_t = e_t n_t \bar{w}_t = \chi^{-\phi}(e_t \bar{w}_t)^{1+\phi}$, which describe how n_t and y_t depend on $\bar{w}_t$. Using also the household budget constraint, equation (3), we obtain the partial derivatives:

$$\begin{aligned} \frac{\partial n_t}{\partial \bar{w}_t} &= \phi \frac{n_t}{\bar{w}_t}, \\ \frac{\partial c_t}{\partial \bar{w}_t} &= e_t n_t. \end{aligned}$$

I.3 Steady State

Given the assumption that variables are stable at the steady state, we can obtain the FOCs at the steady state by simply replacing variables in the FOCs of the transition dynamics at their steady state values. For example, $\bar{r}_t, \bar{r}_{t+1}$ can be replaced by the steady state value $\bar{r}$. Same for $\bar{w}_t, B_t$, and all the aggregate variables. Notice that households' choice variables a_{t+1}, λ_{t+1} are different, because they are not constant variables but depend on the state of the household. Following Straub and Werning (2020), we focus on the recursive formulation of the problem, that is to say, current period variables a and λ are the state variables which summarize the history and decide next period choice variables a' and λ', together with current period productivity shock e. We can then replace a_t and λ_t with a and λ, and replace a_{t+1} and λ_{t+1} with a' and λ'. Now the steady state solution is characterized by a set of FOCs, as equation (18) to (22), together with following constraints:

$$c + a' = a\left(1+\bar{r}\right) + y\left(e, \bar{w}\right) + T, \tag{A1}$$

$$G + \left(1+\bar{r}\right) B + \bar{r} K + \bar{w} N + T \leq F\left(K, N\right) + \delta K + B, \tag{A2}$$

$$K = A - B, \tag{A3}$$

$$A = \int a d\mu, \tag{A4}$$

$$N = \int e n d\mu. \tag{A5}$$

II Proofs of Section 3

Proof of Theorem 1

Equation $\gamma = \beta\left(1 + F_K\left(K', N'\right) - \delta\right)\gamma'$ implies in the steady state since $\gamma = \gamma'$ that $1 = \beta\left(1 + F_K\left(K, N\right) - \delta\right)$, which is the modified golden rule.

Proof of Theorem 2

The idea of the proof is as follows. We first show that the Ramsey problem is generically regular (building on Debreu (1970) and Dierker and Dierker (1972)) which implies that a stationary solution to the the Ramsey problem is locally unique. We then show that the steady state depends continuously on initial conditions such as the initial debt level. Together with the local uniqueness this implies that the steady state does not depend on initial conditions. The proof for now assumes that labor supply is exogenous, $n = N = 1$, and we explain later that the arguments generalize in a straightforward way to the case with endogenous labor supply.

We first show local uniqueness and divide this proof into several steps. As a first step we show that the steady state, which is characterized as a solution to

$$\begin{aligned}
u_c\left(c,n\right) &\geq \beta\left(1+\bar{r}'\right)\mathbb{E}\left[u_c\left(c',n'\right)|e\right] \\
&\quad \text{with equality if } a' > -\underline{a}, && \text{(A6)}\\
u_{cc}\left(c,n\right)\left(\lambda\left(1+\bar{r}\right)-\lambda'\right) &= \beta\left(1+r'\right)\mathbb{E}_t\left[u_{cc}\left(c',n'\right)\left(\lambda'\left(1+\bar{r}'\right)-\lambda''\right)\right] \\
&+ \beta\gamma\left(F_K\left(K',N'\right)-\delta-\bar{r}'\right), \\
&\quad \text{if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0. && \text{(A7)}\\
1 &= \beta\left(1+F_K\left(K,N\right)-\delta\right), && \text{(A8)}\\
\gamma A &= \mathbb{E}\left[u_c\left(c,n\right)\lambda\right. \\
&+ \left. a u_c\left(c,n\right) + a u_{cc}\left(c,n\right)\left(\lambda\left(1+\bar{r}\right)-\lambda'\right)\right], && \text{(A9)}\\
\gamma N &= \gamma\left(w-\bar{w}\right)\frac{\partial N}{\partial \bar{w}} + \mathbb{E}\Bigg[enu_c\left(c,n\right) + \\
&+ \left(enu_{cc}\left(c,n\right) + \frac{\phi n}{\bar{w}}u_{cn}\left(c,n\right)\right)\left(\lambda\left(1+\bar{r}\right)-\lambda'\right)\Bigg], && \text{(A10)}\\
c + a' &= a\left(1+\bar{r}\right) + y\left(e,\bar{w}\right) + T, && \text{(A11)}\\
G + \left(1+\bar{r}\right)B + \bar{r}K + \bar{w}N &\leq F\left(K,N\right) - \delta K + B, && \text{(A12)}\\
K &= A - B, && \text{(A13)}\\
A &= \int a d\mu, && \text{(A14)}
\end{aligned}$$

$$N = \int end\mu. \tag{A15}$$

can be characterized as the solution to two equations $z^{AM}(\bar{r},\bar{w}) = 0$ and $z^{LM}(\bar{r},\bar{w}) = 0$ in the unknowns $\bar{r}$ and $\bar{w}$, the "excess demand" functions in the Asset Market and the Labor Market. Regularity of the steady state then means that these two functions are locally invertible, what we establish in Step 2 below.

Step 1: Characterization of steady state ("excess demand")

To express the steady state as a solution to two equations we first show the existence of steady-state Lagrange Multipliers q.

i) Proof of Existence of steady-state Lagrange Multipliers q

Here we prove the existence and uniqueness of a linear-affine function $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e)$, which solves the steady state equation (A7), which after using (A6), division by γ and defining $q = \lambda/\gamma$ equals
If $a' = -\underline{a}$, $\quad q' = 0$.
If $a' > -\underline{a}$:

$$\text{If } a' > -\underline{a}: \quad u_{cc}(c,n)\left[q(1+\bar{r}) - q'\right] \\ = \beta(1+\bar{r})\mathbb{E}\left[u_{cc}(c',n')\left[q'(1+\bar{r}) - q''\right]|e\right] + 1 - \beta(1+\bar{r}), \tag{A16}$$

where we used that $\frac{\partial c'}{\partial a'} = -(1+\bar{r})\frac{\partial c'}{\partial a''}$.

We establish our results for given interest rate $\bar{r}$ and wage $\bar{w}$, individual saving decisions $a'(a,e)$ and individual consumption decisions $c(a,e)$. Introduce the notation $v := -u_{cc}(c(a,e),n(a,e)) > 0$ and ditto notation for $v' := -u_{cc}(c(a',e'),n(a',e')) > 0$. Rewrite the affine $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e)$ as

$$q'(q,a,e) = \Big[(1+\bar{r})q + H(a,e)/v(a,e)\Big] \cdot K(a,e), \tag{A17}$$

where H, K are nonnegative, $K(a,e) = 0$ for those (a,e) such that $a'(a,e) = -\underline{a}$, so that

$$\alpha_0(a,e) = (1+\bar{r}) \cdot K(a,e) \quad \text{and} \tag{A18}$$

$$\alpha_1(a,e) = K(a,e)H(a,e)/v(a,e). \tag{A19}$$

Similarly

$$q''(q',a',e') = \Big[(1+\bar{r})q' + H'/v'\Big] \cdot K' \tag{A20}$$

for H', K' all nonnegative, $K'(a', e') = 0$ for those (a', e') such that $a''(a', e') = -\underline{a}$.
Insert this into (A16)

$$(1+\bar r)(K-1)vq + KH - 1 + (1+\bar r)\beta = (1+\bar r)\beta \mathbb{E}\left[(1+\bar r)(K'-1)v'q' + H'K'|e\right] \quad \text{(A21)}$$

$$= (1+\bar r)\beta \mathbb{E}\left[(1+\bar r)^2(K'-1)v'Kq + (1+\bar r)(K'-1)HKv'/v + H'K'|e\right] \quad \text{(A22)}$$

Gather the "q" terms:

$$(1+\bar r)(K-1)v = (1+\bar r)^3 \beta \mathbb{E}\left[(K'-1)v'|e\right] K \quad \text{(A23)}$$

and solve out for K, taking into account where it must be zero:

$$K = \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1 + (1+\bar r)^2 \beta \mathbb{E}\left[(1-K')v'/v \middle| e\right]}. \quad \text{(A24)}$$

We define an iteration of functions which converge to the solution. As initialization we set $K_0(a, e) \equiv 0$ and define inductively

$$K_{n+1}(a,e) = \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1 + (1+\bar r)^2 \beta \mathbb{E}\left[\left\{1 - K_n(a'(a,e),e')\right\} v'(a',e')/v(a,e) \middle| e\right]} \quad \text{(A25)}$$

By induction, it follows that $1 \geq K_{m+1} \geq K_m \geq \ldots K_0 = 0$. This is obviously true for $n = 0$. For $m + 1$ it follows from $K_{m+1} \geq K_m$ that

$$\begin{aligned} K_{m+2}(a,e) &= \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1 + (1+\bar r)^2 \beta \mathbb{E}\left[\left\{1 - K_{m+1}(a'(a,e),e')\right\} v'(a',e')/v(a,e) \middle| e\right]} && \text{(A26)} \\ &\geq \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1 + (1+\bar r)^2 \beta \mathbb{E}\left[\left\{1 - K_{m}(a'(a,e),e')\right\} v'(a',e')/v(a,e) \middle| e\right]} && \text{(A27)} \\ &= K_{m+1}(a,e). && \text{(A28)} \end{aligned}$$

We therefore obtain a well-defined measurable function K defined by the pointwise $K(a, e) := \sup_m K_m(a, e)$ ($\in [0, 1]$ and 0 when $a' = -\underline{a}$).
That was the "q" terms. For the constant term:

$$KH - 1 + (1+\bar r)\beta = (1+\bar r)\beta \mathbb{E}\left[(1+\bar r)(K'-1)HKv'/v + H'K' \middle| e\right], \quad \text{i.e.} \quad \text{(A29)}$$

$$H \cdot \underbrace{\left\{1 + (1+\bar r)^2 \beta \mathbb{E}\left[(1-K')v'/v \middle| e\right]\right\} \cdot K}_{=1 \text{ on } \{(a,e);a'>-\underline{a}\} \text{ by (A24)}} = 1 - (1+\bar r)\beta + (1+\bar r)\beta \mathbb{E}\left[H'K' \middle| e\right] \quad \text{(A30)}$$

As we can safely put $H = 0$ on $\{(a,e); a' = -\underline{a}\}$, we can iterate from $H_0(a,e) \equiv 0$ the relation

$$H_{m+1}(a,e) = \left\{1 - (1+\bar{r})\beta + (1+\bar{r})\beta\mathbb{E}\Big[H_m(a'(a,e),e')K'(a'(a,e),e')\Big|e\Big]\right\} \cdot 1_{\{(a,e);a'>-\underline{a}\}} \tag{A31}$$

Now the condition $(1+\bar{r})\beta \sup_e \mathbb{E}K'(a'(a,e),e') < 1$ (recall that $K' \in [0,1]$ and $(1+\bar{r})\beta < 1$ – and except in the trivial case, zero when the credit constraint is binding) is sufficient for a contraction and unique solution H; if we start at 0, then we have bounded monotonicity $1 \geq H_{m+1} \geq H_m \geq 0$, and thus H defined by $H(a,e) := \sup_m H_m(a,e) \in [0,1]$ does the job. We have therefore established the existence a solution $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e) = \Big[(1+\bar{r})q + H(a,e)/v(a,e)\Big] \cdot K(a,e)$.

Note that the complete markets economy does not feature a unique steady state value of q: in complete markets models, $\bar{r} = r = F_K(K,N) - \delta$ and using this in (A7) yields

$$u_{cc}(c,n)\bar{r}\lambda = u_{cc}(c,n)\bar{r}\lambda,$$

so that λ and q are not uniquely determined in a complete markets steady state.

ii) "Excess Demand" Functions

For a given $\bar{w}$, equations (A6) and (A11) describe households consumption and savings behavior as a function of $\bar{r}$, resulting in an aggregate asset supply function $S(\bar{r},\bar{w})$.[27]

Asset demand D, the sum of capital and bonds, follows from the government budget constraint (A12) using (A13) and (A15)

$$D(\bar{r},\bar{w}) := A = K + B = \frac{F\left(K,N\right) - \delta K - \bar{w}N - G}{\bar{r}},$$

which, since we already established that capital K satisfies the modified golden rule (equation (A8)), is actually just describing how many government bonds are demanded. We therefore define

$$z^{AM}(\bar{r},\bar{w}) = D(\bar{r},\bar{w}) - S(\bar{r},\bar{w}).$$

A solution $\bar{r}$ (for given $\bar{w}$) to $z^{AM}(\bar{r},\bar{w}) = 0$ fully characterizes a stationary Aiyagari economy (and solves equation (A14)).

To derive the second equation $z^{LM}(\bar{r},\bar{w})$ we use the remaining equations (A9) and (A10).

[27] While it is conceivable that aggregate asset supply is not unique given $\bar{r}$ and $\bar{w}$, this is not a concern here since we impose the standard assumption that the planner picks the unique welfare maximizing allocation.

After division by γ equation (A9) reads

$$A = \mathbb{E}\left[u_c\left(c,n\right)q + a\frac{u_c\left(c,n\right)}{\gamma} + au_{cc}\left(c,n\right)\left(q\left(1+\bar{r}\right)-q'\right)\right],$$

where q' depends on $\bar{r}$, $\bar{w}$ and other parameters. Solving this equation for γ yields a function $\tilde{\gamma}(\cdot)$:

$$\tilde{\gamma}(\cdot) = \frac{\mathbb{E}\left[\frac{a}{A}u_c\left(c,n\right)\right]}{1-\mathbb{E}\left[\frac{u_c(c,n)q}{A} + au_{cc}\left(c,n\right)\left(q\left(1+\bar{r}\right)-q'\right)\right]}$$

Plugging this function into (A10) (and noting $n=1$) yields

$$\int end\mu = \mathbb{E}\left[e[\frac{u_c\left(c,n\right)}{\tilde{\gamma}} + u_{cc}\left(c,n\right)\left(q\left(1+\bar{r}\right)-q'\right)]\right].$$

We therefore define

$$\tilde{z}^{LM}(\bar{r},\bar{w}) := \mathbb{E}\left[e[\frac{u_c\left(c,n\right)}{\tilde{\gamma}} + u_{cc}\left(c,n\right)\left(q\left(1+\bar{r}\right)-q'\right)]\right].$$

and the excess labor demand

$$z^{LM}(\bar{r},\bar{w}) := \tilde{z}^{LM}(\bar{r},\bar{w}) - \int end\mu.$$

The optimal steady state then satisfies

$$z^{LM}(\bar{r},\bar{w}) = 0.$$

Step 2: Local Invertibility

We first show that the interest rate $\bar{r}$ can generically (in the sense of Debreu (1970)) be expressed locally as a function of $\bar{w}$ (and other parameters). This first step is what is not feasible in complete markets models and is thus the reason why we obtain local uniqueness here but not in complete markets models. After that we show that $\bar{w}$ is also generically locally invertible. This second step holds both in complete and incomplete markets models.

i) Interest Rate

Acemoglu and Jensen (2015) show that a tightening of the borrowing limit leads to an increase in the supply of assets for given $\bar{r}$ and $\bar{w}$ but will not change the modified golden rule level of capital.[28] The transversality theorem (see e.g. Dierker and Dierker (1972), Shannon (2006)) implies then that

$$\frac{\partial z^{AM}(\bar{r}, \bar{w})}{\partial \bar{r}} \neq 0,$$

which implies that $\bar{r}$ is locally invertible and is thus a function of $\bar{w}$ and can be written as $\bar{r}(\bar{w})$. The transversality theorem allows us to not directly compute the derivative with respect to $\bar{r}$ but instead to consider the derivative for some parameter, the exogenous borrowing constraint, and then infer the local invertibility for $\bar{r}$. Clearly, this line of arguments does not apply in complete markets models, which do not have a steady-state asset demand function but only a correspondence, and where the arguments of Acemoglu and Jensen (2015) are not applicable since tightening the borrowing constraint is not a well-defined experiment in the standard complete markets model.

Figure A-1 confirms these results for our incomplete markets economy using the benchmark calibration. The left panel shows the 3-dimensional plot for z^{AM} and the right panel shows the 2-dimensional plot of z^{AM} for $\bar{w}$ fixed at its the optimal level. Both panels establish a unique solution $\bar{r}(\bar{w})$ for all values of $\bar{w}$ consistent with our theoretical proof. The right panel also shows the complete markets counterpart, where $\bar{r} = r = \frac{1}{\beta} - 1$, independently of the value of $\bar{w}$. In a steady state asset supply is then not determined — a standard Ricardian equivalence argument — such that we cannot define a function z^{AM} for this model class. Moreover, as explained above, there is no unique value of q if markets are complete.

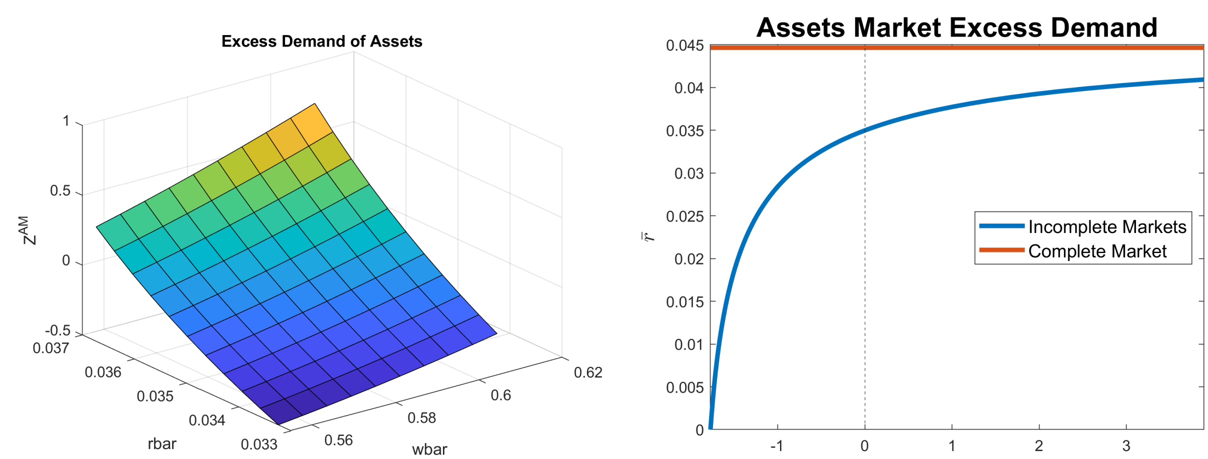


Figure A-1: Asset Market: Excess Demand. For all $(\bar{r}, \bar{w})$ (left) and optimal $\bar{w}$ (right)

[28] Acemoglu and Jensen (2015) call such an experiment a positive shock. Their objective is more demanding then just showing an increase in the supply function. They characterize the response of the equilibrium output per capita which has to take into account the endogeneity of prices.

ii) Wage The after tax wage $\bar{w}$ is determined as the solution to

$$z^{LM}(\bar{r}(\bar{w}), \bar{w}, \mu) = \tilde{z}^{LM}(\bar{r}(\bar{w}), \bar{w}, \mu) - \sum_{e \in E} e\mu(e) = 0,$$

where we have plugged in $\bar{r}(\bar{w})$ and use, consistent with the numerical implementation, a more convenient discrete space $E = \{e_1 < e_2, \ldots < e_N\}$.
We now follow Debreu (1970) and apply Sard's theorem to the function $F : \mathbb{R}^{N+1} \to \mathbb{R}^N$,

$$F(\bar{w}, \{\mu(e_i)\}_{i=1}^N) = (\mu(e_1), \mu(e_2), \ldots, \mu(e_{N-1}), \frac{\tilde{z}^{LM}(\bar{r}(\bar{w}), \bar{w}, \mu) - \sum_{i=1}^{N-1} e_i \mu(e_i)}{e_N}).$$

The optimal solution is characterized as $F(\bar{w}, \{\mu(e_i)\}_{i=1}^N) = (\mu(e_1), \ldots, \mu(e_N))$ and Sard's theorem, which implies that the set of critical values has measure zero, delivers the local invertibility result.[29]

We could also again apply the transversality theorem (which is based on Sard's theorem) and perturb the highly-dimensional whole distribution to show that the derivative with respect to the one-dimensional variable $\bar{w}$ is generically non-zero. Again, the transversality theorem implies that we do not have to calculate the derivative w.r.t. $\bar{w}$ but that instead it is sufficient to show that not for all all $\mu(e_n)$ the derivative of the last element of F is equal to zero. As this case — a function or integral which evaluates to zero at every point would be zero, such that all expectations are zero implying that the resource constraint is not binding, a contradiction — we again obtain local invertibility.

Both the distribution μ and the after-tax wage $\bar{w}$ live on a compact space K, $(\bar{w}, \{\mu(e_i)\}_{i=1}^N) \in K$. This is obvious for $\mu(e_i) \in [0,1]$ and for $\bar{w}$ follows from Aiyagari (1994) who ensures that no-one is willing to work in the market at a wage of 0 and the marginal productivity of labor is bounded since capital and hours (time) are.

The inverse image $F^{-1}(\mu(e_1), \ldots, \mu(e_N))$ of a regular value $((\mu(e_1), \ldots, \mu(e_N))$ is compact since F is continuous and K is compact. Consider now $e := (\bar{w}, \{\mu(e_i)\}_{i=1}^N) \in F^{-1}(\mu(e_1), \ldots, \mu(e_N))$, for a regular $((\mu(e_1), \ldots, \mu(e_N))$ implying that the Jacobian does not vanish. The inverse function theorem implies that for each such e there is an open neighborhood U_e of e such that $F^{-1}(\mu(e_1), \ldots, \mu(e_N)) \cap U_e = \{e\}$. Since $F^{-1}(\mu(e_1), \ldots, \mu(e_N))$ is compact it can be covered by finite number of open sets U_e and therefore is finite.

This implies that the set of $\mu(e_1), \ldots, \mu(e_N)$ for which an infinite number of steady states exists consists of critical values only and has therefore measure zero (and so does its closure).

[29] For a continuously differentiable function $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$, a point $e \in U$ is a critical point if the Jacobian matrix of F at e has rank smaller than n. A point $\mu \in \mathbb{R}^n$ is a critical value if there is a critical point $e \in U$ such that $F(e) = \mu$. A point $\mu \in \mathbb{R}^n$ is a regular value if it is not a critical value.

Vice versa, the number of steady state solution is generically, that is on a measure one, finite.

Again a comparison with the complete markets case is instructive. The left panel of Figure A-2 shows the function $z^{LM}(\bar{r}(\bar{w}), \bar{w}, \mu)$ for our benchmark incomplete markets model. The optimal $\bar{w}$ is determined as the intersection of the z^{LM} curve with the vertical 0 line. The right panel of Figure A-2 shows the same experiment for the complete markets case. As explained above, the value of q is not pinned down in the complete markets economy, implying different functions z^{LM} for different values of q. For a fixed q the function z^{LM} is monotone and we can solve for the unique value of $\bar{w}$ such that $z^{LM} = 0$, establishing that the asset market and not the labor market is the relevant difference between complete and incomplete markets models.

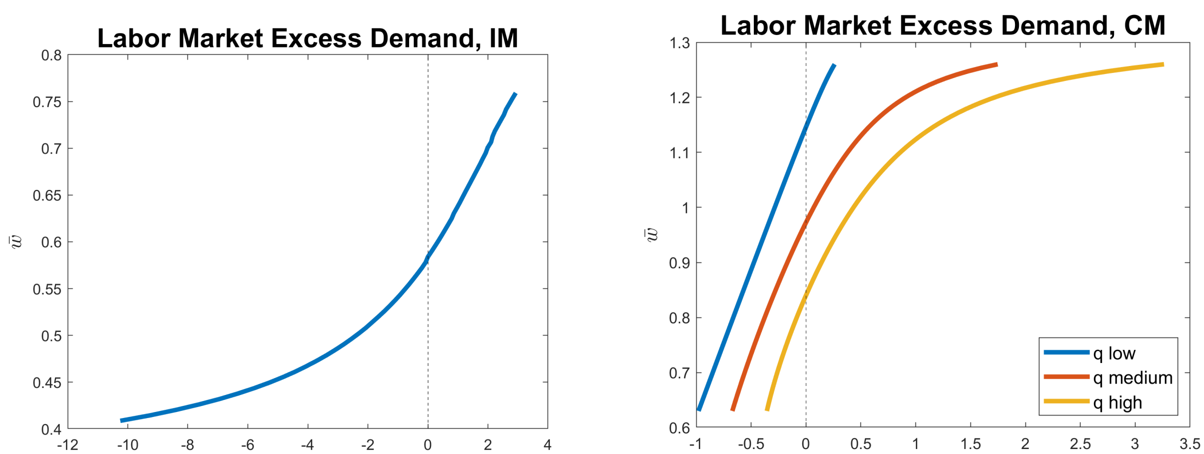


Figure A-2: Labor market: Excess Demand. Incomplete markets (left) and complete markets (right)

Step 3: Continuity w.r.t. Initial Conditions

Berge's maximum theorem (see Theorem 17.31 in Aliprantis and Border (2006) for the infinite-dimensional version) implies that the optimal policy path, and thus in particular the steady state, depends continuously on the initial level of government debt. Here we use the same topology, the product topology, as in Appendix A of Aiyagari (1994). The equations describing the constraints of the Ramsey planner problem are continuous and the constrained set is by Tychonoff's theorem compact. The maximand of the Ramsey problem is continuous as well. These properties imply that a solution to the optimal tax problem exists, as shown in Aiyagari (1994), and allow us to apply Berge's maximum theorem. Finally, note that a function is continuous in the product topology iff all its projections are continuous, implying that the usual real analysis ϵ/δ characterization of continuity holds for all t and in particular for arbitrarily large t.

Therefore, the function $\zeta : \mathbb{R} \to \mathbb{R}^n$ mapping the initial government debt level into the steady state policies (tax rates and debt level) is continuous.

Step 4: Independence of Initial Conditions

We have shown in Step 2 that the set of solutions to the first order conditions is finite. These first-order conditions do not depend on the initial level of government debt. That is the finite set of solutions does not depend on the initial level of debt. The first-order conditions are necessary conditions for an optimum, implying that every optimal policy has to be one of the finite solutions to the first-order conditions. What remains to be shown is that each initial debt levels always yields the same solution to the first-order conditions, that is that there is no selection of these solutions based on initial conditions. Using our results above, this is straightforward.

A continuous function mapping into a discrete set is constant, implying that ζ maps every initial debt level to the same steady-state policy.

Remarks:

Elastic Labor supply

The same arguments hold when labor supply is elastic. We then define

$$z^{LM}(\bar{r}, \bar{w}) := \mathbb{E}\left[en[\frac{u_c(c,n)}{\tilde{\gamma}} + u_{cc}(c,n)(q(1+\bar{r}) - q')]\right] + (F_N(K,N) - \bar{w})\frac{\partial N}{\partial \bar{w}}$$

and the optimal steady state then satisfies

$$z^{LM}(\bar{r}, \bar{w}) = \sum_e \pi_e en(e, \bar{w}).$$

III Computational Algorithms

III.1 Steady State

To numerically compute the steady state, we first need to introduce the steady state distribution of state variables (a, λ, e), represented by a density function $p(a, \lambda, e)$. Moreover, we denote the density function of (a, e) as $m(a, e)$. We discretize e, using the method of Tauchen (1986) and 7 equally spaced values for e in $E = [-3\sigma_e, 3\sigma_e]$. Now the steady state equations involving expectation and integration can be explicitly expressed using p and m. Equation (21), (22), (A4) and (A5) are now:

$$\gamma A = \sum_e \int \int \Big[u_c(c,n)\lambda$$

$$+ a\left(u_c\left(c,n\right)+u_{cc}\left(c,n\right)\left[\lambda\left(1+\bar{r}\right)-\lambda'\right]\right)\Big]p\left(a,\lambda,e\right)dad\lambda, \tag{A32}$$

$$\gamma N = \gamma\left(F_N\left(K,N\right)-\bar{w}\right)\frac{\partial N}{\partial \bar{w}} + \sum_e \int\int \Big[enu_c\left(c,n\right)$$

$$+ \left(enu_{cc}\left(c,n\right)+\frac{\phi n}{\bar{w}}u_{cn}(c,n)\right)\left(\lambda\left(1+\bar{r}\right)-\lambda'\right)\Big]p\left(a,\lambda,e\right)dad\lambda, \tag{A33}$$

$$N = \sum_e \int enm\left(a,e\right)da, \tag{A34}$$

$$A = \sum_e \int am\left(a,e\right)da. \tag{A35}$$

Moreover, the density functions satisfy

$$p\left(a',\lambda',e'\right) = \sum_e \pi_{ee'} \int I\left[g_{a'}\left(a,e\right)=a', g_{\lambda'}\left(a,\lambda,e\right)=\lambda'\right]p\left(a,\lambda,e\right)dad\lambda, \tag{A36}$$

$$m\left(a',e'\right) = \sum_e \pi_{ee'} \int I\left[g_{a'}\left(a,e\right)=a'\right]m\left(a,e\right)da. \tag{A37}$$

Using the steady state equations, i.e., equation 18 to 20, A1, A2, and A32 to A37, we compute the steady state variables according to the below steps.

1. Guess $\bar{w}$. Solve for $\bar{r}\left(\bar{w}\right)$ following Aiyagari (1995):
   (a) Solve for K from (20).
   (b) Guess $\bar{r}$ and solve the household's problem: solve for $c\left(a,e\right), a'\left(a,e\right)$ from (18) and (A1), keeping in mind that $n=\left(\bar{w}e\right)^{\frac{1}{\phi}}, y=\left(\chi^{-1}\bar{w}^{1+\phi}e^{1+\phi}\right)^{\frac{1}{\phi}}$. To solve for $c\left(a,e\right), a'\left(a,e\right)$ we use an endogenous grid apporach, as described by Carrol (2006).
   (c) Compute N from (A34).
   (d) Solve for $m\left(a,e\right)$ or equivalently $\mu\left(.,e\right)$ from (A37).
   (e) Solve for A from (A35).
   (f) Solve for B from (A3).
   (g) Verify $\bar{r}$ using (A2). If the equation is not satisfied, update $\bar{r}$.
2. Define $q\equiv\frac{\lambda}{\gamma}$, and solve for $q'\left(a,q,e\right)$ by iterating on $q'\left(a,q,e\right)$ using (19) until q' converges. Guess $q'\left(a,q,e\right)=g^0_{q'}\left(a,q,e\right)$, and then use equation (19) to find the new $q'\left(a,q,e\right)=g^1_{q'}\left(a,q,e\right)$ as follows:

$$q' = \frac{u_{cc}\left(c,n\right)q\left(1+\bar{r}\right)+\beta(1+\bar{r}')\mathbb{E}\left[u_{cc}\left(c',n'\right)q''|e\right]-1+\beta\left(1+\bar{r}\right)}{u_{cc}\left(c,n\right)+\beta(1+\bar{r}')\mathbb{E}\left[u_{cc}\left(c',n'\right)\left(1+\bar{r}\right)\right]}$$

where $\mathbb{E}\left[u_{cc}\left(c'\right)q''|e\right]$ can be computed using $g^0_{q'}\left(a,q,e\right)$, and the new q' gives us the new policy function, denoted as $g^1_{q'}\left(a,q,e\right)$. Keep updating until $g^i_{q'}\left(a,q,e\right)$ converges to $g_{q'}\left(a,q,e\right)$. It can be proven that the above functional equation is a contraction mapping.

3. Solve for γ from (21)

4. Check whether (22) is satisfied. If so, stop. Otherwise update $\bar{w}$.

III.2 Transition

Below we outline the algorithm for computing the transition from the model calibrated to the U.S. economy to the optimal long run steady state.:

1. Choose a number of transition periods, J.

2. Compute the optimal long run steady state as outlined in III.1 and obtain $a_{t+1}\left(a_t,e_t\right)$, $c_t\left(a_t,e_t\right)$ at time J.

3. Compute the steady state for the economy calibrated to the U.S. and obtain $m_0\left(a_0,e_0\right)$ A_0, B_0, K_0.

4. Guess $\{\bar{w}_t,\bar{r}_t\}_{t=0}^{J}$.

5. Solve households' problems by backward induction and obtain $a_{t+1}\left(a_t,e_t\right)$, $c_t\left(a_t,e_t\right)$.

6. Compute distribution of asset and productivity $m_t\left(a_t,e_t\right)$, using simulation starting from $m_0\left(a_0,e_0\right)$.

7. Compute A_t and N_t from (9) and (10).

8. Compute K_t and B_{t+1} going backwards using (11) and (2), namely,

$$K_t = A_t - B_t$$
$$B_{t+1} = G_t + \left(1+\bar{r}_t\right)B_t + \bar{r}_t K_t + \bar{w}_t N_t + T_t - F\left(K_t,N_t\right) + \delta K.$$

9. Compute γ_t backward using

$$\gamma_t = \beta\left(1+F_K\left(K_{t+1},N_{t+1}\right)-\delta\right)\gamma_{t+1}$$

10. Solve for $\lambda_{t+1}\left(a_t,\lambda_t,e_t\right)$ from 14.

11. Compute p_t forward by simulations using p_0 and the policy functions: $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$ and $\lambda_{t+1}(a_t, \lambda_t, e_t)$.

12. Check the errors implied by the guessed $\{\bar{w}_t, \bar{r}_t\}_{t=0}^{J}$. This means check the equations 16, and 17. If they are not satisfied, update the guess for $\{\bar{w}_t, \bar{r}_t\}_{t=0}^{J}$. In practice we do this by a minimization routine, which minimizes the sum of the squared errors in the equations.

IV Comments on Chen et al. (2018)

In this Section, we explain why several of the arguments in Chen et al. (2018) are incorrect and others are incomplete.

Overview

Chen et al. (2018) argue that in a Ramsey problem with incomplete markets, similar to Aiyagari (1995), there is no long-run steady state. A key step in the proof of the paper is to show that the original Ramsey problem can be simplified by dropping households' Euler equations, and that this simplified relaxed Ramsey problem is equivalent to the original problem. The intuitive argument in Chen et al. (2018) is that if a household's Euler Equation is not satisfied, then the planner can improve this household's welfare by smoothing her consumption without affecting other households and the aggregate economy. Therefore, the planner automatically makes sure that households' Euler equations are satisfied and it is not necessary to include households' Euler equations as constraints in the planner's problem. In particular, if a household's marginal utility of consumption today is higher than her expected marginal utility, then the planner can increase today's consumption and reduce today's savings by a small amount, such that the reduction in consumption in period $t + 1$ leads to a smoother consumption allocation and gives higher utility to that household without affecting others.

Increasing consumption in period t necessarily reduces investment in this period and thus decreases the capital stock and output in period $t + 1$. For the reshuffling to be welfare improving requires that the output decrease due to a lower capital stock in period $t + 1$ is smaller than the drop in consumption in period $t + 1$, as this allows to satisfy the aggregate resource constraint and at the same time leaving the consumption of all other households unchanged.

This is where the proof of Chen et al. (2018) fails. The social cost of reducing a household's savings is larger than her private cost, if the capital income tax rate is positive. The former cost is proportional to the marginal product of capital which is equal to the before-tax interest rate, and the latter is the after-tax counterpart, which is smaller if the capital tax is positive.

Therefore, the smoother consumption allocation for a specific household constructed in the proof violates the aggregate resource constraint.

As Chen et al. (2018) aim to disprove Aiyagari (1995), which features a positive capital income tax rate, making the assumption that the capital income tax rate is zero is circular. Furthermore, Aiyagari (1995) shows that a steady state is inconsistent with a zero capital income tax. Therefore it is not surprising that Chen et al. (2018) find that a Ramsey steady state does not exist since this is what they assume when imposing a zero capital income tax rate. They basically assume their result.

Dávila et al. (2012)

Before explaining these arguments in detail, we relate our previous discussion to the findings in Dávila et al. (2012), which shows from a different perspective that the key step in the proof of Chen et al. (2018) is incorrect.

The seminal article by Dávila et al. (2012) shows that the optimal saving decisions of households are typically not socially constrained efficient in incomplete markets models with capital. Constrained efficiency in this context means that a planner makes saving decisions on behalf of every household — a very different concept than the Ramsey approach adopted in our paper — and Dávila et al. (2012) find that the planner and households save different amounts. The intuition is based on a simple pecuniary externality. A higher capital stock increases wages which in an incomplete markets model has distributional effects and thus affects welfare.

Chen et al. (2018) claim that the Euler equation is binding with a zero capital income tax rate and that the planner has no incentives to distort households' private savings at that point. This is equivalent to claiming that the pecuniary externality identified in Dávila et al. (2012) is always zero and that thus the private and social saving incentives are aligned, contradicting the key finding in Dávila et al. (2012).

Arguments in detail

We adopt the notation of Chen et al. (2018) as much as possible to simplify matters. We therefore define θ^t as the history of idiosyncratic shocks until time t. A key step in their proof is the claim that for an allocation where consumption of a non-credit constrained household with history θ^s in period s satisfies

$$u'\left(c_s\left(\theta^s\right)\right) > \beta R_{t+1} E u'\left(c_{s+1}\left(\theta^{s+1}\right)\right),$$

where R_{t+1} is the after-tax interest rate, a welfare-improving allocation exists. They try to prove this claim in two steps. (1) They show how to construct a different consumption allocation which increases the utility of this household. The marginal utility in state θ^s is decreased by ϵ_1 and thus consumption $c_s(\theta^s)$ is increased and savings are reduced by Ω in state θ^s. Next period's consumption decreases correspondingly so that this new allocation satisfies this household's budget constraint.

(2) They argue that this new allocation is also feasible, that is the consumption allocations for all other households can remain unchanged in all periods without violating the aggregate resource constraint in any period.

We now show in three steps that the new allocation and the capital sequence constructed in the Proof of Proposition 2 in Chen et al. (2018, Appendix Subsection A.3) are not feasible. First, we compute the consumption changes in period s and $s+1$ implied by the variation. Second, we compute the capital changes implied by the consumption variations in the next two periods: $K_t^{\nu} - K_t$ for periods $s+1$ and $s+2$. Third, we show that this difference $K_t^{v} - K_t$ of capital between the new and the old allocations increases exponentially over time. To simplify the notation and without affecting our conclusion, we hold labor fixed and focus on consumption since it is the variation in consumption which is supposed to improve welfare in the first place and which we show not to be feasible. The same arguments hold if labor is elastically supplied.

First, if $u'(c_s(\theta^s)) > \beta R_{t+1} E u'(c_{s+1}(\theta^{s+1}))$ for a household with history θ^s in period s, then the authors suggest to increase consumption $c_s(\theta^s)$ and reduce savings $\hat{a}_{s+1}(\theta^s)$ in state θ^s for this household. The change in savings is denoted as Ω, which is negative. Since consumption in periods $t \geq s+2$ is unchanged for all households, the household budget constraint implies that consumption $c_{s+1}(\theta^{s+1})$ next period decreases by $R_{s+1}\Omega$, for all states θ^{s+1}, where R_{s+1} is the after-tax interest rate.

Denoting savings in state θ^s by $\hat{a}_{s+1}(\theta^s)$ and the new allocations by a ν superscript, they choose

$$\begin{aligned} \hat{a}_{s+1}^{\nu}(\theta^s) &= \hat{a}_{s+1}(\theta^s) + \Omega \\ c_s^{\nu}(\theta^s) &= c_s(\theta^s) - \Omega. \end{aligned}$$

implying that next period's consumption is

$$c_{s+1}^{\nu}\left(\theta^{s+1}\right) = c_{s+1}\left(\theta^{s+1}\right) + R_{s+1}\Omega.$$

as reducing savings by $|\Omega|$ units leads to a decrease of consumption next period by $R_{s+1}|\Omega|$ units.[30]

[30] A previous version of the paper (Chen et al., 2017) picked consumption $c_{s+1}^{\nu}\left(\theta^{s+1}\right)$ in state θ^{s+1} such that the marginal utility increased by the same amount ϵ_2 in all states θ^{s+1}, implying that the consumption change is different in all these states. Such an allocation does not satisfy the measurability constraints, that is it is not implementable with a non-state contingent asset. Therefore the previous proof failed already at

Second, the decrease of savings implies that aggregate capital decreases:

$$K^{\nu}_{s+1} = K_{s+1} - |\Omega|\, m, \tag{A38}$$

where m denotes the measure of households with history θ^s. Aggregate consumption in period $s+1$ also falls,

$$C^{\nu}_{s+1} = C_{s+1} - R_{s+1}\, |\Omega|\, m. \tag{A39}$$

While the consumption increase in period s and the decrease in period $s+1$ by construction satisfy budget constraints of θ^s households, the induced reduction in the capital stock and output do not satisfy the resource constraint, as we show now.

One unit less of capital translates into $1+r = F_K + (1-\delta)$ units of loss in output. Using (A38), a Taylor expansion yields

$$\begin{aligned} F\left(K^{\nu}_{s+1}\right) + (1-\delta)\, K^{\nu}_{s+1} &= F\left(K_{s+1} - |\Omega|\, m\right) + (1-\delta)\left(K_{s+1} - |\Omega|\, m\right) \\ &= F\left(K_{s+1}\right) + (1-\delta)\, K_{s+1} - \left(F_K\left(K_{s+1}\right) + (1-\delta)\right)|\Omega|\, m + O\left(\Omega^2\right) \\ &= F\left(K_{s+1}\right) + (1-\delta)\, K_{s+1} - \left(1 + r_{s+1}\right)|\Omega|\, m + O\left(\Omega^2\right). \end{aligned} \tag{A40}$$

The consumption reduction in period $s+1$ only partly compensates the output loss, such that the aggregate resource constraint implies that the capital stock in period $s+2$ is still lower than in steady state:

$$\begin{aligned} K^{\nu}_{s+2} - K_{s+2} &= F\left(K^{\nu}_{s+1}\right) + (1-\delta)\, K^{\nu}_{s+1} - \left(F\left(K_{s+1}\right) + (1-\delta)\, K_{s+1}\right) - \left(C^{\nu}_{s+1} - C_{s+1}\right) \\ &= -\left(r_{s+1} - \bar{r}_{s+1}\right)|\Omega|\, m + O\left(\Omega^2\right), \end{aligned}$$

where we used $R_{s+1} = 1 + \bar{r}_{s+1}$. If $r_{s+1} > \bar{r}_{s+1}$, i.e. the after-tax return on capital return is lower than the before-tax capital return, or equivalently, the capital tax rate $\tau^k_s > 0$, then $K^{\nu}_{s+2} < K_{s+2}$. Another way to interpret $r_{s+1} > \bar{r}_{s+1}$ is to say that the private cost of reducing savings for households is lower than its social cost.

Finally, we show that the capital loss $K^{\nu}_{t>s+2} - K_t$ increases over time if $r_t > 0$. Since $C^{\nu}_{t\geq s+2} = C_t$, we have

$$\begin{aligned} K^{\nu}_{s+3} - K_{s+3} &= \left(F_K\left(K_{s+2}\right) + 1 - \delta\right)\left(K^{\nu}_{s+2} - K_{s+2}\right) + O\left(\Omega^2\right) \\ &= -\left(1 + r_{s+2}\right)\left(r_{s+1} - \bar{r}_{s+1}\right)\Omega m + O\left(\Omega^2\right). \end{aligned} \tag{A41}$$

this stage. The new version overcomes this problem at the cost of aggravating others.

Iterating yields

$$K^{\nu}_{t>s+2} - K_t = -\left(\prod_{t=s+2}^{t-1} 1 + r_t\right)(r_{s+1} - \bar{r}_{s+1})\,\Omega m + O\left(\Omega^2\right).$$

If $r_{s+1} > \bar{r}_{s+1}$ and $r_t > 0$ then $K^{\nu}_{t>s+2} - K_t$ diverges, showing that welfare cannot be improved. Since both $r_{s+1} > \bar{r}_{s+1}$ and $r_t > 0$ hold in the steady-state considered in Aiyagari (1995), our derivations establish that the re-shuffling of consumption suggested in Chen et al. (2018) does not yields higher welfare than the Aiyagari steady state. We already explained that $r = \bar{r}$, i.e. a zero capital income tax rate, would be inconsistent with a steady state. In other words Chen et al. (2018) consider an empty set when assuming $r = \bar{r}$,

The difference in capital stocks $K^{\nu}_t - K_t$ even diverges if $r_t = 0$. The reason is that $K^{\nu}_{s+1} - K_{s+1} < 0$ implies that $r^{\nu}_t > 0$ in period $s+1$ and all future periods, such that the same arguments as above apply now. Formally we show this by writing the Taylor expansion around K^{ν}_t instead of K_t:

$$\begin{aligned} F(K_{s+1}) + (1-\delta)K_{s+1} &= F\left(K^{\nu}_{s+1} + |\Omega|\, m\right) + (1-\delta)\left(K^{\nu}_{s+1} + |\Omega|\, m\right) \\ &= F\left(K^{\nu}_{s+1}\right) + (1-\delta)K^{\nu}_{s+1} - \left(F_K\left(K^{\nu}_{s+1}\right) + (1-\delta)\right)|\Omega|\, m - O\left(\Omega^2\right) \\ &= F\left(K^{\nu}_{s+1}\right) + (1-\delta)K^{\nu}_{s+1} - (1 + r^{\nu}_{s+1})\,|\Omega|\, m - O\left(\Omega^2\right), \end{aligned}$$

implying that

$$F\left(K^{\nu}_{s+1}\right) + (1-\delta)K^{\nu}_{s+1} = F(K_{s+1}) + (1-\delta)K_{s+1} + (1 + r^{\nu}_{s+1})\,|\Omega|\, m + O\left(\Omega^2\right),$$

which is similar to the Taylor expansion (A40) around K except that now we use $1 + r^{\nu}_{s+1}$ instead of $1 + r_{s+1}$. Notice that due to the concavity of the production function, $r^{\nu}_t > r_t = 0$ if $K^{\nu}_t < K_t$. Then we obtain

$$\begin{aligned} K^{\nu}_{s+2} - K_{s+2} &= -(r_{s+1} - \bar{r}_{s+1})\,|\Omega|\, m + O\left(\Omega^2\right), \\ K^{\nu}_{s+3} - K_{s+3} &= -(1 + r^{\nu}_{s+2})(r_{s+1} - \bar{r}_{s+1})\,\Omega m + O\left(\Omega^2\right), \\ &\dots \\ K^{\nu}_{t>s+2} - K_t &= -\left(\prod_{t=s+2}^{t-1}(1 + r^{\nu}_t)\right)(r_{s+1} - \bar{r}_{s+1})\,\Omega m + O\left(\Omega^2\right), \end{aligned}$$

showing that $K^{\nu}_{t>s+2} - K_t$ diverges since $1 + r^{\nu}_t > 1$.

Comment on the Proof of Proposition 3

A crucial step in the proof of Chen et al. (2018) (page 44) to rule out a steady state with constant marginal utility of consumption μ (Case a), is: "Given that the term $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t)$ (the shadow price of collecting tax revenue via varying L_t) is positive, there is no possibility for Case (a) to uphold."

On page 23 in the main text they claim that "the marginal utility cost of collecting government revenue by changing the aggregate labor supply is positive" , that is $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t) > 0$ because Ljungqvist and Sargent (2012, p. 629) argued that it is positive in a representative agent model, so by analogy, it should also be positive in the incomplete markets model (considered in Chen et al., 2018).

This part of the proof is incomplete. A result valid in one model does not have to hold in a very different model. What would have been necessary is to show that $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t) > 0$ within their model. Since this step is missing, the proof is incomplete.