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JEL Classification: D85, C72, Z13

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Coordination on Networks

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October 23, 2017

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We study a coordination game among agents on a network, choosing whether or not to take an action that yields value increasing in the actions of neighbors. In a standard global game setting, players receive noisy information of the technology's common state-dependent value. We show the existence and uniqueness of a pure equilibrium in the noiseless limit. This equilibrium partitions players into coordination sets, within members take a common cutoff strategy and are path connected. We derive an algorithm for calculating limiting cutoffs, and provide necessary and sufficient conditions for agents to inhabit the same coordination set. The strategic effects of perturbations to players' underlining values are shown to spread throughout but be contained within the perturbed players' coordination sets.

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1 Introduction

Settings with binary actions and positive network effects are ubiquitous: the choice to adopt a technology or platform, such as in social media, where the value of the technology/platform is increasing in the adoption by friends; the choice to partake in crime, when the proficiency of crime, and thus the likelihood of not getting caught, is increasing in the criminality of accomplices; or, the implementation of defense or anti-immigration policies, when the likelihood of attack or the influx of migrants depends on policies employed in neighboring countries. In each of these examples, uncertainty in a common state can also influence the value of adoption: the underlining value of the technology; the strength or presence of the police force; the aggression of attackers or the state of the economy. This paper studies coordination in these uncertain environments.

The ensuing model employs the tools of global games embedded into a network game. Players' positions in the network define their preferences over the action choices of others. Using the language of technology adoption, the total value an agent receives from adopting the technology is increasing in the technology's underlining value (the state) and in the adoption by neighbors. Agents receive noisy signals informing them of the state. In equilibrium, agents further use their private information to infer the observations and actions of neighboring agents, and anticipate the ultimate value they will enjoy from adopting the technology.

The classic equilibrium selection of global games obtains. In our setting with binary actions, the equilibrium selected in the noiseless limit comes in the form of cutoff strategies. Each agent adopts the technology when their private signal exceeds their equilibrium cutoff, a cutoff determined by their position in the network. We explore the role of the network's architecture in determining who coordinates their adoption choices with whom. The analysis begins by providing an algorithm for calculating limiting cutoffs, then provides necessary and sufficient conditions for agents to inhabit the same coordination set, defined as the set of path-connected agents taking a common cutoff.

We first consider the case of homogeneous values, where the network alone introduces (ex ante) heterogeneity across agents. We give an exact condition for which a single coordination set exists. This condition says that the coordination set has to be *balanced*, that is, the average degree of each sub-network (composed of any nonempty subset of agents in the network) is not greater than the average degree of original network. To understand this, consider any core-periphery network, with regular core of degree d_c and size n_c, and with n_p periphery nodes, each connected to k core nodes symmetrically. This graph is balanced if and only if $d_c \leq 2k$, which means that either the core is not very connected (as, for example, in a star network), or the number of links to the core is very large. Otherwise the periphery node will have a strictly higher cutoff than the core nodes and there will be more than one coordination sets.

This characterization to global coordination implies that under homogeneous values, strikingly, agents with arbitrarily different degrees may belong to the same coordination set. For example, in a star network, regardless of the number of peripheral agents, all agents coordinate together in the limit, meaning that they adopt the technology under the same set of states.[1] To understand this result, near the noiseless limit, equilibrium cutoffs for the center and the peripheral agents must lie within each others' noise supports. Therefore, in the limit, the center and periphery agents must take identical cutoff strategies. Upon increasing the size of the core, network effects within the core become sufficiently reinforcing so that the core agents may take a strictly lower cutoff than the peripheral agents. In fact, the selection of an equilibrium that exhibits a common cutoff across all agents within the network is shown to extend to all tree networks and regular-bipartite networks.[2] In the former, the absence of multiple closed walks guarantees existence of a common cutoff. In the latter, the short side of the network modulates the cutoff shared by both sides. Such robust coordination is equally interesting when multiple cutoffs obtain. As an initial illustration of this, we provide conditions under which additional links between coordination sets impose zero influence on the equilibrium play of the coordination set taking a lower cutoff. As a lone peripheral agent sequentially links to a clique, for example, each link influences the lone agent's cutoff while the clique remains unaffected, until a threshold number of links are established, after which the full network begins to coordinate together.

Once introducing heterogeneous values, we can extend and more broadly characterize the robust coordination described under homogeneous values. In particular, we are able to give a more general condition for which the coordination set is *balanced* where both network effects and intrinsic valuations matter. The attainment of a common cutoff within each coordination set is shown to be robust to perturbations to the intrinsic (state independent) value of the technology to any given agent. Holding fixed other parameters of the model, we characterize the range of intrinsic values that support an agent's coordination with her coordination set. The size of this support is shown to be strictly

[1] Indeed, using the notations above, for the star network, $d_c = 0 < 2k = 2$, so this network is balanced.

[2] A formal definition to regular-bipartite networks is provided in Section 5. A common cutoff is also shown to obtain in networks that have a unique cycle and those that have at most four nodes.

increasing in the relative size of network effects: coordination becomes increasingly *sticky* as network effects strengthen. Perturbations are shown to influence equilibrium adoption only across members within the perturbed agent's coordination set. Thus, the contagion of such perturbations extends within coordination sets, but discontinuously drops to zero across coordination sets: contagion in contained.

We explore the welfare and policy implications of the model. We develop closed-form marginal gains to a policy designer aiming to maximize (i) adoption and (ii) ex ante welfare (i.e. the benevolent planner) across the network. Under the former benchmark, the designer faces the following tradeoff. If she subsidizes adoption within large interconnected coordination sets where strategic contagion is relatively broad, the influence of the intervention on the target will be limited due to the stickiness of coordination. That is, while the intervention reaches a large set of agents, the direct impact of the intervention can be dampened by the target's coordination with her coordination set.

The tradeoffs of the benevolent planner are no less complex. Ex ante, policy interventions impose positive externalities on neighboring agents in coordination sets taking lower cutoffs, though having no influence on their equilibrium behaviors. This establishes a fundamental wedge between the objectives of designers aiming to maximize adoption versus ex ante welfare. As such, if the benevolent planner targets a coordination set taking the lowest cutoff (the most interconnected coordination sets under homogeneous values), this excludes the potential for direct externalities enjoyed by coordination sets taking lower cutoffs. As a consequence of this wedge, the optimal targets to our two designers need not coincide.

The paper is organized as follows. In the next section, we relate our paper to the relevant literature. Section 3 introduces the model. Section 4 establishes the limit equilibrium and refines it to cutoff strategies, and derives an algorithm for calculating limiting cutoffs. In Section 5, we characterize the equilibrium under homogeneous valuations and partition players into coordination sets, within members take a common cutoff strategy and are path connected. In Section 6, we generalize our analysis to heterogeneous valuations. Section 7 discusses the welfare and policy implications. Section 8 discusses extensions and applications to platform adoption, crime, and immigration policy. Finally, Section 9 concludes. All proofs can be found in the Appendix.

2 Related Literature

This paper adds to the growing literature on network games.[3] Ballester et al. (2006), and more recently Bramoullé et al. (2014) characterize conditions for equilibrium existence and uniqueness when actions are continuous and best replies are linear.[4] Galeotti et al. (2010) obtain multiplicity of equilibrium in games under more general best replies, assuming incomplete and symmetric information of the extended network structure (beyond own degree). The present paper takes strategic complements under incomplete information. While multiple equilibrium again obtain under complete information, noisy information of a common fundamental state provides a unique equilibrium selection in the noiseless limit of our game.

This paper also adds to a younger literature on network games with incomplete information. Calvó-Armengol et al. (2007) and De Marti and Zenou (2015) study the linear-quadratic setting of Ballester et al. (2006) under the enrichment of a Bayesian game. Calvó-Armengol et al. (2015) and Leister (2017) incorporate endogenous investment in signal precision in these settings. And in a different vein, Hagenbach and Koessler (2010) and Galeotti et al. (2013) study cheap-talk in networks. Golub and Morris (2017a,b) study consistency and convergence in higher order expectations in Bayesian network games under linear best replies. The current paper diverges from these contributions by focusing the analysis near and in the noiseless limit, and taking actions to be binary.

Carlsson and van Damme (1993) first exhibited this selection devise for global games of two players and two actions.[5] Frankel et al. (2003) extend the result to arbitrary games of strategic complements. Our paper sits in the middle, employing the structure of a network game under binary actions toward characterizing the topology of equilibrium coordination. In a two-sided environment, Morris and Shin (1998) provide closed forms to their common limit-equilibrium cutoff, toward studying the interaction of a government defending a currency from a continuum of currency speculators. The ensuing model can be viewed as a network of governments interacting, while abstracting away from the role of speculation within each country.[6] Sákovics and Steiner (2012) study policy impact in

[3] See Jackson (2008) chapter 9, Jackson and Zenou (2015) and Bramoullé and Kranton (2016) for surveys.

[4] These conditions involve bounding eigenvalues of transformations of the network's adjacency matrix.

[5] They show the risk-dominant equilibrium to be selected in these games.

[6] Equilibrium selection (Frankel et al. (2003) Theorem 5) along with all characterizations of Section 4 (excluding θ^* from (8) of Proposition 2) are robust to the inclusion of speculators at each node (country). Related applications include crises and banks runs; see Dasgupta (2004), Goldstein and Pauzner (2004,

a global game with a continuum of agents who value an agent-weighted average action, where a common cutoff obtains in the noiseless limit. In the current paper, network heterogeneity induces multiple limit cutoffs. Our policy analysis contrasts adoption-based with welfare-based policies to establish a basic wedge between the two benchmarks, a wedge which only obtains under multiple cutoffs.[7]

Our results also bare on those of the network contagion literature. Morris (2000) studies a coordination game on a network under complete information, characterizing equilibrium adoption via the property of "cohesion" within subsets of players.[8] While connectivity within agent sets similarly plays an important role in the ensuing model (Proposition 6, below), the global game selection insures a unique prediction of coordination amongst agents.[9] Elliot et al. (2014) and Acemoglu et al. (2015) model the clearing of liabilities between institutions. The contagion of the ensuing model offers an alternative prediction to the spread of perturbations over the network, while incorporating strategic play, be it under a more elemental machinery.

3 Model Setup

A finite set of agents N simultaneously choose whether to adopt a technology.[10] $a_i \in \{0, 1\}$ will denote agent i's choice to adopt. The components of the model are defined as follows.

Payoffs. Payoffs from adopting the technology depend on the underlying fundamental θ, continuously distributed over bounded, interval support $\Theta \subseteq \mathbb{R}$. Moreover, the agents are connected via a network $\mathcal{G} = (N, E)$. E defines the set of edges between unordered pairs ij taken from N. We assume a connected and undirected graph: $i \in N_j$ if and only if $j \in N_i$, where $N_i := \{j : ij \in E\}$ is the set of i's neighbors, and $d_i := |N_i|$ her degree.

2005), and Rochet and Vives (2004).

[7]Sákovics and Steiner find the optimal adoption-minimizing subsidy targets agents with high influence while being relatively uninfluenced by others.

[8]In the present paper, the value of adoption is a function of the total number of neighbors adopting, rather than the fraction of neighbors adopting. Moreover, incomplete information of a common state with equilibrium selection are significant departures from this work.

[9]In a setting similar but more general than Morris (2000) where an infinite population of players interact locally and repeatedly, Oyama and Takahashi (2015) determine when a convention spreads contagiously from a finite subset of players to the entire population in some network.

[10]For the sake of the exposition, we use the example of technology adoption but, of course, any $\{0, 1\}$ binary actions will yield the same results.

Then, each i obtains the following payoff from adopting:

$$u_i(\mathbf{a}_{-i}|\theta) = v_i + \sigma(\theta) + \phi \sum_{j \in N_i} a_j \tag{1}$$

where $v_i \in \mathbb{R}$, $\sigma : \Theta \mapsto \mathbb{R}$, and $\phi > 0$. v_i gives the intrinsic (state independent) value to i from adopting, σ the state dependent value, with each of i's neighbors' adoption positively influencing the technology's value. $\sigma(\theta)$ is assumed to be differentiable and strictly increasing in θ. The network effect ϕa_j in (1) captures the positive externality that j's adoption imposes on i, while ϕ uniformly scales the size of network effects. The value to each agent from not adopting the technology is normalized to zero.

Dominance Regions. For each i, we assume v_i, σ and ϕ are such that there exist $\underline{\theta}_i$ and $\overline{\theta}_i$ such that $v_i + \sigma(\theta) + \phi d_i < 0$ when $\theta < \underline{\theta}_i$ and $v_i + \sigma(\theta) > 0$ when $\theta > \overline{\theta}_i$. Thus, there exist dominant regions $[\min \Theta, \underline{\theta}]$ and $[\overline{\theta}, \max \Theta]$, with $\underline{\theta} := \min_i\{\underline{\theta}_i\}$ and $\overline{\theta} := \max_i\{\overline{\theta}_i\}$, such that not adopting and adopting the technology (respectively) are dominant strategies for all players. When the realization of θ is common knowledge amongst agents, with σ continuous in θ and $\phi > 0$ there can exist a strictly positive measure of θ realizations within $[\underline{\theta}, \overline{\theta}]$ at which multiple pure strategy Nash equilibria occur.

Information Structure. In the perturbed game, θ is observed with noise by all agents. Each i realizes signal $s_i = \theta + \nu\epsilon_i$, $\nu > 0$, where ϵ_i is distributed via density function f and cumulative function F with support $[-1, 1]$. All signals are independently drawn across agents conditional on θ. For each $\nu > 0$, we write $G(\nu)$ the corresponding global game.[11,12]

4 Limit Equilibrium

4.1 Existence and Uniqueness of Limit Equilibrium

$G(\nu)$ gives a Bayesian game of strategic complements between agents. Agent i chooses (possibly mixed) signal-contingent strategy $\pi_i : S \mapsto [0, 1]$, mapping each signal realiza-

[11]The assumption of a common noise structure is without loss of generality as the limit-equilibrium selection is robust to arbitrary, idiosyncratic F_i. All results in the limit hold under Gaussian ϵ_i (unbounded support). See Section 8.1.

[12]As is standard in the global game literature, and without loss of generality, we assume agents do not carry prior information of θ. See Sákovics and Steiner (2012) for discussion.

tion to the likelihood i adopts. We write $\boldsymbol{\pi} := (\pi_1, \ldots, \pi_N)$ and denote $\boldsymbol{\pi}^*$ a Bayesian Nash Equilibrium of $G(\nu)$.

Frankel et al. (2003) Theorem 1 establishes uniqueness of a limiting mixed-strategy equilibrium in general global games of strategic complements. We can refine their result in our setting with binary actions, where a pure limit equilibrium in cutoff strategies is obtained. Formally, define i's cutoff strategy at $s_i^\dagger \in S$ by:

$$\pi_i^\dagger(s_i) := \begin{cases} 1 & \text{if } s_i \geq s_i^\dagger \\ 0 & \text{if } s_i < s_i^\dagger \end{cases} .$$

Let us formulate expected payoffs when all neighbors use cutoff strategies. Given $\boldsymbol{\pi}_{-i}^\dagger$ and conditional on signal realization s_i, i's expected payoff from adopting is:

$$\begin{aligned} U_i(\boldsymbol{\pi}_{-i}^\dagger | s_i) &:= \mathbb{E}_\theta \left[\mathbb{E}_{\mathbf{s}_{-i}} \left[u_i(\mathbf{a}_{-i}|\theta) \,\middle|\, \boldsymbol{\pi}_{-i}^\dagger, \theta \right] \middle|\, s_i \right] \\ &= \mathbb{E}_\theta \left[v_i + \sigma(\theta) + \phi \sum_{j \in N_i} r(\theta, s_j^\dagger; \nu) \,\middle|\, s_i \right], \end{aligned} \tag{2}$$

where the conditional likelihood that $j \in N_i$ adopts is given by:

$$r(\theta, s_j^\dagger; \nu) := \int_{-1}^{1} \pi_j^\dagger(\theta + \nu\epsilon_j) f(\epsilon_j) d\epsilon_j = \begin{cases} 0 & \text{if } \theta \leq s_j^\dagger - \nu \\ F\left(\frac{\theta - s_j^\dagger}{\nu}\right) & \text{if } \theta \in (s_j^\dagger - \nu, s_j^\dagger + \nu] \\ 1 & \text{if } \theta > s_j^\dagger + \nu \end{cases} . \tag{3}$$

Expression (2) can then be written:

$$U_i(\boldsymbol{\pi}_{-i}^\dagger | s_i) = v_i + \int_{-1}^{1} \left(\sigma(s_i - \nu\epsilon_i) + \phi \sum_{j \in N_i} r(s_i - \nu\epsilon_i, s_j^\dagger; \nu) \right) f(\epsilon_i) d\epsilon_i. \tag{4}$$

Lemma 1. *A Bayesian Nash Equilibrium $\boldsymbol{\pi}^*$ of $G(\nu)$ in cutoff strategies exists.*

We have shown that there is a unique signal $s_i^* \in (\underline{\theta} - \nu, \overline{\theta} + \nu)$ that solves: $U_i(\boldsymbol{\pi}_{-i}^\dagger | s_i^*) = 0$, with adoption optimal for i if and only if $s_i \geq s_i^*$. There is therefore a unique limit equilibrium in cutoff-strategies. The next result is straightforward to obtain using Lemma 1 and Theorem 1 in Frankel et al. (2003).

Proposition 1. *There exists an essentially unique strategy profile $\vec{\boldsymbol{\pi}}$, which is in cutoff strategies, such that any $\boldsymbol{\pi}$ surviving iterative elimination of strictly dominated strategies* *in $G(\nu)$ satisfies* $\lim_{\nu\to 0}\boldsymbol{\pi} = \vec{\boldsymbol{\pi}}$.

The unique limit equilibrium $\vec{\boldsymbol{\pi}}$ of $G(0)$ is characterized by $\theta_i^* := \lim_{\nu\to 0} s_i^*$, with each i choosing to adopt if and only if $\theta \geq \theta_i^*$. With Proposition 1, we are free to study cutoff-strategy equilibria of $G(\nu)$, which must converge on $\vec{\boldsymbol{\pi}}$. $U_i(\boldsymbol{\pi}_{-i}^{\dagger}|s_i^*) = 0$ for each $i \in N$ define the system of conditions pinning down such equilibria.

4.2 Calculating the Limit Equilibrium in General Settings

Calculating limit cutoffs $\boldsymbol{\theta}^*$ defining $\vec{\boldsymbol{\pi}}^*$ entails finding a consistent set of limiting expectations, for each agent, on other agents' adoption choices. Denote $\mathbf{w}^*$ the limiting expectations placed on neighbors adopting in equilibrium $\boldsymbol{\pi}^*$, when each agent i realizes signal s_i equal to her equilibrium cutoff s_i^*. Precisely:

$$w_{ij}^* = \lim_{\nu\to 0} \mathbb{E}_{s_j}[\pi_j^*(s_j)|s_i = s_i^*].$$

Moreover, define:

$$\mathcal{W} = \{\mathbf{w} = (w_{ij}, (i,j) \in E)|w_{ij} \geq 0, w_{ji} \geq 0, w_{ij} + w_{ji} = 1; \forall (i,j) \in E\},$$

as the set of *feasible weighting functions* for $\mathcal{G}$. Clearly, $\mathcal{W}$ is compact, convex, and isomorphic to $[0,1]^{e(N)}$, where $e(N)$ is the number of links in $\mathcal{G}$. And as shown in the Appendix (see Lemma 2), $\mathbf{w}^* \in \mathcal{W}$. Given the values $\mathbf{v} = (v_1, \cdots, v_n)'$, we define the affine mapping $\Phi : \mathcal{W} \to \mathbf{R}^n$ as follows:

$$\Phi_i(\mathbf{w}) = v_i + \phi \sum_{ij\in E} w_{ij}, \quad \forall i \in N. \tag{5}$$

Let $\Phi(\mathcal{W})$ denote the image of $\mathcal{W}$ under the mapping Φ. Given linearity of $\Phi(\cdot)$, $\Phi(\mathcal{W})$ is a compact, convex polyhedron. Denote $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^n$ and $||\mathbf{x} - \mathbf{y}|| := \sqrt{\langle \mathbf{x}, \mathbf{y}\rangle}$ the Euclidean norm.

Theorem 1. *For any $\mathbf{v}$, ϕ, and network $\mathcal{G}$, the equilibrium limit cutoffs $\boldsymbol{\theta}^*$ are given by:*

$$\sigma(\theta_i^*) + q_i^* = 0, \quad \forall i, \tag{6}$$

where $\mathbf{q}^* = (q_1^*, \cdots, q_n^*)$ is the unique solution to:

$$\mathbf{q}^* = \underset{\mathbf{z}\in\Phi(\mathcal{W})}{\operatorname{argmin}} ||\mathbf{z}||. \tag{7}$$

Theorem 1 provides a program for calculating equilibrium limit cutoffs, for arbitrary network structure $\mathcal{G}$. The solution $\mathbf{q}^*$ to this program maps one-to-one to and is monotonically decreasing with $\boldsymbol{\theta}^*$, as defined by (6). Strikingly, $\mathbf{q}^*$ solves a simple quadratic program with linear constraints, as defined by (7). $\mathbf{q}^*$ maps back to weighting matrix $\mathbf{w}^*$, via $\Phi(\cdot)$, Theorem 1 can be reformulated using the tools of projections mappings.

Definition 1. *Let K be a closed convex set in $\mathbb{R}^n$. For each $\mathbf{x} \in \mathbb{R}^n$, the* orthogonal projection *(or,* projection*)*[13] *of $\mathbf{x}$ on the set K is the unique point $\mathbf{y} \in K$ such that:*

$$||\mathbf{x} - \mathbf{y}|| \leq ||\mathbf{x} - \mathbf{z}||, \quad \forall \mathbf{z} \in K.$$

We denote $\mathbf{Proj}_K[\mathbf{x}] := \mathbf{y} = \operatorname{argmin}_{\mathbf{z} \in K} ||\mathbf{x} - \mathbf{z}||$.

Observe that the vector $\mathbf{q}^*$ is a projection of the origin onto the compact, convex space $\Phi(\mathcal{W})$, which is the image of $\mathcal{W}$ under the mapping Φ:

$$\mathbf{q}^* = \mathbf{Proj}_{\Phi(\mathcal{W})}[\mathbf{0}_n],$$

for $\mathbf{0}_n$ the vector of zeros in $\mathbb{R}^n$. Denoting $T := \sum_{i \in N} \Phi_i(\mathbf{w})$, and $\mathbf{1}_n$ the unit vector in $\mathbb{R}^n$,[14] observe also that, since the set $\Phi(\mathcal{W})$ lies on the hyperplane $H = \{\mathbf{x} \in \mathbf{R}^n, \sum_i x_i = \sum_i v_i + \phi e(N) = T\}$, which includes the diagonal vector $\frac{T}{n}\mathbf{1}_n$, it does not matter which vector one chooses in the projection provided it is a scaling of $\mathbf{1}_n$ (i.e. it lies on the diagonal). In particular, $\mathbf{q}^*$ is equivalent to the projection of $\frac{T}{n}\mathbf{1}_n$ onto the convex set $\Phi(\mathcal{W})$, with $\mathbf{q}^* = \frac{T}{n}\mathbf{1}_n$ when $\frac{T}{n}\mathbf{1}_n \in \Phi(\mathcal{W})$.[15]

The following example illustrates the unique projection $\mathbf{q}^*$ for the dyad network.

Example 1. *For dyad with agents 1 and 2, $\mathcal{W} = \{w, 1-w : w \in [0,1]\}$, where $w_{12} = w$ and $w_{21} = 1 - w$, and $\Phi(\mathcal{W}) = \{v_1 + \phi w, v_2 + \phi(1-w)\}$. Figure 1 depicts three cases:* (a) $v_1 - v_2 < -\phi$, (b) $\phi \geq v_1 - v_2 \geq -\phi$, *and* (c) $v_1 - v_2 > \phi$.

When the value gap $|v_1 - v_2| > \phi$ in cases (a) *and* (c), *the projection $\mathbf{q}^*$ obtains a corner of $\Phi(\mathcal{W})$. Precisely, $q_1^* < q_2^*$ and $w = 1$ in case* (a), *and $q_1^* > q_2^*$ and $w = 0$ in case* (c). *In case* (b), *$\Phi(\mathcal{W})$ intersects the diagonal, and thus $q_1^* = q_2^*$, with $w \in (0,1)$ when $\phi > v_1 - v_2 > -\phi$.*

[13] See Chapter 1 of Nagurney (1992) for characterization and properties of this projection operator.

[14] Clearly, for any $\mathbf{w} \in \mathcal{W}$, $\sum_{i \in N} \Phi_i(\mathbf{w}) = \sum_{i \in N} v_i + \phi e(N)$.

[15] The mapping $\Phi(\cdot)$ may not be injective. As the dimension of $\mathcal{W}$ is $e(N)$, the image always lies on the hyperplane H, so the dimension of the image is at most $n - 1$. Thus, the inverse image $\mathbf{w}^* = \Phi^{-1}(\mathbf{q}^*)$ need not be unique.

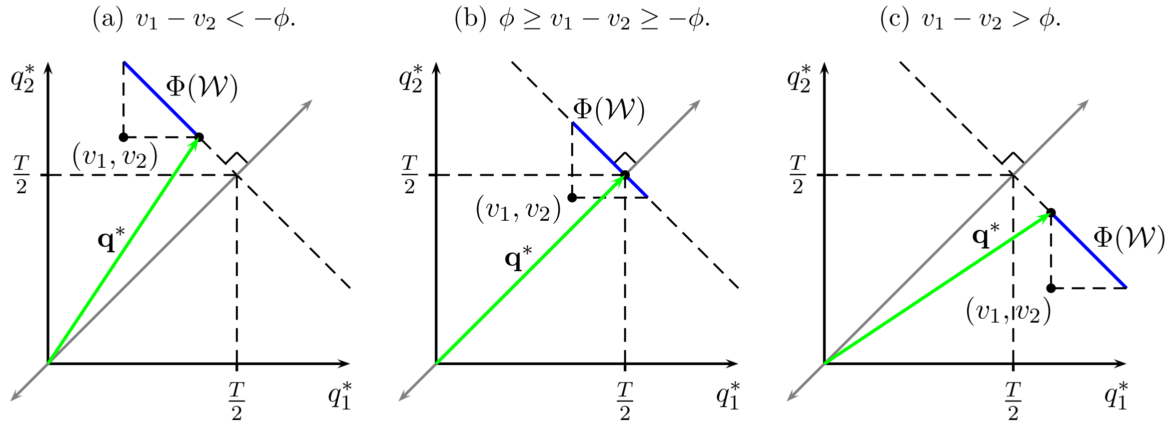


Figure 1: The vector $\mathbf{q}^*$ (green arrow) as the projection of the diagonal (gray arrow) onto $\Phi(\mathcal{W})$ (blue line segment) for the dyad network.

Example 1 shows that $q_1^* = q_2^*$, and thus $\theta_1^* = \theta_2^*$, for a range of value gaps $|v_1 - v_2| \leq \phi$. Provided a sufficient extent of symmetry holds in the limit game $G(0)$, the two agents will take a common cutoff, adopting exactly when the other adopts. The following notion of a *coordination set* will generalize such behavior to general networks.

For agent subset $S \subseteq N$, denote E_S the subset of edges in E corresponding with the subgraph $\mathcal{G}_S := (\hat{N}, E_S)$ of $\mathcal{G}$ restricted to vertices S.[16] The limit equilibrium $\vec{\boldsymbol{\pi}}$ must then define a unique ordered partition $\mathcal{C}^* := (C_1^*, \ldots C_{\bar{m}^*}^*)$ of N (i.e. $\cup_m C_m^* = N$ and $C_m^* \cap C_{m'}^* = \emptyset$ for $m \neq m'$) with the following properties:

Definition 2 (limit partition)**.** *The limit equilibrium $\vec{\boldsymbol{\pi}}$ maps to a unique ordered partition $\mathcal{C}^* := (C_1^*, \ldots C_{\bar{m}^*}^*)$ of N satisfying:*

1. *For each m, $C_m^* \mapsto \theta_m^* \in \Theta$ with $\theta_i^* = \theta_j^* = \theta_m^*$ for each $i, j \in C_m^*$, and $\theta_m^* \leq \theta_{m'}^*$ for each $m < m'$.*
2. *For each m, $\mathcal{G}_{C_m^*}$ is connected.*
3. *For each $m \neq m'$ such that $\theta_m^* = \theta_{m'}^*$, $E_{C_m^* \cup C_{m'}^*} = E_{C_m^*} \cup E_{C_{m'}^*}$.*

Each C_m^* defines a *coordination set* of agents. By condition 1, each agent within a coordination set shares the same cutoff, which we are free to order when defining $\mathcal{C}^*$.[17] By condition 2, these agents are connected via paths within their coordination set. And by condition 3, coordination sets sharing the same cutoff are disconnected. Importantly, the grouping of agents according to Definition 2 is without loss of generality, as the

[16] Precisely, $ij \in E_S$ if and only if $i, j \in S$ and $ij \in E$.

[17] Condition 1 of Definition 2 pins only a partial ordering of $\{C_1, \ldots, C_{\bar{m}*}\}$, and thus there can be multiple orderings satisfying the conditions.

exhaustive partition (i.e. $\mathcal{C} = \{\{i\}; i \in N\}$) satisfies conditions 1, 2 and 3, characterizing (non)coordination when all agents take distinct cutoffs.

Also illustrated with Example 1, when value gap $|v_1 - v_2| > \phi$ agent i, taking higher limit cutoff places limiting likelihood $w^*_{ij} = 1$ on $j \neq i$ adopting when i observes $\theta = \theta^*_i > \theta^*_j$. Conversely, j places likelihood $w^*_{ji} = 0$ on i adopting when j observes $\theta = \theta^*_j$. We extend these insights to arbitrary $\mathcal{G}$ with the next proposition, which provides a calculation of $\boldsymbol{\theta}^*$ solely in terms of counting degrees amongst members of a coordination set. Let $d_i(S) := |N_i \cap S|$ the *within-degree* of i in set S. For any agent sets S and S', $S \cap S' = \emptyset$, we denote:

$$e(S, S') = \sum_{i \in S} d_i(S'),$$

the number of edges from S to S'. And for any S, we denote:

$$e(S) = \frac{1}{2} \sum_{i \in S} d_i(S),$$

the number of edges between members of S, and $v(S) := \sum_{i \in S} v_i$. Finally, for each $C^*_m \in \mathcal{C}^*$ we denote the set of agents $\underline{C}^*_m := \cup_{m' < m} C^*_{m'}$.

Proposition 2 (Coordination sets' limit cutoffs)**.** *For network $\mathcal{G}$ with limit partition $\mathcal{C}^*$ and $C^*_m \in \mathcal{C}^*$, in the limit $\nu \to 0$, θ^*_m is given by:*[18]

$$\theta^*_m = \sigma^{-1}\left(-\frac{v(C^*_m) + \phi(e(C^*_m, \underline{C}^*_m) + e(C^*_m))}{|C^*_m|}\right). \tag{8}$$

Equivalently, $q^*_m := -\sigma(\theta^*_m)$ gives exactly the average (across $i \in C^*_m$) of v_i, plus ϕ times the average number of links to agents taking strictly lower cutoffs, plus ϕ times one-half the average within-degree $d_i(C^*_m)$. When convenient, $m(i)$ will denote i's coordination set: $i \in C^*_{m(i)}$.

Strikingly, Proposition 2 shows that, while $\mathcal{G}$ plays a key rule in determining the limit partition $\mathcal{C}^*$, upon conditioning on $\mathcal{C}^*$ the network structure within coordination sets plays no role in determining limiting cutoffs. Precisely, given $v(C^*_m)$, $e(C^*_m, \underline{C}^*_m)$, and $e(C^*_m)$, moving the position of links within C^*_m carries no impact on θ^*_m. In other words, while the structure of $\mathcal{G}$ plays a global role determining who coordinates with whom, its role is muted at the local level.

[18] Note that $e(C^*_m, \underline{C}^*_m)$ is independent across the admissible orderings in $\mathcal{C}^*$ (see footnote 17).

5 Characterizations under Homogeneous Valuations

Throughout this section, we assume that $v_i = v$ for each i; by imposing such homogeneity, the structure of $\mathcal{G}$ solely determines the limiting coordination amongst agents.

As suggested by Proposition 2, the limit partition defines an essential instrument for describing the properties of $\vec{\boldsymbol{\pi}}$. The first result of the section, which is a corollary of Theorem 1, establishes two basic properties of this partition under homogeneous valuations. Denote $\mathbf{q}_0^*$ to give the $\mathbf{q}^*$ at $\mathbf{v} = \mathbf{0}_n$ and $\phi = 1$.

Corollary 1 (Limit partition homogeneity)**.** *Under homogeneous valuations,* $\mathcal{C}^*$ *is independent of* v *and of* ϕ*. Moreover,* $\mathbf{q}^* = v\mathbf{1}_n + \phi\mathbf{q}_0^*$.

Scaling the size of valuations or network effects has no effects on the limit partition. Moreover, $\mathbf{q}^*$ is linearly augmented by the size of values v and of network effects ϕ.

We can better intuit Proposition 2 and (8) by considering cases when C_m^* exhibits *within set symmetry.* Consider $C_m^* \in \mathcal{C}^*$ such that all $i, j \in C_m^*$ satisfy $d_i(\underline{C}_m^*) + d_i(C_m^*)/2 = d_j(\underline{C}_m^*) + d_j(C_m^*)/2 =: b_m$. As formally shown by (A3) of Lemma 2 used in the proof of Theorem 1, this implies that i and j face the same expected network effect near and in the noiseless limit. Any regular network of degree $d := d_i = d_i(N)$ for each $i \in N$ is clearly within set symmetric, where $\mathcal{C}^* = \{N\}$ and the weighted degree b_1 reduces to $d/2$. When within-set symmetry obtains, θ_m^* takes the following form.

Corollary 2 (Limit cutoffs: within-set symmetry)**.** *If* $C_m^* \in \mathcal{C}^*$ *is within-set symmetric, where:*

$$d_i(\underline{C}_m^*) + \frac{1}{2}d_i(C_m^*) =: b_m, \quad \forall i \in C_m^*,$$

then in the limit $\nu \to 0$, θ_m^* *is given by:*

$$\theta_m^* = \sigma^{-1}(-v - \phi b_m). \tag{9}$$

Near the noiseless limit, expected network effects among each member $i \in C_m^*$, conditional on signal s_i realizing value s_i^*, scale by the agent's *weighted degree*, defined as $d_i(\underline{C}_m^*) + d_i(C_m^*)/2$. With agents in C_m^* coordinating together on a common cutoff in equilibrium, a signal realization of $s_i = s^*$ leaves i placing a fifty-fifty gamble on $a_j = 1$, $j \in C_m^*$, depending on the event $\epsilon_j \geq 0$. This scaling persists in the noiseless limit, as captured by $w_i^* = 1/2$. Thus, the weighted degree b_m in (8) captures the certainty placed on $\underline{C}_m^*$ adopting as well as the uncertainty of each $j \in C_m^*$ adopting, when i realizes her equilibrium cutoff. In the sequel, we will use θ_d^{r*} to denote the limit cutoff of a regular network of degree d. That is, $\theta_d^{r*} := \sigma^{-1}(-v - \phi d/2)$.

We collect the following intuition from Proposition 2 and Corollary 2. From (8), we see that the value to each i from adopting around some limit-cutoff θ_m^* is increasing in the number of neighbors currently adopting. As intuition would suggest, changes to the strategies of agents coordinating on more adoption will in general not influence the cutoff strategies of those coordinating on less adoption. Moreover from (9), the value to each i also increases with the number of neighbors that coordinate around a common limit cutoff. And lastly, the number of neighbors currently not adopting remains absent from (8) and (9), and thus does not influence θ_m^*.

The next proposition describes how orientation within the network influences coordination. The included necessary and sufficient condition informs when a nonempty, connected agent set $S \subseteq N$ coordinates on a common limit cutoff, holding fixed the actions of agents outside of S. For this, consider partition $\{N_0, N_1, S\}$ of N such that $\mathcal{G}_S$ is connected. Denote $\mathcal{C}_S^*$ the limit partition of S in the game constrained by $a_i = c$ for all $i \in N \backslash S$, $c = 0, 1$.

Proposition 3. $\mathcal{C}_S^* = \{S\}$ *if and only if for each nonempty* $S' \subseteq S$*:*

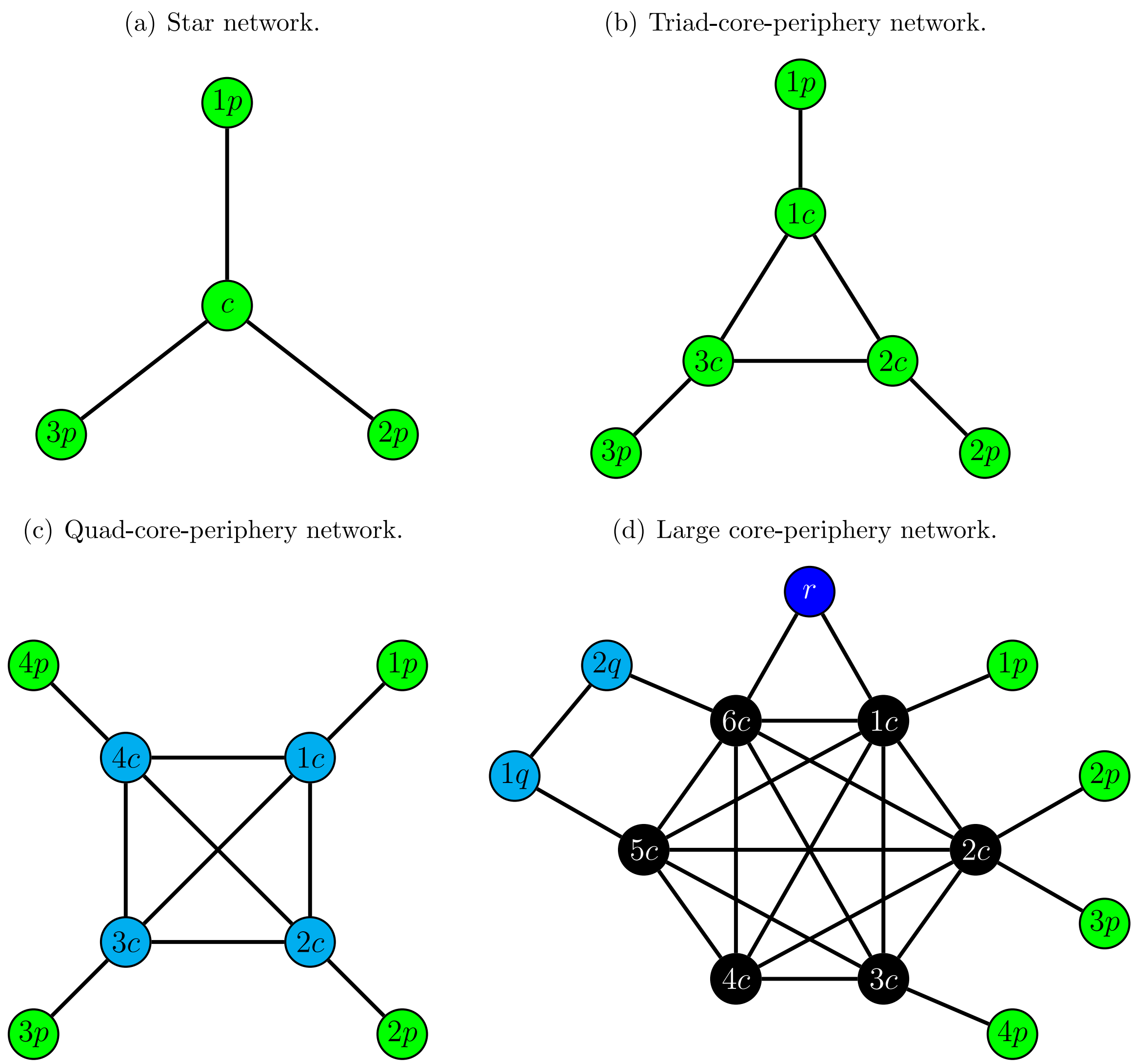
$$\frac{e(S', N_1) + e(S')}{|S'|} \leq \frac{e(S, N_1) + e(S)}{|S|}. \tag{10}$$

The inequalities (10) considers orientations of S', which position these agents in a distinct coordination set to the remaining agents in S. In particular, the left-hand side of (10) gives the average weighted network effect within S' when S' takes a lower cutoff to $S \backslash S'$. When no such S' can achieve a greater expected network effect than the parent set S, then all of S must inhabit one coordination set.

While agent sets N_0 and N_1 are indeed passive, their inclusion provides a partial characterization of limit partition $\mathcal{C}^*$, by counting and comparing degrees within the network $\mathcal{G}$. The following example applies Propositions 2 and 3 to the star and simple core-periphery networks.

Example 2. *Figure 2 gives the star and three core-periphery networks of differing core sizes. In each case, we apply either Proposition 2 or 3, focusing on agent sets which are symmetric over their respective included agents. For these cases,* $e(S, S')$ *reduces to* $d_i(S')$ *and* $e(S)$ *to* $d_i(S)/2$*, for any* $i \in S$*. We normalize* $v = 0$ *and* $\phi = 1$.[19]

[19]Given independence of $\mathcal{C}^*$ and $\mathbf{q}^*$ affine in $v\mathbf{1}_n$ and ϕ by Corollary 1, this normalization is without



(a) Star network.

(b) Triad-core-periphery network.

(c) Quad-core-periphery network.

(d) Large core-periphery network.

Figure 2: Coordination and network structure.

For the star, if multiple coordination sets were to exist, the most natural case is for the center to take a strictly lower cutoff to the periphery. Defining agent sets $S' = \{c\}$ *and* $S = N$ *(and,* $N_0 = N_1 = \emptyset$*), we see that* (10) *is satisfied, with:*

$$\frac{e(\{c\})}{|\{c\}|} = \frac{1}{2} d_c(\emptyset) = 0 < \frac{0+3}{4} = \frac{e(N, \emptyset) + e(N)}{|N|},$$

which implies the center can not have a strictly lower limit cutoff to the periphery. Upon validating (10) *for all other* S, S'*, it is shown that all members of the star coordinates on*

significant loss of generality.

a common cutoff.[20] *Note that the analogous inequalities to the above hold for arbitrary number of peripheral agents, implying that all agents of star networks coordinate together.*

For the triad-core-periphery network depicted, set $S' = \{1p, 2p, 3p\}$ *and* $S = N$ *(again,* $N_0 = N_1 = \emptyset$*).* (10) *is now weakly satisfied:*

$$\frac{e(\{1p, 2p, 3p\})}{|\{1p, 2p, 3p\}|} = \frac{3}{3} = \frac{1}{2}d_c(\emptyset) = \frac{2}{2} \leq \frac{0+6}{6} = \frac{e(N, \emptyset) + e(N)}{|N|}.$$

Once the size of the core exceeds three, as with the quad-core-periphery network, the expected network effects within the core suffice for it to break away from the periphery. Applying Proposition 2, we see:

$$q^*_{ic} = d_{ic}(\emptyset) + \frac{1}{2}d_{ic}(C^*_m) = 0 + \frac{3}{2} > (0+1) + \frac{0}{2} = d_{jp}(\emptyset \cup C^*_m) + \frac{1}{2}d_{jp}(C^*_{m'}) = q^*_{jp},$$

for each $ic \in C^*_m = \{1c, \ldots, 4c\}$ *and* $jp \in C^*_{m'} = \{1p, \ldots, 4p\}$*, and thus* $m = 1$ *and* $m' = 2$ *with* $\theta^*_m < \theta^*_{m'}$*. For the Large core-periphery network, taking sets* $C^*_m = \{1c, \ldots, 6c\}$ *and* $C^*_{m'} := \{1q, 2q\}$ *and applying Proposition 2, we have:*

$$q^*_{ip} = d_{ip}(\emptyset) + \frac{1}{2}d_{ip}(C^*_m) = 0 + \frac{5}{2} > (0+1) + \frac{1}{2} = d_{jq}(\emptyset \cup C^*_m) + \frac{1}{2}d_{jq}(C^*_{m'}) = q^*_{jq},$$

for each $ip \in C^*_m$ *and* $jq \in C^*_{m'}$*, and thus the core and* $\{1q, 2q\}$ *coordinate away from each other in the noiseless limit. Likewise, setting* $C^*_{m'} := \{r\}$*:*

$$q^*_{ip} = d_{ip}(\emptyset) + \frac{1}{2}d_{ip}(C^*_m) = 0 + \frac{5}{2} > (0+2) + \frac{0}{2} = d_r(\emptyset \cup C^*_m) + \frac{1}{2}d_r(C^*_{m'}) = q^*_r,$$

and thus the core and $\{r\}$ *also coordinate away from each other in the noiseless limit. Note that the number of peripheral cliques connecting to the core, which take the local structures depicted in Figure 2, is inconsequential to the equilibrium cutoff of the core.*

To illustrate these results, we provide $\mathbf{q}^*$ *from* (7) *of Theorem 1, for each network structure. We obtain the same coordination derived above. In the star,* $w^*_{ic} = 3/4$ *and* $w^*_{ci} = 1/4$ *for each peripheral node* i*. In the other three networks,* $w^*_{ic} = 1$ *and* $w^*_{ci} = 0$ *for any peripheral agent* i *and core agent* c *(see Figure 3).*

By setting $S = N$, and $N_0 = N_1 = \emptyset$ in Proposition 3, we obtain a necessary and sufficient condition for a single coordination set in the network.

[20] Within-set symmetry clearly obtains for all peripheral cliques and core cliques, in each network structure of this example. For such cases, condition (10) can be shown to reduce to two inequalities, which compares the orientations of the two within-set symmetric sets.

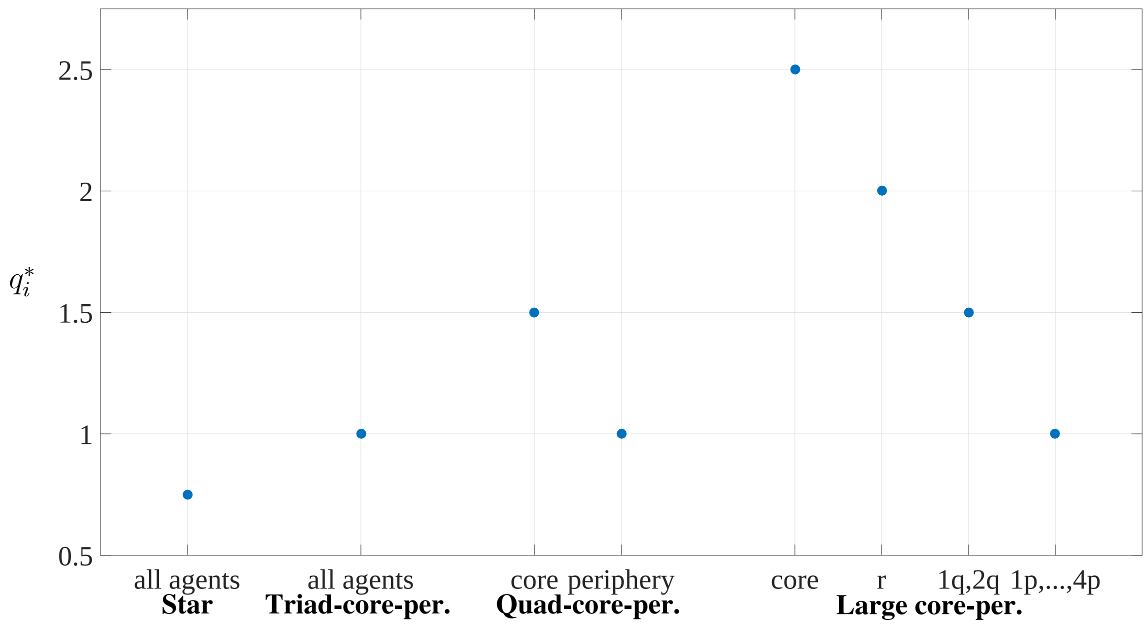


Figure 3: **Coordination and network structure**. Limit weighted-network-effects.

Proposition 4 (Single coordination set)**.** *Under homogeneous valuations, a single coordination set exists (i.e. $\mathcal{C}^* = \{C_1\}$) if and only if it is* balanced*, in the sense that for every nonempty $S \subset N$,*

$$\frac{e(S)}{|S|} \leq \frac{e(N)}{|N|}. \tag{11}$$

When this condition is satisfied, the common cutoff in the network is $\theta_1^* = \sigma^{-1}(-v - \phi\frac{e(N)}{|N|})$.

Condition (11) says that the average degree of each subnetwork $\mathcal{G}_S$ is no greater than the average degree of the original network $\mathcal{G}$. Equivalently:

$$\frac{\sum_{i,j \in S} g_{ij}}{|S|} \leq \frac{\sum_{i,j \in N} g_{ij}}{|N|}, \quad \forall\, \emptyset \neq S \subset N$$

Returning to and extending beyond Example 2, consider any core-periphery structure with regular core of degree d_c and size n_c, and with n_p periphery nodes, each connected to k core nodes symmetrically. This graph is balanced if and only if $d_c \leq 2k$. Either the core is not very connected, or the number of links to the core is very large. Otherwise the periphery node will have a strictly higher cutoff than the core nodes.

We apply Proposition 4 to show a unique coordination set for the following families of network structures. A tree is any connected network without cycles. We say network

$\mathcal{G}$ is a *regular-bipartite network* with disjoint within-set symmetric agent sets B_1 and B_2, with $B_1 \cup B_2 = N$ and of sizes $n_s := |B_s|$ and degrees $d_s := d_i,\ i \in B_s$, for sides $s = 1, 2$. Note that regular-bipartite networks satisfy $e(N) = n_1 d_1 = n_2 d_2$.

Proposition 5 (Single coordination set: examples)**.** *Under homogeneous valuations, there exists a single coordination set if $\mathcal{G}$:*

1. *is a tree network,*

2. *is a regular-bipartite network,*

3. *has a unique cycle,*

4. *has at most four nodes.*

Proposition 5 exhibits the striking extent to which global coordination may obtain. Members of all trees, regardless of their size and complexity, adopt using a common limit cutoff. Parts 1 and 3 establish the existence of at least two distinct cycles in $\mathcal{G}$ as a necessary condition for multiple limit cutoffs to obtain in equilibrium. This establishes trees as the family of network structures exhibiting the highest limit cutoffs. The triad-core-periphery network of Example 2 provides an example of a network with one cycle, illustrating Proposition 5, part 3. Still, regular-bipartite networks (and regular networks) may carry arbitrary numbers of cycles, yet all of these structures yield a unique coordination set.

Proposition 2 provides an exact calculation of each θ^*_m as a function of average degrees across all members of C^*_m. The following result provides bounds on limit cutoffs using only the minimal degree within a given agent set. Again, θ^{r*}_d denotes the limit cutoff of a regular network of degree d.

Proposition 6 (bounding limit cutoffs)**.**

1. *For each agent set $S \subseteq N$, $\max_{i \in S} \theta^*_i \leq \theta^{r*}_d$, setting $d = \min_{i \in S} d_i(S)$.*

2. *For each coordination set $C^*_m \in \mathcal{C}^*$, $\theta^*_m \geq \theta^{r*}_{2d}$, setting $d = \min_{i \in C^*_m} d_i(\underline{C}^*_m \cup C^*_m)$.*

To illustrate Proposition 6, we return to Example 2 under $v = 0$, $\phi = 1$, and expected network effects $b_m = d/2$ (see Proposition 2) to yield $q^{r*}_d := \sigma(\theta^{r*}_d) = d/2$. As observed in Figure 3, and consistent with part 1 of the proposition, the star and triad-core-periphery networks exhibit a common q^*_1 positioned weakly above those of the dyad and triad, $q^{r*}_1 = 0.5$ and $q^{r*}_2 = 1$, respectively. Likewise, the cores of the quad and large core-periphery networks exhibit q_1^* positioned weakly above $q_3^{r*} = 1.5$ and $q_5^{r*} = 2.5$, respectively. For part 2, the peripheral agents of the star and triad-core-periphery networks carry one link within their coordination sets. All members of these networks exhibit q_1^* at or below q_2^{r*}.

The following applies Propositions 2 and 6 to tree and regular-bipartite networks.

Remark 1 (Bounding limit cutoffs: trees and regular-bipartite networks)**.**

1. *For any tree network,* $\theta_1^{r*} \geq \theta_1^* = -\sigma^{-1}(v + \phi\frac{|N|-1}{|N|}) \geq \theta_2^{r*}$.

2. *For any regular-bipartite network,* $\theta_{\min\{d_1,d_2\}}^{r*} \geq \theta_1^* = -\sigma^{-1}(v+\phi\frac{e(N)}{n_1+n_2}) \geq \theta_{2\min\{d_1,d_2\}}^{r*}$.

We see that the limit cutoffs of the dyad and triad bound any tree's limit cutoff from above and below. The common limit cutoff of any regular-bipartite network can also be bounded, both above and below, now by the degree of the network's less-connected side.

The above results take non-singleton coordination sets as cases of interest. The next result shows that under homogeneous valuations, $\mathcal{C}^*$ must always exhibit such coordination.

Proposition 7. *For homogeneous valuations and any $\mathcal{G}$, there exists at least one coordination set with size at least 2. In particular, it is impossible to have n distinct cutoffs.*

The final results of this section establish our first comparative static, which is with respect to the network structure $\mathcal{G}$. Consider network $\mathcal{G}_{+ij}$, defined as the supergraph of $\mathcal{G}$ which includes the additional link ij, and $\mathcal{C}_{+ij}^*$ the limit partition under $\mathcal{G}_{+ij}$. While adding links can affect the limit partition, Proposition 3 can be employed to verify when the limiting coordination is left unchanged: for $\mathcal{C}_{+ij}^* = \mathcal{C}^*$. For these cases, Proposition 8 establishes a disparity in the effects of included links on equilibrium cutoffs. While additional links unambiguously encourage adoption amongst agents taking higher cutoffs, the equilibrium adoption of the agent taking a lower cutoff may not be influenced by the additional link. For the following, and in the sequel, we focus on changes to $\mathbf{q}^*$, again noting the one-to-one correspondence with $\boldsymbol{\theta}^*$ via (6). Let $q_{m,+ij}^*$ correspond to coordination set C_m^* under network $\mathcal{G}_{+ij}$.

Proposition 8 (linkage: limit cutoffs)**.** *Take i, j with $m(i) \geq m(j)$, $ij \notin E$, such that $\mathcal{C}_{+ij}^* = \mathcal{C}^*$. If:*

1. $\theta_{m(i)}^* > \theta_{m(j)}^*$, *then:*

$$q_{m(i),+ij}^* - q_{m(i)}^* = \phi\frac{1}{|C_{m(i)}^*|}, \qquad \textit{and} \qquad q_{m(j),+ij}^* - q_{m(j)}^* = 0;$$

2. $m(i) = m(j) =: m$*, then:*

$$q^*_{m,+ij} - q^*_m = \phi \frac{1}{|C^*_{m(i)}|}.$$

The inclusion of links between members of distinct coordination sets will expand adoption outcomes within the coordination set taking higher cutoff, but carry zero influence on adoption within the coordination set taking lower cutoff. While the inclusion of links between members of the same coordination set directly influences the two members' incentives to adopt, the expansion in adoption outcomes within the coordination set is comparable to that resulting from a single link to an agent taking a lower cutoff.

Example 3. *Consider network structures of the form depicted in Figure 4, under the symmetric conditions* $v_i = v$ *for each* $i \in N$*. Agents 1 through 5 and 7 through 10 form cliques, with agent 6 bridging the two cliques with varying connectivity to each clique. We denote* ℓ_1 *the number of links that 6 has with agents in* $\{1, \ldots, 5\}$*, and* ℓ_2 *the number of links that 6 has with agents in* $\{7, \ldots, 10\}$*. Table 1 summarizes the equilibrium coordination sets, and provides* $\mathbf{q}^*$ *from Theorem 1 for various values of* (ℓ_1, ℓ_2)*, setting* $v = 0$ *and* $\phi = 1$*.*

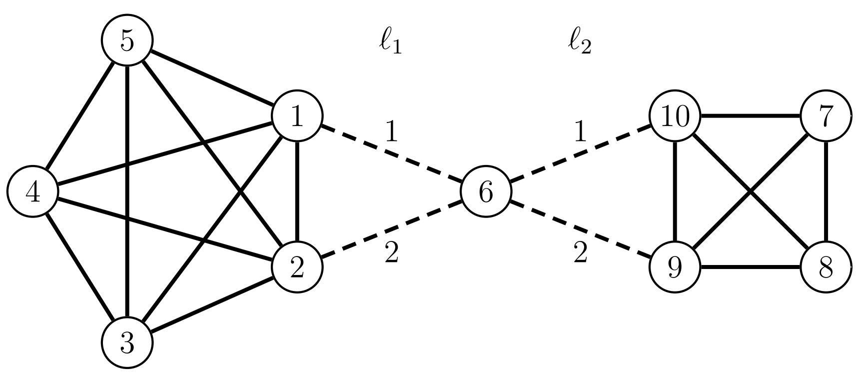


Figure 4: Coordination and bridging.

(ℓ_1, ℓ_2)	$\mathcal{C}^*$	q^*
$(0,0)$		$(2, 1.5, 0)$
$(0,1)$	$\{\{1,\ldots,5\},\{7,\ldots,10\},\{6\}\}$	$(2, 1.5, 1)$
$(1,0)$		$(2, 1.5, 1)$
$(1,1)$		$(2, 1.6)$
$(0,2)$	$\{\{1,\ldots,5\},\{6,\ldots,10\}\}$	$(2, 1.6)$
$(1,2)$		$(2, 1.8)$
$(2,0)$	$\{\{1,\ldots,6\},\{7,\ldots,10\}\}$	$(2, 1.5)$
$(2,1)$		$(2, 1.75)$
$(2,2)$	$\{N\}$	(2)

Table 1: Coordination sets $\mathcal{C}^*$ and $\mathbf{q}^*$ for agent 6 linkage.

As agent 6 forms two links with each of the two cliques, all of the agents coordinate together on a common cutoff in the noiseless limit. While the total number of links that 6 carries with each clique lies strictly below that of the members of each respective clique, 6 functions as a coordination bridge, synchronizing adoption strategies through the economy. When the number of links to either clique drops below two, 6 either coordinates with one of the two cliques, or coordinates with neither when holding only one link. We see that forming one link with either clique increases q_6^* *by exactly* $1 = \phi/|\{6\}|$*, while having no impact on cutoffs of the clique, as predicted by Proposition 8 part 1.*[21] *When agent 6 holds one link with clique* $\{7,\ldots,10\}$ *and adds an additional link to the clique, we see an increase in* q_i^**,* $i = 6,\ldots,10$*, of* $0.2 = \phi/|\{6,\ldots,10\}|$*, that is from* 1.6 *to* 1.8*, as predicted by Proposition 8 part 2.*

6 Characterizations under Heterogeneous Valuations

In what follows, we allow for heterogeneous v_i. The results therefore apply under our general framework. First, there exists an analogous version of Proposition 3 for heterogeneous intrinsic values. Again, $v(S) := \sum_{i\in S} v_i$ for $S \subseteq N$.

Theorem 2. *Under heterogeneous valuations, the condition for any connected subset* $S \subset N$ *to coordinate together (i.e. for* $S = C_S^*$*), when disjoint agents sets* $N_1, N_0 \subseteq N \backslash S$

[21] Likewise, if 6 holds two links with clique $\{1,\ldots,5\}$ and adds a link to clique $\{7,\ldots,10\}$, we see an increase in q_i^*, $i = 7,\ldots,10$ of $0.25 = \phi/|\{7,\ldots,10\}|$, specifically from 1.5 to 1.75, as predicted by Proposition 8, part 1.

do and do-not adopt (respectively) with probability one, is:

$$\frac{v(S') + \phi(e(S', N_1) + e(S'))}{|S'|} \leq \frac{v(S) + \phi(e(S, N_1) + e(S))}{|S|}, \quad \forall\, \emptyset \neq S' \subset S.$$

When this condition is satisfied, the common cutoff is $\theta_1^* = \sigma^{-1}(-\frac{v(N)+\phi e(N)}{|N|})$. Under homogeneous valuations, Theorem 2 reduces to Proposition 3. Setting $S = N$ gives an analogous condition to Proposition 4 under heterogeneous valuations.

Moreover, Proposition 8 extends to heterogeneous valuations whenever additional links do not affect the limit partition.

Remark 2. *Under heterogeneous valuations, if $\mathcal{C}^*_{+ij} = \mathcal{C}^*$ for i, j with $m(i) \geq m(j)$, $ij \notin E$, then Proposition 8 obtains.*

The next result shows that as network effects strengthen with an increase ϕ, the range of intrinsic values that support coordination amongst agents expands. This characterizes a *stickiness* in coordination as a result of network effects. Maintaining the above assumptions for $\mathcal{G}$, take $\mathbf{v}$ such that all $i \in C^*_m \in \mathcal{C}^*$ coordinate on common θ^*_m cutoff in the limit equilibrium $\vec{\boldsymbol{\pi}}$. For each $i \in C^*_m$ denote:

$$\begin{aligned}
\hat{v}_i^* &:= \text{argmax}\{v_i : \theta_i^* = \theta_j^*, j \in C^*_m \backslash \{i\}; \mathbf{v}_{-i}\}, \\
\check{v}_i^* &:= \text{argmin}\{v_i : \theta_i^* = \theta_j^*, j \in C^*_m \backslash \{i\}; \mathbf{v}_{-i}\}.
\end{aligned}$$

That is, $[\check{v}_i^*, \hat{v}_i^*]$ gives the ranges to i's intrinsic values that support i and members of $C^*_m \backslash \{i\}$ (for at least one $j \in C^*_m \backslash \{i\}$) coordinating on the same limiting adoption cutoff.[22] When $C^*_m \backslash \{i\}$ coordinate on a common cutoff θ^*_m for $v_i \in (\hat{v}_i^*, \check{v}_i^*)$, then $\hat{v}_i^* = \text{argmax}\{v_i : \theta_i^* = \theta^*_m; \mathbf{v}_{-i}\}$ and $\check{v}_i^* := \text{argmin}\{v_i : \theta_i^* = \theta^*_m; \mathbf{v}_{-i}\}$.[23] We can bound $\hat{v}_i^* - \check{v}_i^*$ in the noiseless limit.

Proposition 9 (sticky coordination and network effects)**.** *Take $\mathbf{v}$ and $\boldsymbol{\sigma}$ yielding coordination set $C^*_m \in \mathcal{C}^*$ with $|C^*_m| > 1$. Then for each $i \in C^*_m$:*

$$\hat{v}_i^* - \check{v}_i^* \geq \phi d_i(C^*_m). \tag{12}$$

[22] Existence of $\hat{v}_i^*$ and $\check{v}_i^*$ follow from existence of their counterparts near the noiseless limit, which obtain by continuity of equilibrium cutoffs in all parameters for each $\nu > 0$.

[23] The star in Example 4 below satisfies this property. The property can be violated when i is a bridge between two cliques, and with $i = 6$ in Example 3.

When $\mathcal{C}^*$ is constant for $v_i \in (\hat{v}_i^*, \underset{\smile}{v}_i^*)$, then:

$$\hat{v}_i^* - \underset{\smile}{v}_i^* = \frac{|C_m^*|}{|C_m^*| - 1} \phi d_i(C_m^*). \tag{13}$$

Expression (12) with $\phi > 0$ establish that $\hat{v}_i^* - \underset{\smile}{v}_i^*$ is strictly positive. Moreover, coordination amongst agents in C_m^* becomes more robust as network effects grow, with the lower bound to $\hat{v}_i^* - \underset{\smile}{v}_i^*$ proportional to i's degree within C_m^*, $d_i(C_m^*)$. When $\mathcal{C}^*$ is constant for $v_i \in (\hat{v}_i^*, \underset{\smile}{v}_i^*)$, then $\hat{v}_i^* - \underset{\smile}{v}_i^*$ is linearly increasing in $d_i(C_m^*)$. As with the dyad in Example 1, regular networks of n agents with $v_j = v,\ \forall j \neq i$ give $\hat{v}_i^* - \underset{\smile}{v}_i^* = n\phi$.

To interpret (12) and $\phi d_i(C_m^*)$ as an underlining lower bound to $\hat{v}_i^* - \underset{\smile}{v}_i^*$, consider the analogues to $\hat{v}_i^*$ and $\hat{v}_i^*$ near the limit:

$$\begin{aligned} \hat{v}_i^*(\nu) &:= \operatorname{argmax}\{v_i : |\theta_i^* - \theta_j^*| < 2\nu, j \in C_m^* \backslash \{i\}; \mathbf{v}_{-i}\}, \\ \underset{\smile}{v}_i^*(\nu) &:= \operatorname{argmin}\{v_i : |\theta_i^* - \theta_j^*| < 2\nu, j \in C_m^* \backslash \{i\}; \mathbf{v}_{-i}\}, \end{aligned}$$

which obtain $\lim_{\nu \to 0} \hat{v}_i^*(\nu) = \hat{v}_i^*$ and $\lim_{\nu \to 0} \underset{\smile}{v}_i^*(\nu) = \underset{\smile}{v}_i^*$. When $v_i = \hat{v}_i^*(\nu)$ in the perturbed game $G(\nu)$, $s_i^* < s_j^*$ for each $j \in N_i \cap C_m^*$, and thus the likelihoods that i and j place on the other adopting –when realizing signals equal to their respective equilibrium cutoffs– equal zero and one, respectively. When $v_i = \underset{\smile}{v}_i^*(\nu)$, then $s_i^* > s_j^*$, and the likelihoods that i and j place on the other adopting –realizing signals equal to equilibrium cutoffs– revert to equal *one* and *zero*. The difference in $\hat{v}_i^*$ and $\underset{\smile}{v}_i^*$ compensates i's adoption, accounting for these extremal probability weightings placed on i's neighbors in C_m^* adopting.

We next show that changes in intrinsic values to one agent reverberate through that agent's entire coordination set, with each member adjusting their cutoffs in step.

Proposition 10 (local contagion: intrinsic values)**.** *In the limit, the mapping $\mathbf{q}^*(\mathbf{v})$ is piecewise linear, Lipschitz continuous, and monotone. Generically, $\frac{\partial \mathbf{q}^*}{\partial \mathbf{v}}$ exists. Generically, when $i, j \in C_m$ and $k \notin C_m$, then:*

$$\frac{\partial q_j^*}{\partial v_i} = \frac{1}{|C_m|}, \qquad \textit{and} \qquad \frac{\partial q_k^*}{\partial v_i} = 0. \tag{14}$$

A change in the intrinsic value of the technology to agent i has a local effect on the adoption strategies of agents that coordinate with i in C_m^*, but has zero influence on adoption strategies in other coordination sets. The intuition is straight forward: while v_i carries influence on cutoffs within agents in C_m^*, when signals $s_{i'} \approx s_{i'}^*$ are realized the members of C_m^* are either all adopting or all not adopting the technology, depending on

$m < m'$ or $m > m'$, respectively. Thus, marginal changes to v_i, and in turn s_i^*, carry zero repercussions to coordination within $C_{m'}^*$.

The fact that $\frac{\partial q_j^*}{\partial v_i} > 0$ from (14) implies that $\frac{\partial s_j^*}{\partial v_i} < 0$ near the limit, by equilibrium cutoffs $\mathbf{s}^*$ continuously differentiable in $\mathbf{v}$ and in ν. Moreover, we can show that the discontinuous drop-to-zero in contagion across coordination sets persists near the limit.

Remark 3. *Near the limit, for $k \notin C_{m(i)}$, $\frac{\partial s_k^*}{\partial v_i} = 0$ when $\nu < \bar{\nu}$ for some $\bar{\nu} > 0$.*

The proof of Remark 3 is provided in the Appendix.

The following example illustrates Proposition 10, both near and in the noiseless limit.

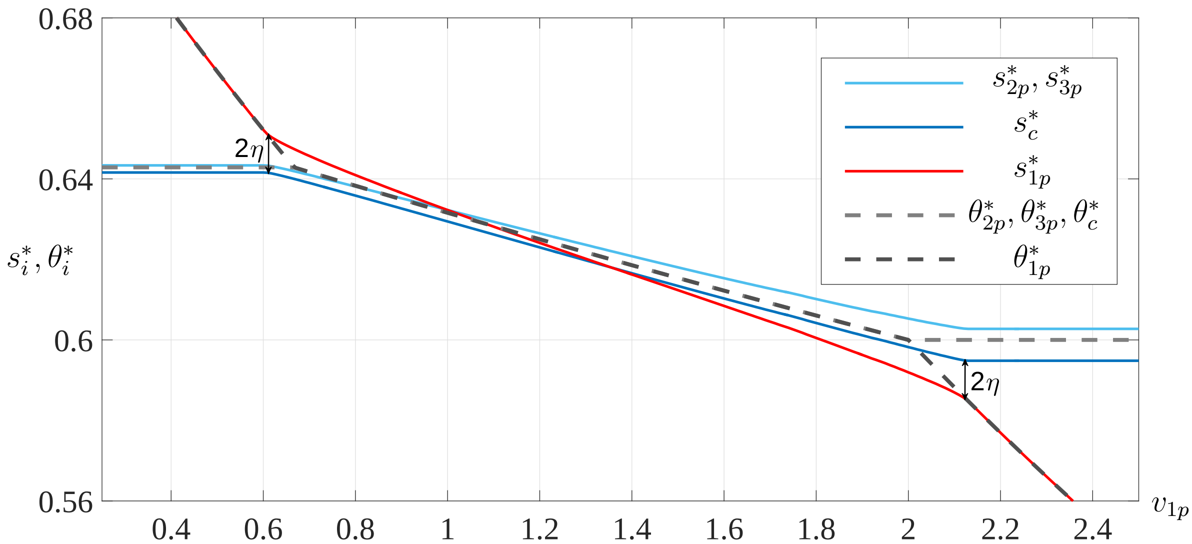


Figure 5: Intrinsic values and local contagion: equilibrium cutoffs, near the limit (solid lines) and in the limit (dashed lines), versus v_{1p} in the star network.

Example 4. *Take the star network with four nodes of Figure 2, Example 2. We take equivalent specifications, but set $v_i = 1$ for $i \neq 1p$, and vary the intrinsic value from adopting of the peripheral node 1, v_{1p}, over $[0.5, 2.5]$. We assume the following specification:*[24]

$$u_i(\mathbf{a}_{-i}|\theta) = v_i - 3\frac{1-\theta}{\theta} + \sum_{j \in N_i} a_j. \tag{15}$$

We consider uniform noise $F(\epsilon) = (\epsilon + 1)/2$, for $\nu = .005$. Figure 5 plots each agent's equilibrium adoption cutoff, both near and in the limit.[25] *For values of v_{1p} below $\underset{\sim}{v}_{1p}^* =$*

[24]Note that $\underline{\theta}, \overline{\theta} \in (0, 1)$ obtain for all $v_i, \phi > 0$.

[25]Equilibria near the limit were calculated via fixed-point method; see online Appendix 3.

0.67*, agent* $1p$ *lies outside of the coordination set* $\{c, 2p, 3p\}$. *As* v_{1p} *rises above* $\underset{\smile}{v}^*_{1p}$, *increases in* v_{1p} *spillover to the other agents' adoption strategies, with agents' adjustment rates inversely proportional to their distances from* $1p$. *When* v_{1p} *rises above* $\hat{v}^*_{1p} = 2$, $1p$ *takes a cutoff strictly below that of the remaining three agents. One can verify from Figure 5 that all agents coordinate together for each* $v_{1p} \in (\underset{\smile}{v}^*_{1p}, \hat{v}^*_{1p})$. *As predicted by Proposition 10,* $\hat{v}^*_i - \underset{\smile}{v}^*_i = \frac{4}{4-1}(0+1) = 1.33$.

The above results establish a stark segmentation across coordination sets. This segmentation, characterized by Proposition 8 (upon adding links) and Proposition 10 (upon adjusting intrinsic values), obtains both in and near the noiseless limit. The next section explores the welfare implications of equilibrium coordination on networks.

7 Welfare and Policy Implications

Proposition 9 reveals an increased robustness in coordination amongst agents to perturbations to intrinsic values, as network effects strengthen. Proposition 10 establishes a discontinuity in effects of such perturbations, with agents outside of the perturbed agent's coordination set remaining unresponsive in equilibrium. The questions remain to any planner: what marginal benefit is realized with adoption subsidies, and which agents' adoption should be subsidized?

To address these questions, below we focus our welfare analysis near the limit, noting that all marginal values are continuous and converge on finite limits as $\nu \to 0$; the limiting analogue to Proposition 11 below obtains. We first derive an expression for the vector of partials $\partial s^*_j / \partial v_i$ for any targeted agent i. Such comparative statics serve an essential ingredient to any optimal policy design targeting players' incentives for adoption. We define the following marginal values:

$$\mu^*_{ij} \quad := \quad \frac{\partial}{\partial s^\dagger_j} U_i(\boldsymbol{\pi}^*_{-i} | s^*_i), \tag{16}$$

We construct expressions for μ^*_{ii} and μ^*_{ij}, $i \neq j$, in the proof of Proposition 11.[26] Denote $\mathbf{M}^* := [\mu^*_{ij}]$ and define $\mathbf{1}_i$ the $|C| \times 1$ column-vectors of zeros with a 1 in row i. The implicit function theorem yields the fundamental equation of comparative statics for the system with respect to v_i:

$$\frac{\partial \mathbf{s}^*}{\partial v_i} = -M^{*-1} \mathbf{1}_i, \tag{17}$$

[26] As consistent with Proposition 10 and Remark 3, $\mu^*_{ij} = 0$ when $j \notin C^*_m$ and $\nu < \bar{\nu}$.

provided $\mathbf{M}^*$ is non-singular. As $\nu \to 0$, the vector (17) converges on their limiting comparative statics:

$$\frac{\partial \mathbf{s}^*}{\partial v_i} \to \frac{\partial \boldsymbol{\theta}^*}{\partial v_i}.$$

Now consider a policy designer with either of the following objectives. In either case, the designer places Pareto weight $\lambda_i \geq 0$ on each agent $i \in N$. First, a designer may aim to maximize the $\boldsymbol{\lambda}$-weighted aggregate adoption likelihood. Precisely, such a designer realizes a marginal increase to this likelihood from increasing v_i of:

$$ma_i^*(\boldsymbol{\lambda}) := \boldsymbol{\lambda}' \frac{\partial \mathbf{s}^*}{\partial v_i} = -\boldsymbol{\lambda}' M^{*-1} \mathbf{1}_i, \tag{18}$$

with $'$ denoting the transpose operator.[27]

Alternatively, a benevolent planner may aim to maximize the $\boldsymbol{\lambda}$-weighted ex-ante welfare amongst agents. To quantify such a designer's marginal gains to policy interventions, we define the following.

$$\bar{\mu}_{ij}^* := \frac{\partial}{\partial s_j^\dagger} \mathbb{E}_{s_i} \left[U_i(\boldsymbol{\pi}_{-i}^\dagger | s_i) \right] \Bigg|_{\boldsymbol{\pi}^\dagger = \boldsymbol{\pi}^*}. \tag{19}$$

Expressions for $\bar{\mu}_{ii}^*$ and $\bar{\mu}_{ij}^*$, $i \neq j$, are provided in the proof of Proposition 11. Denote $\bar{\mathbf{M}}^* := [\bar{\mu}_{ij}^*]$. Then, the benevolent planner realizes a marginal gain from increasing v_i of:

$$mw_i^*(\boldsymbol{\lambda}) := \boldsymbol{\lambda}' \bar{\mathbf{M}}^* \frac{\partial \mathbf{s}^*}{\partial v_i} = -\boldsymbol{\lambda}' \bar{\mathbf{M}}^* \mathbf{M}^{*-1} \mathbf{1}_i. \tag{20}$$

The following obtains.

Proposition 11 (policy impact)**.** *For each $0 < \nu < \bar{\nu}$, and each $i, j \in C_m^*$, and $k \notin C_m^*$:*

$$ma_i^*(\mathbf{1}_j) < 0, \qquad \textit{and} \qquad ma_i^*(\mathbf{1}_k) = 0. \tag{21}$$

For each $j \in C_{m^-}^*$ and $k \in C_{m^+}^*$ with $m^- \leq m < m^+$, and with $jl \in E$ for some $l \in C_m^*$ when $m^- < m$:

$$mw_i^*(\mathbf{1}_j) > 0, \qquad \textit{and} \qquad mw_i^*(\mathbf{1}_k) = 0. \tag{22}$$

That is, a subsidy to i's adoption increases adoption amongst members of i's coordination set, while carrying zero influence amongst members of other coordination sets. On the

[27] Normalizing $\sum_{i \in N} \lambda_i = 1$ allows for $ma^*(\cdot|\boldsymbol{\lambda})$ to be interpreted as an aggregate probability measure.

other hand, the subsidy generates positive welfare gains to each agent k that take adoption cutoff at or below s_i^* –provided these agents are either in or directly connected to k's coordination set– but generates no gains to agents taking cutoffs above s_i^*. With local contagion persisting in the noiseless limit by Proposition 10, (21) and (22) hold as $\nu \to 0$.

We see a separation between the objectives of a designer that aims to maximize adoption likelihoods with one that maximizes ex ante welfare. If the designer targets coordination sets that are most interconnected, agents in C_1^*, she excludes the potential for direct externalities to coordination sets C_m^*, $m > 1$, which have direct links to C_1^*. While agents taking strictly lower cutoffs (to that of the target agent) do not adjust their strategies in response to the policy intervention, there are ex-ante gains to these agents when the target (and those within the target's coordination set) adjust downward their adoption cutoffs. The benevolent planner will weigh-in these positive (expected) externalities when identifying an optimal target.

8 Extensions and Applications

8.1 Extensions and variations

The following extensions of the model are offered. The first two extensions establish that the unique equilibrium selection is broadly robust to the properties of the noise technology of the perturbed game. The subsequent extension and variation of the model, addressing welfare spillovers and miss-coordination costs (respectively), address the potential for additional/alternative externalities, either non-strategic (in the former) or strategic (in the latter).

Unbounded noise

The above model takes agents' noise supports to be contained within the bounded interval $[-\nu, \nu]$.[28] The positive and normative implications of the model maintain in the noiseless limit under unbounded noise. Consider, for example, the perturbed game where θ is observed with Gaussian noise by all agents: each i observes signal $s_i = \theta + \epsilon_i$, where each

[28]This assumption conveniently yields equilibrium properties near the noiseless limit which are commensurate with the properties of $\vec{\boldsymbol{\pi}}$. In particular, local contagion (Remark 3) and the reach of policy interventions (Proposition 11) extend but remain contained within coordination sets, provided ν is sufficiently small.

$\epsilon_i \sim N(0, \nu)$, $\nu > 0$, and all signals independently drawn conditional on θ. The program of Theorem 1 continues to describe the limit equilibrium $\vec{\boldsymbol{\pi}}$.[29] Therefore, all limiting characterizations, including those of sticky coordination, linkage, and local contagion, as well as the model's welfare properties are intrinsic to the equilibrium selected from the complete information game $G(0)$.

Noise-independent selection

The equilibrium selection in the noiseless limit is not sensitive to the commonality of the noise distribution F. Online Appendix 1 extends the model setup to establish noise-independent selection (see Frankel et al. (2003), Section 6).

Spillovers

We can incorporate a spillover function $w_i(\mathbf{a}_{-i}|\theta)$ to augment both $u_i(a_{-i}|\theta)$ and the payoffs to not adopting (now equal to $w_i(\mathbf{a}_{-i}|\theta)$ instead of zero). Under this extension, the equilibrium selected in the limit along with all of the positive results remain. The measure $mw_i^*(\boldsymbol{\lambda})$ will adjust accordingly to incorporate welfare spillovers, positively and negatively so when $w_i(\mathbf{a}_{-i}|\theta)$ is positive and negative, respectively.

Miscoordination costs

As an application of the model under heterogeneous values, we can set $v_i = v - \phi d_i$ to give:

$$u_i(\mathbf{a}_{-i}|\theta) = v + \sigma(\theta) - \phi \sum_{j \in N_i} (1 - a_j). \tag{23}$$

Such a setup may be construed as homogeneous values under miscoordination costs. In this setting, an inverted form of Corollary 2 obtains with more connected (within-symmetric) coordination sets taking *higher* cutoffs. In equilibrium, agents' links to coordination sets taking lower cutoffs carry zero weight, as these miscoordination costs are avoided with probability one. links to others within the coordination set are penalized with weights one-half. And, links to coordination sets taking higher cutoffs are penalized with weights one, with these costs being borne with probability one. Interestingly, despite this inversion, common coordination within the network families of Proposition 5 persists. Online Appendix 2 addresses this setup in more detail.

[29] An analogous proof to Lemma 2 can be constructed. Beyond this, the theorem's proof is identical.

8.2 Applications

Here we map either the basic model or its extensions to the three applications offered in the introduction: Platform adoption, crime, and immigration policy.

Platform and Cryptocurrency Adoption

The adoption of platforms, from currencies and online marketplaces to social media platforms, offer natural applications of our model, provided the value to users is increasing in the adoption by neighbors.[30] Take, for example, the adoption by firms to deal in a given cryptocurrency (e.g. Bitcoin).[31] The efficacy of the currency as a medium of exchange is increasing in its adoption by firms that take counterparty positions in business dealings (e.g. suppliers). Each firm i's idiosyncratic value to using the currency can be captured by $v_i + \sigma(\theta)$ (i.e. heterogeneous values), where θ captures the future stability or inflation of the currency. In addition to this value, i realizes a gain due to neighboring counterparty firms' adoption $\phi \sum_{j \in N_i} a_j$.

Now, consider a third-party payment services provider offering cryptocurrency-based services at price $p > 0$, leaving a net value to i of $v_i - p + \sigma(\theta) + \phi \sum_{j \in N_i} a_j$. The impact of a targeted subsidy by the provider, in the form of a decrease in the price charged to i or an increase to v_i via granting i access to exclusive features, can be measured using ma_i. Precisely, assuming subsidy Δp is provided for i's adoption, the direct increase in revenue to the provider is measured by $p - \Delta p$ in all signal outcomes in which i had not adopted but now does, and by $-\Delta p$ in all signal outcomes in which i adopts regardless of the subsidy. Taking $\mathbf{s}^*$ as the equilibrium cutoff profile near the noiseless limit without the subsidy, the total expected marginal revenue $mr_i(\theta)$ to the provider conditional on θ can then be approximated:

$$\begin{aligned} mr_i(\theta) \quad &\approx \quad \mathbb{E}\Big[\underbrace{p \sum_{j \in N} \chi(s_j \in (s_j^* - \Delta p \cdot ma_i(\mathbf{1}_j), s_j^*])}_{\text{revenue from additional purchases}} - \underbrace{\Delta p \chi(s_i > s_i^* - \Delta p \cdot ma_i(\mathbf{1}_i))}_{\text{subsidy cost}} \Big| \theta \Big] \\ &\xrightarrow{\nu \to 0} \quad p|C_m^*| \chi\left(\theta \in \left(\theta_m^* - \Delta p \frac{\partial \theta_m^*}{\partial v_i}, \theta_m^*\right]\right) - \Delta p \chi\left(\theta > \theta_m^* - \Delta p \frac{\partial \theta_m^*}{\partial v_i}\right); \quad i \in C_m^*, \end{aligned}$$

[30] For products such as software, mobile phones, video game consoles, communication apps and the like, these peer-effects are technological in nature: consumers need to adopt technologies compatible with those of their peers in order to have effective interactions. In particular, in recent observations from the marketing literature, network effects are especially pronounced in product categories with competing technological standards (see e.g., Van den Bulte and Stremersch (2004)).

[31] We thank Ben Golub for suggesting this application.

where χ denotes the indicator function.[32] Thus, the optimal subsidy targets a firm i precisely when θ is slightly below s_i^*, which converges on θ_i^* as $\nu \to 0$. This yields a certain increase in revenue equal to $p|C_m^*| - \Delta p$ as $\nu \to 0$: the limiting optimal subsidy targets the largest coordination set.

Per Proposition 11, $mw_i(\theta)$ further incorporates the ex ante gains that subsidized adoption brings to coordination sets that take (i) lower limit cutoffs and (i) are directly connected to the targeted coordination set. As such, the welfare-maximizing target need not inhabit the largest coordination set.

Crime

It is well-established that delinquency is, to some extent, a group phenomenon, and the source of crime and delinquency is located in the intimate social networks of individuals (see e.g. Sutherland (1947), Warr (2002), Bayer et al. (2009), Dustmann and Piil Damm (2014)). Indeed, delinquents often have friends who have themselves committed several offenses, and social ties among delinquents are seen as a means whereby individuals exert an influence over one another to commit crimes. There are few network models of crime (see e.g. Ballester et al. (2010)) and, to the best of our knowledge, none that combines both explicit network structure and imperfect information on the probability of being caught in a crime model. Let us show how our model captures these different aspects.

Consider a population of potential criminals. Allow $a_i = 1$ to designate agent i's choice to participate in crime. Criminal i's relative experience and criminal competence is increasing in the criminality of neighbors (peer effects). Crime comes with a payoff of $p > 0$. Help from or payments to neighboring criminals comes at cost $c > 0$.[33] We model the probability of getting away with crime by $\rho(\theta) + \tau \sum_{j \in N_i} a_j$ with ρ increasing and yielding values in $[0, 1 - \tau d_i]$. Greater $1 - \rho(\theta)$ (i.e. lower θ) corresponds with a greater presence of police or security. Denoting the cost of being caught by $\kappa > 0$, this gives the conditional payoff function:

$$\begin{aligned} u_i(\mathbf{a}_{-i}|\theta) &= \left(\rho(\theta) + \tau \sum_{j \in N_i} a_j\right) p - \left(1 - \rho(\theta) - \tau \sum_{j \in N_i} a_j\right) \kappa - c \sum_{j \in N_i} a_j \\ &= \underbrace{-\kappa}_{v} + \underbrace{\rho(\theta)(p + \kappa)}_{\sigma(\theta)} + \underbrace{(\tau(p + \kappa) - c)}_{\phi} \sum_{j \in N_i} a_j. \end{aligned}$$

[32] When the provider itself holds a noisy signal of θ, it takes a conditional expectation of $mr_i(\theta)$.

[33] Costs incurred independent of i's criminal activity can be captured by $w_i(\mathbf{a}_{-i}|\theta)$; see Section 8.1.

Provided $\tau(p + \kappa) > c$, the incentive to partake in crime is increasing in the criminal activity of neighbors. Viewed through the lens of the results of Section 5, sparsely connected networks such as trees will exhibit sudden shifts in activity across the community when θ drops below some threshold. For networks with highly interconnected pockets of the community, criminal activity will arise more often amongst these pockets.

Immigration Policy

More than a million migrants and refugees crossed into Europe in 2015, compared with just 280,000 the year before. Most of these migrants, who have fled the Middle East and Africa, pulled by the promise of a better life in Europe, were illegal. The reactions from the European countries were very different. Some countries such as Germany and Sweden were promising (at least in the beginning) to regularize them if they were coming from war-torn countries such as Syria while others, such as Poland and Hungary, were basically slamming doors at migrants and where committing themselves to never regularize them.

We can use our framework to model these different immigration policies by allowing $a_i = 0$ to designate the government of country i's choice to take an anti-immigration (i.e. "isolationist") stance. The relative value of taking an inclusive policy ($a_i = 1$), in the form of political support from electorates, is captured by $\sigma(\theta)$. θ may measure a perceived global need for pro-immigration policies, driven by perceptions of foreign conflict or severity of a refugee crisis. We model the inflow of immigrants into country i by $f + \tau \sum_{j \in N_i}(1 - a_j)$, with $\tau > 0$ capturing the overflow of migrants into neighboring country i when $j \in N_i$ takes an anti-immigration stance. The marginal cost to migrant flow is given by $c > 0$. This gives conditional payoff function:

$$\begin{aligned} u_i(\mathbf{a}_{-i}|\theta) &= \sigma(\theta) - c\left(f + \tau \sum_{j \in N_i}(1 - a_j)\right) \\ &= \underbrace{-cf}_{v} + \sigma(\theta) - \underbrace{c\tau}_{\phi} \sum_{j \in N_i}(1 - a_j). \end{aligned}$$

Thus, miscoordination costs obtain in this model (see Section 8.1). Here, countries in regions with many boarding neighbors are predicted to take anti-immigration stances in more states than countries that are geographically isolated. To avoid the different stances on immigration issues mentioned above, our model suggests that the European Union should have a *common* immigration policy so that all countries belonging to the union could coordinate on a common cut-off strategy. Such a common immigration policy avoids miscoordination costs from excessive migrant flows to pro-immigration countries.

9 Conclusion

This paper offers a first look into the properties of equilibrium selection in global games, within the context of a general network game of strategic complements with binary actions. This selection embodies equilibrium properties far removed from those exhibited in the network games literature. Our equilibrium selection implies that proximal agents similarly connected within the network perfectly coordinate actions over states of the world. The reach of the model's predictions, in particular those of sticky coordination and contained contagion, to applications such as technology, crime or public policy adoption, remains for the lens of empirical investigation.

It is also left for future work to study the effects of signaling (Angeletos at. al (2006)) or signal jamming (Edmond (2013)) on equilibrium properties such as limit uniqueness and coordination partitioning. Dahleh et al. (2012) study information exchange through a social network, under a symmetric global game; the implications of information transmission under a general network game remains an open question. Equilibrium characterizations under more extensive departures from idiosyncratic noise, such as the introduction of a public signal, also remains for future research.[34]

[34] See Weinstein and Yildiz (2007) and Morris et al. (2016) for contributions.

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Appendix

Proof of Lemma 1. We first show that each agent best responds in $G(\nu)$ to a profile of cutoff strategies via a unique cutoff strategy. With $\sigma(\theta)$ strictly and $r(\theta, s_j^\dagger; \nu)$ weakly increasing in θ, it is immediate that the integrand in (4) is strictly increasing in s_i. There must then be a unique signal $s_i^* \in (\underline{\theta} - \nu, \overline{\theta} + \nu)$ that solves:

$$U_i(\boldsymbol{\pi}^\dagger_{-i}|s_i^*) = 0, \tag{A1}$$

with adoption optimal for i if and only if $s_i \geq s_i^*$. By continuity of all payoffs in others' cutoffs, we can applying Brouwer's fixed point theorem giving the result. □

Proof of Theorem 1. We start with two lemmas.

Lemma 2. *For $\nu > 0$ and game $G(\nu)$, for any pair i, j with cutoffs $s_i^\dagger, s_j^\dagger$,*

$$\mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] + \mathbb{E}[\pi_i(s_i; s_i^\dagger)|s_j = s_j^\dagger] = 1 \tag{A2}$$

In particular, when $s_i^\dagger = s_j^\dagger = s^*$,

$$\mathbb{E}[\pi_j(s_j; s^*)|s_i = s^*] = \frac{1}{2} = \mathbb{E}[\pi_i(s_i; s^*)|s_j = s^*]. \tag{A3}$$

In the limit as $\nu \to 0$,

$$\lim_{\nu \to 0} \left\{ \mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] + \mathbb{E}[\pi_i(s_i; s_i^\dagger)|s_j = s_j^\dagger] \right\} = 1.$$

Moreover, if $\lim_{\nu\to 0} s_i^\dagger < \lim_{\nu \to 0} s_j^\dagger$, then

$$\lim_{\nu \to 0} \mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] = 0, \text{ and } \lim_{\nu \to 0} \mathbb{E}[\pi_i(s_i; s_i^\dagger)|s_j = s_j^\dagger] = 1. \tag{A4}$$

Proof. Given $s_i = s_i^\dagger$, the conditional distribution of θ is $s_i^\dagger - \nu\epsilon_i$, so:

$$\Pr(s_i^\dagger - v\epsilon_i \leq \theta) = 1 - F\left(\frac{s_i^\dagger - \theta}{v}\right)$$

Moreover, conditional on θ the distribution of s_j is $\theta + v\epsilon_j$, so:

$$\mathbb{E}[\pi_j(s_j; s_j^\dagger)|\theta] = \Pr(\theta + v\epsilon_j \geq s_j^\dagger) = 1 - F\left(\frac{s_j^\dagger - \theta}{\nu}\right)$$

Using the iterated law of expectation:

$$\mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] = \int_\theta \left\{1 - F\left(\frac{s_j^\dagger - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{s_i^\dagger - \theta}{\nu}\right)\right]$$

Similarly:

$$\mathbb{E}[\pi_i(s_i; s_i^\dagger)|s_j = s_j^\dagger] = \int_\theta \left\{1 - F\left(\frac{s_i^\dagger - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{s_j^\dagger - \theta}{\nu}\right)\right]$$

Taking sum and using the product rule:

$$\mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] + \mathbb{E}[\pi_i(s_i; s_i^\dagger)|s_j = s_j^\dagger] = \left\{\left[1 - F\left(\frac{s_i^\dagger - \theta}{\nu}\right)\right]\left[1 - F\left(\frac{s_j^\dagger - \theta}{\nu}\right)\right]\right\}_{\theta=-\infty}^{\theta=+\infty}$$

$$= (1-0)(1-0) - (1-1)(1-1) = 1$$

The limiting result (A3) follows, since (A2) holds for any cutoff and any v, it continues to hold in the limit as ν goes to zero.

To show (A4), recall that:

$$\mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] = \int_\theta \left\{1 - F\left(\frac{s_j^\dagger - \theta}{\nu}\right)\right\} d\left[1 - F\left(\frac{s_i^\dagger - \theta}{\nu}\right)\right]$$

We change variable by letting $z = s_i^\dagger - \theta$, then:

$$\mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] = \int_\theta \left\{1 - F\left(\frac{s_j^\dagger - s_i^\dagger}{\nu} - z\right)\right\} dF(z).$$

When $\lim_{\nu\to 0} s_i^\dagger < \lim_{\nu\to 0} s_j^\dagger$, for each fixed z:

$$\left\{1 - F\left(\frac{s_j^\dagger - s_i^\dagger}{\nu} - z\right)\right\} \to 0, \text{as } \nu \to 0.$$

So by Dominant Convergence Theorem:

$$\lim_{\nu \to 0} \mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger] = \int_\theta 0 dF(z) = 0.$$

Similarity we show: $\lim_{\nu \to 0} \mathbb{E}[\pi_i(s_i; s_i^\dagger)|s_j = s_j^\dagger] = 1$.

$\square$

Lemma 3. *The unique vector $\mathbf{q}^*$, the projection of $\mathbf{0}_n$ onto the $\Phi(\mathcal{W})$, is uniquely characterized by the following two conditions:*

(C1) $\mathbf{q}^* \in \Phi(\mathcal{W})$, i.e. there exists $\mathbf{w}^*$ such that $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*, \forall i$

(C2) for any edge $(i,j) \in E$ and for any $z_{ij} \in [0,1]$,

$$(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0.$$

Moreover, we can replace (C2) by the equivalent form:

(C2') $(i,j) \in E, (q_i^* - q_j^*) > 0 \implies w_{ij}^* = 0, w_{ji}^* = 1$.

Proof. We first show necessity. Clearly $(C1)$ is just the feasibility condition, hence necessary. For $(C2)$, for any $\mathbf{w}'$, by optimality of $\mathbf{q}^*$, the following must be true:

$$\eta(t) := ||\Phi((1-t)\mathbf{w}^* + t\mathbf{w}')||^2 \geq ||\Phi(\mathbf{w}^*)||^2 = ||\mathbf{q}^*||^2 = \eta(0)$$

for any $t \in [0,1]$.

Linearity of Φ implies $\frac{\partial}{\partial t}\Phi((1-t)\mathbf{w}^* + t\mathbf{w}') = \Phi(\mathbf{w}' - \mathbf{w}^*)$. Taking the derivative of $\eta(t)$ at $t = 0$, we obtain:

$$0 \leq \eta'(0) = \langle \mathbf{q}^*, \Phi(\mathbf{w}' - \mathbf{w}^*) \rangle. \tag{A5}$$

Now for any $z_{ij}' \in [0,1]$, we construct a special $\mathbf{w}'$ by only modifying the weights w_{ij}^* and $w_{ji}^* = 1 - w_{ij}^*$ on the edge between i and j in $\mathbf{w}^*$ to $w_{ij}' = z_{ij}$ and $w_{ji}' = 1 - z_{ij}$. Clearly, $\mathbf{w}'$ is still in $\mathcal{W}$. Inequality (A5) becomes:

$$\phi(q_i^*(z_{ij} - w_{ij}^*) + q_j^*(z_{ji} - w_{ji}^*)) \geq 0.$$

However, $z_{ji} - w_{ji}^* = (1 - z_{ij}) - (1 - w_{ij}^*) = -(z_{ij} - w_{ij}^*)$. So the above inequality is equivalent to:

$$(q_i^* - q_j^*)(z_{ij} - w_{ij}^*) \geq 0.$$

Let us show sufficiency. For any $\mathbf{w}' \in \mathcal{W}$, simple calculation shows that:

$$\langle \mathbf{q}^*, \Phi(\mathbf{w}' - \mathbf{w}^*) \rangle = \phi \sum (q_i^* - q_j^*)(w'_{ij} - w^*_{ij}) \geq 0,$$

as each term in the summation is nonnegative. Therefore, $\eta'(0) \geq 0$, moreover $\eta(\cdot)$ is clearly convex in $t \in [0, 1]$.[35] Therefore,

$$\eta(1) - \eta(0) \geq (1 - 0)\eta'(0) \geq 0,$$

that is:

$$||\Phi(\mathbf{w}')||^2 \geq ||\Phi(\mathbf{w}^*)||^2 = ||\mathbf{q}^*||^2$$

since $\mathbf{w}'$ is arbitrary, and indeed $\mathbf{q}^*$ is the projection of $\mathbf{0}_n$ onto $\Phi(\mathcal{W})$.

Now we need to verify that for any edge ij with $g_{ij} = 1$, $(C2)$ is equivalent to $(C2')$:

$$\left\{(q_i^* - q_j^*)(z_{ij} - w^*_{ij}) \geq 0, \forall z_{ij} \in [0, 1]\right\} \Leftrightarrow \left\{(i, j) \in E, (q_i^* - q_j^*) > 0 \Rightarrow w^*_{ij} = 0, w^*_{ji} = 1\right\}.$$

If so, then $q_i^* > q_j^* \implies w^*_{ij} = 0$ and $w^*_{ji} = 1$; $q_i^* < q_j^* \implies w^*_{ij} = 1$ and $w^*_{ji} = 0$.

From $(C2)$ to $(C2')$: Suppose $q_i^* > q_j^*$, and let $z_{ij} = 0$. We have $(q_i^* - q_j^*)(0 - w^*_{ij}) \geq 0$, and by $w^*_{ij} \geq 0$ it must be the case that $w^*_{ij} = 0$. Similarly, assuming $q_i^* < q_j^*$ and picking $z_{ij} = 1$ shows that $w^*_{ij} = 1$.

From $(C2')$ to $(C2)$: If $q_i^* > q_j^*$ and $w^*_{ij} = 0$, then for any $\forall z_{ij} \in [0, 1]$, $(q_i^* - q_j^*)(z_{ij} - w^*_{ij}) = (q_i^* - q_j^*)(z_{ij}) \geq 0$. Similarly, if $q_i^* < q_j^*$ and $w^*_{ij} = 1$, then for any $\forall z_{ij} \in [0, 1]$, $(q_i^* - q_j^*)(z_{ij} - w^*_{ij}) = -(q_i^* - q_j^*)(1 - z_{ij}) \geq 0$.

□

Let us now prove the theorem. First, we write down a few necessary conditions for the limiting equilibrium. The cutoffs in the limit must satisfy the indifference conditions

$$\sigma(\theta_i^*) + v_i + \phi \sum_{j \in N_i} w^*_{ij} = 0, \forall i.$$

where

$$w^*_{ij} = \lim_{\nu \to 0} \mathbb{E}[\pi_j(s_j; s_j^\dagger)|s_i = s_i^\dagger]$$

Clearly, $w^*_{ij} + w^*_{ji} = 1$ by Lemma 2. Let $q_i^* = -\sigma(\theta_i^*), i \in N$. Then $\theta_i^* < \theta_j$ if and only if $q_i^* > q_j^*$. Then $q_i^* = v_i + \sum_j g_{ij} w^*_{ij}, \forall i$. Moreover, for any connected node i and j, suppose

[35] As Φ is affine and $||\mathbf{x}||^2$ is a convex function of $\mathbf{x}$

$\theta_i^* < \theta_j^*$, then $q_i^* > q_j^*$, and $w_{ij}^* = 0$ and $w_{ji}^* = 1$ by Lemma 2.

As a result, $\mathbf{q}^*$ satisfies the two conditions stated in Lemma 3, therefore $\mathbf{q}^*$ must be the projection of $\mathbf{0}_n$ onto $\Phi(\mathcal{W})$, which proves the theorem.

$\square$

Proof of Proposition 2. Take weighting matrix $\mathbf{w}^*$. Given $\theta_i^* = \theta_j^* = \theta_m^*$ for each $i, j \in C_m^*$ by definition, it must be that:

$$\begin{aligned} |C_m^*|\sigma(\theta_m^*) = \sum_{i \in C_m^*} \sigma(\theta_i^*) &= \sum_{i \in C_m^*} \left(v_i + \phi \sum_{j \in N_i} w_{ij}^* \right) \\ &= \sum_{i \in C_m^*} \left(v_i + \phi \left(\sum_{j \in N_i \setminus C_m^*} w_{ij}^* + \sum_{j \in C_m^*} w_{ij}^* \right) \right) \\ &= v(C_m^*) + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*)), \end{aligned}$$

the final equality following from Lemma 3. Expression (8) follows.

$\square$

Proof of Corollary 1. Take v and ϕ and corresponding $\mathbf{q}^*$ from Theorem 1. For each $v' \neq v$ it must be that $q_i'^* = q_i^* + (v - v')\mathbf{1}_n$, as $\Phi'(\mathcal{W})$ under v' is:

$$\Phi'(\mathcal{W}) = \{\mathbf{q} + (v - v')\mathbf{1}_n : \mathbf{q} \in \Phi(\mathcal{W})\}.$$

Thus, $q_i^* = q_j^*$ if and only if $q_i'^* = q_j'^*$: $\mathcal{C}^*$ is independent of v. This also shows that $\mathbf{q}^*$ is affine in v with $\frac{\partial q_i^*}{\partial v} = 1$.

Setting $v = 0$, again take ϕ and corresponding $\mathbf{q}^*$ from Theorem 1. For each positive $\phi' \neq \phi$ it must be that $q_i'^* = \frac{\phi'}{\phi} q_i^*$, as $\Phi'(\mathcal{W})$ under ϕ' is:

$$\Phi'(\mathcal{W}) = \{\frac{\phi'}{\phi}\mathbf{q} : \mathbf{q} \in \Phi(\mathcal{W})\}.$$

Again, $q_i^* = q_j^*$ if and only if $q_i'^* = q_j'^*$: $\mathcal{C}^*$ is independent of ϕ. Again, this shows that $\mathbf{q}^*$ is affine in ϕ. $\mathbf{q}^* = v\mathbf{1}_n + \phi \mathbf{q}_0^*$ then follows. $\square$

Proof of Proposition 3. We prove Proposition 4, extend to Theorem 2, and treat Proposition 3 as a Corollary.

Clearly, the existence of a single coordination set is equivalent to that

$$\frac{T}{n}\mathbf{1}_n \in \Phi(\mathcal{W})$$

where $T = \sum v_i + \phi e(N)$. This can be re-formulated as a feasibility condition to the following linear programming problem:

$$\begin{aligned}
&v_i + \phi \sum_{j \in N_i} w_{ij} = \frac{T}{n}, \forall i \in N \\
&w_{ij} \geq 0, \forall (i,j) \in E \\
&w_{ij} + w_{ji} = 1, \forall (i,j) \in E,
\end{aligned}$$

given homogeneous valuations, $v_i = v, \forall i$, and $T = \sum v_i + \phi e(N) = nv + \phi e(N)$. So the above system is equivalent to

$$\begin{aligned}
&\sum_{j \in N_i} w_{ij} = \frac{e(N)}{|N|}, \forall i \in N \\
&w_{ij} \geq 0, \forall (i,j) \in E \\
&w_{ij} + w_{ji} = 1, \forall (i,j) \in E.
\end{aligned} \tag{A6}$$

To show the necessity, suppose there exists a solution $\mathbf{w}^*$ to the above system. Then

$$|S|\frac{e(N)}{|N|} = \sum_{i \in S}(\sum_{j \in N_i} w_{ij}^*) \geq \sum_{i \in S, j \in S, (i,j) \in E} w_{ij}^* = e(S) \cdot (1) = e(S)$$

where the first inequality is trivial, and the second inequality follows from the fact that for each edge with two end nodes i, j both in S, $w_{ij}^* + w_{ji}^* = 1$, there are exactly $e(S)$ such links in the summation.

To show sufficiency, we first re-formulate the above condition as a feasibility condition to a network flow problem, and apply Gale's Demand Theorem (see Gale (1957)). From the original network $G = (N, E)$, we construct a specific bipartite network $\tilde{G} = (V, A)$, where the set of nodes is the union $V = V_1 \cup V_2$ where $V_1 = E$ and $V_2 = N$. The arcs (flow) in $\tilde{G}$ are only from V_1 to V_2. In particular, $j \in E = V_1$ is connected to $i \in N = V_2$ in the bipartite graph $\tilde{G} = (V, A)$, if and only if i is one of the end-points of this edge j

in the original network G.

Each vertex $i \in V_2$ is a demand vertex, demanding $d_i = \frac{e(N)}{|N|}$ units of a homogeneous goods. Each vertex in $j \in V_1$ is a supply vertex, supplying $s_j = 1$ unit of the same good. Supply can be shipped to demand nodes only along the arcs A in the constructed bipartite network $\tilde{G}$. Gale's Demand Theorem states that there is a feasible way to match demand and supply if and only if for all $S \subset V_2$:

$$\sum_{i \in S} d_i \leq \sum_{j \in N(S)} s_j,$$

where $N(S)$ is the set of neighbors of vertices in S in $\tilde{G}$. Substituting the values of s_j, d_i yields the following equivalent condition

$$|S| \frac{e(N)}{|N|} \leq |N(S)|, \quad \forall\, \emptyset \subset S \subset V_2.$$

Clearly the above condition holds when S is either empty or the whole set N. For any other case of S, from the construction of $\tilde{G}$, the set $N(S)$ is only the edges in E such that at least one endpoint belongs to S. Therefore:

$$|N(S)| = e(N) - e(S^c)$$

where $S^c = N \backslash S$ is the complement set of S. Recall that:

$$|N| = |S| + |S^c|, e(N) = |N(S)| + e(S^c),$$

It is easy to see that:

$$|S| \frac{e(N)}{|N|} \leq |N(S)| \Longleftrightarrow \frac{e(N)}{|N|} \leq \frac{|N(S)|}{|S|} \Longleftrightarrow \frac{e(N)}{|N|} \leq \frac{e(N) - e(S^c)}{|N| - |S^c|} \Longleftrightarrow \frac{e(S^c)}{|S^c|} \leq \frac{e(N)}{|N|}.$$

So the feasibility condition is equivalent to the following:

$$\frac{e(S^c)}{|S^c|} \leq \frac{e(N)}{|N|}, \quad \forall\, \emptyset \neq S^c \subset N.$$

Since S is an arbitrary subset of N, and S^c is also arbitrary, the sufficiency direction is proved.

To prove Theorem 2, the proof is analogous provided we modify the values of demand

d_j and supply s_i, accounting for v_i, links between S and N_1 and constraining to subgraph $\mathcal{G}_S$. For this, define $\tilde{V}_1 = E_S \cup \{ij \in E_N : i \in S, j \in N \backslash S\}$ and $\tilde{V}_2 = S$. Define:

$$\tilde{s}_j = \phi, \forall j \in \tilde{V}_1, \text{ and } \tilde{d}_i = \frac{v(S) + \phi(e(S, N_1) + e(S))}{|S|} - (v_i + \phi d_i(N_1)), \forall i \in \tilde{V}_2,$$

as each link from i to N_1 carries a weight of one. It is straight forward to check that:

$$\sum_{j \in \tilde{V}_1} \tilde{s}_j = \phi e(S) = \sum_{i \in \tilde{V}_2} \tilde{d}_i.$$

The remark again follows from Gale's Demand Theorem; the condition for $S = C_S^*$ is that for any nonempty subset $S' \subset S$:

$$\frac{v(S') + \phi(e(S', N_1) + e(S'))}{|S'|} \leq \frac{v(S) + \phi(e(S, N_1) + e(S))}{|S|}.$$

□

Proof of Proposition 5. For trees, there are no cycles, so $e(N) = N - 1$, while for each subset S the resulting subnetwork G_S is still cycle-free. Therefore, the number of edges within S is at most $|S| - 1$, so $e(S) \leq |S| - 1$, and thus:

$$\frac{e(S)}{|S|} \leq \frac{|S| - 1}{|S|} \leq \frac{e(N)}{|N|} = \frac{|N| - 1}{|N|}.$$

For regular bipartite networks with two disjoint groups B_1, B_2 with size n_1, n_2, we set $w_{ij}^* = \frac{n_1}{n_1+n_2}$ if $i \in N_1, j \in N_2$, and $w_{ij}^* = \frac{n_2}{n_1+n_2}$ if $i \in N_2, j \in N_1$. Clearly this $\mathbf{w}^*$ is a feasible solution to (A6), with $d_i w_{ij}^* = \frac{e(N)}{n_1+n_2} \in (0, 1)$ for each $i \in N$. Therefore by Lemma 3, $q_i^* = q_j^*$ for all $i, j \in N$.

If G is a network with a unique cycle, then $e(N) = N$. For each subset S, the resulting subnetwork G_S contains at most one cycle, so the number of edges within S is at most $|S|$, so that $e(S) \leq |S|$, and thus:

$$\frac{e(S)}{|S|} \leq \frac{|S|}{|S|} = 1 = \frac{e(N)}{|N|}$$

When G contains at most four nodes, all networks with three or fewer nodes contain at most one cycle. The only network structures over four nodes that contain more than one cycle are the circle with a link connecting one non-adjacent pair i and j (two networks) and the complete network. For the former, we can show these networks to have one coordination set with weights: $w^*_{ij} = w^*_{ji} = 1/2$, $w^*_{ki} = w^*_{kj} = 5/8$ and $w^*_{ij} = w^*_{ik} = 3/8$ for each $k \neq i, j$. Each weight is within $(0, 1)$ and thus by Lemma 3, $q^*_i = q^*_j = q^*_k$ for each $k \neq i, j$. The complete network with four nodes and 6 edges is regular, and clearly has a symmetric equilibrium (i.e. one coordination set). Note, when $N = 5$, there exists a network such that two coordination sets emerges. For example, a core with four nodes plus one periphery node having one link to one of the core node. □

Proof of Proposition 6. For part *1.*, $s^*_i \leq s^*_j$ for all $i \in S$ and j in regular network $\mathcal{G}$ of degree d follows from supermodularity of $G(\nu)$, uniqueness of s^* for ν small, and $d_i \geq d$ for each $i \in S$. By continuity, $\max_{i \in S} \theta^*_i \leq \theta^{r*}_d$.

For part *2.*, take coordination set C^*_m and $\underline{i} \in \text{argmin}_{i \in C^*_m} d_i(\underline{C}^*_m \cup C^*_m)$. $\underline{i}$'s expected network effect in $G(\nu)$ is no greater than $d = d_{\underline{i}}(\underline{C}^*_m \cup C^*_m)$, which equals expected network effect to each k in a regular network of degree $2d$. Thus, $s^*_{\underline{i}} \geq s^*_k$ for all $\nu > 0$ small. By continuity, $\theta^*_{\underline{i}} \geq \theta^{r*}_{2d}$. □

Proof of Proposition 7. Suppose not. If each coordination set is of size one, then consider the node i with the highest q^*_i. Clearly, $q^*_i > q^*_j$, for any $j \in N_i$. It follows that $w_{ij} = 0$ for all $j \in N_i$, so $q^*_i = v + \phi \sum_{j \in N_i} w^*_{ij} = v$. Since $\sum q^*_i = |N|v + e(N) > |N|v$, so $q^*_i = v$ implies i cannot have the highest q^*_i. □

Proof of Proposition 8. Given $\mathcal{C}^*_{+ij} = \mathcal{C}^*$, C^*_m is unchanged upon inclusion of ij. Moreover, if $\theta^*_{m(i)} > \theta^*_{m(j)}$, then this ordering must maintain upon inclusion of ij, else contradicting $\mathcal{C}^*_{+ij} = \mathcal{C}^*$. We may directly apply (8) of Proposition 2:

$$\begin{aligned}
q^*_{m(i)} &= \frac{v(C^*_m) + \phi(e(C^*_m, \underline{C}^*_m) + e(C^*_m))}{|C^*_m|}, \\
q^*_{m(i),+ij} &= \frac{v(C^*_m) + \phi(e(C^*_m, \underline{C}^*_m) + e(C^*_m) + 1)}{|C^*_m|}; \\
q^*_{m(j),+ij} &= q^*_{m(j)} \quad if\ \theta^*_{m(i)} > \theta^*_{m(j)},
\end{aligned}$$

the second equality holding whether $j \notin m(i)$ with $\theta^*_{m(i)} > \theta^*_{m(j)}$(for *1.*) or $j \in m(i)$ (for *2.*). Differencing $q^*_{m(j),+ij}$ and $q^*_{m(j)}$ gives the result.

□

Proof of Proposition 9. To show (12), denote $\hat{q}_j^*$ and $\utilde{q}_j^*$ the limit equilibrium cutoffs of $j \in N$ when $v_i = \hat{v}_i^*$ and $v_i = \utilde{v}_i^*$, respectively. $\hat{q}_i^* > \utilde{q}_i^*$ with $\hat{q}_j^* \geq \utilde{q}_j^*$ for $j \neq i$ given uniqueness of $\boldsymbol{\theta}^*$ and strategic complementarities. For any $\utilde{\mathbf{w}}^*$ of Theorem 1 under $v_i = \hat{v}_i^*$, we can find some $\hat{\mathbf{w}}^*$ under $v_i = \utilde{v}_i^*$ with $\hat{w}_{ij}^* \leq \utilde{w}_{ij}^*$ for each $j \neq i$. Moreover, by construction $\hat{w}_{ij}^* = 0$ and $\utilde{w}_{ij}^* = 1$ for each $j \in C_m^*$. At each v_i, q_i^* must satisfy $q_i^* = v_i + \phi \sum_{j \in N_i} w_{ij}^*$. Evaluating v_i at $\hat{v}_i$ and $\utilde{v}_i$, and taking differences gives:

$$
\begin{aligned}
\hat{v}_i - \utilde{v}_i &= (\hat{q}_i^* - \utilde{q}_i^*) + \phi \sum_{j \in N_i} (\utilde{w}_{ij}^* - \hat{w}_{ij}^*) \\
&= (\hat{q}_i^* - \utilde{q}_i^*) + \phi \left(d_i(C_m^*) + \sum_{j \in N_i \backslash C_m^*} (\utilde{w}_{ij}^* - \hat{w}_{ij}^*) \right) \\
&\geq \phi d_i(C_m^*),
\end{aligned}
$$

giving expression (12).

To show (13), first by Proposition 2, we can write:

$$
q_m^* = \frac{v_i + v(C_m^* \backslash \{i\}) + \phi(e(C_m^*, \utilde{C}_m^*) + e(C_m^*))}{|C_m^*|}. \tag{A7}
$$

At $v_1 = \hat{v}_i^*$ by Proposition 2 and $w_{ij}^* = 0$ for each $j \in C_m^* \backslash \{i\}$, we have $\hat{q}_m^* = \hat{q}_i^* = \hat{v}_i^* + \phi d_i(\utilde{C}_m^*)$, which by equating with (A7) at $v_i = \hat{v}_i^*$ gives:

$$
\hat{v}_i^* = \frac{v(C_m^* \backslash \{i\}) + \phi(-|C_m^*| d_i(\utilde{C}_m^*) + e(C_m^*, \utilde{C}_m^*) + e(C_m^*))}{|C_m^*| - 1}. \tag{A8}
$$

At $v_1 = \utilde{v}_i^*$, $w_{ij}^* = 1$ for each $j \in C_m^* \backslash \{i\}$, giving $\utilde{q}_m^* = \utilde{q}_i^* = \utilde{v}_i^* + \phi(d_i(\utilde{C}_m^*) + d_i(C_m^*))$, which by equating with (A7) at $v_i = \utilde{v}_i^*$ gives:

$$
\utilde{v}_i^* = \frac{v(C_m^* \backslash \{i\}) + \phi(-|C_m^*|(d_i(\utilde{C}_m^*) + d_i(C_m^*)) + e(C_m^*, \utilde{C}_m^*) + e(C_m^*))}{|C_m^*| - 1}. \tag{A9}
$$

Differencing (A8) and (A9) yields expression (13).

□

Proof of Proposition 10.

Lipschitz continuity. Note that $\mathbf{q}^*$ is the projection of $\mathbf{0}_n$ onto the space $\Phi(\mathcal{W})$:

$$
\mathbf{q}^*(\mathbf{v}) = \mathbf{Proj}_{\Phi(W)}[\mathbf{0}_n].
$$

Since Φ depends on $\mathbf{v}$ in a linear way, we let $\mathbf{K} = \Phi(\mathcal{W})$ when $\mathbf{v} = \mathbf{0}_n$. Then for any $\mathbf{v}$:

$$\Phi(\mathcal{W}) = \mathbf{v} + \mathbf{K}.$$

We can rewrite the projection problem as follows:

$$\mathbf{q}^*(\mathbf{v}) = \arg \min_{\mathbf{z} \in \mathbf{v}+\mathbf{K}} ||\mathbf{z}||^2 = \mathbf{v} + \arg \min_{\mathbf{y} \in \mathbf{K}} ||(-\mathbf{v}) - \mathbf{y}||^2 = \mathbf{v} + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]$$

The projection mapping is nonexpansive (see chapter 1 of Nagurney 1992), i.e:

$$||\mathbf{Proj}_{\mathbf{K}}[\mathbf{x}] - \mathbf{Proj}_{\mathbf{K}}[\mathbf{y}]|| \leq ||\mathbf{x} - \mathbf{y}||, \forall \mathbf{x}, \mathbf{y} \in \mathbf{R}_n.$$

So for any $\mathbf{v}$ and $\mathbf{v}'$, we have

$$||\mathbf{q}^*(\mathbf{v}) - \mathbf{q}^*(\mathbf{v}')|| = ||(\mathbf{v} + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]) - (\mathbf{v}' + \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}'])||$$

$$\leq ||\mathbf{v} - \mathbf{v}'|| + ||\mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}]) - \mathbf{Proj}_{\mathbf{K}}[-\mathbf{v}']|| \leq 2||\mathbf{v} - \mathbf{v}'||.$$

Hence, $\mathbf{q}^*(\mathbf{v})$ is Lipschitz continuous in $\mathbf{v}$.

Comparative Statics. By Lipschitz continuity, $\mathbf{q}^*(\mathbf{v})$ is differentiable for almost all $\mathbf{v}$. By Proposition 2, for each coordination set C_m^*, $q_i^* = q_m^* = -\sigma(\theta_m^*)$ for each $i \in C_m^*$, with q_m^* given by:

$$q_m^* = \frac{\sum_{i \in C_m^*} v_i + \phi(e(C_m^*, \underline{C}_m^*) + e(C_m^*))}{|C_m^*|}.$$

Note that the terms $e(C_m^*, \underline{C}_m^*)$ and $e(C_m^*)$ are constant holding $\mathcal{C}^*$ constant. For generic $\mathbf{v}$, $\mathcal{C}^*$ is locally constant, hence $e(C_m^*, \underline{C}_m^*)$ and $e(C_m^*)$ do not depend on $\mathbf{v}$ locally. The derivative results follows directly.

Monotonicity. $\partial \mathbf{q}^* / \partial \mathbf{v}$ is nonnegative, so $\mathbf{q}^*$ is monotone in $\mathbf{v}$.

$\square$

Proof of Remark 3. Near the limit ($\nu > 0$), for $k \notin m(i)^*$ with $\theta_{m(i)}^* \neq \theta_{m(k)}^*$, then $s_k^* \notin (s_i^* - \nu, s_i^* + \nu)$ for $\nu > 0$ sufficiently small (i.e. for $\nu \ll |\theta_{m(i)}^* - \theta_{m(k)}^*|/2$), and thus for all $i' \in C_{m(i)}^*$, $a_{i'}$ either equals one or zero (depending on $m' < m$ or $m' > m$, respectively) with probability one conditioning on $s_k = s_k^*$. Because this is true for arbitrary k, it is also true for all members of any $m' \neq m(i)$ (including $m(j)$) for $\nu > 0$

sufficiently small (i.e. for $\nu \ll \min_{m' \neq m(i)} |\theta^*_{m(i)} - \theta^*_{m'}|/2$). Given no atoms of F, this must hold in a neighborhood of s^*_i, which implies $\partial s^*_j/\partial s^*_i = 0$ for all $j \notin m(i)$. If instead $k \notin m(i)^*$ but $\theta^*_{m(i)} = \theta^*_{m(k)}$, by $\partial s^*_j/\partial v_i = 0$ for each $j \notin m(i)^*$ when $\theta^*_{m(i)} \neq \theta^*_{m(j)}$ and by $C^*_{m(k)}, C^*_{m(j)}$ disjoint by assumption, $\partial s^*_j/\partial s^*_i = 0$ again follows. $\partial s^*_j/\partial s^*_i = 0$ then implies $\partial s^*_j/\partial v_i = 0$.

□

Proof of Proposition 11. (21) follows directly from Proposition 10 and Remark 3.

We derive expressions for μ^*_{ij}, which follow from Leibniz integral rule. For μ^*_{ii}:

$$\begin{aligned}
\mu^*_{ii} \quad &:= \quad \frac{\partial}{\partial s^\dagger_i} U_i(\boldsymbol{\pi}_{-i}|s^*_i) = \int_{-1}^{1} \left(\frac{\partial}{\partial \theta}\sigma(s^*_i - \nu\epsilon_i) + \phi \sum_{j\in N_i} \frac{\partial}{\partial \theta} r(s^*_i - \nu\epsilon_i, s^*_j; \nu) \right) f(\epsilon_i) d\epsilon_i \\
&= \quad \int_{-1}^{1} \left(\frac{\partial}{\partial \theta}\sigma(s^*_i - \nu\epsilon_i) + \phi \sum_{j\in N_i} \frac{1}{\nu} f\left(\frac{(s^*_i - s^*_j) - \nu\epsilon_i}{\nu} \right) \right) f(\epsilon_i) d\epsilon_i \\
&= \quad \int_{-1}^{1} \frac{\partial}{\partial \theta}\sigma(s^*_i - \nu\epsilon_i) f(\epsilon_i) d\epsilon_i + \phi \sum_{j\in N_i} \Lambda(s^*_i, s^*_j; \nu),
\end{aligned}$$

where we denote:

$$\Lambda(s, s'; \nu) := \frac{1}{\nu} \int_{-1}^{1} f\left(\frac{s - s' - \nu\epsilon}{\nu} \right) f(\epsilon) d\epsilon.$$

For μ^*_{ij}, $\mu^*_{ij} = 0$ when $ij \notin E$, and otherwise:

$$\mu^*_{ij} := \frac{\partial}{\partial s^*_j} U_i(\boldsymbol{\pi}_{-i}|s^*_i) = \phi \int_{-1}^{1} \frac{\partial}{\partial s^*_j} r(s_i - \nu\epsilon_i, s^\dagger_j; \nu) f(\epsilon_i) d\epsilon_i = -\phi\Lambda(s^*_i, s^*_j; \nu),$$

which is non-zero when $i, j \in C^*_m$, as consistent Proposition 10.

To show (22), we derive expressions for $\bar{\mu}^*_{ij}$. Write h and H the density and cumulative functions (respectively) of the marginal distribution of each s_i.[36] First, $\bar{\mu}^*_{ii} = 0$ by the envelope theorem: $U_i(\boldsymbol{\pi}_{-i}|s^*_i) = 0$ in equilibrium and thus marginally increasing $s^\dagger_i$ above

[36]This requires the prior distribution of θ, which does not influence equilibrium play in $G(\nu)$ provided the conditional distribution of s_i on θ for each i are common knowledge; see footnote 12.

s_i^* yields zero expected gain to i. For $\bar{\mu}_{ij}^*$, $j \neq i$, $\bar{\mu}_{ij}^* = 0$ when $ij \notin E$. Otherwise:

$$\begin{aligned} \bar{\mu}_{ij}^* &:= \left. \frac{\partial}{\partial s_j^\dagger} \mathbb{E}_{s_i}\left[U_i(\boldsymbol{\pi}_{-i}|s_i)\right] \right|_{\boldsymbol{\pi}^\dagger=\boldsymbol{\pi}^*} = \int_{s_i^*}^{\infty} \frac{\partial}{\partial s_j^\dagger} U_i(\boldsymbol{\pi}_{-i}^*|s_i) dH(s_i) \\ &= -\phi \int_{s_i^*}^{\infty} \Lambda(s_i^*, s_j^*; \nu) h(s_i) ds_i. \end{aligned}$$

We see that $\bar{\mu}_{ji}^* = 0$ when $s_i^* > s_j^* + \nu$, yielding $mw_i^*(\mathbf{1}_j) = 0$ when $j \in C_{m'}^*$, $i \in C_m^*$ with $m' > m$ and ν is sufficiently small. Moreover, $\bar{\mu}_{ji}^* = 0$ for each $j \in C_{m'}^*$, $i \in C_m^*$ when $m' < m$ and $\theta_{m'}^* = \theta_m^*$, as C_m^* and $C_{m'}^*$ are disjoint. This yields $mw_i^*(\mathbf{1}_{k'}) = 0$ of the Proposition by setting $j = k'$. $\bar{\mu}_{ji}^* < 0$ provided $\nu \in (0, \bar{\nu})$ (as consistent with Proposition 10) when either $i, j \in C_m^* = C_{m'}^*$ or $i \in C_m^*$, $j \in C_{m'}^*$ with $m > m'$ and $ij \in E$. Thus, taking $i, j \in C_m^*$ and $k \in C_{m^-}^*$ with $jk \in E$ (per the Proposition statement), $\partial s_j^* / \partial v_i < 0$ by Proposition 10 for ν sufficiently small, and thus $mw_i^*(\mathbf{1}_k) \geq \bar{\mu}_{kj}^* \cdot \partial s_j^* / \partial v_i > 0$.

□

Online Appendix for "Coordination on Networks"

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1 Appendix: Noise-independent selection

Here we consider the robustness of equilibrium selection to heterogeneous noise structures. Consider the following extension.[1]

Information Structure. In the perturbed game, each i realizes signal $s_i = \theta + \nu\epsilon_i$, $\nu > 0$, where ϵ_i is distributed via density function f_i and cumulative function F_i with support within $[-1, 1]$. Signals are independently drawn across agents conditional on θ.

As shown in Theorem 1, the limiting cutoff θ_i^* are fully determined by the parameters $\mathbf{v}, \phi, \sigma(\cdot)$, and $\mathcal{G}$, in particular, the cutoffs are independent of the noise distribution F. In this appendix, we provide an alternative proof of the noise-independent selection result from a potential game approach.

In the simple case with two-player and binary action coordination game (dyad case in our paper), as shown in Carlsson and van Damme (1993), the risk-dominant equilibrium is selected by global game and it is independent of noise distribution. Frankel et al. (2003) generalize this result to n-player supermodular games which yield a potential, which applies to our setting under arbitrary network structures. Recall that in our coordination game, each player has a binary action $a_i \in \{0, 1\}$. Define the following function:

$$P(\mathbf{a}|\theta) := \sum_{i \in N} (v_i + \sigma(\theta)) a_i + \frac{1}{2}\phi \sum_{i,j \in N; i \neq j} a_i a_j, \text{ where } \mathbf{a} \in \{0, 1\}^N. \tag{B1}$$

It is straightforward to check that $P(\mathbf{a}|\theta)$ is a potential function of game $G(0)$ at θ

[1] Frankel et al. (2003) Section 6 addresses such an enrichment.

(Monderer and Shapley 1998), by the following

$$\begin{aligned} P(a_i', \mathbf{a}_{-i}|\theta) - P(a_i, \mathbf{a}_{-i}|\theta) &= (a_i' - a_i)\left(v_i + \sigma(\theta) + \phi \sum_{j \in N_i} a_j\right) \\ &= u_i(a_i', \mathbf{a}_{-i}|\theta) - u_i(a_i, \mathbf{a}_{-i}|\theta). \end{aligned}$$

Moreover, the potential P is supermodular in $(a_i, \mathbf{a}_{-i})$ for fixed θ, and strictly supermodular in (a_i, θ) for fixed $\mathbf{a}_{-i}$. As a result, by Frankel et al. (2003), Oyama and Takahashi (2017) and Basteck et al. (2013), the game $G(0)$ has an exact potential, therefore the maximizer of the potential is selected by the global game, and this selection is independent of noise distribution F.[2]

The connection between the potential game approach and our approach in Theorem 1 can be understood from the following relationship:[3] for generic $\mathbf{v}$,

$$\theta_i^* = \inf\{\theta \in \Theta | \exists \mathbf{a}_{-i} \text{ such that } (1, \mathbf{a}_{-i}) \in \arg\max_{\mathbf{a}} P(\mathbf{a}|\theta)\}.$$

While the potential approach requires solving the maximization of P for each θ, which makes it challenging for comparative statics due to discreteness of $\mathbf{a}$, our approach has the advantage that more precise information about the equilibrium cutoff points θ_i^* is obtained using Theorem 1 and the projection algorithm. Moreover, the information coordination set, i.e., who coordinates with whom, is also directly decoded using the cutoff values, which enables us to conduct comparative statics with respect to network structure and valuations in a much simpler manner.

2 Appendix: Miscoordination costs

Expression (20) of the main text:

$$u_i(\mathbf{a}_{-i}|\theta) = v + \sigma(\theta) - \phi \sum_{j \in N_i} (1 - a_j),$$

yields the following modification to Corollary 2. Define $\bar{C}_m^* := \cup_{m' > m} C_{m'}^*$.

[2]Moreover, Ui (2001) shows that the selected equilibrium is *robust* in the sense of Kajii and Morris (1997). See Morris and Ui (2005), Oyama and Takahashi (2017) for further discussions.

[3]Note that for generic $\mathbf{v}$, the potential P has a unique maximizer.

Corollary C1 (Limit cutoffs under miscoordination costs: within-set symmetry)**.** *If $C_m^* \in \mathcal{C}^*$ is within-set symmetric such that:*

$$d_i(\bar{C}_m^*) + \frac{1}{2}d_i(C_m^*) = d_j(\bar{C}_m^*) + \frac{1}{2}d_j(C_m^*) =: b_m, \quad \forall i \in C_m^*,$$

then in the limit $\nu \to 0$, θ_m^* is given by:

$$\theta_m^* = \sigma^{-1}(-v + \phi b_m). \tag{C1}$$

To see this, we may again apply Proposition 2, now with agents penalized for holding greater degree. Summing over links $ij \in E$ with $i \in C_m^*$ and $j \in \underline{C}_m^*$, $-d_i(\underline{C}_m^*)$ in v_i and $d_i(\underline{C}_m^*)$ cancel for each i in the numerator of expression (8), while $-d_i(\bar{C}_m^*)$ in v_i leave q_m^* linearly *increasing* in $d_i(\bar{C}_m^*)$ with slope one. For each $ij \in E$ with $i, j \in C_m^*$, $v(C_m^*)$ double counts each link while $e(C_m^*)$ does not, leaving q_m^* linearly *increasing* in $d_i(\bar{C}_m^*)$ with slope one-half.

When a single coordination set obtains the common cutoff is $\theta_1^* = \sigma^{-1}(-v + \phi\frac{e(N)}{|N|})$, again by Proposition 2. One can apply Theorem 2 to reconstruct Proposition 4 as the equivalent condition for a single coordination set. To show this, set $N_1 = \emptyset$ with $v_i = v - d_i$ in Theorem 2 to obtain:

$$\frac{|S|v - \sum_{i \in S} d_i + e(S)}{|S|} \leq \frac{|N|v - \sum_{i \in N} d_i + e(N)}{|N|}, \quad \forall \emptyset \neq S \subset N,$$

which, given $-\sum_{i \in S} d_i + e(S) = -e(S, S^c) - e(S)$, is equivalent to:

$$\frac{e(S, S^c) + e(S)}{|S|} \geq \frac{e(N)}{|N|}, \quad \forall \emptyset \neq S \subset N.$$

Because $E = e(S) \cup e(S^c) \cup e(S, S^c)$, for this inequality to hold it must be that:

$$\frac{e(S^c)}{|S^c|} \leq \frac{e(N)}{|N|}, \quad \forall \emptyset \neq S^c \subset N.$$

As this is true for all nonempty $S^c \subset N$, we are free to drop the complement superscripts. With Proposition 4 in hand, Proposition 5 obtains.

3 Appendix: Numerical solutions near the limit

The numerical solution near the limit of Example 4 solves for $\mathbf{s}^*$ using fixed-point method, taking $\sigma = -w(1-\theta)/\theta$:

$$u_i(\mathbf{a}_{-i}|\theta) = v_i - w\frac{1-\theta}{\theta} + \phi \sum_{j \in N_i} a_j.$$

Note that $\underline{\theta}, \overline{\theta} \in (0,1)$ exist for all $v_i, \phi > 0$. All examples take $w = 3$ and $s_i|\theta \sim U[\theta - \nu, \theta + \nu]$, or $f(s) = 1/2\nu$ for $s \in [\theta - \nu, \theta + \nu]$ and zero otherwise. Moreover, we approximate $s_i \sim U[0,1]$.[4]

Expected value to adopting conditional on s_i and $\boldsymbol{\pi}_{-i}$ are then derived as follows. First:

$$\mathbb{E}\left[\frac{1-\theta}{\theta}\middle| s_i\right] = \frac{1}{2\nu}\int_{s_i-\nu}^{s_i+\nu} \frac{1-\theta}{\theta} d\theta = \frac{1}{2\nu}\left(\ln(\theta) - \theta\right)\Big|_{s_i-\nu}^{s_i+\nu} = \frac{1}{2\nu}\ln\left(\frac{s_i+\nu}{s_i-\nu}\right) - 1.$$

i's expectation of a_j conditional on s_i and given $s_j^\dagger$ is derived as follows. If $s_i \leq s_j^\dagger - 2\nu$ then $\mathbb{E}[a_j|s_i] = 0$, if $s_i \geq s_j^\dagger + 2\nu$ then $\mathbb{E}[a_j|s_i] = 1$, and otherwise:

$$\begin{aligned}
\mathbb{E}[a_j|s_i \in [s_j^\dagger - 2\nu, s_j^\dagger]] &= \mathbb{E}_\theta[r(\theta, s_j^\dagger; \nu)|s_i \in [s_j^\dagger - 2\nu, s_j^\dagger]] \\
&= \frac{1}{2\nu}\int_{s_j^\dagger-\nu}^{s_i+\nu} r(\theta, s_j^\dagger;\nu) d\theta \\
&= \frac{1}{4\nu^2}\left(\frac{1}{2}\left((s_i+\nu)^2 - (s_j^\dagger - \nu)^2\right) - (s_j^\dagger - \nu)(s_i - s_j^\dagger + 2\nu)\right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[a_j|s_i \in [s_j^\dagger, s_j^\dagger + 2\nu]\right] &= \mathbb{E}_\theta[r(\theta, s_j^\dagger; \nu)|s_i \in [s_j^\dagger, s_j^\dagger + 2\nu]] \\
&= \frac{1}{2\nu}\left(\int_{s_i-\nu}^{s_j^\dagger+\nu} r(\theta, s_j^\dagger;\nu) d\theta + \int_{s_j^\dagger+\nu}^{s_i+\nu} d\theta\right) \\
&= \frac{1}{2\nu}\left(\frac{1}{2\nu}\left(\frac{1}{2}\left((s_j^\dagger+\nu)^2 - (s_i - \nu)^2\right) - (s_j^\dagger - \nu)(s_j^\dagger - s_i + 2\nu)\right) + (s_i - s_j^\dagger)\right).
\end{aligned}$$

The elements of μ_{ij}^* and $\bar{\mu}_{ij}^*$ for each i, j, provided in the proof of Proposition 11, can

[4] Such a prior is consistent with $s_i|\theta$ uniform as $\nu \to 0$.

be derived as follows. For the elements of μ^*_{ij}:

$$\int_{-1}^{1} \frac{\partial}{\partial\theta}\sigma(s_i^* - \nu\epsilon_i) f(\epsilon_i) d\epsilon_i = \int_{-1}^{1} \frac{-1}{2(s_i^* - \nu\epsilon_i)^2} d\epsilon_i = \left. \frac{1}{2\nu(s_i^* - \nu\epsilon_i)} \right|_{-1}^{1} = \frac{1}{s_i^{*2} - \nu^2}.$$

Moreover, $\Lambda(s, s'; \nu)$ gives the unit-area tent-function with peak of height $1/(2\nu)$ at $s = s'$ and base length of 4ν:

$$\Lambda(s, s'; \nu) := \begin{cases} 0 & if \quad s \leq s' - 2\nu \\ (2\nu + (s - s'))/(4\nu^2) & if \quad s' - 2\nu < s \leq s' \\ (2\nu + (s' - s))/(4\nu^2) & if \quad s' < s \leq s' + 2\nu \\ 0 & if \quad s' + 2\nu \leq s \end{cases}.$$

We can then substitute this closed form to $\Lambda(s, s'; \nu)$ into:

$$\begin{aligned} \mu^*_{ii} &= \frac{1}{s_i^{*2} - \nu^2} + \phi \sum_{j \in N_i} \Lambda(s_i^*, s_j^*; \nu) \\ \mu^*_{ij} &= -\phi \Lambda(s_i^*, s_j^*; \nu). \end{aligned}$$

For $\bar{\mu}^*_{ij}$, $j \neq i$:

$$\begin{aligned} \bar{\mu}^*_{ij} &= -\phi \int_{s_i^*}^{\infty} \Lambda(s_i, s_j^*; \nu) ds_i \\ &= -\phi \begin{cases} 1 & if \quad s_i^* \leq s_j^* - 2\nu \\ 1 - (s_i^* - s_j^* + 2\nu)^2/(8\nu^2) & if \quad s_j^* - 2\nu < s_i^* \leq s_j^* \\ (s_j^* - s_i^* + 2\nu)^2/(8\nu^2) & if \quad s_j^* < s_i^* \leq s_j^* + 2\nu \\ 0 & if \quad s_i^* > s_j^* + 2\nu \end{cases}. \end{aligned}$$

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