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Differentiated Durable Goods Monopoly

A Robust Coase Conjecture

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January 2018

Abstract: The paper analyzes a durable good monopoly problem in which multiple varieties can be produced and sold. A robust Coase conjecture establishes that the market eventually clears, that profits exceed static optimal market-clearing profits, and that profits converge to this lower bound in all stationary equilibria when prices can be revised instantaneously. In contrast to the one-variety case though, equilibrium pricing is neither efficient nor minimal (that is, equal to the maximum between marginal cost an the minimal value). Conclusions apply even when products can be scrapped albeit at possibly smaller mark-ups. If so, a novel motive for selling high cost products naturally emerges. Moreover, with positive marginal costs, cross-subsidization arises as a result of equilibrium pricing. The online appendix delivers insights on product design.

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1 Introduction

While the incentives to set up product-lines are well understood in static environments,[1] not much is known about these incentives in dynamic environments in which a monopolist faces a commitment problem. The purpose of the paper is to give a comprehensive analysis of such incentives. The analysis delivers two main contributions to the literature: it robustly generalizes classical Coasian results to environments in which multiple varieties can be sold; and it shows that a simple static problem characterizes stationary equilibrium pricing in the dynamic game when the monopolist can adjust prices frequently. Our conclusions nest both classical Coasian insights as well as several more modern Coasian failures in a unified framework.

Nobel Laureate Ronald Coase brought the commitment problem of a durable good monopolist to the attention of academic community (Coase 1972). His work has become one of the cornerstones to modern economic theory. Coase conjectured that a monopolist selling a durable good could not credibly sell at a static monopoly price to forward-looking buyers, and that "the competitive outcome may be achieved even with a single supplier". Coase's insight was that the monopolist, lacking the ability to commit to future prices, would face the competition of its own future selves thereby dissipating all of its own monopoly power. Upon selling the initial quantity, the monopolist would necessarily benefit by lowering prices in order to sell to buyers with lower valuations who did not yet purchase. If so, prices would decline after each sale. Forward-looking consumers expecting prices to fall, would then be unwilling to pay the initial high price. Consequently, if the speed of price-revisions was to increase, the opening price would converge to the marginal cost and the competitive quantity would be sold in a *twinkle of an eye*. Formal proofs of this statement appeared in seminal papers by Stokey 1981, Fudenberg, Levine and Tirole 1985, Gul, Sonnenschein and Wilson 1986, among others.

The present work considers the same environment originally studied by Coase, but presumes that the monopolist can produce and sell more than one variety of the durable good. Such a natural extension will not rule out any one of key ingredients required to obtain classical Coasian results which are lack of commitment, deterioration of market conditions, and competition from future selves who are willing to cut prices in the wake of market deterioration. Because of this, conclusions will not give rise to outright failures of classical insights on Coasian dynamics, but rather they will qualify their content. In multi-dimensional settings, the Coasian logic will still prevail in that: (1) prices at which all consumers are willing to purchase a variety will still commit the seller as, upon clearing the market, there would be

[1] See, for example, Mussa and Rosen 1978, Maskin and Riley, 1984, and Deneckere and McAfee, 1996

no incentive to lower prices; (2) in any stationary equilibrium almost all consumers will still purchase a variety at the opening price when the time between offers is small. In multi-variety settings however, these insights will not lead to efficiency, to pricing at minimal valuations, or to zero profits because intratemporal price discrimination will make up for the lack of intertemporal price discrimination. The seller's reservation value in the dynamic game will be pinned down by the simple static problem of maximizing profits subject to clearing the market (that is, selling a variety to every consumer in the support of the measure). Despite the possible loss in bargaining power, the seller's payoff will always exceed such a static optimal market-clearing profit. Also, as in classical Coasian settings, static optimal market-clearing profits will also identify the limit profit in any stationary equilibrium.

Specifically, we consider a monopolist with constant marginal costs who sells two varieties of a durable good to a continuum of consumers with unit-demand for the product.[2] As in classical Coasian settings, buyers' are privately informed of their value for each of the two products. The distribution of values is represented by a measure that can exhibit any arbitrary correlation structure. The set-up accommodates as special cases several commonly used designs such as vertical product differentiation (when consumers' valuations for the two products are positively correlated) and horizontal product differentiation (when consumers' valuations are negatively correlated). The time horizon is infinite; and in every period the monopolist sets a price for each of the two varieties for sale, while the buyers, upon observing prices, choose which variety to purchase (if any). In such environments, the seller always has an incentive to segment consumers between the two varieties so to reduce (or eliminate) its temptation to cut the prices in future, and regain some of its lost profit.

In the baseline setting to favour comparability with classical results, the marginal cost of each variety equals zero, and consumers exit the market never to return upon buying a product. These assumptions are relaxed later on. The analysis begins by characterizing the static problem of maximizing profits subject to clearing the market. Optimal market-clearing profits are always strictly positive when more than one variety is for sale, as it always possible to sell one variety for free (thereby clearing the market) while using the other variety to screen consumers and raise profits. Market-clearing profits can coincide with monopoly profits in the commitment case (for instance, when varieties are horizontally differentiated this is often the case). Thus, the lack of commitment does not necessarily deteriorate the seller's bargaining power. Consumption decisions are generally distorted and inefficient as buyers occasionally purchase their least preferred variety only because it is sold at a cheaper price. Because of intratemporal price discrimination, optimal market-clearing profits often strictly exceed even the lowest valuation of the durable good (that is, the smallest value of

[2] Results easily extend to any finite number of varieties.

the preferred variety) and consequently the lowest valuation of each of the two varieties. [3]

The analysis then extends classical Coasian conclusions to multi-dimensional settings. Preliminary results establish that in any perfect Bayesian equilibrium of the dynamic game: (1) there is skimming as the measure of buyers in the market at any point in time is a truncation of the original measure; and (2) the market clears instantaneously whenever the seller sets static market-clearing prices. The latter immediately implies that optimal market-clearing profits bound equilibrium profits in the dynamic game from below. Results then focus on stationary equilibria in order to relate to classical contributions on Coasian dynamics. Stationary equilibria always exist, and occasionally display mixing along the entire equilibrium path as concealing which variety will be most heavily discounted can increase profits. However as in classical settings, stationary equilibrium profits always converge to optimal market-clearing profits when the time between offers converges to zero. If so, monopoly profits accrue by selling almost instantaneously to almost all consumers.

The spirit of our conclusions is close to classical Coasian results obtained in the one variety case (Stokey 1981, Fudenberg, Levine and Tirole 1985, Gul, Sonnenschein and Wilson 1986, Ausubel and Deneckere 1989). Such results fall in two distinct scenarios. In the gaps case (when the lowest value in the support is strictly positive), the market clears in a finite time, equilibrium profits are positive, unique and converge to the lowest valuation as the time between offers converges to zero. In the no-gaps case (when the lowest value equals zero), the market clears in an infinite time, a Folk Theorem applies to equilibrium profits, and stationary equilibrium profits converge zero. With multiple varieties, results closely resemble the one-variety gaps case even when there are buyers who value both varieties at zero. But, there are some significant differences. In particular, when multiple varieties are sold, equilibrium profits are always positive, and stationary equilibrium profits always converge to optimal market-clearing profits as the time between offers converges to zero. Thus in multi-variety settings, assumptions on gaps no longer pin down the sign of equilibrium profits. Also, having gaps no longer guarantees equilibrium uniqueness, as several market-clearing prices may be optimal at once. However, as in the one-variety case, the presence of gaps still determines the time it takes for the market to clear which is finite with gaps, but not necessarily without. With gaps, as in classical settings, the monopolist always prefers to sell instantaneously to all consumers when few buyers are in the market as price discrimination gains must be small. In contrast to the one-variety case though, stationary equilibria may require mixing take place even after the initial period as the seller conceals future discounts to induce forward-looking consumers to purchase its products at higher prices.

The second part of the analysis considers extensions of the baseline model, and contextu-

[3] For instance, this always occurs when independent varieties share a common minimal valuation.

alizes the contribution within the existing literature. First, it shows how to apply our results to settings with positive marginal costs. When costs are positive, similar conclusions hold, but equilibria may display cross-subsidization (with one variety being sold above marginal cost and the other below). The analysis then establishes why similar results would also apply in settings in which consumers remain in the market after purchasing a variety provided that the definition of static market-clearing is suitably adjusted. Clearing the market would now require: (1) all buyers to purchase a variety; and (2) the marginal costs of varieties to exceed the value of switching between varieties for all buyers. The change however would only bound mark-ups relative to our earlier analysis, but would not restore efficiency when marginal costs are positive. This provides a novel rationale for high cost varieties in durable goods' markets as such products favour intratemporal price discrimination by preventing price reductions to supply buyers close to indifference. The second extension hints at why Coasian dynamics should not only be summarized as optimal market-clearing (or agreement), but rather as renegotiation-proof agreement.[4] The analysis concludes by relating to existing contributions on durable goods pricing and by nesting within our framework both classical Coasian conclusions and some well-known Coasian failures.

The important insight of the analysis consists in relating optimal pricing without commitment to a simple problem, such as static optimal market-clearing. The analysis accomplishes this by showing that optimal market-clearing well-approximates stationary pricing in the dynamic game when frequent price revisions are possible. These observations clarify the nature of many multi-dimensional extensions of the Coase conjecture, and may serve applied economists by delivering predictions about the approximate equilibrium pricing in durable goods' markets. The simple characterization of limiting profits also delivers implications about the design of product-lines. For instance, it is straightforward to observe that increasing the number of varieties favours the seller's ability to intratemporally discriminate buyers by further segmenting the market. In the online appendix, we carry out further comparative statics on optimal market-clearing profits.[5] These product design exercises are, to the best of our knowledge, the first theoretical attempt at analyzing the incentives to develop product-lines in the context of dynamic pricing model.[6]

A vast literature has analyzed tactics to limit the monopolist's commitment problem.

[4]In the classical bargaining settings, *agreement* refers to the seller trading with every buyer whose value exceeds marginal cost. In multi-variety settings, agreement amounts to market-clearing, or equivalently to depletion of gains from trade.

[5]Comparative statics establish that horizontal product differentiation is always optimal and that, in contrast to the one-variety case, volatility in valuations can occasionally benefit the seller.

[6]The seminal reference for related static design questions is Johnson and Myatt 2016. A stylized dynamic exercise appears in House and Ozdenoren 2008 which establishes the optimality of mass products with one variety.

These approaches often prevent the market from deteriorating, and thereby allow the monopolist to sustain higher prices over time. For instance, the seller has been shown to relax its commitment problem and increase its profit by renting the good rather than selling it (Bulow 1982), by introducing best-price provisions (Butz 1990), or by introducing new updated versions of the durable goods over time (Levinthaland and Purohit 1989, Waldman 1993 and 1996, Choi 1994, Fudenberg and Tirole 1998, and Lee and Lee 1998). Other studies have instead analyzed environments which preclude the market from fully deteriorating. These include environments with capacity constraints (Kahn 1986, and McAfee and Wiseman 2008), with entry of new buyers (Sobel 1991), with time-varying buyers' valuations (Biehl 2001, Deb 2011, Garrett 2016), with time-varying costs of production (Ortner 2014), with depreciation (Bond and Samuelson 1984), and with discrete demand (Bagnoli, Salant and Swierzbinski 1989, Fehr and Kuhn 1995, Montez 2013).

Most closely related to this paper are Board and Pycia 2014, Hahn 2006, Inderst 2008, Takeyama 2002, and Wang 1998. The last four provide examples in which a multi-variety durable good monopolist is able to mitigate its commitment problem. Their analysis is restricted to vertically differentiated products, two types of consumers (Hahn 2006, Inderst 2008), and two periods (Takeyama 2002). Their main focus is on the possibility of strategically changing the quality of the goods (through upgrades or downgrades). Their conclusions can be nested in our setting as vertical product differentiation is allowed. In our view however, these results should not be interpreted as failures of the Coase conjecture. Rather, they display the essence of the Coasian insight which is optimal market-clearing (or agreement), and not efficiency or minimal pricing. Similarly, Board and Pycia 2014 shows that a durable good monopolist never cuts its price if an outside option with strictly positive value is available for free. Their conclusions can also be nested within our framework. Indeed, because any price set by the monopolist clears the market when an outside option is freely available, the monopolist not undercutting on its initial price would be consistent with the proposed extension of the Coase conjecture. Of course, by pricing the outside option, the monopolist would be able to achieve an even higher profit as both varieties would be optimally sold at positive prices when there are gaps (which is the case in their setting). Similar considerations apply to Wang 1998 which establishes an instantaneous clearing result (evocative of Board and Pycia 2014) in a two-type model. These and other related results have often been classified as failures of the Coase conjecture. But in fact, these are not failures, rather they capture features of multi-dimensional Coasian generalizations which entail only immediate market-clearing and not minimal profits or efficiency.

The rest of the paper is organized as follows. Section 2 introduces the model and the relevant solution concepts. Section 3 characterizes optimal static pricing subject to market clearing when marginal costs equal zero. Section 4 solves the dynamic pricing game and features our Coasian results when marginal costs equal zero. Section 5 extends conclusions to settings with positive marginal costs and to settings in which buyers remain active upon purchasing a variety. Section 6 relates in detail to classical Coasian results and their failures, and concludes. The proofs of lemmas and propositions are deferred to the appendix, Section 7. Results on the optimal design of varieties and the proofs of remarks can be found in the online appendix.

2 A Market with Differentiated Varieties

A monopolist produces and sells two varieties of a durable good, a and b. A unit measure of non-atomic consumers has unit-demand for the durable good. Time is discrete, the time-horizon is infinite, and all the players discount the future by a common factor δ. A consumer is completely pinned down by its value profile $v = (v_a, v_b)$, where v_i denotes the value of consuming variety $i \in \{a, b\}$. Value profiles are private information of consumers. A measure $\mathcal{F}$, defined on the unit square $[0,1]^2$, describes the distribution of values profiles among buyers. Throughout denote by F its associated cumulative distribution and by V its support.[7] To simplify parts of the discussion, some results require the measure $\mathcal{F}$ to be non-atomic.

Condition 1 *The market is said to be regular if $\mathcal{F}$ is absolutely continuous with respect to the Lesbegue measure on $[0,1]^2$, if the density f satisfies $f(v) \in \left(\underline{f}, \overline{f}\right)$ for any $v \in V$, and if the support V is convex.*

Regularity implies that a bounded and strictly positive density exists on the entire support V. Weaker, albeit more involved, conditions could be imposed to discipline the measure only on the relevant parts of the support.[8] We opted for a stronger, but more elegant, condition while qualifying its role throughout analysis. Denote the marginal cumulative distribution of variety i by F_i, its support by V_i, and its density by f_i, when it exists.

In the baseline setting buyers, having unit-demand for the product and exit the market upon purchasing any one for the two varieties.[9] Thus, the final payoff of a buyer purchasing variety $i \in \{a, b\}$, at date t, at a price p_i simply amounts to $\delta^t (v_i - p_i)$, while the payoff of a buyer never purchasing a variety simply amounts to 0. The monopolist's marginal cost of

[7]The support identifies the smallest closed set whose complement has probability zero.

[8]Our regularity condition differs from the classical assumptions imposed on the single variety case in Gul, Sonnenschein, and Wilson 1986. The two assumptions however cannot be nested as their conditions are stronger but local, whereas we impose weaker conditions but on the entire support. The assumption invoked is a natural extension of assumptions in Fundenberg, Levine, Tirole 1984.

[9]We discuss which conclusions are affected by the permanent exit assumption in Section 4.

producing of variety $i \in \{a,b\}$ is constant and denoted by $c_i \in [0,1]$. Marginal costs are common knowledge. Units are produced when sold so to minimize production costs, and the monopolist's payoff simply amounts to the present discounted value of future profits.

In every period: the firm sets a price in $[\omega, 1]$ for each of the two varieties produced in order to maximize the expected present value of future profits;[10] and consumers, who have not previously purchased a product, choose whether to buy any one of the two varieties at current prices so to maximize their expected present value.

Information Structure and Solution Concepts: Players observe the prices set by the monopolist in every previous period. A t-period seller-history, h^t, specifies for every period $s \in \{0, ..., t-1\}$ the prices that were set by the seller for each of the two varieties of the durable good. A t-period buyer-history for a player who has yet to purchase a variety, $\hat{h}^t$, consists of a history h^t followed by the prices announced by the monopolist at date t. Denote the set of t-period seller-histories by $H^t = [\omega, 1]^{2t}$ and the set of seller histories by $H = \cup_{t=0}^{\infty} H^t$. Similarly, denote the set of t-period buyer-histories by $\hat{H}^t = [\omega, 1]^{2t+2}$ and the set of buyer histories by $\hat{H} = \cup_{t=0}^{\infty} \hat{H}^t$.

As customary in the literature, we impose measurability restrictions on joint consumers' strategies which require the set of consumers purchasing variety $i \in \{a, b\}$ at any possible history to be a measurable set. For a metric space X, denote by $\mathcal{P}(X)$ the set of all probability measures on $(X, \Omega(X))$ where $\Omega(X)$ denotes the Borel sigma-algebra. Similarly, denote by $\mathcal{P}^*(X)$ the set of all measures on $(X, \Omega(X))$. A behavioral pure strategy profile for buyers consists of a function $\alpha : \hat{H} \times V \to \{0, a, b\}$ such that $\alpha(\hat{h}^t, \cdot)$ is measurable for any $\hat{h}^t \in \hat{H}$. Action 0 is to be interpreted as the decision not to buy any product in the current period. Actions a and b respectively denote the decision to purchase variety a or b in the current period. Intuitively, α determines consumption decisions of buyers at every possible history. Behavioral mixed strategies at any history then consist of probability distributions over such measurable functions. A behavioral strategy profile for the monopolist consists of function satisfying $\sigma : H \to \mathcal{P}([\omega, 1]^2)$, where σ determines the probability distribution over prices charged by the monopolist as a function of history of play.

Any strategy profile $\{\sigma, \alpha\}$ generates a path of prices and sales which can be computed recursively. Given a mixed strategy profile $\{\sigma, \alpha\}$, let $\mathcal{D}_i(h^t) \in \mathcal{P}^*(V)$ denote the measure of consumers purchasing variety $i \in \{a, b\}$ at date t in at any buyer-history h^t, and let $D_i(h^t)$ denote its support. A consumer with value profile $v \in V$ is *active* at history h^t if they have not yet purchased a variety of the durable good. Formally, define the measure of *active buyers*

[10] We keep the choice set of the seller compact for technical convenience. Prices are allowed to be negative as $\omega < 0$; but, will always be non-negative in equilibrium.

$\mathcal{A}(h^t)$ at a given history h^t as

$$\mathcal{A}(E|h^t) = \mathcal{F}(E) - \textstyle\sum_{s=0}^{t-1} [\mathcal{D}_a(E|h^s) + \mathcal{D}_b(E|h^s)] \quad \text{for any} \quad E \in \Omega(V),$$

where h^s denotes the sub-history of length s of h^t. Let $A(h^t)$ denote the support of this measure. When clarity is not compromised, we omit the dependence on the history and we denote these measures and supports simply by $\mathcal{D}_i^t$, D_i^t, $\mathcal{A}^t$ and A^t. For any strategy profile $\{\sigma, \alpha\}$, let $\Pi(\sigma, \alpha|h^t)$ be the expected present value of profits generated after history h^t, and let $U(\sigma, \alpha|\hat{h}^t, v)$ be the expected present value of surplus of an active buyer v who chooses not buy any variety at history $\hat{h}^t$. When an equilibrium strategy is fixed, we omit the dependence on strategies and denote by $\Pi(h^t)$ the expected present value of profits and by $U(\hat{h}^t, v)$ the continuation value of player v.

A *perfect Bayesian equilibrium* (equivalently a PBE) consist of a mixed strategy profile $\{\sigma, \alpha\}$ and updated beliefs about the measure of active buyers satisfying the two standard requirements: strategies are optimal given beliefs, and beliefs are derived from strategies according to Bayes rule whenever possible. To guarantee the existence of an equilibrium, players are allowed to mix at any stage of the game.

With the proposed information structure, buyers' deviations cannot be detected by the seller. In this respect, the paper is closest to the classical asymmetric information bargaining model in Fudenberg, Levine and Tirole 1985. Yet, rather than having a single buyer, the model preserves the durable goods interpretation by retaining a measure of buyers. Because buyers' deviations cannot be detected, no further refinements are invoked. If buyers' deviations were detectable however, similar conclusions would hold for equilibria in which deviations by non-atomic subsets of buyers have not effect on future play.[11] In any such equilibrium, the path of play would still coincide with the equilibrium path associated to a perfect Bayesian equilibrium that we characterize given that players' strategies prescribe optimal behavior after all histories with non-atomic deviations. Consequently, unilateral deviations by non-atomic buyers would not affect the actions of the remaining consumers, their beliefs, or the actions of the monopolist.

In general, buyers' equilibrium strategies may not only depend on the current price profile p^t, but on the entire history of play (as the entire history may affect beliefs about future prices). Ausubel and Deneckere 1989 show that a Folk Theorem can hold in this class of games even when a single variety is for sale, if the monopolist can make its pricing decision

[11] In classical durable goods settings (such as Gul, Sonnenschein and Wilson 1986 and Ausubel and Deneckere 1989), every deviation is detectable. To deal with the implied complications, a refinement is invoked restricting attention to equilibria in which deviations by subsets of active buyers with measure zero change neither the actions of the remaining buyers nor those of the seller.

contingent on the full history of play.[12] As a similar logic applies to settings with multiple varieties, it is convenient to consider stationary equilibria in which the monopolist does not exploit changes in beliefs to commit to a given price path. As customary in the literature therefore, the results on Coasian dynamics rely a common class of Markovian equilibria. Define a *weak Markov equilibrium* (equivalently a WME) to be a perfect Bayesian equilibrium in which the strategy of active buyers depends only on the current price profile.[13] In any weak Markov equilibrium, the continuation value of any active buyer only depends on current prices.

3 Optimal Market-Clearing

We begin by defining the set of static market clearing prices and by discussing some of its properties. Such prices will play an important role in the analysis of the dynamic pricing problem at hand. In the following two sections, we focus on the case in which marginal costs equal zero, $c = (0, 0)$. This however does not rule out the classical no-gaps case.

Throughout, when denoting by i a generic variety in $\{a, b\}$, we denote by $j \neq i$ the other variety. A *market clearing price* is a price profile that clears the market when the seller commits to setting such prices for the infinite future. Equivalently, it is a price profile that clears the market in the static version of the model. Formally, the static demand $d_i(p)$ for a variety $i \in \{a, b\}$ given a price profile p satisfies

$$d_i(p) \in [\mathcal{F}(v_i - p_i > \max\{v_j - p_j, 0\}), \mathcal{F}(v_i - p_i \geq \max\{v_j - p_j, 0\})].$$

The demand equation does not impose tie-breaking assumptions for indifferent consumers.[14] The set of market-clearing prices M consists of those prices at which every consumer is willing to purchase at least one of the two varieties,

$$M = \left\{p \in \mathbb{R}^2 \mid \max_{i\in\{a,b\}} \{v_i - p_i\} \geq 0 \text{ for all } v \in V\right\}.$$

Let $\underline{v}_i$ denote the *minimal value* for variety i in the support V. With only one variety, the highest market-clearing price always coincides with the minimal value in the support. With

[12] The Folk theorem holds in the single variety case whenever the stationary equilibrium payoff of the seller converges to zero.

[13] That is, a WME is a PBE in which, at any two histories $\hat{h} = (p, h) \in \hat{H}$ and $\hat{h}' = (p, h') \in \hat{H}$, we have that $\alpha(\hat{h}, v) = \alpha(\hat{h}', v)$ for all $v \in A(\hat{h}) \cap A(\hat{h})$.

[14] When the market is regular, tie-breaking assumptions are entirely inconsequential as demand simplifies to $d_a(p) = \int_0^{\min\{1,1+p_b-p_a\}} \int_{\max\{v_b+p_a-p_b,0\}}^1 f(v)dv$.

more than one variety, any price profile p in which one variety is sold at a price that does not exceed its minimal value is a market-clearing price,

$$p_i \leq \underline{v}_i \quad \Rightarrow \quad p \in M.$$

When the valuations of the two varieties are independently distributed, at any market-clearing price $p \in M$ at least one of the two varieties must be sold at a price below its minimal value (Figure 1, Panel 2). But, this need not be the case in general. For instance, when the values of the two varieties display perfect negative correlation, market-clearing prices exist in which both varieties are sold at prices that strictly exceed their respective minimal values (Figure 1, Panel 3). For a consumer $v \in V$, define the *value of the durable good* as the value of the preferred variety, $v_g = \max_{i \in \{a,b\}} v_i$. If so, the *minimal value of the durable good* amounts to

$$\underline{v}_g = \min_{v \in V} \max_{i \in \{a,b\}} v_i.$$

The minimal value of the durable good always exceeds the minimal value of every variety, $\underline{v}_g \geq \max_{i \in \{a,b\}} \underline{v}_i$. Moreover, at any market-clearing price $p \in M$, at least one variety is sold at a price below $\underline{v}_g$ as the market clears, $\min_{i \in \{a,b\}} p_i \leq \underline{v}_g$.

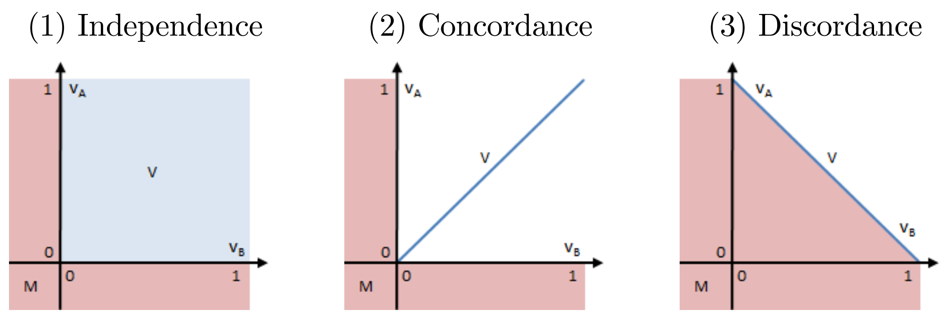

Figure 1: In pink the set of market-clearing M and in blue the support V for three possible distributions.

Optimal market-clearing prices, $\bar{p}$, are market-clearing prices that maximize static monopoly profits. Formally, an optimal market-clearing price (equivalently an OMC price) is defined as a solution to the following static profit maximization problem

$$\bar{p} \in \arg\max_{p \in M} [d_a(p)p_a + d_b(p)p_b]. \tag{1}$$

Optimal market-clearing prices may fail to exist when regularity is violated as extrema may never be attained. So, define the supremum of this problem as the *optimal market-clearing profit*, $\bar{\pi}$. Optimal market-clearing profits always exist as profits are nonnegative and necessarily bounded above by 1. The first result bounds optimal market-clearing profits from below when more than one variety is for sale. The proof does not rely on assumptions on the measure $\mathcal{F}$, and includes scenarios in which the market is not regular. We say that *varieties are identical* if their values coincide for all buyers (that is, if $v_a = v_b$ for any $v \in V$). We say that *varieties are unranked* if not all buyers weakly prefer one variety to the other (that is, if for any i there is $v \in V$ such that $v_i > v_j$). Clearly, if varieties are unranked, they cannot be identical.

Proposition 1 *Optimal market-clearing profits:*

(1) weakly exceed $\underline{v}_g$*;*

(2) strictly exceed $\max_{i \in \{a,b\}} \underline{v}_i$ *if varieties are unranked;*

(3) equal $\min_{i \in \{a,b\}} \underline{v}_i$ *if and only if varieties are identical;*

(4) equal 0 *if and only if varieties are identical and* $(0,0) \in V$*;*

(5) strictly exceed $\underline{v}_g$ *if varieties are not identical,* $\underline{v}_a = \underline{v}_b$ *and* $(\underline{v}_a, \underline{v}_b) \in V$*.*

The monopolist can always clear the market by selling both varieties at a price $\underline{v}_g$. So, optimal profits must weakly exceed the minimal value of the durable good, and consequently the minimal value of every variety. When varieties are unranked, the seller can raise higher profits while clearing the market by selling both varieties at prices that exceed the largest minimal value, one of them strictly so. When varieties are identical, optimal market-clearing profits must be equal to $\min_{i \in \{a,b\}} \underline{v}_i = \underline{v}_g$ as all buyers purchase the cheapest variety. But otherwise, profits always strictly exceed the smallest of the two minimal values since market-clearing prices exist in which both varieties are sold and in which one variety is sold at a price that strictly exceeds the smallest minimal value. As $\underline{v}_i \geq 0$ for any variety i, optimal market-clearing profits can therefore be equal to 0 if and only if varieties are identical and $(0,0) \in V$. By a similar logic, optimal market-clearing profits strictly exceed even the minimal value of the durable good when varieties are differentiated, minimal values coincide, and a single buyer has the smallest possible value for both varieties.[15]

[15] A model in which willingness to pay can be determined by budget constraints could deliver $\underline{v}_a = \underline{v}_b$ and $(\underline{v}_a, \underline{v}_b) \in V$.

(1) Independence (2) Concordance (3) Discordance

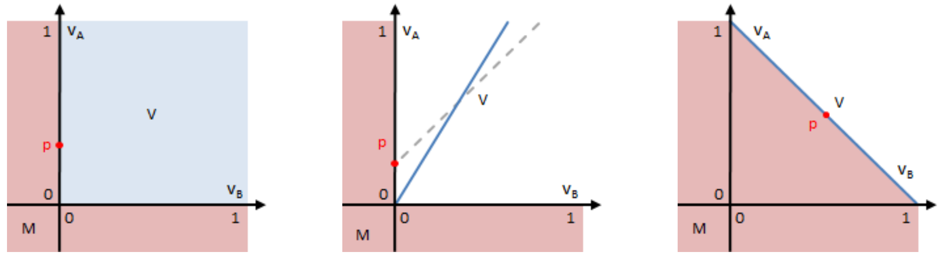

Figure 2: In red, optimal market-clearing prices for three possible distributions.

Proposition 1 hints at why the Coasian intuition about the seller eventually depleting the market does not necessarily lead to zero profits or to pricing at minimal values when differentiated varieties can be produced and sold. Although a monopolist lacking commitment may still have to clear the market, market-clearing no longer requires that profits coincide with minimal valuations. Indeed, when the market is regular, optimal market-clearing profits always strictly exceed the smallest minimal valuation; and under mild conditions, such profits strictly exceed even the minimal valuation of the durable good.

A market-clearing price profile is said to be *efficient* if every buyer purchases its preferred variety. Formally, the set of efficient price profiles simply amounts to

$$M^* = \{p \in M \mid v_i \geq v_j \quad \Rightarrow \quad v_i - p_i \geq v_j - p_j \;\; \text{for all} \;\; v \in V\}\,.$$

We refer to such prices as efficient as they maximize utilitarian social welfare.[16] Any market-clearing price $p \in M$ such that $p_a = p_b$ is always obviously efficient. Furthermore, no other price can be efficient when $v_a = v_b$ for some $v \in V$. Although efficient market-clearing prices always exist, optimal market-clearing need not be efficient. For instance, when a single buyer has the smallest possible value for both varieties and the minimal values coincide, optimal market-clearing prices are necessarily inefficient provided that varieties are differentiated.

Remark 1 *Optimal market-clearing prices are inefficient if V is connected, varieties are not identical, $\underline{v}_a = \underline{v}_b$ and $(\underline{v}_a, \underline{v}_b) \in V$.*

[16] As utility is transferrable and costs equal zero, an efficient price always maximizes surplus since all buyers get to purchase their preferred variety.

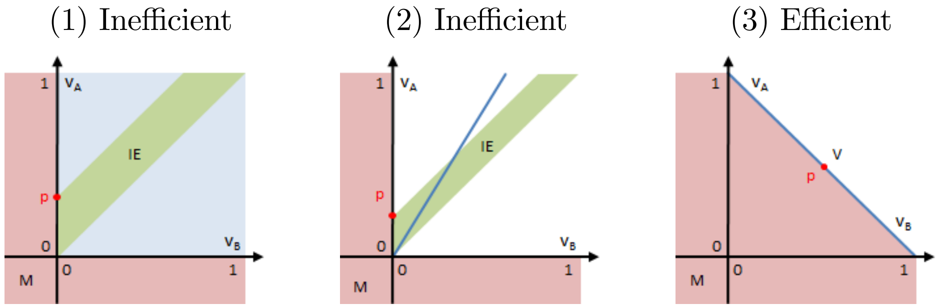

Figure 3: In green, buyers purchasing the inefficient variety at the market-clearing price p.

In fact, for suitable definitions it would be possible to show that the inefficiency of optimal market-clearing prices is indeed a generic phenomenon. This hints at why the generalizations of Coasian logic may not necessarily lead to efficiency if the main force disciplining dynamic pricing was shown to be optimal market-clearing. Of course, if varieties were identical, optimal market-clearing prices would necessarily be efficient simply because the market clears. But this is not the case with differentiated products. Figure 3 depicts two instances of inefficiency with independent and vertically differentiated products (in Panels 1 and 2), while Panel 3 shows why we had to assume $(\underline{v}_a, \underline{v}_b) \in V$ to guarantee frictions.

4 Coasian Dynamics as Market-Clearing

This section extends classical Coasian results to settings with multiple varieties. It establishes that the market for the durable good must eventually clear even when multiple varieties can be sold. The intuition coincides with that of seminal Coasian results. As the monopolist cannot commit to future prices, the market must clear or else clearing the market would become a profitable deviation as soon as the seller no longer expects to trade. In a multi-variety setting, however, market-clearing no longer implies that profits are minimal. Indeed, perfect Bayesian equilibrium profits always exceed optimal market-clearing profits, and converge to such profits in any weak Markovian equilibrium as price revisions become arbitrarily frequent. These results highlight why lack of commitment and Coasian pricing do not necessarily lead to minimal-valuation pricing or efficiency, but only to market-clearing and agreement. When price revisions are frequent, the monopolist will simply choose the profit maximizing way to supply all the buyers. As in the single-variety case, having gaps will guarantee that all buyers are supplied in a finite time. But in contrast to the classical case, the market can clear in a finite time even when there are no gaps.

We begin the analysis with a few preliminary results that unveil some important features of equilibrium strategies in this dynamic pricing game. As in the classical single-variety setting, the measure of active buyers must be a truncation of the original measure in any equilibrium. Furthermore, equilibrium play displays a specific form of top-down *skimming* of the market. In particular, for all values of $k \in \mathbb{R}$, a cutoff identifies, at every possible history, the smallest value buyer who is willing to purchase one of the two varieties on the line

$$L_k = \left\{ v \in \mathbb{R}^2 \mid v_a - v_b = k \right\}.$$

Thus, every subset of buyers with a given value difference k will be skimmed from the top down as a result of equilibrium play. To show this, we introduce a general notion of multidimensional truncation. We say that a measure $\mathcal{F}'$ is a *truncation* of $\mathcal{F}$ if for some set $A \subset \Omega(V)$

$$\mathcal{F}'(E) = \mathcal{F}(E \cap A) \text{ for any } E \in \Omega(V).$$

Lemma 1 *In any perfect Bayesian equilibrium, at any buyer-history h:*
(1) if buyer v strictly prefers to buy variety i, so does any active buyer v' such that

$$v'_i - v_i \geq \max\{0, v'_j - v_j\};$$

(2) if buyer v prefers to buy a variety, any active buyer $v' > v$ strictly prefers to buy if

$$\delta \max_{i \in \{a,b\}}\{v'_i - v_i\} < \min_{i \in \{a,b\}}\{v'_i - v_i\};$$

(3) if buyer v prefers not to buy, any active buyer $v' < v$ strictly prefers not to buy if

$$\delta \max_{i \in \{a,b\}}\{v_i - v'_i\} < \min_{i \in \{a,b\}}\{v_i - v'_i\};$$

(4) if the market is regular, the measure of active buyers is a truncation of $\mathcal{F}$.

The proof of the lemma is intuitive. If a buyer with value v was willing to purchase variety i at current prices, the same should hold for any active buyer $v' > v$ provided that relative value for the two varieties is similar. In fact, by delaying buyer v' could capture at most $\delta \max_i\{v'_i - v_i\}$ on top of the continuation value of buyer v. But if so, buying now should be preferable as they would capture $\min_i\{v'_i - v_i\}$ more surplus than v. This naturally obtains as delay costs are higher for high value consumers, and implies that the measure of active buyers must be a truncation whenever $\mathcal{F}$ is non-atomic. However, stronger notions of skimming would not apply. For instance, it is not the case in general that v buying a variety and $v' > v$ together imply that v' buys a variety. The active player set in Figure 4 would violate this more stringent skimming requirement as there are values v who purchase variety a and values $v' > v$ who do not purchase any variety. This occurs naturally in equilibrium when buyer v' prefers to wait to purchase good b at a lower price in the future. Still, whenever buyer v strictly prefers to purchase variety i, so do all the buyers v' who have a higher value for i provided that the change in value for variety i exceeds that for variety j (equivalently, $v'_i - v_i \geq v'_j - v_j$). A similar logic also implies that, if a buyer v does not buy any variety, so does any buyer $v' < v$ with a similar relative value for the two varieties. Parts (1) and (3) of the Lemma are depicted in right panel in Figure 4: if a dot belongs to one of the three regions, then its triangle must also belong to it.

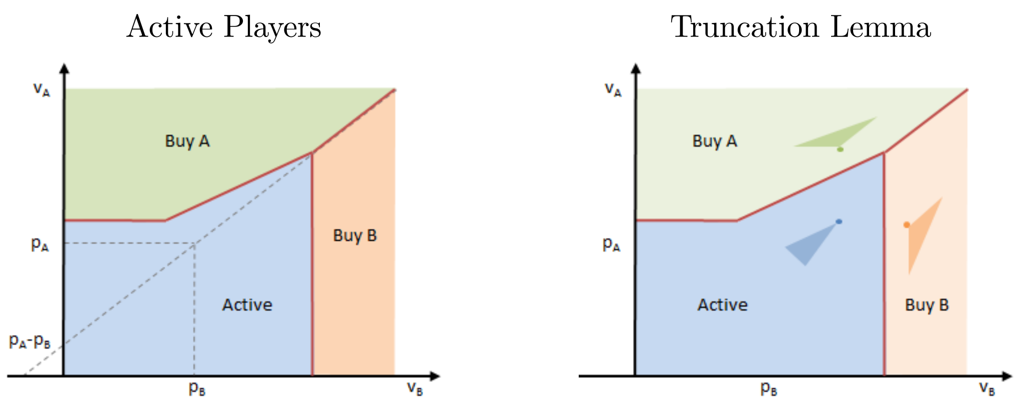

Figure 4: For a market clearing within two periods: in blue A^{t+1} the active buyer set; in green D_a^t those who purchase variety a; in orange D_b^t.

The next lemma relates features of equilibrium pricing to static market-clearing. The key observation establishes why static market-clearing prices must also clear the market in any equilibrium of the dynamic model. This immediately delivers two central conclusions. First, the monopolist never sets prices in the interior of the static market-clearing set. Second, optimal market-clearing profits always bound profits from below in any equilibrium, at any history and for any possible discount factor. To state the result, let $\bar{M}$ denote the "interior" of the market-clearing price set M, or equivalently

$$\bar{M} = \left\{ p \in \mathbb{R}^2 \mid \max_{i \in \{a,b\}} \{v_i - p_i\} > 0 \text{ for all } v \in V \right\}.$$

Given any history h and its associated active player set A, let $\bar{\pi}(A)$ denote the optimal market-clearing profit for the residual measure of buyers $\mathcal{F}(A)$. When the market is regular, $\bar{\pi}(A)$ simply amounts to

$$\bar{\pi}(A) = \max_{p \in M} \sum_{i \in \{a,b\}} p_i \mathcal{F}(v_i - p_i > v_j - p_j | A).$$

Lemma 2 *In any perfect Bayesian equilibrium, at any seller-history h:*
(1) all active buyers purchase a variety if prices are in $\bar{M}$*;*
(2) the monopolist never sets prices $\bar{M}$*;*
(3) the present value of profits satisfies

$$\Pi(h) \geq \bar{\pi}(A).$$

Equilibrium profits must weakly exceed optimal market-clearing profits for any $\delta < 1$. In multi-variety settings, the inability to intertemporally price discriminate forward-looking buyers may still hurt the seller. Market-clearing and intratemporal price discrimination, however, shield the seller from further profit declines. In fact, even when the minimal value for the durable good equals zero $\underline{v}_g = 0$, equilibrium profits cannot be competitive and the allocation may be inefficient. This contrasts with a classical interpretation of the Coase conjecture for single-variety settings with no-gaps which requires equilibrium pricing to be approximately competitive and approximately efficient when buyers are arbitrarily patient. As the rest of the analysis clarifies however, the Coasian logic persists to the extent that agreement and market-clearing still dictate equilibrium pricing. An immediate implication of the lemma is that the seller cannot lose bargaining power whenever optimal market-clearing profit coincides with the monopoly profit. The proof of the lemma identifies the set of price profiles which are immediately accepted by all the buyers regardless of their beliefs. It then establishes by contradiction that such a set of prices must include all static market-clearing prices because of consumers' discounting.[17]

To grasp the full connection between known Coasian results and their failures, it is instructive to consider a few more observations. The next lemma establishes that the market must eventually clear and that market-clearing must take place in a finite time whenever the minimal valuation for the durable good is positive, $\underline{v}_g > 0$ (call this the *gaps* case). The same result holds with a single variety when the smallest buyer's valuation is strictly positive. But in contrast to the one-variety setting: the market clears in a finite time even when the minimal value of both varieties equals zero provided that $\underline{v}_g > 0$; and it may take an infinite time to clear the market even when equilibrium profits are positive (which is generically the case by Lemma 2 and Proposition 1). The result, however, does not imply that the market cannot clear in a finite time when there are no gaps, $\underline{v}_g = 0$.

Lemma 3 *If the market is regular, in any perfect Bayesian equilibrium:*

[17] This is a common feature of many bargaining models dating back to Rubinstein's 1982 seminal work.

(1) every buyer $v \in V$ purchases a variety as time diverges to infinity;
(2) if $\underline{v}_g > 0$, every buyer $v \in V$ purchases a variety in a finite time.

When the measure of active buyers is small and $\underline{v}_g > 0$, the monopolist benefits from clearing the market instantaneously. This is the case as $\underline{v}_g > 0$ implies that the loss caused by discounting future profits outweighs any possible price-discrimination gain if few buyers remain active. If so, the monopolist clears the market at once. In general, the monopolist must eventually sell to all buyers because otherwise instantaneously clearing the market would be profitable whenever the measure of active buyers is close to a limit.

As in the single variety case, it is possible to show that perfect Bayesian equilibria exist and that at least one of these equilibria is weakly Markovian. The next result proves directly the existence of a weak Markov equilibrium which implies the existence of perfect Bayesian equilibria. The proof applies also to the no-gaps case, $\underline{v}_g = 0$.

Proposition 2 *If the market is regular, a weak Markov equilibrium exists.*

The proof strategy is classical and evocative of the single variety case. When there are gaps, the equilibrium is finite, and thus backward induction and a suitable variant of the Kakutani-Fan-Glicksberg fixed point theorem suffice to establish existence. When there are no-gaps, the equicontinuity of the equilibrium correspondence is further exploited to deliver the result. Stationary equilibria are not necessarily unique in multi-variety settings as optimal market clearing prices are not unique in general.

In contrast to classical results on the single-variety case, mixed strategy equilibria exist in which the seller randomizes even in the final stage upon clearing the market. This is the case as the monopolist may have an incentive to conceal future price reductions if buyers delay purchasing those varieties that are going to be more heavily discounted. To show an instance of this phenomenon consider atomic measure of buyers with support

$$V = (1,1) \cup \left\{ v \in [0,1]^2 \mid v_j = (1 - v_i)/3 \text{ for any } v_i \in [1/4, 1] \ \& \text{ any } i \in \{a, b\} \right\}.$$

The dark blue region in the left panel of Figure 5 depicts this support. Although the measure fails regularity, similar conclusion would hold in the regular market in which the support is the convex hull of V (the light blue shaded region in the left plot of Figure 5) and in which almost all the measure is on V. Consider the following joint distribution on V

$$F(v) = \begin{cases} 1 & \text{if} \quad v_i = 1 \quad \& \quad v_j = 1 \\ (6v_a + 6v_b - 3)/10 & \text{if} \quad v_i \in [1/4, 1] \quad \& \quad v_j \in [1/4, 1) \\ (18v_j + 6v_i - 6)/10 & \text{if} \quad v_i \in [1/4, 1] \quad \& \quad v_j \in [1 - 3v_i, 1/4] \\ 0 & \text{if} \quad \text{otherwise} \end{cases}.$$

Intuitively, such distribution has a measure $1/10$ on the atom at $(1,1)$, while the $9/10$ of the measure is uniformly distributed on the other component of V.

Optimal market-clearing profits in this market amount to 0.272 (approximately) and can be secured by two symmetric market-clearing price profiles in which one variety is sold at $5/12$, while the other at $7/36$. If the seller had the ability to commit to the price profile instead, it would optimally sell both varieties at a price of $13/24$ thereby raising a profit of 0.352 (approximately). As in the one variety case, the seller's bargaining power is diminished by the inability to commit to the price path (at least for sufficiently high values of δ). Optimal market-clearing profits however exceed the minimal value of the durable good, as $\underline{v}_g = 0.25$, and the minimal value of both varieties, as $\underline{v}_a = \underline{v}_b = 0$. Intratemporal price discrimination partly off-sets the inability to intertemporally price discriminate.

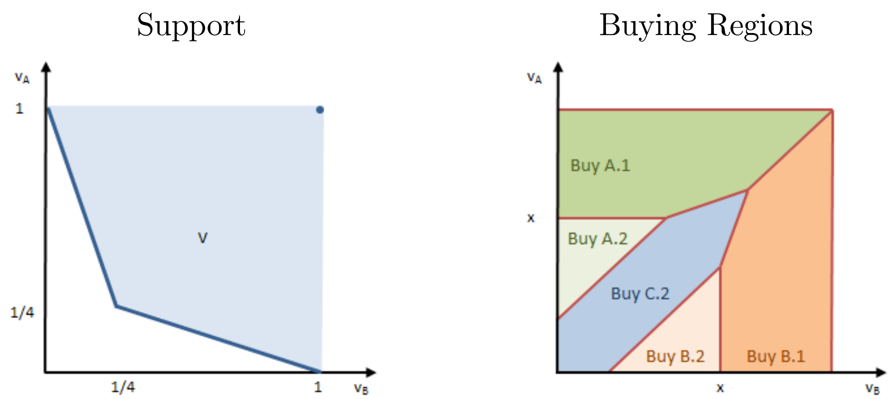

Figure 5: On the left, the support of a measure with late mixing; on the right, values partitioned into buying regions based on date of purchase and variety.

When $\delta = 3/4$, the stationary equilibrium that maximizes the payoff of the seller in the dynamic pricing game entails stochastically clearing the market in exactly two periods. In this equilibrium, the monopolist sells both varieties in the first period at a price equal to $311/864$, and clears the market in the second period by setting one of two market-clearing price profiles, $(73/216, 143/648)$ and $(143/648, 73/216)$, with equal probability. By doing so, the seller secures a profit of 0.292 (approximately). The right panel in Figure 5 partitions values into buying regions for a mixed strategy equilibrium in which the market clears in two periods. In each of these regions, the letter stands for the variety purchased (where c denotes the cheapest variety) and the number stands for the date in which the variety is purchased.

Mixed strategy equilibrium profits exceed the profit that the seller can secure in any stationary pure strategy equilibrium which only amounts to 0.288 (approximately). The loss in profit stems from the following intuition. When $\delta = 3/4$, in all stationary equilibria the market clears in two periods and buyers with value $(1,1)$ necessarily purchase in the first period. But if so, buyers with value $(1,1)$ are unwilling to pay much more than the lowest market-clearing price set in the second period if the equilibrium is pure, while they are unwilling to pay much more than the average market-clearing price if the equilibrium is mixed. For suitably chosen values of δ, this effect depresses the price that can be charged on the cheaper variety in the first period in a pure strategy equilibrium, and implies that clearing the market stochastically benefits the seller. The details of the example are reported in the online appendix.

The final result on this dynamic pricing game delivers a generalization of the classical Coasian insight to multi-variety settings. In any stationary equilibrium, the seller's profit must always converge to the optimal market-clearing profit as the discount factor converges to unity. As in the single variety setting, patience deteriorates the seller's bargaining power and decreases its equilibrium profit. Because of Lemma 2 though, the inability to intertemporally price discriminate buyers does not fully erode the seller's bargaining power when more than one variety can be sold.

Proposition 3 *If the market is regular, profits converge to optimal market-clearing profits in any weak Markov equilibrium as δ converges to* 1.

The proof establishes that, when the discount factor is close to 1, prices must be close to market-clearing after any real time T. Thus, profits will be close to market-clearing as patient consumers would wait any finite amount of time for a price reduction, and consequently varieties will only ever be sold at prices that are close to market clearing. The intuition for this result is as follows. Consider a time period t in which the demand for both products is small. A possible deviation for the monopolist in period t consists of setting prices according to its mixed strategy in period $t+1$ rather than setting the equilibrium price p^t. Such a deviation would have three effects on the profit of the monopolist. Firstly, it would reduce profit by lowering the price paid by those who were expecting to consume a variety i at date t and continue to do so. Secondly, it would increase profits by anticipating the stream of future revenue on all units to be sold at later stages. Thirdly, it would have an ambiguous effect on profits by inducing some consumers to change their demand from one to the other product. The first effect, however, is small as price changes must be small if a patient consumer is unwilling to wait one period to purchase the product. Similarly, the third effect must be small (if positive), because the set of buyers contemplating to switch varieties is a small subset of those contemplating to purchase when price changes are small (by absolute continuity). Thus, for such a deviation not to be profitable profits must be arbitrarily small after a finite time T. If so, prices must be close to market clearing after date T given that PBE profits exceed market-clearing profits by Lemma 2 and given that optimal market-clearing profits can be small only if the measure of active buyers is small by Proposition 1. If buyers are patient, the latter implies that prices must be close to market-clearing from the beginning of the game for sales to take place before date T.

As in classical settings, a monopolist lacking commitment extracts no more than optimal market-clearing profits in any stationary equilibrium when price revisions can be arbitrarily frequent. But, stationary pricing without commitment only amounts to optimal market-clearing and global agreement, and not to minimal pricing or efficiency.

5 Extensions: Costs and Market Exit

Positive Marginal Costs: The key insights discussed in the previous sections carry over to settings with strictly positive marginal costs. But, a few significant differences arise. In general, our notion of market clearing only required that gains from trade be depleted. Consequently, market clearing no longer requires selling to all buyers when marginal costs are positive. But rather, it requires selling a variety at the current prices to all those buyers who value at least one variety more than its marginal cost. In particular, denote by V^+ the set of values with positive gains from trade,

$$V^+ = \left\{ v \in V \mid \max_{i \in \{a,b\}} \{v_i - c_i\} \geq 0 \right\}.$$

When marginal costs are positive, the set of market-clearing prices then amounts to

$$M^+ = \left\{ p \in \mathbb{R}^2 \mid \max_{i \in \{a,b\}} \{v_i - p_i\} \geq 0 \text{ for all } v \in V^+ \right\}.$$

A price p in the interior of the support V can now clear the market, but no price in the interior of V^+ does so. As displayed in Figure 6, any price $p \leq c$ clears the market even when it belongs to the interior of V. Market-clearing prices (even optimal ones) may display *cross-subsidization* which amounts to selling one variety below marginal costs while the other above marginal cost. Figure 6 depicts such an instance.

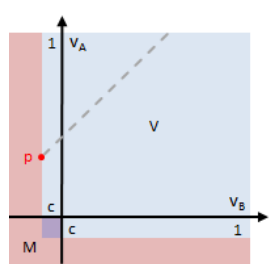

Figure 6: With costs, market-clearing prices may belong to the support (in purple), and cross-subsidization may be optimal market-clearing.

When costs are positive, optimal market-clearing profits no longer need to be strictly positive. When the minimal valuation of each product within V^+ is much below its marginal cost, it may be hard to clear the market at a positive profit. The next remark provides a simple sufficient condition for optimal market-clearing profits to be strictly positive. As before, let $\underline{v}_i^+$ denote the *minimal value* for variety i in the support V^+. With costs, we say that *varieties are ranked* if they are not identical and all buyers prefer one variety (that is, if $v_i - c_i \geq v_j - c_j$ for some i for any $v \in V$). We say that *varieties are unranked* if they are not ranked (that is, if for any i there is $v \in V$ such that $v_i - c_i > v_j - c_j$).

Remark 2 *Optimal market-clearing profits are strictly positive if varieties are unranked and $\underline{v}_i^+ \geq c_i$ for some variety $i \in \{a, b\}$.*

The remark intuitively holds as it is always possible to clear the market and make positive profits by setting prices $p_i = c_i$ and $p_j > c_j$ whenever varieties are not ranked.

Lemma 2 immediately extends to settings with positive marginal costs as its proof did not impose much discipline on the seller's preferences. Thus, even with positive costs, equilibrium profits remain bounded below by optimal market-clearing profits. Furthermore, as in Proposition 3, stationary equilibrium profits still converge to optimal market-clearing profits as δ converges to 1. As before when δ is close to 1, the measure of active buyers will be arbitrarily small after any finite time T (for the seller not to profitably deviate by selling units sooner), and profits will not exceed by much the static optimal market-clearing profits (when buyers are patient). We summarized the two key Coasian observations in the following remark.

Remark 3 *If the market is regular, optimal market-clearing profits:*
(1) are a lower bound on perfect Bayesian equilibrium profits;
(2) coincide with the limit of weak Markov equilibrium profits as δ converges to 1.

Relaxing the Permanent Exit Assumption: It may seem that our interpretation of Coase's seminal result as market-clearing relies on the assumption (implicit in some of the literature) requiring buyers to permanently exit the market upon purchasing a variety. Such an assumption is without loss: (i) if every buyer purchases its preferred variety; (ii) if goods are consumed when purchased thereby dissipating their need forever; or (iii) if players commit to stay out of the market upon purchasing the good. The first scenario is not so uncommon when the measure is symmetric (for instance, for discordant symmetric distributions). In these circumstances, limit pricing in the baseline model may be efficient and clear the market while strictly exceeding minimal values. The second scenario is compelling for goods that are durable, but that are consumed once purchased (such as many services). After all, in these models durability simply amount to sales permanently depleting the demand for the good. In other markets however, it may be more plausible to assume that buyers remain active in the market until they purchase their preferred variety. If so, they may be able to scrap the variety they have purchased at an earlier round when a preferred variety is sufficiently cheap.

To analyze this setting, we postulate that when a buyer $v \in V$ purchases any variety $i \in \{a, b\}$ of the durable good then their value for each varieties transitions to

$$v'_i = 0 \text{ and } v'_j = v_j - v_i.$$

Thus upon purchasing a variety, the value of that variety fully depletes, whereas the value for the other variety amounts to the difference in values between the two. The latter is natural as the change in value from scrapping variety i to purchase j amounts to $v_j - v_i$.

As pointed out in the section on costs, our notion of market-clearing simply amounts to full depletion of gains from trade. Applying this definition to settings in which buyers remain active upon purchasing a variety, changes the shape of the market clearing set as follows

$$M^* = \left\{p \in M \mid v_i - p_i \geq v_j - p_j \;\Rightarrow\; c_j \geq v_j - v_i \text{ for all } v \in V^+\right\}.$$

The definition states that a price clears the market if: (i) every buyer purchases a variety; (ii) if the marginal cost of supplying variety j to any buyer purchasing variety i exceeds their change in value. Therefore, the market must clear whenever the change in price is smaller than the cost of every variety,

$$M^* \supseteq \left\{p \in M^+ \mid -c_b \leq p_a - p_b \leq c_a\right\}.$$

Moreover, the latter must hold with equality whenever $c_i \leq \max_{v \in V^+} v_i - v_j$ for every variety i. Market clearing prices will thus be efficient (as $p_a = p_b$) when the marginal cost of each product equals zero, but not otherwise. As before, provided that $v \geq c$ for all values in the support V, optimal market-clearing profits equal zero if and only if products are identical and there are no gaps ($c \in V$). More generally, as in the previous part of the section, optimal market-clearing profits are strictly positive if varieties are unranked and $\underline{v}_i^+ \geq c_i$ for some variety $i \in \{a, b\}$.

As shown in the online appendix, Lemma 2 also extends to this setting. Equilibrium profits remain bounded below by static optimal market-clearing profits as the argument establishing 2 readily applies to all the prices in M^*. As was the case with permanent exit, stationary equilibrium limit profits remain uniquely pinned down by static optimal market-clearing profits. The intuition is again similar to that of Proposition 3 and relies on the measure of switchers remaining small when players are patient and prices are close to market clearing. We summarize these conclusions in the following remark proven in the online appendix.

Remark 4 *If buyers remain active and the market is regular, optimal market-clearing profits:*
(1) are a lower bound on perfect Bayesian equilibrium profits;
(2) coincide with the limit of weak Markov equilibrium profits as δ converges to 1.

When marginal costs equal zero and players remain active, limiting stationary equilibria are efficient as all buyers eventually purchase the preferred variety when prices belong to M^* (this is the case in the top three panels in Figure 8). In such efficient limiting stationary equilibria, varieties are not necessarily sold at their minimal value, but rather at the minimal value of the durable good. For instance, with discordant valuations, there can be scenarios in which pricing is efficient and in which every variety is sold above its minimal value (see the top right panel in Figure 8).

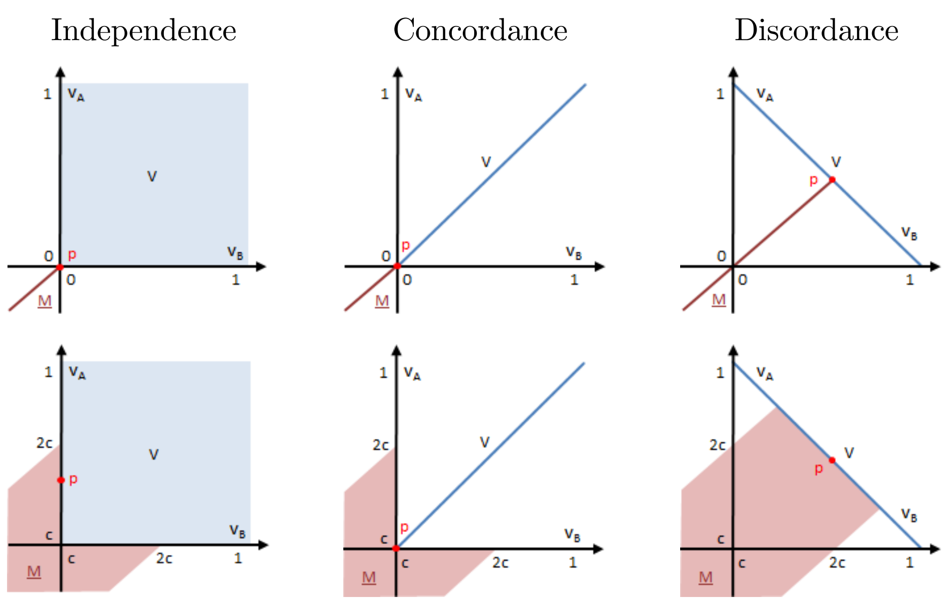

Figure 8: Market-clearing set without exit $\underline{M}$ for three possible distributions.

When marginal costs are positive though, results are closer to the baseline setting in which buyers committed to exit. If so, the seller retains its ability to intratemporally price discriminate and efficiency seldom obtains as marginal costs prevent the seller from undercutting to supply buyers who were sold the inefficient variety for sake of price discrimination (this is the case in the bottom three panels in Figure 8). This logic offers a novel rationale for selling high marginal costs varieties, namely intratemporal price discrimination in durable goods markets.

6 Classical Results and Coasian Failures

Relationship to Classical Coasian Results: The paper discussed a dynamic monopoly problem in which multiple varieties of a product were produced and sold. It extended classical conclusions on equilibrium pricing and established that, in any robust generalization of Coasian results, static optimal market-clearing would play a role similar to that of the minimal valuation in the single-variety case. This insight considerably simplified the analysis of the dynamic game and enabled meaningful generalizations of classical Coasian dynamics. With more than one variety, intratemporal price discrimination was shown to make up at least in part for absence of intertemporal price discrimination caused by the lack of commitment. Although the paper was presented for two varieties analogous conclusions would obtain with more than two varieties. The table below summarizes classical contributions on dynamic monopoly pricing with one variety, and highlights which conclusions are specific to this scenario and which generalize to multi-dimensional settings.

Number of Varieties	Single		Multiple		
OMC Profit	0	+	0	+	
Gaps	No	Yes	No	No	Yes
Market Clearing	Yes	Yes	Yes	Yes	Yes
Bound on PBE Profit	OMC	OMC	OMC	OMC	OMC
Limit WME Profit	OMC	OMC	OMC	OMC	OMC
Time to Clear	Infinite	Finite	Infinite	Infinte	Finite
Efficiency	Yes	Yes	$\exists$WME	Rare	Rare
Minimal Value Pricing	WME	Yes	$\exists$WME	No	No
PBE Late Mixing	No	No	–	Yes	Yes
PBE Uniqueness	No	Yes	No	No	No

We would like to argue that the three consistent phenomena across Coasian settings are: eventual market-clearing; optimal market-clearing providing a lower-bound on equilibrium profits; and optimal market-clearing identifying stationary equilibrium profits. Thus, one could consider these three aspects as the essence of the Coase conjecture. Other phenomena, instead, are not robust and depend on specific assumptions invoked on the durable good environment. These include: the time it takes for the market to clear; the efficiency of equilibrium pricing; whether goods are eventually sold at minimal valuations; whether mixing can take place after the initial period; and equilibrium uniqueness.

Multiplicity of perfect Bayesian equilibria contrasts with the uniqueness result obtained in the single-variety case with gaps. Multiplicity naturally arises when more then one variety is for sale as optimal market-clearing prices need not be unique. This observation alone does not imply that a Folk Theorem holds as in the Ausubel and Deneckere 1989. Their seminal contribution establishes that with one variety and no-gaps a Folk theorem obtains. In such settings, the seller can extract the full static monopoly surplus by following a strategy with a slow price descent. Such a strategy is incentive compatible for the seller if consumers' beliefs about future prices revert to the stationary equilibrium path upon observing any deviation. However, for the slow price descent to be incentive compatible, the stationary equilibrium limit profit must equal 0 and thus there must be no gaps. In multi-variety settings, it unclear the no-gaps assumption would suffice to deliver a full-fledged Folk theorem as equilibrium profits may be strictly positive even when there are no gaps.[18] If a Folk theorem was to hold, our analysis would identify the lowest perfect Bayesian equilibrium profit and the stationary limit payoff.

As usual, it is possible to interpret our setting as a two-player model of bargaining with one-sided incomplete information in which the uninformed party always proposes. In this interpretation varieties amount to alternative prospects that the proposer can offer to the receiver to screen their type. If so, our conclusions establish that the uninformed party regains some bargaining power by statically screening consumers as it can extract surplus even if it has to agree with every possible type of the informed player. Our bargaining interpretation of the Coase conjecture would then amount to immediate agreement in limiting stationary equilibria and would essentially coincide with optimal market-clearing. If players were to stay in the market upon purchasing product, agreement would have to be renegotiation-proof to guarantee that no player would not want to switch varieties at any price exceeding marginal cost.

Approximating stationary equilibrium profits (with frequent price revisions) with optimal market-clearing may not just amount to a theoretical curiosity. Instead, such an approximation could in principle deliver a concrete stepping-stone to inform applied research durable good pricing and to develop product design implications for such markets.

Relationship to Some Coasian Failures: The analysis is closely related to some known violations of the classical Coase conjecture. Board and Pycia 2014 considers a durable good monopoly problem in which buyers have the option to commit to stay out of the market by taking an outside option. The outside option amounts to a second variety of the durable good

[18] We defer the full-blown analysis of non-stationary equilibria to future work.

that must be sold at a price of zero. They consider settings in which the value of the outside option is strictly positive for all players (there are gaps) and independent of the value of the durable good (the left plot of Figure 7 depicts such an environment). Their main contribution establishes that the monopolist sets a strictly positive price for the durable good, and never undercuts on the initial price as the market clears at once. In our setting, their result holds immediately by Lemma 2. As the price of the outside option is zero, any price for the durable good is a market-clearing price. Thus, the monopolist would never undercut. Furthermore, this holds even without gaps and even with an arbitrary correlation structure. Of course, setting the price of the outside option to zero would be suboptimal in our environment, as any such price profile would belong to the interior of the market-clearing price set. Still in our view, Board and Pycia's novel contribution should not be classified as a failure of the Coase conjecture. Rather our results aim to highlight that the essence of the Coasian intuition is market-clearing, and not necessarily minimal pricing, zero profits, or efficiency. Similar considerations apply to Wang 1998 who establishes a result evocative of Board and Pycia in the context of a two type model.

Likewise, Hahn 2006 expands on classical conclusions by showing that selling damaged products can increase the profit of a durable good monopolist. A damaged product acts like a second variety with a lower value. In particular, their analysis considers settings in which the valuations of the two varieties are binary and perfectly correlated (the right plot of Figure 7 depicts such an environment). Similar conclusions obtain in our setting independently of the joint measure of valuations. However, these are again no failure of the Coase conjecture, but rather its essence, as limit profits again amount to optimal market-clearing profits in our formulation of the problem. Other similarly classified Coasian failures fit this bill.

(1) Outside Options (2) Vertical Differentiation

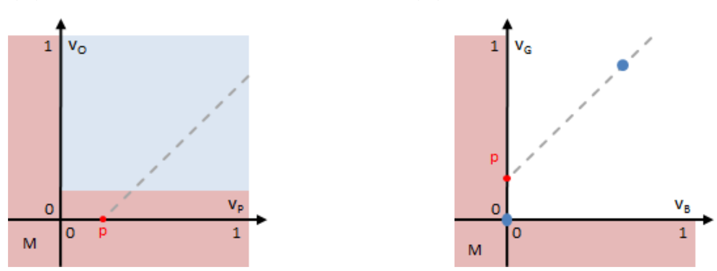

Figure 7: The left plot depicts the environment studied in Board and Pycia 2014, the right plot depicts alleged Coasian failures with vertical differentiation.

Our conclusions hold even when no buyer values one of the two varieties. In essence, if the monopolist could pay a penny (or any small amount) to buyers for them to permanently exit the market, the Coasian profit would no longer amount to the smallest valuation in the support. Instead, the monopolist would be able to approximately extract the full static monopoly profit, as any price would again clear the market.

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7 Appendix

Proof Proposition 1. (1) Consider the price $p^g = (\underline{v}_g, \underline{v}_g)$. Such a price belongs to M, as for all $v \in V$

$$\max_{i\in\{a,b\}}\{v_i - p_i^g\} \geq \max_{i\in\{a,b\}} v_i - \underline{v}_g \geq 0.$$

By setting price p^g, the monopolist obviously achieves a profit of $\underline{v}_g$. Thus, when $\underline{v}_a < \underline{v}_b$, optimal market-clearing profits weakly exceed $\underline{v}_b$ and strictly exceed $\underline{v}_a$. (2) If varieties are unranked, further consider the MC price profile $p^\varepsilon = (\underline{v}_b + \varepsilon, \underline{v}_b)$ for some small number $\varepsilon > 0$. By the definition of the static demand, $\lim_{\varepsilon\to 0} d_i(p^\varepsilon) \geq \mathcal{F}(v_i > v_j)$ for any $i \in \{a, b\}$. As products are unranked, $\mathcal{F}(v_i > v_j) > 0$. But if so, optimal MC profits must strictly exceed $\underline{v}_b$ since $d_a(p^\varepsilon) > 0$ for ε sufficiently small.

(3) If $\underline{v}_a = \underline{v}_b = \underline{v}$, again consider the market-clearing price profile $p^\varepsilon = (\underline{v}+\varepsilon, \underline{v})$. Again, it must be that $\lim_{\varepsilon\to 0} d_i(p^\varepsilon) \geq \mathcal{F}(v_i > v_j)$ for any $i \in \{a,b\}$. If at p^ε some consumers were to purchase variety a, optimal market-clearing profits would strictly exceed $\underline{v}$. If instead optimal market-clearing profits were equal to $\underline{v}$ for any $\varepsilon > 0$, then $\lim_{\varepsilon\to 0} d_a(p^\varepsilon) = \mathcal{F}(v_a > v_b) = 0$, and thus $v_a \leq v_b$ for any $v \in V$. A symmetric argument would then establish that $v_b \leq v_a$ for any $v \in V$. So, OMC profits could be equal to $\underline{v}$, only if $v_a = v_b$ for any $v \in V$. Similarly, if varieties were identical, any market-clearing price would set one of the two prices to $\underline{v}$ as $(\underline{v}, \underline{v}) \in V$. (4) But if so, optimal market-clearing profits would amount $\underline{v}$, as all players would purchase the cheapest variety. Thus profits equal 0 if and only if varieties are identical and $\underline{v} = 0$.

(5) If $\underline{v}_a = \underline{v}_b = \underline{v}$ and $(\underline{v}, \underline{v}) \in V$, then $\underline{v}_g = \underline{v}$. If varieties are not identical, it must be that $\mathcal{F}(v_i > v_j) > 0$ for some $i \in \{a,b\}$. If so, the price $(p_i, p_j) = (\underline{v}+\varepsilon, \underline{v}) \in M$ would raise more profits than $\underline{v}_g$ for ε sufficiently small. ■

Proof Lemma 1. Consider any PBE and any buyer-history $h^t \in \hat{H}^t$. To establish (1), observe that, since v strictly prefers buying variety i,

$$v_i - p_i > \max\{v_j - p_j, \delta U(h^t, v)\}.$$

where $U(h^t, v)$ denotes equilibrium continuation value of player v at date $t+1$ after history h^t. As buyer v can mimic the strategy of buyer v' from period $t+1$ onwards (by accepting and rejecting the very same offers) and since $v'_j - v_j < v'_i - v_i$, it follows that

$$U(h^t, v') - U(h^t, v) \leq \textstyle\sum_{s=0}^{\infty} \delta^s \left[\sum_{k\in\{a,b\}} \alpha_k^s(h^t, v')\,(v'_k - v_k)\right] \leq v'_i - v_i,$$

where $\alpha_j^s(h^t, v')$ denotes the probability conditional on h^t that variety j is purchased by v' at time $t+s+1$. But if so, buyer v' strictly prefers buying variety i and part (1) follows since

$$\begin{aligned} v'_i - p_i = v_i - p_i + (v'_i - v_i) &> \max\{v_j - p_j, \delta U(h^t, v)\} + (v'_i - v_i) \\ &\geq \max\{v'_j - p_j, \delta U(h^t, v')\}. \end{aligned}$$

To prove (2) similarly observe that, since buyer v weakly prefers to buy a variety,

$$\max_{i\in\{a,b\}}\{v_i - p_i\} \geq \delta U(h^t, v).$$

As buyer v can mimic the strategy of buyer v' from period $t+1$ onwards, it follows that

$$U(h^t, v') - U(h^t, v) \leq \max_{i\in\{a,b\}} \{v'_i - v_i\}.$$

But if $\delta \max_i\{v_i - v_i'\} < \min_i\{v_i - v_i'\}$, buyer v' strictly prefers buying a variety since

$$\begin{gathered}\max_i\{v_i' - p_i\} \geq \max_i\{v_i - p_i\} + \min_i\{v_i' - v_i\} \\ > \delta U(h^t, v) + \delta \max_{i\in\{a,b\}}\{v_i' - v_i\} \geq \delta U(h^t, v').\end{gathered}$$

Similarly, to prove (3) observe that, since buyer v weakly prefers not to buy any variety,

$$\max_{i\in\{a,b\}}\{v_i - p_i\} \leq \delta U(h^t, v),$$

As buyer v' can mimic the strategy of buyer v from period $t+1$ onwards, it follows that

$$U(h^t, v) - U(h^t, v') \leq \max_{i\in\{a,b\}}\{v_i - v_i'\}.$$

But if so, buyer v' strictly prefers not buying any variety

$$\begin{gathered}\max_i\{v_i' - p_i\} \leq \max_i\{v_i - p_i\} - \min_i\{v_i - v_i'\} \\ < \delta U(h^t, v) - \delta \max_i\{v_i - v_i'\} \leq \delta U(h^t, v').\end{gathered}$$

Also, note that (2) immediately implies (4) because on any ray $v_a = v_b + k$ there exists a cut-off valuation identifying the marginal buyer and because indifferent consumer have measure zero when the market is regular. ■

Proof Lemma 2. To prove the result it suffices to show that in any PBE all consumers accept any price in $\bar{M}$ at any information set. Suppose this were not the case. Select any equilibrium, let P denote the set of prices that will be accepted by all buyers in any possible subgame,

$$P = \left\{ p \in \mathbb{R}^2 \;\middle|\; \max_{i\in\{a,b\}}\{v_i - p_i\} > \delta U((p,h), v) \text{ for all } (h,v) \in H \times V \right\}$$

By contradiction suppose that $\bar{M}$ is not contained in P, that is $\bar{M}\backslash P \neq \emptyset$.

Observe that $p \in P$ whenever $\min_{i\in\{a,b\}} p_i < -1$. To show the latter observe that the proof of Lemma 1 implies that the buyers' value functions at any buyer-history $\hat{h} \in \hat{H}$ are non-decreasing in v and have modulus of continuity less than 1 since for $v' \geq v$

$$U(\hat{h}, v') - U(\hat{h}, v) \leq \max_{i\in\{a,b\}}\{v_i' - v_i\}.$$

But then, in any PBE we have that $U(\hat{h}, v) \leq 1$ for all $v \in V$. In turn, this implies that all buyers strictly prefer to purchase a variety of the durable good when $\min_{i\in\{a,b\}} p_i < -1$, as

$$\max_{i\in\{a,b\}} \{v_i - p_i\} > 1 > \delta U(\hat{h}, v) \ \text{ for all } \ \hat{h} \in \hat{H}. \tag{2}$$

As $\min_{i\in\{a,b\}} p_i < -1$ implies $p \in P$, for any $\varepsilon > 0$ there is a price $\hat{p} \in \bar{M} \backslash P \neq \emptyset$ such that:

$$\begin{array}{ll} \text{(i)} & p \leq \hat{p} - (\varepsilon, 0) \ \text{ implies } \ p \in P; \\ \text{(ii)} & p \leq \hat{p} - (0, \varepsilon) \ \text{ implies } \ p \in P. \end{array}$$

To find such a price $\hat{p}$, let $\tilde{p}_a = \inf_{q\in\bar{M}\backslash P} q_a$ and for some $\eta \in (0, \varepsilon)$ let

$$\tilde{p}_b = \inf_{q\in\bar{M}\backslash P} q_b \ \text{ s.t. } \ q_a \leq \tilde{p}_a + \eta,$$

where $\min_{i\in\{a,b\}} \tilde{p}_i \geq -1$ by (2). Then set a $\hat{p}$ to be any price in $\bar{M}\backslash P$ such that $\hat{p} \leq \tilde{p} + (\eta, \eta)$. Such a price must exist by definition of $\tilde{p}$ for all sufficiently small η. Moreover, (i) holds as $p \in P$ whenever $p_a \leq \hat{p}_a - \varepsilon < \tilde{p}_a$ by definition of $\tilde{p}_a$; while (ii) holds as $p \in P$ for any $p_a \leq \hat{p}_a$ when $p_b \leq \hat{p}_b - \varepsilon$ by definition of $\tilde{p}_b$.

But, when ε is sufficiently small,

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i\} > \delta \max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \quad \Leftrightarrow \quad \varepsilon < \frac{1-\delta}{\delta} \max_{i\in\{a,b\}} \{v_i - \hat{p}_i\}.$$

If so, all consumers would accept $\hat{p}$ at any seller-history $h \in H$. If a type was to reject an offer, they could agree no sooner than tomorrow, and the most they could expect any one price to drop is ε as any further drop would lead to acceptance by all buyers. Thus, for all $v \in V$

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \geq U((\hat{p}, h), v).$$

But in turn this would imply that

$$\max_{i\in\{a,b\}} \{v_i - \hat{p}_i\} > \delta \max_{i\in\{a,b\}} \{v_i - \hat{p}_i + \varepsilon\} \geq \delta U((\hat{p}, h), v) \ \text{ for any } \ h \in H.$$

As $\hat{p} \notin P$, the latter however contradicts the definition of P, and consequently establishes (1) and (2). As every consumer buys when prices belong to $\bar{M}$, the seller can secure a payoff arbitrarily close to the optimal market-clearing profits $\bar{\pi}(A) > 0$ (where $A = A(h)$ denotes the active player set associated with history h) by choosing a price in $\bar{M}$. Part (3) then follows. ■

Proof Lemma 3. To prove (1) fix a PBE. Let $A^t = A(h^t)$ denote the support of the measure of active players associated to a history $h^t \in H$ of length t. Suppose that there exists a history $h^s \in H$ with $\mathcal{F}(A^s) > 0$ and such that at any date $t > s$ for any active player set A^t that may arise with positive probability as a result of equilibrium play after history h^s we have that

$$\mathcal{F}(A^s) - \mathcal{F}(A^t) < \eta.$$

At such a history, the equilibrium profit of the seller must then be bounded by

$$\Pi(h^s) < \eta,$$

as no variety is ever sold at price higher than 1 (the highest value in the initial support). However, when the market is regular and η is sufficiently small,

$$\bar{\pi}(A^s) > \eta,$$

as optimal market clearing profits are strictly positive whenever $\mathcal{F}(A^s) > 0$. If so, a contradiction would emerge for a sufficiently small η as the seller would prefer to immediately clear the market,

$$\Pi(h^s) < \eta < \bar{\pi}(A^s).$$

Thus, for η sufficiently small, at any history h^s there always exists a continuation-history h^t that arises with positive probability from equilibrium play such that

$$\mathcal{F}(A^s) - \mathcal{F}(A^t) > \eta.$$

But then, for any infinitely long history $h^\infty \in H$ consistent with equilibrium play we have that

$$\mathcal{F}(A^\infty) = 0,$$

where $A^\infty = A(h^\infty)$.

To prove (2), begin with a few preliminary observations. For any sequence of sets $\{A^t\}_{t=0}^\infty$ satisfying $A^{t+1} \subseteq A^t$ and $A^t \subseteq V$ denote by A^∞ the limit of this sequence, $A^\infty = \cap_{t=0}^\infty A^t$. Consider any such sequence satisfying the following additional requirements:

(i) $\mathcal{F}(A^\infty) = 0$;

(ii) A^t satisfies Lemma 1 for any $t \geq 0$;

As the market is regular, condition (i) implies that $\mathcal{F}(A^\infty) = f\mathcal{L}(A^\infty)$ for some $f \in (\underline{f}, \overline{f})$ and thus that $\mathcal{L}(A^\infty) = 0$. Condition (ii) implies that $v' \geq v$ for any $v \in A^\infty$ and any $v' \in V$ such that

$$v_a - v_b = v'_a - v'_b.$$

If $v' < v$, then $v' \in A^\infty$ by part (3) of Lemma 1. But, this would lead to a contradiction when the market is regular since $\mathcal{L}(A^\infty) > 0$ given that all values in a neighborhood of v' would also remain active as $v' \in V$.

Moreover, A^∞ must be an increasing set (formally, $v'_i > v_i$ implies $v'_j \geq v_j$ for any $v, v' \in A^\infty$). If this were not the case, the buyer

$$v'' = (\max\{v_a, v'_a\}, \max\{v_b, v'_b\})$$

would also prefer not to buy any variety by part (1) of Lemma 1 (formally, $v'' \in V \;\; \Rightarrow v'' \in A^\infty$). But if so, by part (3) of the Lemma 1, any buyer $\hat{v} \in V$ such that $\hat{v} < v''$ would strictly prefer not to buy (formally $\hat{v} \in A^\infty$) provided that

$$\delta \max_{i\in\{a,b\}}\{v''_i - \hat{v}_i\} < \min_{i\in\{a,b\}}\{v''_i - \hat{v}_i\}. \tag{3}$$

But, when the market is regular, V is convex and hence

$$v(\kappa) = \kappa v + (1-\kappa)v' \in V \text{ for all } \kappa \in (0,1)\,.$$

But for some value $\hat{\kappa}$

$$v''_a - v''_b = v_a(\hat{\kappa}) - v_b(\hat{\kappa}),$$

and therefore $v(\hat{\kappa}) \in A^\infty$. But, this would again lead to a contradiction since $\mathcal{L}(A^\infty) > 0$ given that a positive measure of buyers would fulfill (3) if $v(\hat{\kappa}) \in V$.

Not only is A^∞ an increasing set, but there exists a variety $i \in \{a,b\}$ such that $v_i = \underline{v}_i(A^\infty)$ for all $v \in A^\infty$. If at some history $h \in \hat{H}$ a buyer v expects not to buy any variety within the next $T+1$ periods

$$U(v|h) \leq \delta^T,$$

as the buyer-surplus is bounded above by 1 given that $p^t \notin \bar{M}$ for all t. If so, $U(v'|h) < \delta^T$ for all $v' < v$ as continuation values are monotone by the proof of Lemma 1. Consider any buyer-history $h \in \hat{H}$ associated to an active player set A at which the buyer $v' = \underline{v}(A)+(\eta,\eta)$ strictly prefers to purchase a variety

$$\max_i\{v'_i(A) - p_i\} = \max_i\{\underline{v}_i(A) - p_i\} + \eta > \delta U(v'|h) \geq 0.$$

Such a history always exists when the market is regular as A^∞ is an increasing set with Lesbegue measure zero (which must be the case if the market eventually clears by the previous part of the argument). If so, any buyer with value v such that $\min_i\{v_i - \underline{v}_i(A)\} > 2\eta$ must buy within $[\log\eta/\log\delta]+1$ periods as a contradiction would emerge otherwise as the buyer v

would prefer to buy immediately. If v was not expecting to purchase within $[\log \eta / \log \delta] + 1$,

$$\begin{aligned} \max_i \{v_i - p_i\} &\geq \max_i \{\underline{v}_i(A) - p_i\} + \min_i \{v_i - \underline{v}_i(A)\} \\ &> \max_i \{\underline{v}_i(A) - p_i\} + 2\eta \geq \eta > \delta\eta > \delta U(v|h), \end{aligned}$$

where: the first inequality holds by construction as $v > \underline{v}(A)$; the second holds because $\max_i \{\underline{v}_i(A) - p_i\} \geq -\eta$; the third holds as $\delta < 1$; and the fourth holds as $T > [\log \eta / \log \delta]$ implies $\eta > \delta^T \geq U(v|h)$. The latter argument establishes that if buyers arbitrarily close $\underline{v}(A)$ purchase at some history h, all buyers with value $v > \underline{v}(A)$ must eventually purchase a variety as well. Hence, there exists $i \in \{a, b\}$ such that $v_i = \underline{v}_i(A^\infty)$ for all $v \in A^\infty$.

For any set $A \subseteq V$ define $\underline{v}_g(A) = \min_{v \in A} v_g$ and $\overline{v}_g(A) = \max_{v \in A} v_g$. Immediately observe that $\underline{v}_g > 0$ implies $\underline{v}_g(A) > 0$ for any $A \subseteq V$. Fix an equilibrium. Consider any infinitely long history h^∞ consistent with equilibrium play. Recall that $A^\infty = A(h^\infty)$. Next we establish that $\underline{v}_g > 0$ implies $\overline{v}_g(A^\infty) = \underline{v}_g(A^\infty)$. If this was not the case, the previous arguments then imply that

$$\overline{v}_j(A^\infty) = \overline{v}_g(A^\infty) > \underline{v}_g(A^\infty) = \underline{v}_i(A^\infty).$$

But, the active player set $A^t = A(h^t)$ at any sub-history h^t of h^∞ converges to A^∞. Thus, for any $\varepsilon > 0$ there must exist t such that:

(a) $|\overline{v}_j(A^t) - \overline{v}_g(A^\infty)| < \varepsilon$;

(b) $|\overline{v}_i(A^t) - \underline{v}_i(A^t)| < \varepsilon$;

(c) $1 - F_j(\overline{v}_g(A^\infty)|A^t) < \varepsilon$.

If so however, a contradiction would emerge for ε sufficiently small since

$$\begin{aligned} \Pi\left(h^t\right)/\mathcal{F}(A^t) &\leq (1 - F_j(\overline{v}_g(A^\infty)|A^t))\overline{v}_j(A^t) + F_j(\overline{v}_g(A^\infty)|A^t)\overline{v}_i(A^t) \\ &\leq \varepsilon\overline{v}_j(A^t) + (1-\varepsilon)\overline{v}_i(A^t) \leq \varepsilon\overline{v}_g(A^\infty) + (1-\varepsilon)\underline{v}_i(A^t) + \varepsilon \\ &< \max_{p_j}(1 - F_j(p_j|A^t))p_j + F_j(p_j|A^t)\underline{v}_i(A^t) \leq \overline{\pi}(A^t)/\mathcal{F}(A^t). \end{aligned}$$

The first inequality holds as the seller instantaneously sells all products at the highest possible value given that buyers with a value v such that $v_j < \overline{v}_j(A^\infty)$ never purchase variety j. The second and the third inequalities are immediate consequences respectively of (c) and of (a) and (b). The fourth relies on the fact that when ε is sufficiently small the optimal $p_j \in (\underline{v}_i, \overline{v}_g(A^\infty))$, as by choosing such a price it is possible to bound profits away from $\underline{v}_i(A^t)$. The final inequality is trivial and relies on the fact that we have checked for maximized profits only on a subset of M. Thus, for any $\varepsilon > 0$, there exists t such that $\overline{v}_g(A^t) - \underline{v}_g(A^t) < \varepsilon$.

To conclude lets establish that if $\overline{v}_g(A) - \underline{v}_g(A) \leq \varepsilon$ the seller prefers to immediately clear the market. Denote residual surplus given A by

$$S(h) = \int_A \max\{v_a, v_b\}\, dF(v|A) \leq \overline{v}_g(A)\mathcal{F}(A).$$

But if so there exists $\alpha \in (0,1)$ such that

$$\Pi(h) < \alpha S(h) + \delta(1-\alpha)S(h),$$

as the seller cannot instantaneously extract the full surplus with a linear price. Finally, recall that Proposition 1 implies that $\bar{\pi}(A) \geq \underline{v}_g(A)\mathcal{F}(A)$. But if so,

$$\begin{aligned}
\Pi(h) - \bar{\pi}(A) &< \alpha S(h) + \delta(1-\alpha)S(h) - \underline{v}_g(A)\mathcal{F}(A) \\
= (\alpha + \delta - \alpha\delta)S(h) - \underline{v}_g(A)\mathcal{F}(A) &\leq \mathcal{F}(A)\left[(\alpha+\delta-\alpha\delta)\overline{v}_g(A) - \underline{v}_g(A)\right] \\
= \mathcal{F}(A)\left[\gamma\overline{v}_g(A) - \underline{v}_g(A)\right] &\leq \mathcal{F}(A)\left[\gamma 2\varepsilon - (1-\gamma)\underline{v}_g(A)\right],
\end{aligned}$$

where $\gamma = \alpha + \delta - \alpha\delta \in (0,1)$ for all $\alpha, \delta < 1$. However, the latter cannot be positive whenever

$$\varepsilon \leq \underline{v}_g(A)\,(1-\gamma)\,/2\gamma.$$

But if so, at history h the seller prefers setting a price arbitrarily close to an optimal market-clearing price to setting any price outside M. Thus if $\underline{v}_g > 0$, there exists a period T such that a measure 1 of buyers purchases a variety of the durable good before date T in any PBE, because price discrimination gains must eventually be sufficiently small $\overline{v}_g(A^{T-1}) - \underline{v}_g(A^{T-1}) < \varepsilon$. ■

Proof Proposition 2. To begin, assume that there are gaps and $\underline{v}_g > 0$. If so, by Lemma 3 there exists a period T such that a measure $\mathcal{F}(V)$ of buyers purchases a variety of the durable good before date T in any PBE. For all values of $z \in \mathbb{N}$, we inductively construct a WME and prove its existence in a corresponding game in which the seller must forever set prices in M from some date z onwards. Then, we argue that this establishes existence of WME even in unrestricted games provided that $T < z$ as the market must clear in a finite time by Lemma 3. Finally, we establish that the proof also generalizes to the case in which $\underline{v}_g = 0$ by equicontinuity.

Let $\mathcal{K}(V) = \{A \subseteq V \mid A \text{ is non-empty and compact}\}$. Let $s \in \{0, ..., z\}$ denote the number of periods before prices must belong to M. When proving existence of WME in this setting we allow buyers' beliefs (and thus the seller's strategy) to depend on the time to market-clearing s. We then show that this is without loss as current prices fully pin down the time it takes for the market to clear. When $s = 0$, the seller must instantaneously set

$p \in M$. If so, denote the payoff of a buyer with value v by $U^0(p,v) = \max_i \{v_i - p_i\}$ for any $p \in M$. Similarly, for any mixed strategy of the seller $\rho \in \mathcal{P}(M)$, denote the expected payoff by

$$U^0(\rho, v) = \int_M \max_i \{v_i - p_i\} \, d\rho(p).$$

For any variety $i \in \{a, b\}$ and any $A \in \Omega(V)$, let

$$d_i^0(p|A) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, 0\}|A).$$

With a minor abuse of notation, let $pd^0(p|A) = p_a d_a^0(p|A) + p_b d_b^0(p|A)$. By Lemma 1, when the market is regular, the seller's beliefs about the measure of active buyers $\mathcal{A}$ are fully pinned down by the beliefs about the support of the measure A. Denote the best response of the seller who believes that only players in A are active when $s = 0$ by

$$B^0(A) = \arg\max_{\rho \in \mathcal{P}(M)} \int_M pd^0(p|A) d\rho(p).$$

Let $\Pi^0(A) = \bar{\pi}(A)$ denote the value of this program (or profit that the seller would make if the market had to clear and only buyers in A were active).

The best response correspondence $B^0(A)$ is upper-hemicontinuous[19] in A and has non-empty, compact, convex values by Berge's maximum theorem.[20] The theorem applies here because both $\mathcal{K}(V)$ and $\mathcal{P}(M)$ are Hausdorff, because the objective function is continuous in ρ and in A (as $d_i^0(p|A)$ is continuous in A by regularity), and because the solution belongs to $\mathcal{P}\left(M \cap [0,1]^2\right)$ which is non-empty and compact.[21] The convexity of the correspondence $B^0(A)$ follows by the linearity in ρ. For any $\rho \in \mathcal{P}(M)$ and any $p \in [0,1]^2$, let $\underline{A}^0(p, \rho)$ identify those valuations which prefer not to purchase a variety today at price p if in the following period $s = 0$ and prices are drawn from ρ,

$$\underline{A}^0(p, \rho) = \left\{ v \in V \mid \max_i \{v_i - p_i\} \leq \delta U^0(\rho, v) \right\}.$$

Next observe that for any $p \in [0,1]^2$, there exists $\sigma^0 \in \mathcal{P}\left(M \cap [0,1]^2\right)$ such that

$$\sigma^0 \in B^0(\underline{A}^0(p, \sigma^0)). \tag{4}$$

The latter follows because $\underline{A}^0(p, \rho)$ is continuous in ρ as $U^0(\rho, v)$ is linear and thus continuous in ρ. Moreover, $\underline{A}^0(p, \rho)$ is single-valued in space $\mathcal{K}(V)$ as a function of ρ, and is thus convex-

[19] More specifically, the best response correspondence is upper-hemicontinuous when Hausdorff metric is applied to its domain $\mathcal{K}(V)$.

[20] For the relevant statement of the Maximum Theorem see Aliprantis and Border 2006 page 570.

[21] $\mathcal{P}(M \cap [0,1]^2)$ is compact, because $M \cap [0,1]^2$ is compact. See Aliprantis and Border 2006 page 513.

valued as a function of ρ. Therefore, the correspondence $B^0(\underline{A}^0(p,\rho))$ has closed graph and convex values, since $B^0(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values. As $\mathcal{P}\left(M \cap [0,1]^2\right)$ is a non-empty, compact, convex subset of a locally convex Hausdorff space, the Kakutani-Fan-Glicksberg fixed point theorem applies.[22] Thus, equation 4 has a non-empty compact set of fixed points. For any initial price quoted by the seller p, these fixed points identify the WME strategies of the seller when $z = 1$ and $s = 0$. For any $p \in [0,1]^2$ label any one of such fixed points as $\sigma^0(p) \in \mathcal{P}(M)$.

Next by induction, suppose that an equilibrium exists in the game in which prices must belong to M after $s-1$ periods to show that an equilibrium exists when prices must belong to M after s periods. Denote the payoff of buyer v when prices must belong to M after $s \geq 1$ periods and the current prices equal p by

$$U^s(p,v) = \max\{\max_i \{v_i - p_i\}, \delta U^{s-1}(\sigma^{s-1}(p), v)\},$$

where $\sigma^{s-1}(p)$ denotes the seller's WME strategy in the subgame in which the market must clear in at most $s-1$ periods (which exists by the induction hypothesis). For any mixed strategy of the seller $\rho \in \mathcal{P}\left([0,1]^2\right)$, let $U^s(\rho, v)$ denote the expected continuation value,

$$U^s(\rho,v) = \int_{[0,1]^2} U^s(p,v) d\rho(p).$$

For any variety $i \in \{a,b\}$ and any $A \in \Omega(V)$, let

$$d_i^s(p|A) = \mathcal{F}(v_i - p_i > \max\{v_j - p_j, \delta U^{s-1}(\sigma^{s-1}(p), v)\}|A).$$

For any $\rho \in \mathcal{P}\left([0,1]^2\right)$ and any $p \in [0,1]^2$, let $\underline{A}^s(p,\rho)$ identify the subset of buyers who prefer not to purchase a variety today at price p if the market clears in $s-1$ periods and prices are drawn from ρ in the following period,

$$\underline{A}^s(p,\rho) = \{v \in V \mid \max_i \{v_i - p_i\} \leq \delta U^s(\rho, v)\}.$$

Denote the best response of the seller who believes that only players in A are active when the market clears in at most s periods by

$$B^s(A) = \arg\max_{\rho \in \mathcal{P}\left([0,1]^2\right)} \int_{[0,1]^2} p d^s(p|A) + \delta \Pi^{s-1}(\underline{A}^{s-1}(p, \sigma^{s-1}(p))) d\rho(p),$$

where $\Pi^s(A)$ denotes the value of this program. The best response correspondence $B^s(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values by Berge's maxi-

[22]For the statement of the relevant Fixed Point Theorem see Aliprantis and Border 2006 page 583.

mum theorem. The theorem applies here because both $\mathcal{K}(V)$ and $\mathcal{P}\left([0,1]^2\right)$ are Hausdorff. because the objective function is continuous in ρ and in A (as $d_i^s(p|A)$ is continuous in A by regularity), and because the solution belongs to $\mathcal{P}\left([0,1]^2\right)$ which is non-empty and compact. The convexity of the correspondence $B^s(A)$ follows again by the linearity in ρ. As before, for any p there exists $\sigma^s \in \mathcal{P}\left([0,1]^2\right)$ such that

$$\sigma^s \in B^s(\underline{A}^s(p, \sigma^s)). \tag{5}$$

The latter follows because $\underline{A}^s(p, \rho)$ is continuous ρ given that $U^s(\rho)$ is linear and thus continuous in ρ. Moreover, $\underline{A}^s(p, \rho)$ is single-valued in space $\mathcal{K}(V)$ as a function of ρ, and is thus convex-valued. Therefore, the correspondence $B^s(\underline{A}^s(p, \rho))$ has closed graph and convex values, since $B^s(A)$ is upper-hemicontinuous in A and has non-empty, compact, convex values. As $\mathcal{P}\left([0,1]^2\right)$ is a non-empty, compact, convex subset of a locally convex Hausdorff space, the Kakutani-Fan-Glicksberg fixed point theorem again applies. Thus, equation 5 has a non-empty compact set of fixed points. For any p, label any one of such fixed points as $\sigma^s(p) \in \mathcal{P}([0,1]^2)$. These fixed points identify the on-path WME pricing strategies of the seller after price p has been quoted when the market has to clear in at most s periods.

Moreover, by picking $\sigma^z \in B^z(V)$, the latter construction proves WME existence and identifies strategies and beliefs in the restricted game for all values of z. In any such equilibrium, for any history $h \in H^t$ of length $t \in \{0, ..., z\}$

$$\sigma(h) = \sigma^{z-t}(p^{t-1}).$$

Moreover, the seller's strategy and consequently the buyers' strategy are independent of the time it takes to clear the market z provided that $z > T$. To show this, define $X^0 = M$ and for any $s \in \{1, ..., z\}$ let

$$X^s = \left\{[0,1]^2 \setminus X^{s-1} \mid pd^{s+1}(p|V) + \delta\Pi^s(\underline{A}^s(p, \sigma^s(p))) \leq pd^s(p|V) + \delta\Pi^{s-1}(\underline{A}^{s-1}(p, \sigma^{s-1}(p)))\right\}.$$

For $z > T$, the collection $\{X^s\}_{s=0}^z$ partitions $[0,1]^2$ by Lemma 3. Intuitively, X^s identifies those prices at which the seller does not benefit from having $s+1$ rather than s periods to clear the market, but at which the seller would suffer by having to clear the market in less than s periods. If so, the strategy of the seller only depends on the price posted in the previous period as for any history $h \in H$

$$\sigma(p, h) = \sigma(p) = \begin{cases} \sigma^z & \text{if} \quad p = \emptyset \\ \sigma^s(p) & \text{if} \quad p \in X^{s+1} \end{cases}.$$

Naturally, the strategy would then be an equilibrium of the restricted game as: (i) the seller maximizes the present value of profits given the beliefs about the active player set; (ii) buyers maximize the present value of surplus given the expected pricing path; (iii) buyers' beliefs are consistent with the seller's strategy; (iv) the seller's and the buyers' beliefs about the active player set are correct (in any subgame in which at most a measure zero of buyers has deviated).

The previous argument also establishes existence in the unrestricted game $z = \infty$ whenever $\underline{v}_g > 0$. The latter holds by Lemma 3 because the seller's strategy cannot be affected by the constraint $\sigma(h) \in \mathcal{P}(M)$ when $z \geq T$ as prices will necessarily belong to M in at most T periods and because all buyers purchase when prices belong to M. Furthermore, the equilibrium is indeed stationary as the buyers' strategy α only depends on only the current prices p given that these prices fully pin down the entire price evolution.

Stationarity further implies that all of the operators defined in the previous part of the proof are independent of s,

$$\Pi^s(A) = \Pi(A); \;\; B^s(A) = B(A); \;\; U^s(p, v) = U(p, v); \;\; \underline{A}^s(p, \rho) = \underline{A}(p, \rho).$$

Next observe that the map $\Pi(A)$ is equicontinuous in $A \subseteq V$. Consider any two active player sets $A' \subseteq A \subseteq V$. Let $p(A)$ denote any pure strategy best response (or any degenerate distributions in $B(A)$). If so, we have that

$$\Pi(A) - \Pi(A') \leq (\mathcal{F}(A) - \mathcal{F}(A')) \max_i p_i(A) \leq \bar{f}(\mathcal{L}(A) - \mathcal{L}(A')),$$

where the first inequality follows as the seller could always set price $p(A)$ even when the beliefs about the active player set amount to A'; and where the second inequality follows as the market is regular and $v \leq (1, 1)$ for all $v \in V$. Also, the operator $\underline{A}(p, \rho)$ is equicontinuous in p and in ρ as $U(\rho|v)$ is continuous in ρ by the proof of Lemma 1.

When $\underline{v}_g = 0$, consider a sequence of games where the measure of buyers $\{\mathcal{F}_n\}_{n=0}^{\infty}$ satisfies

$$f_n(v) = \begin{cases} f(v) & \text{if} \quad v \geq \underline{v}^n \\ 0 & \text{otherwise} \end{cases}$$

for a strictly positive decreasing sequence $\{\underline{v}^n\}_{n=0}^{\infty}$ such that $\lim_{n \to \infty} \underline{v}^n = \underline{v}$. As each of these games displays gaps ($\underline{v}_g^n \geq \max_i \underline{v}_i^n > 0$), there exists a stationary equilibrium $\{\sigma_n, \alpha_n\}$ in each of these games by the previous part of the argument.

Since the sequence of functions $\{\Pi_n, \underline{A}_n\}_{n=0}^{\infty}$ is equicontinuous, it has a uniformly convergent sub-sequence converging to continuous functions $\{\Pi, \underline{A}\}$. For notational ease, posit that the sequence actually converges to this limit. For any $A \in \mathcal{K}(V)$, label by $J(A, \Pi, \underline{A}, \sigma)$ the limit problem of the seller

$$\max_{\rho \in \mathcal{P}\left([0,1]^2\right)} \int_{[0,1]^2} pd(p|A) + \delta\Pi(\underline{A}(p, \sigma(p)))d\rho(p)$$

and by $J_n(A, \Pi_n, \underline{A}_n, \sigma_n)$ the same problem for the n^{th} element of the sequence of games. Since J_n is Lipschitz continuous in $(\Pi_n, \underline{A}_n, \sigma_n)$ in the uniform topology, J_n converges to J, and by the theorem of the maximum the limit points of σ_n must converge to an equilibrium of the limit game σ at any continuity point of σ. Thus, for all $p \in [0,1]^2$, we have that

$$\lim_{n\to\infty} \sigma_n(p) = \sigma(p) \quad \Rightarrow \quad \sigma(p) \in B(\underline{A}(p, \sigma(p)))).$$

Consequently, σ is an equilibrium of the limit game, and thus a WME exists even when $\underline{v}_g = 0$. ■

Proof Proposition 3. Fix a weak Markovian equilibrium $\{\sigma, \alpha\}$. We shall omit the dependence on $\{\sigma, \alpha\}$ to simplify notation. Let $\delta = e^{-r\Delta}$ and consider what happens when Δ converges to 0. As the buyers' strategy is stationary, denote by $\hat{U}(p, v)$ the equilibrium expected payoff of a buyer with value v when p was the last price quoted by the monopolist. As in the proof of Proposition 2, for any quoted price p define the equilibrium-path set active buyers on the as

$$\hat{A}(p) = \left\{ v \in V \mid \max_i \{v_i - p_i\} \leq \delta\hat{U}(p, v) \right\}.$$

For any price p^t, with a minor abuse of notation, denote by $D_i(p^t)$ the set of buyers who purchase variety i at such a price,

$$D_i(p^t) = \left\{ v \in \hat{A}(p^{t-1}) \mid v_i - p_i^t > \max\{v_j - p_j^t, \delta\hat{U}(p^t, v)\} \right\}$$

and denote by $d_i(p^t)$ the measure of this set. Let $\hat{\sigma}(p^t)$ denote the equilibrium mixed strategy of the seller when the set of active buyers is $\hat{A}(p^t)$, and let $\hat{E}[\cdot \mid p^t]$ denote the expectation with respect to such distribution. If $v \in D_i(p^t)$, buyer v prefers purchasing variety i immediately than purchasing the preferred variety tomorrow,

$$v_i - p_i^t \geq \delta\hat{E}[\max\{v_i - p_i^{t+1}, v_j - p_j^{t+1}\} \mid p^t].$$

Thus, the expected price reductions at histories in which $d_i^t > 0$ satisfy

$$v_i(1-\delta) \geq \hat{E}[\max\{p_i^t - \delta p_i^{t+1}, p_i^t - \delta p_j^{t+1} + \delta(v_j - v_i)\} \mid p^t]. \tag{6}$$

Let $\Pi(p^t, p^{t-1})$ denote the present discounted value of equilibrium profits when the active player set is $\hat{A}(p^{t-1})$ and the price set by the seller is p^t,

$$\Pi(p^t, p^{t-1}) = p^t d(p^t) + \delta \hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right].$$

Because of stationary of buyers' strategies, the seller's present discounted value of equilibrium profits depends only on the distribution of active buyers (which is summarized by its support $\hat{A}(p^{t-1})$) and not on the entire history of play h^t. For a strategy to be an equilibrium, setting a price p^t in the support of $\hat{\sigma}(p^{t-1})$ at date t and selling according to $\hat{\sigma}(p^t)$ at date $t+1$ must be more profitable than selling according to $\hat{\sigma}(p^t)$ directly at date t. Formally,

$$\textstyle\sum_i p_i^t d_i(p^t) + \delta \hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] \geq \hat{E}\left[\Pi(p^{t+1}, p^{t-1}) \mid p^t\right]. \tag{7}$$

For any price p^{t+1} in the support of $\hat{\sigma}(p^t)$, denote by $K_i(p^{t+1})$ the set of buyers who were expected to purchase variety i at price p^t and that keep consuming the variety i at price p^{t+1}. Similarly, denote by $S_i(p^{t+1})$ the set of buyers who instead switch variety from variety i to variety j. Because of the equilibrium is weak Markovian and the active player set only depends on the current prices p^{t+1}, these sets simplify to

$$\begin{aligned} K_i(p^{t+1}) &= \left\{v \in D_i(p^t) \mid v_i - v_j \geq p_i^{t+1} - p_j^{t+1}\right\}, \\ S_i(p^{t+1}) &= \left\{v \in D_i(p^t) \mid v_i - v_j \leq p_i^{t+1} - p_j^{t+1}\right\}, \end{aligned}$$

where indifference is unimportant by absolute continuity. Denoting the measures of the two sets by $k_i(p^{t+1})$ and $s_i(p^{t+1})$ respectively, condition (7) can then be rewritten as follows,

$$R = \textstyle\sum_i \hat{E}[\underbrace{(p_i^t - p_i^{t+1})k_i(p^{t+1})}_{\text{Discrimination Gain}} + \underbrace{(p_i^t - p_j^{t+1})s_i(p^{t+1})}_{\text{Substitution Effect}} \mid p^t] \;\geq\; \underbrace{(1-\delta)\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right]}_{\text{Deferral Loss}}. \tag{8}$$

Fix any real time $\hat{T}$, the number of periods between 0 and $\hat{T}$ amounts to $\hat{T}/\Delta$ and diverges to infinity as $\Delta \to 0$. Because of this, for any value $\eta > 0$ there exist Δ sufficiently small such that $d_a(p^t) + d_b(p^t) \leq \eta$ in almost every period $t \leq \hat{T}/\Delta$. In particular, $d_a(p^t) + d_b(p^t) > \eta$ for at most $1/\eta$ periods as the market would clear by date $\hat{T}$ otherwise. Let H^* denote the set of histories of length $\hat{T}/\Delta$ that can occur with positive probability when players comply with the equilibrium strategies. We aim to show that for any $h \in H^*$ there exists $p^t \in h$ such that the expected stationary equilibrium profit of the seller $\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] \leq \kappa\eta$ for some constant $\kappa > 0$ independent of δ. The latter would imply that after any history in H^* only few players could be active as $\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right]$ always exceed optimal market-clearing profits by Proposition 1 and as optimal market clearing profits can be small only when the measure of active buyers is small by Lemma 2. But, if almost all buyers were to purchase before date $\hat{T}$, prices would necessarily be close to market clearing by date $\hat{T}$ (as buyers would never pay more than their value for a product). Thus, by choosing $\hat{T}$ sufficiently small, the cost of delaying consumption for any real time $\hat{T}$ would vanish, and no buyer would purchase a variety until prices are close to the expected market-clearing price. The latter would then imply that the seller's initial profit would be close to a static market-clearing profit for any Δ sufficiently small.

To conclude, we show that profits must become small before date $\hat{T}/\Delta$ when Δ is small. Fix any history $h \in H^*$. The conclusion obviously holds when there exists $p^t \in h$ such that $d_a(p^t) + d_b(p^t) = 0$ as $\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] = 0$ by (8). So, begin by considering prices $p^t \in h$ such that $d_a(p^t) + d_b(p^t) \leq \eta$. The left hand side of (8) can be rewritten as follows

$$R = \sum_i \hat{E}[p_i^t - p_i^{t+1} \mid p^t] d_i(p^t) + \hat{E}[(p_i^{t+1} - p_j^{t+1})(s_i(p^{t+1}) - s_j(p^{t+1})) \mid p^t].$$

By (6), we have that whenever $d_i(p^t) > 0$,

$$\hat{E}[p_i^t - p_i^{t+1} \mid p^t] \leq (1 - \delta).$$

Thus, at such a history the desired conclusion would hold if for all δ there would exist some $\kappa' < K$ such that

$$\hat{E}[(p_i^{t+1} - p_j^{t+1})(s_i(p^{t+1}) - s_j(p^{t+1})) \mid p^t] \leq (1 - \delta)\eta\kappa'$$

as $\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] \leq (2 + \kappa')\eta$. If instead the converse inequality held for all $\kappa' < K$, there would exist prices p^{t+1} in the support of $\hat{\sigma}(p^t)$ such that

$$(p_i^{t+1} - p_j^{t+1})(s_i(p^{t+1}) - s_j(p^{t+1})) > (1 - \delta)\eta K.$$

At such prices p^{t+1}, we would have that

$$\begin{aligned} p_i^{t+1} > p_j^{t+1} &\implies s_i(p^{t+1}) > (1 - \delta)\eta K, \\ p_i^{t+1} < p_j^{t+1} &\implies s_j(p^{t+1}) > (1 - \delta)\eta K. \end{aligned}$$

Moreover, at such prices, some players would necessarily switch their demand decision as $s_i(p^{t+1}) - s_j(p^{t+1}) \neq 0$. If so, at p^{t+1} there would exist a type $\bar{v} \leq (1, 1)$ that is indifferent between the two varieties $\bar{v}_i - p_i^{t+1} = \bar{v}_j - p_j^{t+1}$ and willing to purchase at the current price by Lemma 1.[23]

[23]If $s_i(p^{t+1}) > 0$, the latter would follow by taking any type $v \in S_i(p^{t+1})$ and then considering $\bar{v} =$

When there is a buyer $\bar{v}$ that is indifferent between the two varieties and willing to purchase, condition (6) implies that

$$(1-\delta) \geq \hat{E}[\max\{p_i^t - \delta p_i^{t+1}, p_j^t - \delta p_j^{t+1} + (1-\delta)(p_j^t - p_i^t)\} \mid p^t].$$

As $(p_a^t - p_b^t) \in [-1, 1]$ for the market not clear, the latter in turn implies that

$$2(1-\delta) \geq \hat{E}[\max\{p_i^t - p_i^{t+1}, p_j^t - p_j^{t+1}\} \mid p^t].$$

With minor manipulations, the left hand side of (8) can be rewritten and bounded as follows

$$\begin{aligned} R = \textstyle\sum_i \hat{E}[(p_i^t - p_i^{t+1})(k_i(p^t) + s_j(p^t)) \mid p^t] + (p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) \mid p^t] \leq \\ \eta\hat{E}[\max\{p_i^t - p_i^{t+1}, p_j^t - p_j^{t+1}\} \mid p^t] + (p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) \mid p^t]. \end{aligned}$$

If so, either $\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] \leq (2+\kappa')\eta$ for some $\kappa' < K$ (as desired) or we must have that

$$(p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) | p^t] > (1-\delta)\eta K.$$

For the latter to be the case, $p_i^t - p_j^t \notin [-(1-\delta)K, (1-\delta)K]$ as by assumption, we must have that

$$\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) | p^t] \in [-\eta, \eta]$$

Thus if $p_i^t > p_j^t$, there exists prices p^{t+1} in the support of $\hat{\sigma}(p^t)$ such that

$$s_i(p^{t+1}) > \frac{1-\delta}{p_i^t - p_j^t}\eta K.$$

But, regularity further implies that at such prices

$$\begin{aligned} \frac{1-\delta}{p_i^t - p_j^t}\eta K < s_i(p^{t+1}) < \overline{f}\mathcal{L}(S_i^t(p^{t+1})) \leq \overline{f}\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)\mathcal{L}(D_i(p^t)) \\ \leq (\overline{f}/\underline{f})\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right) d_i(p^t) \leq (\overline{f}/\underline{f})\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)\eta, \end{aligned}$$

where the second inequality holds by regularity as $s_i(p) \leq \overline{f}\mathcal{L}(S_i^t(p))$; the third holds by absolute continuity given that $S_i(p^{t+1})$ is a subset of $D_i(p^t)$ with height bounded by $\left(p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t\right)$; the fourth holds by $d_i(p^t) \geq \underline{f}\mathcal{L}(D_i(p^t))$; and the final inequality holds ad $d_i(p^t) \leq \eta$. Thus at such histories, the price difference $(p_i^t - p_j^t)$ increases by τ with

$(\bar{v}_i, \bar{v}_i - p_i^{t+1} + p_j^{t+1})$.

strictly positive probability whenever it is positive

$$p_i^{t+1} - p_j^{t+1} - p_i^t + p_j^t > \frac{\underline{f}(1-\delta)K}{\overline{\overline{f}}(p_i^t - p_j^t)} = \tau,$$

and similarly declines by τ whenever it is negative. Moreover, when $p_i^t > p_j^t$, the probability of such an increase in the price difference would necessarily satisfy

$$\begin{aligned}
&\Pr(p_i^{t+1} - p_j^{t+1} - p_i^t - p_j^t > \tau \mid p^t) \\
&\geq \Pr((p_i^t - p_j^t)(s_i(p^{t+1}) - s_j(p^{t+1})) > (1-\delta)\eta K \mid p^t) \\
\geq\ &\frac{(p_i^t - p_j^t)\hat{E}[s_i(p^{t+1}) - s_j(p^{t+1}) \mid p^t]/\eta K - (1-\delta)}{(p_i^t - p_j^t) - (1-\delta)} > 0,
\end{aligned}$$

by a simple variant of Markov inequality. But the previous arguments imply that, whenever demand is small ($d_a(p^t) + d_b(p^t) \leq \eta$) and profits are large ($\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] > (2+K)\eta$), the price difference $(p_i^t - p_j^t)$ has to increase with strictly positive probability by at least τ if it is positive, and has decline with strictly positive probability by at least τ if it is negative. But, if that was the case in every period in which demand is small (which is almost every period when Δ is small), the price difference would eventually fall outside the set $[-1, -(1-\delta)K] \cup [(1-\delta)K, 1]$ as the number of periods diverges to infinity,

$$\lim_{\Delta \to 0} \Pr\left(p_i^t - p_j^t \notin [-1, -(1-\delta)K] \cup [(1-\delta)K, 1] \text{ for some } t \leq \hat{T}/\Delta\right) = 1.$$

If so, the market would be close to clearing before date $\hat{T}/\Delta$ with probability 1, as $p_i^t - p_j^t \in [-(1-\delta)K, (1-\delta)K]$ would imply that profits are small while $p_i^t - p_j^t \notin [-1, 1]$ would imply that one of the two prices is smaller or equal to zero (which implies market-clearing by the proof of Lemma 2). But if so, buyers would expect the market to almost clear before date $\hat{T}/\Delta$, and so profits would be small $\hat{E}\left[\Pi(p^{t+1}, p^t) \mid p^t\right] \leq (2+K)\eta$ with probability 1 on the equilibrium path before date $\hat{T}/\Delta$. ■