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JEL Classification: C72, D72, D82

Keywords: tournament, optimal allocation of prizes, unimodality, failure rate, comparative statics, dispersive order, convex transform order

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Optimal Tournaments

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1 Introduction

In this paper, we obtain a novel solution to the canonical problem of optimal allocation of prizes in rank-order tournaments. We also provide a set of new comparative statics results for general multi-prize rank-order tournaments with respect to the number of players, prizes, and properties of the distribution of noise.

Rank-order tournaments are incentive schemes in which agents, who expend effort or other resources, are rewarded based on the rank of their output. Such rankings are often used in organizations to assign discretionary bonuses or "merit raises" to employees. Typically, a manager is provided a fixed budget for bonuses and has to decide on a rank-based allocation rule. Our results inform on the effects of such allocation rules on workers' performance, and on the properties of optimal (effort-maximizing) allocation rules. An extreme version of rank-based compensation is promotion, which in most cases is limited to one employee in the group and thus constitutes an example of a "winner-take-all" (WTA) prize allocation, i.e., an incentive schemes rewarding only the top performer. In many organizations, promotion has become the only way to receive a significant permanent raise (Prendergast, 1999); it is, therefore, important to understand under what circumstances the WTA compensation structure is optimal.

Despite the existence of a substantial literature on rank-order tournaments following the framework of Lazear and Rosen (1981), relatively little is known about their general properties. Ryvkin and Drugov (2017) recently provided a comprehensive analysis of comparative statics for WTA tournaments. In this paper, we extend their methodology to study tournaments with multiple prizes. We address two related but distinct problems. First, we provide a solution to the optimal prize allocation problem under symmetry and risk-neutrality in a setting where the principal allocates a fixed budget. This setting is similar to the one studied by Krishna and Morgan (1998) and Moldovanu and Sela (2001), and different from the more traditional setting of Lazear and Rosen (1981) and most later studies in that it does not impose a binding zero-profit constraint on the principal or a participation constraint on the agents. As a result, we obtain a unique optimal allocation of prizes, unlike the more traditional settings where a continuum of prize allocations can be optimal under risk-neutrality. Our major result is that the WTA prize schedule is optimal when the distribution of noise has an increasing failure rate (IFR). More generally, for noise distributions with unimodal failure rates the optimal prize allocation moves closer to WTA as the distribution becomes smaller in the convex transform order. Additionally, we show that "winner tournaments," i.e., prize schedules awarding a positive prize to less than a half of all agents, are optimal when the pdf of noise is symmetric.

Second, we provide a set of comparative statics results for multi-prize tournaments, which generalize some of the results for WTA tournaments by Ryvkin and Drugov (2017) and rely on the methods developed in that paper. We show that when the pdf of noise is log-concave the equilibrium effort changes monotonically in response to shifts in the allocation of prizes to and away from the WTA schedule, in a well-defined sense. For tournaments with k fixed equal prizes, we show that the equilibrium effort is unimodal in k and in the number of players, n, when the pdf of noise is unimodal. Under an additional log-supermodularity condition, the equilibrium effort is also unimodal in the number of players when it is stochastic and changes in the first-order stochastic dominance (FOSD) sense. Finally, we discuss how the equilibrium effort is affected by noise dispersion.

Related literature

Starting with the groundbreaking paper by Lazear and Rosen (1981), multiple studies have explored optimal tournament contracts in an organizational context, comparing their efficiency to other incentive schemes, such as piece rates (Nalebuff and Stiglitz, 1983; Green and Stokey, 1983), and identifying their various properties pertaining to the number, risk-aversion and heterogeneity of agents (Krishna and Morgan, 1998; Akerlof and Holden, 2012; Balafoutas et al., 2017).[1]

The paper that is closest to ours is Krishna and Morgan (1998), who consider a Lazear-Rosen tournament with risk-averse agents and a principal allocating a fixed budget. Their analysis, however, is restricted to tournaments with $n \leq 4$ agents and symmetric unimodal noise distributions. In contrast, we focus on risk-neutral agents but provide very general results regarding the effects of the number of agents and noise. In particular, we generalize the (risk-neutral version of the) "winner-take-all principle" formulated by Krishna and Morgan (1998) for small tournaments.[2] We show that it is the symmetry, and not unimodality, of the pdf of noise that leads to the optimality of WTA when $n \leq 4$, and the IFR property provides an independent sufficient condition for the optimality of WTA that holds for any $n \geq 2$. Schweinzer and Segev (2012) demonstrate the optimality of the WTA schedule for multi-prize Tullock contests.[3]

[1]For a review of the earlier literature see, e.g., McLaughlin (1988), Lazear (1999), Prendergast (1999), Connelly et al. (2014).

[2]Under their assumptions, Krishna and Morgan (1998) demonstrate the WTA principle for $n \leq 3$ for risk-averse agents and for $n \leq 4$ for risk-neutral agents. Both bounds are tight.

[3]The symmetric equilibrium in the Tullock (1980) contest model is the same as in an appropriately transformed Lazear-Rosen tournament with additive noise following the Gumbel (extreme value type-I) distribution; thus, the Tullock contest model can be thought of as a special case of the Lazear-Rosen

There is a parallel literature on the optimal allocation of prizes in all-pay auctions with private information (e.g., Moldovanu and Sela, 2001). Sisak (2009) provides a review of the existing results on multi-prize Tullock contests and all-pay auctions.

The rest of the paper is organized as follows. Section 2 sets up the model. The optimal prize allocation is characterized in Section 3. Section 4 provides comparative statics results and looks at some extensions and applications. Section 5 concludes. All missing proofs are collected in Appendix.

2 Model setup

There are $n \geq 2$ risk-neutral players indexed by $i = 1, \ldots, n$. The players simultaneously and independently choose effort levels $e_i \in \mathbb{R}_+$. The cost of effort e_i to player i is $c(e_i)$, where $c(\cdot)$ is strictly increasing, strictly convex and twice differentiable on $(0, c^{-1}(1)]$, and continuous at zero with $c(0) = 0$. The output of player i is stochastic and given by $y_i = e_i + X_i$,[4] where shocks X_i are i.i.d. across players, with a cumulative density function (cdf) F and probability density function (pdf) f defined on an interval support $U = [\underline{x}, \overline{x}]$ (finite or infinite). The pdf f is atomless, continuous and piece-wise differentiable in the interior of U. Let $F^{-1}(z) = \inf\{t \in U : F(t) \geq z\}$ denote the corresponding quantile function, and $m(z) = f(F^{-1}(z)) : [0, 1] \to \mathbb{R}_+$ denote the inverse quantile density (Parzen, 1979). We assume that $m(z)$ is continuous and piece-wise differentiable on $(0, 1)$ and integrable on $[0, 1]$.

A risk-neutral principal observes the ranking of outputs and allocates nonnegative rank-dependent prizes $v_1, \ldots, v_n$ to the n players. Specifically, a player whose output is ranked r (where $r = 1$ corresponds to the highest output, $r = 2$ to the second highest, etc.) receives a prize $v_r \geq 0$.[5] By default, throughout the paper we assume that prizes are decreasing[6] in rank, $v_1 \geq v_2 \geq \ldots \geq v_n$, and satisfy the budget constraint $\sum_{r=1}^{n} v_r = 1$. On several occasions, we discuss the consequences of relaxing these assumptions.

Consider a symmetric pure-strategy Nash equilibrium where all players exert some

model (Jia, Skaperdas and Vaidya, 2013; Ryvkin and Drugov, 2017). The Gumbel distribution is IFR.

[4]Via a change of variables, this model can also accommodate tournaments with multiplicative noise, with $y_i = e_i X_i$ (see Jia, 2008; Jia, Skaperdas and Vaidya, 2013; Ryvkin and Drugov, 2017).

[5]Ties in the ranking are broken randomly, but occur with zero probability for an atomless f.

[6]Throughout this paper, "increasing" means nondecreasing and "decreasing" means nonincreasing. Whenever the distinction is important, we use the terms "strictly increasing" and "strictly decreasing."

effort $e^* > 0$. The payoff of player i from some deviation effort e_i is

$$\pi_i(e_i, e^*) = \sum_{r=1}^{n} P^{(i,r)}(e_i, e^*) v_r - c(e_i), \tag{1}$$

where $P^{(i,r)}(e_i, e^*)$ is the probability that player i's output is ranked r. This probability is given by

$$P^{(i,r)}(e_i, e^*) = \binom{n-1}{r-1} \int_U F(e_i - e^* + t)^{n-r} [1 - F(e_i - e^* + t)]^{r-1} dF(t).$$

Indeed, in order to be ranked r, player i's output must be higher than the output of exactly $n - r$ other players, and there are $\binom{n-1}{r-1}$ ways to choose those players. The symmetric first-order condition, $\left.\frac{\partial \pi_i(e_i, e^*)}{\partial e_i}\right|_{e_i = e^*} = 0$, produces the equation

$$\sum_{r=1}^{n} \beta_{r,n} v_r = c'(e^*), \tag{2}$$

where $\beta_{r,n} \equiv \left.\frac{\partial P^{(i,r)}(e_i, e^*)}{\partial e_i}\right|_{e_i = e^*}$, the marginal probabilities of reaching rank r, are given by

$$\beta_{r,n} = \binom{n-1}{r-1} \int_U F(t)^{n-r-1} [1 - F(t)]^{r-2} [n - r - (n-1)F(t)] f(t) dF(t). \tag{3}$$

In what follows, we assume that the e^* given by Eq. (2) is a symmetric equilibrium effort level.[7] Coefficients $\beta_{r,n}$ are determined entirely by the distribution of noise. The following properties follow immediately from Eq. (3) (cf. also Akerlof and Holden, 2012): (i) $\sum_{r=1}^{n} \beta_{r,n} = 0$, $\beta_{1,n} > 0$, $\beta_{n,n} < 0$; (ii) if pdf f is symmetric then $\beta_{r,n} = -\beta_{n-r+1,n}$ for all r.

After the probability integral change of variable, $z = F(t)$, Eq. (3) becomes

$$\beta_{r,n} = \binom{n-1}{r-1} \int_0^1 z^{n-r-1} (1-z)^{r-2} [n - r - (n-1)z] m(z) dz. \tag{4}$$

The representation (4), a special case of which for WTA tournaments was introduced by

[7] That is, we assume that the e^* solving (2) is a global maximum of $\pi_i(e_i, e^*)$, Eq. (1), in e_i. Equilibrium existence and uniqueness issues are outside the scope of this paper. These are still open questions for the Lazear-Rosen tournament model. Qualitatively, the existence of a pure-strategy equilibrium requires that the distribution of noise be sufficiently dispersed and/or the cost of effort be sufficiently large and convex (Nalebuff and Stiglitz, 1983).

Ryvkin and Drugov (2017), is useful in that it separates the effects of the distribution of noise, contained entirely in the inverse quantile density $m(z)$, from the effects of the number of players, n, and performance rank, r. Note that $\beta_{1,n} = (n-1)\int_0^1 z^{n-2}m(z)dz = \mathrm{E}(m(Z_{(n-1:n-1)}))$; that is, in a WTA tournament with $v_1 = 1$ and $v_2 = \ldots = v_n = 0$ the equilibrium effort is determined by the expectation of $m(z)$ over the highest order statistic among $n-1$ i.i.d. draws from the uniform distribution on $[0,1]$ (cf. Ryvkin and Drugov, 2017). As we show in the next section, more generally, for an arbitrary prize schedule $(v_1, \ldots, v_n)$ the left-hand side of (2) can be written as a linear combination of expectations of $m(z)$ over the set of uniform order statistics $Z_{(n-r:n-1)}$, $r = 1, \ldots, n-1$.

3 Optimal prize allocations

In this section, we characterize the optimal allocation of prizes. In Section 3.1, we rewrite the problem in a more convenient form and identify the general structure of the solution. In Section 3.2, we relate the properties of the solution to the shape of the distribution of noise.

3.1 Restating the problem

We consider a principal whose objective is to maximize total expected output, ne^*. It follows immediately from (2), and the assumption that $c(\cdot)$ is strictly convex, that the principal's prize allocation problem has the form

$$\max_{v_1,\ldots,v_r} \sum_{r=1}^{n} \beta_{r,n} v_r \quad \text{s.t. } v_1 \geq \ldots \geq v_n \geq 0, \quad \sum_{r=1}^{n} v_r = 1. \tag{5}$$

Problem (5) is a linear programming problem. If prize schedules were not restricted to be monotone, its solution would simply be to allocate the entire prize to the rank r that maximizes $\beta_{r,n}$. Therefore, if $\beta_{r,n}$ is maximized at $r = 1$, the optimal prize allocation is a *winner-take-all* (WTA) tournament with $v_1 = 1$ and $v_r = 0$ for $r = 2, \ldots, n$, for which the monotonicity constraint is satisfied automatically. If, however, $\beta_{r,n}$ is (strictly) maximized at some $r > 1$, it is impossible to allocate a high prize to rank r without also allocating at least the same prizes to ranks $1, \ldots, r-1$. The larger r is, the smaller these prizes will be, thereby diminishing incentives to compete for higher ranks. A different prize allocation, therefore, may be optimal in this case.

Recall that $\beta_{n,n} < 0$. This implies that it is never optimal to assign a positive prize to rank $r = n$. That is, $v_n = 0$ in any optimal prize schedule. It is convenient to introduce nonnegative variables $d_r = v_r - v_{r+1}$ for $r = 1, \ldots, n-1$, from which the original prizes can be recovered as $v_r = \sum_{k=r}^{n-1} d_k$. Further, let $B_{r,n} = \sum_{k=1}^{r} \beta_{k,n}$ denote the cumulative version of coefficients $\beta_{r,n}$. It is straightforward to verify that (cf. also Balafoutas et al., 2017)

$$B_{r,n} = \frac{(n-1)!}{(n-r-1)!(r-1)!} \int_0^1 z^{n-r-1}(1-z)^{r-1} m(z) dz. \tag{6}$$

Note that $B_{r,n} > 0$ for all $r = 1, \ldots, n-1$ and $B_{n,n} = 0$. Using summation by parts, rewrite the objective function of problem (5) as $\sum_{r=1}^{n} \beta_{r,n} v_r = \sum_{r=1}^{n-1} B_{r,n} d_r$. Taking into account that $v_n = 0$, we can also rewrite the budget constraint in the new variables as $\sum_{r=1}^{n} v_r = \sum_{r=1}^{n-1} r d_r = 1$. Thus, problem (5) reduces to

$$\max_{d_1,\ldots,d_{n-1}} \sum_{r=1}^{n-1} B_{r,n} d_r \quad \text{s.t. } d_1, \ldots, d_{n-1} \geq 0, \quad \sum_{r=1}^{n-1} r d_r = 1. \tag{7}$$

The representation (7) can be understood intuitively as follows. Recall that $\beta_{r,n}$ is the marginal probability of reaching rank r in the symmetric equilibrium, cf. Eqs. (2) and (3). Therefore, coefficient $B_{r,n} = \sum_{k=1}^{r} \beta_{k,n}$ can be interpreted as the marginal probability of reaching the rank of *at least* r. At the same time, prize differential d_r is the premium for reaching the rank of at least r (as compared to reaching the rank of at least $r+1$). Indeed, a player who reached the rank of at least r earns at least $d_{n-1} + \ldots + d_r = v_r$. Since exactly r players reach the rank of at least r, the budget constraint takes the form as in (7).

Problem (7) is a standard linear utility maximization problem in $\mathbb{R}_+^{n-1}$ with a budget constraint where the r-th commodity has price r. "Commodities" in this problem are the differentials $d_r = v_r - v_{r+1}$ between adjacent prizes. Such commodities indeed become more expensive for the principal as r increases because a positive prize differential once introduced for some r has to be carried over to all lower r to preserve the monotonicity of prizes. Coefficients $B_{r,n}$ play the role of (constant) marginal utilities of these commodities. A generic solution to problem (7) is a vertex solution where the entire budget is allocated to one commodity that yields the highest marginal utility per dollar spent. We, therefore, arrive at the following result.

Proposition 1 *The optimal prize allocation is a two-prize schedule, with* $v_1 = \ldots = v_{r^*} = \frac{1}{r^*}$ *and* $v_{r^*+1} = \ldots = v_n = 0$, *where the location* r^* *of a positive prize differential is* *given by*

$$r^* \in \arg\max_{r\in\{1,\dots,n-1\}} \frac{B_{r,n}}{r}. \tag{8}$$

The optimal prize schedule described by Proposition 1 allocates equal prizes to the top r^* players. From Eq. (2), the resulting equilibrium effort is given by the equation $\frac{B_{r^*,n}}{r^*} = c'(e^*)$. The location of r^* is determined entirely by coefficients $B_{r,n}$, Eq. (6), i.e., by the properties of the distribution of noise. Generically, the optimal prize schedule is unique.

The optimal location r^* of the prize differential, Eq. (8), maximizes the objective $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r} = \frac{1}{r}\sum_{k=1}^{r}\beta_{k,n}$, which is the running *average* of coefficients $(\beta_{1,n},\dots,\beta_{r,n})$. As discussed previously, if $\beta_{r,n}$ is maximized at $r=1$, this implies immediately that $r^*=1$. Similarly, as long as $\beta_{r,n}$ is increasing, $\bar{\beta}_{r,n}$ is also increasing. Thus, if $\beta_{r,n}$ is increasing for all $r=1,\dots,n-1$, we have $r^*=n-1$. However, if $\beta_{r,n}$ is interior unimodal, r^* is located at or to the right of the maximum of $\beta_{r,n}$. The exact location of r^* for a nonmonotone $\beta_{r,n}$ depends on the details of the distribution of noise. This dependence is studied in the next section.

3.2 A characterization of optimal prize allocations

From Eq. (6), coefficients $B_{r,n}$ can be written in the form

$$B_{r,n} = \int_0^1 m(z)dF^B(z;n-r,r) = \mathrm{E}(m(Z_{(n-r:n-1)})), \tag{9}$$

where $F^B(z;x,y)$ is the cdf of the beta distribution with parameters (x,y),[8] and $Z_{(n-r:n-1)}$ is the $(n-r)$-th order statistic among $n-1$ i.i.d. draws from the uniform distribution on $[0,1]$. As shown by Ryvkin and Drugov (2017), expectations of the form (9) preserve the unimodality of function $m(z)$ under additional log-supermodularity conditions. As we show, these conditions are satisfied by the regularized incomplete beta function, leading to the following properties of $B_{r,n}$ (all missing proofs are relegated to Appendix).

Lemma 1 *For any $1\le r\le n-1$,*
(i) If $f(t)$ is increasing (decreasing) then $B_{r,n}$ is decreasing (increasing) in r and increasing (decreasing) in n.
(ii) If $f(t)$ is unimodal (U-shaped) then $B_{r,n}$ is unimodal (U-shaped) in r and in n.

[8]Function $F^B(z;x,y)$ is also known as the regularized incomplete beta function (see, e.g., Paris, 2010).

(iii) If $f(t)$ is log-concave then $B_{r,n}$ is concave in r.
(iv) If $f(t)$ is symmetric then $B_{r,n}$ is symmetric in r, with $B_{r,n} = B_{n-r,n}$.

Part (i) of Lemma 1 follows immediately from (9) by the FOSD ordering of order statistics. The most general property, part (ii), is implied by Lemma 1 of Ryvkin and Drugov (2017) on the preservation of unimodality of expectations of unimodal functions.[9] In order to prove part (iii), we note that the log-concavity of $f(t)$ implies that $m(z)$ is concave; and via integration by parts it can be shown that $\beta_{r,n} = B_{r,n} - B_{r-1,n}$ is decreasing in r. Finally, part (iv) follows directly from Eq. (6) (cf. also Balafoutas et al., 2017).

Next, we turn to the analysis of $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r}$, the objective of problem (8) that determines the location r^* of the positive prize differential in the optimal allocation of prizes. Using (6), we write it in the form

$$\bar{\beta}_{r,n} = \frac{1}{n}\int_0^1 \frac{m(z)}{1-z} dF^B(z; n-r, r+1) = \frac{1}{n}\mathrm{E}(h(Z_{(n-r:n)})), \tag{10}$$

where $h(z) = \frac{m(z)}{1-z}$ is the hazard quantile function (Nair, Sankaran and Balakrishnan, 2013), which is a quantile representation of failure (or hazard) rate $\frac{f(t)}{1-F(t)}$ in duration analysis. We will refer to distributions as having increasing failure rate (IFR) if $h(z)$ is increasing and decreasing failure rate (DFR) if $h(z)$ is decreasing. The exponential distribution with pdf $f(t) = \lambda \exp(-\lambda t)$ is the only distribution with a constant failure rate (equal λ). Representation (10) immediately leads to the following properties of $\bar{\beta}_{r,n}$.

Lemma 2 *(i) If $f(t)$ is IFR (DFR) then $\bar{\beta}_{r,n}$ is decreasing (increasing) in r for $r = 1, \ldots, n-1$.*
(ii) If $f(t)$ has a unimodal (U-shaped) failure rate then $\bar{\beta}_{r,n}$ is unimodal (U-shaped) in r for $r = 1, \ldots, n-1$.

Parts (i) and (ii) of Lemma 2 follow from the same arguments as the corresponding parts of Lemma 1 applied to the failure rate. Part (i) leads to the first major result of this section, which is the next proposition.

Proposition 2 *(i) If $f(t)$ is IFR then $r^* = 1$, i.e., the WTA tournament is optimal.*
(ii) If $f(t)$ is DFR then $r^* = n-1$, i.e., it is optimal to award prize $\frac{1}{n-1}$ to all but the very last player.
(iii) If $f(t)$ is an exponential distribution, any allocation of prises with $v_n = 0$ is optimal.

[9] For completeness, we provide the lemma and accompanying definitions in the Appendix, cf. Lemma 3.

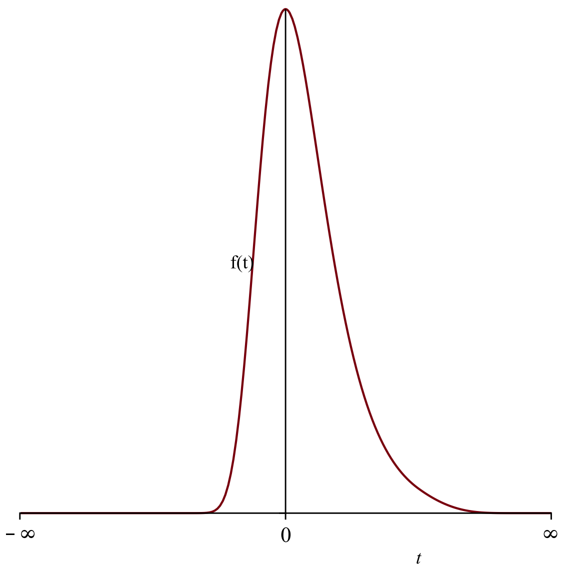
Many standard distributions fall into one of the monotone failure rate classes covered by Proposition 2. In particular, distributions with increasing or log-concave pdfs are IFR.

Corollary 1 *The WTA tournament is optimal if $f(t)$ is increasing or log-concave.*

Note that Corollary 1 follows independently also from parts (i) and (iii) of Lemma 1. Indeed, when $B_{r,n}$ is decreasing or concave in r, the ratio $\frac{B_{r,n}}{r}$ is decreasing and hence the WTA tournament is optimal.

The role of failure rate in the optimal allocation of prizes can be understood intuitively as follows. Coefficients $B_{r,n}$, Eq. (9), can be rewritten in the form $B_{r,n} = \int_U f(t) f_{(n-r:n-1)}(t)dt = f_{X-X_{(n-r:n-1)}}(0)$, i.e., they represent the density at zero of the difference between idiosyncratic noise X and the r-th largest order statistic out of $n-1$ i.i.d. copies of X. Indeed, in order to achieve a rank of at least r in the symmetric equilibrium, a player's noise realization must exceed $X_{(n-r:n-1)}$. When top r ranks are awarded with equal prizes, it is sufficient to achieve a rank of at least r to win the prize; however, the value of the prize, $\frac{1}{r}$, is decreasing in r. Suppose first that pdf $f(t)$ is increasing. In this case, as r increases and $X_{(n-r:n-1)}$ shifts (probabilistically) to the left, the density $f_{X-X_{(n-r:n-1)}}(0)$ decreases. Combined with the reduction in the prize value, this implies the overall negative effect of r on the equilibrium effort; hence, $r = 1$ is optimal. At the same time, when pdf $f(t)$ is decreasing or has a decreasing upper tail, an increase in r leads to two competing effects: while the prize value still decreases, the density $f_{X-X_{(n-r:n-1)}}(0)$ increases as it becomes easier to surpass $X_{(n-r:n-1)}$. Which of the two effects dominates is determined by the behavior of $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r}$ given by (10). Further, Eq. (10) can be rewritten as $\bar{\beta}_{r,n} = \frac{1}{n}\int_U f(t|X \geq t) f_{(n-r:n)}(t)dt$, where the failure rate $\frac{f(t)}{1-F(t)}$ is written as the density at t of variable X conditional on $X \geq t$. Thus, $\bar{\beta}_{r,n}$ is determined by the density at zero of the difference between X and $X_{(n-r:n)}$ conditional on $X \geq X_{(n-r:n)}$. Indeed, the probability of reaching a rank of at least r can be expressed as the probability of surpassing the r-th highest noise realizations out of n conditional on X being among the top r realizations, multiplied by the probability that X is in the top r (equal $\frac{1}{r}$). This explains why, as r increases, $\bar{\beta}_{r,n}$ changes according to the shape of the failure rate, similar to how $B_{r,n}$ changes according to the shape of the pdf.

As an example of an IFR distribution (which is also log-concave), consider the Gumbel distribution with parameter 1, whose pdf $f(t) = \exp[-t - \exp(-t)]$ is shown in the left panel of Figure 1. This distribution generates the contest success function (CSF) of the Tullock contest (Jia, Skaperdas and Vaidya, 2013), for which the symmetric equilibrium with multiple prizes was identified by Clark and Riis (1996) and Fu and Lu (2012). The



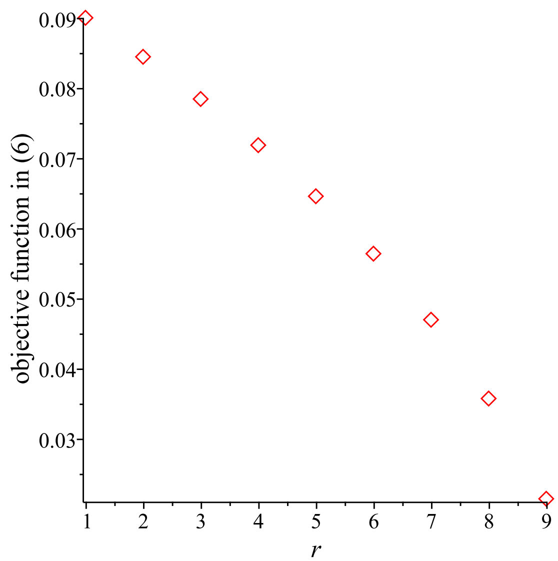


Figure 1: Gumbel distribution with parameter 1. *Left*: Pdf $f(t)$. *Right*: The objective $\bar{\beta}_{r,n}$ for $n = 10$ is maximized at $r = 1$.

right panel in Figure 1 shows the objective $\bar{\beta}_{r,n}$ of problem (8) as a function of r. As seen from the figure, $r = 1$ is indeed optimal. The optimality of the WTA prize schedule for Tullock contests was demonstrated by Schweinzer and Segev (2012). Proposition 2 generalizes this result to arbitrary tournaments with IFR distributions of noise.

For an example of a DFR distribution, consider the Pareto distribution with parameters $(1, 1)$, with pdf $f(t) = \frac{1}{t^2} I_{t \geq 1}$, for which $m(z) = (1-z)^2$ and $\bar{\beta}_{r,n} = \frac{r+1}{n(n+1)}$ is increasing in r. Hence, it is optimal to reward every player but the very last, see Figure 2.

Finally, for an example of a distribution with an interior unimodal failure rate consider the Cauchy distribution with parameters $(0, 1)$ which has pdf $f(t) = \frac{1}{\pi(t^2+1)}$ and the inverse quantile density $m(z) = \frac{1}{\pi[\tan^2(\pi(z-\frac{1}{2}))+1]}$. Then, a two-prize schedule with an interior r^* is optimal, see Figure 3.

When pdf $f(t)$ is symmetric (but not necessarily unimodal or with a well-behaved failure rate), part (iv) of Lemma 1 shows that $B_{r,n}$ is symmetric. In this case, the optimal location r^* of the positive prize differential can be restricted as described in following proposition.

Proposition 3 *The optimal prize allocation is a two-prize winner tournament, with $r^* < \frac{n}{2}$, if $f(t)$ is symmetric.*

We refer to prize schedules with $r^* < \frac{n}{2}$ as "winner tournaments" (cf. Akerlof and Holden, 2012) since they reward relatively few top-performing players. In contrast, "loser

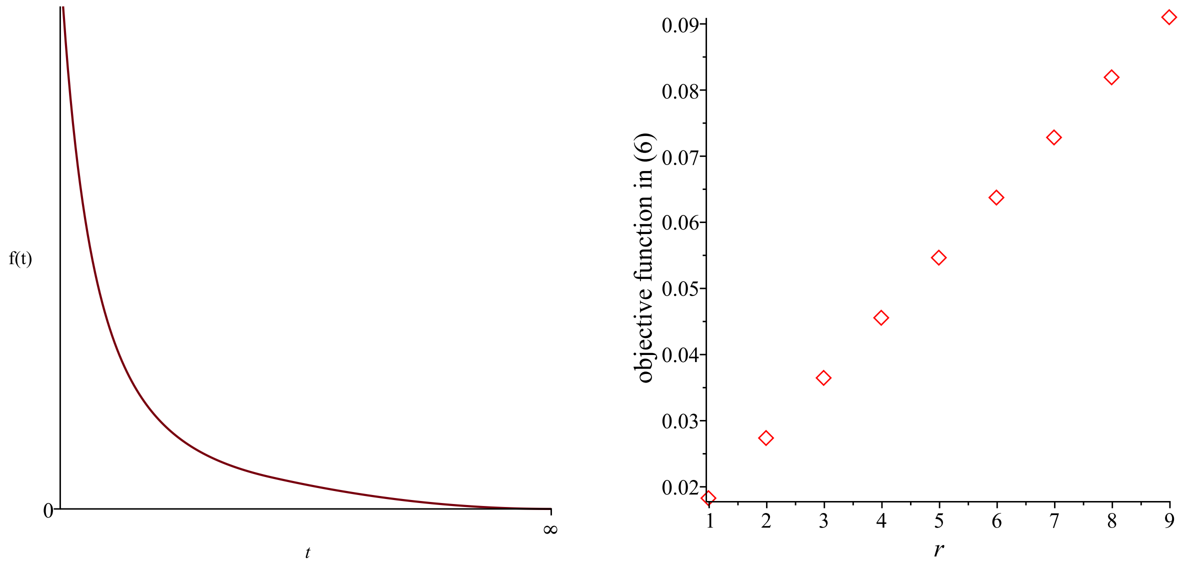


Figure 2: The Pareto distribution with parameters $(1, 1)$. *Left*: Pdf $f(t)$. *Right*: The objective $\bar{\beta}_{r,n}$ for $n = 10$ is maximized at $r = 9$.

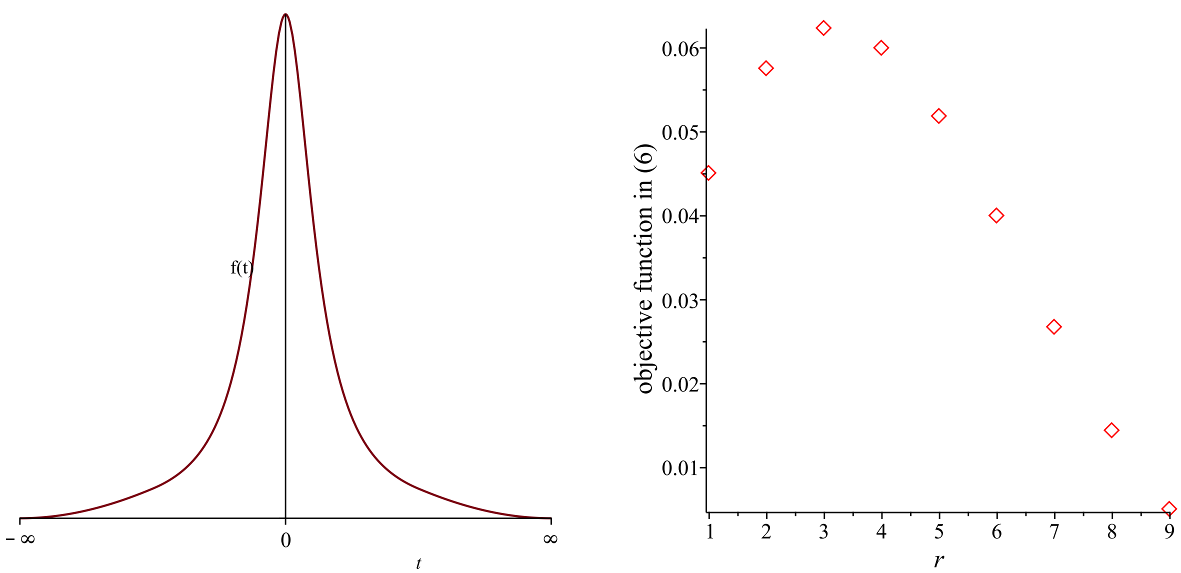


Figure 3: The Cauchy distribution with parameters $(0, 1)$. *Left*: Pdf $f(t)$. *Right*: The objective $\bar{\beta}_{r,n}$ for $n = 10$ is maximized at $r = 3$.

tournaments," i.e., those with $r^* > \frac{n}{2}$, reward most players and hence can be thought of as emphasizing punishment of low performers. One extreme example is the case of Pareto distribution, Figure 2, where only the bottom performer is punished.

Lemmas 1 and 2 also allow us to characterize the optimal allocation of prizes in the absence of the monotonicity constraint $v_1 \geq \ldots \geq v_n$. Recall that in this case it is optimal to allocate the entire prize to the rank r that maximizes $\beta_{r,n}$.

Corollary 2 *When prizes are not restricted to be monotone in rank, it is optimal to allocate the entire prize to rank $\hat{r}$ with the following properties.*
(i) If $f(t)$ is IFR then $\hat{r} = 1$.
(ii) If $f(t)$ is unimodal and symmetric then $\hat{r} < \frac{n}{2}$.
(iii) If $f(t)$ is U-shaped and symmetric then either $\hat{r} = 1$ or $\hat{r} > \frac{n}{2}$.

Part (i) of Corollary 2 follows from part (i) of Lemma 2. Indeed, suppose $f(t)$ is IFR and hence $\bar{\beta}_{r,n}$ is decreasing in r for $r \leq n-1$. Notice that $\bar{\beta}_{r,n} \leq \bar{\beta}_{r-1,n}$ implies $\beta_{r,n} \leq \bar{\beta}_{r-1,n}$. But $\bar{\beta}_{r-1,n} \leq \bar{\beta}_{1,n} = \beta_{1,n}$ for any $r = 2, \ldots, n$, which gives $\beta_{r,n} \leq \beta_{1,n}$. Part (ii) of the corollary follows from parts (ii) and (iv) of Lemma 1 because $\beta_{r,n}$ is single-crossing $+-$ when $B_{r,n}$ is unimodal. Part (iii) follows similarly because $\beta_{r,n}$ is single-crossing $-+$ for $r = 2, \ldots, n-1$ when $B_{r,n}$ is U-shaped.

Propositions 2 and 3 can be compared to the results of Krishna and Morgan (1998), who explored tournaments of $n \leq 4$ players assuming that the pdf of noise is symmetric and unimodal. For risk-neutral players, Krishna and Morgan (1998) show that the WTA schedule is optimal for $n = 2, 3$ and 4. These results follow directly from our Proposition 3, since for $n = 4$ fewer than $\frac{n}{2} = 2$ prizes are to be awarded. Note that the unimodality of $f(t)$ is not needed.[10] Additionally, Proposition 2 shows the optimality of the WTA schedule for arbitrary n when the distribution of noise is IFR (such distributions are not necessarily unimodal or symmetric).

Next, we identify an ordering of noise distributions in terms of their effect on the location r^* of the optimal prize differential. We will use $m_X(z)$ and r^*_X to denote, respectively, the inverse quantile density and r^* corresponding to a random variable X.

Proposition 4 *Suppose Y has a unimodal failure rate. Consider another random variable X such that $\rho(z) = \frac{m_X(z)}{m_Y(z)}$ is increasing. Then $r^*_X \leq r^*_Y$.*

[10] It can be shown that $n = 4$ is a tight bound for when the symmetry of $f(t)$ leads to the optimality of WTA. That is, already for $n = 5$ a symmetric (and unimodal) distribution of noise can be found such that the WTA schedule is not optimal. For example, $r^* = 2$ for the t-distribution with 0.5 degrees of freedom.

Proposition 4 states conditions under which the location of the optimal prize differential shifts to higher ranks. When Y is DFR, we have $r_Y^* = n-1$ and the result holds automatically. For the case when Y is IFR or has an interior unimodal failure rate, it can be shown that if $\bar{\beta}_{r,n}$ decreases in r for some r when the noise is Y then it also decreases in r when the noise is X (see Appendix). This implies, since $\bar{\beta}_{r,n}$ is unimodal for Y due to Lemma 2, that the maximum of $\bar{\beta}_{r,n}$ for X is (weakly) to the left of the maximum for Y. In particular, under the conditions of the proposition, when Y is IFR and the WTA tournament is optimal for Y then it is also optimal for X.

Corollary 3 *Suppose Y is IFR and $\rho(z) = \frac{m_X(z)}{m_Y(z)}$ is increasing. Then the WTA tournament is optimal for both X and Y.*

An increasing ratio $\rho(z) = \frac{m_X(z)}{m_Y(z)}$ is equivalent to X being smaller than Y in the *convex transform order*, defined by the property that $F_Y^{-1} \circ F_X$ is convex (Shaked and Shanthikumar, 2007). Note that being smaller than the exponential distribution in the convex transform order is equivalent to IFR; therefore, Corollary 3 is related to Proposition 2. However, in general, neither X nor Y have to be IFR for the convex transform order to lead to a shift in r^*, as described in Proposition 4.

4 Comparative statics and applications

In this section we apply the results from Section 3 to different but related settings. In Section 4.1, we identify a natural ordering of general prize schedules (i.e., not necessarily two-prize ones) that leads to the ordering of equilibrium effort. In section 4.2, we analyze a two-prize but not budget-balanced schedule. In Section 4.3, we show that awarding a stochastic number of prizes is never (strictly) optimal. In Section 4.4, we characterize the optimal prize allocation when the number of players is stochastic. In Section 4.5 we apply our findings to tournaments for status, i.e., tournaments in which, following Moldovanu, Sela and Shi (2007), prizes are proportional to the difference between the number of players ranked strictly below and above a given player. Finally, in Section 4.6 we analyze the effect of noise dispersion.

4.1 General prize schedules

As we showed in the previous Section 3, two-prize schedules of the form $v_1 = \ldots = v_r = \frac{1}{r}$, $v_{r+1} = \ldots = v_n = 0$ are optimal for a principal maximizing aggregate effort. It may be, however, that two-prize schedules are not feasible for institutional or political reasons. For example, in tournaments for status the number of status categories can be predetermined by an organizational or social structure. More generally, the principal may be restricted to only using prize schedules from a certain class, $\Gamma = \{v(\theta)\}_{\theta\in\Theta}$, where $v(\theta) = (v_1(\theta), \ldots, v_n(\theta))$ denotes a prize schedule, and parameter θ can be continuous or discrete and takes values in some set $\Theta \subseteq \mathbb{R}$. For example, Γ may be all linearly decreasing schedules such that $v_{r+1} = v_r - \theta$, or exponentially decreasing schedules such that $v_{r+1} = \frac{v_r}{1+\theta}$. Indeed, bonuses and pay raises in organizations may be structured as linear or percentage increases between ranks.

In this section, we explore how individual (and hence aggregate) equilibrium effort is affected by changes in a general prize schedule, i.e., how e^* given by Eq. (2) responds to changes in θ. We maintain the budget constraint and monotonicity assumptions, $\sum_{r=1}^{n} v_r(\theta) = 1$ and $v_1(\theta) \geq \ldots \geq v_n(\theta) \geq 0$, for all $\theta \in \Theta$.

Under our assumptions, a vector of prizes $v(\theta)$ can be thought of as a probability mass function (pmf) of a random variable K_θ taking values $1, \ldots, n$ such that $\Pr(K_\theta = r) = v_r(\theta)$. Let $V_r(\theta) = \sum_{k=1}^{r} v_k(\theta)$ denote the corresponding cumulative mass function (cmf), which is increasing in r and satisfies $V_n(\theta) = 1$. Equation (2) then can be written as $c'(e^*) = A(\theta) \equiv \mathrm{E}(\beta_{K_\theta,n})$, where the expectation is taken with respect to random variable K_θ.

From part (iii) of Lemma 1, $\beta_{r,n} = B_{r,n} - B_{r-1,n}$ is decreasing if $f(t)$ is log-concave. The following result is a direct application of first-order stochastic dominance (FOSD).

Proposition 5 *Suppose $f(t)$ is log-concave and $V_r(\theta)$ is increasing in θ for all $r = 1, \ldots, n$. Then $A(\theta)$ (and hence e^*) is increasing in θ.*

Indeed, consider some $\theta' > \theta$ and suppose $V_r(\theta') \geq V_r(\theta)$ for all $r = 1, \ldots, n$. This implies that K_θ dominates $K_{\theta'}$ in the FOSD order and, therefore, $\mathrm{E}(\beta_{K_{\theta'},n}) \geq \mathrm{E}(\beta_{K_\theta,n})$ when $\beta_{r,n}$ is decreasing in r. This result is rather intuitive. We already know from Corollary 1 that the WTA tournament maximizes effort for log-concave distributions. The (downward) FOSD shift in K_θ produces a more sharply decaying prize schedule with larger differentials at the top, making it more similar to the WTA schedule and thus leading to a higher equilibrium effort.

To illustrate Proposition 5, consider a class of linearly decreasing prize schedules, $v_r(\theta) = \frac{1}{n} + \theta\left(\frac{n+1}{2} - r\right)$, where $\theta \in \Theta = \left(0, \frac{2}{n(n-1)}\right]$. A higher θ makes the decline in prizes steeper, the normalization $\sum_{r=1}^{n} v_r(\theta) = 1$ is maintained, and the restriction $\theta \leq \frac{2}{n(n-1)}$ ensures that $v_n \geq 0$. The cmf for this prize schedule, $V_r(\theta) = \frac{r}{n} + \frac{\theta r(n-r)}{2}$, is increasing in

θ, implying that an increase in θ leads to a downward FOSD shift in the distribution of prizes across ranks. It then follows from Proposition 5 that, assuming $f(t)$ is log-concave, $A(\theta)$ (and hence e^*) is increasing in θ. The results are similar for a class of exponentially decreasing prize schedules, with $v_r(\theta) = \frac{\theta(1+\theta)^{n-r}}{(1+\theta)^n-1}$, where cmf $V_r(\theta) = \frac{(1+\theta)^n-(1+\theta)^{n-r}}{(1+\theta)^n-1}$ is also increasing in θ.

4.2 Schedules with k fixed equal prizes

Consider now a setting where the principal awards a fixed number, k, of equal prizes $w > 0$ to the top $k \geq 1$ performers in a tournament of $n > k$ players. For example, a Ph.D. program may offer a certain number of scholarships each year, or a certain number of top performers in an organization may receive a fixed bonus. In this section, we assume that there is no budget constraint and explore the effects of the number of prizes, k, as well as of the total number of tournament participants, n, on the equilibrium effort.

The schedule with $k < n$ equal prizes w is a two-prize schedule with $v_1 = \ldots = v_k = w$ and $v_{k+1} = \ldots = v_n = 0$.[11] The first-order condition (2) for the symmetric equilibrium effort in this case takes the form $c'(e^*) = wB_{k,n}$. Lemma 1 then immediately leads to the following comparative statics.

Corollary 4 *In tournaments with $k < n$ fixed equal prizes,*
(i) If $f(t)$ is increasing (decreasing) then e^* is decreasing (increasing) in k and increasing (decreasing) in n.
(ii) If $f(t)$ is unimodal then e^* is unimodal in k and in n.
(iii) If $f(t)$ is symmetric then e^* is symmetric in k (that is, e^* with k prizes is the same as with $n-k$ prizes).
(iv) If $f(t)$ is log-concave then $c'(e^*)$ is concave in k.

Suppose n is fixed, and the principal decides how many equal prizes to award in order to maximize the aggregate effort. As shown above, the optimal number of prizes is $k^* \in \arg\max_{r\in\{1,\ldots,n-1\}} B_{r,n}$. This leads to the following results.

Corollary 5 *Suppose the principal chooses how many fixed equal prizes to award. Then,*
(i) If $f(t)$ is increasing (decreasing) then it is optimal to award one top prize ($n-1$ top prizes).

[11] The loser prize is set to zero without loss of generality. Any fixed loser prize $w' < w$ would lead to the same results with w replaced by $w - w'$.

(ii) If $f(t)$ is unimodal and symmetric then it is optimal to award $\frac{n}{2}$ top prizes if n is even and $\frac{n-1}{2}$ or $\frac{n+1}{2}$ top prizes if n is odd.

It may also be of interest to consider a more general setting without the monotonicity constraint, when the principal can award k equal fixed prizes to any ranks $r = 1, \ldots, n-1$. The equilibrium effort is given by (2); therefore, it is optimal to award a prize to all ranks r such that $\beta_{r,n} > 0$. Recall that $\beta_{r,n} = B_{r,n} - B_{r-1,n}$; hence, parts (i) and (ii) of Corollary 5 still hold in this case. Moreover, whenever $f(t)$ is unimodal, the optimal prize schedule is monotone; that is, if rank r is rewarded then all ranks $1, \ldots, r-1$ are rewarded as well.

For example, for the Gumbel distribution with parameter λ, i.e., the Tullock contest with discriminatory power λ,

$$\beta_{r,n} = \frac{\lambda}{n}\left[\frac{n^2 - nr + r}{n(n-r)} - \sum_{k=0}^{r-1} \frac{1}{n-r+k}\right].$$

The Gumbel distribution is log-concave and hence, by Lemma 1, $\beta_{r,n}$ is decreasing in r. Moreover, $\beta_{\frac{n}{2},n} > 0$ implying that at least half of the ranks should be rewarded. Moreover, $\beta_{\frac{n}{2}+1,n} > 0$ for $n \geq 6$, implying that more than half of the ranks should be rewarded if there are at least six players.

As seen from the example in Section 3, $n-1$ prizes should be awarded if the distribution of noise is Pareto.

4.3 Schedules with a stochastic number of prizes

Can it be optimal to make the number of prizes, k, stochastic? Let K denote the corresponding random variable taking values $1, \ldots, n-1$ and following some probability mass function (pmf) $q = (q_1, \ldots, q_{n-1})$, where $q_k = \Pr(K = k)$ is the probability that k prizes are awarded, with $\sum_{k=1}^{n-1} q_k = 1$.

Suppose first that the budget constraint $\sum_{r=1} v_r = 1$ is imposed and the principal awards k equal prizes. The first-order condition for equilibrium effort, Eq. (2), then becomes

$$\sum_{k=1}^{n-1} q_k \frac{B_{k,n}}{k} = c'(e^*). \tag{11}$$

In order to maximize the left-hand side of (11), we need to set $q_k = 1$ at k that maximizes $\frac{B_{k,n}}{k}$; that is, r^* from Proposition 1 is the solution, and there should be r^* prizes given with certainty. We conclude that there is no reason to introduce uncertainty over the number of prizes in this setting.

Consider now a setting without a budget constraint, assuming k equal fixed prizes w are awarded. The first-order condition then takes the form

$$w\sum_{k=1}^{n-1} q_k B_{k,n} = c'(e^*).$$

Again, the effort-maximizing choice of q involves no uncertainty, with $q_k = 1$ for k that maximizes $B_{k,n}$. Corollary 5 then applies regarding the choice of k.

4.4 Two-prize schedules with a stochastic number of players

Suppose the principal awards k equal prizes but the number of players in the tournament, n, is stochastic. Let N denote the corresponding random variable taking values $0, 1 \ldots, M$, where the upper limit M can be finite or infinite. We will assume that $M > k$; that is, with a positive probability the tournament is competitive. Let $p = (p_0, p_1, \ldots, p_M)$ denote the pmf of N, where $p_n = \Pr(N = n)$ and $\sum_{n=0}^{M} p_n = 1$.

Consider first a setting with the budget constraint where top k performers receive prize $\frac{1}{k}$. From the perspective of a participating player, the distribution of the number of players in the tournament is updated as $\tilde{p}_n = \frac{p_n n}{\bar{n}}$, where $\bar{n} = \sum_{n=0}^{M} p_n n$ is the expected number of participants in the tournament (which we assume is finite).[12] The first-order condition (2) for symmetric equilibrium effort then takes the form

$$\frac{1}{k}\sum_{n=1}^{M} \tilde{p}_n B_{k,n} = c'(e^*). \tag{12}$$

Let $\{\tilde{p}(\theta)\}_{\theta\in\Theta}$ denote a family of tournament size pmfs parameterized by $\theta \in \Theta \subseteq \mathbb{R}$. We will assume that an increase in θ leads to an FOSD increase in N; that is, the cmf $\tilde{P}_n(\theta) = \sum_{l=1}^{n} \tilde{p}_l(\theta)$ is decreasing in θ for all n. We will use $\tilde{P}_n'(\theta)$ to denote the derivative (if θ is a continuous parameter; in this case we will assume that $\tilde{P}_n(\theta)$ is differentiable) or the first difference (if θ is a discrete index) of $\tilde{P}_n(\theta)$ with respect to θ.

Proposition 6 *Suppose $P_n'(\theta) \leq 0$. Then, in the setting with a budget constraint,*
(i) If $f(t)$ is increasing then e^* is increasing in θ.

[12]See, e.g., Harstad, Kagel and Levin (1990), Myerson and Wärneryd (2006), Ryvkin and Drugov (2017).

(ii) If $f(t)$ is decreasing and $\tilde{P}_k(\theta) = 0$ (that is, the tournament has at least $k+1$ players) then e^* is decreasing in θ.
(iii) If $f(t)$ is unimodal and $|\tilde{P}'_n(\theta)|$ is log-supermodular in (n, θ) then e^* is unimodal in θ.
(iv) If $f(t)$ is IFR then the optimal number of prizes is $k^* = 1$, i.e., the WTA tournament is optimal for any p.

Consider now a setting without a budget constraint where some number k of fixed prizes w are awarded. The first-order condition for the equilibrium effort then takes the form

$$w \sum_{n=1}^{M} \tilde{p}_n B_{k,n} = c'(e^*). \tag{13}$$

As seen from Eq. (13), the dependence of e^* on n in this setting follows the same patterns as in the setting with the budget constraint; that is, parts (i)-(iii) of Proposition 6 apply.

4.5 Tournaments for status

Consider tournaments for status where a player's reward is modeled as the difference between the number of players ranked below and above her (Moldovanu, Sela and Shi, 2007; Dubey and Geanakoplos, 2010). Assuming the total amount of status in the group is fixed (and normalized to one), and the status rewards are nonnegative, incentives in a symmetric tournament for status can be described by a first-order condition of the form (2) with appropriately defined prizes. The main question in the literature on status is how to partition the set of available ranks $\{1, \dots, n\}$ into categories such that within each category players' ranks are not distinguishable, and the total effort is maximized. For example, in the extreme case of the most refined partition, when each rank r is its own category, the status prize of a player ranked r is $v_r = \alpha[(n-r)-(r-1)]+\beta$. Indeed, $n-r$ is the number of players ranked below the player, while $r-1$ is the number of players ranked above. Here, α and β are constants ensuring that $\sum_{r=1}^{n} v_r = 1$ and $v_n \geq 0$.[13] In the other extreme case when the partition consists only of two categories separated by rank $\hat{r} \leq n-1$, the status prize of players in the top category is $v_1 = \alpha(n-\hat{r}) + \beta$ while in the bottom category $v_2 = \alpha(-\hat{r}) + \beta$.

[13]Thus, our prizes are an affine transformation of the status prizes used by Moldovanu, Sela and Shi (2007), who allow status to be negative and normalize the sum of status prizes is zero. In our model, the nonnegativity of prizes is an important condition; without it, since $\beta_{n,n} < 0$, an arbitrarily high effort can be reached by choosing a large negative bottom prize v_n. Under the nonnegativity assumption, the normalization of the total budget to one is without loss og generality.

The results of this section imply the following properties of effort-maximizing partitions in the tournament for status. Proposition 1 implies that the optimal partition consists of two categories. Proposition 2 shows that if $f(t)$ is IFR (DFR) then the top category includes one $(n-1)$ players. Proposition 3 shows that if $f(t)$ is symmetric then the top category should include fewer than $\frac{n}{2}$ players. Finally, Proposition 4 implies that if Y has a unimodal failure rate and X is smaller than Y in the convex transform order then the optimal partition under X places (weakly) fewer players in the top category than the optimal partition under Y; in particular, if the top category for Y includes one player then so does the top category for X.

Interestingly, Moldovanu, Sela and Shi (2007) also identify the IFR property and the convex transform order of distributions as the main determinants of the structure of optimal partitions, although in their model these properties pertain to the distribution of (heterogeneous and private) types. The reason is that the IFR property is related to the FOSD ordering of spacings – the distances between adjacent order statistics of a distribution. Moldovanu, Sela and Shi (2007) use an all-pay auction model with private i.i.d. types, similar to Moldovanu and Sela (2001), where incentives are determined by expectations of spacings in the distribution of types. In our model, players are symmetric but subjected to i.i.d. shocks, whereas incentives are determined by "quasi-spacings" $\beta_{r,n} = B_{r,n} - B_{r-1,n}$, which are distances between expectations of random variable $m(Z)$ over adjacent order statistics $Z_{(n-r:n-1)}$ and $Z_{(n-r+1:n-1)}$.

4.6 The effect of noise dispersion

In this section, we explore how the equilibrium effort e^* is affected by changes in the noise distribution. Throughout this section, we will use $f_X(t)$, $F_X(t)$ and $m_X(z)$ to denote the pdf, cdf and inverse quantile density of a random variable X. We will also use $B_{r,n}[f, v]$ and $e^*[f, v]$ to denote, respectively, the coefficient $B_{r,n}$ and symmetric equilibrium effort e^* generated by a distribution of noise with pdf f and prize allocation $v = (v_1, \ldots, v_n)$.

Coefficients $B_{r,n}$, Eq. (6), depend on the shape of the distribution of noise through the inverse quantile density function $m(z)$. Ryvkin and Drugov (2017) showed that for WTA tournaments the equilibrium effort decreases with noise when noise increases in the sense of the dispersive order (Lehmann, 1988).

Definition 1 *Random variable X is more dispersed than random variable Y if $F_X^{-1}(z') - F_X^{-1}(z) \geq F_Y^{-1}(z') - F_Y^{-1}(z)$ for all $z, z' \in [0, 1]$, $z' > z$.*

As shown by Shaked and Shanthikumar (2007), if X is more dispersed than Y then $m_X(z) \leq m_Y(z)$ for all $z \in [0,1]$. The first-order condition in the form $c'(e^*) = \sum_{r=1}^{n-1} B_{r,n} d_r$ and Eq. (9) then immediately imply the following result.

Proposition 7 *If X is more dispersed than Y then $e^*[f_Y, v] \geq e^*[f_X, v]$ for any monotone prize schedule v.*

Proposition 7 is a generalization of a similar result by Ryvkin and Drugov (2017) for WTA tournaments. Indeed, for any monotone allocation of prizes, the equilibrium effort increases in $\sum_{r=1}^{n-1} B_{r,n} d_r$, where $d_r = v_r - v_{r+1} \geq 0$ are the differentials between adjacent prizes. An increase in noise, in the sense of the dispersive order, leads to a reduction in all coefficients $B_{r,n}$ and hence to a reduction in e^*.

Moreover, if optimization over prizes is carried out separately for X and Y, the dispersion-based ranking of equilibrium efforts is preserved. Let v_X^{opt} denote the prize allocation that maximizes effort when the noise is X. From Proposition 7, if X is more dispersed than Y then $e^*[f_X, v_X^{\text{opt}}] \leq e^*[f_Y, v_X^{\text{opt}}] \leq e^*[f_Y, v_Y^{\text{opt}}]$.

Corollary 6 *Suppose X is more dispersed than Y and the prize allocations are optimal in each case. Then $e^*[f_Y, v_Y^{\text{opt}}] \geq e^*[f_X, v_X^{\text{opt}}]$.*

A prominent special case satisfying the dispersive order is when additional dispersion is generated by a "stretching" transformation.

Definition 2 *Function $\phi : U \to \mathbb{R}$ is a stretching transformation if it is strictly monotone and for any $x, x' \in U$ such that $x' > x$*

$$|\phi(x') - \phi(x)| \geq x' - x.$$

For differentiable functions, Definition 2 is equivalent to the requirement that $|\phi'(x)| \geq 1$ for all $x \in U$. In this case, if $X = \phi(Y)$, where ϕ is a stretching transformation, we have (cf. Ryvkin and Drugov, 2017)

$$m_X(z) = \frac{1}{|\phi'(F_Y^{-1}(z))|} m_Y(z) \leq m_Y(z), \tag{14}$$

and hence the following result.

Corollary 7 *Suppose $X = \phi(Y)$ where ϕ is a stretching transformation. Then X is more dispersed than Y and hence $e^*[f_Y, v] \geq e^*[f_X, v]$ for any monotone prize schedule v.*

A special case of stretching is scaling, $X = \sigma Y$, with $\sigma > 1$. Let $F(t, \sigma)$ and $f(t, \sigma)$ denote the cdf and pdf of a random variable with a scale parameter $\sigma > 0$ such that $F(t, \sigma) = F\left(\frac{t}{\sigma}, 1\right)$ and $f(t, \sigma) = \frac{1}{\sigma} f\left(\frac{t}{\sigma}, 1\right)$. For the inverse quantile densities, Eq. (14) gives $m_X(z) = \frac{1}{\sigma} m_Y(z)$, implying the following result.

Corollary 8 *Suppose the pdf of noise is $f(t, \sigma)$ where σ is a scale parameter. Then $B_{r,n}[f(t, \sigma), v] = \frac{1}{\sigma} B_{r,n}[f(t, 1), v]$, and $e^*[f(t, \sigma), v]$ is decreasing in σ for any monotone prize schedule v.*

Examples of scale parameters include the width of the uniform distribution, the standard deviation of the zero-mean normal distribution, the expected value of the exponential distribution, and the scale of the Gumbel distribution (and hence $\frac{1}{\lambda}$, where λ is the "discriminatory power" of the Tullock contest).

5 Conclusions

The exploration we have undertaken in this paper can be viewed as a study of several aspects of tournament design. Tournaments are a common occurrence in organizations, political institutions and other social settings; it is, therefore, important to understand how various parameters of the tournament mechanism, most notably the allocation of prizes, affect agents' behavior in tournaments. Our results contribute to the discussion of tournament prize allocation rules in the literature, as well as in the business world, revolving around the issue of whether or not, and under what circumstances, the winner-take-all (WTA) rule is optimal, or, more generally, how "steep" the profile of bonuses must be at the top.

We use a version of the canonical model of Lazear and Rosen (1981) where symmetric and risk-neutral agents exert costly efforts distorted by idiosyncratic shocks, and the principal allocates a fixed budget to award prizes based solely on output ranks. We show that, generically, a unique optimal (effort-maximizing) prize allocation emerges, which involves equal prizes awarded to some number of top performers. For a wide class of noise distributions known as increasing failure rate (IFR), the optimal allocation of prizes is WTA. More generally, we characterize optimal prize allocations for noise distributions with unimodal failure rates and show that the optimal prize schedule becomes "more like WTA" when noise distributions can be ranked by the convex transform order.

The methodology used in this paper is an extension of the techniques developed by Ryvkin and Drugov (2017) for the analysis of WTA tournaments. Note that while the robust comparative statics of WTA tournaments revolve mainly around the unimodality of the pdf of noise and its preservation under uncertainty, the robust properties of optimal prize allocations identified in this paper are related to the unimodality of the failure (or hazard) rate. Nevertheless, once the appropriate class of unimodal functions has been found, the techniques turned out to be easily transferable. The methods, therefore, show promise in applications to other problems as well.

References

Akerlof, Robert J., and Richard T. Holden. 2012. "The nature of tournaments." *Economic Theory*, 51(2): 289–313.

Balafoutas, Loukas, E. Glenn Dutcher, Florian Lindner, and Dmitry Ryvkin. 2017. "The optimal allocation of prizes in tournaments of heterogeneous agents." *Economic Inquiry*, 55(1): 461–478.

Clark, Derek J., and Christian Riis. 1996. "A multi-winner nested rent-seeking contest." *Public Choice*, 87(1-2): 177–184.

Connelly, Brian L., Laszlo Tihanyi, T. Russell Crook, and K. Ashley Gangloff. 2014. "Tournament Theory: Thirty Years of Contests and Competitions." *Journal of Management*, 40(1): 16–47.

Dubey, Pradeep, and John Geanakoplos. 2010. "Grading exams: 100, 99, 98,... or A, B, C?" *Games and Economic Behavior*, 69(1): 72–94.

Fu, Qiang, and Jingfeng Lu. 2012. "Micro foundations of multi-prize lottery contests: a perspective of noisy performance ranking." *Social Choice and Welfare*, 38(3): 497–517.

Green, Jerry R., and Nancy L. Stokey. 1983. "A comparison of tournaments and contracts." *The Journal of Political Economy*, 349–364.

Harstad, Ronald M., John H. Kagel, and Dan Levin. 1990. "Equilibrium bid functions for auctions with an uncertain number of bidders." *Economics Letters*, 33(1): 35–40.

Jia, Hao. 2008. "A stochastic derivation of the ratio form of contest success functions." *Public Choice*, 135(3-4): 125–130.

Jia, Hao, Stergios Skaperdas, and Samarth Vaidya. 2013. "Contest functions: Theoretical foundations and issues in estimation." *International Journal of Industrial Organization*, 31(3): 211 – 222.

Krishna, Vijay, and John Morgan. 1998. "The winner-take-all principle in small tournaments." *Advnaces in Applied Microeconomics*, 7: 61–74.

Lazear, Edward P. 1999. "Personnel economics: Past lessons and future directions. Presidential address to the society of labor economists, San Francisco, May 1, 1998." *Journal of Labor Economics*, 17(2): 199–236.

Lazear, Edward P., and Sherwin Rosen. 1981. "Rank-order tournaments as optimum labor contracts." *Journal of Political Economy*, 89(5): 841–864.

Lehmann, Erich L. 1988. "Comparing location experiments." *The Annals of Statistics*, 16(2): 521–533.

McLaughlin, Kenneth J. 1988. "Aspects of tournament models: A survey." *Research in Labor Economics*, 9(1): 225–56.

Moldovanu, Benny, and Aner Sela. 2001. "The optimal allocation of prizes in contests." *American Economic Review*, 542–558.

Moldovanu, Benny, Aner Sela, and Xianwen Shi. 2007. "Contests for status." *Journal of Political Economy*, 115(2): 338–363.

Myerson, Roger B, and Karl Wärneryd. 2006. "Population uncertainty in contests." *Economic Theory*, 27(2): 469–474.

Nair, N. Unnikrishnan, P. G. Sankaran, and N. Balakrishnan. 2013. "Quantile-based reliability concepts." In *Quantile-Based Reliability Analysis*. 29–58. Springer.

Nalebuff, Barry J., and Joseph E. Stiglitz. 1983. "Prizes and incentives: towards a general theory of compensation and competition." *The Bell Journal of Economics*, 21–43.

Paris, R.B. 2010. "Incomplete beta functions." *NIST Handbook of Mathematical Functions*, , ed. Frank W. J. Olver, Daniel M. Lozier, Ronald F. Boisvert and Charles W. Clark. Cambridge University Press. http://dlmf.nist.gov/8.17.

Parzen, Emanuel. 1979. "Nonparametric statistical data modeling." *Journal of the American Statistical Association*, 74(365): 105–121.

Prendergast, Canice. 1999. "The Provision of incentives in firms." *Journal of Economic Literature*, 37(1): 7–63.

Ryvkin, Dmitry, and Mikhail Drugov. 2017. "Winner-take-all tournaments." *Working paper.* http://myweb.fsu.edu/dryvkin/LR_stochastic_22.pdf.

Schweinzer, Paul, and Ella Segev. 2012. "The optimal prize structure of symmetric Tullock contests." *Public Choice*, 153(1-2): 69–82.

Shaked, Moshe, and J. George Shanthikumar. 2007. *Stochastic orders.* Springer-Verlag New York.

Sisak, Dana. 2009. "Multiple-prize contests – the optimal allocation of prizes." *Journal of Economic Surveys*, 23(1): 82–114.

Tullock, Gordon. 1980. "Efficient rent seeking." *Toward a Theory of the Rent-Seeking Society*, , ed. James M. Buchanan, Robert D. Tollison and Gordon Tullock, 97–112. College Station:Texas A&M University Press.

Appendix

A.1 Preservation of unimodality under uncertainty

In this section, we present a condensed version of the relevant result from Ryvkin and Drugov (2017).

Definition 3 *Function $\phi : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$, is unimodal if there exists a $\hat{t} \in U$ such that $\phi(t)$ is increasing (decreasing) for $t \in U \cap (-\infty, \hat{t}]$ ($t \in U \cap [\hat{t}, \infty)$). Function ϕ is interior unimodal if it is unimodal and nonmonotone. Function ϕ is (interior) U-shaped if $-\phi$ is (interior) unimodal.*

Definition 4 *Function $\psi : S_1 \times S_2 \to \mathbb{R}$, where $S_1, S_2 \subseteq \mathbb{R}$, is log-supermodular if for any $t_1, t_1' \in S_1$, $t_2, t_2' \in S_2$ such that $t_1' > t_1$, $t_2' > t_2$,*

$$\psi(t_1, t_2')\psi(t_1', t_2) \leq \psi(t_1, t_2)\psi(t_1', t_2'),$$

or, equivalently, the ratio $\frac{\psi(t_1', t_2)}{\psi(t_1, t_2)}$ is increasing in t_2.

Suppose $u(z) : [0, 1] \to \mathbb{R}$ is an integrable, continuous and piece-wise differentiable function and $H(z, \theta)$ is a cdf of a random variable $Z|\theta$ defined on $[0, 1]$ and parameterized by $\theta \in \Theta \subseteq \mathbb{R}$. Suppose $Z|\theta$ increases in θ in the FOSD order; that is, $H(z, \theta)$ is decreasing in θ. Let $H_\theta(z, \theta) \leq 0$ denote the derivative of $H(z, \theta)$ with respect to θ if θ is a continuous parameter (in which case we assume that $H(z, \theta)$ is differentiable) or the first difference, $H(z, \theta + d) - H(z, \theta)$, if θ is a discrete index with step size $d > 0$.

Lemma 3 (Ryvkin and Drugov (2017)) *The expectation* $\gamma(\theta) = \int_0^1 u(z)dH(z, \theta)$ *is unimodal for all unimodal functions* $u(z)$ *if and only if* $|H_\theta(z, \theta)|$ *is log-supermodular.*

A.2 Proofs

Proof of Lemma 1 We will use the following properties of the regularized incomplete beta function (Paris, 2010):

$$F^B(z; x + 1, y) = F^B(z; x, y) - \frac{z^x(1 - z)^y}{x\mathcal{B}(x, y)} \tag{15}$$

$$F^B(z; x, y + 1) = F^B(z; x, y) + \frac{z^x(1 - z)^y}{y\mathcal{B}(x, y)}. \tag{16}$$

Here, $\mathcal{B}(x, y)$ is the beta function.

By construction, order statistics $Z_{n-r:n-1}$ are FOSD-decreasing in r. Property (15) implies that they are also FOSD-increasing in n. Part (i) then follows directly from (9).

For part (ii), we will first show that $F_r^B(z; n-r, r) = F^B(z; n-r-1, r+1) - F^B(z; n-r, r)$ is log-supermodular in $(-r, z)$. Using properties (15) and (16), obtain

$$\begin{aligned}
&F^B(z; n - r - 1, r + 1) - F^B(z; n - r, r) \\
&= F^B(z; n - r - 1, r + 1) - F^B(z; n - r, r + 1) + F^B(z; n - r, r + 1) - F^B(z; n - r, r) \\
&= \frac{z^{n-r-1}(1 - z)^{r+1}}{(n - r - 1)\mathcal{B}(n - r - 1, r + 1)} + \frac{z^{n-r}(1 - z)^r}{r\mathcal{B}(n - r, r)} \\
&= \frac{z^{n-r-1}(1 - z)^{k+1}(n - 1)!}{(n - r - 1)(n - r - 2)!r!} + \frac{z^{n-r}(1 - z)^r(n - 1)!}{r(n - r - 1)!(r - 1)!\mathcal{B}(n - r, r)} \\
&= \binom{n - 1}{r} z^{n-r-1}(1 - z)^r.
\end{aligned}$$

Thus, for any $r' < r$ the ratio $\frac{F_k^B(z;n-r',r')}{F_k^B(z;n-r,r)} \propto \left(\frac{z}{1-z}\right)^{r-r'}$ is increasing in z; therefore, $F_r^B(z; n - r, r)$ is log-supermodular in $(-r, z)$.

Second, we will show that $|F_n^B(z; n-r, r)| = |F^B(z; n+1-r, r) - F^B(z; n-r, r)|$ is log-supermodular in (n, z). Indeed, from (15) $|F_n^B(z; n-r, r)| = \frac{z^{n-r}(1-z)^k}{(n-r)\mathcal{B}(n-r,r)}$, which gives, for some $n' > n$,

$$\frac{F_n^B(z; n'-r, r)}{F_n^B(z; n-r, r)} = \frac{(n'-r)\mathcal{B}(n'-r, r)z^{n'-n}}{(n-r)\mathcal{B}(n-r, r)}.$$

The above ratio is increasing in z, i.e., indeed $|F_n^B(z; n-r, r)|$ is log-supermodular in (n, z). The results then follow from Lemma 1 of Ryvkin and Drugov (2017), cf. Lemma 3.

For part (iii), note that if $f(t)$ is log-concave then $m(z)$ is concave. We will show that in this case $\beta_{r,n} = B_{r,n} - B_{r-1,n}$ is decreasing in r. Integrating Eq. (4) by parts, obtain

$$\beta_{r,n} = \binom{n-1}{r-1} \int_0^1 m(z) z^{n-r-1}(1-z)^{r-2}[n-r-(n-1)z]dz \tag{17}$$

$$= \binom{n-1}{r-1} \int_0^1 m(z) d[z^{n-r}(1-z)^{r-1}] \tag{18}$$

$$= \binom{n-1}{r-1} \left[m(z) z^{n-r}(1-z)^{r-1}\Big|_0^1 - \int_0^1 z^{n-r}(1-z)^{r-1} m'(z) dz \right]$$

$$= \binom{n-1}{r-1} [m(1) I_{r=1} - m(0) I_{r=n}] - \frac{1}{n} \frac{n!}{(n-r)!(r-1)!} \int_0^1 z^{n-r}(1-z)^{r-1} m'(z) dz$$

$$= m(1) I_{r=1} - m(0) I_{r=n} - \frac{1}{n} \int_0^1 m'(z) dF^B(z; n-r+1, r).$$

Here, $F^B(z; n-r+1, r)$ is the cdf of order statistic $Z_{n+1-r:n}$. These order statistics are FOSD-decreasing in r; therefore, given that $m'(z)$ is decreasing, the integral is increasing in r. The first term in the expression above is equal to $m(1)$ for $r = 1$, $-m(0)$ for $r = n$ and 0 otherwise; hence, it is decreasing in r. Thus, combined we have a sequence that is decreasing in r.

The symmetry of $f(t)$ implies that $m(1-z) = m(z)$. Part (iv) then follows directly from (9). ■

Proof of Proposition 3 For a symmetric $f(t)$, $B_{r,n} = B_{n-r,n}$ by Lemma 1(iv). This implies $\bar{\beta}_{r,n} = \frac{B_{r,n}}{r} > \frac{B_{n-r,n}}{n-r} = \bar{\beta}_{n-r,n}$ for $r < \frac{n}{2}$. To rule out the case $r^* = \frac{n}{2}$ when n is even and $n \geq 4$, we will show that $\bar{\beta}_{\frac{n}{2}-1,n} > \bar{\beta}_{\frac{n}{2},n}$. Indeed, using (9),

$$
\begin{aligned}
&\bar{\beta}_{\frac{n}{2}-1,n} - \bar{\beta}_{\frac{n}{2},n} = \frac{B_{\frac{n}{2}-1,n}}{\frac{n}{2}-1} - \frac{B_{\frac{n}{2},n}}{\frac{n}{2}} \\
&= \binom{n-1}{\frac{n}{2}-1} \int_0^1 z^{n-(\frac{n}{2}-1)-1}(1-z)^{\frac{n}{2}-1-1} m(z)dz - \binom{n-1}{\frac{n}{2}} \int_0^1 z^{n-\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} m(z)dz \\
&= \binom{n-1}{\frac{n}{2}} \int_0^1 z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1} \frac{2z-1}{1-z} m(z)dz.
\end{aligned}
$$

We now show that the last integral is positive, by writing it as

$$
\begin{aligned}
&-\int_0^{\frac{1}{2}} z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1}\frac{1-2z}{1-z}m(z)dz + \int_{\frac{1}{2}}^1 z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1}\frac{2z-1}{1-z}m(z)dz \\
&> -\int_0^{\frac{1}{2}} z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1}\frac{1-2z}{\frac{1}{2}}m(z)dz + \int_{\frac{1}{2}}^1 z^{\frac{n}{2}-1}(1-z)^{\frac{n}{2}-1}\frac{2z-1}{\frac{1}{2}}m(z)dz = 0.
\end{aligned}
$$

The inequality follows by replacing the term $\frac{1}{1-z}$ by its maximum value in the first integral and minimum value in the second integral. The resulting expression is equal to zero due to the symmetry of $m(z)$ around $z = \frac{1}{2}$. ■

Proof of Proposition 4 If Y is DFR, $r_Y^* = n-1$ and the result holds automatically. Suppose Y is IFR or with an interior unimodal failure rate. It is sufficient to show that if $\bar{\beta}_{r,n,Y}$ is decreasing for some r then $\bar{\beta}_{r,n,X}$ is decreasing for that same r; that is, we will show that if $\frac{B_{r-1,n,Y}}{r-1} \geq \frac{B_{r,n,Y}}{r}$ then $\frac{B_{r-1,n,X}}{r-1} \geq \frac{B_{r,n,X}}{r}$.

Suppose $\frac{B_{r-1,n,Y}}{r-1} \geq \frac{B_{r,n,Y}}{r}$, then $\frac{B_{r-1,n,Y}}{B_{r,n,Y}} \geq \frac{r-1}{r}$. It is sufficient to show that $\frac{B_{r-1,n,X}}{B_{r,n,X}} \geq \frac{B_{r-1,n,Y}}{B_{r,n,Y}}$, or, equivalently,

$$
\frac{\int_0^1 z^{n-r}(1-z)^{r-2}\rho(z)m_Y(z)dz}{\int_0^1 z^{n-r-1}(1-z)^{r-1}\rho(z)m_Y(z)dz} \geq \frac{\int_0^1 z^{n-r}(1-z)^{r-2}m_Y(z)dz}{\int_0^1 z^{n-r-1}(1-z)^{r-1}m_Y(z)dz}.
$$

Define function $a(z) = z^{n-r-1}(1-z)^{r-1}m_Y(z)$. The inequality above then can be rewritten as

$$
\frac{\int_0^1 \frac{z}{1-z}a(z)\rho(z)dz}{\int_0^1 a(z)\rho(z)dz} \geq \frac{\int_0^1 \frac{z}{1-z}a(z)dz}{\int_0^1 a(z)dz}.
$$

Note that $\frac{z}{1-z}$ is an increasing function; therefore, the inequality will hold if the distribution with pdf $\frac{a(z)\rho(z)}{\int_0^1 a(z')\rho(z')dz'}$ FOSD the distribution with pdf $\frac{a(z)}{\int_0^1 a(z')dz'}$. Thus, we need to show that for any $z \in [0,1]$ we have

$$\frac{\int_0^z a(z')\rho(z')dz'}{\int_0^1 a(z')\rho(z')dz'} \leq \frac{\int_0^z a(z')dz'}{\int_0^1 a(z')dz'}$$

or

$$\int_0^z a(z')\rho(z')dz' \int_0^1 a(z')dz' \leq \int_0^z a(z')dz' \int_0^1 a(z')\rho(z')dz'.$$

Subtracting $\int_0^z a(z')\rho(z')dz' \int_0^z a(z')dz'$ from both sides, obtain

$$\int_0^z a(z')\rho(z')dz' \int_z^1 a(z')dz' \leq \int_0^z a(z')dz' \int_z^1 a(z')\rho(z')dz'.$$

But $\rho(z)$ is an increasing function; therefore, the left-hand side of this inequality does not exceed $\rho(z) \int_0^z a(z')dz' \int_z^1 a(z')dz'$, whereas the right-hand side is not less than the same number, and hence the inequality holds. ■

Proof of Proposition 6 For part (i), note that $B_{n,k} = 0$ for $n \leq k$. From Lemma 1(i), if $f(t)$ is increasing then $B_{n,k}$ is increasing in n for $n > k$. This implies that $B_{n,k}$ is increasing for all n and hence the expectation $\mathrm{E}_{\tilde{p}(\theta)}(B_{N,k}) = \sum_{n=1}^M \tilde{p}_n(\theta) B_{k,n}$ is increasing in θ.

For part (ii), by Lemma 1(i), $B_{n,k}$ is decreasing in n for $n > k$. However, $B_{n,k} = 0$ for $n \leq k$; therefore, over the entire range of $n = 1, \ldots, M$ $B_{n,k} = 0$ is interior unimodal with a mode at $n = k+1$, and the monotonicity of $\mathrm{E}_{\tilde{p}(\theta)}(B_{N,k})$ as a function of θ cannot be established without an additional restriction. Namely, from the assumption that $\tilde{P}_k(\theta) = 0$ we obtain

$$\mathrm{E}_{\tilde{p}(\theta)}(B_{N,k}) = \sum_{n=k+1}^{M} \tilde{p}_n(\theta) B_{k,n} = \mathrm{E}_{\tilde{p}(\theta)}(B_{N,k} | N > k),$$

which is decreasing in θ because $B_{n,k}$ is decreasing in n for $n > k$.

For part (iii), by Lemma 1(ii), $B_{k,n}$ is unimodal in n for all $n > k$. Given that $B_{n,k} = 0$ for $n \leq k$, this implies that $B_{k,n}$ is unimodal for all n. Thus, $\mathrm{E}_{\tilde{p}(\theta)}(B_{N,k})$ is an expectation of a unimodal sequence. The result then follows by Lemma 2 from Ryvkin and Drugov (2017) on the preservation of unimodality under uncertainty (the formulation is similar to Lemma 3).

For part (iv), Proposition 1 shows that $\frac{B_{k,n}}{k}$ is maximized by $k^* = 1$ for any $n = 1, \ldots, M$. This implies that the sum in the left-hand side of (12) is maximized by $k^* = 1$. ■