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Abstract

Equilibria where altruistic generations are linked via positive bequests are indeterminate and subject to sunspot variables when each individual's utility in non-separable in her own age contingent consumption and sufficiently biased towards old age. This result does not require strong income effects and it applies if individuals select their own savings and bequests by taking the decisions of their offsprings and successors as given. In this case, the equivalence with the Dynastic Equilibria of a Ramsey-type model envisaged in Barro (1974) fails. I show that the structure of equilibria of the olg model with altruism is more similar to the one generated in a canonical olg economy with two goods.

JEL Classification: E10, E21, E32

Keywords: Bequets, parental altruism, Indeterminacy

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Abstract. Equilibria where altruistic generations are linked via positive bequests are indeterminate and subject to sunspot variables when each individual's utility in non-separable in her own age contingent consumption and sufficiently biased towards old age. This result does not require strong income effects and it applies if individuals select their own savings and bequests by taking the decisions of their offsprings and successors as given. In this case, the equivalence with the Dynastic Equilibria of a Ramsey-type model envisaged in Barro (1974) fails. I show that the structure of equilibria of the olg model with altruism is more similar to the one generated in a canonical olg economy with two goods.

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1. Introduction

Real indeterminacy of equilibria in pure exchange overlapping generations (olg) economies is a consequence of a backward bending offer curve (strong income effect). In this contribution I show that, by introducing some however small degree of parental altruism, indeterminacy can be obtained under more general conditions, and in the absence of a strong income effect.

In order to get direct comparisons with the existing literature, I consider a model very similar to Calvo (1978), *i.e.*, a standard olg economy with two-period lived individuals using a long-run real asset as a store of value, a type of Lucas tree with positive dividends. This specification of the model is motivated by the intention to

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stress that indeterminacy is not a consequence of nominal assets o asset bubbles. In the canonical version of this type of economy, individuals are totally selfish and real indeterminacy can only arise if the offer curve is backward bending, *i.e.*, when the savings function is decreasing in the real interest rate. In the reformulation of the model considered here, individuals are characterized by some degree of altruism with respect to their offsprings. In particular, their welfare is defined by a utility function of their own age-contingent consumption (which I call *selfish utility*) and the welfare of their immediate successors (*i.e.*, the next generation of individuals) multiplied by a discount rate $\beta \in (0, 1)$. The notion of equilibrium adopted in this paper is taken from Bernheim (1989): all households select their own savings and bequests by taking the decisions of their offsprings and successors as given.

I show that local indeterminacy, sunspot equilibria and even global indeterminacy (when altruism is one sided and the non negativity constraints on bequests are occasionally binding) may arise under the assumption that the selfish utility is non-separable, linearly homogeneous and such that the elasticity of substitution between young and old age consumption is greater than one. The latter assumption guarantees that the canonical olg model obtained by setting $\beta = 0$ has a unique equilibrium. More specifically, the basic requirement for indeterminacy under this set of assumptions is that the selfish utility is sufficiently biased toward old age. An interesting consequence of these findings is that the dynamic patterns of households' wealth with *inter vivos* bequests may have very different characteristics irrespective of the degree of altruism.

It is commonly argued that, when parental altruism (in the form just described) is allowed in the canonical olg model with no market frictions, each individual behaves as though it is a single infinitely lived individual and the set of competitive equilibria (*dynastic equilibria*[1]) inherit some of the typical properties of the Ramsey model with a single representative individual. In particular, they should be Pareto optimal and locally determined. More precisely, equilibria correspond to the Pareto optimum derived from a social welfare function assigning zero weights to all generations except for the initial one. The present contribution shows that this conclusion is unwarranted, in the sense that it requires a notion of equilibrium

[1]This is the term used by Bernheim (1989).

whereby the initial generation choses her own consumptions and bequests as well as the consumptions and bequests of all future generations.

In the final section of the paper I present a different variation of the canonical olg model in Calvo (1978), where individuals are selfish ($\beta = 0$), but the old derive utility from an extra good that is produced through the labor effort of the young. Similarly to the case of parental altruism, the competitive equilibria of this model display a large degree of indeterminacy despite the absence of a strong income effect (backward bending offer curve) in the canonical olg model. Hence, bequests play a role that is somewhat similar to the presence of an extra good that is offered by the young and demanded by the old.

The remainder of the paper is organized as follows. Section 2 presents the model, section 3 discusses the structure of equilibria, section 4 provides conditions for indeterminacy, section 5 discusses the relation of this model with the model by Barro (1974), section 6 presents the olg model with two goods and section 7 concludes.

2. The Model

Time is discrete with periods of equal length $t = 0, 1, ..$ and the economy is endowed with an exogenous sequence of a unique consumption good, $\{y_t\}_{t=0}^{\infty}$. Each period $t \geq 0$ a new two period lived individual (household) is born, having utility function

$$U^t = u(c_t^y, c_{t+1}^o) + \beta U^{t+1}, \tag{1}$$

where c_t^j for $j = y, o$ represents the time-t consumption of the good at young and old age, respectively, and the discount factor, $\beta \in (0, 1)$, represents the degree of altruism of generation t. As is standard in olg models, the individual born at time $t = 0$ coexist with an old individual, whose utility, U^{-1}, has the same characterization provided in (20), for some exogenously given young age consumption determined in the unrepresented previous period.

The selfish utility function, $u(.)$, has standard properties, *i.e.*, it is assumed to be strictly increasing, strictly concave and twice continuously differentiable. I also assume that $u(.)$ is linearly homogeneous. This last assumption is mostly made for expositional convenience. The property that is important for my results is that $u(c^y, c^o)$ in non-separable, which is ruled out by strict concavity and linear homogeneity.

I assume perfect financial markets allowing for unlimited lending and borrowing and zero endowment in old age, and consider market arrangements such that any young individual born at time $t \geq 0$ is endowed with some amount $w_t > 0$ of the available consumption good, whereas the old individuals derive resources from capital income only. Then, the household born at $t \geq 0$ has budget constraints

$$c_t^y + s_t = w_t + b_t, \qquad c_{t+1}^o = R_{t+1}s_t - b_{t+1},$$

where s_t represents the individual's real savings to be invested in a real asset, with gross real return R_{t+1}, and b_t, b_{t+1} are the (*inter vivos*) bequests received and provided by the household. To provide a representation of the households decisions in the market economy I follow Bernheim (1989)'s idea of viewing the model as a game played by the succession of generations. In particular, I assume that any household selects her own savings and bequests for the next household by taking all the decision variables selected by the other households (*i.e.*, the successors savings and planned bequests) as given. More formally, for any given sequence of endowments and interest rates, a generic household born at $t \geq 0$ solves

$$(\mathrm{P}(t)) \qquad \max_{s,b\geq 0} u(w_t + b_t - s, R_{t+1}s - b) + \beta u(w_{t+1} + b - s_{t+1}, R_{t+2}s_{t+1} - b_{t+2})$$

given the previous and next generations' choices, (b_t, s_{t+1}, b_{t+2}), whereas the initial old household solves

$$(\mathrm{P}(-1)) \qquad \max_{b\geq 0} u(w_{-1} + b_{-1} - s_{-1}, R_0 s_{-1} - b) + \beta u(w_0 + b - s_0, R_1 s_0 - b_1)$$

given her own (previous period) savings, s_{-1}, and the first young generation's choices $(s_{-1}, b_{-1}, s_0, b_1)$. This assumption implies that households cannot make the amount of their bequests conditional on their offsprings' actions. Notice that parental altruism is assumed to be one-sided, *i.e.*, $b_{t+1} \geq 0$. This appears to be a reasonable assumption, although it is not needed for all of the following results.

I provide a first order characterization of the households' best actions by introducing some new notation. In particular, by linear homogeneity, we can write

$$u(c^y, c^o) = c^y u(1, c^o/c^y).$$

Then, by defining the old-to-young consumption ratio, $x = c^o/c^y$, and the *intensive form* utility, $v(x) = u(1, c^o/c^y)$, I introduce the marginal utilities with respect to young and old age consumption

$$\begin{aligned}\psi_1(x) &= u_1(c^y, c^o) = u_1(1, x) = v(x) - xv'(x),\\ \psi_2(x) &= u_2(c^y, c^o) = u_2(1, x) = v'(x),\end{aligned}$$

as functions of the ratio x only. Using this notation, the first order characterization of the households' best actions is given by the following three equations

$$R_{t+1} = R(x_{t+1}), \tag{2}$$

$$\psi_2(x_t) \geq \beta\psi_1(x_{t+1}), \tag{3}$$

$$\psi_2(x_t)b_t = \beta\psi_1(x_{t+1})b_t, \tag{4}$$

where

$$R(x) \equiv \frac{\psi_1(x)}{\psi_2(x)}$$

is a continuous increasing function of x. Observe that $\psi_1'(x) = -xu_{22}(1, x) > 0$ and $\psi_2'(x) = u_{22}(1, x) < 0$. In particular, by the non-separability of $u(c^y, c^o)$ (strict concavity), the functions $\psi_j(x)$ are non constant for $j = 1, 2$.

The above equations can be combined with the lifetime budget constraint to determine the age-contingent consumptions of the individual born at time t as

$$c_t^y = (1 - \phi(x_{t+1}))I_t, \tag{5}$$

$$c_{t+1}^o/R_{t+1} = \phi(x_{t+1})I_t, \tag{6}$$

where

$$I_t = w_t + b_t - b_{t+1}/R_{t+1}.$$

is the present value of the lifetime income net of the end of period bequests and

$$\phi(x) = \frac{x}{R(x) + x} \in [0, 1]$$

is a continuous function representing the old age consumption expenditure share. Observe that $\phi'(x) \geq 0$ if and only if

$$\gamma \equiv \frac{R(x)}{xR'(x)} \geq 1, \tag{7}$$

i.e., if and only if the elasticity of substitution between young and old age consumption is greater than one. Hence, condition (7) insures that the saving function would be non decreasing in the interest rate with zero bequests, *i.e.*, the substitution effect dominates over the income effect. Defining the time-t savings in the more general case of bequests as

$$s_t(x_{t+1}, b_t, b_{t+1}) = \phi(x_{t+1})(w_t + b_t) + (1 - \phi(x_{t+1}))b_{t+1}/R(x_{t+1}), \tag{8}$$

we can see that $s_1(x, b, b') > 0$ obtains only if ϕ' is sufficiently large relative to the size of bequests.

We close the model by assuming that households save at any time t by exchanging a real asset with per unit price q_t that promises to the owner a per unit dividend ρ_{t+1} the next period (in units of consumption). Then,

$$R(x_{t+1}) = (q_{t+1} + \rho_{t+1})/q_t, \tag{9}$$

at all $t \geq 0$.

By normalizing the asset aggregate supply to one and exploiting Walras' Law, I can define a competitive equilibrium as a sequence of non negative consumption ratios, prices and bequests, $\{x_t, R_t, q_t, b_t\}_{t=0}^{\infty}$, satisfying equations (2), (3), (4), (9) and

$$q_t = s_t(x_{t+1}, b_t, b_{t+1}), \tag{10}$$

where $s_t(.)$ is defined in equation (8). Evidently, the sequence of age-contingent consumptions can be retrieved from $c_t^o = x_t c_{t-1}^y$ and the resource constraint

$$c_t^y + x_t c_{t-1}^y = y_t, \tag{11}$$

where $y_t = w_t + \rho_t$.

3. Alternative Equilibrium Configurations

The model that I have just laid out lends itself to various possible equilibrium configurations, two of which have attracted most of the attention in the existing literature. The first is exactly equivalent to the one that we would derive from the canonical olg model considered by Calvo (1978) and it is obtained as a special case of the present model when the non-negativity constraint on bequests is binding at all $t \geq 0$. For this reason, I call this a *canonical olg equilibrium.* The second configuration is the one that we obtain when the non-negativity constraints are slack at all periods. I call this a *dynastic equilibrium.* However, I will show in a moment that slackness of the non-negative bequests constraints are not sufficient to pin down a unique dynastic equilibrium.

For the sake of simplicity, from now on I assume that the economy is stationary, in the sense that $w_t = w$, $\rho_t = \rho$ for all $t \geq 0$. In this case, a canonical olg equilibrium is characterized by the following law of motion:

$$q_t = \phi \left(R^{-1} \left(\frac{q_{t+1} + \rho)}{q_t} \right) \right) w. \tag{12}$$

It is a simple exercise showing that, when $\phi'(x) \geq 0$ (*monotonic offer curve*), *i.e.*, $\gamma \geq 1$, the above implies that the unique equilibrium with zero bequests is the *canonical olg steady state* (CSS), $(x^*, R^*, q^*, 0)$, such that

$$R^* \leq 1/\beta, \qquad q^* = \rho/(R^* - 1) = \phi(R^*)w.$$

Now consider a dynastic equilibrium. In this case, the equilibrium restrictions provide a unique steady state, (x^d, R^d, q^d, b^d), to be called *dynastic steady state* (DSS), such that equation (3) is verified with equality and bequests are positive. The DSS exists under the condition

$$\frac{\beta}{1-\beta}\rho > \phi\left(x^d\right) w, \tag{13}$$

and it is such that

$$R^d = 1/\beta, \qquad q^d = \frac{\beta}{1-\beta}\rho, \qquad b^d = \frac{q^d - \phi(x^d)w}{\beta + (1-\beta)\phi(x^d)}.$$

It is clear that, if $\phi'(x) > 0$ (*i.e.*, $\gamma > 1$), the two steady states are mutually exclusive, since, if $R^* \leq 1/\beta$, then $q^d = (\beta/(1-\beta))\rho < \phi(x^d)w$. Are there other (non stationary) dynastic equilibria? Are there *mixed equilibria*, *i.e.*, equilibria that do not fall into either one of these categories (*i.e.*, such that the non-negative bequests constraints are occasionally binding)?

I postpone the analysis of mixed equilibria to section 4.3, while I devote the remaining part of this section to a discussion of dynastic equilibria. These can be more easily characterized in terms of a sequence of consumptions, $\{c_t^y, x_t\}_{t=0}^{\infty}$ satisfying the first order condition (3) with equality, *i.e.*,

$$\psi_2(x_t) = \beta\psi_1(x_{t+1}), \tag{14}$$

and the resource constraint (11). The first of these two restrictions establishes a link between successive generations through bequests. The latter, together with the sequence of interest rates and asset prices, can be retrieved from equations (2), (9) and (10). The key observation is that the two restrictions (3) and (11) are not sufficient to pin down a unique equilibrium. In fact, observe that equation (3) is an autonomous difference equation defining the evolution of the non predetermined state variables x_t. Then, suppose that the properties of $\psi_1(.)$ and $\psi_2(.)$ are such that the dynamics induced by (3) generate a sequence $\{x_t^d\}_{t=0}^{\infty}$ such that $1 > x_t^d > 0$ for all $t \geq 0$ and some arbitrary initial value $x_0^d \in (0,1)$ (conditions guaranteeing this possibility will be provided in the next section). Then, since $c_{-1}^y < y$, resource feasibility implies that, for all $t \geq 0$,

$$c_t^y = y - x_t^d c_{t-1}^y > y(1 - x_t^d) > 0.$$

The above is sufficient to prove that the solution $\{x_t^d\}_{t=0}^{\infty}$ to the difference equation induced by (3) with initial condition x_0^d, and the sequence $\{c_t^y\}_{t=0}^{\infty}$ satisfying the resource constraint (11) for $x_t = x_t^d$ for all $t \geq 0$ is a dynastic equilibrium.

4. Indeterminacy

4.1. **Local Indeterminacy.** Here I analyze the set of non stationary dynastic equilibria characterized by a time invariant sequence of dividends and incomes, *i.e.*, $\rho_t = \rho$, $w_t = w$ for all $t \geq 0$, assuming that a DSS exists (condition (25)). Remember that these equilibria verify equation (14). Then, by linearizing equations (9), (10) and (14) near the DSS and defining $\hat{x} = x - x^o$, $\hat{q} = q - q^o$, $\hat{b} = b - b^o$, we obtain

$$(15) \quad \hat{x}_{t+1} = -\mu \hat{x}_t,$$

$$(16) \quad \hat{q}_{t+1} = -(q^o \mu / \gamma^2)\hat{x}_t + \beta^{-1}\hat{q}_t,$$

$$(17) \quad \hat{b}_{t+1} = -(\mu^2/\gamma)(\gamma\beta b^o + (1-\gamma)q^o)\hat{x}_t + (\beta(1-\phi))^{-1}\hat{q}_t - (\mu\beta)^{-1}\hat{b}_t,$$

where $\mu = (1 - \phi(x^o))/\phi(x^o)$.

The three eigenvalues of the Jacobian matrix derived from the above system are the real values

$$\lambda_1 = -\mu, \qquad \lambda_2 = \beta^{-1}, \qquad \lambda_3 = (\beta\mu)^{-1}.$$

Since there are no predetermined variables, competitive equilibria near the DSS display local indeterminacy if at least one eigenvalue is inside the unit circle. This possibility occurs for $\mu < 1$, or, equivalently, for $\phi(x^o) > 1/2$. This inequality implies that individuals allocate a larger share of their expenses to old than to young age consumption.

4.2. **Sunspot Equilibria.** Observe that $\mu < 1$ is also a sufficient condition for the existence of sunspot equilibria. In particular, assume that the sunspot variable is a stochastic Markov process, $\{s_t\}_{t=0}^{\infty}$, with s_t taking two possible values, *i.e.*, $s_t \in \{\sigma^1, \sigma^2\}$ at all $t \geq 0$. The Markov property implies that the realizations of s_{t+1} only depend on s_t and that there is a transition probability matrix,

$$\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$$

with $\pi_{ij} \in [0,1]$, $\sum_{j=1}^{2} \pi_{ij} = 1$, for $i, j = 1, 2$, and the following interpretation

$$\pi_{ij} = Pr\left(s_{t+1} = \sigma^j | s_t = \sigma^i\right).$$

The economy is exactly identical to the one described above, *i.e.*, the endowments and the dividends are time invariant and totally unaffected by the stochastic process $\{s_t\}_{t=0}^{\infty}$. However, individuals believe that asset prices are affected by the sunspot and these beliefs are validated by the actual behaviour of prices. We concentrate on the case of a stationary allocation such that the gross interest rate can take only two possible values, x^1, x^2, with $x^1 > x^2$. Then, recasting the first order conditions (2), (3) under uncertainty and non binding bequest limits, we can characterize the individuals' optimal consumption (and asset) allocation through

$$\psi_2(x^1) = \beta(\pi_{11}\psi_1(x^1) + (1-\pi_{11})\psi_1(x^2)), \tag{18}$$

$$\psi_2(x^2) = \beta((1-\pi_{22})\psi_1(x^1) + \pi_{22}\psi_1(x^2)). \tag{19}$$

A stationary equilibrium exhibits *extrinsic uncertainty* (*i.e.*, there exists a *sunspot equilibrium*) if equations (18), (19) are solved by an array $(x^1, x^2, \pi_{11}, \pi_{22})$ such that $\pi_{jj} \in [0,1]$, $x^j > 0$ and $\pi_{jj} \in (0,1)$ for at least one j.

Proposition 1. *If $\phi^o > 1/2$ there is a sunspot equilibrium near the DSS.*

Proof. From (18), (19) and for $x^1 \neq x^2$, we derive

$$\pi_{11} = \frac{\psi_2(x^1) - \beta\psi_1(x^2)}{\beta(\psi_1(x^1) - \psi_1(x^2))}, \qquad \pi_{22} = \frac{\psi_2(x^2) - \psi_1(x^1)}{\beta(\psi_1(x^2) - \psi_1(x^1))}.$$

Now let

$$x^1 = x^o + a_1\epsilon, \qquad x^2 = x^o - a_2\epsilon$$

for some $\epsilon > 0$ and a pair of positive numbers, (a_1, a_2). Then, define

$$\begin{aligned}\pi_{11}(\epsilon) &= \frac{\psi_2(x^o + a_1\epsilon) - \beta\psi_1(R^o - a_2\epsilon)}{\beta(\psi_1(x^o + a_1\epsilon) - \psi_1(x^o - a_2\epsilon))},\\ \pi_{22}(\epsilon) &= \frac{\psi_2(x^o - a_2\epsilon) - \beta\psi_1(x^o + a_1\epsilon)}{\beta(\psi_1(x^o - a_2\epsilon) - \psi_1(x^o + a_1\epsilon))}.\end{aligned}$$

Evidently, a Sunspot Equilibrium exists if, for some $\epsilon \neq 0$ and $(a_1, a_2) \neq (0, 0)$, we have $\pi_{jj}(\epsilon) \in [0, 1]$ for $j = 1, 2$ and $\pi_{jj}(\epsilon) \in (0, 1)$ for at least some j in $\{1, 2\}$. Using l'Hôspital Rule and observing that

$$\psi_2' x/\psi_2 = -(R'x/R)(1 - \phi(x)), \qquad R(x^o) = 1/\beta,$$

we derive

$$\lim_{\epsilon\to 0} \pi_{11}(\epsilon) = \frac{a_2 - a_1\mu}{a_1 + a_2}, \qquad \lim_{\epsilon\to 0} \pi_{22}(\epsilon) = \frac{a_1 - a_2\mu}{a_1 + a_2},$$

where, we recall, $\mu = (1 - \phi(x^o))/\phi(x^o)$. Then, by continuity of $\pi_{11}(\epsilon)$, $\pi_{22}(\epsilon)$ for ϵ near zero, a sufficient condition for a sunspot equilibrium is $\mu < \min\{a_1/a_2, a_2/a_1\}$, which requires $\mu < 1$, *i.e.*, $\phi^o > 1/2$. $\square$

4.3. **Global Indeterminacy.** For simplicity, the utility function is assumed to have the CES specification

$$u(c^y, c^o) = \left(\theta (c^y)^{\frac{\gamma-1}{\gamma}} + (1-\theta)(c^o)^{\frac{\gamma-1}{\gamma}}\right)^{\frac{\gamma}{\gamma-1}}, \tag{20}$$

with $\gamma > 1$.

Proposition 2. *Assume $\gamma > 1$ and*

$$\left(\frac{1-\theta}{\theta}\right) > \beta^{\frac{\gamma-1}{\gamma}}.$$

Then, for all $x_t > 0$, there exists a unique value $f(x_t) > 0$ such that equation (3) is verified, i.e., $\psi_2(x_t) = \beta\psi_1(f(x_t))$ and $\psi_2(x_t) \geq \beta\psi_1(x_{t+1})$ for all $x_{t+1} \in [0, f(x_t)]$.

Proof. For some $x \geq 0$, let

$$g(x, y) = \beta\psi_1(y) - \psi_2(x), \qquad \Gamma(x) = \{y \geq 0,\ g(x, y) \leq 0\}.$$

Notice that, by the properties of the utility function, $g_2(x,y) > 0$ and by the assumption $\gamma > 1$,

$$g(x,0) = \beta\theta^{\frac{\gamma}{\gamma-1}} - \psi_2(x), \qquad \lim_{y\to\infty} g(x,y) = \infty.$$

Then, if

$$\beta\theta^{\frac{\gamma}{\gamma-1}} < \min_x \psi_2(x) = (1-\theta)^{\frac{\gamma}{\gamma-1}},$$

we derive that, for all $x > 0$, there exists a unique value $f(x) > 0$ such that $\Gamma(x) = [0, f(x)] \neq \emptyset$. □

By the above proposition we derive that, for all initial values, $x_0 > 0$, we can construct a sequence $\{x_t\}_{t=0}^{\infty}$ satisfying equation (3) for all $t \geq 0$. From this sequence we derive, in turn, a sequence of bequests, $\{b_t\}_{t=0}^{\infty}$, such that, for all $t \geq 0$, $b_t = 0$ if $x_{t+1} < f(x_t)$ and $b_t \geq 0$ otherwise. Finally, the equilibrium sequence of asset prices, $\{q_t\}_{t=0}^{\infty}$ is recovered from (9) and (10). Observe, however, that this procedure requires some restrictions on initial conditions. In particular, by the Euler equation (2) and by (9), we derive, for all $t \geq 0$,

$$\psi_2(x_t)q_t = \psi_1(x_t)q_{t-1} - \rho\psi_2(x_t).$$

Using the first order condition for bequests, (3),

$$\psi_2(x_t)q_t \leq \psi_2(x_{t-1})q_{t-1}/\beta - \rho\psi_2(x_t).$$

Then, solving backward, we derive, for all $t > 0$,

$$\beta^t\psi_2(x_t)q_t \leq \psi_2(x_0)q_0 - \rho\sum_{k=0}^{t}\beta^k\psi_2(x_k).$$

Hence, the existence of an equilibrium with positive asset prices requires

$$\psi_2(x_0)q_0 \geq \rho\lim_{t\to\infty}\sum_{k=0}^{t}\beta^k\psi_2(x_k). \tag{21}$$

Finally, observe that indeterminacy may not be related to the failure of the transversality condition, which I write as

$$\lim_{t\to\infty}\frac{b_t}{\Pi_{k=0}^{t}R(x_k)} = 0.$$

To see this, notice that, by (21), present value prices must be converging to zero. In fact, by using the Euler equation (2) repeatedly and assuming (21), we have

$$\lim_{t\to\infty} \frac{1}{\Pi_{k=0}^{t} R(x_k)} = \lim_{t\to\infty} \beta^t \left(\frac{\psi_2(x_t)}{\psi_1(x_0)} \right) = 0.$$

Hence, if the equilibrium sequence of bequests is bounded, transversality holds.

5. Comparison with the Barro-Type Dynastic Equilibrium

An alternative notion of equilibrium, which I call *Barro-type dynastic equilibrium*, is based on the idea that, for a given sequence of endowments and interest rates and for a given initial bequest just received from the previous generation, each generation born at $t \geq 0$ chooses her own consumption and bequests, as well as the consumption and bequests of all future generations. This can be interpreted as an equilibrium where the initial household maximizes her own (altruistic) utility by imposing her own choices about the present and future savings, consumptions and bequests on all future generations (see Barro (1974), Bernheim (1989) and de la Croix and Michel (2002)). In this case, the inter-temporal consumption allocations correspond to a *planning optimum* defined as

$$\max_{\{c_t^y, c_t^o\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{t-1}^y, c_t^o), \qquad \text{s.t.: } c_t^y + c_t^o \leq y_t,$$

for some given c_{-1}^y. By strict concavity, the solution to a planning optimum is unique and can be characterized using standard dynamic programming. Namely, let $V(c_{t-1}^y)$ be the value function from the Planner's problem at time $t \geq 0$. Then, by the Bellman equation we obtain

$$V(c_{t-1}^y) = \max_{x \geq 0} \left\{ v(x) c_{t-1}^y + \beta V(y - c_{t-1}^y x) \right\}.$$

By the first order conditions of the above problem and the envelope theorem, the Planner's solution is completely characterized by equations (11), (14) and

$$\psi_2(x_t) = \beta V'(y - c_{t-1}^y x_t), \tag{22}$$

for all $t \geq 0$. Observe that the first two of these three restrictions are the same restrictions that are sufficient to characterize a dynastic equilibrium, but equation (22) is an additional requirement providing a unique solution $\{c_t^y, x_t\}_{t=0}^{\infty}$ for the given initial value c_{-1}^y. As usual, the equilibrium sequence of interest rates and bequests can be retrieved from equations (2), (9) and (10).

6. Comparison with a Two-Goods olg model

To provide some intuition about the reason why the olg model considered in this paper displays indeterminacy of equilibria despite the absence of a backward bending offer curve, I will consider now an example of a two-goods economy that inherits some key features of the olg model *cum* altruistic bequests. The basic idea is that *inter vivos* bequests can be considered as an alternative consumption good that enters the households' utility.

Suppose that individuals are totally selfish, the economy is endowed with a time invariant amount, $y = \rho + w$, of a consumption good (fruit), just like in the previous example. However, the young are now endowed with a unit of labor. The latter is able to produce a consumption good (chocolate) providing utility in old age only. The technology for the production of this good requires labor only, it is linear and it is characterized by a unit labor productivity. Finally, I assume, for simplicity, that utility is separable, *i.e.*,

$$U^t = u(c_t^y, c_{t+1}^o) + v(z_{t+1}),$$

where z denotes the units of chocolate consumed, $u(.)$ satisfies the same assumptions considered so far and $v(z)$ is increasing and concave. Letting p_t the t-period unit price of chocolate, and w be the young's endowment of fruit, the budget constraints of any individual born at time t is

$$c_t^y + s_t = w + p_t, \qquad c_{t+1}^o + p_{t+1} z_{t+1} = R_{t+1} s_t.$$

The resource feasibility constraints now are

$$c_t^{t-1} + c_t^t = w + \rho, \qquad z_t = 1,$$

and the gross real interest rate, R_{t+1}, may be derived from the right hand side of equation (9). By applying the same logic used in the analysis of the model in section 2 and incorporating the above resource constraints, we may characterize a competitive equilibrium of the present model as a positive sequence, $\{x_t, q_t\}_{t=0}^{\infty}$, satisfying, at all $t \geq 0$,

$$(23) \qquad R(x_{t+1}) = (q_{t+1} + \rho)/q_t,$$

$$(24) \qquad q_t = \phi(x_{t+1})(w + p(x_t)) - (1 - \phi(x_{t+1}))p(x_{t+1})/R(x_{t+1}), ,$$

where

$$p(x) = v'(1)/\psi_2(x).$$

Notice that the dynamical system (23), (24) defining a competitive equilibrium of the present economy boils down to the one-dimensional system defining the competitive equilibrium of the canonical olg economy of Calvo (1978) for $v'(1) = 0$, *i.e.*, $p_t = 0$ for all $t \geq 0$. For the remaing part of this section I will assume that condition (7) holds, *i.e.*, the elasticity of substitution between young and old age consumption (of fruit), γ, is greater than one. This implies that the offer curve for the canonical olg model has a regular upward sloping shape so that savings is an increasing function of the interest rate.

A positive steady state (x^*, q^*) is characterized by

$$R^* = R(x^*) = 1 + \frac{\rho}{q^*}, \quad q^* = \phi(x^*) + p^* \left(\phi(x^*) - \frac{1 - \phi(x^*)}{R^*} \right), \quad p^* = \frac{v'(1)}{\psi_2(x^*)}.$$

The Jacobian matrix evaluated at steady state of the system (23), (24) provides the following characteristic equation

$$P(\lambda) = \lambda^2 - (R^*(1+\alpha) - \alpha\eta)\lambda - R^*\alpha\eta,$$

where

$$\alpha = \frac{R^* q^*}{(1-\phi^*)((\gamma-1)R^* q^* + p^*((\gamma-1)+\phi^*)}, \qquad \eta = \left(\frac{\phi^*(1-\phi^*)}{q^*} \right) p^*,$$

and $\phi^* = \phi(x^*)$. Observe that, for $p^* = 0$, the characteristic equation $P(\lambda)$ generates two eigenvalues $\lambda_1 = 0$, $\lambda_2 = R^*(1+\alpha) > 1$. Since the highest eigenvalue is greater than one and the lowest is zero, it is clear that, in this case, the only feasible trajectory is the steady state value (x^*, q^*).

Consider now the more general case of $v'(1) > 0$ and positive prices of chocolate. We can immediately verify that the largest eigenvalue of $P(\lambda)$, λ_2, is greater than one. In fact, since $\gamma > 1$, it is $P(0) < 0$ for all $p^* > 0$, so that the two eigenvalues are real with opposite sign. Furthermore, because $R^* > 1$,

$$P(1) = 1 + \alpha\eta - R^*(1 + \alpha(1+\eta)) < -\alpha < 0,$$

so that $\lambda_2 > 1$. Finally, since $P(0) < 0$ for all $p^* > 0$ and $P(0) \to 0$ as $p^* \to 0$, by continuity we derive that the smaller eigenvalue, λ_1, is in $(-1, 0)$ for all small enough values of p^*. More precisely, we can make the following claim.

Proposition 3. *Assume that the elasticity of substitution between the age-contingent consumption of fruit,* γ*, is greater than one. Then, the characteristic equation* $P(\lambda)$ *has two real eigenvalues of opposite signs,* $\lambda_1 < 0$*,* $\lambda_2 > 1$*, with* $\mid \lambda_1 \mid < 1$ *if the steady state price of chocolate,* p^**, is low enough. If, on the other hand,*

$$\gamma \geq 1 + \phi^*(R^* - 1), \tag{25}$$

it is $\mid \lambda_1 \mid < 1$ *for all* $p^* > 0$*.*

Proof. It is sufficient to show that, for $p^* > 0$, we have $\lambda_1 \in (-1, 0)$ if $P(-1) > 0$, *i.e.*,

$$1 - \alpha\eta + R^*(\alpha + 1 - \alpha\eta) > 0,$$

which is always verified for $\alpha\eta \leq 1$, *i.e.*, for

$$(\gamma - 1)R^* q^* \geq p^*(\phi^*(R^* - 1) - (\gamma - 1)).$$

The above is always verified under condition (25). □

7. Conclusions

I have shown that indeterminacy of equilibria arises in the olg model with parental altruism under a wide range of parameter configurations which include the absence of strong income effects. This phenomenon crucially depends on the assumption that individuals' decisions about their own savings and bequests are taken for a given set of decisions about the same variables taken by all future generations of individuals. A consequence of this finding is that economies may display different patterns of consumption, wealth and asset prices even when individuals' preferences and endowments are identical. In a final section of the paper I have shown that a comparable degree of indeterminacy arises also when the canonical olg model is amended to include an extra good that is produced through the labor effort of the young and desired by the old. This example suggests that these two different ways to amend the olg model (altruistic bequests and two goods) have similar impacts of the structure of equilibria.

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