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Abstract

We introduce 'Balanced Voting', a voting scheme tailored to fundamental societal decisions. It works as follows: Citizens may abstain from voting on a fundamental direction in a first stage. This guarantees the voting right in a second voting stage on the variants of the fundamental direction chosen in the first. All losers from the first stage also obtain voting rights in the second stage, while winners do not. We develop a model with two fundamental directions and variants of these directions. Information about the preferences is private. We identify circumstances under which Balanced Voting performs well with regard to utilitarian welfare and Pareto dominance. We discuss the robustness of the results, procedural rules to implement the voting scheme, and extensions. Moreover, we provide several examples, such as the US presidential election, for which the scheme could be applied.

JEL Classification: D7

Keywords: Balanced Voting, fundamental decision, tyranny of majority, minority protection

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Balanced Voting*

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1 Introduction

Motivation and Idea

Polities are repeatedly confronted with the need to take decisions that are fundamental for a large group of citizens. Decisions such as exiting nuclear power, reversing the course in public indebtedness, enacting comprehensive labor market reforms or joining the European Union have a large and long-lasting impact on the direction a society takes. When a democratic society takes such decisions, some subgroups strongly favor or oppose a fundamental new direction, while others may care much less.

Typically, democratic societies use simple majority voting to take such fundamental decisions. Yet, it is well-known that this rule does not allow agents to express the *intensity* of their preferences. This may be particularly problematic for fundamental decisions. The majority overrules the minority, even if this minority is much more concerned about the fundamental direction than the majority. Allowing the minority to veto the resulting decision might mitigate the problem, but may also be detrimental: The protection of the minority would then turn into tyranny.

In this paper, we introduce a scheme, that we call 'Balanced Voting'. It is particularly suited for taking fundamental decisions and aims at striking a balance between the intensity of preferences and the protection of minorities. We will compare Balanced Voting with several existing voting schemes in a collective-decision problem involving two related decisions. We will identify circumstances under which Balanced Voting performs better than other voting schemes and examples for which the scheme could be applied.

The idea underlying Balanced Voting is conveniently explained in a two-stage set-up. Suppose a society or a committee of individuals votes on two related binary decisions. The first decision determines the fundamental direction, which is assumed to be irreversible. An example is the decision whether to use nuclear power for energy production or discontinuing its use. The second decision establishes the variant or the way in which the choice of the fundamental direction should be realized. For example, either investing in new nuclear power plants or improving the safety and efficiency of existing ones are possible variants, if the fundamental decision was to keep using nuclear power. Typical non-nuclear variants are the investment in more solar power, hydro power or the building

of new coal or gas power stations.

Under Balanced Voting, agents have the option of either voting for the fundamental direction or of abstaining in this first stage. Those who abstain 'save' their voting rights for the second stage. The agents who are 'losers' in the first voting also obtain voting rights for the second stage, while the 'winners' in the first stage are not allowed to vote in the second stage.

Balanced Voting thereby allows individuals who do not feel strongly about the fundamental decision to trade off their voting rights in the first stage for a guaranteed voting right in the second stage. Thus, individuals who are only weakly-inclined towards a particular fundamental direction have the opportunity to exert their influence on this second-stage decision. This allows, for instance, strong advocates or opponents of nuclear power to have more influence on the first-stage decision, i.e. to have a stronger say on their 'favorite' decision. Those agents that voted for a fundamental direction, but belong to the 'losers', are compensated by the right to vote in the second stage. Hence, if nuclear power is chosen in the first stage, the opponents of nuclear power will be in a better position to limit the number of nuclear power plants to be built in the future and/or to impose higher safety and efficiency standards on existing plants. Similarly, if the decision to discontinue nuclear power is taken in the first stage, strong proponents of nuclear power will have a better chance of selecting the one non-nuclear alternative they prefer.

Examples and Results

Procedurally, Balanced Voting can be applied to any sequential collective voting problem. In this paper, we focus on its application to two consecutive collective decisions in which the first decision is much more important for a subgroup of the society than the second one. In addition to the example of nuclear power already mentioned above we will provide four examples for which Balanced Voting could be applied.[1] In one case, the *spirit* of Balanced Voting is applied implicitly.

It may be useful to illustrate the working of Balanced Voting in an example of everyday life. Consider a group of five individuals who spend the weekend together. They first

[1] One might also apply Balanced Voting in court decisions taken by a group of judges, with the first vote indicating the basic ruling and the second vote the specific application.

vote on whether to go to France or Italy and then on a bicycle tour versus hiking in the chosen country.

A voting process under Balanced Voting with five individuals A, B, C, D, E could look as follows:

1st Round	**2nd Round**
Country	Type of Tour
France/Italy	Bicycle/Hiking
A, B. C choose to vote in 1st round	D, E choose to vote in 2nd round + loser of 1st round
2:1	2:1
{A,B}:{C}	{C,D}:{E}
2 winners	2 winners
(A and B)	(C and D)

Balanced Voting allows voters A and B choose to have their say on the country choice, since this seems more important for them than the second choice. The loser C from the first round obtains his preferred choice in the second ballot, at least. C is the only person who can vote twice. D and E, who care about the tour, abstain in the first ballot and one of them, D, obtains his preferred tour. E is the only one who really loses.

Of course many other examples of voting result patterns for such set-ups can be constructed under Balanced Voting. The second example are parliamentary decisions on the size of the budget deficit and the composition of expenditures. Often a parliament decides first on a debt ceiling—as in the US for instance—and only then on the mix of expenditures. Suppose that in the first stage the decision is whether public debt should be increased or not. If the parliament decides to increase public debt, the second decision could involve determining the mix of expenditures for this additional funding. If the parliament decides not to increase the debt level, the second decision could involve the choice about the mix of tax increases or expenditure cuts to meet the debt ceiling.

The third example is a recent vote in Switzerland on the limitation of second homes in the Alps. The issue at hand was whether there should be a limitation on the share of

second homes. The popular vote on the corresponding constitutional initiative in 2012[2] has been accepted, but the precise implementation was left to a subsequent decision as to what extent the laws should specify, whether the *planned* second homes had to be stopped or whether they could be completed. The alpine cantons cared a lot about such a limitation as they feared an economic-downturn through a collapse of their construction industry. In non-alpine cantons one subgroup cared a lot about imposing the limitation since they favored untouched landscapes. After the voting of the citizenry of the entire country decided to limit second homes, the parliament invited the alpine cantons to help craft the corresponding legislation and to draft a variant that was less painful for them. This follows closely the spirit of Balanced Voting as in the second decision, strongly affected-losers and weakly-inclined individuals were instrumental in specifying the precise implementation of the first decision.

The fourth example concerns the application of Balanced Voting to elections and suggests a balanced election procedure that might even be used for US presidential elections. Suppose that in a majority rule political system with two parties, an election for a public office takes place. Then, in a first election, citizens could decide to which party the office-holder should belong. In a second vote, an office-holder would be selected from a set of candidates named by the party who won the first vote.[3] In such circumstances, under Balanced Voting, citizens would need to choose whether to vote or abstain in the first election, when the winning party is selected.

We will examine Balanced Voting for a particular scenario in which individuals have either high stakes or low stakes in the first decision. The first group cares most about the first collective decision: for instance, people who are sufficiently afraid of nuclear power may wish to exit from this source of electrical power. For individuals in the second group, the preferences on the use of nuclear energy might be conditional on the type of energy that is used instead of nuclear, say solar, wind or coal, for instance. Similarly, a fraction of the electorate may care most about whether a Democrat or a Republican is in the White House, while for an other part of the electorate, the preference mostly depends on the person elected. We show that in equilibrium, individuals with strong

[2]See www.zweitwohnungsinitiative.ch (retrieved on March 8, 2016).

[3]The set of candidates has to be determined in advance, e.g. all candidates who collect a number of signatures from party members above some threshold could be declared eligible, or parties may organize primary elections to select the set of candidates.

preferences participate in the first voting stage, while those who are weakly-inclined abstain, provided that the stakes of strongly-inclined individuals are sufficiently high and those of weakly-inclined individuals are sufficiently low. We identify the circumstances under which Balanced Voting performs well with regard to utilitarian welfare and Pareto dominance. These circumstances involve large utility differences between strongly-inclined and weakly-inclined individuals towards the fundamental direction, a sizeable group of individuals that cares strongly about the fundamental direction, and a sufficiently large voting body.

Relation to the Literature

Several innovative voting schemes that lead to higher social welfare have been developed in recent years. In the Storable Votes scheme developed by Casella (2005), agents can either use their vote in the current decision or store it for future use. This allows agents to concentrate their votes on decisions dealing with issues over which they have intense preferences. Hortala-Vallve (2012) introduced Qualitative Voting, in which each individual is granted a stock of votes that can be freely distributed over a series of binary choices, whereby all votes are cast simultaneously. The Minority Voting scheme introduced by Fahrenberger and Gersbach (2010) and further analyzed in Fahrenberger and Gersbach (2012) focuses on protecting individuals from the accumulation of utility losses over time if they wind up in the minority repeatedly. In a two-stage voting procedure, Minority Voting gives the losers in the first stage exclusive voting rights in the second stage. All these voting schemes link *separate* voting decisions. Jackson and Sonnenschein (2007) show that in general, linking repeated voting decisions leads to welfare gains, and that it is possible to achieve full efficiency as the number of decisions grows large.

Other mechanisms that attempt to capture agents' preference intensity are Cumulative Voting and Vote-trading. In Cumulative Voting, individuals are endowed with a stock of votes they can freely distribute over the candidates standing for election, whereby allocating more than one to a candidate is allowed (see Sawyer and MacRae (1962), Gerber, Morton, and Rietz (1998) and Cox (1990)). Under Vote-trading, an agent is permitted to hand over his voting right to another, in return for monetary compensation (see, for example, Coleman (1966) and Philipson and Snyder (1996)).

Balanced Voting uses the ideas from Storable Votes and Minority Voting. It is tailored

to fundamental societal decisions and examined under incomplete information. Consequently, Balanced Voting not only allows agents to use their vote on issues about which they feel strongly, but it also provides better protection to minorities with strong preferences, in keeping with the argument of Guinier (1994) and Casella, Palfrey, and Riezman (2008) that such minorities indeed to *deserve* some special protection.

Our paper is related to the proportionality principle which requires that in collective decision-making processes, the power should be proportional to individual stakes (Brighouse and Fleurbaey (2010)). Under Balanced Voting, people having high stakes on one issue are selected endogenously into the group of people that takes the decision, which fosters proportionality.

The rest of the paper is organized as follows: Section 2 describes the model. Section 3 characterizes the equilibria under Balanced Voting, simple majority voting, Storable Votes and Minority Voting. In Section 5, we compare social welfare under Balanced Voting with social welfare under simple majority voting, Storable Votes and Minority Voting, and we present our main result. In Section 6, we discuss procedural rules and solutions of organizational issues when Balanced Voting is applied. Morover, we precent various extensions and a discussion of the robustness of the results. Section 7 concludes. In Appendix B, we provide the proofs. Appendix C contains detailed welfare comparisons. In Appendix D, we summarize the properties of the expressions used in the paper. Appendix A outlines detailed calculations of key expressions.

2 The Model

We consider a society of N ($N \geq 3$) individuals who are indexed by i or i' ($i, i' \in \{1, ..., N\}$). They first take a fundamental decision and then decide on variants of the outcome of the first decision.

2.1 The Setting

There are two fundamental directions, denoted by A and B. An alternative in the first stage is denoted by $\Omega \in \{A, B\}$. Let each Ω have two variants which we denote by x^j_Ω ($j = 1, 2$).

2.2 Utilities

Each individual derives non-negative utility from the final decision x^j_Ω ($\Omega \in \{A, B\}$, $j = 1, 2$). We denote by $U_i(x^j_\Omega)$ the utility individual i obtains if Ω is selected in the first stage and x^j_Ω in the second stage.

2.3 Preference Groups

Every agent has an inclination towards one of the fundamental directions. This inclination may be weak or strong. Furthermore, no agent is indifferent between the variants of a fundamental direction. We denote the group of strongly-inclined individuals by SI and the set of weakly-inclined agents by WI. N_s and N_w represent the number of individuals belonging to groups SI and WI respectively. Thus, $N_s + N_w = N$. The utility of agent i is as follows:

$$i \in SI \quad \text{iff} \quad \begin{cases} U_i(x^j_\Omega) = 1 + H & \text{for some } \Omega \in \{A, B\} \text{ and some } j \in \{1, 2\} \\ U_i(x^k_\Omega) = 0 + H & \text{for } k \in \{1, 2\}, j \neq k \\ U_i(x^{j'}_{\Omega'}) = 1 & \text{for some } j' \in \{1, 2\}, \Omega' \in \{A, B\}, \Omega' \neq \Omega \\ U_i(x^{k'}_{\Omega'}) = 0 & \text{for } k' \in \{1, 2\}, k' \neq j' \end{cases}$$

$$i \in WI \quad \text{iff} \quad \begin{cases} U_i(x^j_\Omega) = 1 + \epsilon & \text{for some } \Omega \in \{A, B\} \text{ and some } j \in \{1, 2\} \\ U_i(x^k_\Omega) = 0 + \epsilon & \text{for } k \in \{1, 2\}, j \neq k \\ U_i(x^{j'}_{\Omega'}) = 1 & \text{for some } j' \in \{1, 2\}, \Omega' \in \{A, B\}, \Omega' \neq \Omega \\ U_i(x^{k'}_{\Omega'}) = 0 & \text{for } k' \in \{1, 2\}, k' \neq j' \end{cases}$$

H and ϵ are positive constants representing the agents' preference intensity. The values H and ϵ are the same across strongly-inclined and weakly-inclined individuals, respectively. Our main assumption is

Assumption 1

$$H > 1 > \epsilon > 0.$$

Detailed assumptions on the lower bounds for H and upper bounds for ϵ will follow. While SI individuals derive higher utility from all variants of their preferred fundamental direction, the utility of WI agents is not guaranteed to be higher if their preferred

proposal is chosen in the first stage. Depending on which variant is selected in the second stage, there may be instances where WI individuals are better off if their less-favored alternative has been selected in the first stage.

Suppose, for example, that individual i is strongly-inclined towards A, while individual i' is weakly-inclined towards B. Then for individual i, variants have second-order importance, while for individual i', the chosen variant is of first-order importance. Table 1 provides an example of a possible realization of their utilities.

	$j=1$	$j=2$
$U_i(x_A^j)$	$1+H$	$0+H$
$U_i(x_B^j)$	0	1
$U_{i'}(x_A^j)$	1	0
$U_{i'}(x_B^j)$	$1+\epsilon$	$0+\epsilon$

Table 1: *An Example of Realization of Utilities*

At this stage, we can provide an initial intuition for the plausible behavior of agents under Balanced Voting. An SI agent highly benefits if his fundamental direction is chosen and thus he has an incentive to vote on the fundamental direction. For a WI individual, if ϵ is very small, the additional utility of 1 he will derive from his preferred variant may seem more important compared to ϵ. Therefore, WI agents may have an incentive to exert their vote on the decision of the second stage.

2.4 Informational Assumptions

We assume that each individual faces a probability $p \in (0,1)$ of having a strong inclination towards one direction, while he has a weak inclination with probability $1-p$. There is a probability of $\frac{1}{2}$ for this inclination to be towards A. Moreover, $\frac{1}{2}$ is the probability that an individual has an additional utility of 1 for one of the variants of a fundamental direction over the other. We assume that the associated random variables are stochastically independent. Utilities are stochastically independent across individuals. We assume that individual utilities are privately observed prior to the start of

voting. The probability p and the values of H and ϵ are common knowledge. The previous informational assumptions determine the underlying probability measure of our model, which we denote by $\mathbb{P}(\cdot)$ from now on. The corresponding expectation operator is denoted by $\mathbb{E}[\,\cdot\,]$. We will use the subscript $\mathbb{P}_m(\cdot)$ to denote that in the current voting round, there are m participants.

2.5 Voting Rules

We consider secret voting under the following voting regimes:

1. Simple Majority Voting (SM)
2. Balanced Voting (BV)
3. Storable Votes (ST)
4. Minority Voting (MV)

The specific rules under each voting scheme are as follows:

Under SM, individuals decide whether to vote and how to vote on the fundamental proposals A and B in the first stage, and the proposal receiving the higher number of votes is selected. If both proposals receive the same number of votes, the winner is determined by flipping a fair coin, giving each alternative a probability of $\frac{1}{2}$ to be selected. In the second stage, individuals once again can participate in a simple majority vote on variants of the selected fundamental proposal. The winner is chosen using the same rules as in the first stage.

Under BV, each individual must first decide whether he participates in or abstains from voting in the first stage. The losers and the absentees[4] of the first stage have the right to vote in the second stage. The decision is taken by the simple majority rule in both stages, with a similar tie-breaking procedure as in SM. If every member decides to abstain from voting in the first stage, or if there are neither losers nor absentees from the first stage, all individuals are eligible to vote in both stages.

[4]We use the term 'absentees' in the spirit of Fahrenberger and Gersbach (2010) to denote the individuals who abstain from voting in the first stage.

We note that BV requires voting to be secret and anonymous, but also that the public authority can identify whether a voter is a winner or a loser. There are several technical methods to achieve this. For instance, when voting takes place electronically or with voting machines, each individual obtains a code that allows to vote at least once. Then, each individual is given a second code, depending on which alternative they selected in the first round. Once the winning alternative in the first round is known, the codes given to voting losers would allow them to vote again in the second round. Such procedures ensure that voting is anonymous and secret with regard to other citizens.[5]

Under ST, agents receive one vote each, which they can either use in the first stage or store for use in the second stage.[6] Following Casella (2005), the proposal receiving a simple majority of votes is selected in both stages, with a tie-breaking procedure similar to the one in SM. If, however, all agents use their votes on the first decision or store them for the second decision, every individual will be allowed to vote in both stages.

Under MV, all citizens can vote in the first stage, while only the minority of the first stage has a voting right in the second stage. An individual may abstain from voting in the first stage, in which case he will not be allowed to vote in the second stage. If there are no losers from the first stage, all agents retain their right to vote in the second stage. The decision is taken by the simple majority rule in both stages, with a tie-breaking procedure similar to the one in SM.

3 Equilibria

3.1 Equilibrium Concept and Probability of Winning

In the following, we look for perfect Bayesian Equilibria in pure strategies under each voting rule.[7] We adopt the standard refinement for voting games to exclude implausible voting behavior. In particular, we assume that all agents participating in the second stage eliminate weakly-dominated strategies at this stage. If this procedure yields unique

[5]The coding system can also ensure that—except for the person or machine collecting the votes—the public authority does not know how a citizen has voted.

[6]The interpretation of ST we use is slightly different from the scheme introduced in Casella (2005), where every agent obtains a vote in each stage. The reason is to allow direct comparison with BV.

[7]Simple examples of the equilibria are worked out in our working paper Ger.

strategies for the second stage, we assume that agents also eliminate weakly-dominated strategies in the first stage, if there are any.

Finally, we look for symmetric equilibria where voters with the same intensity of preference, i.e. SI and WI individuals, behave similarly with respect to the sincerity of their vote and their possible abstention from the first stage. A perfect Bayesian Equilibrium with these properties will simply be called an 'equilibrium' in the remainder of the paper. A justification of the refinement is given in our working paper Ger.

In Section 6.3, we will show that equilibria under BV remain unaffected even if we impose less demanding symmetry requirements.

To derive the equilibria, it is useful to express an individual's probability of winning in an isolated vote on two alternatives. Consider the following isolated voting problem:

- n ($n \leq N$) agents voting,
- two alternatives,
- ex-ante probability of $\frac{1}{2}$ that an individual favors one alternative,
- private information on utilities.

Then, if n individuals cast their vote, the probability of winning for one individual if all other individuals vote sincerely is given by[8]

$$P(n) = \frac{1}{2^n}\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} + \frac{1}{2}. \tag{1}$$

The above expression states that an individual faces a probability greater than $\frac{1}{2}$ of having his preferred option elected by participating in the vote. This probability increases if the number of voting agents decreases. Furthermore, $P(n-1) = P(n)$ when n is an odd number. Moreover, $\lim_{n\to\infty} P(n) = \frac{1}{2}$, as the weight of individual vote becomes negligible. The plot of $P(n)$ is given in Figure 1.

Before characterizing the properties of each voting scheme, we introduce some useful notation. We have already shown that if n agents participate in the first stage, each

[8]See e.g. Fahrenberger and Gersbach (2010). Note that for $a \in \mathbb{R}$, $\lfloor a \rfloor = \max \{ b \in \mathbb{N} : b \leq a \}$.

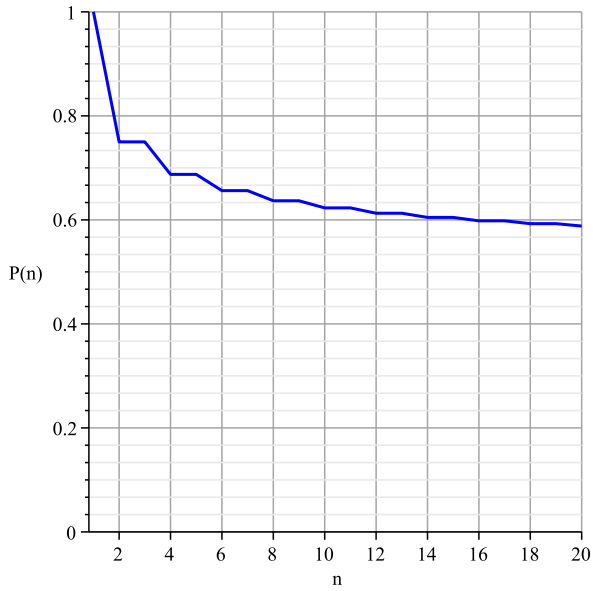

Figure 1: The Probability of Winning $P(n)$ vs. the Number of Voting Agents n.

participating agent wins with probability $P(n)$ and loses with probability $(1 - P(n))$. The number of agents allowed to vote in the second stage, however, varies with the voting scheme. Hence, from an ex-ante point of view, i.e. before voting begins, but after agents have privately observed their own preferences, an agent faces different probabilities of winning in the second stage under each voting scheme. It is thus important that these probabilities be formally introduced.

Under SM, all agents are allowed to vote in the second stage. In Section 3.4, we show that no agent will abstain from voting in either stage under SM. Hence, no matter whether he wins or loses in the first stage, every individual expects to win in the second stage with probability $P(N)$ if all members of the committee vote in the second stage.

Under BV, the winners will not be allowed to vote in the second stage. Hence, their preferred variant in the second stage will be selected with probability $\frac{1}{2}$. The losers will participate in the second stage, along with the absentees. Suppose $n \in (0, N)$ agents participated in the first stage and $N_L \in [0, \lfloor \frac{n}{2} \rfloor]$ agents lost the vote. Then, $N - n + N_L$ individuals will vote in the second stage. Ex-ante, a voter in the second stage wins with a probability that depends on whether he was allowed to vote as an absentee or as a loser from the first stage. In order to derive these probabilities, we define the following

events:

$$N_L := \text{a voting round results in } N_L \text{ losers;}$$
$$Y := \text{a voting round results in at least one loser.}$$

Note that in a round with m voters, there must be at least 0 and at most $\lfloor \frac{m}{2} \rfloor$ losers. In Appendix A, we show that for $N_L \in \{0, 1, ..., \lfloor \frac{m}{2} \rfloor\}$, we have

$$\mathbb{P}_m(N_L) = \begin{cases} \binom{m}{N_L} \frac{1}{2^m} & \text{if} \quad N_L = m/2; \\ \binom{m}{N_L} \frac{1}{2^{m-1}} & \text{if} \quad \text{else,} \end{cases}$$

whereas

$$\mathbb{P}_m(Y) = 1 - \mathbb{P}_m(N_L = 0) = \frac{2^{m-1} - 1}{2^{m-1}}.$$

We now define the ex-ante probability

$$\begin{aligned} \Phi(N, n) \quad &:= \text{Ex-ante probability of winning in second stage under BV, for losers of} \\ &\quad \text{the first stage, given population size } N \text{ and } n \text{ voters in the first round.} \end{aligned}$$

In Appendix A, we use Bayes' Theorem and the events N_L and Y to show that [9]

$$\Phi(N, n) = \sum_{N_L=1}^{\lfloor \frac{n}{2} \rfloor} P(N - n + N_L) \cdot \frac{\mathbb{P}_n(N_L)}{\mathbb{P}_n(Y)}. \tag{2}$$

Note that this is only well-defined if there is at least one loser in the first round, so if $1 < n \leq N$. For $n = 1$, we set $\Phi(N, 1) = 0$.

Similarly, we calculate an ex-ante probability of winning for individuals who abstain from voting in the first round:

$$\begin{aligned} \Lambda(N, n) \quad &:= \text{Ex-ante probability of winning in second stage under BV, for absentees} \\ &\quad \text{of first stage, given population size } N \text{ and } n \text{ voters in the first round} \\ &= \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} P(N - n + N_L) \cdot \mathbb{P}_n(N_L). \end{aligned} \tag{3}$$

[9] The expression for $\Phi(N, n)$ follows from the formula for $\Theta(N)$, given in (4). Detailed derivations of $\Theta(N)$, $\Psi(N, n)$ and $\Lambda(N, n)$ can be found in Appendix A. Further properties of these probabilities are given in Table D.1.

Under ST, losers from the first stage are not allowed to vote in the second stage. Therefore, all participants in the first stage have their preferred variant in the second stage selected with probability $\frac{1}{2}$. All $N-n$ absentees vote in the second stage and win with probability $P(N-n)$.

In Section 3.4, we show that no agent with voting rights will abstain from voting in either stage under MV. As in BV, all winners of the first stage win their preferred option in the second stage with probability $\frac{1}{2}$. The losers vote in the second stage and win with the following probability, which is derived in detail in Appendix A:[10]

$$\begin{aligned}\Theta(N) \;&:= \text{Ex-ante probability of winning in second stage under MV for losers} \\ &\quad\; \text{of the first stage, given a population of } N \text{ voters} \\ &= \sum_{N_L=1}^{\lfloor \frac{N}{2} \rfloor} P(N_L) \cdot \frac{\mathbb{P}_N(N_L)}{\mathbb{P}_N(Y)}. \end{aligned} \tag{4}$$

Using the properties of the binomial coefficient and the number of agents expected to vote in each case, we derive the following fact:

Fact 1

For all $N \geq 3$, we have

- $\Theta(N) > \Lambda(N,n) \geq P(N)$ *when* $0 < n < N$, *where* $\Lambda(N,n) = P(N)$ *if and only if* $n = 1$ *and* N *is odd.*
- $\Theta(N) > \Lambda(N,n) \geq \Phi(N,n) \geq P(N)$ *when* $1 < n < N$, *where* $\Phi(N,n) = P(N)$ *if and only if* $n = 2$ *and* N *is odd.*

The proof of Fact 1 is given in Appendix B. The conditions in the fact are intuitive: When the number of agents participating in a vote is less than N, the participating individuals win with a probability greater than or equal to $P(N)$. Thus, min $\{\Theta(N), \Lambda(N,n), \Phi(N,n)\} \geq P(N)$. When only losers from the first stage vote in the second, the voting body in the second stage is always smaller than or equal to $\lfloor \frac{N}{2} \rfloor$. However, when absentees from the first stage have guaranteed voting rights in the second, the voting body of the second stage is greater than $\lfloor \frac{N}{2} \rfloor$ with positive probability. Thus, $\Theta(N) >$ max $\{\Lambda(N,n), \Phi(N,n)\}$. Interestingly, although under BV, both losers and

[10]Further properties of $\Theta(N)$ are given in Table D.1.

absentees of the first stage vote in the second stage, we note that their ex-ante probability of winning can differ. That is, $\Lambda(N,n) \geq \Phi(N,n)$, which can be explained as follows: N_L could be zero and the voting body could merely consist of absentees in the second stage. This occurs with positive probability. If losers from the first stage exist, such an individual faces a voting body that is strictly larger than the number of absentees, which makes his probability of winning in the second stage weakly lower than that of the absentees. To illustrate this property with an example, we plot the probabilities for $N = 20$ in Figure 2.

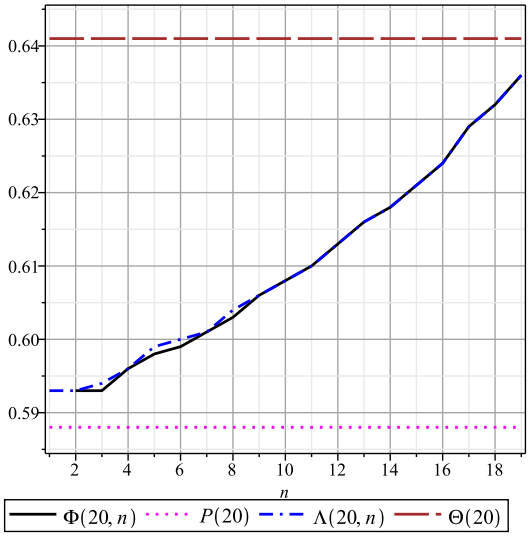

Figure 2: $\Theta(N), \Lambda(N,n), \Phi(N,n), P(N)$ vs. n for $N = 20$.

We further recall the following three special cases and the corresponding voting rules.

1. If all agents abstain ($n = 0$), then all individuals can vote in both stages under BV or ST.

2. If all agents participate in the first stage ($n = N$), then all agents can vote in both stages under ST.

3. Suppose all agents participate in the first stage ($n = N$) under BV and MV. If individuals are unanimously inclined towards A or B, then all agents can vote in both stages. If there are losers in the first stage, they receive exclusive voting rights in the second stage.

We note that in these three special cases, whenever all agents can vote in both stages, no individual has an incentive to abstain from voting, as participation strictly increases his probability of winning.

Now, we are ready to examine the equilibria under each voting scheme.

3.2 Equilibria under BV

Under BV, winners of the first stage are not allowed to vote in the second stage, while losers are compensated for their loss with a vote in the second stage. However, with privately-observed preferences, it is ex-ante unclear whether an agent would belong to the losers or not. Thus, individuals face a trade-off in the first stage: abstention lowers the probability of winning in the first stage, while it secures voting rights in the second stage. We characterize the agents' voting behavior in the following proposition:

Proposition 1
Let

$$\epsilon^{crit}(N) := \min\left\{\frac{P(N)-\frac{1}{2}}{1-P(N)},\ 4\Lambda(N,1)-\Phi(N,2)-\frac{3}{2}\right\}$$

and suppose that $H > 1$ *and* $\epsilon \leq \epsilon^{crit}(N)$*. Then, there exists a unique equilibrium under BV characterized by*

(i) all votes in both stages are cast sincerely;

(ii) every individual $i \in SI$ *participates in the first stage. All individuals* $i \in WI$ *abstain.*

The proof of Proposition 1 is given in Appendix B. Proposition 1 establishes a threshold on the intensity of preferences of WI individuals regarding the fundamental direction the society should take. If ϵ is below $\epsilon^{crit}(N)$, an individual $i \in WI$ would always be inclined to abstain from voting in the first stage. As shown in the proof, with our main assumption $H \gg 1$, SI individuals are always better off by voting in the first stage. This is in line with the main motivation behind BV, which is to allow individuals that care more about an issue to exert more influence on the corresponding decision, while

those who care less can choose to abstain to maintain the possibility to influence future decisions.

We observe from Proposition 1 that BV generates a different voting pattern (and potentially different outcomes) than SM, ST or MV. A plot of $\epsilon^{crit}(N)$ is given in Figure 3.[11] We note that $\epsilon^{crit}(N)$ is monotonically decreasing in N. As the electorate grows larger, the expected voting body in the first and second stage also grows larger if the equilibrium in Proposition 1 is employed. However, it becomes less attractive to abstain from the first stage for a better chance of winning in the second. The reasons are as follows: Abstention only opens up the chance to vote in the second stage in the expected large voting body. Voting in the first stage allows to potentially influence the voting outcome in the first stage and with some probability in the second stage. Moreover, if the voting body is large in the first round, it will be smaller in the second stage, as there are more winners in the first round and vice versa. The value of ϵ should thus be smaller for WI individuals to still prefer abstention, which is confirmed by Figure 3. For the remainder of the paper, we assume that ϵ is such that Proposition 1 holds.

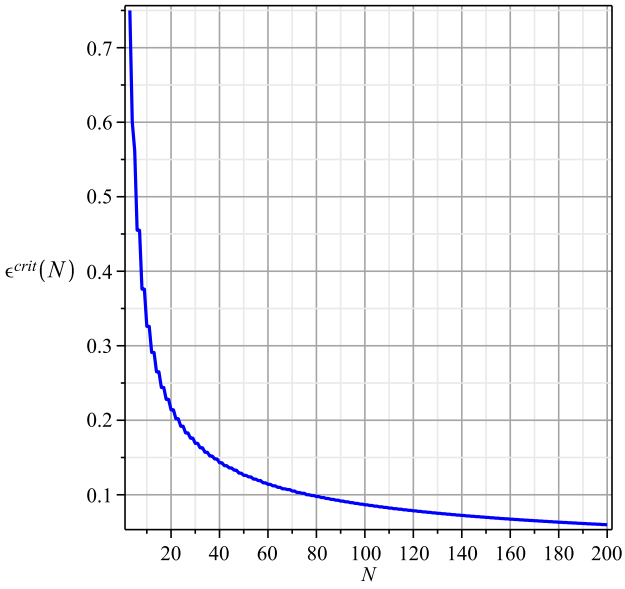

Figure 3: $\epsilon^{crit}(N)$ vs. the number of agents N.

[11]Further properties of $\epsilon^{crit}(N)$ are given in Table D.1.

3.3 Equilibria under ST

Suppose the society adopts ST as the voting scheme, where each agent will receive one vote, which he can either use in the first stage or store for use in the second stage. We characterize the voting behavior of agents under ST in the following lemma:

Lemma 1

There exists a unique equilibrium under ST, in which all individuals $i \in SI$ cast their vote sincerely in the first stage, while all $i \in WI$ store it and vote sincerely in the second stage, for H and ϵ satisfying the following conditions:

- $H \geq H_{st}^{crit}(N) := \frac{1-P(N)}{P(N)-\frac{1}{2}}$,
- $\epsilon \leq \epsilon_{st}^{crit}(N) := \frac{P(N)-\frac{1}{2}}{1-P(N)}$.

In particular, we have $\epsilon_{st}^{crit}(N) \geq \epsilon^{crit}(N)$.

The proof of Lemma 1 follows the proof of Proposition 1 and is given in Appendix B. We plot $H_{st}^{crit}(N)$ and $\epsilon_{st}^{crit}(N)$ in Figure 4.[12]

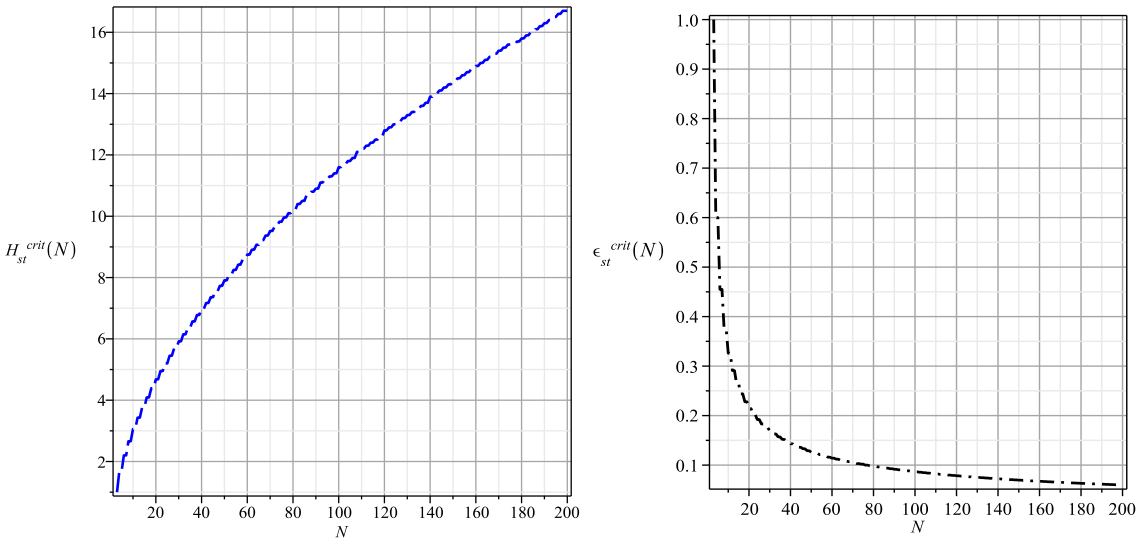

Figure 4: $H_{st}^{crit}(N)$ and $\epsilon_{st}^{crit}(N)$ vs. N.

We note that the thresholds on ϵ for the existence of a unique equilibrium under BV and ST are almost the same. Moreover, if $\epsilon \leq \epsilon^{crit}(N)$, $\epsilon \leq \epsilon_{st}^{crit}(N)$ also holds.

[12]Further properties of $H_{st}^{crit}(N)$ and $\epsilon_{st}^{crit}(N)$ are given in Table D.1.

Under ST, $H_{st}^{crit}(N)$ imposes a lower bound on H that was not required under BV to ensure that all SI individuals are voting in the first stage. Under ST, only absentees of the first stage vote in the second. Under BV, however, losers from the first stage also vote in the second, making the voting body in the second stage larger with positive probability. Thus, voting under ST is less attractive and H should be high enough for participation in the first stage to be more desirable than abstention.

Moreover, we observe that $H_{st}^{crit}(N)$ is increasing in N. If most of the agents choose to vote in the first stage, the voting body of the first stage grows larger with increasing N, while the voting body of the second becomes comparatively smaller. An individual voting in the first stage should, therefore, be willing to renounce the much higher probability of winning in the second stage, which is possible only if the strength of his preference is sufficiently high.

3.4 Equilibria under SM and MV

Voting equilibria under SM and MV are straightforward to derive. Under SM, voting sincerely weakly-dominates voting strategically, i.e. voting for the option that gives one lower utility, as no member can realize any utility gain by voting against his true preference in a binary collective decision when the game ends in the second stage.

Thus, after sequentially eliminating weakly-dominated strategies, all individuals participate and vote sincerely in both stages under SM. Hence, there exists a unique Bayesian Equilibrium in pure strategies, where all agents vote sincerely in both stages.

Now suppose that MV is the voting scheme used to reach the collective decision. Under MV, all agents will vote in the first stage and the resulting losers will receive exclusive voting rights in the second stage. In contrast with the setting with complete information analyzed by Fahrenberger and Gersbach (2010), we focus on a society with incomplete information. Incomplete information makes it impossible to gain by voting strategically. The resulting voting behavior of agents is summarized in the following lemma:

Lemma 2
There exists a unique equilibrium under MV where all individuals vote sincerely in both stages.

The proof of Lemma 2 is given in Appendix B. We note that since abstention from voting in the first stage under MV means being excluded from voting in both stages, no agent has an incentive to abstain and therefore, there are no restrictions on H and ϵ for the existence of the equilibrium under MV.

4 An Example

We illustrate BV by the example of a society with three agents ($N = 3$). They vote in a setting of incomplete information, where each agent observes his own utility but not the utility of the other two. We consider the following cases:

Case 1: $N_s = 3$ and $N_w = 0$

First, consider the scenario where all three individuals are SI. Then, as shown in Section 3, all individuals will choose to vote in the first stage.

If option Ω ($\Omega \in \{A, B\}$) achieves a majority of two votes, it will be selected as the fundamental direction. The agent who preferred Ω' ($\Omega' \in \{A, B\}, \Omega \neq \Omega'$) will obtain the sole voting right in the second stage, and the variation x^1_Ω or x^2_Ω that gives him the highest utility will be selected.

If Ω receives all three votes in the first stage, under the rules of BV, all agents will be eligible to vote in the second stage, which makes BV equivalent to SM.

Case 2: $N_s = 2$ and $N_w = 1$

In this case, two of the three individuals are SI, while the remaining agent is WI. Thus, the SI individuals will choose to vote in the first stage, while the WI individual will abstain.

If the SI agents agree on a fundamental decision Ω, it will be selected unanimously in the first stage. The WI individual will be the sole decision-maker in the second stage, and will select x^1_Ω or x^2_Ω, according to his preference.

If the SI agents cannot agree on a fundamental decision, A or B will be chosen with a probability of $\frac{1}{2}$ by flipping a fair coin. Suppose option Ω ($\Omega \in \{A, B\}$) was selected

this way. Then, the SI agent who preferred Ω', together with the WI individual, will be eligible to vote in the second stage. Variation x^1_Ω or x^2_Ω will be then selected, either unanimously or by flipping a fair coin.

Case 3: $N_s = 1$ and $N_w = 2$

Now suppose there is only one SI individual, while the other two are WI. The SI individual will be the only voter of the first stage, as the WI agents will choose to abstain in the first stage. Hence, the desired fundamental direction of the SI agent, Ω, will be the outcome of the first stage. The WI individuals will then select one of the variations x^1_Ω or x^2_Ω, either unanimously or by flipping a fair coin.

Case 4: $N_s = 0$ and $N_w = 3$

Finally, consider the case where all three individuals are WI. Then, all individuals will choose to abstain in the first stage. Under the rules of BV, all individuals will then be allowed to vote in both stages, making BV equivalent to SM in this particular case.

5 Welfare Comparison

We compute social welfare from an ex-ante perspective when the citizens' utilities have not been realized yet, under the four voting schemes considered, i.e. SM, BV, ST and MV. We then compare the welfare under SM, ST and MV to welfare under BV to derive the conditions for which BV can be better than the other voting schemes. For this purpose, we denote the aggregate expected utility by

$$W^V := \mathbb{E}\left[\sum_{i=1}^{N} U_i(\cdot)\right], \tag{5}$$

where $V \in \{BV, SM, ST, MV\}$ denotes the current voting system. The welfare comparison is based on the utilitarian criterion, which means that V is strictly superior to V' ($V' \in \{BV, SM, ST, MV\}$, $V \neq V'$) if and only if $W^V > W^{V'}$.

We now provide the results of detailed welfare comparisons between BV and SM, BV and ST and BV and MV, and identify circumstances under which BV dominates the other

voting schemes in terms of welfare. Moreover, we establish the conditions under which BV Pareto-dominates SM and MV. As these comparisons are lengthy and involved, they are assembled in Appendix C. Our main result is:

Theorem 1

BV yields higher expected social welfare than SM, ST and MV if the following conditions are satisfied:

- $\epsilon \leq \epsilon^{crit}(N)$,
- $H \geq max\{H_{st}^{crit}(N), M(N,p)\,\epsilon + C(N,p)\}$, *and*
- $N \geq N_{st}(p)$,

where $M(N,p)$, $C(N,p)$, *and* $N_{st}(p)$ *are given in Equations* (28), (29), *and* (32) *in Appendix C, respectively, and* $H_{st}^{crit}(N)$ *and* $\epsilon^{crit}(N)$ *are defined in Lemma 1 and Proposition 1, respectively.*

Theorem 1 shows that when the stakes, expressed by the value of H, are sufficiently high in comparison with ϵ, and when the society is sufficiently large, BV performs well in comparison to other common voting rules which could be used in such circumstances. Essentially, BV splits the society endogenously into those agents who have large stakes and those agents with small stakes regarding the first vote. Individuals with high stakes endogenously decide on the fundamental societal issue. Still, to allow as many citizens as possible to participate in the decision on variants of the chosen fundamental direction, and thus to increase the aggregate welfare that can be achieved in the second stage, BV gives losers from the first stage voting rights for the second stage. Hence, BV aims that individuals self-select into those decisions mattering most for them while simultaneously the likelihood of inefficient collective decisions remains small when those decisions are taken by a subgroup of the citizenry.

Theorem 1 imposes three conditions on the model parameters for the result to hold. It is useful to demonstrate these conditions numerically for a better understanding of our main result.

First, suppose $N = 10$ and $p = 0.66$. Then, $\epsilon \leq 0.215$ and $H \geq 1.3$. Moreover, for $p = 0.66$, $H_{st}^{crit}(N)$ determines the floor on H and $\epsilon^{crit}(N)$ determines the ceiling on ϵ. The third condition amounts to $N > 3$. As p increases, the restriction on H is imposed

by the term $M(N,p)\epsilon + C(N,p)$, which is the condition that needs to be satisfied for welfare under BV to be higher than welfare under SM. For example, suppose $N = 10$ and $p = 0.9$. Then the conditions for Theorem 1 to hold are $\epsilon \leq 0.215$, $H \geq 2.7$ and $N > 8$.

Second, suppose that N is higher and also that p is lower than $\frac{1}{2}$. For example, consider the case $N = 200$ and $p = 0.33$. Then, the conditions translate to $\epsilon \leq 0.065$, $H \geq 16.7$ and $N > 3$. As p increases, for example to $p = 0.66$, the conditions remain the same except for $N > 4$.

We observe that the most restrictive conditions on the model parameters for Theorem 1 to hold are on ϵ and H, especially as the size of the voting body grows large. The condition on N is less demanding, as even for very large p, a small voting body suffices to fulfil the third condition $N \geq N_{st}(p)$ (e.g. $N > 8$ for $p = 0.9$ and $N > 51$ for $p = 0.99$).

In addition to the welfare comparison in Appendix C, the utility differences for the groups of agents identified below in specific cases are also computed, which illustrates how different groups of citizens fare under different voting rules.

(i) WI agents when $N_s = 0$,

(ii) WI agents when $0 < N_s < N$,

(iii) SI agents when $N_s = N$,

(iv) SI agents when $0 < N_s < N$.

6 Discussion and Extensions

The main focus of the paper was to introduce BV, to establish its properties, and to examine its potential to improve welfare when stakes are high. Several issues emerge when BV is considered. Besides various extensions, applications of BV require further procedural rules and solutions of organizational issues. These points are detailed in this section.

6.1 Procedural Rules

To apply BV, several additional procedural rules must be established.

First, a *pre-decision* is necessary to assess whether an issue is decisive for a fundamental direction. In many cases, this appears to be intuitive. Examples are exiting nuclear power, legalizing abortion, joining a currency union or reducing public debt significantly. Still, there is a need for a procedure that defines an issue as fundamental. There are several possibilities to clarify this matter. First, BV could simply be used as the voting rule for some recurrent and well-defined issues such as the election example or the sequential decision about debt levels and government expenditures. Second, whether a sequence of two interdependent collective decisions should be subject to BV could itself be decided by a majority decision in parliament. It could be a question a small and selected committee decides upon, e.g. a special committee in parliament.

Second, one might be concerned that the group of citizens allowed to vote on the variants of a fundamental decision may be too small, which would reduce the efficiency of such votes. Such cases could be avoided by requiring a minimal size for the electorate of the second stage. This can be achieved by randomly-selecting additional voters from the group of individuals who belonged to the majority in the first stage, for instance.

Third, the composition of the society might change from one stage to the other. In situations where the final decision depends on the votes of two separate voting bodies—the US Congress, for example—BV would have to be modified and applied to each body separately. In other situations, one committee may be a subset of the other. In the democracies that comply with the Westminster System, the Cabinet of Ministers is appointed out of the current Members of Parliament, and the Ministers decide on the implementation details of the legislature that has been decided by the parliament. In such a scenario, it is possible to apply BV to those agents who are allowed to vote in both stages. It is important, however, to define procedural rules that avoid voting by very small decision-making bodies, as discussed above.

Fourth, BV excludes some individuals from voting, either in the first stage or in the second stage. Losing their right to vote may seem inequitable for some individuals. Thus, it is important that the adoption of BV as a decision rule be supported by a large consensus preceding its implementation. Requiring an almost unanimous vote in favor

of using BV would give individuals the freedom to veto the adoption of a voting scheme in which they might lose their voting right in one of the two stages.

6.2 Practical Implementation

Once the necessary procedural rules are established, the practical implementation of BV requires close examination.

First, the voting process associated with BV can be organized in two different ways. BV may be implemented practically in two voting rounds or in a single voting round without changing the results we have established. In two-round voting, on the one hand, the votes of the two stages should be cast sequentially, i.e., individuals vote on the fundamental direction first and once a fundamental direction is selected, they vote on its variants. In one-round voting, on the other hand, individuals could cast all their choices simultaneously, where their vote on the variants would be valid conditional on the outcome of the first stage and the voters having a right to vote in the second stage or not. As an example, an individual may cast his vote as *'First Stage: vote for A. Second Stage:* x_A^1 *if A is selected,* x_B^2 *if B is selected (and I have a right to vote for the second decision)'.*

Second, we have assumed that voting is secret. Despite secret ballots, one might wonder whether anonymity can be preserved, as some agents lose their voting rights after the first stage. One could infer that such agents were in the majority in the first stage. With paper ballots or electronic ballots, anonymity can be preserved, as other citizens cannot observe whether a particular citizen votes in the first stage or obtains a ballot in the second stage. Thus preferences remain private information.

Third, BV can also be applied to open ballots. As long as voting takes place simultaneously in both stages, our results apply. Individual voting strategies will remain the same in the second stage regardless of the secrecy of the vote. Deviating from sincere voting will not reap any benefits for any individual as the game ends with the second stage. Equilibrium voting strategies will remain best responses if voting occurs simultaneously and openly in the first stage. But our results may need to be modified if the votes are publicly observable and are not strictly simultaneous. For example, an SI individual

would have an incentive to abstain from the first stage if he observes that his preferred fundamental direction has received sufficient votes to be selected.

6.3 Extensions

We have limited our analysis to a very simple framework. A variety of interesting extensions emerges as soon as we deviate from our basic framework.

Publicly-observable agent preferences
An interesting extension is the case where the preferences of agents are common knowledge. If individuals can observe each other's preferences, voters may coordinate to reach a minimal majority in the first stage. This would lead to a large group of individuals (close to 50% of voters in the first stage) being eligible to vote in the second stage. This would further enhance the welfare properties under BV.

Deliberation before voting
An intriguing new dimension to our framework would be to introduce pre-vote deliberation, i.e. agents engaging in communication before voting. Since fundamental societal decisions have significant long-term impact on the members of society, an opportunity to share information and views and engage in a relevant public debate may be desirable. How deliberation – including the possibility to misinterpret one's preferences or the presence of persuasive agents – affects the performance of BV is an important avenue of further research.

Alternative models
It is possible to consider alternative models with different informational assumptions. One could examine the equilibria (if any) when ϵ and H violate their critical thresholds, for example. The intuition would be that this would yield mixed strategy equilibria, as discussed in more detail in Section **??**. Moreover, there could be a fraction of the society that has a fixed type while the remaining agents' preferences would be random. One could also analyze situations where a society has a higher tendency to lean towards one of the fundamental directions, by considering unequal probabilities for an individual to prefer A over B and *vice versa*. The same can be extended to an agent's preference over variants of the fundamental directions. While these constellations require new and

elaborate analyses, the thrust of our analysis so far suggests that BV continues to be performing well as long as the stakes in the first decision are high.

Extension to several stages

BV can also be applied to sequential voting processes involving several stages. For example, suppose that the society decides on a fundamental direction in the first stage and determines its implementation through a sequence of smaller decisions to follow. BV can be applied to this scenario in two ways. First, if the number of stages is even, BV can be applied on segments of two consecutive stages, as described in this paper. Second, agents who abstain from voting in the first stage and the losers of the first stage, can be guaranteed participation in all stages to follow.

Other equilibrium refinements

We have restricted our analysis to symmetric equilibria where individuals with the same preference intensity display the same voting behavior. This excludes the possible symmetric equilibria that may arise if we allow agents who prefer the same fundamental direction or the same variant of a fundamental direction to behave in different ways. Whether such equilibria exist under ST and MV is an open question. It is clear that no other equilibria would arise under SM if we impose less restrictive symmetry assumptions. Under BV, no further equilibria arise if we impose less demanding symmetry requirements.[13]

7 Conclusion

We have introduced Balanced Voting as a voting scheme that aims at striking a balance between expressing the intensity of preferences and protecting minorities. It appears to be particularly suitable for fundamental societal decisions, which generally foster intense preferences or high stakes for some groups of citizens. We have shown that capturing this preference intensity and providing additional protection to a minority with high stakes through Balanced Voting lead to welfare gains. We have provided a summary of procedural rules, and practical aspects of implementation, and we outline several avenues to extend Balanced Voting beyond the framework studied in the main body of the paper. Hence, the work on the voting rule has just begun.

[13]Details are available upon request.

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Appendix A: Detailed Calculations

A.1 Derivation of $\Theta(N)$, $\Phi(N,n)$ and $\Lambda(N,n)$

Recall that we denote by $\mathbb{P}$ the probability measure underlying or model and that we use the subscript $\mathbb{P}_m$ to denote that in the current voting round, m voters are participating. Moreover, recall the events

$$\begin{aligned} N_L &:= \text{first round results in } N_L \text{ losers,} \\ Y &:= \text{first round results in at least one loser.} \end{aligned}$$

We wish to derive precise expressions for $\Theta(N)$, $\Phi(N,n)$ and $\Lambda(N,n)$. For this purpose, we first show how Bayes' rule can be used to express these probabilities in terms of the events above. After that, we will give expressions for $\mathbb{P}_m(N_L)$ and $\mathbb{P}_m(Y)$.

First of all, consider $\Theta(N)$:

$$\begin{aligned} \Theta(N) := \quad & \text{Ex-ante probability of winning in the second stage for losers} \\ & \text{of the first stage under MV, given a population of } N \text{ voters.} \end{aligned}$$

Within the definition of $\Theta(N)$, we assume that there is at least one loser in the first round. By definition of Y, we can thus write:

$$\Theta(N) = \mathbb{P}\left(\text{loser of the 1st round wins in the 2nd round} \mid Y\right).$$

Recall from (1) that we defined $P(n)$ as the probability that an individual wins an isolated voting round with n voters. Applying Bayes' Rule and some basic properties of the probability measure, we derive:

$$\begin{aligned} \Theta(N) &= \sum_{N_L=0}^{\left\lfloor \frac{N}{2} \right\rfloor} \mathbb{P}\left(\text{ loser 1st round wins 2nd round} \,|\, Y, N_L\right) \cdot \mathbb{P}_N\left(N_L \,|\, Y\right) \\ &= \sum_{N_L=0}^{\left\lfloor \frac{N}{2} \right\rfloor} P(N_L) \cdot \frac{\mathbb{P}_N\left(N_L, Y\right)}{\mathbb{P}_N(Y)} \\ &= \sum_{N_L=1}^{\left\lfloor \frac{N}{2} \right\rfloor} P(N_L) \cdot \frac{\mathbb{P}_N\left(N_L\right)}{\mathbb{P}_N\left(Y\right)}, \end{aligned}$$

where the last equation follows from the fact that

$$\mathbb{P}_N(N_L, Y) = 0 \qquad \text{if } N_L = 0 \tag{6}$$

$$\mathbb{P}_N(N_L, Y) = \mathbb{P}_N(N_L) \qquad \text{if } N_L \geq 1. \tag{7}$$

We proceed to $\Phi(N, n)$. Recall that

$\Phi(N, n) :=$ Ex-ante probability of winning in the second stage under BV for losers of the first stage if n out of N individuals vote in the first round.

Again, the definition of $\Phi(N, n)$ implies the assumption that the first round result in at least one loser. Therefore, the derivation of $\Phi(N, n)$ follows the same steps as the derivation of $\Theta(N)$, with the following changes:

- Substituting n for N, since only n individuals vote in the first stage under BV;
- Substituting $P(N - n + N_L)$ for $P(N_L)$, since the $N - n$ agents who abstain from the first stage are also given the right to vote in the second stage under BV.

We conclude that

$$\Phi(N, n) = \sum_{N_L=1}^{\left\lfloor \frac{n}{2} \right\rfloor} P(N - n + N_L) \cdot \frac{\mathbb{P}_n(N_L)}{\mathbb{P}_n(Y)}.$$

Finally, recall that

$\Lambda(N, n) :=$ Ex-ante probability of winning in the second stage under BV for absentees of the first stage if n out of N individuals vote in the first round.

Note that as we consider abstantees, there might be no losers in the first round, i.e. we do not condition on Y and therefore, N_L may be 0. Hence, in this case Bayes' rule yields

$$\begin{aligned}\Lambda(N, n) &= \sum_{N_L=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \mathbb{P}\left(\text{abstantee wins 2nd round} \,\middle|\, N_L\right) \cdot \mathbb{P}_n(N_L) \\ &= \sum_{N_L=0}^{\left\lfloor \frac{n}{2} \right\rfloor} P(N - n + N_L) \cdot \mathbb{P}_n(N_L).\end{aligned}$$

As $P(n)$ is already given in (1), it remains to determine $\mathbb{P}_m(N_L)$ and $\mathbb{P}_m(Y)$ for any integer $0 \leq m \leq N$ in order to derive expressions for $\Theta(N)$, $\Phi(N,n)$ and $\Lambda(N,n)$.

A.2 Derivation of $\mathbb{P}_m(N_L)$ and $\mathbb{P}_m(Y)$

Consider an isolated voting round in the game described in Section 2, and suppose that m voters choose to participate in this round. In order to derive $\mathbb{P}_m(N_L)$, we consider two cases:

- Case 1: m is odd.
  In this case, the vote cannot result in a tie. The loosing group can either prefer A or B. Therefore,
  $$\mathbb{P}_m(N_L) = \binom{m}{N_L}\left(\frac{1}{2}\right)^m \cdot 2 = \binom{m}{N_L}\frac{1}{2^{m-1}}.$$

- Case 2: m is even.
  As the vote may now result in a tie, we have to account for the assumed tie-breaking rule. Again, the loosing group can either prefer A or B. We obtain
  $$\mathbb{P}_m(N_L) = \begin{cases} \binom{m}{N_L}\frac{1}{2^{m-1}} & \text{if} \quad 0 \leq N_L < m/2; \\ \binom{m}{N_L}\frac{1}{2^{m-1}} \cdot \frac{1}{2} & \text{if} \quad N_L = m/2. \end{cases}$$

As there can be at most $\lfloor \frac{m}{2} \rfloor$ losers in a round with m voters, $\mathbb{P}_m(N_L) = 0$ for any $N_L < 0$ or $N_L > \lfloor \frac{m}{2} \rfloor$. From Case 1 and 2 we conclude that for $N_L \in \left[0, \lfloor \frac{m}{2} \rfloor\right]$,

$$\mathbb{P}_m(N_L) = \begin{cases} \binom{m}{N_L}\frac{1}{2^m} & \text{if} \quad N_L = m/2; \\ \binom{m}{N_L}\frac{1}{2^{m-1}} & \text{if} \quad \text{else}. \end{cases}$$

Finally, by definition of N_L and Y we have

$$\mathbb{P}_m(Y) = 1 - \mathbb{P}_m(N_L = 0) = 1 - \binom{m}{0} \cdot \frac{1}{2^{m-1}} = \frac{2^{m-1} - 1}{2^{m-1}}.$$

Appendix B: Proofs

B.1 Proof of Fact 1

Recall from equation (1) that the probability $P(n)$ is non-increasing in n. More precisely, $P(n-1) = P(n)$ if n is odd and $P(n-1) > P(n)$ if n is even. Furthermore, by definition of Y and N_L, we have

$$\sum_{N_L=1}^{\lfloor \frac{m}{2} \rfloor} \frac{\mathbb{P}_m(N_L)}{\mathbb{P}_m(Y)} = \frac{\sum_{N_L=1}^{\lfloor \frac{m}{2} \rfloor} \mathbb{P}_m(N_L)}{\mathbb{P}_m(Y)} = 1,$$

for any integer $m \geq 0$. Moreover, as in a voting round with m participants, we have $N_L \in \left[0, \left\lfloor \frac{m}{2} \right\rfloor\right]$, it follows that

$$\sum_{N_L=0}^{\lfloor \frac{m}{2} \rfloor} \mathbb{P}_m(N_L) = 1.$$

We use these observations in order to prove Fact 1. Now let $N \geq 3$. We split the proof into four steps:

Step 1: $\Phi(N,n) \geq P(N)$ for $1 < n < N$, where $\Phi(N,n) = P(N)$ if and only if $n = 2$ and N is odd.
By the observations above, we find that

$$P(N) = \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} P(N) \cdot \frac{\mathbb{P}_n(N_L)}{\mathbb{P}_n(Y)} \leq \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \frac{\mathbb{P}_n(N_L)}{\mathbb{P}_n(Y)} = \Phi(N,n),$$

where equality occurs if and only if $n = 2$ and N is odd.

Step 2: $\Lambda(N,n) \geq P(N)$ for $0 < n < N$, where $\Phi(N,n) = P(N)$ if and only if $n = 1$ and N is odd.
Again, by the observations above, we have

$$P(N) = P(N) \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} \cdot \mathbb{P}_n(N_L) \leq \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \mathbb{P}_n(N_L) = \Lambda(N,n),$$

where equality occurs if and only if $n = 2$ and N is odd.

Step 3: $\Lambda(N,n) \geq \Phi(N,n)$ **for** $1 < n < N$**.**
Recall that for $1 < n < N$ we have

$$\Lambda(N,n) = \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \mathbb{P}_n(N_L),$$
$$\Phi(N,n) = \sum_{N_L=1}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \frac{\mathbb{P}_n(N_L)}{\mathbb{P}_n(Y)}.$$

Hence, as $\mathbb{P}_n(N_L = 0) = 1 - \mathbb{P}_n(Y)$ and the probability $P(n)$ is non-increasing in n, we find that for $1 < n < N$

$$\begin{aligned}
\Lambda(N,n) - \Phi(N,n) &= P(N-n) \cdot \mathbb{P}_n(N_L = 0) + \sum_{N_L=1}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \mathbb{P}_n(N_L) \left[1 - \frac{1}{\mathbb{P}_n(Y)}\right] \\
&= P(N-n) \cdot \mathbb{P}_n(N_L = 0) - \frac{\mathbb{P}_n(N_L = 0)}{\mathbb{P}_n(Y)} \sum_{N_L=1}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \mathbb{P}_n(N_L) \\
&\geq P(N-n) \cdot \mathbb{P}_n(N_L = 0) - \frac{P(N-n) \cdot \mathbb{P}_n(N_L = 0)}{\mathbb{P}_n(Y)} \cdot \sum_{N_L=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{P}_n(N_L) \\
&= 0.
\end{aligned}$$

We conclude that for $1 < n < N$ we have $\Lambda(N,n) \geq \Phi(N,n)$.

Step 4: $\Theta(N) > \Lambda(N,n)$ **for** $0 < n < N$**.**
It suffices to show that $\Lambda(N,n)$ is increasing in n for fixed $N > 1$ and $0 < n < N$, and $\Lambda(N, N-1) < \Theta(N)$. Indeed, if both assertions hold for $0 < n < N$, then

$$\Lambda(N,n) \leq \Lambda(N, N-1) < \Theta(N)$$

for $0 < n < N$. We first prove monotonicity of $\Lambda(N,n)$ and then proceed to the case where $n = N - 1$.

- **Step 4a: Monotonicity**.

Fix some $N \geq 3$ and let $0 < n \leq m < N$ be given. Then,

$$\begin{aligned}\Lambda(N,m) - \Lambda(N,n) &= \sum_{N_L=0}^{\lfloor \frac{m}{2} \rfloor} P(N-m+N_L) \cdot \mathbb{P}_m(N_L) - \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} P(N-n+N_L) \cdot \mathbb{P}_n(N_L) \\ &\geq \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} \Big[P(N-m+N_L) \cdot \mathbb{P}_m(N_L) - P(N-n+N_L) \cdot \mathbb{P}_n(N_L) \Big],\end{aligned}$$

as we assumed that $m \geq n$. Now notice that for each N_L, the probability $\mathbb{P}_m(N_L)$ is decreasing in m if $m \geq 2N_L$. Indeed, for any $m \geq 2N_L$, we have

$$\mathbb{P}_{m+1}(N_L) = \frac{m+1}{m-N_L+1} \cdot \frac{1}{2} \cdot \mathbb{P}_m(N_L) < \mathbb{P}_m(N_L).$$

It follows that

$$\Lambda(N,m) - \Lambda(N,n) \geq \sum_{N_L=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{P}_n(N_L) \Big[P(N-m+N_L) - P(N-n+N_L) \Big] \geq 0,$$

since $P(N-m+N_L) \geq P(N-n+N_L)$, by monotinicity of $P(n)$. We conclude that $\Lambda(N,n)$ is increasing in n for $0 < n < N$.

- **Step 4b:** $\Lambda(N, N-1) < \Theta(N)$**.**
  Now suppose that $n = N-1$. We have:

$$\begin{aligned}\Theta(N) &= \sum_{N_L=1}^{\lfloor \frac{N}{2} \rfloor} P(N_L) \cdot \frac{\mathbb{P}_N(N_L)}{\mathbb{P}_N(Y)}, \\ \Lambda(N, N-1) &= \sum_{N_L=0}^{\lfloor \frac{N-1}{2} \rfloor} P(N_L+1) \cdot \mathbb{P}_{N-1}(N_L) = \sum_{N_L=1}^{\lfloor \frac{N-1}{2} \rfloor + 1} P(N_L) \cdot \mathbb{P}_{N-1}(N_L - 1).\end{aligned}$$

In order to compare these probabilities, we will make use of the following laws for binomial coefficients:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \qquad \binom{n}{k} = \binom{n}{n-k}. \tag{8}$$

We distinguish two cases:

Case 1: N **even.** In this case $\lfloor \frac{N-1}{2} \rfloor + 1 = \lfloor \frac{N}{2} \rfloor = \frac{N}{2}$. Hence, as $\mathbb{P}_N(Y) < 1$

$$\begin{aligned}\Theta(N) - \Lambda(N, N-1) &= \sum_{N_L=1}^{\frac{N}{2}} P(N_L) \cdot \left[\frac{\mathbb{P}_N(N_L)}{\mathbb{P}_N(Y)} - \mathbb{P}_{N-1}(N_L - 1) \right] \\ &> \sum_{N_L=1}^{\frac{N}{2}} P(N_L) \cdot \left[\mathbb{P}_N(N_L) - \mathbb{P}_{N-1}(N_L - 1) \right].\end{aligned}$$

We look at the second factor within the sum. For $N_L < \frac{N}{2}$, we have

$$\begin{aligned}\mathbb{P}_N(N) - \mathbb{P}_{N-1}(N-1) &= \binom{N}{N_L} \frac{1}{2^{N-1}} - \binom{N-1}{N_L - 1} \frac{1}{2^{N-2}} \\ &= \frac{1}{2^{N-2}} \cdot \binom{N-1}{N_L - 1} \cdot \left[\frac{N}{2N_L} - 1 \right].\end{aligned}$$

For $N_L = \frac{N}{2}$, we obtain

$$\begin{aligned}\mathbb{P}_N(N) - \mathbb{P}_{N-1}(N-1) &= \binom{N}{N_L} \frac{1}{2^N} - \binom{N-1}{N_L - 1} \frac{1}{2^{N-2}} \\ &= \frac{1}{2^{N-2}} \cdot \binom{N-1}{N_L - 1} \cdot \left(\frac{N}{4N_L} - 1 \right) \\ &= \frac{1}{2^{N-2}} \cdot \binom{N-1}{N_L - 1} \cdot \frac{-1}{2} \\ &= \frac{-1}{2^{N-1}} \cdot \binom{N-1}{\frac{N}{2} - 1}.\end{aligned}$$

Finally, recall that $P(n)$ is decreasing. We find that

$$\begin{aligned}\Theta(N) - \Lambda(N, N-1) &> \sum_{N_L=1}^{\frac{N}{2}} P(N_L) \cdot [\mathbb{P}_N(N_L) - \mathbb{P}_{N-1}(N_L - 1)] \\ &= \sum_{N_L=1}^{\frac{N}{2}-1} \frac{P(N_L)}{2^{N-2}} \binom{N-1}{N_L - 1} \left[\frac{N}{2N_L} - 1 \right] - \frac{P(\frac{N}{2})}{2^{N-1}} \binom{N-1}{\frac{N}{2} - 1} \\ &\geq \frac{P(\frac{N}{2})}{2^{N-2}} \left[\sum_{N_L=1}^{\frac{N}{2}-1} \binom{N-1}{N_L - 1} - \frac{1}{2} \cdot \binom{N-1}{\frac{N}{2} - 1} \right].\end{aligned}$$

Now, from (8) the fact that N is even, we deduce that

$$\begin{aligned}\sum_{N_L=1}^{\frac{N}{2}-1}\binom{N-1}{N_L-1} &= \sum_{z=0}^{\frac{N}{2}-1}\binom{N-1}{z} - \binom{N-1}{\frac{N}{2}-1} \\ &= \frac{1}{2}\sum_{z=0}^{N-1}\binom{N-1}{z} - \binom{N-1}{\frac{N}{2}-1} \\ &= 2^{N-2} - \binom{N-1}{\frac{N}{2}-1}.\end{aligned}$$

Plugging this into the inequality above, we find that if N is even,

$$\Theta(N) - \Lambda(N, N-1) > \frac{P(\frac{N}{2})}{2^{N-2}}\left[2^{N-2} - \frac{3}{2}\cdot\binom{N-1}{\frac{N}{2}-1}\right] \geq 0,$$

where the last inequality holds for $N \geq 5$. For $N = 4$ we have

$$\Theta(4) = 0.829 > 0.8125 = \Lambda(4, 3).$$

We conclude that for $N \geq 3$ even, we have $\Theta(N) > \Lambda(N, N-1)$.

Case 2: N odd. Now $\lfloor\frac{N-1}{2}\rfloor + 1 = \lfloor\frac{N}{2}\rfloor + 1 = \frac{N+1}{2}$. Hence,

$$\begin{aligned}\Theta(N) - \Lambda(N, N-1) &= \sum_{N_L=1}^{\frac{N-1}{2}} P(N_L)\cdot\left[\frac{\mathbb{P}_N(N_L)}{\mathbb{P}_N(Y)} - \mathbb{P}_{N-1}(N_L-1)\right] - P\left(\tfrac{N+1}{2}\right)\mathbb{P}_{N-1}\left(\tfrac{N-1}{2}\right) \\ &> \sum_{N_L=1}^{\frac{N-1}{2}} P(N_L)\cdot[\mathbb{P}_N(N_L) - \mathbb{P}_{N-1}(N_L-1)] - P\left(\tfrac{N+1}{2}\right)\mathbb{P}_{N-1}\left(\tfrac{N-1}{2}\right)\end{aligned}$$

As N is odd, $N-1$ is even and therefore,

$$\mathbb{P}_{N-1}(N_L-1) = \begin{cases} 0 & \text{if} \quad N_L - 1 > \frac{N-1}{2}; \\ \binom{N-1}{N_L-1}\frac{1}{2^{N-1}} & \text{if} \quad N_L - 1 = \frac{N-1}{2}; \\ \binom{N-1}{N_L-1}\frac{1}{2^{N-2}} & \text{if} \quad \text{else.}\end{cases}$$

We find that

$$\begin{aligned}\sum_{N_L=1}^{\frac{N-1}{2}} P(N_L)\cdot[\mathbb{P}_N(N_L) - \mathbb{P}_{N-1}(N_L-1)] &= \sum_{N_L=1}^{\frac{N-1}{2}} P(N_L)\cdot\left[\frac{1}{2^{N-1}}\cdot\binom{N}{N_L} - \frac{1}{2^{N-2}}\cdot\binom{N-1}{N_L-1}\right]\\ &= \sum_{N_L=1}^{\frac{N-1}{2}} \frac{P(N_L)}{2^{N-2}}\cdot\binom{N-1}{N_L-1}\left[\frac{N}{2N_L}-1\right]\\ &> \frac{P\left(\frac{N+1}{2}\right)}{2^{N-2}}\sum_{N_L=1}^{\frac{N-1}{2}}\binom{N-1}{N_L-1},\end{aligned}$$

by monotonicity of $P(n)$. Now (8) implies that

$$\begin{aligned}\sum_{N_L=1}^{\frac{N-1}{2}}\binom{N-1}{N_L-1} &= \sum_{z=0}^{\frac{N-1}{2}}\binom{N-1}{z} - \binom{N-1}{\frac{N-1}{2}}\\ &= \frac{1}{2}\left[\sum_{z=0}^{N-1}\binom{N-1}{z} - \binom{N-1}{\frac{N-1}{2}} - \binom{N-1}{\frac{N-3}{2}}\right] - \binom{N-1}{\frac{N-1}{2}}\\ &= 2^{N-2} - \frac{3}{2}\binom{N-1}{\frac{N-1}{2}} - \frac{1}{2} - \binom{N-1}{\frac{N-3}{2}}.\end{aligned}$$

Combining everything above, we find that

$$\begin{aligned}\Theta(N) - \Lambda(N,N-1) &> \frac{P\left(\frac{N+1}{2}\right)}{2^{N-2}}\left[2^{N-2} - \frac{3}{2}\binom{N-1}{\frac{N-1}{2}} - \frac{1}{2} - \binom{N-1}{\frac{N-3}{2}} - \frac{1}{2}\binom{N-1}{\frac{N-1}{2}}\right]\\ &> \frac{P\left(\frac{N+1}{2}\right)}{2^{N-2}}\left[2^{N-2} - 2\binom{N-1}{\frac{N-1}{2}} - \frac{1}{2}\binom{N-1}{\frac{N-3}{2}}\right] \geq 0,\end{aligned}$$

where the last inequality holds for $N \geq 17$. This shows that $\Theta(N) > \Lambda(N, N-1)$ for all odd $N \geq 17$. For the odd numbers $3 \leq N < 17$ we have

$$\begin{aligned}&\Theta(3) = 1 > 0.8750 = \Lambda(3,2),\\ &\Theta(5) = 0.833 > 0.7813 = \Lambda(5,4),\\ &\Theta(7) = 0.778 > 0.7383 = \Lambda(7,6),\\ &\Theta(9) = 0.7279 > 0.6871 = \Lambda(11,10),\\ &\Theta(13) = 06810 > 0.6709 = \Lambda(13,12),\\ &\Theta(15) = 0.6681 > 0.6586 = \Lambda(15,14).\end{aligned}$$

Hence, $\Theta(N) > \Lambda(N, N-1)$ for all odd $N \geq 3$.

By Step 4a and 4b, we have shown that $\Theta(N) > \Lambda(N, n)$ for all $N \geq 3$ and $0 < n < N$. This completes the proof.

■

B.2 Proof of Proposition 1

We split the proof into two parts: we first show in Part A the existence of an equilibrium and secondly, we show in Part B that the derived equilibrium is unique.

A) **Existence**

We start with the proof of *(i)* and show that all agents vote sincerely in both stages if they have – and use – the right to vote. We then analyze an individual's decision to abstain from the first stage, given this voting behavior.

Step A1: Sincere voting in the second stage
As voting in the second stage is governed by the simple majority rule, every member with the right to vote supports his preferred variant upon elimination of weakly-dominated strategies. Voting insincerely would lower the probability that his preferred variant is selected.

Step A2: Sincere voting in the first stage
We assume that individual i votes in the first stage. We compare i's expected utility when voting sincerely and insincerely, and show that voting sincerely is his best response.

Regardless of how i votes, his utility depends on the voting behavior of other agents. We assume that all individuals voting in the first stage, except i, vote sincerely. It is, however, unclear how many agents might choose to abstain. Let x_i denote the number of agents, excluding i, who are voting in the first stage, that is, $0 \leq x_i \leq N - 1$. Since the probability distribution of x_i is unknown at this stage, as no equilibrium has been identified yet, it is not possible to compute i's expected utility explicitly. Therefore, we derive i's expected utility conditional on x_i, and for all possible values of x_i.

We start by formulating the probability of winning or losing in each stage, expressed as functions of x_i. Let the indices WW, WL, LW and LL represent the following outcomes for individual i, respectively: winning in both stages, winning in the first stage but losing in the second, losing in the first stage but winning in the second, and losing in both stages. Then, i's expected probabilities of winning are as follows:

$$\Gamma^{Vo}_{WW}(N,x_i) = \begin{cases} P(x_i+1)\frac{1}{2} & \text{if } 0 \leq x_i < N-1 \\ P(N)\left(\frac{P(N)}{2^{N-1}} + \left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2}\right) & \text{if } x_i = N-1 \end{cases}$$

$$\Gamma^{Vo}_{WL}(N,x_i) = \begin{cases} P(x_i+1)\frac{1}{2} & \text{if } 0 \leq x_i < N-1 \\ P(N)\left(\frac{1-P(N)}{2^{N-1}} + \left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2}\right) & \text{if } x_i = N-1 \end{cases}$$

$$\Gamma^{Vo}_{LW}(N,x_i) = \begin{cases} (1-P(x_i+1))\Phi(N,x_i+1) & \text{if } 0 < x_i < N-1 \\ (1-P(N))\Theta(N) & \text{if } x_i = N-1 \end{cases}$$

$$\Gamma^{Vo}_{LL}(N,x_i) = \begin{cases} (1-P(x_i+1))(1-\Phi(N,x_i+1)) & \text{if } 0 < x_i < N-1 \\ (1-P(N))(1-\Theta(N)) & \text{if } x_i = N-1, \end{cases}$$

where the expressions of $\Theta(N)$ and $\Phi(N, x_i+1)$ are given by (4) and (2), respectively, and where the index Vo stands for voting in the first stage. We note that $\Gamma^{Vo}_{LW}(N,x_i)$ and $\Gamma^{Vo}_{LL}(N,x_i)$ are not defined for $x_i = 0$, since losing in the first stage entails more than one agent participating in the vote.

Now we examine i's decision to vote either sincerely or insincerely in the first stage. By voting sincerely, i derives an expected utility of

$$\Gamma^{Vo}_{WW}(N,x_i)(1+M) + \Gamma^{Vo}_{WL}(N,x_i)(0+M) + \Gamma^{Vo}_{LW}(N,x_i), \tag{9}$$

where $0 \leq x_i \leq N-1$ and M represents an agent's strength of preference, with $M = H$ for SI individuals and $M = \epsilon$ for WI agents.

When voting insincerely, the expected utility of i amounts to

$$\Gamma^{Vo}_{WW}(N,x_i) + \Gamma^{Vo}_{LW}(N,x_i)(1+M) + \Gamma^{Vo}_{LL}(N,x_i)(0+M). \tag{10}$$

Voting sincerely is a best response if for all $0 \leq x_i \leq N-1$,

$$M\big(\Gamma^{Vo}_{WW}(N,x_i) + \Gamma^{Vo}_{WL}(N,x_i) - \Gamma^{Vo}_{LW}(N,x_i) - \Gamma^{Vo}_{LL}(N,x_i)\big) > 0. \tag{11}$$

Since $P(x_i+1) > \frac{1}{2} > 1-P(x_i+1)$, for all $x_i \in [0, N-1]$, every individual is better off voting sincerely in the first stage, irrespective of whether they are strongly – or weakly – inclined, given that other agents will also vote sincerely.

In summary, voting sincerely is a best response in the first stage.

Step A3: Decision to abstain from first stage

Next, we evaluate i's decision to abstain in the first stage. We assume again that all agents participating in the first stage, except i, vote sincerely. In Step A2, we have shown that should i decide to participate in the first stage, he will also vote sincerely. Now we calculate i's expected utility by abstaining from voting in the first stage.

Suppose i has decided to abstain from voting in the first stage. Then, his expected probabilities of winning are as follows, where x_i and the indices WW, WL, LW and LL represent the same as in Step A2, the index Ab stands for abstention from voting in the first stage, and $\Lambda(N, x_i)$ is given in (3):

$$\Gamma^{Ab}_{WW}(N, x_i) = \begin{cases} P(N)P(N) & \text{if } x_i = 0 \\ \frac{1}{2}\Lambda(N, x_i) & \text{if } 0 < x_i \leq N-1 \end{cases}$$

$$\Gamma^{Ab}_{WL}(N, x_i) = \begin{cases} P(N)\big(1-P(N)\big) & \text{if } x_i = 0 \\ \frac{1}{2}\big(1-\Lambda(N, x_i)\big) & \text{if } 0 < x_i \leq N-1 \end{cases}$$

$$\Gamma^{Ab}_{LW}(N, x_i) = \begin{cases} \big(1-P(N)\big)P(N) & \text{if } x_i = 0 \\ \frac{1}{2}\Lambda(N, x_i) & \text{if } 0 < x_i \leq N-1 \end{cases}$$

$$\Gamma^{Ab}_{LL}(N, x_i) = \begin{cases} \big(1-P(N)\big)\big(1-P(N)\big) & \text{if } x_i = 0 \\ \frac{1}{2}\big(1-\Lambda(N, x_i)\big) & \text{if } 0 < x_i \leq N-1. \end{cases}$$

For any $x_i \in [0, N-1]$, the expected utility of i by abstaining from the first stage and voting sincerely in the second stage is

$$\Gamma^{Ab}_{WW}(N, x_i)(1+M) + \Gamma^{Ab}_{WL}(N, x_i)(0+M) + \Gamma^{Ab}_{LW}(N, x_i). \tag{12}$$

If i derives a higher expected utility from abstaining, as compared to voting sincerely in the first stage, he will be inclined to abstain.

Step A4: Critical Conditions

To determine which, of participation or abstention in the first stage, gives i a higher expected utility, we need to compare his expected utility from each decision, weighted over the possible values of x_i. We obtain this by weighing the expected utility calculated in (9) and (12) with the probability distribution of x_i, and taking the sum over all possible values of x_i. That is, i will participate in voting in the first stage if and only if

$$\sum_{x_i=0}^{N-1} Pr(x_i)\Big[\Gamma^{Vo}_{WW}(N,x_i)(1+M)+\Gamma^{Vo}_{WL}(N,x_i)M+\Gamma^{Vo}_{LW}(N,x_i)\Big]$$
$$\geq \sum_{x_i=0}^{N-1} Pr(x_i)\Big[\Gamma^{Ab}_{WW}(N,x_i)(1+M)+\Gamma^{Ab}_{WL}(N,x_i)M+\Gamma^{Ab}_{LW}(N,x_i)\Big], \quad (13)$$

where $Pr(x_i)$ denotes the probability that the number of agents voting in the first stage, except i, is x_i. For a given N, this condition imposes an lower bound on M, such that i is better off participating in the first stage if and only if

$$M \geq \frac{\sum_{x_i=0}^{N-1} Pr(x_i)\left[\Gamma^{Ab}_{LW}(N,x_i)-\Gamma^{Vo}_{LW}(N,x_i)+\Gamma^{Ab}_{WW}(N,x_i)-\Gamma^{Vo}_{WW}(N,x_i)\right]}{\sum_{x_i=0}^{N-1} Pr(x_i)\left[\Gamma^{Vo}_{WW}(N,x_i)-\Gamma^{Ab}_{WW}(N,x_i)+\Gamma^{Vo}_{WL}(N,x_i)-\Gamma^{Ab}_{WL}(N,x_i)\right]}. \quad (14)$$

Therefore, SI individuals with sufficiently large H prefer to participate in the first stage and WI individuals with sufficiently small ϵ refrain from voting in the first stage.

Step A5: Sufficient Conditions

Since we do not know the probability distribution of x_i at this stage, it is not yet possible to explicitly calculate the critical value of M such that participation is more attractive than abstention. Therefore, we derive a sufficient condition on M such that if satisfied, it ensures that the critical condition is also satisfied.

We obtain the sufficient condition by calculating the value of M such that Expression (13) is satisfied for each term in the summation from $x_i = 0$ to $N-1$. That

is, for each $x_i \in [0, N-1]$, we have

$$\begin{aligned} 0 \leq \quad & \Big(\Gamma^{Vo}_{WW}(N,x_i) - \Gamma^{Ab}_{WW}(N,x_i)\Big)(1+M) + \Gamma^{Vo}_{LW}(N,x_i) \\ & - \Gamma^{Ab}_{LW}(N,x_i) + \Big(\Gamma^{Vo}_{WL}(N,x_i) - \Gamma^{Ab}_{WL}(N,x_i)\Big)M \\ \Rightarrow M \geq \quad & \frac{\Gamma^{Ab}_{LW}(N,x_i) - \Gamma^{Vo}_{LW}(N,x_i) + \Gamma^{Ab}_{WW}(N,x_i) - \Gamma^{Vo}_{WW}(N,x_i)}{\Gamma^{Vo}_{WW}(N,x_i) - \Gamma^{Ab}_{WW}(N,x_i) + \Gamma^{Vo}_{WL}(N,x_i) - \Gamma^{Ab}_{WL}(N,x_i)}. \end{aligned} \tag{15}$$

This condition simplifies to the following inequalities:

$$M \geq \begin{cases} M_0(N) := \frac{P(N)-\frac{1}{2}}{1-P(N)} & \text{for} \quad x_i = 0, \\ M_1(N,x_i) := \frac{\Lambda(N,x_i) - \frac{P(x_i+1)}{2} - (1-P(x_i+1))\Phi(N,x_i+1)}{P(x_i+1)-\frac{1}{2}} & \text{for} \quad 0 < x_i < N-1, \\ M_2(N) := \frac{\Lambda(N,N-1) - P(N)\left[\frac{P(N)}{2^N-1} + (1-\frac{1}{2^N-1})\frac{1}{2}\right] - (1-P(N))\Theta(N)}{P(N)-\frac{1}{2}} & \text{for} \quad x_i = N-1. \end{cases} \tag{16}$$

We observe that for any given value of N, $M_1(N,x_i)$ is monotonically increasing in x_i.[14] As an illustration, in Figure B.1 we plot $M_1(N,x_i)$ against x_i for $N = 200$.

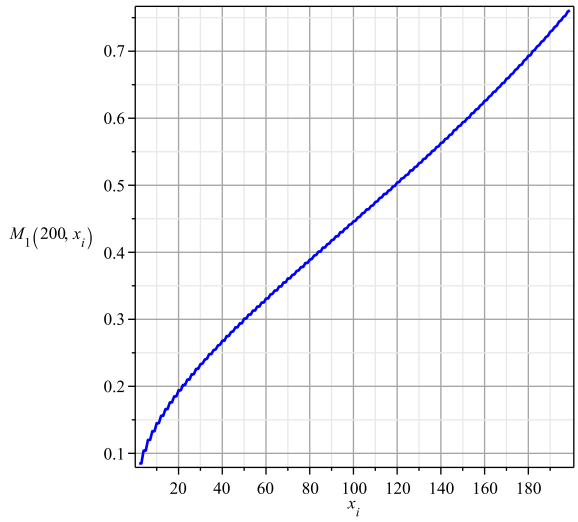

Figure B.1: $M_1(N,x_i)$ vs. x_i for $N = 200$.

As $M_1(N,x_i)$ is monotonically increasing in x_i, it reaches its maximum for $x_i = N-2$ and minimum for $x_i = 1$. In Figure B.2, we plot $M_0(N)$, $M_1(N,N-2)$,

[14] It was verified numerically for $N \in [3,1000]$ and $0 < x_i < N-1$.

$M_1(N,1)$ and $M_2(N)$.

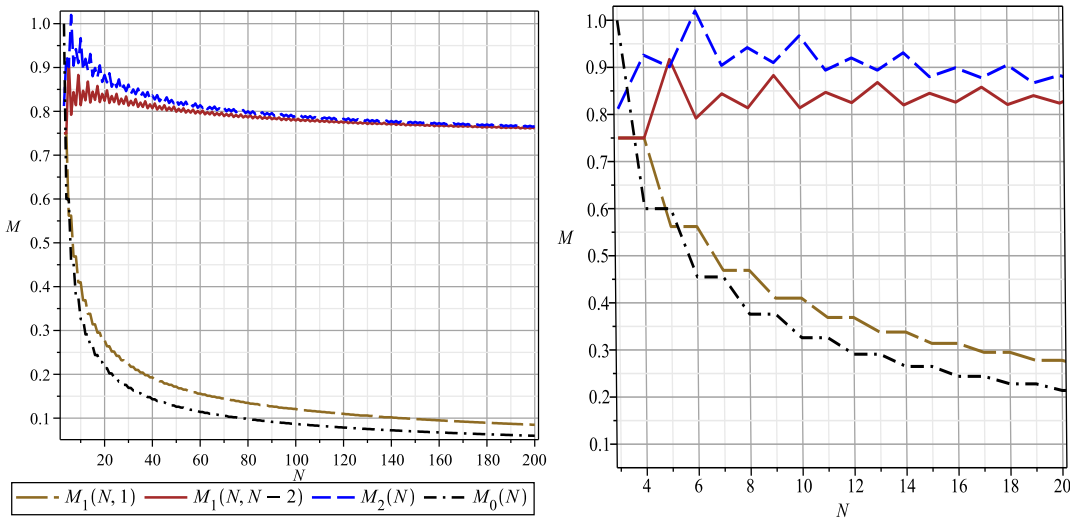

Figure B.2: $M_0(N)$, $M_1(N,N-2)$, $M_1(N,1)$ and $M_2(N)$ vs. N

We note that if x_i is very low (e.g. $x_i = 0$ or 1), the threshold on M monotonically decreases with N. This result is intuitive because for low values of x_i, an increase in N is directly proportional to the increase in the number or agents voting in the second stage. Therefore, as N grows large, it becomes less attractive to abstain from the first stage, and participation becomes attractive even for low M.

If x_i is very large (e.g. $x_i = N-2$ or $N-1$), the threshold on M fluctuates for small N, but decreases asymptotically to 0.7 as $N \to \infty$.[15] It is, however, strictly higher than if x_i is very small for the same N. This is because when x_i is very large, the voting body will always be smaller in the second stage than when x_i is very small – even very small voting bodies being possible with positive probability in the second stage. Therefore, there is a higher incentive to abstain in the first stage and to vote in the second stage, and M has to be sufficiently high for agents to still prefer participation over abstention.

The downward trend of the curve when x_i is high can be explained as follows. As the size of the committee becomes substantially large, the voting body in the second

[15]Numerically verified for the range $N \in [3, 1000]$.

stage also grows larger with positive probability. The desirability of abstaining from the first stage, therefore, shrinks with N, and agents find participation more attractive, even for low M. Note, however, that the threshold on M is still higher than when x_i is very low for the same N. A low value of x_i represents a large voting body in the second stage with certainty, whereas a high x_i indicates that a large voting body in the second stage can occur with high probability.

We set

$$
\begin{aligned}
H \geq H^{crit}(N) \quad &:= \max\Big[M_0(N), M_1(N, N-2), M_2(N)\Big] && (17)\\
\epsilon \leq \epsilon^{crit}(N) \quad &:= \min\Big[M_0(N), M_1(N, 1), M_2(N)\Big] \\
&= \min\Big[M_0(N), M_1(N, 1)\Big] \\
&= \min\Big[\frac{P(N) - \frac{1}{2}}{1 - P(N)},\ 4\Lambda(N, 1) - \Phi(N, 2) - \frac{3}{2}\Big], && (18)
\end{aligned}
$$

so that all SI agents participate in voting in the first stage, while all WI agents abstain. In Figure B.3, we plot $H^{crit}(N)$ and $\epsilon^{crit}(N)$.

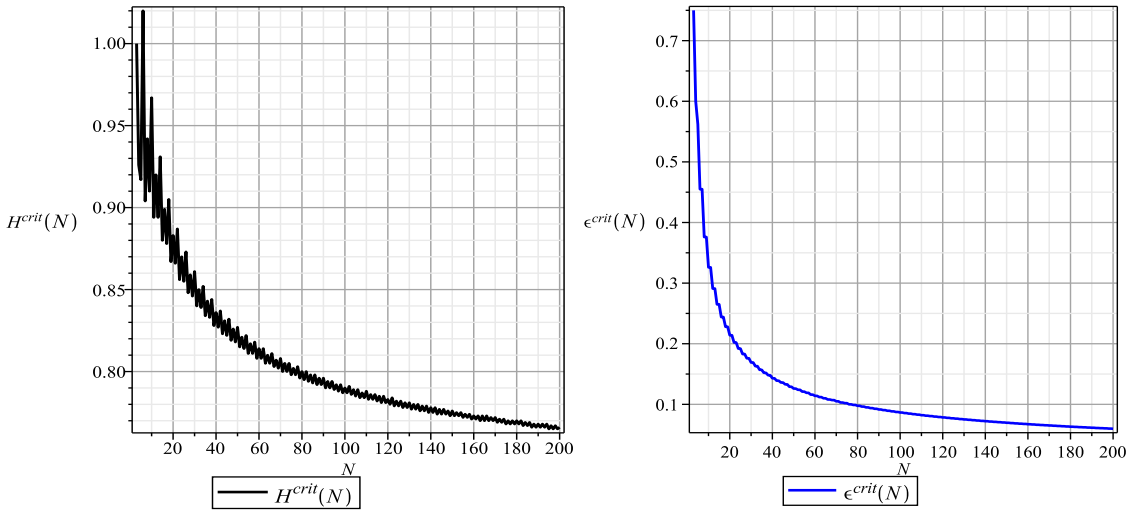

Figure B.3: $H^{crit}(N)$ and $\epsilon^{crit}(N)$ vs. the number of agents N.

We observe that $H^{crit}(N) \leq 1$ for all $N \neq 6$,[16] and $H^{crit}(6) \approx 1$. Under our main assumption that $H >> 1$, all SI individuals will participate in voting in the first stage. Therefore, (18) and $H > 1$ are sufficient conditions for all SI agents to participate in voting in the first stage, and for all WI agents to abstain.

B) **Uniqueness**

We analyze the uniqueness of the constructed equilibrium by first considering sincere voting in the second stage, then sincere voting in the first stage, and finally, the decision to abstain from the first stage.

Step B1: Sincere voting in second stage

After eliminating weakly-dominated strategies, it is clear that agents voting in the second stage have no incentive to deviate from sincere voting in any conceivable equilibrium.

Step B2: Sincere voting in first stage and the role of $P(n)$

Now let us consider the first stage. We start by analyzing individual i's decision to vote sincerely or insincerely, given that he has chosen to participate in voting in the first stage. Recall that $P(n)$ is the probability of winning in the first stage for i, if $n-1$ agents, excluding i, vote in the first stage and if they all vote sincerely. That is, $P(n)$ is based on the critical assumption that all $n-1$ agents vote for a fundamental direction with an ex-ante probability of $\frac{1}{2}$.

Further recall from Step A2 that given $P(n)$, voting sincerely is the best response of i regardless of the number of agents voting in the first stage (x_i). Therefore, i's decision to vote sincerely or insincerely in the first stage is not affected by the other agents' decision to abstain from the first stage or not. Rather, it depends only on $P(n)$, which, in turn, depends on whether other voters vote sincerely or not. Therefore, we will recalculate $P(n)$ in Steps B3 and B4 *without* the assumption that other voters will vote sincerely, and then analyze if voting sincerely is still a dominant strategy for i.

Step B3: Possible symmetric equilibria

[16] It was verified numerically for $N \in [3, 1000]$.

We next consider all conceivable symmetric equilibria where the voters with the same preference intensity behave the same way. From Step B2, we know that the decision of individuals to abstain from the first stage or not can be disregarded for the question whether or not an individual votes sincerely in the first stage. Thus, any equilibrium under BV must belong to one of the following categories:

1. SI agents vote sincerely and WI agents vote insincerely,
2. SI agents vote insincerely and WI agents vote sincerely,
3. All agents vote sincerely,
4. All agents vote insincerely.

Suppose individual j, $j \neq i$, has also chosen to vote in the first stage. We introduce the following notation where Ω, $\Omega' \in \{A, B\}$, $\Omega \neq \Omega'$, VS (VI) denotes voting sincerely (insincerely) and $k \in \{1, 2, 3, 4\}$ denotes the equilibrium category above:

- $j_\Omega :=$ the event that j will vote for Ω,
- $j_{P\Omega} :=$ the event that j prefers the direction Ω,
- $j^k_{VS} :=$ the event that j will vote sincerely in equilibrium k,
- $j^k_{VI} :=$ the event that j will vote insincerely in equilibrium k,
- $Pr[x] :=$ the probability of event x.

For each equilibrium category $k \in \{1, 2, 3, 4\}$, we have $Pr[j^k_{VI}] = 1 - Pr[j^k_{VS}]$ and

$$\begin{aligned} Pr[j^1_{VS}] &= p, \\ Pr[j^2_{VS}] &= 1 - p, \\ Pr[j^3_{VS}] &= 1, \\ Pr[j^4_{VS}] &= 0. \end{aligned}$$

Step B4: Probability of voting for a fundamental direction

For each equilibrium category $k \in \{1, 2, 3, 4\}$, we now have

$$\begin{aligned} Pr[j_\Omega] &= Pr[j_\Omega | j_{P\Omega}] Pr[j_{P\Omega}] + Pr[j_\Omega | j_{P\Omega'}] Pr[j_{P\Omega'}] \\ &= Pr[j^k_{VS} | j_{P\Omega}] Pr[j_{P\Omega}] + Pr[j^k_{VI} | j_{P\Omega'}] Pr[j_{P\Omega'}]. \end{aligned}$$

Due to our definition of symmetric equilibria, the event that j votes sincerely is related to whether he is strongly – or weakly – inclined, and therefore, stochastically independent from the event that his preference is towards Ω or Ω'. Hence, we now have

$$\begin{aligned} Pr[j_\Omega] &= Pr[j^k_{VS}]Pr[j_{P\Omega}] + Pr[j^k_{VI}]Pr[j_{P\Omega'}] \\ &= Pr[j^k_{VS}]\frac{1}{2} + (1 - Pr[j^k_{VS}])\frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

Thus, in all possible symmetric equilibria under consideration from the perspective of individual i, individual j will vote for a fundamental direction with probability $\frac{1}{2}$. Therefore, even without the assumption of sincere voting by other agents, the construction of $P(n)$ remains unchanged. Hence, the analysis in Step A2 is still valid, and allows to conclude that in all possible equilibria, voting sincerely is a dominant strategy for individual i if he participates in voting in the first stage, no matter how other agents vote.

Step B5: Decision to abstain

Given that all agents vote sincerely in both stages, and by construction of H and ϵ, it is clear that the only constellation in which no agent has an incentive to deviate with respect to his choice whether or not to participate in the first stage is the behavior described in Proposition 1. Therefore, the constructed equilibrium is unique.

■

B.3 Proof of Lemma 1

The proof of Lemma 1 follows the exact steps of the proof of Proposition 1. The only difference is in the utility calculations which are presented below.

Step 1: Sincere voting in both stages

It is clear that agents who save their vote for the second stage vote sincerely. Suppose agent i decides to use his vote in the first stage and all other agents voting in the first

stage vote sincerely. Then, i's expected probabilities of winning, derived in the proof of Proposition 1, are modified as follows:

$$\Gamma^{Vo}_{WW}(N,x_i) = \begin{cases} P(x_i+1)\frac{1}{2} & \text{if } 0 \le x_i < N-1 \\ P(N)P(N) & \text{if } x_i = N-1 \end{cases}$$

$$\Gamma^{Vo}_{WL}(N,x_i) = \begin{cases} P(x_i+1)\frac{1}{2} & \text{if } 0 \le x_i < N-1 \\ P(N)\big(1-P(N)\big) & \text{if } x_i = N-1 \end{cases}$$

$$\Gamma^{Vo}_{LW}(N,x_i) = \begin{cases} \big(1-P(x_i+1)\big)\frac{1}{2} & \text{if } 0 < x_i < N-1 \\ \big(1-P(N)\big)P(N) & \text{if } x_i = N-1 \end{cases}$$

$$\Gamma^{Vo}_{LL}(N,x_i) = \begin{cases} \big(1-P(x_i+1)\big)\frac{1}{2} & \text{if } 0 < x_i < N-1 \\ \big(1-P(N)\big)\big(1-P(N)\big) & \text{if } x_i = N-1. \end{cases}$$

As in the proof of Proposition 1, voting sincerely is a best response if and only if Condition (11) is satisfied. After applying the modified probabilities of winning, it is clear that Condition (11) is satisfied as $P(x_i+1) > 1-P(x_i+1)$ for all $x_i \in [0, N-1]$. Therefore, if an individual decides to use his vote in the first stage, voting sincerely is a dominant strategy, no matter whether he is strongly or weakly-inclined.

Step 2: Decision to store the vote

Now suppose that i has decided to store his vote for the second stage. Then, his probabilities of winning are modified as follows:

$$\Gamma^{Ab}_{WW}(N,x_i) = \begin{cases} P(N)P(N) & \text{if } x_i = 0 \\ \frac{1}{2}P(N-x_i) & \text{if } 0 < x_i \le N-1 \end{cases}$$

$$\Gamma^{Ab}_{WL}(N,x_i) = \begin{cases} P(N)\big(1-P(N)\big) & \text{if } x_i = 0 \\ \frac{1}{2}\big(1-P(N-x_i)\big) & \text{if } 0 < x_i \le N-1 \end{cases}$$

$$\Gamma^{Ab}_{LW}(N,x_i) = \begin{cases} \big(1-P(N)\big)P(N) & \text{if } x_i = 0 \\ \frac{1}{2}P(N-x_i) & \text{if } 0 < x_i \le N-1 \end{cases}$$

$$\Gamma^{Ab}_{LL}(N,x_i) = \begin{cases} \big(1-P(N)\big)\big(1-P(N)\big) & \text{if } x_i = 0 \\ \frac{1}{2}\big(1-P(N-x_i)\big) & \text{if } 0 < x_i \le N-1. \end{cases}$$

As in the proof of Proposition 1, voting in the first stage is a best response if and only if critical Condition (14) is satisfied. Since the probability distribution of x_i is unknown at this stage, we once again derive a sufficient condition on M, as in the proof of Proposition 1. Applying the modified probabilities of winning to Expression (13), we

obtain that an individual i will vote in the first stage if

$$M \geq \begin{cases} M_0(N) := \frac{P(N)-\frac{1}{2}}{1-P(N)} & \text{for} \quad x_i = 0, \\ M_3(N, x_i) := \frac{P(N-x_i)-\frac{1}{2}}{P(x_i+1)-\frac{1}{2}} & \text{for} \quad 0 < x_i < N-1, \\ M_4(N) := \frac{1-P(N)}{P(N)-\frac{1}{2}} & \text{for} \quad x_i = N-1. \end{cases} \tag{19}$$

Therefore, SI individuals with sufficiently large H use their vote in the first stage, while WI individuals with sufficiently small ϵ store their vote for the second stage.

Step 3: Existence of equilibrium

We note that $M_3(N, x_i)$ is monotonically increasing in x_i. Therefore, it reaches its maximum value for $x_i = N-2$ and minimum for $x_i = 1$. In Figure B.4, we plot $M_0(N)$, $M_3(N, N-2)$, $M_3(N, 1)$ and $M_4(N)$ against N.[17]

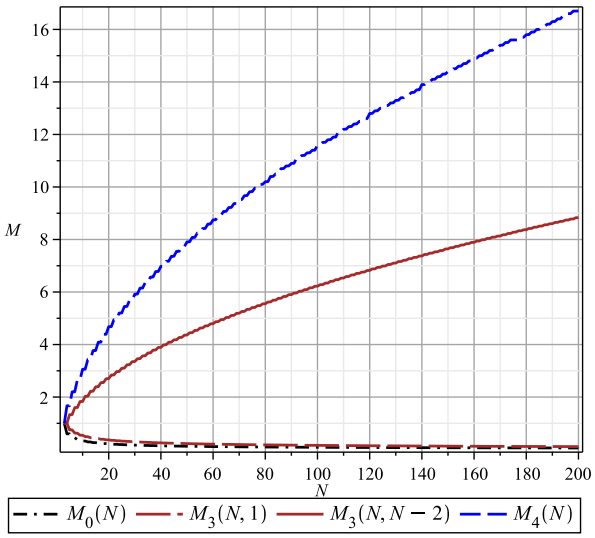

Figure B.4: $M_0(N)$, $M_3(N, N-2)$, $M_3(N, 1)$ and $M_4(N)$ vs. N.

We note that if x_i is very low (e.g. $x_i = 0$ or 1), the threshold on M monotonically decreases with N for the same reason as with BV. The curve for $x_i = 0$ is, in fact, identical to the one under BV.

[17]Further properties of these expressions are given in Table D.2.

For x_i very large, however, the threshold on M is higher than under BV. A very large x_i under ST means a very small voting body in the second stage, as only those who abstain will vote in the second stage. Therefore, it is more attractive for individuals to abstain under ST, and M has to be higher than under BV for participation to be more attractive than abstention.

Moreover, for very large x_i (e.g. $x_i = N-1$ or $N-2$), the threshold on M is increasing with N. Under ST, as N increases, the number of agents voting in the first stage increases proportionately. The probability of winning in the first stage thus grows smaller, making the very high probability of winning in the second stage more attractive. Therefore, M has to increase analogously to keep participation more attractive than abstention.

We set

$$\begin{aligned} H &\geq H_{st}^{crit}(N) := \max\left[M_0(N), M_3(N, N-2), M_4(N)\right] = \frac{1-P(N)}{P(N)-\frac{1}{2}} \\ \epsilon &\leq \epsilon_{st}^{crit}(N) := \min\left[M_0(N), M_3(N, 1), M_4(N)\right] = \frac{P(N)-\frac{1}{2}}{1-P(N)} \geq \epsilon^{crit}(N), \end{aligned}$$

so that all SI agents will participate in voting in the first stage. We refer to Figure 4 for the plot of $H_{st}^{crit}(N)$ and $\epsilon_{st}^{crit}(N)$ against N.

Step 4: Uniqueness
Following the same logic as in the proof of Proposition 1, the constructed equilibrium is unique.

■

Proof of Lemma 2:

We proceed in two steps.

Step 1: Sincere voting in second stage
As in the other voting schemes, there are no gains from voting strategically in the second stage. Hence, all individuals with voting rights cast their votes sincerely in the second stage, as they have a positive probability to be pivotal.

Step 2: Sincere voting in first stage
We note that no agent has an incentive to abstain from voting under MV. Abstention

under MV means being excluded from voting in both stages, while participation in a vote strictly increases an individual's chances of winning. Furthermore, following the logic outlined in Proposition 1, in equilibrium, an agent i assumes that other individuals vote sincerely in the first stage because in all symmetric equilibria under consideration, an agent will vote for a fundamental direction with probability $\frac{1}{2}$. As a consequence, given that all N agents will vote in the first stage, i estimates that by voting sincerely, he will derive an ex-ante expected utility of

$$\begin{aligned} & P(N)\left[\frac{1}{2}(1+M)+\frac{1}{2}(0+M)\right]+\Big(1-P(N)\Big)\Theta(N) \\ = \; & P(N)\left[\frac{1}{2}+M\right]+\Big(1-P(N)\Big)\Theta(N), \end{aligned}$$

where $M \in \{H, \epsilon\}$ and $\Theta(N)$ is the ex-ante probability of winning in the second stage for losers of the first stage under MV, given by (4).

If i votes strategically, his expected utility is

$$\begin{aligned} & P(N)\frac{1}{2}+\Big(1-P(N)\Big)\left[\Theta(N)(1+M)+(1-\Theta(N))\,(0+M)\right] \\ = \; & \frac{1}{2}P(N)+\Big(1-P(N)\Big)\left[\Theta(N)+M\right]. \end{aligned}$$

Since $P(N) > \frac{1}{2} > 1 - P(N)$, voting sincerely is a dominant strategy for both SI and WI individuals.

■

Appendix C: Welfare Comparisons

C.1 Welfare Expressions

We begin our welfare comparisons by calculating W^{BV}, W^{SM}, W^{ST}, and W^{MV}, the expected aggregate utility at equilibrium under BV, SM, ST, and MV respectively. Recall that in any of these voting systems, an individual i belongs to the set SI with probability p, independently of the other voters. By Bayes' rule we thus have

$$W^V = \mathbb{E}\left[\sum_{i=1}^{N} U_i\right] = \sum_{N_s=0}^{N}\sum_{i=1}^{N}\binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)}\,\mathbb{E}\left[U_i \,\middle|\, |SI| = N_s\right]. \tag{20}$$

for any $V \in \{BV, SM, ST, MV\}$. For simplicity, we denote $\mathbb{E}_{N_s}[U_i] \equiv \mathbb{E}\left[U_i \,\middle|\, |SI| = N_s\right]$ from now on.

By assumption, we know that the expected utility of an individual voter given N_s depends solely on the type of inclination they have about the outcome of the first voting round. Moreover, we know that at equilibrium, if voters can abstain from voting, they do so in the first round if and only if they belong to WI. We have

$$\sum_{i=1}^{N} \mathbb{E}_{N_s}[U_i] = N_s\,\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in SI\right] + (N - N_s)\,\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in WI\right], \tag{21}$$

under any of the four voting systems. In Subsections C.1.1 to C.1.4, we use (20) and (21) to determine W^{BV}, W^{SM}, W^{ST}, and W^{MV}.

C.1.1 Welfare under BV

We first consider the voting system BV. Recall that if everyone abstains from voting (i.e. if $N_s = 0$), or if there are neither abstantees nor losers in the first round (i.e. if $N_s = N$ and $N_L = 0$), then all voters are allowed to participate in both rounds. Moreover, the realized utility of the voters depends on the voting rounds won. For ease of reference, we summarize the utilities and ex-ante probabilities of winning for each group of agents given the number of strongly inclined voters N_s in Table C.1.[18]

[18] We note that the probability of winning in the first stage only and the probability of losing in both stages can be directly derived from the expressions in Table C.1.

		$\mathbb{P}(\cdot)$	U_i
$i \in WI$	Win in both stages	$P(N)P(N)$	$1+\epsilon$
and $N_s = 0$	Win only in 2^{nd} stage	$(1-P(N))P(N)$	1
$i \in WI$	Win in both stages	$\frac{1}{2}\Lambda(N,N_s)$	$1+\epsilon$
and $0 < N_s < N$	Win only in 2^{nd} stage	$\frac{1}{2}\Lambda(N,N_s)$	1
$i \in SI$	Win in both stages	$P(N)\left(\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2} + \left(\frac{1}{2^{N-1}}\right)P(N)\right)$	$1+H$
and $N_s = N$	Win only in 2^{nd} stage	$(1-P(N))\Theta(N)$	1
$i \in SI$	Win in both stages	$P(N_s)\frac{1}{2}$	$1+H$
and $0 < N_s < N$	Win only in 2^{nd} stage	$(1-P(N_s))\Phi(N,N_s)$	1

Table C.1: *Utilities and ex-ante probabilities under BV given inclination and N_s.*

First suppose that $0 < N_S < N$. Using Table C.1, we find that

$$\begin{aligned}
\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in SI\right] &= P(N_s)\tfrac{1}{2}(1+H) + P(N_s)\tfrac{1}{2}H + \big(1-P(N_s)\big)\Phi(N,N_s);\\
\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in WI\right] &= \frac{1}{2}\Lambda(N,N_s)(1+\epsilon) + \frac{1}{2}\big(1-\Lambda(N,N_s)\big)\epsilon + \frac{1}{2}\Lambda(N,N_s)\\
&= \Lambda(N,N_s) + \frac{\epsilon}{2}.
\end{aligned}$$

Now assume that $N_s = 0$. In this case, all voters are allowed to vote in both rounds:

$$\begin{aligned}
\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in WI\right] &= P(N)P(N)(1+\epsilon) + P(N)(1-P(N))\cdot\epsilon + (1-P(N))P(N)\\
&= P(N)(1+\epsilon).
\end{aligned}$$

Finally, suppose that $N_s = N$. As everyone is voting in the first round now, we have to

account for the possibility that there is not any loser in this round. We have

$$\begin{aligned}\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in SI\right] =& P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2}+\frac{1}{2^{N-1}}P(N)\right](1+H)+\\ & P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2}+\frac{1}{2^{N-1}}\left(1-P(N)\right)\right]H+\\ & \left(1-P(N)\right)\Theta(N)\\ =& P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\left(\frac{1}{2}+H\right)+\frac{1}{2^{N-1}}\left(P(N)+H\right)\right]+\\ & \left(1-P(N)\right)\Theta(N).\end{aligned}$$

Combining these three cases with (20) and (21) yields

$$\begin{aligned}W^{BV} = & \sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\left[N_s\left(P(N_s)\left(\frac{1}{2}+H\right)+\big(1-P(N_s)\big)\Phi(N,N_s)\right)\right.\\ & \qquad\qquad \left.+(N-N_s)\left(\Lambda(N,N_s)+\frac{\epsilon}{2}\right)\right]\\ & +(1-p)^N NP(N)(1+\epsilon)\\ & +p^N N\left(P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\left(\frac{1}{2}+H\right)+\frac{1}{2^{N-1}}\left(P(N)+H\right)\right]\right.\\ & \qquad\qquad \left.+\big(1-P(N)\big)\Theta(N)\right).\end{aligned}$$

C.1.2 Welfare under SM

Recall that under SM, all voters vote in both rounds. The utilities and ex-ante probabilities of winning given the inclination can be found in Table C.2. Applying (20) and (21) then yields

$$\begin{aligned}W^{SM} = & \sum_{N_s=0}^{N}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\Big[N_s\Big(P(N)\Big(P(N)+H\Big)+[1-P(N)]P(N)\Big)\\ & +(N-N_s)\Big(P(N)\Big(P(N)(1+\epsilon)+[1-P(N)]\epsilon\Big)+[1-P(N)]P(N)\Big)\Big]\\ = & \sum_{N_s=0}^{N}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\Big[N_sP(N)(1+H)+(N-N_s)P(N)(1+\epsilon)\Big].\end{aligned}$$

		$\mathbb{P}(\cdot)$	U_i
$i \in WI$	Win in both stages Win only in 2^{nd} stage	$P(N)P(N)$ $(1-P(N))P(N)$	$1+\epsilon$ 1
$i \in SI$	Win in both stages Win only in 2^{nd} stage	$P(N)P(N)$ $(1-P(N))P(N)$	$1+H$ 1

Table C.2: *Utilities and ex-ante probabilities under SM given inclination.*

C.1.3 Welfare under ST

Under ST, all voters can vote in exactly one round, unless everyone abstains from voting in the first round. That is, if $|SI| = N_s = 0$ at equilibrium, all voters are allowed to vote in both rounds. We summarize the ex-ante probabilities of winning and the utilities in Table C.3.

		$\mathbb{P}(\cdot)$	U_i
For WI and $N_s = 0$	Win in both stages Win only in 2^{nd} stage	$P(N)P(N)$ $(1-P(N))P(N)$	$1+\epsilon$ 1
For WI and $0 < N_s < N$	Win in both stages Win only in 2^{nd} stage	$\frac{1}{2}P(N-N_s)$ $\frac{1}{2}P(N-N_s$	$1+\epsilon$ 1
For SI and $N_s = N$	Win in both stages Win only in 2^{nd} stage	$P(N)P(N)$ $(1-P(N))P(N)$	$1+H$ 1
For SI and $0 < N_s < N$	Win in both stages Win only in 2^{nd} stage	$P(N_s)\frac{1}{2}$ $(1-P(N_s))\frac{1}{2}$	$1+H$ 1

Table C.3: *Utilities and ex-ante probabilities under ST given inclination and N_s*

.

By (20), (21), and Table C.3, we have

$$W^{ST} = \sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} \left[N_s \left(P(N_s)H + \frac{1}{2} \right) + (N-N_s) \left(P(N-N_s) + \frac{\epsilon}{2} \right) \right] + p^N N\big(P(N)\,(H+1)\,\big) + (1-p)^N N\big(P(N)\,(\epsilon+1)\,\big).$$

C.1.4 Welfare under MV

Under MV, all voters vote in the first round, and only the losers have the right to vote again in the second round. Recall that if however, there were no losers in the first round, all voters are allowed to vote in the second round again. The ex-ante probabilities of winning, given N_s and the inclination, can be found in Table C.4.

		$\mathbb{P}(\cdot)$	U_i
$i \in WI$	Win in both stages	$P(N)\left(\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2} + \left(\frac{1}{2^{N-1}}\right)P(N)\right)$	$1+\epsilon$
and $0 \leq N_s < N$	Win only in 2^{nd} stage	$(1-P(N))\Theta(N)$	1
$i \in SI$	Win in both stages	$P(N)\left(\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2} + \left(\frac{1}{2^{N-1}}\right)P(N)\right)$	$1+H$
and $0 < N_s \leq N$	Win only in 2^{nd} stage	$(1-P(N))\Theta(N)$	1

Table C.4: *Utilities and ex-ante probabilities under MV, given inclination and N_s.*

From Table C.4 we find that if $0 < N_s \leq N$,

$$\begin{aligned}
\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in SI\right] &= P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2} + \frac{1}{2^{N-1}}P(N)\right](1+H) + \\
&\quad P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\frac{1}{2} + \frac{1}{2^{N-1}}\left(1-P(N)\right)\right]H + \\
&\quad (1-P(N))\,\Theta(N) \\
&= P(N)\left[\left(1-\frac{1}{2^{N-1}}\right)\left(\frac{1}{2}+H\right) + \frac{1}{2^{N-1}}\left(P(N)+H\right)\right] + \\
&\quad (1-P(N))\,\Theta(N) \\
&= P(N)\left[\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + H\right] + (1-P(N))\,\Theta(N).
\end{aligned}$$

Similarly, we know that for $0 \leq N_s < N$ we have

$$\mathbb{E}_{N_s}\left[U_i \,\middle|\, i \in WI\right] = P(N)\left[\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + \epsilon\right] + (1 - P(N))\,\Theta(N).$$

Hence, by (20) and (21), welfare under MV is given by

$$W^{MV} = \sum_{N_s=0}^{N} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} \cdot \left[N_s\left(P(N)\left[\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + H\right] + (1-P(N))\,\Theta(N)\right) + (N-N_s)\left(P(N)\left[\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + \epsilon\right] + (1-P(N))\,\Theta(N)\right)\right].$$

C.2 Welfare Comparison of BV and SM

We summarize the results of welfare comparison between BV and SM in the following proposition:

Proposition 2
The comparison of ex-ante expected utility under SM and BV in equilibrium yields the following insights:

- *The welfare under BV is higher than the welfare under SM if*

$$H \geq M(N,p)\,\epsilon + C(N,p),$$

  where $M(N,p)$ and $C(N,p)$ are given by Equations (28) and (29), respectively.

- *WI individuals are indifferent between BV and SM if $N_s = 0$.*

- *When $0 < N_s < N$, WI agents prefer BV to SM, provided that*

$$\epsilon \leq \min\{\epsilon^{crit}(N),\ \epsilon^{SM}(N,N_s)\}.$$

- *When $N_s = N$, SI individuals strictly prefer SM to BV.*

- *When $0 < N_s < N$, SI agents prefer BV to SM if*

$$H \geq H^{SM}(N,N_s).$$

- *When $N_s = N-1$ and N is odd, SI agents strictly prefer SM to BV.*

We also summarize the conditions under which BV Pareto-dominates SM.

Proposition 3

The equilibrium outcome under BV Pareto-dominates the outcome under SM if and only if the following conditions are satisfied:

(i) $0 < N_s < N$ *when* N *is even and* $0 < N_s < N-1$ *when* N *is odd,*

(ii) $\epsilon \leq \min\{\epsilon^{crit}(N),\ \epsilon^{SM}(N, N_s)\}$, *and*

(iii) $H \geq H^{SM}(N, N_s)$.

In order to prove Proposition 2 and 3, we need to compare W^{BV} and W^{SM}. Recall their expressions from Subsection C.1. BV yields a higher welfare than SM if and only if $W^{BV} \geq W^{SM}$. That is, if and only if

$$\begin{aligned}
&\sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} N_s \left[P(N_s)\left(\frac{1}{2}+H\right) + (1-P(N_s))\Phi(N,N_s) - P(N)(H+1) \right] \\
&+ \sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} (N-N_s) \left[\Lambda(N,N_s) + \frac{\epsilon}{2} - P(N)(\epsilon+1) \right] \\
&+ p^N N \left[P(N) \left\{ \frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} - 1 \right\} + (1-P(N))\Theta(N) \right] \geq 0.
\end{aligned} \tag{22}$$

The first expression in square brackets is the expected utility difference for an SI individual when $0 < N_s < N$, while the second set of square brackets captures the same for a WI agent. The third expression in square brackets shows the expected utility difference for an SI agent when $N_s = N$. We note that there is no welfare difference between the two voting schemes when $N_s = 0$, as SM and BV coincide in that case.

We prove the two propositions in Subsections C.2.1 and C.2.2.

C.2.1 Utility Comparison for Agents

We first compare the expected utility under BV with the one under SM for different groups of agents, before proceeding with the comparison of overall welfare in the next subsection. Recall that the probabilities of winning under BV and SM in both stages,

as well as the winning probabilities for the second stage only, are summarized in Tables C.1 and C.2.

For WI agents if $N_s = 0$

Since

$$\mathbb{E}^{BV}_{N_s=0}\left[U_i \,|\, i \in WI\right] = P(N)\,(1+\epsilon) = \mathbb{E}^{SM}_{N_s=0}\left[U_i \,|\, i \in WI\right],$$

any WI agents is indifferent between the two voting schemes if $N_s = 0$.

For WI agents if $0 < N_s < N$

In this scenario, WI individuals prefer BV to SM if and only if

$$\begin{aligned} \mathbb{E}^{SM}_{N_s}\left[U_i \,|\, i \in WI\right] &\leq \mathbb{E}^{BV}_{N_s}\left[U_i \,|\, i \in WI\right] \\ P(N)(\epsilon+1) &\leq \Lambda(N,N_s) + \frac{\epsilon}{2} \\ \epsilon &\leq \frac{\Lambda(N,N_s) - P(N)}{P(N) - \frac{1}{2}} =: \epsilon^{SM}(N,N_s). \end{aligned} \tag{23}$$

In Figure C.1, we plot $\epsilon^{SM}(N,N_s)$ against N_s for $N = 20$ and $N = 200$.[19] We also plot the corresponding $\epsilon^{crit}(N)$ value for comparison.

We note that WI agents face a trade-off with BV when $0 < N_s < N$, compared to SM. They have a better probability of winning in the second stage, although their probability of winning in the first stage is lower. Their inclination for a fundamental direction ϵ must therefore be adequately low for them to prefer abstention in the first stage in exchange for a guaranteed vote in the second, thereby preferring BV over SM.

As N_s becomes larger for a given value of N under BV, the body of voters in the second stage becomes smaller with positive probability. The benefits of using BV thus become more substantial and $\epsilon \leq \epsilon^{crit}(N)$ is a sufficient condition for WI agents to prefer BV over SM. For very small N_s, however, $\epsilon^{SM}(N,N_s)$ is more restrictive than $\epsilon^{crit}(N)$. When most of the society consists of WI agents, these agents' probability of winning in the second stage becomes very similar under BV and SM, thus weakening the advantage presented by BV over SM. Hence, for WI agents to prefer BV over SM, the inclination towards A or B must be correspondingly small.

[19] Further properties of $\epsilon^{SM}(N,N_s)$ are given in Table D.1.

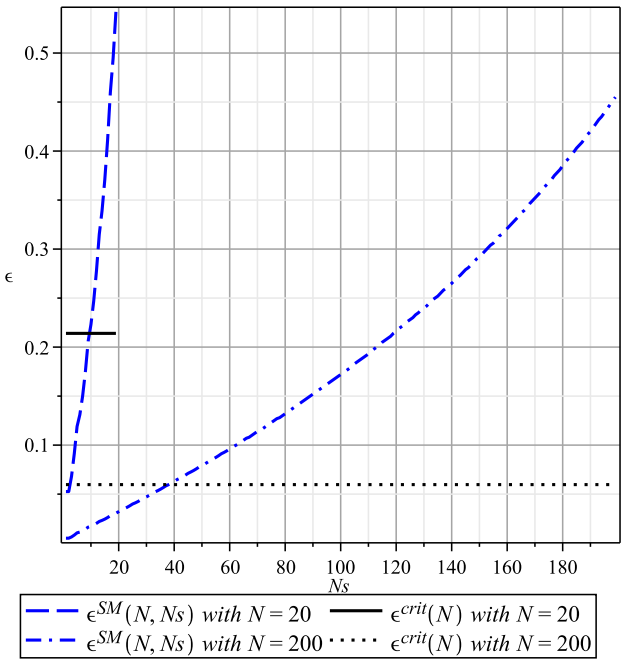

Figure C.1: $\epsilon^{SM}(N, N_s)$ and $\epsilon^{crit}(N)$ vs. the number of SI agents N_s.

We observe that for $N_s = 1$ and N odd, $\Lambda(N, 1) = P(N)$ from Fact 1 and therefore, $\epsilon^{SM}(N, N_s) = 0$. WI agents face the same probability of winning in the second stage as with SM, but have a lower probability of winning in the first stage compared to SM. Thus, in this case WI agents strictly prefer SM to BV.

For SI agents if $N_s = N$

In this case, SI agents prefer BV to SM if and only if

$$0 \leq P(N)\left\{\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} - 1\right\} + \big(1 - P(N)\big)\Theta(N) =: Q(N) \tag{24}$$

To examine whether (24) holds, we plot $Q(N)$ in Figure C.2. We note that $Q(N)$ is negative for all N.[20] Therefore, when the whole society is comprised of SI agents, these agents strictly prefer SM to BV. We note that the probability of winning in the first stage is the same under SM and BV. In the second stage, losers of the first stage have a higher probability of winning under BV, while this probability is lower for winners.

[20]This follows from the observation that $P(N)$ is a monotonically-decreasing sequence bounded by $\frac{1}{2}$. We further note that $Q(N)$ converges to 0. This follows from the fact that $P(N)$ and $\Theta(N)$ converge to $\frac{1}{2}$ for $N \to \infty$ while preserving $\Theta(N) > P(N)$ for all N.

The welfare gains from protecting the minority, however, do not offset the welfare losses resulting from the loss of voting rights for the majority.

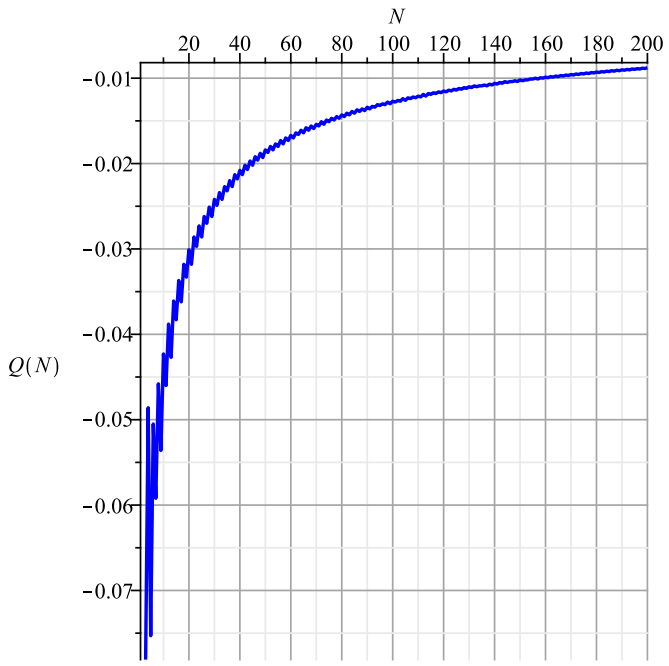

Figure C.2: $Q(N)$ vs. the number of agents N.

For SI agents if $0 < N_s < N$

SI individuals prefer BV to SM if and only if

$$P(N)(H+1) \quad \leq P(N_s)\left(\frac{1}{2}+H\right)+(1-P(N_s))\Phi(N,N_s),$$

which is equivalent to

$$(P(N_s)-P(N))H \quad \geq P(N)-\frac{P(N_s)}{2}-(1-P(N_s))\Phi(N,N_s). \tag{25}$$

When $P(N_s) \neq P(N)$, this yields

$$H \geq H^{SM}(N,N_s) := \frac{P(N)-\frac{P(N_s)}{2}-(1-P(N_s))\Phi(N,N_s)}{P(N_s)-P(N)}. \tag{26}$$

The plots of $H^{SM}(N,N_s)$ for $N=20$ and $N=200$ are given in Figure C.3.[21]

[21]Further properties of $H^{SM}(N,N_s)$ are given in Table D.1.

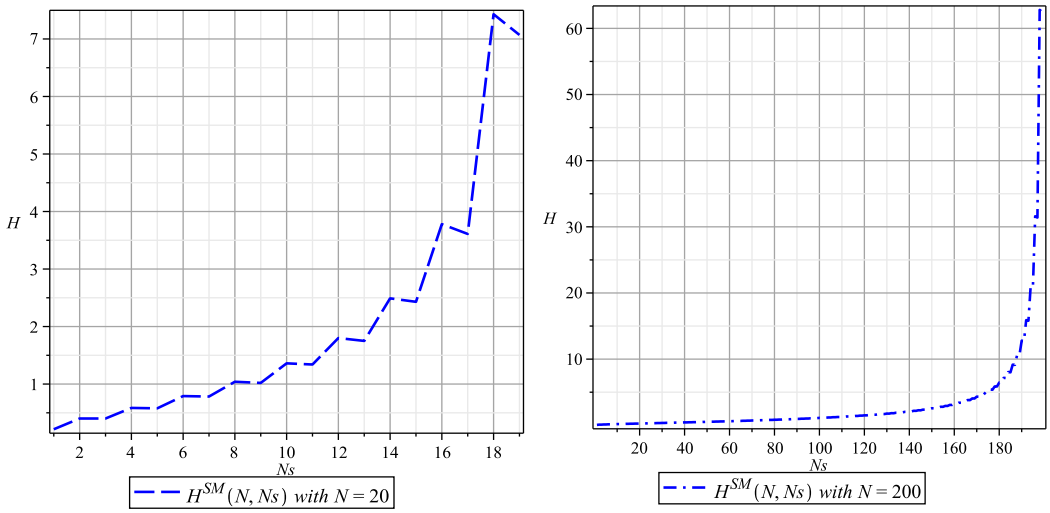

Figure C.3: $H^{SM}(N, N_s)$ vs. the number of SI agents N_s for $N = 20$ and $N = 200$.

We observe that for small N_s, SI individuals prefer BV over SM, since $H > 1$. BV restricts the number of voters in the first stage to N_s, thus yielding a higher probability of winning for SI agents, which is particularly significant for small groups of SI individuals. The majority of these agents, however, face a lower probability of winning in the second stage. This probability becomes more undesirable as N_s increases and the probability of winning in the first stage approaches the one under SM. Therefore, SI agents must have a stronger inclination towards a fundamental direction to find BV more favorable despite the risk of losing their voting right for the second stage.

We note that $H^{SM}(N, N_s)$ is undefined for $P(N_s) = P(N)$ as the denominator becomes zero. This occurs when $N_s = N - 1$ and N is odd. In this case, Inequality (25) does not hold, as the right-hand side is positive for all N.[22] Hence, if $N_s = N - 1$ and N is odd, SI agents strictly prefer SM over BV. The result is intuitive, because in such circumstances, BV does not offer any additional benefits for SI individuals in the first stage, as they face the same probability of winning as under SM. A large fraction of the society, however, loses its voting rights for the second stage, and this utility loss is not compensated by the increase in the probability of winning for the minority.

[22]This conclusion follows from the observation that $Q(N) + P(N) - \frac{P(N_s)}{2} - (1 - P(N_s))\Phi(N, N_s) > 0$ while $Q(N) < 0$ for all N.

C.2.2 Aggregate Utility Comparison

We now compare the aggregate expected utility under BV to the one under SM. Rearranging the Expression (22) yields

$$\begin{aligned}
&\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\left[N_s\left\{P(N_s)-P(N)\right\}H-(N-N_s)\left(P(N)-\frac{1}{2}\right)\epsilon\right]\\
\geq\ &\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\Bigg[(N-N_s)\left\{P(N)-\Lambda(N,N_s)\right\}\\
&\qquad\qquad+N_s\left\{P(N)-\frac{P(N_s)}{2}-(1-P(N_s))\Phi(N,N_s)\right\}\Bigg]\\
&+p^N N\left[P(N)\left\{1-\frac{2^{N-1}-1}{2^N}-\frac{P(N)}{2^{N-1}}\right\}-\big(1-P(N)\big)\Theta(N)\right].
\end{aligned}$$

This is equivalent to

$$H\geq M(N,p)\,\epsilon+C(N,p), \tag{27}$$

where $M(N,p)$ and $C(N,p)$ are given by

$$M(N,p):=\ \frac{\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}(N-N_s)\left(P(N)-\frac{1}{2}\right)}{\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}N_s(P(N_s)-P(N))} \tag{28}$$

and

$$C(N,p):=\ \frac{\overline{C}(N,p)}{\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}N_s\left(P(N_s)-P(N)\right)}. \tag{29}$$

Here,

$$\begin{aligned}
\overline{C}(N,p):=\ &\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\Bigg[(N-N_s)\left\{P(N)-\Lambda(N,N_s)\right\}\\
&\qquad\qquad+N_s\left\{P(N)-\frac{P(N_s)}{2}-(1-P(N_s))\Phi(N,N_s)\right\}\Bigg]\\
&+p^N N\left[P(N)\left\{1-\frac{2^{N-1}-1}{2^N}-\frac{P(N)}{2^{N-1}}\right\}-\big(1-P(N)\big)\Theta(N)\right].
\end{aligned}$$

We next verify that $M(N,p)$ and $C(N,p)$ are finite and positive. For $M(N,p)$, both the nominator and the denominator are positive and finite sums. The reason is that for N

odd, $N-1$ is even and thus $N-N_s>0$ and $P(N_s)-P(N)>0$. Moreover, $P(N)>\frac{1}{2}$. The same logic, together with the property $P(N)>\Lambda(N,N_s)$ from Fact 1, establishes that $C(N,p)\in(0,\infty)$.[23] To illustrate this with an example, we plot the equality of (27) in Figure C.4 for $N=20$ and for different values of p. We also denote the corresponding $\epsilon^{crit}(N)$ and $H=1$ values, and shade the area in which Proposition 1 is satisfied.

The reason why H has to increase when ϵ increases for BV to be welfare-superior to SM is as follows: An increase in ϵ means that WI agents may prefer SM over BV when N_s is small. To maintain that welfare under BV is superior to that under SM, H must increase analogously to ensure that more SI agents prefer BV over SM when N_s is large.

Moreover, we observe that for small values of p (e.g. $p=0.3$), welfare is superior under BV to welfare under SM. As p increases, H must increase as well to ensure the superiority of BV over SM. An increase in p indicates an expected increase in the number of SI agents in the society. As explained in the preceding analysis of Section C.2.1, as N_s increases, H has to simultaneously increase for SI agents to maintain these agents' preference for BV over SM.

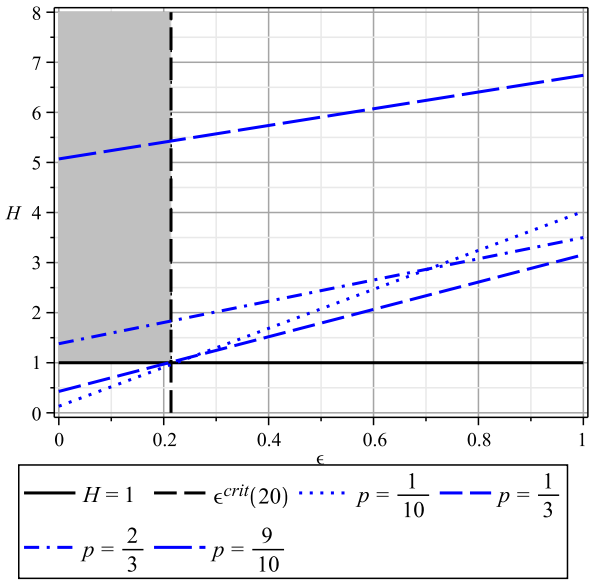

Figure C.4: H vs. ϵ for $W^{BV}=W^{SM}$ to hold when $N=20$, according to (27).

[23] It has also been verified numerically for the ranges $N\in[3,1000]$ and $p\in[0.01,0.99]$ in steps of 0.01.

C.3 Welfare Comparison of BV and ST

We summarize the results of welfare comparison of BV and ST in the following proposition:

Proposition 4

The comparison of expected utility under ST and BV in equilibrium yields the following insights:

- *The welfare under BV is higher than the welfare under ST for sufficiently large N, i.e. if*

$$N \geq N_{st}(p),$$

  where $N_{st}(p)$ is defined in (32).

- *WI individuals are indifferent between BV and ST if $N_s = 0$.*
- *When $0 < N_s < N$, WI individuals strictly prefer ST to BV.*
- *When $N_s = N$, SI agents strictly prefer ST to BV.*
- *When $0 < N_s < N$, SI individuals strictly prefer BV to ST.*

In order to prove Proposition 4, we need to compare W^{BV} and W^{ST}, which were calculated in Subsection C.1. BV yields a higher welfare than SM if and only if $W^{BV} \geq W^{ST}$. That is, if and only if

$$\begin{aligned}
&\sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} N_s \left[P(N_s)\left(\frac{1}{2}+H\right) - \left(P(N_s)H + \frac{1}{2}\right) \right. \\
&\qquad\qquad \left. + \big(1-P(N_s)\big)\Phi(N,N_s) \right] \\
&+ \sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} (N-N_s) \left[\Lambda(N,N_s) + \frac{\epsilon}{2} - \left(P(N-N_s) + \frac{\epsilon}{2}\right) \right] \\
&+ (1-p)^N N \left[P(N)(\epsilon+1) - P(N)(\epsilon+1) \right] \\
&+ p^N N \left[P(N) \left\{ \frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} - 1 \right\} + \big(1-P(N)\big)\Theta(N) \right] \geq 0. \qquad (30)
\end{aligned}$$

The first expression in square brackets captures the welfare difference for SI individuals if $0 < N_s < N$, while the second set of square brackets indicates the welfare difference

for WI agents for the same situation. The third expression in square brackets is the welfare difference for WI individuals if $N_s = 0$, which is zero. The remaining expression displays the welfare difference for SI agents if $N_s = N$.

The proof of Proposition 4 is established in Subsections C.3.1 and C.3.2.

C.3.1 Utility Comparison for Agents

We first compare the expected utility under BV to the one under ST for different groups of agents.

For WI agents if $N_s = 0$

WI individuals are indifferent between ST and BV in this case.

For WI agents if $0 < N_s < N$

We note that $\Lambda(N, N_s) < P(N - N_s)$, as $\Lambda(N, N_s)$ gives the ex-ante probability of winning when the committee comprises the WI agents and the losers from the first stage, while $P(N - N_s)$ is the probability of winning when the committee consists only of WI agents. Therefore, WI individuals strictly prefer ST to BV if $0 < N_s < N$. Their probability of winning is higher under ST, as there are no losers from the first stage who take part in voting in the second stage.

For SI agents if $N_s = N$

In this case, SI agents prefer BV to ST if and only if

$$0 \leq P(N) \left\{ \frac{2^{N-1} - 1}{2^N} + \frac{P(N)}{2^{N-1}} - 1 \right\} + \big(1 - P(N)\big)\Theta(N) = Q(N).$$

From Figure C.2, we observe that $Q(N)$ is negative for all N. Thus, when $N_s = N$, SI agents strictly prefer ST to BV. ST is identical to SM when $N_s = N$, and, as explained in Section C.2, BV is inferior to SM when $N_s = N$. As N becomes larger, $P(N)$ tends to $\frac{1}{2}$, and the gap between losses and gains is reduced, driving $Q(N)$ closer to zero.

For SI agents if $0 < N_s < N$

The SI agents are strictly better off with BV – as opposed to ST – if $0 < N_s < N$. The reason is that BV provides them with the same benefits as ST, with the additional feature of better protection, should they belong to the minority of the first stage.

C.3.2 Aggregate Utility Comparison

Next we compare the aggregate expected utility under BV to the one under ST. We rearrange (30) to

$$\begin{aligned}\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}&\left[N_s\left\{\big(1-P(N_s)\big)\Phi(N,N_s)+\frac{P(N_s)}{2}-\frac{1}{2}\right\}\right.\\&\left.-(N-N_s)\left\{P(N-N_s)-\Lambda(N,N_s)\right\}\right]\\+p^N N&\left[P(N)\left\{\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}-1\right\}+\big(1-P(N)\big)\Theta(N)\right]\qquad\geq 0.\end{aligned}\tag{31}$$

We denote the entire expression on the left-hand side of Inequality (31) by $K(N,p)$. Hence, we have

$$W^{BV}\geq W^{ST}\iff\quad K(N,p)\geq 0.$$

In Figure C.5, we plot $K(N,p)$ against p for different values of N.

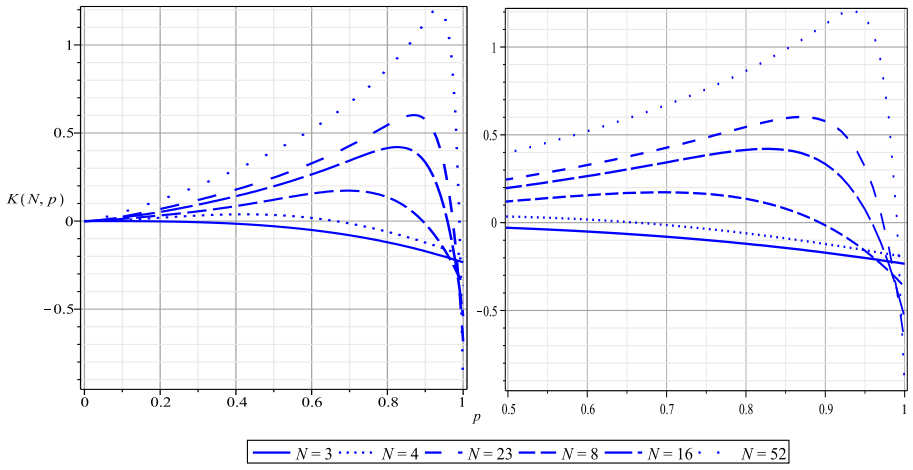

Figure C.5: $K(N,p)$ vs. p.

We observe that (31) holds for sufficiently large N or sufficiently small p. We have

verified numerically that for each $N > 3$, there is some p which yields $K(N, p) = 0$.[24] For $N = 3$, welfare properties of ST are superior to BV for all values of p. From Figure C.2, we note that if $N_s = N$, the social losses under BV are rather large if $N = 3$, contributing to the poor welfare performance. We also note that the welfare losses for $N_s = N$ rapidly become smaller as N increases. Hence, for $N > 3$, BV is superior to ST as long as $p \leq 0.6$. As p increases, however, N must increase analogously to keep welfare under BV more desirable than under ST. For $p = 0.9$ and $p = 0.99$, for example, $N > 8$ and $N > 51$ must be satisfied, respectively, for BV to be superior to ST. An increase in p means there is a larger probability that the committee will be comprised of SI agents. Hence, the case $N_s = N$ has a higher impact on the overall welfare. Thus, as p increases, N must also increase to ensure $W^{BV} \geq W^{ST}$.

We define the threshold on N that yields $K(N, p) \geq 0$ for a given value of p as $N_{st}(p)$, i.e.

$$K(N, p) \gtreqless 0 \quad \text{if} \quad N \gtreqless N_{st}(p). \tag{32}$$

For $p \in [0.6, 0.99]$ in steps of 0.01, we calculate $N_{st}(p)$, which determines the threshold on N that yields $K(N, p) \geq 0$. In Figure C.6, we plot $N_{st}(p)$ against p.

Thus, the welfare under BV is higher than the welfare under ST for $N \geq N_{st}(p)$. We note that the welfare comparison between BV and ST imposes a restriction on the number of agents that depends on p for BV to be superior to ST. The intuition is as follows: A large p indicates a high probability that the entire society consists of SI agents. Consider the case where the entire society does consist of SI agents indeed. Then, ST allows all individuals to vote in both stages, while under BV, the majority is not allowed to vote in the second stage. When N is low, this effectively means that the majority loses voting rights on a decision for which it would have had a high chance of being pivotal. This leads to large welfare losses under BV. As N increases, however, the chances of being pivotal in the second stage decrease, and the utility losses under BV diminish. Thus, for sufficiently large N, BV is superior to ST from a utilitarian welfare perspective.

[24] The analysis was conducted for $N \in [4, 1000]$ and $p \in [0.01, 0.99]$ in steps of 0.001.

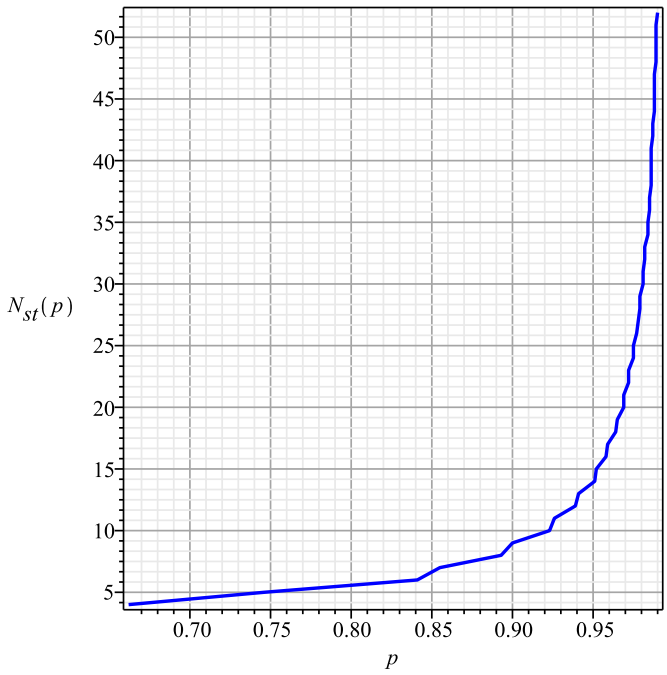

Figure C.6: $N_{st}(p)$ vs. p.

C.4 Welfare Comparison of BV and MV

We summarize the outcomes of the welfare comparison between BV and MV in the following proposition:

Proposition 5

Suppose that $H >> 1$ *and* $\epsilon \leq \epsilon^{crit}(N)$. *Then the comparison of ex-ante expected utility under MV and BV in equilibrium yields the following results.*

- *The welfare under BV is strictly higher than the welfare under MV for all* $N \notin \{3, 5, 7\}$.
- *For* $N \in \{3, 5, 7\}$, *welfare under BV is superior to welfare under MV if*

$$H \geq M(N, p)\,\epsilon + D(N, p),$$

  where $M(N, p)$ *and* $D(N, p)$ *are given by Equations* (28) *and* (40), *respectively.*
- *WI individuals strictly prefer BV to MV if* $N_s = 0$.

- *When $0 < N_s < N$, WI agents strictly prefer BV to MV, provided that*

$$\epsilon \leq \min\{\epsilon^{crit}(N),\ \epsilon^{MV}(N, N_s)\}.$$

- *When $N_s = N$, SI individuals are indifferent between BV and MV.*
- *When $0 < N_s < N$, SI agents strictly prefer BV to MV.*
- *When $N_s = N - 1$ and N is odd, SI agents strictly prefer MV to BV.*

We also summarize the conditions under which BV Pareto-dominates MV.

Proposition 6
The equilibrium outcome under BV Pareto-dominates the outcome under MV if and only if the following conditions are satisfied:

(i) $N_s < N$ *when* N *is even and* $N_s < N - 1$ *when* N *is odd, and*

(ii) $\epsilon \leq \min\{\epsilon^{crit}(N),\ \epsilon^{MV}(N, N_s)\}$.

We prove Proposition 5 and 6 by comparing W^{BV} and W^{MV}, which we calculated in Subsection C.1. Welfare under BV is higher than welfare under MV if and only if $W^{BV} \geq W^{MV}$, i.e. if and only if

$$\begin{aligned}
&\sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} N_s \left[P(N_s)\left(\frac{1}{2} + H\right) + (1 - P(N_s))\Phi(N, N_s) \right. \\
&\qquad \left. - P(N)\left(\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + H\right) - (1 - P(N))\Theta(N) \right] \\
&+ \sum_{N_s=1}^{N-1} \binom{N}{N_s} p^{N_s}(1-p)^{(N-N_s)} (N - N_s) \left[\Lambda(N, N_s) + \frac{\epsilon}{2} - (1 - P(N))\Theta(N) \right. \\
&\qquad \left. - P(N)\left(\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + \epsilon\right) \right] \\
&-(1-p)^N N \left[P(N)\left(\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} - 1\right) + (1 - P(N))\Theta(N) \right] \geq 0. \qquad (33)
\end{aligned}$$

The first expression in square brackets captures the welfare difference for SI individuals if $0 < N_s < N$, while the second pair of square brackets comprises the welfare difference for WI agents in the same case. The third expression in square brackets shows the welfare difference for WI individuals if $N_s = 0$.

The proof of Propositions 5 and 6 is spread out over Subsection C.4.1 and C.4.2.

C.4.1 Utility Comparison for Agents

We first compare the expected utility under BV with the one under MV for different groups of agents.

For WI agents if $N_s = 0$

WI individuals prefer BV to MV in this case if and only if

$$P(N) \geq \quad P(N)\left(\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}}\right) + (1-P(N))\Theta(N),$$

or equivalently,

$$0 \geq \quad P(N)\left(\frac{P(N)}{2^{N-1}} - \frac{1}{2} - \frac{1}{2^N}\right) + (1-P(N))\Theta(N) = Q(N). \quad (34)$$

Recall that Figure C.2 gives the plot of $Q(N)$. Hence, Inequality (34) holds for all N. When the entire society consists of WI agents, everyone votes in both stages under BV, while under MV, only losers of the first stage can vote in the second stage. For the majority, the loss of voting rights on the decision they would have preferred to exert more influence on results in a welfare loss under MV, thus making BV strictly more desirable.

For WI agents if $0 < N_s < N$

In this case, BV is preferred to MV by WI individuals if and only if

$$\Lambda(N, N_s) + \frac{\epsilon}{2} - P(N)\left(\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}} + \epsilon\right) - (1-P(N))\Theta(N) \geq 0 \quad (35)$$

$$\epsilon \leq \underbrace{\frac{1}{P(N)-\frac{1}{2}}\left[\Lambda(N, N_s) - P(N)\left(\frac{2^{N-1}-1}{2^N} + \frac{P(N)}{2^{N-1}}\right) - (1-P(N))\Theta(N)\right]}_{=:\, \epsilon^{MV}(N, N_s)}. \quad (36)$$

We note that for a given N, $\Lambda(N, N_s)$ is monotonically increasing in N_s. Therefore, so is the right-hand side of Inequality (36). Since we have assumed $\epsilon \leq \epsilon^{crit}(N)$ throughout the paper, we examine if Inequality (36) holds for the least possible value of N_s, which is $N_s = 1$, when $\epsilon = \epsilon^{crit}(N)$. In Figure C.7, we plot $\epsilon^{MV}(N, 1) - \epsilon^{crit}(N)$ for $N \in [3, 200]$ and observe that it is positive if N is not too small, i.e. if $N > 7$.[25]

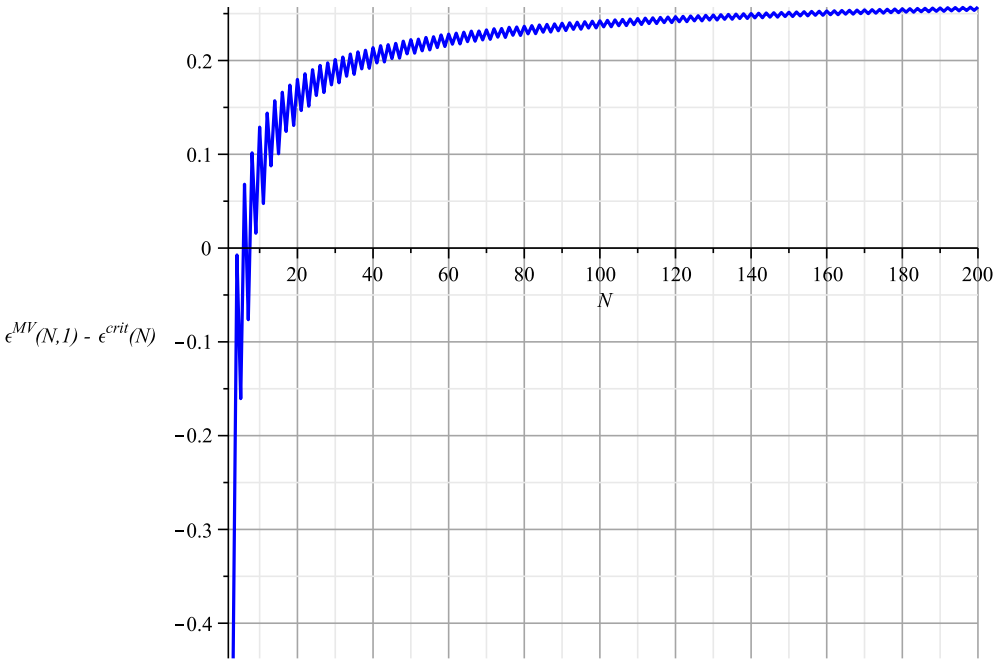

Figure C.7: $\epsilon^{MV}(N, 1) - \epsilon^{crit}(N)$ vs. N.

Thus, our assumption $\epsilon \leq \epsilon^{crit}(N)$ is sufficient for WI agents to prefer BV over MV when $N > 7$. Under MV, WI individuals vote with the rest of the society in the first stage, but face the possibility of losing their right to vote in the second stage when the voting body is much smaller. This leads to a welfare loss that is not offset by the comparatively higher probability of winning in the first stage, which is small if the society is large.

When $N \leq 7$, the probability of winning in the first stage, $P(N)$, is much larger than $\frac{1}{2}$, which renders participation in the first stage more attractive. Therefore, ϵ must satisfy the more stringent criteria $\epsilon \leq \epsilon^{MV}(N, 1) < \epsilon^{crit}(N)$ for WI individuals to find BV more attractive than MV.

[25]The generalization of the result to all N follows from the observation that for all N_s, the left-hand side of Inequality (35) is positive and tends to zero as $N \to \infty$.

For SI agents if $N_s = N$

SI individuals are indifferent between BV and MV if $N_s = N$, as BV and MV coincide in this case.

For SI agents if $0 < N_s < N$

BV is preferred to MV by SI individuals in this case if and only if

$$P(N_s)\left(\frac{1}{2}+H\right)+(1-P(N_s))\Phi(N,N_s)-P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}+H\right) - (1-P(N))\Theta(N)\geq 0$$

$$\Rightarrow \quad H\big(P(N_s)-P(N)\big)\geq P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}\right)+(1-P(N))\Theta(N) - (1-P(N_s))\Phi(N,N_s)-\frac{P(N_s)}{2}. \tag{37}$$

When $P(N_s)\neq P(N)$ this means,

$$H\geq \frac{1}{P(N_s)-P(N)}\left[P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}\right)+(1-P(N))\Theta(N) - (1-P(N_s))\Phi(N,N_s)-\frac{P(N_s)}{2}\right] =: H^{MV}(N,N_s). \tag{38}$$

Since the relationship between $H^{MV}(N,N_s)$ and N_s is not straightforward, we numerically calculate the maximum value of $H^{MV}(N,N_s)$ for each $N\in[3,200]$ and plot it in Figure C.8.

We note that (38) holds if $H > H^{MV}(N,N_s)\leq 1.26$.[26] Since we have assumed $H >> 1$, it is therefore possible to conclude that (38) holds for all N. Under MV, the entire society participates in the first stage, while only SI agents do so under BV. This gives them a better chance of winning in the decision that matters more to them. MV, however, offers a higher probability of winning in the second stage. Yet, the benefits of BV still outweigh those of MV. Therefore, SI agents strictly prefer BV over MV when $0 < N_s < N$.

[26]The generalization of the result to all N follows from the observation that the right-hand side of Inequality (38) tends to zero as $N\to\infty$, since $\Phi(N,N_s)\approx\Theta(N)$ for large N and N_s. A large value of N_s is considered, since the trend of $H^{MV}(N,N_s)$ is increasing with N_s.

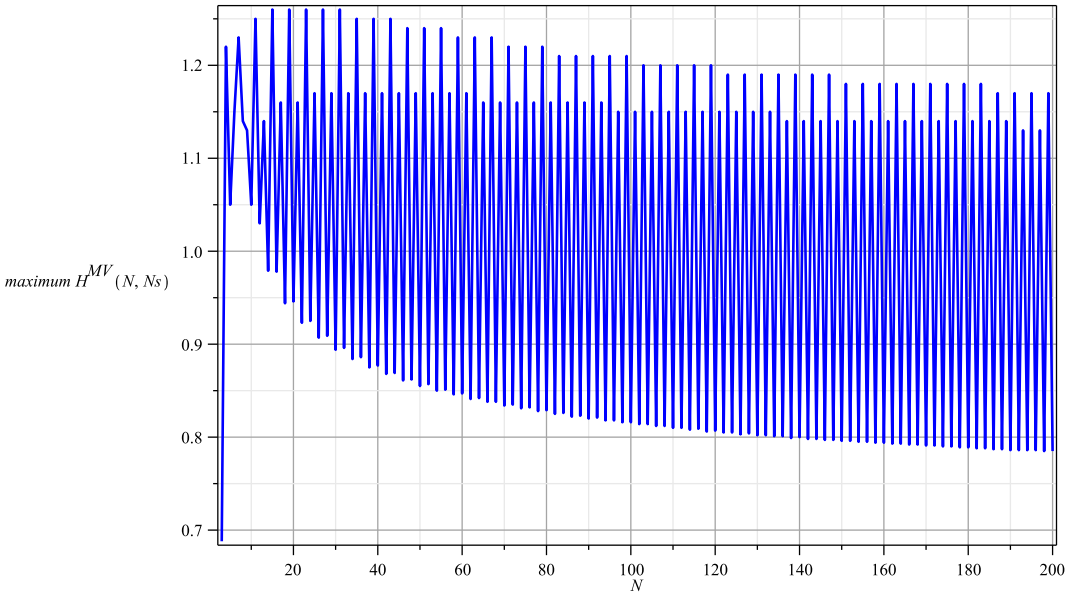

Figure C.8: $\max_{N_s} H^{MV}(N, N_s)$ vs. the number of agents N.

We note that $H^{MV}(N, N_s)$ is undefined for $P(N_s) = P(N)$, that is, when $N_s = N - 1$ and N is odd. In this case, Inequality (37) does not hold, as the right-hand side is strictly positive for all N and tends to zero as $N \to \infty$. Hence, if $N_s = N - 1$ and N is odd, SI agents strictly prefer MV over BV. Although there is one agent less who votes in the first stage under BV, SI individuals face the same probability of winning as the one under MV, since $P(N-1) = P(N)$. In the second stage, however, the WI agent participates in the voting under BV, which decreases the SI minority's chances of winning.

C.4.2 Aggregate Utility Comparison

Next, we compare the aggregate expected utility under BV to the one under MV. We rearrange (33) to obtain that $W^{BV} \geq W^{MV}$ if and only if

$$
\begin{aligned}
&\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\left[N_sH\big(P(N_s)-P(N)\big)-(N-N_s)\left(P(N)-\frac{1}{2}\right)\epsilon\right]\\
\geq\ &\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}N_s\Bigg[(1-P(N))\Theta(N)-\big(1-P(N_s)\big)\Phi(N,N_s)\\
&\qquad\qquad+P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}\right)-\frac{P(N_s)}{2}\Bigg]\\
&+\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}(N-N_s)\left[P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}\right)\right.\\
&\qquad\qquad\left.+(1-P(N))\Theta(N)-\Lambda(N,N_s)\right]\\
&+(1-p)^NN\left[(1-P(N))\Theta(N)-P(N)\left(1-\frac{2^{N-1}-1}{2^N}-\frac{P(N)}{2^{N-1}}\right)\right],
\end{aligned}
$$

which is equivalent to

$$
\iff\quad H\geq M(N,p)\,\epsilon+D(N,p), \tag{39}
$$

where $M(N,p)$ is given in (28) and $D(N,p)$ is defined by

$$
D(N,p):=\frac{\overline{D}(N,p)}{\displaystyle\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}N_s\left(P(N_s)-P(N)\right).} \tag{40}
$$

Here,

$$
\begin{aligned}
\overline{D}(N,p):=\ &\sum_{N_s=1}^{N-1}\binom{N}{N_s}p^{N_s}(1-p)^{(N-N_s)}\Bigg[(N-N_s)\left\{P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}\right)\right.\\
&\qquad\left.+(1-P(N))\Theta(N)-\Lambda(N,N_s)\right\}\\
&\qquad+N_s\left\{P(N)\left(\frac{2^{N-1}-1}{2^N}+\frac{P(N)}{2^{N-1}}\right)+(1-P(N))\Theta(N)\right.\\
&\qquad\left.-\frac{P(N_s)}{2}-(1-P(N_s))\Phi(N,N_s)\right\}\Bigg]\\
&+(1-p)^NN\left[\big(1-P(N)\big)\Theta(N)-P(N)\left(1-\frac{2^{N-1}-1}{2^N}-\frac{P(N)}{2^{N-1}}\right)\right].
\end{aligned}
$$

We note that $D(N,p)$ is finite since the denominator is strictly positive and we have a finite sum. Moreover, $D(N,p) < C(N,p)$.[27] To illustrate, we plot the Equality (39) in Figure C.9 for $N = 20$ and for different values of p. We also plot the values of $H = 1$ and $\epsilon^{crit}(20)$, and mark the area in which H and ϵ must lie to satisfy Proposition 1. We have verified that for all $N \notin \{3,5,7\}$, welfare under BV is always superior to the welfare under MV.[28] In Section C.4.1, we showed that in almost all situations, individuals either strictly or weakly prefer BV over MV, given $H >> 1$ and $\epsilon \leq \epsilon^{crit}(N)$ are satisfied. This results in the welfare under BV being strictly better than under MV.

For $N \in \{3,5,7\}$, welfare under BV is superior to welfare under MV only if H and ϵ satisfy Expression (39). In Section C.4.1, we illustrated that $\epsilon \leq \epsilon^{crit}(N)$ is not sufficient for WI agents to prefer BV over MV for $N \leq 7$. Furthermore, we showed that when N is odd and $N_s = N - 1$, SI agents strictly prefer MV to BV. The combination of these two effects yields the result that welfare under BV is not always superior to welfare under MV for $N \in \{3,5,7\}$. Welfare properties of BV are better only if ϵ is sufficiently low, or analogously, if H is sufficiently high, such that Expression (39) is satisfied.

[27] This observation follows from the fact that the denominator is the same for both $C(N,p)$ and $D(N,p)$, and $\overline{D}(N,p)$ is finite and is strictly smaller than $\overline{C}(N,p)$, since $Q(N)$ is negative but finite for all N.

[28] This was verified numerically for the ranges $N \in [3,1000]$ and $p \in [0.01, 0.99]$ in steps of 0.01.

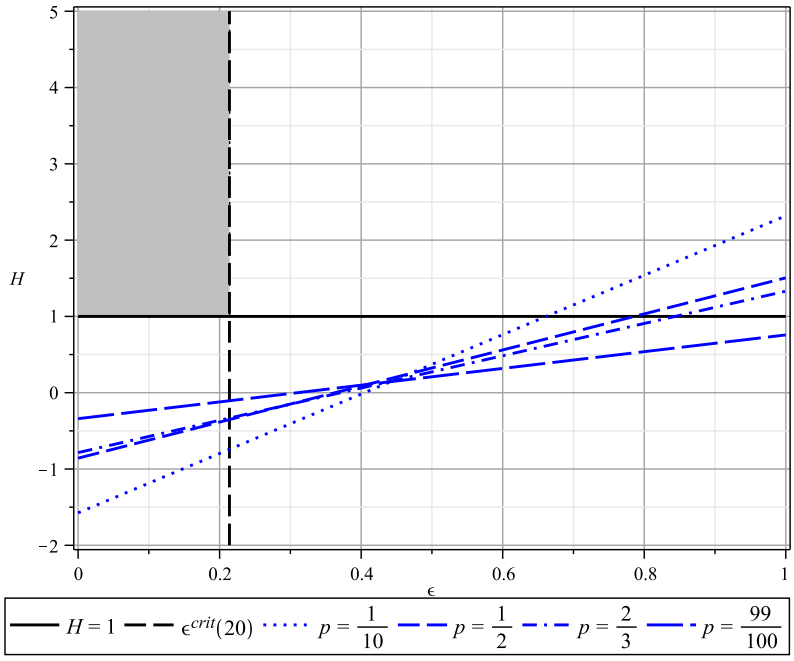

Figure C.9: H vs. ϵ to satisfy $W^{BV} \geq W^{MV}$ for $N = 20$.

Appendix D: Summary of Properties of Key Expressions

In this Appendix, we summarize the properties of the key expressions used in the main text. Specifically, we analyze each expression at the limit, and state the results obtained. When analytical verification is not possible, we provide the ranges for which an expression has been numerically verified to fulfil the properties considered in the main text.

Expression	**Formula**	**Analytical Results**	**Numerical Verifications**	**Illustration**
$P(N)$	Equation (1)	$\lim_{n\to\infty} P(n) = \frac{1}{2}$		vs N for $N \in [1, 20]$ Figure 1
$\Theta(N), \Lambda(N,n), \Phi(N,n)$	$\Theta(N) =$ (4) $\Lambda(N,n) =$ (3) $\Phi(N,n) =$ (2)	$\lim_{N\to\infty} \Theta(N) = \frac{1}{2}$ $\forall n, \lim_{N\to\infty} \Lambda(N,n) = \frac{1}{2}$ $\forall n, \lim_{N\to\infty} \Phi(N,n) = \frac{1}{2}$		vs n for $N = 20$ Figure 2
$\epsilon^{crit}(N)$	Proposition 1	$\lim_{N\to\infty} \epsilon^{crit}(N) = 0$		vs N for $N \in [3, 200]$ Figure 3
$H_{st}^{crit}(N)$, $\epsilon_{st}^{crit}(N)$	Lemma 1	$\lim_{N\to\infty} \epsilon^{crit}(N) = 0$ $\lim_{N\to\infty} H^{crit}(N) = \infty$		vs N for $N \in [3, 200]$ Figure 4
$\epsilon^{SM}(N, N_s)$	Equation (23)	$\forall N_s$ when N even and $\forall N_s \neq 1$ when N odd, $\lim_{N\to\infty} \epsilon^{SM}(N, N_s) = \infty$		vs N_s for $N = 20$ and $N = 200$ Figure C.1
$Q(N)$	Equation (24)	$\lim_{N\to\infty} Q(N) = 0$		vs N for $N \in [3, 200]$ Figure C.2
$H^{SM}(N, N_s)$	Equation (26)	$\forall N_s$ when N even and $\forall N_s \neq N-1$ when N odd, $\lim_{N\to\infty} H^{SM}(N, N_s) = \infty$		vs N_s for $N = 20$ and $N = 200$ Figure C.3
$H = M(N,p)\,\epsilon + C(N,p)$	Equation (27)		$N \in [3, 1000]$ $p \in [0.01, 0.99]$, in steps of 0.01	H vs ϵ for $N = 20$, $p = \frac{1}{10}, \frac{1}{3}, \frac{2}{3}, \frac{9}{10}$ Figure C.4
$K(N,p)$	Equation (31)		$N \in [4, 1000]$ $p \in [0.01, 0.99]$, in steps of 0.001	vs $p \in [0.01, 0.99]$ for $N = 3$, $N = 4$, $N = 8$, $N = 16$, $N = 23$, $N = 52$ Figure C.5

Table D.1: Summary of Properties of Key Expressions.

Expression	**Formula**	**Analytical Results**	**Numerical Verifications**	**Illustration**
$N_{st}(p)$	Equation (32)		$p \in [0.6, 0.99]$ in steps of 0.01	vs $p \in [0.6, 0.99]$ Figure C.6
$\epsilon^{MV}(N,1) - \epsilon^{crit}(N)$	Equation (36)	$\forall N_s$, $\lim_{N\to\infty} \epsilon^{MV}(N, N_s) = \infty$		vs N for $N \in [3, 200]$ Figure C.7
$\max_{N_s} H^{MV}(N, N_s)$	Equation (38)	for large N_s, $\lim_{N\to\infty} H^{MV}(N, N_s) = 0$		vs N for $N \in [3, 200]$ Figure C.8
$H = M(N,p)\,\epsilon + D(N,p)$	Equation (39)		$N \in [3, 1000]$ $p \in [0.01, 0.99]$, in steps of 0.01	H vs ϵ for $N = 20$, $p = \frac{1}{10}, \frac{1}{2}, \frac{2}{3}, \frac{99}{100}$ Figure C.9
$M_1(N, x_i)$	Equation (16)		$N \in [3, 1000]$ $0 < x_i < N-1$	vs x_i for $N = 200$ Figure B.1
$M_0(N)$, $M_1(N, N-2)$, $M_1(N,1)$, $M_2(N)$	Equation (16)		$N \in [3, 1000]$	vs N for $N \in [3, 200]$ Figure B.2
$H^{crit}(N)$ and $\epsilon^{crit}(N)$	Equation (17) Equation (18)		$N \in [3, 1000]$ for $H^{crit}(N)$	vs N for $N \in [3, 200]$ Figure B.3
$M_0(N)$, $M_3(N, N-2)$, $M_3(N,1)$, $M_4(N)$	Equation (19)	$\lim_{N\to\infty} M_0(N), M_3(N,1) = 0$ $\lim_{N\to\infty} M_3(N, N-2), M_4(N) = \infty$		vs N for $N \in [3, 200]$ Figure B.4

Table D.2: Summary of Properties of Key Expressions