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**THE CONSISTENCY OF OPTIMAL POLICY IN
STOCHASTIC RATIONAL EXPECTATIONS MODELS**

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The Consistency of Optimal Policy in
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ABSTRACT

This paper extends the work of Barro and Gordon (1983) to general linear models with rational expectations. We examine the question whether the optimal policy rule, i.e. the one that a government which could pre-commit itself would use, can be sustained as a consistent rule in the sense defined by Kydland and Prescott (1977) if the reputational effects similar to those described by Barro and Gordon operate. The analysis is carried out in the context of an infinite horizon game between the government and private sector agents in the economy. We are able to show that, providing the support of the distribution of shocks hitting the economy is bounded, and providing the discount rate in the government objective function is low enough, the optimal policy rule can be sustained as a consistent policy in general.

We obtain solutions for the optimal policy rule and the consistent policy rule using straightforward recursive methods and dynamic programming, and show that control theory can be made applicable to economic planning even when expectations are rational (pace Kydland and Prescott, 1977).

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NON-TECHNICAL SUMMARY

The rational expectations revolution in macroeconomics has emphasized the way in which the current behaviour of economic agents - households and firms - depends on their expectations about future events. And among these events are the future policy actions of the government. For example, the likely future path of the money supply and the government's budget deficit have immediate effects on the exchange rate and interest rates, as recent events in the US clearly demonstrate. Thus expectations concerning future government actions affect the current state of the economy.

The key point of the rational expectations hypothesis is that expectations of future events should be modelled as if they are formed by making predictions which use available information in the best way. This implies that knowledge of how the economy works should be used in calculations involving expected future events. So, for example, if inflation were known to be related to the money supply, rational agents would not forecast inflation by looking at past inflation and extrapolating this behaviour into the future, but would instead forecast the future path of the money supply, and compute the future inflation implied by it, using the known relationship between the two.

This aspect of rational expectations raises questions concerning optimal policy for the government. An optimal policy for monetary and fiscal policy can still be computed, but the relevance of such optimal policies has been questioned. This issue was first highlighted by Kydland and Prescott in 1977 in their discussion of the "time-inconsistency" of optimal plans. There are a number of famous examples of such time inconsistency. In the field of macroeconomics, the most straightforward is the example of monetary policy and inflation. Suppose the government desires lower rather than higher inflation, and higher rather than lower employment and output. In the short run money wages

are fixed, but in the medium and long run the rate of wage inflation depends, indirectly, on the expected rate of monetary expansion. Then the government has an incentive to announce a monetary policy which is initially quite expansionary, since with fixed money wages a higher money stock will cause higher employment and output, but which is subsequently much tighter, since in the long run faster money growth merely induces more rapid growth of wages and prices and no long-run change in output and employment. Thus the government may want to have a monetary policy which is "loose" now, but which is tight in the medium and long run. This may be the optimal policy for the government. And indeed, if the government were able to commit itself to a particular policy, this may be the one to which it would commit itself.

However this policy would be difficult for the government to sustain without such a precommitment because there would always be a temptation to introduce an unexpected but temporary loosening of monetary policy to produce a little more output and employment in the short run. If the policy were unsustainable for this reason, people would always expect inflation, and the government would simply run the appropriate loose monetary policy consistent with high inflation.

One consideration which may affect a government is concern for its "reputation". This may persuade it to stick to the optimal tight money policy if the private sector may interpret a deviation from it as a sign of weakening which presages loose money in the future, thereby inducing expectations of future inflation, high wage increases and high interest rates. Once lost, a government's "reputation" for sticking to its (announced) tight money policy may be difficult to recover. Thus a "temporary" relaxation of policy, whilst attractive when the

economy believes it to be merely temporary, may become very unattractive when the price is a long or even indefinite period of high inflation or higher unemployment in the future.

The purpose of this paper is to formalize this argument and show that the "reputational effect" can exercise sufficient discipline on the government so that it will stick to its optimal macroeconomic policy. In recent years this idea has been applied to simple examples, and this paper generalizes the analysis to any linear macroeconomic model and quadratic objective function.

The paper shows that the existence of a sequence of random shocks hitting the economy (such as OPEC price increases, unexpected financial innovations, or foreign fiscal and monetary policy shocks) may be important if reputation is to discipline the government sufficiently. The paper also shows that the government must have a sufficiently long-term view in order that reputation effects influence its choice of policy in this fashion.

Similar results have been obtained by Currie and Levine (CEPR Discussion Paper No. 94, February 1986) using a continuous-time model. The analysis in this paper uses discrete time methods, and so is complementary to the work of Currie and Levine.

1. Introduction

In 1977, Kydland and Prescott asserted that "there is no way control theory can be made applicable to economic planning when expectations are rational." The purpose of this paper is to show by contrast (1) that control theory can be used to design optimal policy and (2) that, in a wide class of dynamic models with rational expectations, the optimal policy is consistent on Kydland and Prescott's definition of consistency.

This paper extends the work of Barro and Gordon (1983) to a wider class of economic models and obtains results for general linear dynamic rational expectations models which may have additive shocks. We use the framework of dynamic games played over an infinite horizon. By contrast with our earlier work on macroeconomic policy (Backus and Driffill, 1985a and b) we here assume that the objectives of all the players in the game are common knowledge.

The planning problem can usefully be thought of as a game played between a government and agents in the private sector of the economy. In each time period the government selects values for some policy instruments, and the private sector agents make some decisions. In a model where expectations are important, the private sector decisions will depend on the current and expected future decisions made by the government. For example, the government's policy instrument might be the nominal money stock in each period, and the private sector might have to make a decision about the allocation of its portfolio of wealth between

money and other assets, investment in real capital, and so on. Typically, the private sector's optimal decision in the current period will depend on both the current money stock and its expectations about the entire future path of the money stock, based perhaps on beliefs about the government's policy rule.

In this paper we consider linear macroeconomic models with stochastic shocks in which the current state of the economy depends on expectations of future values of some variables. The government has a quadratic objective function which is well known to all the players in the game.

If the game is played a finite number of periods, there is a unique subgame-perfect equilibrium solution, and this is the same as the solution which Kydland and Prescott called the "consistent" solution. This is the only solution of the game which the government can achieve in the absence of being able to make a binding commitment at the start of the game to follow a particular strategy throughout. This solution will tend to be worse than the "optimal" solution which is the best one the government could achieve with the help if necessary of a binding commitment at the start of the game.

If however the game is played an infinite number of times, there is no longer a unique consistent policy for the government to follow, as Barro and Gordon (1983a and b) have shown for the case of one particular model. In addition to the policy which is the counterpart in the infinitely repeated game to the unique consistent policy of the finitely repeated game (which they call the "discretionary" policy), there are other policies ("rules") which are better from the government's point of view and which are also consistent. Suppose the agents in the private sector of the economy play the following strategy. If they have observed

the government following a "rule" in the past, they act in the expectation that the government will forever continue to follow it. If however they have observed the government to deviate from the rule, they act on the expectation of discretionary policy by the government from then onwards. In this case, the government may wish at each stage of the game to stick with the rule, because if it deviates from the rule its next best alternative is the discretionary policy. Providing it is indeed the case that the government always under these circumstances wishes to stick with the rule, then the expectations of agents in the private sector are rational, and the rule is consistent. In this paper we show that in a general class of macroeconomic models the optimal policy itself is sustainable as a consistent rule.

Kydland and Prescott in 1977 did not consider the possibility of consistent policies other than the one which is analogous to that which Barro and Gordon call the "discretionary" policy. The other consistent policies ("rules") are supported by a reputational mechanism (again in Barro and Gordon's terminology). The government is induced to follow the rule each period because any deviation would destroy its reputation for so doing. By contrast, when the private sector expects the government to pursue the discretionary policy in each future period, the best the government can do is to pursue the discretionary policy. There is no temptation for the government to deviate from it, and the private sector's expectations are rational. The discretionary policy does not require any reputational effect to sustain it.

The plan of the rest of the paper is as follows. In part 2 the optimal policy rule is derived for discrete-time stochastic rational expectations models. This draws on the work of Levine and Currie (1984). The derivation used gives a very simple way of calculating optimal policy

rules for such models. In part 3 the discretionary policy rule is derived, and in part 4 the policy rules are compared to see whether the optimal rule is sustainable. Part 5 contains some numerical examples which illustrate general results, and part 6 a summary and conclusions.

2. Optimal policy in a stochastic linear model

The general framework is as follows. z_t is an $n \times 1$ vector of state variables, of which the first n_1 (x_t) are predetermined variables and the remaining n_2 (y_t) are non-predetermined ($n_1 + n_2 = n$). u_t is an m vector of policy variables. The equations of motion of z are

$$x_{t+1} = A_{11}x_t + A_{12}y_t + B_1u_t + v_{x,t+1} \quad (2.1a)$$

$$E_t y_{t+1} = A_{21}x_t + A_{22}y_t + B_2u_t \quad (2.1b)$$

where A and B are $n \times n$ and $n \times m$ matrices of coefficients and $v_{x,t+1}$ is an n_1 -vector of white noise innovations.

This structure is used as a general representation of most dynamic linear rational expectations models. The structure assumed is sufficiently general to allow for models in which anticipated monetary policy can have real effects, and those in which it is ineffective.

The government's objective is assumed to be the minimization of the following expected loss function

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} (z_{\tau}' Q z_{\tau} + z_{\tau}' U u_{\tau} + u_{\tau}' U' z_{\tau} + u_{\tau}' R u_{\tau}). \quad (2.2)$$

The minimization of (2.2) subject to (2.1) would be straightforward but for the fact that (2.1b) has the expected value of y_{t+1} on the left hand side rather than the actual values, and at any given time t the values of y_t are not predetermined. Equation (2.1b) may be replaced with

$$y_{t+1} = A_{21}x_t + A_{22}y_t + B_2u_t + v_{t+1} \quad (2.1b')$$

where v_{t+1} is by construction the innovation in y_{t+1} , and its value is unconstrained except by the requirement that it should be uncorrelated with any variables dated t or earlier. It is not exogenously given, however, unlike v_{t+1} , and indeed it is a consequence of decisions taken by the private sector in their expectations of y_{t+1} . The value of y_t for $t=1$ (denoted by y_1) is determined in the same way.

The optimization problem can be solved in the following way. First, y_1 and v_{t+1} , $t=2,3,\dots$, are treated as exogenous, and an optimum feedback rule for u_t is computed. Then, values of y_1 and v_{t+1} which are optimal from the government's point of view are found. Finally we check that the optimal feedback rule for u_t can be expressed in such a way that, given the rule, the private sector of the economy makes its decisions in such a way that the realizations of y_1 and v_{t+1} are those found in stage two above.

We first consider the optimization with respect to u_t taking y_1 and v_{t+1} , $t=2,3,\dots$, as exogenous. If the problem has been solved for time $t+1$ and future periods with the control vector expressed as a linear function of the state vector at each point in time, the cost-to-go function at time $t+1$ can be written as a quadratic form in the state vector at $t+1$, plus a constant. The cost-to-go at $t+1$ can be written as $(z_{t+1}'V_{t+1}z_{t+1} + k_{t+1})$. Thus at any time t , $t=2,\dots,B$, minimize with respect to u_t the cost function

$$E_t(\beta(z_{t+1}'V_{t+1}z_{t+1} + k_{t+1}) + z_t'Qz_t + z_t'Uu_t + u_t'U'z_t + u_t'Ru_t) \quad (2.3)$$

taking z_t as given. This gives the standard result for u_t ,

$$u_t = -F_t z_t \quad (2.4)$$

where

$$F_t = (\beta E'V_{t+1}B + R)^{-1}(B'V_{t+1}A + U') \quad (2.5)$$

and thus z_t evolves according to

$$z_{t+1} = (A - BF)z_t + v_{t+1} \quad (2.6)$$

where v_{t+1} is an n -vector whose first n_1 elements are $v_{1,t+1}$ and the rest are $v_{2,t+1}$. The cost to go at t is thus

$$z_t' (\beta(A-BF_t)'V_{t+1}(A-BF_t) + Q - UF_t - U'F_t' + F_t'R'F_t)z_t + \beta k_{t+1} + \text{tr}(\beta V_{t+1}X) = z_t'V_t z_t + k_t. \quad (2.7)$$

Providing V_t converges, it satisfies the equation

$$V = \beta(A-BF)'V(A-BF) + Q - UF - U'F' + F'R'F \quad (2.8)$$

and k_t converges to

$$k = \text{tr}(V\Omega)\beta/(1-\beta) \quad (2.9)$$

where $\Omega = \text{cov}(V_t)$.

A condition for convergence of V_t is that the system being controlled should be stabilizable (Kwakernaak and Sivan, 1972, chapter 6). A stronger condition which is sufficient is that the system be completely controllable. If either of these hold, then V_t converges for any discount factor $\beta \leq 1$.¹

¹ The linear difference equation system

$$z_{t+1} = Az_t + Bu_t$$

is completely controllable if the state of the system can be transferred from any initial state z_0 at initial time t_0 to any terminal state z_1 at time t_1 in a finite time $t_1 - t_0$. A necessary and sufficient condition for complete controllability is that the column vectors of the controllability matrix

$$P = (B, AB, A^2B, \dots, A^{n-1}B)$$

span the n -dimensional space.

Stabilizability is a weaker condition, which ensures that, even if the system is not completely controllable, we can find a feedback matrix F such that $A-BF$ is stable. Kwakernaak and Sivan define a stabilizable system as one whose unstable subspace is contained in its controllable subspace. Any completely controllable system is stabilizable.

The expected discounted cost at $t=1$, the initial point, is obtained as a quadratic form in the whole z vector. And likewise, the control rule for u_t in equation 2.4 is expressed as a feedback on both x_t and y_t . And the equation of motion for the system under control, equation 2.6, generates values of the state vector which depend on realizations of v_{1t} which are not exogenous. Therefore we now proceed to the second stage of the procedure and find values of y_1 and v_{1t} which are optimal from the government's viewpoint. y_1 is not predetermined at the initial period of time. It can therefore be chosen so as to minimize the initial expected cost. By partitioning V conformably with $(x \ y)$ and minimizing we get the optimal starting value for y :

$$y_1 = -V_{22}^{-1}V_{21}x_1. \quad (2.10a)$$

To minimize the stochastic part of the cost function, v_{1t} should be chosen so as to minimize $E_{t-1}v_t'Vv_t$ and this implies setting

$$v_{1t} = -V_{22}^{-1}V_{21}v_{x,t}, \quad t=2,3,\dots \quad (2.10b)$$

The minimized value of the expected cost function is now

$$y_1'V^*y_1 + \text{tr}V^*\Omega_1\beta/(1-\beta),$$

where $V^* = V_{11} - V_{12}V_{22}^{-1}V_{21}$.

This expression now gives the initial expected discounted cost of the optimal policy. The values of the non-predetermined variables y_t and v_{1t} for $t=2,3,\dots$ have now been expressed in terms of exogenous variables, x_t and $v_{x,t}$. However it does not give the optimal policy feedback rule in an operational form since the system of equations (2.1) with policy given by (2.4) does not identify a unique rational expectations solution for the model. The reason is that the resulting system, 2.6 above,

$$z_{t+1} = (A-BF)z_t + v_{t+1}$$

is stable -- this is a consequence of the conditions for the convergence of the recursion in V_t in equations (2.7) and (2.8) -- and therefore any value of v_t is consistent with a stable rational expectations path for the economy. The usual criterion of choosing the path that is stable does not in this case enable private sector agents to make a unique choice of $E_t v_{t+1}$. The same arguments apply to choice of v_{t+1} .

We now go to the third part of the argument in which we show that it is possible to represent the optimal policy rule in such a way as to induce the private sector to make decisions on $E_t v_{t+1}$ which will support the values of y_t and v_t desired by the government. The technique is to represent the optimal policy rule so that the path of the economy implied by the optimizations above is the unique stable rational expectations path.

It is clear from equations (2.10a) and (2.10b) above that a variable defined as a linear combination of x_t and y_t ,

$$p_{y,t} = V_{21}x_t + V_{22}y_t,$$

will have an initial value of zero ($p_{y1}=0$) and will have zero innovations. $p_{y,t}$ is the shadow price on the non-predetermined variables. The system of equations (2.6) can be transformed to give a system of equations in x_t and $p_{y,t}$ rather than x_t and y_t , and the control rule 2.4. can be transformed so that u_t feeds back on x_t and $p_{y,t}$.

Define

$$p_t = Vz_t \tag{2.11}$$

so that p_t is the shadow price of z_t .

Consider the motion of p_t through time. By virtue of (2.11) we have

$$\begin{aligned} p_{2,t+1} &= V_{21}x_{t+1} + V_{22}y_{t+1} \\ &= V_{21}E_t x_{t+1} + V_{22}E_t y_{t+1} + V_{21}(x_{t+1} - E_t x_{t+1}) \end{aligned}$$

$$\begin{aligned}
 & + V_{22}(y_{t+1} - E_t y_{t+1}) \\
 = & V_{21}E_t x_{t+1} + V_{22}E_t y_{t+1}.
 \end{aligned} \tag{2.12}$$

Thus as stated above, p_{yt} contains no innovations and can be treated as a predetermined variable. The system of equations under control can be transformed so that they give the following equations in x_t and p_{yt} :

$$\begin{aligned}
 \begin{matrix} x_{t+1} \\ p_{yt+1} \end{matrix} & = T(A-BF)T^{-1} \begin{bmatrix} x_t \\ p_{yt} \end{bmatrix} + \begin{bmatrix} v_{x_t} \\ 0 \end{bmatrix}
 \end{aligned} \tag{2.13}$$

where the matrix T is defined by

$$T = \begin{bmatrix} I & 0 \\ V_{21} & V_{22} \end{bmatrix} \tag{2.14}$$

Now the value of the control variable can be expressed in terms of the predetermined variables x_t and p_{yt} as follows

$$\begin{aligned}
 u_t & = -FT^{-1} \begin{bmatrix} x_t \\ p_{yt} \end{bmatrix} \\
 & = -(F_1 - F_2 V_{22}^{-1} V_{21})x_t - F_2 V_{22}^{-1} p_{yt}
 \end{aligned} \tag{2.15}$$

The optimal policy rule derived here is equivalent to one derived by Whiteman (1984) for an example with one non-predetermined variable and an exogenous stochastic driving variable, using frequency domain methods. It is of course the same rule as discussed by Currie and Levine (1985) for continuous time stochastic models.

It now remains to be shown that the transformed system has a unique rational expectations equilibrium along the optimum path. As far as the private sector of the economy are concerned, the control variable u_t is given by (2.15), the motion of p_{yt} is given by the last n_2 equations in (2.13) and the equations of motion of x_t and y_t are given by (2.1a) and (2.1b). Thus the system viewed from the private sector's point of view can be written as

$$x_{t+1} = A_{11}x_t + A_{12}y_t + B_1u_t + v_{x,t+1} \quad (2.16a)$$

$$E_t y_{t+1} = A_{21}x_t + A_{22}y_t + B_2u_t \quad (2.16b)$$

$$p_{y,t+1} = (V_{21} \ V_{22})(A-BF)T^{-1} \begin{bmatrix} x_t \\ p_{y,t} \end{bmatrix} \quad (2.16c)$$

with u_t given by (2.15) above. This system of equations has been constructed in such a way that a path of x_t , y_t and $p_{y,t}$ which satisfies

$$p_{y,t} = V_{21}x_t + V_{22}y_t \quad (2.17)$$

is a solution. However, there may be other stable solutions. If that were so, then, given values of $p_{y,t}$ and x_t , there would be other values of y_t than that given in (2.17) above which were consistent with a stable solution of the system. To check whether this is so, investigate the roots of the system (2.15), (2.16) associated with the eigenvector $(V_{21} \ V_{22} \ -I)$. (2.18)

The vector w_t can be defined as

$$w_t = (V_{21} \ V_{22} \ -I) \begin{bmatrix} x_t \\ y_t \\ p_{y,t} \end{bmatrix}$$

By multiplying (2.18) into (2.16) and taking expectations we get the following relation:

$$E_t w_{t+1} = V_{22}(V_{22}^{-1}V_{21}A_{12} + A_{22})V_{22}^{-1}w_t.$$

Thus the condition for a unique stable path given x_t and $p_{y,t}$ is that the eigenvalues of $(V_{22}^{-1}V_{21}A_{12} + A_{22})$ have modulus greater than one. (Note that this condition is not the same as the condition that the uncontrolled system have exactly n_2 unstable roots. That condition can be stated in the following way. If G is a matrix whose rows are the left-hand eigenvectors of A , and ordered in such a way that the last n_2 rows of G are associated with unstable eigenvalues, we have the relation

$$G_{22}^{-1}G_{21}A_{12} + A_{22} = G_{22}^{-1}G_{22}G_{22}$$

where G is the matrix whose diagonal elements are the eigenvalues of A , and G_{22} is the last n_2 rows and columns of that matrix.)

2.2. Alternative representation of the optimal policy rule

The procedure used above for finding an operational representation of optimal policy is not the only one possible. The analysis of Levine and Currie (1984) suggests alternatives. For example they derive optimal policy rules using Lagrangian methods. In the following paragraphs, their derivation is used to get a different representation from the one above.

The optimization problem is as above. (2.2) is minimized subject to (2.1) with a given value of x_1 . Set up the Lagrangian

$$L = E_1 \sum_{t=0}^{\infty} \beta^t (z_t' Q z_t + 2z_t' U u_t + u_t' R u_t + p_{t+1}' (A z_t + B u_t + v_{t+1} - z_{t+1})) \quad (2.19)$$

The first order conditions for an optimum are

$$U' z_t + R u_t + B' E_t p_{t+1} = 0 \quad (2.20a)$$

$$Q z_t + U u_t + A' E_t p_{t+1} - \beta^{-1} p_t = 0 \quad (2.20b)$$

$$A z_t + B u_t + v_{t+1} = z_{t+1} \quad (2.20c)$$

Substitute for u_t in (2.20b) and (2.20c) using (2.20a) to get an autonomous system of equations in z_t and p_t

$$\begin{bmatrix} 0 & A' - UR^{-1}B' \\ I & BR^{-1}B' \end{bmatrix} \begin{bmatrix} z_{t+1} \\ E_t p_{t+1} \end{bmatrix} = \begin{bmatrix} -(Q - U'R^{-1}U) & I \\ (A - BR^{-1}U') & 0 \end{bmatrix} \begin{bmatrix} z_t \\ p_t \end{bmatrix} + \begin{bmatrix} 0 \\ v_{t+1} \end{bmatrix} \quad (2.21)$$

This is a system of $2n$ equations, and when $\beta=1$ it has always n stable and n unstable roots. In fact if λ_j is a root, then so is $1/\lambda_j$. (Kwakernaak and Sivan, 1972, ch. 6, section 6.4.4). Along a stable path for the variables we must have $p_t = V z_t$, where V is given by the equation (2.8) above. (If $p_t = V z_t$ is imposed, for some arbitrary V , and substituted into (2.21), it is found that V must satisfy equation (2.8).) Now (2.8) implies

$$p_{xt} = V_{11} x_t + V_{12} y_t \quad (2.22a)$$

$$p_{yt} = V_{21} x_t + V_{22} y_t \quad (2.22b)$$

(2.22a) can be imposed on the system (2.20) or (2.21) above, so that it is reduced to equations in x_t , y_t and p_{yt} , and u_t . In other words, the government announces the policy rule for u_t as a function of x_t , y_t and p_{yt} , and the equation of motion of p_{yt} as a function of x_t , y_t and p_{yt} . The private sector then makes a decision about its expectations for y_{t+1} . The conditions for the implied system of equations to give a unique path are the same as for the original system (2.21). (2.22a) has already been imposed, and (2.22b) must hold along the stable path.

Thus when the optimal policy rule and the associated equation of motion of p_{yt} is specified in this way -- as a particular function of x_t , y_t , and p_{yt} -- then there is a unique stable path under mild conditions. Namely, for $\beta=1$, there is a unique stable path for the system under control if it is stabilizable, quite independently of the stability of the uncontrolled system.

This procedure is different, however from the one described in section (2.1). There, the policy rule for u_t and the equation of motion for p_{yt} were expressed in terms of x_t and p_{yt} only. y_t had been substituted out of them by, effectively, imposing the equilibrium relation (2.22b) which holds along the stable path. Doing that changes the dynamic structure of the system in a way which alters behaviour off the desired path and consequently changes the conditions for saddlepoint stability.

The second procedure may seem better in the sense that stability is assured under weaker conditions, but it has the disadvantage that the value of the policy instruments is dependent on the current values of nonpredetermined variables which may include exchange rates or stock market prices. It may seem desirable to anchor current policy variables to something firmer in practice. In the model discussed here, of course,

assuming stability under both representations, it makes no difference to the evolution of the system.

Yet another way of representing optimal policy would be as a function of current and past values of the innovations v_{xt} . By applying equation (2.13) repeatedly it is possible to produce an expression for x_t and p_{yt} and consequently u_t as a function of v_{x1}, \dots, v_{xt} . If the optimal policy rule is expressed in this way, then the system under control will have saddlepoint stability if and only if the uncontrolled system had.

It is not surprising that the form in which optimal policy rule is expressed affects the stability of the system. Aoki (1985) makes the observation that it matters whether the rule is expressed as a feedback on state variables or as a function of exogenous innovations only, or as a mixture of the two, as far as the stability of the system is concerned. This is particularly important in the context of rational expectations models where it is important to have saddlepoint stability, so that there is a unique stable rational expectations path.

Currie and Levine (1985) discuss alternative representations of optimal policy rules in some detail for continuous time stochastic models.

2.3 Cost-to-go under optimal policy

The expected cost of the optimal policy was given above in equation (2.10) as

$$x_1' V^* x_1 + \text{tr}(V^* X_{11}) B / (1-B)$$

where $V^* = (V_{11} - V_{12} V_{22}^{-1} V_{21})$. This is the optimal policy at the start of the plan when $t=1$, and $p_{yt}=0$. However, later on during the course of time, p_{yt} will not in general be zero. Since the cost will in general be

$$(x_t' \quad y_t') V \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \text{tr}(V^* \Omega_{11}) B / (1-B)$$

we can write the expected cost as a function of the predetermined variables as

$$x_t' V^* x_t + p_{yt}' V_{22}^{-1} p_{yt} + \text{tr}(V^* \Omega_{11}) \beta / (1-\beta) \quad (2.23)$$

The second term in this expression picks out the additional cost caused by having to continue with an old plan, and to carry out the accumulated commitments, rather than to start afresh, in which case p_{yt} would be reset to zero and the term would be deleted.

2.4 Evolution of system under optimal policy

We can write the dynamic system for x_t and p_{yt} as

$$\begin{matrix} x_{t+1} \\ p_{yt+1} \end{matrix} = T(A-BF)T^{-1} \begin{bmatrix} x_t \\ p_{yt} \end{bmatrix} + \begin{bmatrix} v_{xt+1} \\ 0 \end{bmatrix} \quad (2.24)$$

where T is the transformation matrix defined above in equation (2.14).

Let G be the matrix of left-hand eigenvectors of $(A-BF)$ and Λ be the matrix of eigenvalues. Then the above system can be written as

$$\begin{matrix} x_{t+1} \\ p_{yt+1} \end{matrix} = TGA(TG)^{-1} \begin{bmatrix} x_t \\ p_{yt} \end{bmatrix} + \begin{bmatrix} v_{xt+1} \\ 0 \end{bmatrix} \quad (2.25)$$

or as

$$\begin{bmatrix} x_t \\ p_{yt} \end{bmatrix} = \sum_0^{t-\tau} TGA^i (TG)^{-i} \begin{bmatrix} v_{xt-i} \\ 0 \end{bmatrix} \quad (2.26)$$

where $v_{xt-\tau}$ can be defined as being $x_{t-\tau}$, i.e., $t-\tau$ being the date at which the optimization was carried out. So the variables can each be written as a distributed lag on past innovations in x_t .

$$\begin{aligned} x_t &= \sum_0^{t-\tau} h_{x,i} v_{xt-i} \\ p_{yt} &= \sum_1^{t-\tau} h_{p,i} v_{xt-i} \end{aligned} \quad (2.27)$$

Thus the cost to go in equation (2.23) above can also be written as a quadratic form in the innovations. Writing $v_{xt,\tau}$ as the vector of innovations from t back to τ , and h_x as the matrix $(h_{xt} \ h_{xt-1} \ h_{xt-2} \ \dots)$

$h_{x,t} = \tau$, and defining h_p similarly, we can write the cost-to-go as

$$v_t \tau' (h_x' V^{-1} h_x + h_p' V_{22}^{-1} h_p) v_t \tau + \text{tr}(V_{22}^{-1} \Sigma_{11}) \beta / (1 - \beta) \quad (2.28)$$

3. The discretionary policy rule.

If the government is optimizing in a discretionary way, then at each period of time it optimizes taking the state of the economy x_t as given, and knowing that it will apply the same procedure in subsequent periods. It cannot take as given either the non-predetermined variables y_t or the artificial variables p_t which enabled it to sustain the commitments needed for the optimal programme. The following derivation yields a policy rule which is the same as that derived in Backus and Driffill (1985c) and is equivalent to the time-consistent solution derived by Cohen and Michel (1985) for a single variable continuous-time model. The derivation used here draws on that by Oudiz and Sachs (1985).

Consider the optimization problem at time t . Since x_t is the only state variable, expectations of y_{t+1} can depend only on x_{t+1} . We may write

$$E_t y_{t+1} = H_{t+1} E_t x_{t+1} \quad (3.1)$$

where H_{t+1} is a matrix to be defined later. Suppose the cost-to-go from $t+1$ is $x_{t+1}' W_{t+1} x_{t+1} + w_{t+1}$, where w_{t+1} is some constant. Then the optimization problem for the government is to choose u_t to minimize

$$\beta [x_{t+1}' W_{t+1} x_{t+1} + w_{t+1}] + z_t' Q z_t + 2 z_t' U u_t + u_t' R u_t$$

subject to (1) and (3.1). Using (1) and (3.1), y_t can be eliminated from the maximand and the equations of motion. The problem can be rewritten

$$\beta E_t (x_{t+1}' \bar{W}_{t+1} x_{t+1} + \bar{w}_{t+1}) + x_t' \bar{Q} x_t + 2 x_t' \bar{U} u_t + u_t' \bar{R} u_t$$

subject to

$$x_{t+1} = \bar{A} x_t + \bar{B} u_t + v_{t+1}$$

where the variables are defined as follows

$$\bar{J}_t = (A_{22} - H_{t+1} A_{12})^{-1} (H_{t+1} A_{11} - A_{21})$$

$$\begin{aligned}
K_t &= (A_{22} - H_{t+1}A_{12})^{-1}(H_{t+1}B_1 - B_2) \\
\bar{A}_t &= A_{11} + A_{12}J_t \\
\bar{B}_t &= B_1 + A_{12}K_t \\
\bar{Q}_t &= Q_{11} + J_t'Q_{21} + Q_{12}J_t + J_t'Q_{22}J_t \\
\bar{U}_t &= Q_{12}K_t + J_t'Q_{22}K_t + U_1 + J_t'U_2 \\
\bar{R}_t &= (R + K_t'Q_{22}K_t + U_2'K_t + K_t'U_2)
\end{aligned}$$

Then optimization gives the standard result that the optimal choice of u_t can be expressed as a feedback on x_t

$$U_t = -\bar{F}_t x_t$$

where

$$\bar{F}_t = (\bar{R}_t + \beta_t \bar{B}' W_{t+1} \bar{B}_t)^{-1} (\bar{U}_t' + \beta_t \bar{E}_t' W_{t+1} \bar{A}_t)$$

Consequently the values of W_{t+1} , w_{t+1} and H_{t+1} can be updated as follows

$$\begin{aligned}
W_t &= \beta (\bar{A}_t - \bar{B}_t \bar{F}_t)' W_{t+1} (\bar{A}_t - \bar{B}_t \bar{F}_t) + \bar{Q}_t - \bar{F}_t' \bar{U}_t - \bar{U}_t' \bar{F}_t + \bar{F}_t' R_t \bar{F}_t \\
w_t &= \beta w_{t+1} + \text{trace}(W_{t+1} Q_{11}) \\
H_t &= J_t - K_t \bar{F}_t.
\end{aligned}$$

Oudiz and Sachs remark that there exist no general proofs of convergence of this recursion; but found no convergence problems in numerical examples. Cohen and Michel (1985) prove convergence for an example with one predetermined and one non-predetermined variable. Providing the scheme converges we get a stationary value of W , and H , and \bar{F} . The beliefs of private agents about the relationship between x_{t+1} and y_{t+1} given by H are then consistent with rational expectations.

The way in which the solution is derived makes it clear that it is the beliefs of private agents about future policy, as summarized in H , which makes the government unable to manipulate the economy with threats of future actions, and so restricts it to a time-consistent policy. Oudiz and Sachs (1985), Cohen and Michel (1985), and Levine and Currie discuss it as the result of a strategic interaction between current and future governments, and that each government uses the feedback rule F because

that is the best one, given that future governments will use F . But it is also important to note the role of private sector expectations. By adopting a particular value of H , the private sector constrains the ability of the government to affect the state of the economy. The problem with the game-between-governments interpretation is that there is no reason for a later government to want to punish a preceding one, if it is costly to do so. (It will be shown below that it may be costly. In such cases only beliefs held by the private sector would sustain the consistent policy.)

This policy rule is also a multidimensional generalization of the one derived by Whiteman (1985) using frequency domain methods.

The cost-to-go under discretionary policy is given by

$$x_t' W x_t + \text{trace}(W \Omega_1 \beta / (1-\beta)) \quad (3.15)$$

4. Comparing optimal and discretionary policy rules.

It is clear that if a government had a completely free hand, with no outstanding commitments left behind from earlier periods, then the optimal feedback rule derived in section 2 above would be better than the discretionary rule, assuming that the optimal policy rule would be carried out without reoptimization or cheating. That is to say that for any given value of x_t ,

$$x_t' V x_t + \text{tr}(V \Omega_1 \beta / (1-\beta)) \quad (4.1)$$

is less than the value of

$$x_t' W x_t + \text{tr}(W \Omega_1 \beta / (1-\beta)). \quad (4.2)$$

Thus, if the government had the choice, it would clearly use the optimal policy. However, the optimal policy has the problem that it may be difficult to sustain without some form of precommitment to it. This is the famous time-inconsistency problem formulated by Kydland and Prescott

(1977). In period 2 and in subsequent periods the cost-to-go under the continuation of the optimal plan devised at $t=1$ is

$$x_t'Vx_t + p_{yt}'Vz_2^{-1}p_{yt} + \text{tr}(V\Omega_1)\beta/(1-\beta) \quad (4.3)$$

where p_{yt} will in general not be equal to zero. By contrast the cost-to-go at $t=2,3,\dots$ if old commitments are abandoned is the same expression with p_{yt} set to zero. This is clearly an improvement and a government which had the choice would take it.

Much of the discussion of dynamic inconsistency has been based on the assumption, made either explicitly or implicitly, that the government could abandon old plans and reoptimize. This involves the government in repeatedly making plans or announcements about future policy in one period which are repudiated in the next period but which are nevertheless believed by the private sector. This is an unrealistic situation. Of course, as Kydland and Prescott observe, this is likely to lead to the private sector expecting period-by-period reoptimization, and this will lead the government to being stuck with the discretionary policy as the best it can do. In some cases this may not be too bad relative to the optimum policy, and in others it may be terrible. Kydland and Prescott discussed the case where the policy was a tax on capital, and there the discretionary policy is terrible. Whiteman (1985) discusses various macroeconomic examples where its relative demerits depend on the time-series properties of the exogenous variables driving the system.

The idea that the government will not be able to reoptimize, repudiate old plans, and at the same time have the private sector continue to believe its plans (except insofar as they are consistent with period by period optimization) is embodied in recent work by Barro and Gordon (1983a and b), Backus and Driffill (1985a and b), and Barro (1985). Barro and

Gordon in a simple macro model assume that the government's announcement that it will stick to the optimal policy will be believed so long as it does that, but that if it ever deviates from playing the optimal policy, then for some period of time afterwards the private sector will "punish" the government by expecting that the government will use the time consistent policy. Backus and Driffill rationalize a similar situation by using Kreps and Wilson's (1982) analysis of the "chain-store paradox" and imposing on the private sector beliefs about the government's behaviour which are updated as the game is played.

These assumptions about private sector behaviour seem more plausible than the degree of gullibility implied in the traditional comparison on which the "time-inconsistency problem" is based. It can be incorporated in the present analysis as follows. Assume that the private sector agents play as follows: if the government has always played the optimal policy, the private sector will act in the belief that the government will forever continue to do so. But if the government ever deviates from the optimal policy in any period, the private sector will from then onwards expect discretionary policy. Thus the government in any period has a choice not between (4.3) and (4.1), but between (4.3) and (4.2).

Comparing the two expressions, the cost-to-go of sticking to the old optimal plan is no greater than the cost of going over to the discretionary policy if

$$x_t'(V^*-W)x_t + p_{vt}'V_{22}^{-1}p_{vt} + \text{tr}((V^*-W)\alpha_{11})\beta/(1-\beta) \leq 0 \quad (4.4)$$

The first and last terms of (4.4) are non-positive, but the middle term is non-negative, and so it is not clear whether the whole expression is non-positive. Consider two cases, first a deterministic model, and the a stochastic model.

In the deterministic case, the third term in (4.4) is removed. There is no temptation to abandon the old plan if for all $t=1,2,\dots$

$$x_t'(V^* - W)x_t + p_{yt}'V_{22}^{-1}p_{yt} \leq 0 \quad (4.5)$$

where x_t and p_{yt} are given by (13) with v_{1t} set to zero. From (26) we have

$$x_t = ((TG)_{11}\Lambda_1^{t-1}(TG)^{-1}_{11} + (TG)_{12}\Lambda_2^{t-1}(TG)^{-1}_{21})x_1 \quad (4.6)$$

$$p_{yt} = ((TG)_{21}\Lambda_1^{t-1}(TG)^{-1}_{11} + (TG)_{22}\Lambda_2^{t-1}(TG)^{-1}_{21})x_1 \quad (4.7)$$

((TG)₁₁ is the matrix consisting of the first n_1 rows and columns of (TG), (TG)⁻¹₁₁ is the first n_1 rows and columns of the inverse of (TG), and Λ_i is the matrix consisting of the first n_i rows and columns of the matrix of eigenvalues of ($A-BF$).)

There do not appear to be any general results on conditions when the inequality 4.5 will be satisfied. It depends on the model and the objective function, and also maybe on the initial value of the system x_1 .

Because of this, Currie and Levine suggested that the presence of uncertainty may help to support the optimal policy. Now consider the case where the model is stochastic. If the variance-covariance matrix of v_{xt} is non-zero, there is the third term in (4.4) which is non-positive, and this may help to tip the scales in favour of the optimal policy. However, if the model is stochastic, then x_t and p_{yt} are random variables whose distribution depends on the value of the variance-covariance matrix X_{11} . It will not as a rule be possible to exclude the possibility of the expression in (4.4) becoming positive. For example, suppose the distribution of v_{xt} is non-degenerate multivariate normal, then at any time $t=2,3,\dots$ it is possible that the innovation v_{xt} is such as to make x_t zero or sufficiently close to zero whilst leaving p_{yt} large enough for the whole expression to be positive.

In a similar exercise using continuous-time analysis, Currie and Levine (1986) show that if the support of the distribution of shocks is bounded and the discount factor is large enough, then quite generally the optimal policy rule is consistent. The argument is basically the same as that of James Friedman (1977, chapter 8) in the context of oligopoly (in a deterministic framework). The argument is as follows. If the support of the distribution of v_{xt} is bounded, then it will be possible to place bounds on the size of p_{yt} via equation 2.27, and hence to place a bound on the size of the term $p_{yt}'V_{22}^{-1}p_{yt}$ in 4.4. This bound is a continuous function of the discount factor β , $\beta \in [0,1]$.

Define $f(\beta) = \text{maximum}_{v_{xt-1}}$

v_{xt-1}

subject to $p_{yt} = \sum_0^{\infty} \beta^i h_{pi} v_{xt-1}$

and $v_{xt-1} \in [v_x, \bar{v}_x]$

Similarly V^* and W are continuous functions of β , $\beta \in [0,1]$. Consider the expression

$$(1-\beta)f(\beta) - (\text{tr}(W(\beta)-V^*(\beta))X_{11})\beta. \quad (4.8)$$

This is negative when $\beta = 1$, since $(\text{tr}(W(\beta)-V^*(\beta))X_{11})$ is strictly positive. The whole expression (4.8) is continuous in β , $\beta \in [0,1]$, and thus there exists a region $\bar{\beta} \leq \beta \leq 1$ in which it is non-positive. For these values of β , (4.4) is also non-positive. Thus we can state as a

Theorem

In any stochastic linear model, if (i) the discount factor is sufficiently close to one, and (ii) the support of the distribution of stochastic shocks hitting the economy is bounded, and (iii) the private sector responds to a deviation from the optimal policy by expecting discretionary government behaviour from then onwards, then optimal policy is time consistent.

5. Numerical Examples

The points made above can be illustrated using some small models. Those discussed by Whiteman and Cohen and Michel will be used in what follows.

Models with one predetermined variable, one non-predetermined variable, and one control variable are easy to deal with and at the same time rich enough to illustrate some of the possibilities mentioned above. The examples of Whiteman and Cohen and Michel fall into this class. In this case, the equations for the evolution of the system simplify somewhat, since x_t , y_t , and p_{yt} are all scalars. In the deterministic case, equations (4.6) and (4.7) above can be written as

$$x_t = (ag^{t-1} + (1-a)g_1^{t-1})x_1 \quad (5.1)$$

$$p_{yt} = b(g^{t-1} - g_2^{t-1})x_1 \quad (5.2)$$

where a and b are defined by

$$a \equiv (TG)_{11}(TG)^{-1}_{11}$$

$$b \equiv (TG)_{21}(TG)^{-1}_{11}$$

The movements of x_t and p_{yt} over time will depend on the values of a , b , g_1 and g_2 . g_1 and g_2 must be less than one in absolute value for the system to converge. But within that limitation are quite a number of possibilities, depending on the sizes and signs of these variables.

Whiteman (1985) uses the following model. A non-predetermined variable e_t is affected by expectations of its own future value, by an exogenous driving variable x_t and by a control variable u_t :

$$E_t e_{t+1} = a e_t + x_t + u_t \quad (5.1)$$

The variable e could be something like the exchange rate, for example. The driving variable is some ARMA process, here AR(1), for simplicity:

$$x_{t+1} = b x_t + v_{t+1} \quad (5.2)$$

where v_{t+1} is a serially uncorrelated innovation. The government is assumed to want to control the exchange rate, say, and also is concerned

about the cost of using the control variable. It attempts to minimize a cost function

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} (e\tau + cu\tau) \quad (5.3)$$

This is an example which falls into the class of models discussed in the above sections. Here we have $y_t = e_t$. In terms of the symbols used above

$$A = \begin{bmatrix} b & 0 \\ 1 & a \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad R = c$$

This system is not completely controllable because x cannot be controlled at all. But it is however stabilizable, since the control variable u can be chosen so as to produce any desired value of e_{t+1} starting from any point e_t .

Consider an example with the following numerical values: $b=.95$, $a = 1.2$, $c=1.0$, $\beta = 0.96$. In this case, if this is considered as a deterministic model, the cost to go under the optimal policy is $6.59838x_t^2 + .516449p_{yt}^2$, while under the consistent policy it is $10.8068x_t^2$. The equations of motion of x_t and p_{yt} under control are

$$x_t = (.95)^{t-1} x_1$$

$$p_{yt} = 1.0688((.95)^{t-1} - (.419749)^{t-1}) x_1.$$

Thus as the driving variable decays towards zero, the co-state variable p_{yt} at first increases and then itself begins to decline towards zero. At no time does the cost of the optimal policy exceed that of the consistent policy. Over time, the ratio p_{yt}/x_t increases towards 1.0688, and so the cost-to-go of the optimal policy at no time exceeds $7.18834x_t^2$. Thus in this particular instance there is never any temptation for the government to renege on the optimal policy under the conditions stated above.

Consider the same model with slightly different parameter values. Suppose that the value of b is reduced to .75 (from .95). Now the cost-to-go under optimal policy is $1.49928x_t^2 + .516449p_{yt}^2$ while the cost under consistent policy is $2.23661x_t^2$. The equations of motion of x_t and p_{yt} are now

$$x_t = (.7)^{t-1}x_1$$

$$p_{yt} = 1.51792((.7)^{t-1} + (.41975)^{t-1})x_1$$

In this instance, the ratio p_{yt}/x_t approaches 1.51792 as t increases, and the cost of continuing with optimal policy approaches $2.68922x_t^2$ which exceeds the cost of the consistent policy.

So this numerical example illustrates that just by changing parameter values it is possible in the case of deterministic models to have on the one hand a case where the optimal policy is sustainable, and on the other hand a case where it is not. This supports the general point made above that there are no general conditions which define when optimal policy can be sustained.

An interesting feature of this example is that the dynamic inconsistency arises only because the driving variable x_t is non zero. If there are no shocks in this model and x_t is equal to zero at all times, optimal and consistent policy coincide. There is nothing to do of course: u_t should be set to zero in both case.

The results above bear out a remark made by Whiteman that the optimal policy does better relative to the discretionary policy the more highly autoregressive is the driving variable.

Consider now another example. This is described by Cohen and Michel (1985). In this a predetermined state variable x_t is determined by its own past value, that of another (non-predetermined) state variable, v_t , which C-M derive as the control variable of private sector agents, and the

government's control variable u_t . y_t behaves like a non-predetermined variable such as an exchange rate. The equations are as follows:

$$x_{t+1} = ax_t + y_t + u_t$$

$$y_{t+1} = b_1x_t + b_2y_t,$$

where $b_2 > 1$ and $a < 1$. The objective is to minimize a discounted sum of squared values of x_t and u_t :

$$E_t \sum_{t=0}^{\infty} (qx_t^2 + ru_t^2).$$

This falls into the same pattern as before, this time with the parameter values

$$A = \begin{bmatrix} a & -1 \\ b_1 & b_2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $R = r$, and $\beta = 1$. With $a=.5$, $b_1=-.4$, $b_2=1.3$, $q=1$, $r=1$, and $b=1$, the system behaves much in the same way as above. x_t converges monotonically on zero. $p_{y,t}$ at first increases, and then falls back toward zero. After the first period, the optimal policy is more costly to continue than it is costly to use consistent policy.

With the same numerical values except for a change in b_1 from $-.4$ to $+.4$, the results are a little different. The feedback equation for u_t is

$$u_t = -.7310374x_{1t} - 0.427435x_{2t}$$

and the roots of the controlled system are complex. The equations of motion of x_t and $p_{y,t}$ are

$$x_t = -10.16066(.545424)^{t-1} \sin(r(t-1) - \theta) x_1$$

$$p_{y,t} = 16.7567(.545424)^{t-1} \sin(r(t-1)) x_1$$

where $r = .047353$ and $\theta = .09858$ (radians). The system has heavily damped oscillations. Starting out with $x_1 = 1.0$, x_t is at first positive, passes through zero between $t=3$ and $t=4$, and then it is negative until after $t=69$, when it becomes positive again. $p_{y,t}$ starts out at zero when $t=1$ and is thereafter positive for 66 periods, peaking around $t=3$. This is an example where x_t passes through zero along the optimal path, and at

the same time p_{2t} is non-zero. Clearly at this point the discretionary policy has to be better than the optimal policy. In fact, in this example, it turns out that the cost to go under discretionary policy is less than under optimal policy from $t=2$ until $t=25$ at least. (At that point the simulation stops.)

Consider now some examples of stochastic models. The following example shows that the stochastic element can enable the optimal policy rule to be consistent. Take the "Whiteman" example above with parameters

$$A = \begin{bmatrix} .7 & 0 \\ 1.0 & 1.2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad R = 1 \\ \beta = .96.$$

so the autoregressive parameter on the predetermined variable is .7, and the discount rate is 4%. In this instance the cost-to-go under optimal policy is

$$1.23018x_t^2 + .516449p_{yt}^2 + 29.52432s_t^2$$

and under discretionary policy it is

$$1.79971(x_t^2 + 24s_t^2).$$

The solution for x_t and p_{yt} under optimal policy turns out to be

$$x_t = \sum_0^{\infty} .7^i v_{xt-i}$$

$$p_{yt} = 1.738524 \sum_0^{\infty} (.7^i - .419749^i) v_{xt-i}.$$

If this is treated as a deterministic model, with $s_t^2=0$, and an initial perturbation v_{x1} , v_{xt} being zero at all other times, then after $t=3$ the discretionary policy would dominate the optimal policy.

If v_{xt} is a sequence of i. i. d. normal random variables with zero mean and variance s_t^2 , then it is not possible to rule out the possibility that at any time the value of p_{yt} is large enough (in absolute value) and x_t small enough (in absolute value) so that the discretionary policy appears cheaper than the optimal policy. In this case the optimal policy cannot be sustained as a consistent policy.

But if the distribution of v_{xt} is such as to limit the maximum possible value of p_{yt} that could ever occur, then this may not be true. If v_{xt} is uniformly distributed over the (closed) interval from -1 to $+1$, its variance s_v^2 is $1/3$, and the maximum possible value of p_{yt} is 2.79892. x_t^2 is non-negative, and examination of the above expressions for cost-to-go under the two policies shows that there can never arise a situation where the government would wish to switch to a discretionary policy. In this case, the optimal policy is consistent. The private sector's strategy in this game is that if the government uses the policy prescribed by the optimal rule in a period, the private sector acts as if they believe the government will stick to the rule forever, but if the government deviates from it, the private sector acts as if they expect period-by-period reoptimization from then onwards.

6. Conclusions

This paper has applied the lessons of Barro and Gordon and Backus and Driffill to general dynamic stochastic macroeconomic models with rational expectations. It shows that optimal policy rules in these models may be time-consistent when the private sector agents form expectations about future government behaviour which effectively "punishes" government deviations from the optimal policy rule. There is not always a need for constitutional restrictions on government action or some other form of formal precommitment to achieve optimal outcomes. The game specified in the above analysis uses the idea that if the government deviates from the optimal policy rule, the private sector will cease to believe that such a rule will guide policy in the future, and the government takes this into account when deciding on its policy actions at each point in time.

~~Time inconsistency has been used as an objection to policy optimization as a theory of policy. But time inconsistency itself is~~

Previous criticism of optimal policy rules for being "time-inconsistent" has been based on the analysis of a particularly simple model of private sector behaviour. It has been asked whether there exists temptation for a government to "renege" on an optimal policy rule, under the assumption that the private sector would believe that any new policy rule or plan would be adhered to as announced. If such a temptation exists, the optimum policy rule has been said to be time-inconsistent.

This is based on a surely implausible model of private sector behaviour, in which it is assumed that the private sector has rational expectations of the workings of the economy, except for a blind spot when it comes to forecasting the behaviour of the government. In the case of the government, they simply believe the government's policy announcements even though they might thereby make systematic errors.

We must model private sector agents' behaviour in such a way that their expectations are rational with respect to government behaviour as with other aspects of the economy. In addition we show that "rational expectations" does not identify a unique equilibrium of the infinitely repeated game. Both the "discretionary" policy rule and the optimal policy rule may be consistent solutions. Which policy rules are consistent depends on the way in which the private sector agents play.

Although a decentralised private sector has no strategic power -- individual agents behave only in accordance with individual self interest and act as if their actions do not affect the evolution of the aggregate economy -- it is nevertheless necessary to specify the policy problem as a game.

The analysis shows that the stochastic element in models may be important in letting optimal policy be sustained in equilibrium, as suggested by Levine and Currie (1984). It matters whether there are shocks or not, and the distribution of the shocks matters.

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