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**LIE SYMMETRIES AND ESSENTIAL
RESTRICTIONS IN ECONOMIC
OPTIMIZATION**

Gadi Perets and Eran Yashiv

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Abstract

In optimization problems in Economics there are explicit or implicit assumptions about the underlying structure. In particular, one typically assumes that optimal behavior will remain invariant, in a sense to be precisely defined. Inter alia, this relates to the rationale for empirical estimation of invariant structural models. This paper has three goals. First, to show how relevant restrictions pertaining to this invariance can be derived using the algebraic technique of Lie symmetries of differential equations. Importantly, the symmetries provide solutions of the optimization problems in question or generate rich information with respect to the properties of the solutions, when no closed-form solutions exist. Second, to provide an example of implementation of this algebra, using an issue of substance, thereby gaining insight on a key topic in utility theory and consumer-investor choice. Third, to outline topics that are at the research frontier, which would be amenable to such analysis, thereby providing a road map for a potentially important new literature.

JEL Classification: N/A

Keywords: economic optimization, invariance, Lie symmetries, differential equations, consumption and portfolio choice, HARA utility, Structural Econometrics, Macroeconomics, Finance.

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Lie Symmetries and Essential Restrictions in Economic Optimization

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Abstract

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This paper has three goals. First, to show how relevant restrictions pertaining to this invariance can be derived using the algebraic technique of Lie symmetries of differential equations. Importantly, the symmetries provide solutions of the optimization problems in question or generate rich information with respect to the properties of the solutions, when no closed-form solutions exist. Second, to provide an example of implementation of this algebra, using an issue of substance, thereby gaining insight on a key topic in utility theory and consumer-investor choice. Third, to outline topics that are at the research frontier, which would be amenable to such analysis, thereby providing a road map for a potentially important new literature.

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Lie Symmetries and Essential Restrictions in Economic Optimization¹

1 Introduction

In optimization problems in Economics there are explicit or implicit assumptions about the underlying structure. In particular, one typically assumes that optimal behavior will remain invariant, in a sense to be defined precisely below. Inter alia, this relates to the rationale behind the empirical estimation of invariant structural models.² This paper has three goals. First, to show how relevant restrictions in economic optimization problems can be derived using the algebraic technique of Lie symmetries of differential equations. Second, to provide an example of implementation of this algebra, using an issue of substance, thereby gaining insight on a key topic in utility theory and consumer-investor choice. Third, to outline issues that are at the research frontier, which would be amenable to such analysis. The first point seeks to provide the economic profession with a very useful algebraic methodology; the second provides both an example of implementation and an insight, which is important in and of itself; the third provides a road map for a potentially important new literature.

For the first point, note that a symmetry is an invariance under transformation. This concept is usually known for the case of the invariance of functions, the homothetic utility or production functions being the most well known special cases. In this paper we present Lie symmetries, which are symmetries of differential equations. Importantly, the symmetries provide the solutions of these equations or generate rich information with respect to the properties of the solutions, when no closed-form solutions exist. In particular, we discuss the prolongation methodology to derive the symmetries. We reference reviews of software codes, that allow for the symbolic computation of the symmetries.

The second point is an implementation of the methodology of Lie symmetries to an economic question of substance, pertaining to consumer-investor choice and utility theory. The background for this exploration is the fact that many models in Economics and Finance assume a CRRA utility function or some other form belonging to the HARA family. This assumption has proved useful for the tractability of the analysis and has squared well with empirical work. It also facilitates aggregation and the con-

¹We thank Zvi Artstein, Lucien Foldes, Tzachi Gilboa, and Ryuzo Sato for useful conversations. Any errors are our own. We dedicate this paper to the memory of Bill Segal.

²We discuss this connection in the literature review below.

struction of equilibrium models. Through the use of Lie symmetries this paper shows that HARA utility is fundamental to economic analysis. The idea is that the HARA functional form is the unique form which satisfies basic economic principles related to optimization. Hence, using HARA is not just a matter of convenience or tractability, but rather emerges from economic reasoning.

Specifically, we demonstrate how to apply Lie symmetries to a differential equation, which expresses the relevant optimality condition in the Merton (1969, 1971) consumer-investor model. We use the fact that optimal consumption and portfolio choice is subject to variations in scale. Thus, agents have resources of different scale, such as different levels of wealth. This could be the result of growth over time or the effects of policy, such as taxation. Lie symmetries derive the conditions whereby the optimal solution remains invariant under scale transformations of wealth. Doing so, the symmetries impose restrictions on the model, with the key restriction being the use of HARA utility. Not using HARA precludes the stable characterization of agents' behavior, for example in empirical work. This is not to claim that agents necessarily need to have HARA utility as their form of preference; what it does say is that economic modelling of optimization, which obeys certain economically-relevant invariance requirements, implies these restrictions.

This result has broad implications, as the Merton (1969,1971) model is a fundamental one in Macroeconomics and Finance. For example, the stochastic growth model, which underlies New Classical business cycle modelling, can be thought of as a variant of this model. Hence this paper provides the theoretical basis for the use of HARA, and the special case of CRRA. We show what happens when invariance does not hold in this model, using a case of non-HARA utility. We also note that the highly influential intertemporal CAPM model of Merton (1973) uses the afore-cited consumer-investor model to set up a market equilibrium.

The third point this paper makes is to offer a road map for studying invariance issues in models at the research frontier. Generally, there is an abundance of optimality equations, which are PDE with complicated structures,³ that can be constructively explored using the Lie symmetries methods presented in this paper. Section 4 below focuses on what we see as the most promising and important ones currently.

The paper proceeds as follows: in Section 2, we address the first point, namely we present the mathematical concept of Lie symmetries of differential equations. In Section 3 we address the second point and show implementation. We discuss the Merton (1969, 1971) model of consumer-investor choice under uncertainty and present the application

³For a review of PDE equations in Macroeconomics see Achdou, Buera, Lasry, Lions, and Moll (2014).

of Lie symmetries to this model. We derive the main results with respect to HARA utility and discuss their economic implications. Section 4 relates to the third point, discussing specific, possible uses at the research frontier. Section 5 concludes.

2 Lie Symmetries

We introduce the mathematical concept of Lie symmetries of differential equations. We begin with a brief reference to the literature and then present the algebraic methodology. The appendix provides an overview of the general concepts of symmetries and their intuition.

2.1 A Brief Note on the Literature

The following references key papers relevant for the current analysis.

From the Mathematics literature, Olver (1993) and Steeb (2007) offer extensive formal discussions of the concept and use of Lie symmetries, including applications. In particular, the prolongation methodology, which is key to this paper and presented in sub-section 2.3 below, is discussed at length.

There are a number of software codes for symbolic analysis of Lie symmetries; see reviews in Filho and Figueiredo (2011) and Vu, Jefferson and Carminati (2012).

The pioneering contributions to economic applications of Lie algebra were made by Sato (1981) and Sato and Ramachandran (1990). For further developments, see Sato and Ramachandran (2014) and references therein. For applications in Finance, see Sinkala, Leach, and O'Hara (2008 a,b). To the best of our knowledge, the prolongation method presented here was not discussed hitherto.

More specifically, one should note a distinction between two concepts: (i) restrictions on the utility function for scale invariance of the preference relation, which is a topic that is not treated here, and was dealt with in seminal work by Skiadas (2009 (Chapters 3 and 6), 2013); (ii) scale invariance of an optimality equation, in the form of a differential equation, which is the object of inquiry of this paper.

Econometric work with invariant structures, in the context of causal analysis and policy evaluation, is reviewed and discussed in Heckman and Vytlačil (2007, see Section 4), Heckman (2008), and Heckman and Pinto (2014). The seminal work on these topics can be traced back to Frisch (1938) and Haavelmo (1943), with further developments in the work of the Cowles Commission in the 1940s.⁴ In Macroeconomics and

⁴An interesting fact to note is a Norwegian or an Oslo University “connection”: the fundamental math-

Finance, a major turning point in econometric studies was associated with the Lucas (1976) critique of reduced-form empirical models, which was built on the afore-cited early insights. This approach was advanced by the development of structural estimation and Rational Expectations Econometrics, mostly associated with the work of Sargent and Hansen (see, for example, Hansen and Sargent (1980) and Hansen (2014)). The relationship of this econometric literature with the current paper is that we show how the application of Lie symmetries to economic optimization problems yields restrictions on the model, which can then be estimated using Structural Econometrics. In this context the following definition by Heckman and Vytlačil (2007, p.4848) is pertinent:

“A more basic definition of a system of structural equations, and the one featured in this chapter, is a system of equations invariant to a class of modifications. Without such invariance one cannot trust the models to forecast policies or make causal inferences.”

2.2 Defining Lie Symmetries

Lie symmetries of differential equations are the transformations which leave the space of solutions invariant. We begin by explaining the concept of invariance of differential equations, culminating by the derivation of the prolongation equation, which is key in deriving the Lie symmetries of a differential equations system. In making the exposition here we are attempting to balance two considerations: the need to explain the mathematical derivation and the constraint that an overload of mathematical concepts may be burdensome to the reader.

Consider the differential equation:

$$L(t, x, y, p) = 0 \tag{1}$$

where $x = x(t)$, $y = y(x)$, $p = \frac{dy}{dx}$ and t is time.

The transformation:

emathical work on symmetries of differential equations was undertaken by Sophus Lie (1842-1899), a Norwegian, who got his PhD at the University of Christiania, now Oslo, in 1871. The econometric approach making use of invariant structures was proposed by Ragnar Frisch (1895-1973), a Norwegian, who got his PhD at the University of Oslo, in 1926. Frisch was editor of *Econometrica* for over 20 years (1933-1954) and won the 1969 Nobel prize (shared with Jan Tinbergen). Important developments of the econometric work on this topic were introduced by Trygve Haavelmo (1911-1999), a Norwegian, who was professor of Economics and Statistics at the University of Oslo for more than 30 years (1948-79). He was awarded the Nobel prize in 1989.

$$x' = \phi(x, y, t) \quad (2)$$

$$y' = \psi(x, y, t)$$

implies the transformation of the derivative $p = \frac{dy}{dx}$ to:

$$p' = \frac{dy'}{dx'} = \frac{\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy}{\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy} = \frac{\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} p}{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} p} \quad (3)$$

The differential equation (1) will be invariant under the transformation $x \rightarrow x'$ and $y \rightarrow y'$ (i.e., one integral curve is mapped to another) if and only if it is invariant under:

$$\begin{aligned} x' &= \phi(x, y, t) \\ y' &= \psi(x, y, t) \\ p' &= \chi(x, y, p, t) \end{aligned} \quad (4)$$

The condition for transformation (4) to leave the differential equation (1) invariant is:

$$H'L \equiv \zeta \frac{\partial L}{\partial x} + \eta \frac{\partial L}{\partial y} + \eta' \frac{\partial L}{\partial p} = 0 \quad (5)$$

where:

$$\begin{aligned} H &= \left(\frac{\partial \phi}{\partial t}\right)_0 \frac{\partial}{\partial x} + \left(\frac{\partial \psi}{\partial t}\right)_0 \frac{\partial}{\partial y} \\ &= \zeta \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \\ \zeta &\equiv \left(\frac{\partial \phi}{\partial t}\right)_0 \quad \eta \equiv \left(\frac{\partial \psi}{\partial t}\right)_0 \\ \eta' &\equiv \frac{\partial \eta}{\partial x} + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \zeta}{\partial x}\right)p - \frac{\partial \zeta}{\partial y} p^2 \end{aligned}$$

and the subscript 0 denotes the derivative at $t = 0$; the notation $\frac{\partial}{\partial x}$ is used for a directional derivative i.e., the derivative of the function in the direction of the relevant coordinate axis, assuming space is coordinated. For this and other technical concepts, see Chapter 1 in Sato and Ramachandran (1990).

Below we use an equation like (5) to derive the symmetries of the optimality condition of the Merton (1969,1971) model. To see the intuition underlying equation (5) consider the invariance of a function (a generalization of homotheticity) rather than that of a differential equation: a function $f(x, y)$ is invariant under a transformation $x \rightarrow x'$ and $y \rightarrow y'$ if $f(x, y) = f(x', y')$. Using a Taylor series and infinitesimal transformations we can write:

$$f(x', y') = f(x, y) = f(x, y) - sHf + \frac{s^2}{2}H^2f + \dots$$

It is evident that the necessary and sufficient condition for invariance in this case is:

$$Hf = 0 \tag{6}$$

Equation (5) is the analog of equation (6) for the case of a differential equation. It is called the prolongation equation and it is linear in ξ and η .⁵ Finding the solution to it gives the infinitesimal symmetries from which the symmetries of the differential equation itself may be deduced.

2.3 The Prolongation Methodology

As noted above, the power of this theory lies in the notion of infinitesimal invariance: one can replace complicated, possibly highly non-linear conditions for invariance of a system by equivalent linear conditions of infinitesimal invariance. This is analogous to the use of derivatives of a function at a point to approximate the function in the neighborhood of this point. Likewise, the infinitesimal symmetries are “derivatives” of the actual symmetries and the way to go back from the former to the latter is through an exponentiation procedure.

The Lie symmetries are derived by calculating their infinitesimal generators, which are vector fields on the manifold composed of all the invariance transformations. Finding these generators is relatively easy, as it is more of an algebraic calculation, while finding the invariance transformations directly amounts more to an analytic calculation. After finding the infinitesimal generators, we “exponentiate” them to get the actual invariant transformations.

A general infinitesimal generator is of the form:

$$v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u} \tag{7}$$

⁵The full prolongation formula is presented in Olver (1993) in Theorem 2.36 on page 110.

We determine all the possible functions ξ, τ, ϕ through the prolongation equation, which puts together all the possible constraints on the functions ξ, τ, ϕ . We show how this is implemented in the next section, when applying Lie symmetries to a fundamental optimization problem in Economics.

3 Implementation

We demonstrate the application of Lie symmetries by implementing them in the Merton (1969, 1971) model of consumer-investor choice. After a short exposition of the model (3.1), we derive the symmetries in detail, using the prolongation methodology (3.2), and discuss their economic interpretation (3.3). The key result is that HARA utility is the only form of the utility function which satisfies invariance with respect to agents' wealth. Finally, we show (3.4) an example, whereby invariance does not hold true in a case of a non-HARA utility function.

3.1 Merton's Model of Optimal Consumption and Portfolio Selection

We briefly present the main ingredients of the consumer-investor optimization problem under uncertainty as initially formulated and solved by Merton (1969, 1971)⁶. We choose this model as it is a fundamental model of consumer-investor choice and is akin to other fundamental models, such as the Ramsey model or the stochastic growth model. A key point, which merits emphasis, is that in what follows we do not just show that this model can be solved in a different way. Rather, we shall use Lie symmetries to solve it and show in what sense HARA utility is fundamental to the economic optimization problem and "comes out" of the analysis.

The essential problem is that of an individual who chooses an optimal path of consumption and portfolio allocation. The agent begins with an initial endowment and during his/her lifetime consumes and invests in a portfolio of assets (risky and riskless). The goal is to maximize the expected utility of consumption over the planning horizon and a "bequest" function defined in terms of terminal wealth.

Formally the problem may be formulated in continuous time, using Merton's notation, as follows: denote consumption by C , financial wealth by W , time by t (running from 0 to T), utility by U , and the bequest by B . There are two assets used for invest-

⁶For a discussion of developments since the initial exposition of these papers see Merton (1990, chapter 6), Duffie (2003) and Skiadas (2009, Chapters 3,4 and 6).

ment,⁷ one of which is riskless, yielding an instantaneous rate of return r . The other asset is risky, its price P generated by an Ito process as follows:

$$\frac{dP}{P} = \alpha(P, t)dt + \sigma(P, t)dz \quad (8)$$

where α is the instantaneous conditional expected percentage change in price per unit time and σ^2 is the instantaneous conditional variance per unit time.

The consumer seeks to determine optimal consumption and portfolio shares according to the following:

$$\max_{(C, w)} E_0 \left\{ \int_0^T U[C(t), t]dt + B[W(T), T] \right\} \quad (9)$$

subject to

$$dW = w(\alpha - r)Wdt + (rW - C)dt + wW\sigma dz \quad (10)$$

$$W(0) = W_0 \quad (11)$$

where w is the portfolio share invested in the risky asset. All that needs to be assumed about preferences is that U is a strictly concave function in C and that B is concave in W . See Kannai (2004, 2005) for discussions of utility function concavity as expressing preference relations.

Merton (1969, 1971) applied stochastic dynamic programming to solve the above problem. In what follows we repeat the main equations; see Sections 4-6 of Merton (1971) for a full derivation.

Define:

(i) An "indirect" utility function:

$$J(W, P, t) \equiv \max_{(C, w)} E_t \left\{ \int_t^T U(C, s)ds + B[W(T), T] \right\} \quad (12)$$

where E_t is the conditional expectation operator, conditional on $W(t) = W$ and $P(t) = P$.

(ii) The inverse marginal utility function:

⁷The problem can be solved with n risky assets and one riskless asset. As in Merton (1971) and for the sake of expositional simplicity, we restrict attention to two assets. Our results apply to the more general case as well.

$$G \equiv [\partial U / \partial C]^{-1} \equiv U_C^{-1}(C) \quad (13)$$

The following notation will be used for partial derivatives: $U_C \equiv \partial U / \partial C$, $J_W \equiv \partial J / \partial W$, $J_{WW} \equiv \partial^2 J / \partial W^2$, and $J_t \equiv \partial J / \partial t$.

A sufficient condition for a unique interior maximum is that $J_{WW} < 0$ i.e., that J be strictly concave in W .

Merton assumes “geometric Brownian motion” holds for the risky asset price, so α and σ are constants and prices are distributed log-normal. In this case J is independent of P , i.e., $J = J(W, t)$.

Time preference is introduced by incorporating a subjective discount rate ρ into the utility function:

$$U(C, t) = \exp(-\rho t) \tilde{U}(C, t) \quad (14)$$

The optimal conditions are given by:

$$\exp(-\rho t) \tilde{U}_C(C^*, t) = J_W \quad (15)$$

$$(\alpha - r)WJ_W + J_{WW}w^*W^2\sigma^2 = 0 \quad (16)$$

where C^* , w^* are the optimal values.

Combining these conditions results in the so-called Hamilton-Jacobi-Bellman (HJB) equation, which is a partial differential equation for J , one obtains:

$$U(G, t) + J_t + J_W(rW - G) - \frac{J_W^2}{J_{WW}} \frac{(\alpha - r)^2}{2\sigma^2} = 0 \quad (17)$$

subject to the boundary condition $J(W, T) = B(W, T)$. Merton (1971) solved the equation by restricting preferences, *assuming* that the utility function for the individual is a member of the Hyperbolic Absolute Risk Aversion (HARA) family of utility functions. The optimal C^* and w^* are then solved for as functions of J_W and J_{WW} , the riskless rate r , wealth W , and the parameters of the model (α and σ^2 of the price equation and the HARA parameters).

3.2 The Symmetries of the Consumer-Investor Optimality Equation

We now derive the symmetries of the HJB equation (17) using the prolongation methodology. Two issues should be emphasized: (i) the symmetries are derived with no as-

sumption on the functional form of the utility function except its concavity in C , a necessary condition for maximization; (ii) the optimal solution depends on the derivatives of the indirect utility function J , which, in turn, depends on wealth W and time t . The idea is to derive transformations of t and W that would leave the optimality equation invariant. These transformations do not require imposing any restrictions on the end points, i.e., transversality conditions, of the type usually needed to obtain a unique solution to optimal control problems.⁸

In economic terms, this means that if wealth varies, say because of taxation or because of intertemporal growth, the optimal solution remains invariant. The underlying interest in the invariance of the optimality equations is that we would like to have invariance of the structure of the solution across different levels of wealth.

3.2.1 Application of the Prolongation Methodology

In order to calculate the symmetries of the HJB equation (17), which is a p.d.e., we first calculate the infinitesimal generators of the symmetries, and then exponentiate these infinitesimal generators to get the symmetries themselves. An infinitesimal generator ν of the HJB equation has the following form, as in equation (7) above:

$$\nu = \xi(W, t, J) \frac{\partial}{\partial W} + \tau(W, t, J) \frac{\partial}{\partial t} + \phi(W, t, J) \frac{\partial}{\partial J} \quad (18)$$

Here ξ, τ, ϕ are functions of the variables W, t, J . The function J , as well as its partial derivatives, become variables in this method of derivation of the symmetries. In order to determine explicitly the functions ξ, τ, ϕ we prolongate the infinitesimal generator ν according to the prolongation formula of Olver (1993, page 110) and the equations thereby obtained provide the set of constraints satisfied by the functions ξ, τ, ϕ (see details in Olver (1993, pages 110-114), whose notation we use throughout).

The prolongation equation applied to ν yields:

$$\begin{aligned} & \left[r\xi J_W - \rho\tau e^{-\rho t} U(G(e^{\rho t} J_W)) + (rW - G(e^{\rho t} J_W))\phi^W + \phi^t \right] J_{WW}^2 \\ & + 2A\phi^W J_W J_{WW} - A\phi^{WW} J_W^2 \\ & = 0 \end{aligned} \quad (19)$$

where $\phi^W, \phi^t, \phi^{WW}$ are given by:

⁸The symmetries, however, do not restrict the optimal solution to be unique.

$$\phi^W = \phi_W + (\phi_J - \xi_W)J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t$$

$$\phi^t = \phi_t - \xi_t J_W + (\phi_J - \tau_t)J_t - \xi_J J_W J_t - \tau_J J_t^2$$

$$\begin{aligned} \phi^{WW} = & \phi_{WW} + (2\phi_{WJ} - \xi_{WW})J_W - \tau_{WW}J_t + (\phi_{JJ} - 2\xi_{WJ})J_W^2 \\ & - 2\tau_{WJ}J_W J_t - \xi_{JJ}J_W^3 - \tau_{JJ}J_W^2 J_t + (\phi_J - 2\xi_W)J_{WW} \\ & - 2\tau_W J_{Wt} - 3\xi_J J_W J_{WW} - \tau_J J_t J_{WW} - 2\tau_J J_W J_{Wt} \end{aligned}$$

Plugging these expressions in the prolongation formula applied to the HJB equation (17) yields:

$$\begin{aligned} & r\xi J_W - \rho\tau e^{-\rho t}U(G(e^{\rho t}J_W)) \tag{20} \\ & + (rW - G(e^{\rho t}J_W))(\phi_W + (\phi_J - \xi_W)J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t) \\ & + (\phi_t - \xi_t J_W + (\phi_J - \tau_t)J_t - \xi_J J_W J_t - \tau_J J_t^2)J_{WW} \\ & + 2A(\phi_W + (\phi_J - \xi_W)J_W - \tau_W J_t - \xi_J J_W^2 - \tau_J J_W J_t)J_W J_{WW} \\ & - A \left(\begin{array}{c} \phi_{WW} \\ + (2\phi_{WJ} - \xi_{WW})J_W \\ - \tau_{WW}J_t \\ + (\phi_{JJ} - 2\xi_{WJ})J_W^2 \\ - 2\tau_{WJ}J_W J_t \\ - \xi_{JJ}J_W^3 \\ - \tau_{JJ}J_W^2 J_t \\ + (\phi_J - 2\xi_W)J_{WW} \\ - 2\tau_W J_{Wt} \\ - 3\xi_J J_W J_{WW} \\ - \tau_J J_t J_{WW} \\ - 2\tau_J J_W J_{Wt} \end{array} \right) J_W^2 \\ & = 0 \end{aligned}$$

Note that the variables J_W, J_{WW}, J_{Wt}, J_t are algebraically independent. This implies that the coefficients of the different monomials in those variables are equal to zero. We

therefore proceed as follows.

(i) We first look at the different monomials in the above equation in which J_{WW} does not appear. Equating the coefficients of these monomials to 0 implies that:

$$\begin{aligned}
\tau_J &= \tau_W = 0 \\
\zeta_{JJ} &= 0 \\
\phi_{WW} &= 0 \\
\phi_{JJ} - 2\zeta_{JW} &= 0 \\
2\phi_{WJ} - \zeta_{WW} &= 0
\end{aligned} \tag{21}$$

(ii) Next we look at monomials in which J_{WW} appears in degree one. This gives (noting that $r \neq \alpha$ implies that $A \neq 0$) the following equation (in which we gathered only those monomials with their coefficients):

$$2(\phi_W + (\phi_J - \zeta_W)J_W - \zeta_J J_W^2)J_W J_{WW} + 3\zeta_J J_W^3 J_{WW} - (\phi_J - 2\zeta_W)J_{WW} J_W^2 = 0$$

From this we deduce that

$$\begin{aligned}
\phi_W &= 0 \\
\zeta_J &= 0 \\
\phi_J &= 0
\end{aligned} \tag{22}$$

(iii) Now we look at the monomials containing J_{WW}^2 which give the following equation:

$$r\zeta_J J_W - \rho\tau e^{-\rho t} U(G(e^{\rho t} J_W)) - (rW - G(e^{\rho t} J_W))\zeta_W J_W + \phi_t - \zeta_t J_W - \tau_t J_t = 0 \tag{23}$$

From which we deduce that

$$\begin{aligned}
\phi_t &= 0 \\
\tau_t &= 0
\end{aligned} \tag{24}$$

From all the constraints above on the functions ζ, τ, ϕ we gather so far that $\tau = \text{Constant}$ and $\phi = \text{Constant}$ and we are left with the following equation for the ζ function:

tion:

$$e^{\rho t}(r\tilde{\zeta} - \tilde{\zeta}_t - rW\tilde{\zeta}_W)J_W + \tilde{\zeta}_W G(e^{\rho t}J_W)e^{\rho t}J_W - \rho\tau U(G(e^{\rho t}J_W)) = 0 \quad (25)$$

From this we deduce that $\tilde{\zeta}_W = 0$ unless the following functional equation is satisfied

$$G(e^{\rho t}J_W)e^{\rho t}J_W - \gamma U(G(e^{\rho t}J_W)) = 0 \quad (26)$$

in which γ is a constant scalar. The last statement is of great importance in the current context, as will be shown below.

We end up with the following constraints for the infinitesimal generators:

$$\phi_t = 0 \quad (27)$$

$$\rho\tau = \phi \quad (28)$$

$$\tilde{\zeta}_W = 0 \quad (29)$$

The last equation holds true unless equation (26) is satisfied.

3.2.2 The Symmetries

The constraints, which we have derived, on the functions $\tilde{\zeta}$, τ , ϕ and their derivatives completely determine the infinitesimal symmetries, which are given by:

Symmetry 1 $\phi_t = 0$

If $J(W, t)$ is a solution to the HJB equation, then so is $J(W, t) + k$ for any $k \in R$.

Symmetry 2 $\rho\tau = \phi$

If $J(W, t)$ is a solution of the HJB equation, so is $e^{-\rho\tau}J(W, t + \tau)$ for any $\tau \in R$.

Symmetry 3 $\tilde{\zeta}_W = 0$

If $J(W, t)$ is a solution of the HJB equation, so is $J(W + ke^{\rho t}, t)$ for any $k \in R$.

For a general specification of the utility function, the HJB equation of the model admits only the above three symmetries. However, from the constraints above we also

get that in case that the utility function satisfies the functional equation (26), and only in that case, there is an extra symmetry for the equation.

We now consider the implications of equation (26). For this we need first the following.

Lemma 1 *The functional equation*

$$G(x)x - \gamma U(G(x)) = 0 \quad (30)$$

where $G = (U')^{-1}$, is satisfied by a utility function U iff U is of the HARA form.

Proof. Upon plugging in the equation a utility $U(x)$ of the HARA form we see the functional equation above is satisfied. Going the other way, after differentiating the equation with respect to x , we get the ordinary differential equation:

$$G'x + G - \varepsilon(xG') = 0$$

The solutions of this equation form the HARA class of utility functions. Then we take the inverse function to get U' and after integration we get that U is of the HARA form.

■

When the functional equation (26) holds true and Lemma 1 is relevant, the exponentiation of the infinitesimal generators yields a fourth symmetry as follows.

Symmetry 4

If $J(W, t)$ is a solution, then so is $e^{k\gamma} J(e^{-k}\{W + \frac{(1-\gamma)\eta}{\beta r}\} - \eta \frac{(1-\gamma)}{\beta r}, t)$ for any $k \in \mathbb{R}$.

In words, this fourth symmetry says that if $J(W, t)$ is a solution then a linear function of J is also a solution. Calculation of the solutions to the functional equation in Lemma 1 shows that only utility functions of the HARA class satisfy it. The HARA function is expressed as follows:

$$\tilde{U}(C) = \frac{1-\gamma}{\gamma} \left(\frac{\beta C}{1-\gamma} + \eta \right)^\gamma \quad (31)$$

The special case of the CRRA function $\frac{C^\gamma}{\gamma}$ has $\beta = (1-\gamma)^{-\frac{(1-\gamma)}{\gamma}}$, $\eta = 0$. The cases of logarithmic utility ($\ln C$) and exponential utility ($-e^{-\beta C}$) are limit cases.

3.3 The Economic Interpretation of the Symmetries

There are four symmetries all together. The first three formulate “classical” principles of utility theory; the fourth places restrictions on the utility function, and is the main point of interest here.

Symmetry 1

If $J(W, t)$ is a solution to the HJB equation, then so is $J(W, t) + k$ for any $k \in R$.

This symmetry represents a formulation of the idea that utility is ordinal and not cardinal.

Symmetry 2

If $J(W, t)$ is a solution of the HJB equation, so is $e^{-\rho\tau}J(W, t + \tau)$ for any $\tau \in R$.

This symmetry expresses the property that displacement in calendar time does not change the optimal solution.

Symmetry 3

If $J(W, t)$ is a solution for the HJB equation, then so is $J(W + ke^{rt}, t)$ for any $k \in R$.

This symmetry expresses a property with respect to W that is similar to the property of Symmetry 2 with respect to t : if the solution is optimal for W then it is also optimal for an additive re-scaling of W ; the term e^{rt} keeps the additive k constant in present value terms.

As noted, for a general specification of the utility function, the HJB equation of the model admits only the above three symmetries.

Symmetry 4

If $J(W, t)$ is a solution, then so is $e^{k\gamma}J(e^{-k}\{W + \frac{(1-\gamma)\eta}{\beta r}\} - \eta\frac{(1-\gamma)}{\beta r}, t)$ for any $k \in R$.

As noted, this holds only if equation (26) is satisfied.

This fourth symmetry is the key point of this part of the paper.

3.3.1 The Implications of Symmetry 4 for Economic Analysis

Symmetry 4 has the following major implications for economic analysis:

(i) *The form of the utility function and invariance of the optimal solution.* Because k is completely arbitrary any multiplicative transformations of W , i.e., $e^{-k}W$, apply. Such transformations are the most natural ones to consider when thinking of wealth growth

or policy effects. Thus, this symmetry states the following: the optimum, expressed by the J function, i.e., maximum expected life-time utility, will remain invariant under multiplicative transformations of wealth W *if and only if* HARA utility is used. Hence the HARA form is determined by the symmetry. Note well that HARA utility is implied by this symmetry, not assumed a-priori. This does not imply, though, that there is a unique such J but it does express a property of any J which solves the HJB equation.⁹

The idea, then, is that there is an interdependence between the functional form of preferences (the form of the utility function) and the requirement that the optimal solution will remain invariant under multiplicative wealth transformations. This interdependence takes specific form in Symmetry 4. This kind of invariance underpins empirical undertakings that aim at estimating stable structural relationships, which were referenced in the literature review in Sub-section 2.1 above.

(ii) *Scale Invariance and Linear Optimal Rules.* In Merton (1971, p. 391) the following theorem is presented and proved:

THEOREM III. Given the model specified... $C^* = aW + b$ and $w^*W = gW + h$ where $a, b, g,$ and h are, at most, functions of time if and only if $U(C, t) \subset HARA(C)$.

Note that this theorem states an *iff* property with respect to linear optimal rules, which emerge from the solution, and HARA utility, which Merton (1971) assumes. In his paper, he does not discuss the notion of wealth scaling invariance.

The result we obtain above – Symmetry 4 – can then be stated as follows:

Theorem 2 *Given the model specified in this section, then Symmetry 4 (the scaling symmetry) is satisfied if and only if $U(C, t) \subset HARA(C)$.*

Combining the last theorem with Merton’s theorem III above, we get:

Corollary 3 *Given the model specified in this section, then $C^* = aW + b$ and $w^*W = gW + h$ where $a, b, g,$ and h are, at most, functions of time, if and only if Symmetry 4 (the scaling symmetry) is satisfied.*

Note that in the above corollary no reference is made to any specific form of the utility function. The proof of this corollary does not necessitate a specific solution to

⁹If, together with the multiplicative transformation, there is also an additive transformation of W , expressed by the term $e^{-k} \left(\frac{(1-\gamma)\eta}{\beta r} \right) - \eta \frac{(1-\gamma)}{\beta r}$, then it is further restricted by the parameters of the HARA utility function, and features the same arbitrary constant k used for the multiplicative transformation.

the HJB equation in the model and in any case it is impossible to give a precise solution when the utility function is not specified.

This means that wealth scale invariance implies linear optimal solutions to the control variables (C^*, w^*) and linear optimal rules imply scale invariance. Scale invariance determines the relevant linear parameters of optimal behavior.

It should be emphasized that this is not simply a re-statement of Merton's (1969, 1971) results. The latter papers have assumed HARA utility and then solved the HJB equation.¹⁰ Here Symmetry 4 shows that utility has to be HARA, so that the consumer-investor problem be invariant for economic plausibility and for structural empirical investigation. This is established even without solving the HJB equation.

(iii) *Comparisons Across Consumers/Investors.* If we know that we can compare the outcome of two different consumers/investors as a linear function of the ratio of their wealth stocks, then necessarily the utility function of the agents is of the HARA form. This can also be stated as follows: if the outcome of two different consumers cannot be compared as a linear function of the ratio of their wealth, then their utility function is not HARA. Again, we can state this even without solving the model explicitly, as it emerges from the analysis of the symmetries, i.e., the invariance properties of the HJB equation. This result stems from the formulation of Symmetry 4 whereby if J is a solution then a linear function of J is a solution and from the 'if and only if' property discussed above.

(iv) *Aggregation and Equilibrium Modelling.* With Symmetry 4, utility is of the HARA form if invariance, in the sense described above, is to hold true. It implies linear consumption and portfolio choice, as postulated in the cited Theorem III by Merton (1971). This linearity facilitates aggregation and the use of representative agent modelling. It is highly important for the construction of an equilibrium model, such as the seminal Merton (1973) intertemporal CAPM model, which embeds this set-up. We turn now to examine the linearity issues in more depth.

3.3.2 Linear Connections: Implications for Risk Aversion

A utility function $U(C)$ is said to be HARA *iff* it has a risk tolerance $T(C)$ which is a linear function of its argument. That is:

$$T(C) = \frac{1}{A(C)} = \frac{C}{1-\gamma} + \frac{\eta}{\beta} \quad (32)$$

¹⁰See, for example, pp. 388-391 in Merton (1971).

where absolute risk aversion $A(C)$ is given by:

$$A(C) = -\frac{U''(C)}{U'(C)}$$

There are three linear formulations here which are inter-connected:¹¹

a. Scale invariance is established through linear transformations of wealth; see the transformation $e^{-k}\{W + \frac{(1-\gamma)\eta}{\beta r}\} - \eta\frac{(1-\gamma)}{\beta r}$ in Symmetry 4 above and the ensuing discussion.

b. Optimal consumption C^* and portfolio shares w^* are linear functions of wealth; see Corollary 3 above.

c. Risk aversion is such that its reciprocal, risk tolerance, is linear in consumption, as in equation (32).

Symmetry 4 establishes the equivalence between these three linear formulations, all with an *iff* property: scaling invariance – as in point a – generates HARA, which yields optimal behavior, as in point b, and which risk aversion is defined in point c.¹²

Note that the functional equation (30), which holds true *iff* there is scale invariance, is the same as the functional equation which has to be satisfied by the utility function in order for it to feature linear risk tolerance (equation (32)). This fundamental functional equation is implicit in the model itself and the way to uncover it is through the calculation of the Lie symmetries.

One can then interpret the results above also as follows: a requirement of risk tolerance to be linear in consumption as an “economic fundamental” means that utility has to be HARA, and through the equivalence implied by Symmetry 4, that the indirect utility function be scale invariant.¹³

3.4 The Case of Non-HARA Utility

A natural question arises from the afore-going discussion – what happens when Symmetry 4 does not hold true and utility is non-HARA? In this sub-section we discuss such a case. Chen, Pelsser, and Vellekoop (2011) have solved the Merton (1969,1971) model for a more general utility function, which they call Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA). This analysis nests a non-HARA utility

¹¹Out of the relations described here, Merton (1971) has shown – in the cited Theorem III – the connection between HARA utility and linear optimal rules.

¹²For good expositions of the relationships between the functional form of the utility function, attitudes towards risk and intertemporal substitution, and optimal consumption choice, see Weil (1989, 1990).

¹³For discussions of the role of HARA utility in the two-fund separation paradigm, and the linear connections embodied there, see the seminal contribution of Cass and Stiglitz (1970).

function as a special case. We begin by briefly introducing their analysis and then turn to discuss this special case.

The SAHARA utility functions are defined for all wealth levels, with the key feature of allowing absolute risk aversion to be non-monotone. The domain of all functions in this class is the whole real line and for every SAHARA utility function there exists a level of wealth, which the authors call ‘threshold wealth,’ where absolute risk aversion reaches a finite maximal value. Risk aversion increases as threshold wealth is approached from above. A focal point for this analysis is that there is decreasing risk aversion for increasingly lower levels of wealth *below* the threshold wealth. Note, though, that SAHARA utility, being concave everywhere, implies that it still has higher disutility for the same loss at low levels of wealth than at high levels of wealth. The authors derive optimal investment strategies, which are explicit functions of the key parameters and the level of wealth.

Formally, using our notation, given $A(W) = -\frac{U''(W)}{U'(W)}$, then SAHARA utility is defined by a U function which satisfies:¹⁴

$$A(W) = \frac{\gamma}{\sqrt{a^2 + (W - d)^2}} > 0 \quad (33)$$

where $a > 0$ is a scale parameter, $\gamma > 0$ is the risk aversion parameter, and $d \in \mathbb{R}$ is threshold wealth.

Hence, risk tolerance, discussed above, is given by:

$$T(W) = \frac{1}{A(W)} = \frac{\sqrt{a^2 + (W - d)^2}}{\gamma} \quad (34)$$

This is a non linear function of W when $d \neq 0$. Comparing equation (34) to equation (32), and in light of the discussion around it above, it is clear that in this case utility is non-HARA. It is also apparent that wealth re-scaling, as in Symmetry 4, will not leave the $A(W)$ function invariant.

Theorem 3.2 (on page 2083) of Chen, Pelsser, and Vellekoop (2011) delineates the optimal solution for SAHARA utility as follows (again, using our notation). Investment in the risky asset at time t is given by:

¹⁴We follow Chen, Pelsser, and Vellekoop (2011) in discussing an investor problem, whereby the agent maximizes a utility function defined over wealth.

$$\begin{aligned}
w_t^* W_t &= p \sqrt{W_t^2 + b(t)^2} \\
b(t) &= a \exp \left(-\left(r - \frac{1}{2} \frac{\lambda^2}{\gamma^2}\right)(T - t) \right)
\end{aligned} \tag{35}$$

where w_t^* is the optimal share, and the prudence function $p(W)$ and the market price of risk λ are given by:

$$p(W) = -\frac{U'''(W)}{U''(W)} \tag{36}$$

$$\lambda = \frac{\alpha - r}{\sigma} \tag{37}$$

To compare, consider Merton's (1971) analog solution for $w_t^* W_t$, using HARA, in his equation 49 (on page 390) as follows, with the same notation:

$$w_t^* W_t = \frac{\lambda}{\delta \sigma} W_t + \frac{\eta \lambda}{\beta r \sigma} (1 - \exp(-r(T - t))) \tag{38}$$

Visual inspection of equation (35) vs. equation (38) shows that scaling invariance (in terms of W) does not hold in the former equation and does hold true in the latter. That is, multiplying wealth by an arbitrary constant k , namely kW , leaves the HARA solution invariant but not the SAHARA one.

It should be noted that Symmetry 4 establishes the conditions for invariance without solving the model. The analysis of Lie symmetries does not necessitate taking the steps of this sub-section, namely assuming particular forms for the utility functions, solving the optimality equations, and comparing the optimal solutions across the assumed functions. As we have stressed above, Symmetry 4 establishes the HARA requirement without assuming any functional form for the utility function and without solving the model in closed form.

4 Applications at the Research Frontier

The preceding analysis has demonstrated the use of Lie symmetries as a tool to deal with economic optimization problems. Lie symmetries techniques are useful for finding the set of functional forms implied by a given economic requirement. While there are likely to be many different optimization problems that would yield restrictions of

the type explored here, consider the following four topics, currently at the research frontier.

First, the problem of modelling optimal behavior in an economy with heterogeneous agents is amenable to such analysis. This kind of problem is an important one in complex DSGE models with heterogeneous agents, which have become very pervasive in business cycle modelling; see Krueger, Mitman, and Perri (2016) for a recent overview. Symmetries can provide conditions for aggregator functions and restrictions on the multitude of functions in the model, such as the utility, production, or costs functions. The latter can include price, labor, capital, and financial frictions. The symmetries inform the researcher on the properties of the solution, even when closed-form solutions do not exist. These conditions and restrictions may be very useful in generating insights on key issues, such as the marginal propensity to consume across heterogeneous consumers, the response of consumption behavior to monetary policy and to fiscal policy, and the response of heterogeneous firms to these policies. As mentioned above for the Merton (1973) intertemporal CAPM model, equilibrium characterizations are facilitated by the invariant structure uncovered by the symmetries.¹⁵

Second, the use of Lie symmetries can greatly extend the scope of models examined using the principle of counterfactual equivalence, suggested by Beraja (2017), for macroeconomic models. His idea is as follows: counterfactuals in structural models are a leading way to analyze policy rule changes, because they are immune to the Lucas Critique. But there are issues as to the appropriate choice of model primitives for these structural models. For example: how do the effects of policy change under variations in the policy rule for different primitives? How does the modeler decide on these primitives? Beraja (2017) proposes methods to deal with these issues. The methods rest on the insight that many models, which are well approximated by a linear representation, are both observationally equivalent under a benchmark policy and yield an identical counterfactual equilibrium under alternative policy. These are called “counter-factually equivalent models.” They can be found through analysis of linear restrictions. One can then know which models will be observationally equivalent under both benchmark and alternative (counterfactual) policy rules, and which will not be. As an example,

¹⁵Another class of heterogeneous agents models studies income and wealth distributions; Achdou, Han, Lasry, Lions, and Moll (2017) make an important contribution. These authors boil down the model to systems of two coupled partial differential equations, the Hamilton-Jacobi-Bellman (HJB) equation for the optimal choices of a single atomistic individual, who takes the evolution of the distribution, and hence prices, as given; and the Kolmogorov Forward (KF) equation characterizing the evolution of the distribution, given optimal choices of individuals. In complementarity with the mathematical tools proposed by Achdou et al (2017), Lie symmetries can be used to provide the entire set of relevant restrictions on the HJB and KF equations.

consider one application examined by Beraja (2017). He shows that search models are counter-factually equivalent across those DMP-type models, which change the primitives of firms' incentives or the job creation technology structure. But models, which change the primitives of wage setting and bargaining, are not counter-factually equivalent.

The algebraic method of the current paper can be used in this context as follows. Write the relevant model equilibrium equations as differential equations. Note that these do not have to be linear, as in Beraja's case. Derive the Lie symmetries for these differential equations. Use the symmetries to identify restrictions on the relevant structural model primitives such that the model remains invariant under policy rules variations. Those models that satisfy these restrictions may be counter-factually equivalent.

Third, for problems of the time-invariance of preferences, starting from those analyzed by Barro (1999), all the way to recent treatments, such as Millner and Heal (2017), who provide an overview, Lie symmetries can provide restrictions on the time preference function. Halevy (2015) points out that one needs to distinguish between stationarity, time invariance, and time consistency of preferences. He shows that any two of these properties imply the third. Hence, time invariance is an important feature related to the time consistency of preferences. There has been great interest in Macroeconomics in the latter issue, predominantly for consumption decisions and for policymaker plans (monetary and fiscal, including debt). Lie symmetries can provide invariance restrictions both on the time dimension and on the cross-sectional dimension, as done above, in Symmetries 2,3, and 4. Rather than assume certain functional forms of time preferences, these could be derived using the tools presented above. The idea, here too, is to employ invariance restrictions based on economic reasoning.

Fourth, Lie symmetries can address issues of indeterminacy and sunspots, such as those that arise in RBC models; see Pintus (2006, 2007) for a discussion. These topics have re-emerged given recent experience with various "bubbles" phenomena. This can be done by deriving the Lie symmetries of the optimality equations of the model (such as equations (8) in Pintus (2006) or in Pintus (2007), formulated in continuous time). The symmetries would then yield conditions relating the production and utility functions to agents' optimal behavior, as done in this paper for the utility function and the HJB equation of the Merton (1969, 1971) model. Thereby the analysis would yield restrictions that need to be placed on these functions. The restrictions, required to insure invariance of the optimality equations, would shed light on the problems of indeterminacy.

5 Conclusions

This paper has introduced an algebraic methodology that could be very useful for the analysis of economic optimization problems. Specifically, it discussed the use of the prolongation methodology to derive the symmetries of differential equations. It implemented this methodology to a well-known model, the consumer-investor problem analyzed by Merton (1969, 1971). Emphasizing that Merton had *assumed* HARA utility to derive optimal rules, the analysis of this paper has shown that HARA utility is not just a convenient assumption, yielding a tractable economic structure. Rather it has derived HARA utility as an inherent feature of economic optimization, uncovering a restriction which applies when invariance of the optimal solution is to obtain under multiplicative transformations of wealth. While the model itself is widely used, including its HARA utility function, this formal derivation was not attempted previously. Subsequently, we have pointed to classes of models for possible similar applications on topical research issues.

Lie symmetries can be used to determine invariance and thereby characterize the solutions of differential equations, even when these cannot be represented in closed-form. However, as far as we can see, there is no general principle here which could be formulated. Models need to be tackled on a case by case basis. In some the symmetries will yield trivial structural characterizations, while in others they could generate significant insights.

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Appendix

General Concepts of Symmetries and Basic Intuition

To give some general intuition to the concept of symmetries consider first a symmetry of a geometric object. This is a transformation of the space in which it is embedded, which leaves the object invariant. The symmetries of an equilateral triangle, for example, are the rotations in angles $\pi/3$, $2\pi/3$, 2π and the three reflections with respect to the bisectors. The symmetries of a circle centered in the origin are all the possible rotations (angles $0 \leq \alpha \leq 2\pi$) and all the reflections with respect to axes passing through the origin. In each case the symmetries form a group with respect to composition. In the first case this is a discrete (finite) group of six elements (it is actually isomorphic to the group of permutations of three letters) and in the second case this is a continuous group which contains the circle itself as a subgroup. Now consider reversing the order: first, fix the set of symmetries and then see which geometric object 'obeys' this set. In the example of the triangle, if we fix the set of symmetries to be the rotation of angle $\pi/3$ and the reflections around the y -axis (and indeed all the possible combinations of these symmetries, hence the group generated by the two symmetries), we obtain that the only geometric object preserved by these two symmetries is an equilateral triangle. This establishes a dual way to 'see' a triangle, i.e., through its symmetries. The same could be done with the circle, the only difference being that the set of symmetries preserving the circle consists of all possible rotations (with angles $0 \leq \alpha \leq 2\pi$) and all possible reflections with respect to axes passing through the origin. This difference, however, is a significant one as it introduces continuity: the group of symmetries being a continuous group, we are now permitted to use concepts of continuous mathematics in order to understand the interplay between the geometric object and the set of symmetries preserving it.

A similar continuous approach may be used to analyze a system of differential equations, which is what is to be done in this paper. We can view a system of partial differential equations as a description of a geometric object which is the space of solutions of the system. The symmetries of the differential equations are thus the transformations which leave the space of solutions invariant. Determining the group of symmetries of the space of solutions may give valuable insights with respect to the solution itself. For example, for the p.d.e $\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$, the heat equation in Physics, the group of symmetries of the equation not only gives insight to the problem in question but actually provides a way to get to the solution itself.

The power of this theory lies in the notion of infinitesimal invariance: one can re-

place complicated, possibly highly non-linear conditions for invariance of a system by equivalent linear conditions of infinitesimal invariance. Infinitesimal symmetries are elements of the tangent space to symmetries of the system. To employ familiar concepts, it is analogous to the use of derivatives of a function at a point to approximate the function in the neighborhood of this point. Likewise the infinitesimal symmetries are “derivatives” of the actual symmetries. The way to go back from the former to the latter is through an exponentiation procedure. To use familiar terminology again, the latter is analogous to the use of a Taylor series.

A crucial point is that if one is looking for smooth symmetries and the equations in question satisfy some non-degeneracy conditions (as is the case analyzed in this paper) then *all* the smooth symmetries of the equation system are derived through the infinitesimal symmetries. We stress this point as the symmetries we derive express diverse aspects of the consumer-investor optimization problem. The afore-cited property assures us of extracting all the possible symmetries.