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DP12354

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Discussion Paper DP12354

Published 06 October 2017

Submitted 06 October 2017

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www.cepr.org

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Abstract

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JEL Classification: N/A

Keywords: N/A

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Acknowledgements

We would like to thank Kfir Eliaz, Collin Raymond, Larry Samuelson, Joel Sobel, Rani Spiegler, Tomasz Strza-lecki, and Neslihan Uler. Parts of this research were conducted while Masatlioglu was visiting the Institute for Advanced Studies at Hebrew University. Nakajima appreciates the financial support provided by Japan Society for the Promotion of Science (JSPS KAKENHI Grant Number 26780113).

Willpower and Compromise Effect*

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Abstract

This paper provides a behavioral foundation for the willpower as limited cognitive resource model which bridges the standard utility maximization and the Strotz models. Using the agent's ex ante preferences and ex post choices, we derive a representation that captures key behavioral traits of willpower constrained decision making. We use the model to study the pricing problem of a profit-maximizing monopolist who faces consumers with limited willpower. We show that the optimal contract often consists of three alternatives and the consumer's choices reflect a form of the "compromise effect" which is induced endogenously.

1 Introduction

Standard theories of decision making assume that people choose what they prefer and prefer what they choose. However, introspection suggests that implementation of choice may not be automatic and there is often a wedge between preferences and actual choices. Recently psychologists and economists have emphasized the lack of self control in decision making as an important reason for this wedge.¹ When people face temptation, they make choices that are in conflict with their commitment preferences.

People do not always succumb to temptation and are sometimes able to overcome temptations by using cognitive resources. This ability is often called willpower.² There is a growing experimental

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¹Models of self-control problems include quasi-hyperbolic time discounting (e.g., Laibson (1997); O'Donoghue and Rabin (1999)), temptation costs (e.g., Gul and Pesendorfer (2001, 2004)), and conflicts between selves or systems (e.g., Shefrin and Thaler (1988); Bernheim and Rangel (2004); Fudenberg and Levine (2006)).

²Loewenstein (2000) emphasizes the role of both positive and negative visceral urges in generating the wedge between choice and preference. Following self-control literature we focus on temptations or positive visceral urges. However similar issues are relevant when individuals face fear or guilt inducing alternatives that cause negative

psychology literature demonstrating that willpower is a limited resource, and it is more than a mere metaphor (e.g., Baumeister and Vohs (2003); Faber and Vohs (2004); Muraven et al. (2006)) and motivated by these experiments economists have used limited willpower to explain patterns of consumption over time (Ozdenoren et al. (2012); Fudenberg and Levine (2012)).

Our goal is to characterize a simple and tractable model of limited willpower that is suitable to study a wide range of economic problems. In many applications a key issue is whether the agents are naive or sophisticated about anticipating their future choices. The evidence suggests that consumers who have self control problems are often, at least partially, naive about this fact. For example, there are “hot-cold empathy gaps” where individuals are not able to recognize the intensity of temptation, or other visceral urges at an ex ante state. Loewenstein and Schkade (1999) review several studies that find people tend to underestimate the influence on their behavior of being in a hot state such as hunger, drug craving, curiosity, sexual arousal, etc. This fact poses a challenge for self-control models that presume that agents can correctly predict their future choices and makes them unsuitable for many applications. An important feature of our set-up is that, by using a novel data set, it allows the modeller to remain agnostic about whether the consumer is sophisticated or naive about anticipating his future choices.

In this paper we propose and characterize two models. We refer to the simpler of the two as the constant willpower model. This model is based on three ingredients.³ The first, commitment utility u , represents the agent’s commitment preferences. The other two ingredients are temptation values v and the willpower stock w which jointly determine how actual choices depart from what commitment utility would dictate. The key to determining the actual choice is the willpower constraint. This constraint is determined by the most tempting available alternative and the willpower stock. The agent is able to consider an alternative x in A if he can overcome the temptation, that is, $\max_{y \in A} v(y) - v(x) \leq w$. Otherwise, he does not have enough willpower to choose this alternative. He then picks the alternative that maximizes his commitment utility from the set of alternatives that satisfies the willpower constraint. Formally, the ex post choice from a set A is the outcome of the following maximization problem:

$$\max_{x \in A} u(x) \quad \text{subject to} \quad \max_{y \in A} v(y) - v(x) \leq w$$

The constant willpower model bridges the standard utility maximization and the Strotz models. When the willpower stock is very large, the willpower-constrained agent behaves like a standard agent who chooses the most preferred alternative (according to u). When the willpower stock is

visceral reactions. In those cases willpower might be necessary to motivate oneself to choose an alternative that causes a more intense negative urge than another feasible alternative.

³We refer to the second model as the limited willpower model. This model is similar to the constant willpower model but it is more general because it allows the willpower stock to depend on the chosen alternative.

lower, the constraint starts to bind and a wedge between preferences and choices appears – the agent can only choose alternatives that are close enough, in terms of temptation, to the most tempting one. In the other extreme, when the willpower stock is very low, the agent behaves like a Strotzian agent who always succumbs to temptation. Notice that the agent’s choice will satisfy WARP in the two extreme cases for different reasons. While in the former choices reflect the ex ante preference alone, in the latter, temptation ranking solely determines the choices. In comparison, in the limited willpower case, when the willpower stock is not too high or too low, choices reflect a compromise between the ex ante preference and the temptation ranking and violate WARP. Similar examples of WARP violations feature in Fudenberg and Levine (2006); Dekel et al. (1998); Noor and Takeoka (2010).⁴ Such behavior, resembling the compromise effect, occurs when the agent does not have enough willpower to choose the least tempting alternative, but has enough willpower to choose the moderately tempting alternative.

To derive the limited and constant willpower representations, we use a novel data set given by the agent’s ex ante preferences (\succsim) and ex post choices (c).⁵ Temptation and self-control have been studied using the preference over menus framework pioneered by Kreps (1979) and Gul and Pesendorfer (2001) where the agent’s second period choices are inferred from his preferences over menus.⁶ In contrast, in our framework the modeller *directly* observes both components of the data, namely ex ante preferences and ex post choices. Menu preferences framework provides a powerful tool to elicit a very rich set of behaviors at the ex ante stage. However, its power relies on the assumption that the agent can predict the degree of his future temptations and his future choices. Although it is useful to study the sophisticated benchmark, many interesting applications with policy related implications emerge from the assumption of naivety. For example, consumers sign contracts believing that they will choose a basic option but they choose tempting upgrades at a later stage. By relying on directly observed rather than predicted behavior our model is tailored to study these applications. This is illustrated in Section 4 where we apply the model to monopolistic contracting – one of the leading applications of self-control – where consumers have limited willpower but are unaware of their willpower problems.

Using this data, we provide two representation theorems in an environment with finitely many alternatives. Our first axiom is that the agent’s ex ante preferences are complete and transitive. The second axiom, Independence from Preferred Alternative (IPA), says that dropping any alternative that is strictly preferred to the chosen one should not affect the actual choice.⁷ The third axiom,

⁴In a menu preference framework, Dillenberger and Sadowski (2012) point out that when agents anticipate experiencing guilt or shame when they deviate from a social norm, their choices can also violate WARP.

⁵Ahn and Sarver (2013) also utilized two kinds of behavioral data. As opposed to ours, to derive their result, they utilize both the entire menu preferences and the *random* ex post choice from menus.

⁶For a survey see Lipman and Pesendorfer (2011).

⁷This axiom can be viewed as a relaxation of WARP that says that any unchosen alternative can be dropped without affecting actual choices. In contrast, IPA allows only strictly preferred alternatives to be dropped without

Choice Betweenness, states that the choice from the union of two sets is “between” the choices made separately from each set with respect to preference.⁸ Our first representation theorem shows that these three axioms are necessary and sufficient for (\succsim, c) to be represented by a limited willpower model in which the willpower stock depends on the chosen alternative. We then turn to the constant willpower case where willpower stock is independent of the chosen alternative. To characterize the constant willpower model we need an additional axiom. This fourth axiom, Consistency, formalizes the intuition that if y is more tempting than z and the agent prefers x but cannot choose it against z , then the agent should not be able to choose x against y either. Our second representation shows that these four axioms are necessary and sufficient for (\succsim, c) to be represented by the constant willpower model.

In Section 4, we solve the pricing problem of a profit-maximizing monopolist who faces consumers with constant willpower. Monopolistic contracting is a key application of self control models and was first studied by DellaVigna and Malmendier (2004). Our treatment is similar to Eliaz and Spigler (2006) who study unconstrained contracting with dynamically inconsistent consumers. This application illustrates the tractability of the constant willpower model in economic applications, and more importantly demonstrates that the constant willpower model has policy related implications that are distinct from those of other models of self-control.

We consider a two-period model of contracting between a monopolist and a consumer. In the first period, the monopolist offers the consumer a contract which the consumer can accept or reject. If the consumer accepts the contract, in the second period he chooses an offer from the contract and pays its price to the monopolist. We assume that both parties are committed to the contract once accepted. This framework fits into many real world situations. For example, when signing up for a phone plan, gym-membership, or a credit card, purchasing a holiday package, or making a hotel reservation consumers often sign a contract that specifies a basic level of consumption but can be “upgraded” at the time of consumption. It has been pointed out that these contracts can be exploitative. The model also captures certain situations where the consumer knows the alternatives that the monopolist offers (as well as the associated prices), but can only choose an alternative after committing to make a purchase. In these cases, our analysis would apply without any changes even though the consumer does not explicitly sign a contract. An example of this would be shopping for a car. The consumer shows up at the car dealer knowing the models the dealer offers and their prices. Once the consumer arrives at the dealership, the consumer might feel committed to make the purchase. At this stage, the consumer typically takes a test drive and experiences the different models and it might require self control to buy an economy car after test driving a better and more

affecting actual choices.

⁸Although at first glance this axiom seems like a translation of Gul and Pesendorfer’s Set Betweenness axiom to our domain, the two axioms are independent. We discuss this point in Section 3.

luxurious model. Under this interpretation our analysis applies to exploitative sales tactics where the monopolist designs its line up of products and their prices to attract the consumer to the store and upsell once the consumer arrives.

We show that when the consumer has positive willpower the optimal contract consists of three alternatives and the consumer’s choices reflect a form of the “compromise effect” which is induced endogenously by the contract offered by the monopolist. The model has several other unique features. For example, the optimal contract includes a tempting alternative that neither the consumer nor the firm believes would be chosen from an ex ante perspective, and indeed is not chosen ex post. Profits are lower and consumer is generally better off if he has more willpower, but we show that a low willpower stock does not protect the agent from exploitation at all.

The costly self control models in Fudenberg and Levine (2006) and Noor and Takeoka (2010) are closely related to the constant willpower model. In the costly self control models, the agent experiences a utility cost which can be a convex function of the difference between the temptation values of the most tempting and the chosen alternatives from a menu. One can express a willpower constraint as the limiting case of a convex cost function that is zero if the temptation difference is less than the willpower stock and infinite otherwise. Hence just like the Strotz model can be viewed as a limiting case of costly self control with linear costs (Gul and Pesendorfer (2001)), the constant willpower model is a limiting case of costly self control with convex costs. Just like the Strotz model, constant willpower is an important special case both methodologically and for applications. Although the functional form is a limiting case of costly convex self control, the two papers use different domains and axioms, and the characterization in Noor and Takeoka (2010) does not apply to this limiting case. Hence, our paper is the first to provide behavioral foundations for this important limiting case. Finally, as we illustrate in Section 4, the constant willpower model is parametric and lends itself easily to applications.

2 Model

The agent’s ex ante preferences \succsim are over a finite set of alternatives X . These preferences can be interpreted as the agent’s commitment preferences. The agent’s ex post choices are captured by a choice correspondence c that assigns a non-empty subset of A to each $A \in \mathcal{X}$ where \mathcal{X} is the set of all non-empty subsets of X .

We say that (\succsim, c) has a *limited willpower representation* if there exists (u, v, w) where $u : X \rightarrow R$

represents preference \succsim and c is given by

$$c(A) = \operatorname{argmax}_{x \in A} u(x) \text{ subject to } \max_{y \in A} v(y) - v(x) \leq w(x)$$

where $v : Z \rightarrow R$ captures the temptation values and $w : X \rightarrow \mathbb{R}_+$ is the willpower function. If w is a constant function, we call it simply a *constant willpower representation*.

In the standard model where there is no willpower problem, a decision maker chooses the alternative that maximizes the commitment utility, u , from any menu. An agent who has constant willpower also maximizes u but faces a constraint. The willpower requirement of alternative x is given by the difference between the temptation value of the most tempting alternative on the menu, $\max_{y \in A} v(y)$, and the temptation value of x . The agent can choose x only if its willpower requirement is less than the willpower stock, w . Otherwise, he does not have enough willpower to choose this alternative. Notice that the willpower requirement is menu dependent. This is because willpower depletion not only depends on how tempting the chosen alternative is but also on the most tempting alternative on the menu.

As a simple example consider three alternatives: an economy car (e), a mid-size sedan (m) and a luxury car (l). Suppose ex ante $e \succ m \succ l$. Suppose $v(e) = 0$, $v(m) = 2$, $v(l) = 4$. Table 1 shows the agent's choices from two sets, $\{e, m, l\}$ and $\{e, m\}$ for varying levels of willpower stock.⁹ When willpower stock is high, $w = 5$, the agent chooses according to his ex ante preferences. When willpower stock is low the agent also behaves like a standard preference maximizer, except that he chooses the most tempting alternative. When the willpower stock is intermediate, $w = 3$, then the model has interesting implications – decisions can be driven by a compromise between the ex ante preference and temptation. To see this suppose all three alternatives are available. The agent is not able to choose e since $v(l) - v(e) = 4 > 3 = w$. In this case he chooses the compromise alternative m since $v(m) - v(e) = 2 < 3$. However when only e and m are available, there is no need to compromise (since $v(m) - v(e) = 2 < 3$) and the agent chooses e .

	$w = 1$	$w = 3$	$w = 5$
$c(e, m, l)$	l	m	e
$c(e, m)$	m	e	e

Table 1: Choices for different levels of the willpower stock

⁹We will abuse the notation and write $c(x, y, \dots)$ instead of $c(\{x, y, \dots\})$. Similarly, we omit braces and write $A \cup x$ instead of $A \cup \{x\}$.

3 Behavioral Characterization

In this section we introduce the axioms and provide two representation theorems. Our first axiom is standard.

A 1. \succ is a linear order.¹⁰

By Axiom 1, the choice is unique, $|c(S)| = 1$ for all S . The second axiom is Independence from (Unchoosable) Preferred Alternative (IPA); better options that are not chosen can be removed without affecting the actual choice.

A 2. (IPA) If $x \succ y$ and $y \in c(A \cup x)$ then $c(A) = c(A \cup x)$.

This axiom can be viewed as a relaxation of WARP. Recall WARP allows any unchosen alternative to be dropped without affecting actual choices. In contrast, IPA allows only preferred unchosen alternatives to be dropped without affecting actual choices.

IPA is based on the intuitive notion that when a tempting alternative is also the most preferred available alternative, it should be chosen. Hence any unchosen alternative that is strictly preferred to the chosen one must have a relatively low temptation value. IPA says that dropping such alternatives should not affect the actual choice. Let's revisit the example in Section 2 with three alternatives, e , m and l with $e \succ m \succ l$. Suppose a mid-size sedan is chosen when all three options are available, i.e. $m = c(e, m, l)$. This means the most preferred alternative (e) is not chosen, and hence, is not the most tempting alternative and is irrelevant in the sense that dropping it from the menu should not affect the choice behavior of the agent. That is, we must have $c(e, m, l) = c(m, l)$. On the other hand, it is possible that removing l , the least preferred alternative, might influence the choice. If m is not as tempting as l , the agent can choose the best alternative e when l is removed, i.e. $e = c(e, m) \neq c(e, m, l)$. Hence, WARP is not satisfied in the presence of limited willpower.

The next axiom is Choice Betweenness (CB): the choice from the union of two sets is “between” the choices made separately from each set with respect to preference.

A 3. (Choice Betweenness) If $c(A) \succ c(B)$ then $c(A) \succ c(A \cup B) \succ c(B)$.

To understand this axiom take the union of two choice sets $A \cup B$ and w.l.o.g. suppose A contains one of the chosen alternatives from $A \cup B$. Consider two (not necessarily mutually exclusive) cases. First, suppose A contains the most tempting item in $A \cup B$. In this case, the agent should not be able to choose a strictly better alternative from A (since he needs to overcome the same temptation from $A \cup B$ as from A) but should still be able to choose the alternative originally chosen from

¹⁰A linear order is a complete, transitive and asymmetric binary relation.

$A \cup B$, i.e., $c(A) \sim c(A \cup B)$.¹¹ Note that in this case the axiom is automatically satisfied since $c(A \cup B)$ must be in between $c(A)$ and $c(B)$ in terms of preference. As a second case suppose B contains the most tempting item in $A \cup B$. In this case the agent should be able to choose at least as preferred an alternative from A as he can from $A \cup B$ since he needs to overcome a weaker temptation from A . Moreover, the alternative chosen from B cannot be strictly preferred since the most tempting alternative is contained in B . Thus, the axiom should be satisfied in this case as well.¹²

A closely related axiom is Gul and Pesendorfer's Set Betweenness (SB). Although at first glance CB seems like a translation of SB to our domain, the two axioms are independent. To make this point precise, denote non-empty subsets of X by \mathcal{X} and suppose \succ_0 is a preference relation over \mathcal{X} . We say \succ_0 satisfies SB if $A \succ_0 B$ implies $A \succ_0 A \cup B \succ_0 B$. We let $x \succ y$ if and only if $\{x\} \succ_0 \{y\}$. We will now provide two examples that show that SB and CB are indeed independent axioms.

In the first example, \succ_0 satisfies SB, but (\succ, c) violates CB. For this example, we use the *costly self-control representation* axiomatized by Noor and Takeoka (2010) in the menu preference framework.¹³ We say that \succ_0 has a costly self-control representation if it can be represented by $V : \mathcal{X} \rightarrow \mathbb{R}$ given by

$$V(A) = \max_{x \in A} u(x) - \varphi(\max_{y \in A} v(y) - v(x))$$

where $u, v : X \rightarrow \mathbb{R}$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. The agent's choices, naturally implied by the model, are given by

$$c(A) = \operatorname{argmax}_{x \in A} u(x) - \varphi(\max_{y \in A} v(y) - v(x)).$$

It is easy to see that if \succ_0 has a costly self-control representation then it satisfies SB. To see that (\succ, c) can violate CB let $X = \{x, y, z\}$, $\varphi(a) = a^5$, $u(x) = 2$, $u(y) = 1$, $u(z) = 0$, and $v(x) = 0$, $v(y) = 1.5$, $v(z) = 3$. In this case direct calculation shows that $x = c(x, z) = c(x, y, z) \succ y = c(x, y) \succ z = c(y, z)$. Hence, (\succ, c) does not satisfy CB since $c(x, y, z) \succ c(x, y) \succ c(y, z)$.

In the second example, (\succ, c) satisfies CB but \succ_0 violates SB. Suppose \succ_0 is represented by a function $W : \mathcal{X} \rightarrow \mathbb{R}$ defined as follows. If A has 2 or more elements:

$$W(A) = \max_{x \in A} u(x) - \left(\max_{y, z \in A, y \neq z} (v(y) + v(z)) - v(x) \right)$$

¹¹Implicit in these arguments is that only the most tempting alternatives matter in influencing the agent's choices. Clearly, this is also the case in the representation since only the alternative with the highest v value matters in determining which alternatives are choosable from a choice set.

¹²In fact, $c(A \cup B)$ can be strictly between $c(A)$ and $c(B)$. Continuing with our earlier example, let $A = \{e, m\}$ and $B = \{l\}$. Recall that both e and m are strictly better than l , so $c(A) \succ c(B)$. The choice from all three options, m , is strictly better than l , the worst alternative, so $c(A \cup B) \succ c(B)$. Moreover, from the set A , e is chosen, thus $c(A) \succ c(A \cup B) \succ c(B)$.

¹³Noor and Takeoka (2010)'s axiomatization is in the lottery domain. Here we adopt their representation to the finite environment.

and for singleton sets $W(\{x\}) = u(x)$ where $u, v : X \rightarrow \mathbb{R}$. The above model is a variation of Gul and Pesendorfer (2001) where the self-control cost is linear but the agent is tempted by not just the most tempting but also the second most tempting alternative in the set. The agent's choices are given by

$$c(A) = \operatorname{argmax}_{x \in A} (u(x) + v(x)).$$

It is easy to see that (\succ, c) satisfies CB. To see that \succ_0 violates SB, let $X = \{x, y, z\}$, $u(x) = 7$, $u(y) = 3$, $u(z) = 2$, $v(x) = 0$, $v(y) = 1$ and $v(z) = 2$. Then, $\{x, y\} \succ_0 \{x, z\} \succ_0 \{x, y, z\}$.

Next, we present our first representation theorem.

Theorem 1. *(\succ, c) satisfies satisfies A1-A3 if and only if it admits a limited willpower representation.*

Obviously the limited willpower representation contains the constant willpower representation as a special case. Less obvious is that the limited willpower representation is also closely related to the costly self-control representation that we discussed earlier. If the cost function φ is linear, this is the model of Gul and Pesendorfer, which satisfies WARP. More interestingly when the cost function is not linear, the model generates WARP violations. The previous literature focused especially on the cases where the cost function is either convex or concave. Theorem 1 sheds light on a distinction between these cases. The convex cost function representation satisfies our A1-A3, hence it is a special case of the limited willpower model. The concave cost function representation, on the other hand, is not a special case since, as shown earlier, it violates CB. This shows that although the set of limited willpower representations is large, it excludes some choice patterns.

Our next goal is to characterize the constant willpower model. To do this we need one more assumption. Consider four alternatives $x, y, z, t \in X$. Suppose, $y \succ c(y, z)$, that is the agent prefers y to z but is unable to choose it. Intuitively this means that z is more tempting than y . If, in addition, $c(t, z) = t$, then t must be more tempting than y as well, otherwise the agent would not be able to choose t . If $x \succ c(x, y)$, then the agent prefers x but cannot choose it against y because y is too tempting. Since t is even more tempting than y , the agent should not be able to choose x against t either. This intuitive conclusion would hold for the limited willpower model but it is not implied by IPA and CB. This is our next axiom, Consistency.

A 4. *(Consistency) Let $y \succ c(y, z)$ and $c(t, z) = t$. If $x \succ c(x, y)$ then $c(x, t) = t$.*

Now, we are ready to state the main representation theorem.

Theorem 2. *(\succ, c) satisfies A1-A4 if and only if it admits a constant willpower representation.*

Remark 1. *The costly self-control model with convex cost function might violate the Consistency axiom. To see this consider the following example. Suppose $\varphi(a) = a^2$, $u(x) = 9$, $u(t) = 4.9$,*

$u(y) = .9$, $u(z) = 0$, and $v(x) = 0$, $v(t) = 2$, $v(y) = 3$, $v(z) = 4$. In this case direct calculation shows that $x \succ y = c(x, y) \succ z = c(y, z)$, $c(t, z) = t$ and $c(x, t) = x$, hence Consistency is violated.

We close this section with a discussion of how we identify the underlying temptation ranking. In our model, whenever $y \succ x = c(x, y)$, we can conclude that x is more tempting than y by at least amount of $w(y)$, i.e., $v(x) - v(y) \geq w(y) > 0$. Hence, in these cases, it is safe to claim that x is more tempting than y ($v(x) > v(y)$). This revelation is true for both of our models.¹⁴

Due to the functional form imposed in the limited willpower model, there are in fact more non-trivial revelations about the temptation ranking.¹⁵ To illustrate this, consider the following data: $y = c(y, t) \succ x = c(x, y) \succ z \succ t = c(z, t)$. This data immediately reveals that x is more tempting than y and t is more tempting than z . In addition, in the general model, x must be more tempting than t . To see this, assume $v(t) \geq v(x)$. Then we have

$$v(t) \geq v(x) > v(y) + w(y)$$

which implies we must have $t = c(y, t)$, which is a contradiction. Therefore we can conclude that

$$v(x) > v(t) > v(z) \text{ and } v(x) > v(y)$$

This example also illustrates the limits of what the data is able to reveal about the temptation ranking. Here, we cannot reveal the temptation ranking of y compared to t or z .

In the constant willpower model we can further identify the temptation ranking. For example, $y = c(y, t) \succ x \succ z = c(x, z) \succ t = c(z, t)$ immediately reveals that t is more tempting than z , which is more tempting than x . However, in the general model, this data cannot reveal the temptation ranking of y relative to all other alternatives. Either y is the most tempting alternative or y is the least tempting alternative but with high $w(y)$. On the other hand, in the constant willpower model, it is revealed that y is more tempting than both z and x . To see this, assume $v(z) \geq v(y)$. Then we have

$$v(t) - w > v(z) \geq v(y)$$

which implies we must have $t = c(y, t)$, which is a contradiction. Therefore we can conclude that

$$v(y), v(t) > v(z) > v(x)$$

Almost all rankings are identified except the ranking between y and t . While there are 24 different possible temptation rankings in this example, only two of them are consistent with our data.

¹⁴In the constant willpower representation, we have $v(x) - v(y) \geq w > 0$ instead.

¹⁵These arguments can be verified in the proof of Theorem 1.

4 Monopoly Pricing

In this section, we apply our representation to the pricing problem of a profit-maximizing monopolist who faces consumers with constant willpower. Through this application we illustrate the tractability of the willpower model in economic applications, and obtain results that contribute to the existing literature on contracting with consumers with self-control problems (DellaVigna and Malmendier (2004), Eliaz and Spigler (2006), Heidhues and Koszegi (2010)).

Like the previous literature we focus on naive consumers who do not necessarily recognize the extent of their self-control problem. In our model, this means that consumers believe that they have more willpower than they actually do. We assume that the monopolist knows that consumers have limited willpower and characterize the contract that best exploits the consumers' naivety about their willpower limitation.

Let's denote the finite set of alternatives available to the monopolist by A . A contract C is a menu of offers where each offer is an alternative with an associated price, i.e., $C = \{(s, p_s) : s \in S \subset A\}$.¹⁶ We consider a two-period model of contracting between a monopolist and a consumer. In the first period, the monopolist offers the consumer a contract C . The consumer can accept or reject the contract. If the consumer accepts the contract, in the second period he chooses an offer from the contract and pays its price to the monopolist. If the consumer rejects the contract then he receives his outside option normalized to zero. We assume that both parties are committed to the contract once accepted.

We denote the cost to the monopolist of providing alternative s by $c(s)$, its utility to the consumer by $u(s)$, and its temptation value by $v(s)$. We assume that the consumer's utility and temptation values are both quasilinear in prices.¹⁷ We denote $U(s, p_s) = u(s) - p_s$ and $V(s, p_s) = v(s) - p_s$.

The monopolist's profit from selling alternative s at price p_s is $p_s - c(s)$.¹⁸ Following Eliaz and Spiegler (2006) and Spiegler (2011) we assume that the consumer is naive in the sense that he believes he has no self-control problem, i.e., he believes that from a contract C he will choose the offer (s, p_s) that maximizes $U(s, p_s)$. In reality, the consumer's second period choices are governed by the constant willpower model, that is, he might be tempted by the other offers in the contract C . This means that from C the consumer chooses the offer (s, p_s) that maximizes $U(s, p_s)$ subject

¹⁶We assume that a contract cannot offer the same alternative with two different prices. That is, if (x, p) and (x, p') are both in C , then p must be equal to p' .

¹⁷Broadly speaking, the idea that temptation would decrease in price seems reasonable in many situations. When the price of a good increases, the consumer must forego other potentially tempting goods. Moreover, when the price is sufficiently high the good might become unaffordable. Quasilinearity of temptation values in prices is clearly a partial equilibrium way of capturing the impact of prices on temptation and a restrictive assumption. Yet it provides tractability and is implicitly invoked in the literature on changing tastes where it is usually assumed that both the present and future utilities are quasilinear in prices.

¹⁸We assume that the production cost is incurred only for the service that the consumer chooses from the menu.

to

$$\max_{(s', p_{s'}) \in C} V(s', p_{s'}) - V(s, p_s) \leq w$$

where w is the consumer's willpower stock. We assume that the monopolist knows that the consumer has limited willpower and can predict perfectly the consumer's second period choices.¹⁹

To simplify the analysis we assume that $u - c$ and $v - c$ have unique maximizers x^u and x^v in A . In other words, x^u and x^v are the most efficient alternatives with respect to u and v . To make the problem interesting we assume that $x^u \neq x^v$. We define the difference between the temptation value and the utility value of an alternative s as its *excess temptation*, and denote it by $e(s) \equiv v(s) - u(s)$. We further assume that e has a unique maximizer, z^* , and a unique minimizer, y^* , in A . Then it is easy to see that

$$\begin{aligned} e(x^v) = v(x^v) - u(x^v) &= v(x^v) - c(x^v) - [u(x^v) - c(x^v)] \\ &> v(x^u) - c(x^u) - [u(x^v) - c(x^v)] \\ &> v(x^u) - c(x^u) - [u(x^u) - c(x^u)] = v(x^u) - u(x^u) = e(x^u). \end{aligned}$$

Therefore, we have

$$e(z^*) \geq e(x^v) > e(x^u) \geq e(y^*).$$

Optimal Contract with Sophisticated Consumers

As a benchmark case, consider a consumer who perfectly understands what he chooses once he accepts the contract. In this case, monopolist's maximization problem is

$$\max_{s \in A, p_s \geq 0} p_s - c(s)$$

subject to

$$u(s) - p_s \geq 0. \tag{1}$$

Clearly, due to the participation constraint, the firm sets the price of s equal to $u(s)$. In other words, the monopolist extracts the entire surplus (in terms of u .) Then the optimal contract offers only the efficient alternative $x^u = \operatorname{argmax} u(s) - c(s)$ at price $u(x^u)$.

¹⁹More precisely, we solve for the optimal contract for the monopolist given its beliefs about the consumer's behavior. To do this we do not need to know whether the monopolist (or the consumer) holds correct beliefs about the consumer's second period behavior.

Optimal Contract with Naive Consumers

Next, we solve for the optimal contract with a naive consumer. We do this in three steps. First, we show that, without loss of generality, we can restrict attention to contracts that offer at most three alternatives. That is, for any contract that sells x at price p_x , there is another contract that sells x at the same price which contains at most three of the alternatives from the original contract. Second, for each alternative $x \in X$, we find a contract that sells x at the highest possible price. Third, we identify the profit maximizing alternative and the associated optimal contract that sells this alternative.

To establish the first step, let $C = \{(s, p_s) : s \in S \subset A\}$ be an arbitrary contract that sells x at price p_x . This means that (i) (x, p_x) is offered in C , (ii) the consumer accepts the contract, i.e., there exists an offer (s^*, p_{s^*}) in C such that $u(s^*) - p_{s^*} \geq 0$ and, (iii) (x, p_x) is the best offer for the consumer given his willpower constraint. Now identify two offers from the contract C such that

$$y = \operatorname{argmax}_{(s, p_s) \in C} u(s) - p_s \quad \text{and} \quad z = \operatorname{argmax}_{(s, p_s) \in C} v(s) - p_s$$

We illustrate that the contract $C' = \{(x, p_x), (y, p_y), (z, p_z)\}$ sells x at price p_x . First of all, this contract is a subset of C and (x, p_x) is offered. The consumer accepts this contract since $u(y) - p_y \geq u(s^*) - p_{s^*} \geq 0$. Finally since the most tempting offer is the same, the constraint he faces is unaffected. Therefore, (x, p_x) is still the best offer for the consumer.

When x, y and z are distinct, the consumer believes he will choose (y, p_y) , but cannot because he does not have enough willpower to choose (y, p_y) when (z, p_z) is available. However, rather than completely indulging in (z, p_z) , the consumer chooses the second best (x, p_x) in terms of U . We refer to a contract with these features as a *compromising contract*. If x is equal to y , then the contract could be simpler. Indeed, the contract $\{(x, p_x)\}$ sells x .²⁰ We refer to a contract that includes a single offer as a *commitment contract*. If x is equal to z , then the contract reduces to $\{(x, p_x), (y, p_y)\}$. In this case, the consumer believes he will choose (y, p_y) , which provides the highest utility. But he actually chooses (x, p_x) because the consumer does not have enough willpower to choose (y, p_y) when (x, p_x) is available. We refer to this type of contract as an *indulging contract*.

We now investigate the revenue-maximizing contract that sells x by focusing on the three types of contracts we identified.

THE COMMITMENT CONTRACT: The monopolist's problem is to maximize p_x subject to $u(x) \geq p_x$. Then the monopolist sets the price p_x equal to $u(x)$. Hence the highest revenue from selling x

²⁰Note that the the contract $\{(x, p_x)\}$ does *not* always sell x because it may not be acceptable at time 0 by the consumer. However, when x is equal to y , given that the original contract is acceptable, $\{(x, p_x)\}$ is also acceptable.

using a commitment contract is $u(x)$.

THE INDULGING CONTRACT: The monopolist's problem is to choose p_x, y and p_y to maximize p_x subject to

$$u(y) - p_y \geq 0 \quad (2)$$

$$v(x) - p_x \geq v(y) - p_y + w \quad (3)$$

Constraint (2) guarantees that the naive consumer is willing to accept the contract. Constraint (3) implies that the consumer does not have enough willpower to resist the temptation to choose (x, p_x) . Clearly, both constraints are binding so the maximization problem is equivalent to choose p_x, y and p_y to maximize $v(x) - e(y) - w$. The monopolist sets $y = y^*$ to minimize $e(y)$, which implies $p_{y^*} = u(y^*)$ by constraint (2). Then constraint (3) implies $p_x = v(x) - e(y^*) - w$. Hence the highest revenue from selling x using an indulging contract is

$$v(x) - e(y^*) - w$$

THE COMPROMISING CONTRACT: The monopolist's problem is now to choose p_x, y, p_y, z and p_z to maximize p_x subject to

$$u(y) - p_y \geq 0 \quad (4)$$

$$v(z) - p_z \geq v(y) - p_y + w \quad (5)$$

$$v(x) - p_x + w \geq v(z) - p_z \quad (6)$$

$$u(x) - p_x \geq u(z) - p_z \quad (7)$$

Constraints (4) and (5) guarantee that the consumer signs the contract believing that he will choose y but does not actually do so. Constraint (6) implies that the consumer has enough willpower to choose x over z and constraint (7) means that choosing x over z is also desirable. It is easy to see that the first two constraints (4) and (5) are binding, which implies

$$p_y = u(y) \quad \text{and} \quad p_z = v(z) - e(y) - w$$

The remaining two constraints become

$$p_x \leq v(x) - e(y) \quad \text{and} \quad p_x \leq u(x) - e(y) + e(z) - w$$

Clearly both constraints can be relaxed by choosing $y = y^*$, and the second constraint can be

relaxed by choosing $z = z^*$.²¹ Thus the optimal price for x is

$$p_x = \min \{v(x) - e(y^*), u(x) - e(y^*) + e(z^*) - w\}. \quad (8)$$

Therefore, if $e(z^*) - e(x) \geq w$, constraint (6) is binding and the revenue is $v(x) - e(y^*)$. On the other hand, if $e(z^*) - e(x) \leq w$, constraint (7) is binding and the revenue is $u(x) - e(y^*) + e(z^*) - w$. Hence the highest revenue from selling x using a compromising contract is

$$\begin{aligned} v(x) - e(y^*) & \quad \text{if } e(z^*) - e(x) \geq w \\ u(x) + e(z^*) - e(y^*) - w & \quad \text{if } e(z^*) - e(x) \leq w \end{aligned}$$

In the former case, when constraint (6) is binding, the monopolist sells a product that has low excess temptation (an item that has relatively good u value but is not too tempting). The monopolist must lower its price sufficiently below its temptation value $v(x)$ to make sure that the consumer has enough willpower to choose it. In the latter case, when constraint (7) is binding, the monopolist sells an item with high excess temptation, and the consumer has enough willpower to choose it. The monopolist must now lower its price sufficiently below its utility value $u(x)$ to make sure that the consumer finds it worthwhile to buy.

The next proposition summarizes the revenue maximizing contract by comparing the revenue generated from the three contracts analyzed above.

Proposition 1. *If $e(z^*) - e(y^*) < w$, the revenue-maximizing contract for any alternative is the best commitment contract. If $e(z^*) - e(y^*) \geq w$, then the revenue-maximizing contract is the best compromising contract.*

4.1 The Optimal Contract

Now that we have identified the revenue-maximizing contract for each alternative, the remaining task is to find which alternative the monopolist should sell to maximize its profit. From Proposition 1, we know that if $e(z^*) - e(x) \leq w$ then monopolist's revenue is either $u(x)$ or $u(x) + e(z^*) - e(y^*) - w$. Hence, from the set $\{x : e(z^*) - e(x) \leq w\}$, it is optimal to sell the maximizer of $u(x) - c(x)$. If, on the other hand, $e(z^*) - e(x) \geq w$ then the monopolist's revenue is $v(x) - e(y^*)$. Hence, from the set $\{x : e(z^*) - e(x) \geq w\}$, it is optimal to sell the maximizer of $v(x) - c(x)$. Hence the optimal

²¹When the second constraint is not binding it may not be necessary to set $z = z^*$ in the contract. In this case monopolist can choose any z that satisfies $e(z) \geq e(x) + w$.

contract sells either

$$\operatorname{argmax}_{x:e(z^*)-e(x)\leq w} u(x) - c(x) \quad \text{or} \quad \operatorname{argmax}_{x:e(z^*)-e(x)\geq w} v(x) - c(x)$$

whichever generates the higher profit shown by Proposition 1. These observations lead to the following theorem.

Proposition 2. *1. For any $e(z^*) - e(x^v) \geq w$, the optimal contract is the best compromising contract selling x^v at $v(x^v) - e(y^*)$. The consumer's welfare is the same as when he had no willpower at all.*

2. For any $e(z^) - e(x^v) < w < e(z^*) - e(x^u)$, the optimal contract is the best compromising contract, which might sell an alternative other than x^u or x^v .*

3. For any $e(z^) - e(x^u) \leq w < e(z^*) - e(y^*)$, the optimal contract is the best compromising contract that includes y^* and z^* but actually sells the efficient service x^u at a price exceeding $u(x^u)$. The consumer is exploited but the degree of the exploitation drops as his willpower goes up.*

4. For any $e(z^) - e(y^*) \leq w$, the optimal contract is the commitment contract selling the efficient service x^u at price $u(x^u)$. The consumer is not exploited even though he is naive.*

4.2 Comparative Statics

Now, we shall consider how the optimal contract, the profit, and the naive consumer's welfare changes as the consumer's willpower changes. Before providing general results, we first illustrate the comparative statics in an example.

Example 1. *Suppose the monopolist sells a product where the quality level can vary and the set of alternatives, $A = [\bar{\alpha}, \bar{\beta}]$, correspond to different quality levels. Here an alternative with a higher index corresponds to a higher quality level. Suppose $u(s) = \alpha s$ and $v(s) = \beta s$ where $\bar{\beta} > \beta > \alpha > \bar{\alpha} \geq 0$, hence both utility and temptation levels increase as the product's quality increases but higher quality levels also have higher excess temptation. This is because, in this example, temptation increases faster than utility. For example, a luxury car has various extra features such as leather seats, heated driving wheel, etc. These features do not influence the basic service from the car, which is transportation, too much, but can be quite tempting. The cost of producing quality level s is $c(s) = \frac{s^2}{2}$.*

The service with lowest excess temptation is $y^ = \bar{\alpha}$ where $e(y^*) = (\beta - \alpha)\bar{\alpha}$. Similarly, the service with highest excess temptation is $z^* = \bar{\beta}$ where $e(z^*) = (\beta - \alpha)\bar{\beta}$. The service that maximizes $u - c$*

is $x^u = \alpha$ with $e(x^u) = (\beta - \alpha)\alpha$. Similarly, the service that maximizes $v - c$ is $x^v = \beta$ with $e(x^v) = (\beta - \alpha)\beta$.

Next, we derive the optimal contracts for different values of w in the above example. We generalize the insights from this example as comparative statics results at the end of this section. To illustrate the solution we utilize Figure 1 where the two lines describe the two upper bounds on the monopolist's profit from using the best compromising contract. One of the upper bounds (blue line) which is peaked at x^v is $v(s) - c(s) - e(y^*)$. The other one (red line), $u(s) - c(s) + e(z^*) - e(y^*) - w$ is maximized at x^u . As we have already shown, the former is greater if $e(z^*) - e(s) \geq w$ and smaller otherwise. Notice that the first one (blue line) is independent of w while the latter shifts down as the consumer's willpower gets stronger.

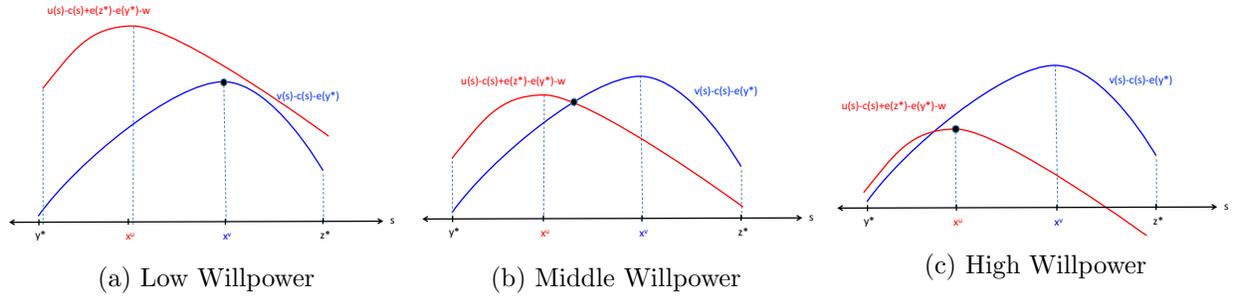


Figure 1: Best Compromising Contract

The optimal contract is given by the maximal value of the lower envelope of the two functions. That is, the monopolist should compare the profits by selling the $u - c$ maximizer in the left of $w - e(z^*)$ and the $v - c$ maximizer in the right of $w - e(z^*)$.

When $w < (\beta - \alpha)(\bar{\beta} - \beta) = e(z^*) - e(x^v)$, the lower envelope is maximized at x^v as illustrated in Figure 1a. Thus, the monopolist should sell x^v by using the best compromising contract $C = \{(x^v, p_{x^v}), (y^*, p_{y^*}), (z^*, p_{z^*})\}$ and earns the profit of $v(x^v) - c(x^v) - e(y^*) = \beta^2/2 - (\beta - \alpha)\bar{\alpha}$. We highlight an interesting feature of our model. When the willpower stock is below a certain level, the optimal contract remains the same, which indicates that a small amount of willpower does not help the consumer at all.

When $(\beta - \alpha)(\bar{\beta} - \beta) < w < (\beta - \alpha)(\bar{\beta} - \alpha)$, the monopolist should sell $x_ = = \bar{\beta} - w/(\beta - \alpha)$ (i.e. $e(x_ =) = e(z^*) - w$) by using the best compromising contract $C = \{(x_ =, p_{x_ =}), (y^*, p_{y^*}), (z^*, p_{z^*})\}$ as illustrated in Figure 1b. The monopolist earns the profit of $v(x_ =) - c(x_ =) - e(y^*)$, which is also equal to $u(x_ =) - c(x_ =) + e(z^*) - e(y^*) - w$. As the willpower stock goes up, the actually sold alternative approaches the efficient level x^u . The consumer's welfare goes up.

When $(\beta - \alpha)(\bar{\beta} - \alpha) < w < (\beta - \alpha)(\bar{\beta} - \bar{\alpha})$, now the monopolist's optimal choice is to sell $x^u = \alpha$ by using the best compromising contract $C = \{(x^u, p_{x^u}), (y^*, p_{y^*}), (z^*, p_{z^*})\}$ and earns the profit of

$u(x^u) - c(x^u) + e(z^*) - e(y^*) - w = \alpha^2/2 - (\beta - \alpha)(\bar{\beta} - \bar{\alpha}) - w$ as illustrated in Figure 1c. Although the provided alternative is efficient, its price exceeds $u(x^u)$ and goes down as the consumer gets more willpower.

When $w > (\beta - \alpha)(\bar{\beta} - \bar{\alpha})$, the monopolist simply sells the efficient service x^u at the price of $u(x^u)$ with the commitment contract (without any exploitation). When the consumer's willpower stock exceeds a certain level, the naivety does not hurt him at all.

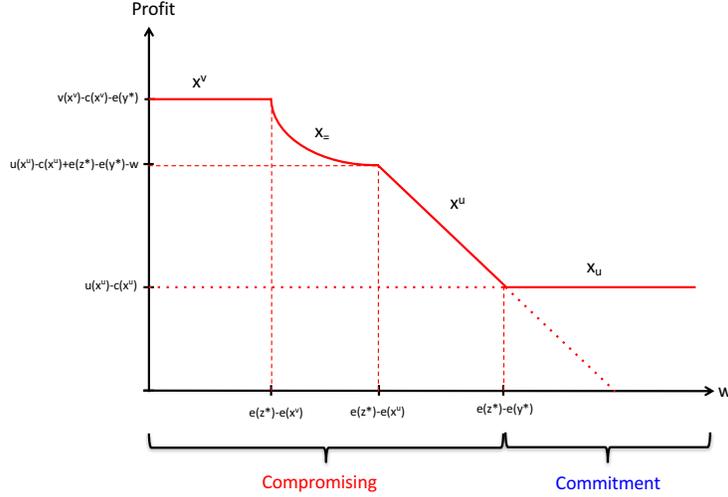


Figure 2: Monopolist's Profit

Figure 2 summarizes these observations: (i) The monopolist's profit is weakly decreasing in consumer's willpower, (ii) the consumer's welfare is weakly increasing in his willpower, (iii) if the consumer's willpower is very small, the monopolist can earn the same amount of the profit when he has no willpower at all, and (iv) when the consumer has a strong enough willpower, he is not exploited at all even though he is naive.

We generalize the comparative statics implications of the above example in the next proposition.

Proposition 3. *Suppose $w > w'$ and any optimal contract under w and w' sells the alternatives x and x' respectively. Then,*

1. *the monopolist's profit is weakly lower under w than under w' ,*
2. *the consumer's welfare is weakly higher under w than under w' ,*
3. *the monopolist sells a weakly more efficient alternative. That is, $u(x) - c(x) \geq u(x') - c(x')$.*

4.3 Partial Naivety About the Willpower Stock

In this section, we briefly consider the case where the consumer is partially naive about his future behavior. Specifically, we assume that the consumer understands that he needs willpower to resist the offers that are more tempting than the one he initially plans to choose but *overestimates* his willpower stock. We assume that the firm knows the true willpower stock. It turns out that the optimal contract with a partially naive consumer is the same as the one with a fully naive consumer model. The key observation is that partial naivety does not affect how the consumer behaves after signing the contract, since his behavior is determined by the true willpower stock. However, it will affect how the consumer evaluates a contract before signing it which depends on how much willpower he believes he has. The optimal contract for fully naive consumers sets prices so that y^* is marginally not choosable over z^* . Thus, anyone who overestimates his amount of willpower still believes that he can choose y^* . Thus, he is willing to sign the contract. Hence, the optimal contract does not change and all previous results go through as long as the consumer overestimates his willpower.

5 Conclusion

Starting from Kreps (1979), researchers have been studying a two-period choice model, in which an agent picks a menu among several menus in the planning period under the assumption that he is going to make a choice from each menu in the consumption period. This new and rich data set allows researchers to study phenomena like temptation and self-control. Menu preferences are not only useful but also necessary to study self-control within Gul and Pesendorfer's model because consumption choices alone cannot reveal whether the agent has a self-control problem. However, the reliability of menu preferences depends on the agent's ability to predict his own future behavior (i.e. sophistication).

In this paper, to derive the limited willpower representation, we use a novel data set: ex ante preferences and ex post choices. Revealing the ex ante preferences over *alternatives* is a simpler and more natural task than revealing ex ante preferences over *all menus of alternatives*. More importantly, our data set allows us to remain agnostic about whether the agent is sophisticated or naive about anticipating his ex post choices. To derive the representation, we introduce a new axiom called Choice Betweenness. We show that this axiom is independent of the Set Betweenness axiom that is commonly invoked in the menu preferences domain.

Although the model is simple and tractable, it is rich enough to generate new insights in applications. We demonstrate this in an application to monopolistic contracting. Finally, we would

like to highlight an important avenue for exploration in future work which is the implications of limited willpower in a dynamic setting with multiple tasks. In the current manuscript, we consider a model where willpower is needed in a single choice task. In fact, people often use willpower in multiple tasks, and using more willpower in one task might mean less willpower is left for another. Moreover, the model is static. In reality, there are dynamic effects in the sense that the amount of willpower used in one period can affect the willpower stock in the next period. Incorporating these considerations in an axiomatic framework can lead to new insights about behavior and a rich set of testable implications.

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A Proofs

Proofs of Theorem 1 and 2

Before we provide the proofs of Theorem 1 and 2, we provide a brief sketch. To prove Theorem 1, we first define a binary relation \triangleright' . We say that $x \triangleright' y$ if $y \succ x = c(xy)$. In words, x blocks y if x is worse than y but agent cannot choose y when x is available. Next, we define a second binary relation \triangleright'' . We say that $x \triangleright'' y$ if $x \succsim y$ and there exist a and b such that $a \triangleright' y$, $x \triangleright' b$, and $a \not\succeq b$. We say that $x \triangleright y$ if $x \triangleright' y$ or $x \triangleright'' y$. Next we show that \triangleright is an interval order, i.e. it is irreflexive and $x \triangleright b$ or $a \triangleright y$ holds whenever $x \triangleright y$ and $a \triangleright b$. The binary relation \triangleright is an interval order if and only if there exist functions v and w such that

$$\Gamma_{\triangleright}(S) = \{x \in S : \max v(y) - v(x) \leq w(x)\}.$$

Finally, to complete the proof of the first step, we show that S is indifferent to the \succsim -best element in $\Gamma_{\triangleright}(S)$.

In the proof of Theorem 2 we use consistency to show that we can construct a semi order $\bar{\triangleright}$ (i.e., $\bar{\triangleright}$ is an interval order and if $x \bar{\triangleright} y \bar{\triangleright} z$ then $x \bar{\triangleright} t$ or $t \bar{\triangleright} z$ for any t) by properly modifying \triangleright such that S is indifferent to the \succsim -best element in $\Gamma_{\bar{\triangleright}}(S)$. To complete the proof we note that the binary relation $\bar{\triangleright}$ is a semi order if and only if there exist a function v and a scalar w such that

$$\Gamma_{\bar{\triangleright}}(S) = \{x \in S : \max v(y) - v(x) \leq w\}.$$

Proof of Theorem 1

We first show that Axiom 1-3 imply an important implication of our model.

Claim 1. *Suppose (\succ, c) satisfies Axiom 1-3. Then, If $x \succ c(A \cup x)$ then $c(B) = c(B \cup x)$ for all $B \supset A$.*

Proof. Let $L(n)$ stand for the statement of Claim 1 that is restricted to when $|B - A| \leq n$. Notice that Axiom 2 is $L(0)$. First, we shall show $L(1)$. That is, $x \succ c(A \cup x)$ (so $c(A) = c(A \cup x)$ by Axiom 2) implies $c(A \cup y) = c(A \cup x \cup y)$ for any y .

Case 1: $y \succ c(A \cup x \cup y)$: By Axiom 2, $c(A \cup x) = c(A \cup x \cup y)$. By the assumption, we have $x \succ c(A \cup x) = c(A \cup x \cup y)$. By applying Axiom 2, we get $c(A \cup y) = c(A \cup x \cup y)$.

Case 2: $y \prec c(A \cup x \cup y)$: By Axiom 3, $y \prec c(A \cup x \cup y) \preceq c(A \cup x) \prec x$. By Axiom 1 we get $c(A \cup x \cup y) \prec x$. Then by Axiom 2, we get the desired result, $c(A \cup y) = c(A \cup x \cup y)$.

Case 3: $y \sim c(A \cup x \cup y)$: We have three sub-cases:

- If $y = c(A \cup y)$, then $c(A \cup y) = c(A \cup x \cup y)$.
- If $y \succ c(A \cup y)$, then Axiom 2 implies $c(A \cup y) = c(A) (= c(A \cup x))$. Applying Axiom 3, we get $c(A \cup x \cup y) = c(A \cup y)$, which is a contradiction because $c(A \cup x \cup y) = y \succ c(A \cup y)$.
- If $y \prec c(A \cup y)$, then Axiom 3 implies $(c(A \cup x) =) c(A) \succsim c(A \cup y)$. Applying Axiom 3 again, it must be $c(A \cup x \cup y) \succsim c(A \cup y) \succ y$, which is a contradiction because $c(A \cup x \cup y) = y$.

Now suppose that $L(k)$ is true up when $1 \leq k \leq n-1$. We shall prove $L(n)$. Assume $x \succ c(A \cup x)$ and let $B = A \cup \{y_1, y_2, \dots, y_n\}$ where all of y_i 's are distinct and excluded from A . Our goal is to show $c(B) = c(B \cup x)$. Without loss of generality, assume $y_1 \succ y_2 \succ \dots \succ y_n$.

Case 1: $y \succ c(A \cup x \cup y)$ for some $y \in \{y_1, y_2, \dots, y_n\}$: Since $(B \setminus y) \cup x \supset A \cup x$ and the difference of their cardinality is $n-1$, we can utilize $L(n-1)$. Then we get $c((B \setminus y) \cup x) = c((B \setminus y) \cup x \cup y) (= c(B \cup x))$. Applying $L(1)$ to $x \succ A \cup x$, we have $(y \succ)c(A \cup x \cup y) = c(A \cup y)$. Applying $L(n-1)$ to this yields $c(B \setminus y) = c((B \setminus y) \cup y) = c(B)$. Notice that $c(B \setminus y) = c((B \setminus y) \cup x)$ because $x \succ c(A \cup x)$ and $L(n-1)$. These three equalities imply $c(B) = c(B \cup x)$.

Case 2: $y \prec c(A \cup x \cup y)$ for some $y \in \{y_1, y_2, \dots, y_n\}$: By Axiom 3 we have $c(A \cup x) \succsim c(A \cup x \cup y) \succ y$. Since $x \succ c(A \cup x)$ and Axiom 1, we have $x \succ c(A \cup x \cup y)$. Because $|B \setminus (A \cup y)| = n-1$, by applying $L(n-1)$ we have $c(B) = c(B \cup x)$.

Case 3: $y_i = c(A \cup y_i \cup x)$ for all $i = 1, \dots, n$: In this case, we have

$$y_1 = c(A \cup y_1 \cup x) \succ y_2 = c(A \cup y_2 \cup x) \succ \dots \succ y_n = c(A \cup y_n \cup x)$$

Since $c(A \cup y_i \cup x) = c(A \cup y_i)$ by $L(1)$, the above relations still hold when x is removed:

$$y_1 = c(A \cup y_1) \succ y_2 = c(A \cup y_2) \succ \dots \succ y_n = c(A \cup y_n)$$

Recursively applying Axiom 3 implies

$$(c(A \cup y_1 \cup x) =) y_1 \succsim c(A \cup \{y_1, y_2, \dots, y_n\}) (= c(B)) \succsim y_n (= c(A \cup y_n \cup x))$$

In other words,

$$c(A \cup y_1 \cup x) \succsim c(B) \succsim c(A \cup y_n \cup x)$$

Since $(A \cup y_1 \cup x) \cup B = B \cup x$, Axiom 3 implies $c(B \cup x) \succsim c(B)$. Similarly, since $(A \cup y_n \cup x) \cup B = B \cup x$, Axiom 3 implies $c(B) \succsim c(B \cup x)$. Therefore, by Axiom 1, $c(B) = c(B \cup x)$. \square

For any binary relation R , let $\Gamma_R(S)$ be the set of R -undominated elements in S , that is,

$$\Gamma_R(S) = \{x \in S : \text{there exists no } y \in S \text{ such that } yRx\}$$

Instead of constructing v and w , we shall construct a binary relation over X , denoted by \triangleright such that $c(S)$ is the \succ -best element in $\Gamma_{\triangleright}(S)$.²² It is known (Fishburn (1979)) that, if (and only if) \triangleright is an interval order²³, there exist functions v and w such that

$$\Gamma_{\triangleright}(S) = \{x \in S : v(y) - v(x) \leq w(x) \forall y \in S\} = \{x \in S : \max_{y \in S} v(y) - v(x) \leq w(x)\}$$

so that we can get the desired representation.

Now, for any $x \neq y$, we define $x \triangleright y$ when either $x \triangleright' y$ or $x \triangleright'' y$ where \triangleright' and \triangleright'' are defined as follow:

1. $x \triangleright' y$ if $y \succ x = c(xy)$

²²In our framework, the \succ -best element is equal to the \succsim -best element.

²³ \triangleright is called an interval order if it is irreflexive and $x \triangleright b$ or $a \triangleright y$ holds whenever $x \triangleright y$ and $a \triangleright b$.

2. $x \triangleright'' y$ if $x \succ y$ and there exist a and b such that $a \triangleright' y$, $x \triangleright' b$, and $a \not\triangleright' b$.

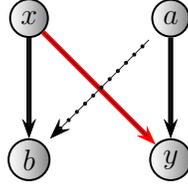


Figure 3: Black and Red arrows represent \triangleright' and \triangleright'' , respectively. Solid and dashed arrows indicate the existence and non-existence of relations, respectively.

Note that $x \triangleright' y$ and $x \triangleright'' y$ cannot happen at the same time. In addition, \triangleright' and \triangleright'' are both irreflexive.

We need to show that (i) \triangleright is an interval order and (ii) the \succ -best element in $\Gamma_{\triangleright}(S)$ is equal to $c(S)$.

Claim 2. \triangleright' is asymmetric and transitive.

Proof. By construction, $x \triangleright' y$ and $y \triangleright' x$ cannot happen at the same time. Suppose $x \triangleright' y$ and $y \triangleright' z$, i.e., $z \succ c(yz) = y \succ c(xy) = x$. Then by Claim 1, $c(xyz) = c(xz)$ because $y \succ c(xy)$. By Axiom 3, $(z \succ)c(yz) \lesssim c(xyz) \lesssim c(xy)$. Hence, we have $z \succ c(xyz) = c(xz)$. Hence we have $z \succ x = c(xz)$, so $x \triangleright' z$. \square

Claim 3. If $x \triangleright' y$ and $a \triangleright' b$ but neither $x \triangleright' b$ or $a \triangleright' y$, then it must be $x \triangleright'' b$ or $a \triangleright'' y$ but not both.

Proof. First we shall show that $x \triangleright'' b$ and $a \triangleright'' y$ cannot happen at the same time. Suppose it does. Then by definition of \triangleright' and \triangleright'' , we have $y \succ x \succ b \succ a \succ y$. Axiom 1 is violated.

Now, we shall show that either $x \triangleright'' b$ or $a \triangleright'' y$ must be defined. Suppose not. Then, along with the definition of \triangleright' , we have $b \succ x = c(xy)$, and $y \succ a = c(ab)$. Therefore, $c(xyab)$ must be weakly worse than x or a because it must be weakly worse than $c(xy)$ or $c(ab)$ by Axiom 3.

Since neither (x, b) nor (a, y) belongs to \triangleright' or \triangleright'' , we have $c(xb) = b \succ x$, and $c(ay) = y \succ a$. By Axiom 3, $c(xyab)$ must be weakly better than $c(xb)$ or $c(ay)$ so it must be weakly better than y or b .

Hence, either x or a must be weakly better than either y or b . Since we have already seen $b \succ x$ and $y \succ a$, the only possibilities are $a \lesssim b$ or $x \lesssim y$, neither of which is possible because $a \triangleright' b$ and $x \triangleright' y$. \square

Claim 4. \triangleright is an interval order.

Proof. We need to show that \triangleright is irreflexive. By definition, we cannot have (i) $x \triangleright' y$ and $y \triangleright' x$, (ii) $x \triangleright' y$ and $y \triangleright'' x$, or (iii) $x \triangleright'' y$ and $y \triangleright'' x$. Hence \triangleright is irreflexive.

Next we show that $x \triangleright b$ or $a \triangleright y$ holds whenever $x \triangleright y$ and $a \triangleright b$. We shall prove this case by case:

Case 1: $x \triangleright' y$ and $a \triangleright' b$: If we have $x \triangleright' b$ or $a \triangleright' y$, then we are done. Assume not, then Claim 3 implies we must have $x \triangleright'' b$ or $a \triangleright'' y$ (not both). Then $x \triangleright b$ or $a \triangleright y$.

Case 2: $x \triangleright' y$ and $a \triangleright'' b$: In this case, by definition of \triangleright'' and Claim 3, there exist s and t such that $a \triangleright' t$ and $s \triangleright' b$ but not $s \triangleright t$. Focus on $x \triangleright' y$ and $a \triangleright' t$, we must have either $a \triangleright y$ (it is done in this case) or $x \triangleright t$ (so either $x \triangleright' t$ or $x \triangleright'' t$). If $x \triangleright' t$, then by looking at $x \triangleright' t$ and $s \triangleright' b$ Claim requires $x \triangleright b$ because it is not $s \triangleright t$. Thus, we consider the final sub-case: $x \triangleright'' t$. If so, we have $x \triangleright' y$ and $s \triangleright' b$ so it must be either $x \triangleright b$ (then done) or $s \triangleright y$. If $s \triangleright y$, then it must be $s \triangleright' y$ (i.e. not $s \triangleright'' y$) because $y \succ x \lesssim t \succ a \lesssim b \succ s$. Therefore, we have $s \triangleright' y$ and $a \triangleright' t$ with not $s \triangleright t$. Hence it must be $a \triangleright y$.

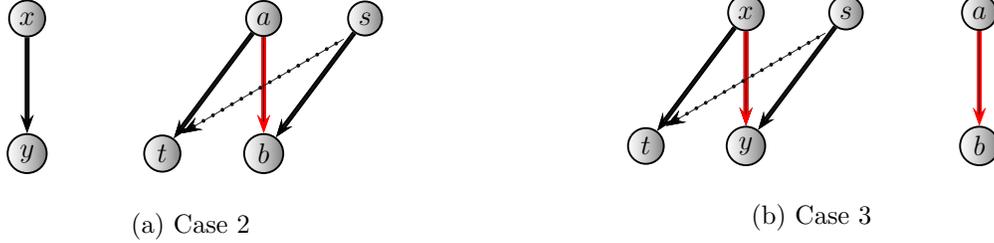


Figure 4: The Proof of Claim 4

Case 3: $x \triangleright'' y$ and $a \triangleright'' b$: By definition of \triangleright'' , there exist s and t such that $x \triangleright' t$ and $s \triangleright' y$ with not $s \triangleright t$. Then by focusing on $x \triangleright' t$ and $a \triangleright'' b$, we must have either $x \triangleright b$ (done) or $a \triangleright t$. Suppose the latter. Then we have $s \triangleright' y$ and “ $a \triangleright' t$ or $a \triangleright'' t$,” so the previous two cases are applicable so we conclude $a \triangleright y$ because it is not $s \triangleright t$. □

Claim 5. $c(S)$ is equal to the \succ -best element in $\Gamma_{\triangleright'}(S)$.

Proof. First, we prove that $\Gamma_{\triangleright'}(S)$ does not include any element that is strictly better than $c(S)$. Suppose $x \in \Gamma_{\triangleright'}(S)$. Let S' and S'' be the subsets of $S \setminus x$ consisting of elements that are better than x and strictly worse than x , respectively. That is,

$$S' := \{y \in S : y \succ x\} \text{ and } S'' := \{y \in S : x \succ y\}.$$

Then, we have $c(S' \cup x) \succ x$ by definition of c and $x = c(xy) \succ y$ for all $y \in S''$ by the definition of \triangleright' . Then by applying Axiom 3 we get $c(S'' \cup x) = x$. Thus, $c(S' \cup x) \succ c(S'' \cup x)$ implies $c((S' \cup x) \cup (S'' \cup x)) = c(S) \lesssim x$ again by Axiom 3.

Next, we shall show that $c(S) \in \Gamma_{\triangleright'}(S)$. Suppose not. Then, there exists $y \in \Gamma_{\triangleright'}(S)$ such that $y \triangleright' c(S)$ by Claim 2 (especially \triangleright' is transitive). That is $c(S) \succ c(\{c(S), y\}) = y$. Thus, by Claim 1, we have $c(S \setminus c(S)) = c((S \setminus c(S)) \cup c(S)) = c(S)$, a contradiction.

Combining the first and second results, the \succ -best element in $\Gamma_{\triangleright'}(S)$ is equal to $c(S)$. □

Claim 6. $c(S)$ is equal to the \succ -best element in $\Gamma_{\triangleright}(S)$.

Proof. Since $\triangleright \supseteq \triangleright'$ by construction, we have $\Gamma_{\triangleright}(S) \subseteq \Gamma_{\triangleright'}(S)$. Therefore, by Claim 5, it is enough to show is that the \succ -best elements in $\Gamma_{\triangleright'}(S)$ (which is $c(S)$) is included in $\Gamma_{\triangleright}(S)$. Suppose $c(S) \notin \Gamma_{\triangleright}(S)$. Since \triangleright is an interval order, it is automatically transitive. Therefore, there exists $y \in \Gamma_{\triangleright}(S)$ such that $y \triangleright c(S)$ but not $y \triangleright' c(S)$. Therefore, it must be $y \triangleright'' x$ so $y \succ c(S)$. Since $y \in \Gamma_{\triangleright'}(S)$, y cannot be strictly better than $c(S)$ (see the proof of Claim 5). □

(The Representation \Rightarrow The Axioms)

Showing that the first axiom is necessary is straightforward. For the second axiom, if $x \succ c(A \cup B)$ then A must have an element y with $v(y) > v(x) + w(x)$, so its superset B also includes y so $\Gamma(A) = \Gamma(B)$, so $c(A) = c(B)$.

The third axiom: Let x^* be the u -best element in $\Gamma(A \cup B)$. Then it must be in $\Gamma(A)$ or $\Gamma(B)$ as well so it is not possible that $A \cup B$ is strictly preferred to both A and B . Now we show that the union cannot be strictly worse than both. Let x_A and x_B be the u -best elements in A and B , respectively, and take v_A and v_B be the maximum values of v in A and in B , respectively. Then we have

$$v_A \leq u(x_A) + \varepsilon(x_A) \text{ and } v_B \leq u(x_B) + \varepsilon(x_B)$$

Therefore the maximum value of v in $A \cup B$ is the higher one between v_A and v_B , either x_A or x_B must be in $\Gamma(A \cup B)$ so $c(A \cup B)$ must be weakly better than either $c(A)$ or $c(B)$.

Proof of Theorem 2

We are now done proving the sufficiency of the axioms for the representation in Theorem 1. Next, we show the sufficiency of Axioms 1-4 for the representation in Theorem 2.

Claim 7. *If $x \succ c(xy) \succ c(yz)$ then, for all t , $c(xyzt)$ is either $c(xt)$ or $c(yz)$.*

Proof. Assume $x \succ c(xy) \succ c(yz)$, then it must be $x \succ y \succ z$. Consider $c(zt)$. If $c(zt) = t$ then by Axiom 4 we get $c(xt) = t$. Since $y \succ c(yz)$, by Claim 1, we have $c(yz) = c(yzt)$. By Axiom 3 we have $c(xt) = c(xyzt) = c(yzt)$. Hence $c(xyzt) = c(xt)$.

Now assume $c(zt) = z$. Since $x \succ c(xy)$, by Claim 1, we have $(z =)c(yz) = c(xyz)$. By Axiom 3, we have $c(zt) = c(xyzt) = c(xyz)(= c(yz))$. Hence $c(xyzt) = c(yz)$. \square

Again as in the proof of Theorem 1, instead of defining $v(\cdot)$ and $w > 0$, we shall construct a binary relation over X , denoted by $\bar{\triangleright}$ such that $c(S)$ is equal to the $\bar{\triangleright}$ -best element in $\Gamma_{\bar{\triangleright}}(S)$ (i.e. the set of $\bar{\triangleright}$ -undominated elements in S). It is known (Fishburn (1979)) that if (and only if) $\bar{\triangleright}$ is a semi order²⁴, which is a special type of an interval order, there exist function v and positive number w such that

$$\Gamma_{\bar{\triangleright}}(S) = \{x \in S : \max_{y \in S} v(y) - v(x) \leq w\}$$

so we get the desired representation.

Next we define the (i, j) -representation for an arbitrary interval order P .

Claim 8. *Any interval order, P , has an (i, j) -representation (see Figure 5) if there exist two functions $i : X \rightarrow \mathbb{N}$ and $j : X \rightarrow \mathbb{N}$ such that*

- i) For all $x \in X$, $i(x) \geq j(x)$,*
- ii) The ranges of i and j have no gap: That is if there exist x and y such that $i(x) > i(y)$ then for any integer n between $i(x)$ and $i(y)$ there is z with $i(z) = n$. Similarly for $j(\cdot)$,*

²⁴ $\bar{\triangleright}$ is a semi order if it is an interval order and if $x \bar{\triangleright} y \bar{\triangleright} z$ then $x \bar{\triangleright} t$ or $t \bar{\triangleright} z$ for any t .

iii) xPy if and only if $i(x) < j(y)$.

Proof. The following proof is based on Mirkin (1979). Given an interval order, P , (xPy and zPw imply xPw or zPy) we can show that, for all x and y in X , $L(x) \subseteq L(y)$ or $L(y) \subseteq L(x)$, and, $U(x) \subseteq U(y)$ or $U(y) \subseteq U(x)$, where $L(x)$ and $U(x)$ are lower and upper contour sets of x with respect to P , respectively. That is, $L(x) = \{y \in X \mid xPy\}$ and $U(x) = \{y \in X \mid yPx\}$. Irreflexivity indicates that there is a chain with respect to lower contour sets (this is also true for upper contour sets), i.e., relabel elements of X , $|X| = n$ such that $L(x_j) \subseteq L(x_i)$ for all $1 \leq i \leq j \leq n$. Moreover, we can strict inclusions such as there exists $s \leq n$ such that $\emptyset = L(x_s) \subset L(x_{s-1}) \dots L(x_2) \subset L(x_1)$ where $\{x_1, x_2, \dots, x_s\} \subseteq X$. For all $k \leq s$, Define

$$I_k = \{x \in X \mid L(x_k) = L(x)\}$$

I_k is not empty for any k since $x_k \in I_k$ by construction. Clearly, the system $\{I_k\}_1^s$ is a partition of the set X , i.e. $\bigcup_{k=1}^s I_k = X$, $I_k \cap I_l = \emptyset$ when $k \neq l$. Define

$$i(x) := k \text{ if } L(x) = L(x_k) \text{ for some } x_k \text{ in } X.$$

Now construct another family of non-empty sets $\{J_m\}_1^s$, as follows

$$J_s = L(x_{s-1}) \setminus L(x_s), \dots, J_2 = L(x_1) \setminus L(x_2), J_1 = X \setminus L(x_1)$$

Clearly, the system $\{J_m\}_1^s$ is another partition of the set X . Most importantly, we have $\emptyset = U(y_1) \subset U(y_2) \dots U(y_{s-1}) \subset U(y_s)$ where $y_i \in J_i$ for all $i \leq s$. Define

$$j(x) := k \text{ if } x \in J_k.$$

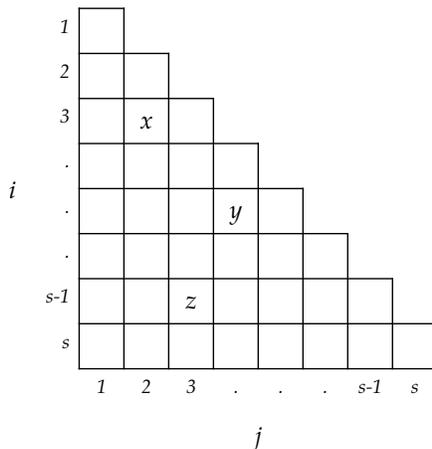


Figure 5: The graph of the (i, j) -representation. Condition *ii*) implies every row and column (not every cell) includes at least one alternative. Condition *iii*) implies $(x, y) \in P$ but $(x, z) \notin P$.

To see Condition *i*) holds, let $i(x) = i$. That means $x \in I_i$. If there exists no element z such that zPx , i.e. $U(x) = \emptyset$, then $j(x) = 1 \leq i(x)$. Otherwise find the largest integer j such that $x \in L(x_j)$. Note that j must be strictly less than i . Then by definition, $j(x) = j + 1$, which is less than $i = i(x)$.

Since both $\{I_k\}_1^s$ and $\{J_k\}_1^s$ are partitions of X , there is no gap (Condition *ii*). Finally, we have Condition *iii*) since $xPy \Leftrightarrow y \in L(x) \Leftrightarrow j(y) \geq i(x) + 1 > i(x)$. \square

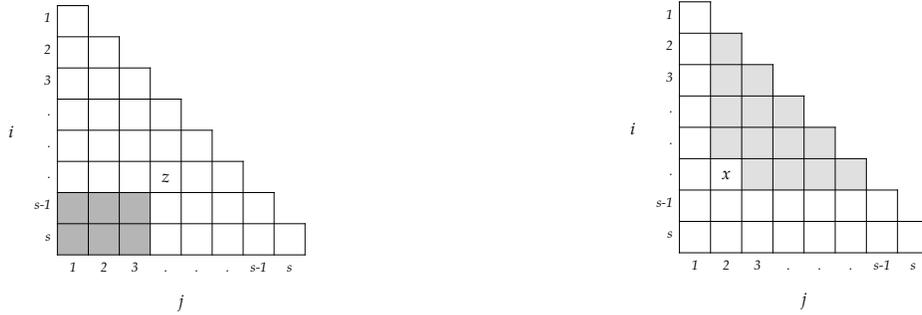
Let \triangleright be the interval order that is defined in the proof of Theorem 1. By Claim 8, it has an (i, j) -representation. We now modify the (i, j) -representation of \triangleright so that the resulting binary relation is a semiorder, say \triangleright , such that $c(S)$ is equal to the \succ -best element in $\Gamma_{\triangleright}(S)$. In other words, we construct a semiorder based on the interval order we created without affecting the representation. To do this, we prove several claims relating the (i, j) -representation with the preference \succ .

Claim 9. *If $i(x) = j(y) - 1$, it must be $y \succ x$.*

Proof. Since $i(x) < j(y)$ we know that $x \triangleright y$. If $x \triangleright' y$ then we are done since in that case $y \succ x = c(xy)$. So suppose that $x \triangleright'' y$. Then by definition of \triangleright'' , there exist α and β such that $\alpha \triangleright' y$ and $x \triangleright' \beta$ and $\alpha \not\triangleright' \beta$. Moreover, by Claim 3 $\alpha \not\triangleright'' \beta$. So $\alpha \not\triangleright \beta$. Since $\alpha \triangleright y$ and $x \triangleright \beta$, $i(\alpha) < j(y)$ and $i(x) < j(\beta)$. Since $\alpha \not\triangleright \beta$, $i(\alpha) \geq j(\beta)$. Therefore it must be $i(x) \leq j(y) - 2$, a contradiction. \square

Definition 1. *(i, j) is called a prohibited cell if there exists z such that $i(z) < i$ and $j(z) > j$. Otherwise, it is called a safe cell (see Figure 6a).*

To obtain a semi-order representation, we need to move each alternative that is in a prohibited cell to a safe cell and still the representation holds. The next definition describes a way in which alternatives can be moved.



(a) Prohibited cells because of z

(b) Movable cells for x

Figure 6: Prohibited and movable cells

Definition 2. *An alternative x can be moved to the cell (i, j) where $i \geq j$ if (a) $i \leq i(x)$ and $j \geq j(x)$, (b) $x \succ y$ for all y with $i < j(y) \leq i(x)$, (c) $z \succ x$ for all z with $j(x) \leq i(z) < j$.*

Definition requires that the alternatives in prohibited cells must move up and right (Condition (a)). As an outcome x is moved a new cell, (i, j) , it is possible that there exists y such that $i(x) \geq j(y)$ but $i < j(y)$. Condition (b) requires that in this case $x \succsim y$. Suppose to the contrary that $y \succ x$. Since $i < j(y)$, in the new representation $x \triangleright' y$. But in the original representation we have $x \not\triangleright y$. So the two representations must represent different preferences. Condition (c) can be understood similarly.

To understand this definition, we provide three examples (Figure 7). In Figure 7a, we have $x \succ y, z$. Since we have $z \not\triangleright x$ and $x \succ z$, x cannot be moved a cell where z will eliminate x

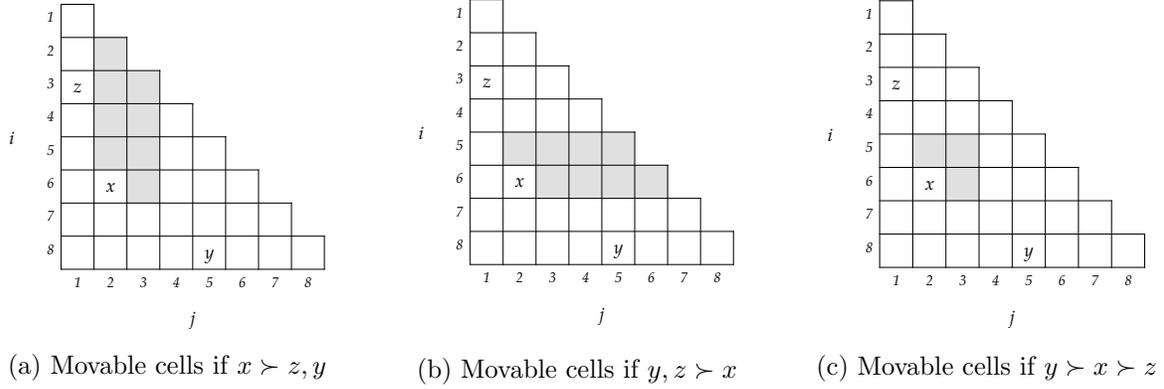


Figure 7: Examples of movable cells with different preferences

(Condition (c)). That is, $j \leq i(z) = 3$. On the other hand, since $x \succ y$, there is no restriction on movement on i . In Figure 7b, we have completely opposite situation $y, z \succ x$. Since we have $x \not\succeq y$ and $y \succ x$, x cannot be moved a cell where x will eliminate y (Condition (b)). That is, $i \geq j(y) = 5$. On the other hand, since $z \succ x$, there is no restriction on movement on j . Finally, we provide an example where both Condition (b) and (c) induce restrictions because we have $y \succ x \succ z$.

Claim 10. Suppose $\beta \succ y \succ \alpha$ and $\alpha \triangleright y \triangleright \beta$. If there exists x such that $x \not\succeq \beta$ and $\alpha \not\succeq x$, then $x \succ \beta$ or $\alpha \succ x$.

Proof. Suppose $\beta \succ x \succ \alpha$ and we shall get a contradiction. Then we have $\beta = c(\beta x)$ and $c(x\alpha) = x$ because $x \not\succeq \beta$ and $\alpha \not\succeq x$. By the assumption, we have $\beta \succ c(\beta y) \succ c(\alpha y)$. By Claim 7, $\alpha\beta xy$ must be equal to either $c(\beta x) = \beta$ or $c(\alpha y) = \alpha$. Consider $c(\beta y)$ and $c(x\alpha)$, both of which are strictly worse than β and strictly better than α . Axiom 3 dictates that $\beta \succ \beta y \succsim c(\alpha\beta xy) \succsim x\alpha \succ \alpha$. Hence, $c(\alpha\beta xy)$ cannot be equal to β or α , which is a contradiction. \square

Given the assumptions of Claim 10, we have $j(x) \leq i(\alpha) < j(y)$ and $i(y) < j(\beta) \leq i(x)$. This means that $(i(x), j(x))$ is a prohibited cell because of y . This means that x needs to be moved. Claim 10 illustrate that x can be moved because $x \succ \beta$ or $\alpha \succ x$. The next claim shows that there is a unique way to move x . That is, x can be moved to either $(i(y), j(x))$ or $(i(x), j(y))$ but not to both.



Figure 8

Claim 11. *Let exist two alternatives x and y such that $i(x) > i(y)$ and $j(x) < j(y)$. Then x can be moved to either $(i(y), j(x))$ or $(i(x), j(y))$ but not to both.*

Proof. There exist two alternatives α and β such that $i(\alpha) = j(y) - 1$ and $j(\beta) = i(y) + 1$.²⁵ By Claim 8 and 9, we have $\beta \succ y \succ \alpha$ and $\alpha \triangleright y \triangleright \beta$. Since $j(\beta) \leq i(x)$ and $j(x) \leq i(\alpha)$, $x \not\triangleright \beta$ and $\alpha \not\triangleright x$ by Claim 8. Thus, by Claim 10, we have $x \succ \beta \succ \alpha$ (so x cannot be moved to $(i(x), j(y))$ because of α) or $\beta \succ \alpha \succ x$ (so x cannot be moved to $(i(y), j(x))$ because of β). Therefore, all we need to show is that x can be moved to either of them.

Case I: $x \succ \beta$. We show that x can be moved to $(i(y), j(x))$. First, Condition (a) holds trivially: $i(y) \leq i(x)$ and $j(x) \geq j(y)$. For Condition (b), take an element z such that $i(y) < j(z) \leq i(x)$ (so $y \triangleright z$ but $x \not\triangleright z$). Then, it must be either $y \succ z$ (which implies $x \succ z$) or $z \succ y = c(yz)$ in which case we have $z \succ y \succ \alpha$ and $\alpha \triangleright y \triangleright z$ (with $x \not\triangleright z$ and $\alpha \not\triangleright x$). By Claim 10, we should have $x \succ z$ or $\alpha \succ x$. Since we are considering the case $x \succ \beta(\succ \alpha)$, it must be $x \succ z$. Condition (c) is trivially satisfied because $j = j(x)$.

Case II: $\alpha \succ x$: Condition (a) and (b) will be now trivial while Condition (c) can be proven in the same way how we prove Condition (b) in case I. \square

Claim 12. *Let*

$$U_x = \{y : i(x) > i(y), j(x) < j(y) \text{ and } x \text{ can be moved to } (i(y), j(x))\} \cup \{x\}$$

$$R_x = \{y : i(x) > i(y), j(x) < j(y) \text{ and } x \text{ can be moved to } (i(x), j(y))\} \cup \{x\}$$

and let

$$i_x = \min_{y \in U_x} i(y) \text{ and } j_x = \max_{y \in R_x} j(y)$$

Then (i) x can be moved to (i_x, j_x) , and (ii) (i_x, j_x) is a safe cell. That is, there is no z with $i(z) < i_x$ and $j(z) > j_x$.

Proof. Notice that by the definitions of movability, $i_x \leq i(x)$ and $j_x \geq j(x)$.

(i) Clearly, $i_x \leq i(x)$ and $j_x \geq j(x)$ as $x \in U_x, R_x$. First, we show that $i_x \geq j_x$. Take an alternative $y \in U_x$ such that $i(y) = i_x$. Since $y \in U_x$, x cannot be moved to $(i(x), j(y))$ by Claim 11. By the definition of movability, x cannot be moved to $(i(x), j)$ if $j \geq j(y)$. Hence for all $z \in R_x \setminus \{x\}$, $j(z) < j(y)$, which means $j_x = \max_{z \in R_x} j(z) \leq j(y)$. Since $i(y) \geq j(y)$, we have $j_x \leq j(y) \leq i(y) = i_x$.

Since x can be moved to $(i_x, j(x))$, then the second condition of the movability is satisfied. Similarly, we can prove the third requirement as well. Therefore, x can be moved to (i_x, j_x)

(ii) If $z \notin U_x, R_x$, then by Claim 11, it must be $(i_x \leq) i(x) \leq i(z)$ or $j(z) \leq j(x) (\leq j_x)$. If $z \in U_x$, then $i(z) \geq i_x$. If $z \in R_x$ then $j(z) \leq j_x$. \square

Now, define $x \bar{\triangleright} y$ if and only if $j_y > i_x$.

Claim 13. $\bar{\triangleright}$ is a semi-order.

²⁵This is because since neither $i(y)$ is the smallest nor $i(y)$ is the largest integer within the range of i .

Proof. Since $i_x \geq j_x$ by Claim 12 for all x , $\bar{\triangleright}$ is an interval order.

Next, we shall show that if (i, j) is a safe cell, there is no element x such that $i_x < i$ and $j_x > j$. Suppose there is such x . Notice that it must be $i \leq i(x)$ or $j \geq j(x)$ because (i, j) is a safe cell so it must be $i_x < i(x)$ or $j > j(x)$. Suppose $i_x < i(x)$. Then there exists y such that $i(y) = i_x$ and $j(x) < j(y)$ such that x can be moved to $(i(y), j(x))$. By Claim 11, x cannot be moved to $(i(x), j(y))$, so it cannot be moved to $(i(x), j')$ for any $j' \geq j(y)$. Since (i, j) is a safe cell and $i(y) = i_x < i$, it must be $j(y) \leq j(< j_x)$. Hence, x cannot be moved to $(i(x), j_x)$, which contradicts the definition of j_x unless $j_x = j(x)$. But if so, $j(y) > j_x > j$ but this contradicts that (i, j) is a safe cell. Analogously, we can show a contradiction if $j > j(x)$.

By Claim 12, all elements have been moved to safe cells, so there is no pair of elements x and y such that $i_x < i_y$ and $j_x > j_y$. Therefore, if $i_x < j_y \leq i_y < j_z$ (i.e. $x \bar{\triangleright} y \bar{\triangleright} z$) then for any w , it must be either $j_w > j_y$ or $i_w \leq i_y$, which implies $j_w > i_x$ or $i_w < j_z$ (i.e. $x \bar{\triangleright} w$ or $w \bar{\triangleright} z$). Therefore, $\bar{\triangleright}$ is a semiorder. \square

Claim 14. *If $x \triangleright y$ then $x \bar{\triangleright} y$.*

Proof. By definitions of i' and j' , $i_x \leq i(x)$ and $j_x \geq j(x)$ for all x . Therefore, if $x \triangleright y$, then $i_x \leq i(x) < j(y) \leq j_y$ so we have $x \bar{\triangleright} y$. \square

Claim 15. *If $x \bar{\triangleright} y$ but not $x \triangleright y$, then $x \succ y$.*

Proof. First, we shall note that both x and y must be in prohibited cells. If neither of them is in, $i_x = i(x)$ and $j_y = j(y)$ so $x \bar{\triangleright} y$ and not $x \triangleright y$ cannot happen at the same time. If only x is in a prohibited cell, then $i_x < j(y) \leq i(x)$ so x cannot be moved to (i_x, j_x) . Similarly we can prove that it is not possible that only y is in a prohibited cell.

Next we shall show that $i_x < i(x)$ and $j_x > j(x)$. Since x can be moved to (i_x, j_x) while $y \succ x$, it must be $i_x \geq j(y)$ because $j(y) \leq i(x)$ (i.e. not $x \triangleright y$). Combined with $x \bar{\triangleright} y$, we get $j_y > j(y)$. Flipping x and y , one can prove $i_x < i(x)$.

Therefore, there must exist z and z' with $i(z) \in [j(y), j_y - 1]$ and $j(z') \in [i_x - 1, i(x)]$ (notice that these intervals are non-empty). Furthermore, we can take such z and z' so that $i(z) = j(z') - 1$ because $i_x - 1 < j_y - 1$ and $i(x) > j(y)$. Thus, $z' \succ z$ by Claim 9. Since x is movable to (i_x, j_x) , we have $x \succ z'$. Similarly, we have $z \succ y$. Therefore, we conclude $x \succ y$. \square

Claim 16. *$c(S)$ is equal to the \succ -best element in $\Gamma_{\bar{\triangleright}}(S)$.*

Proof. We know $\bar{\triangleright}$ is transitive, $\bar{\triangleright} \supseteq \triangleright$ and $x \succ y$ whenever $x \bar{\triangleright} y$ but not $x \triangleright y$. It is easy to see that this claim can be proven in the exactly same way as Claim 6. \square

(Representation \Rightarrow Axioms 1-4) Showing that the first axiom is necessary is straightforward. Let

$$\Gamma(A) = \{x \in A : \max_{y \in A} v(y) - v(x) \leq \bar{w}\}$$

For Axiom 2, if $x \succ c(A \cup x)$ then A must have an element y with $v(y) > v(x) + \bar{w}$, so it is clear that $\Gamma(A) = \Gamma(A \cup x)$ so $c(A) = c(A \cup x)$.

Axiom 3: Let x^* be the u -best element in $\Gamma(A \cup B)$. Then it must be in $\Gamma(A)$ or $\Gamma(B)$ so it is not possible that $c(A \cup B)$ is strictly preferred to both $c(A)$ and $c(B)$. Now we show that the

union cannot be strictly worse than both. Let x_A and x_B be the u -best elements in $\Gamma(A)$ and $\Gamma(B)$, respectively, and take v_A and v_B be the maximum values of v in A and in B , respectively. Then we have

$$v_A \leq v(x_A) + \bar{w} \text{ and } v_B \leq v(x_B) + \bar{w}$$

Therefore the maximum value of v in $A \cup B$ is the higher one between v_A and v_B , either x_A or x_B must be in $\Gamma(A \cup B)$ so $c(A \cup B)$ must be weakly better than either $c(A)$ or $c(B)$.

Finally we show that the representation implies Axiom 4. Suppose $x \succ c(xy) \succ c(yz)$, then it must be $x \succ y \succ z$, $v(y) - v(x) > \bar{w}$ and $v(z) - v(y) > \bar{w}$. Therefore, $v(z) - v(x) > 2\bar{w}$.

Since $c(tz) = t$, we must either “ $z \succ t$ and $v(t) - v(z) > \bar{w}$ ” or “ $t \succ z$ and $v(z) - v(t) \leq \bar{w}$.” In both cases, we have $v(t) - v(x) > \bar{w}$, hence we have $c(xt) = t$.

Proof of Proposition 1

Proof. We first show that, for any x , the best compromising contract is strictly better than the best indulging contract as long as $w > 0$ and $x \neq z^*$. Indeed, it is easy to see that, if $w = 0$ then the two contracts generate the same revenue and if $x = z^*$, the compromising contract reduces to an indulging contract. Now suppose $w > 0$ and $x \neq z^*$. If the revenue from the compromising contract is $v(x) - e(y^*)$, the result is immediate. Suppose the revenue from the best compromising contract is $u(x) + e(z^*) - e(y^*) - w$. Since, by definition of z^* , $u(x) + e(z^*) \geq v(x)$, it exceeds $v(x) - e(y^*) - w$ which is the revenue from the best indulging contract.

Now, let us compare the best commitment contract and the best compromising contract. If $e(z^*) - e(x) > w$, then the best compromising contract yields $v(x) - e(y^*)$, which is weakly greater than $u(x)$ since $e(x) \geq e(y^*)$. If $e(z^*) - e(y^*) > w \geq e(z^*) - e(x)$, the best compromising contract yields $u(x) + e(z^*) - e(y^*) - w$. Since $e(z^*) - e(y^*) - w > 0$, the best compromising contract is strictly better than the best commitment contract.

If $e(z^*) - e(y^*) \leq w$, then $e(z^*) - e(x) \leq e(z^*) - e(y^*) \leq w$. This means the best compromising contract yields $u(x) + e(z^*) - e(y^*) - w$. Since $e(z^*) - e(y^*) - w \leq 0$, the best commitment contract is better. \square

Proof of Proposition 2

Proof. 1. Assume $e(z^*) - e(x^v) \geq w$. By Proposition 1, the optimal contract must be a compromising contract. Since $e(z^*) - e(x^v) \geq w$, x^v provides the highest revenue among the alternative satisfying $e(z^*) - e(x) \geq w$. Now consider x such that $e(z^*) - e(x) < w$. Then we have $e(z^*) - v(x) + u(x) < w$. This implies that $u(x) + e(z^*) - e(y^*) - w < v(x) - e(y^*)$. By definition, $u(x) + e(z^*) - e(y^*) - w < v(x) - e(y^*) \leq v(x^v) - e(y^*)$ for all x such that $e(z^*) - e(x) < w$. This establishes the fact that x^v provides the highest revenue overall.

2. Assume $e(z^*) - e(x^v) < w < e(z^*) - e(x^u)$. In Example 1, the monopolist sells neither x^u nor x^v when w falls in the corresponding region.

3. Assume $e(z^*) - e(x^u) \leq w < e(z^*) - e(y^*)$. By Proposition 1, the optimal contract must be a compromising contract. Since $e(z^*) - e(x^u) \leq w$, x^u provides the highest revenue among the alternative satisfying $e(z^*) - e(x) \leq w$. Now consider x such that $e(z^*) - e(x) > w$. Then

we have $e(z^*) - v(x) + u(x) > w$. This implies that $u(x) + e(z^*) - e(y^*) - w > v(x) - e(y^*)$. By definition, $u(x^u) + e(z^*) - e(y^*) - w > u(x) + e(z^*) - e(y^*) - w > v(x) - e(y^*)$ for all x such that $e(z^*) - e(x) > w$. This establishes the fact that x^u provides the highest revenue overall. The price, which is $u(x^u) + e(z^*) - e(y^*) - w$, is strictly higher than $u(x^u)$ since $w < e(z^*) - e(y^*)$. Therefore, the efficient service x^u is sold at a price exceeding $u(x^u)$.

4. Assume $e(z^*) - e(y^*) \leq w$. By Proposition 1 the optimal contract is the commitment contract selling the efficient service x^u at price $u(x^u)$.

□

Proof of Proposition 3

Proof. To see part (i), recall that, by Equation 8, the maximum revenue for selling x is $\min[v(x) - e(z^*), u(x) - e(y^*) + e(z^*) - w]$. Hence, the monopolist's revenue for selling any alternative is weakly decreasing in w . This implies that the monopolist's optimal profit is weakly decreasing in w .

Next we show part (ii). That is, we show total surplus is weakly increasing in w . Suppose $w > w'$. If $w \geq e(z^*) - e(y^*)$, the optimal contract sells x^u at price $u(x^u)$. This maximizes the total surplus $(u - c)$ and gives 0 to the consumer. Clearly, the optimal contract under w' does not generate more total surplus or consumer's ex ante welfare. Thus, we focus on the case where $w < e(z^*) - e(y^*)$. Let

$$E_1 = \{s : v(s) - e(y^*) - c(s) \leq u(s) - e(y^*) + e(z^*) - w - c(s)\}$$

$$E_2 = \{s : u(s) - e(y^*) + e(z^*) - w - c(s) \leq v(s) - e(y^*) - c(s) \leq u(s) - e(y^*) + e(z^*) - w' - c(s)\}$$

$$E_3 = \{s : u(s) - e(y^*) + e(z^*) - w' - c(s) \leq v(s) - e(y^*) - c(s)\}$$

Clearly, these three sets cover the entire alternative set. With this definition, notice that the monopolist's highest profits from selling s is given by

when the willpower is w :

$$v(s) - e(y^*) - c(s) \text{ when } s \in E_1 \quad u(s) - e(y^*) + e(z^*) - w - c(s) \text{ when } s \in E_2 \cup E_3$$

and when the willpower is w' :

$$v(s) - e(y^*) - c(s) \text{ when } s \in E_1 \cup E_2 \quad u(s) - e(y^*) + e(z^*) - w' - c(s) \text{ when } s \in E_3$$

Let x and x' be the alternative sold in the optimal contract and π and π' the profit generated by the optimal contract under w and w' , respectively.

Case 1: $x, x' \in E_1$

In this case, the both of them maximize $v(s) - e(y^*) - c(s)$. By the assumption (the uniqueness of the optimal alternative), $x = x'$. Thus the total surplus must be equal.

Case 2: $x \in E_1, x' \in E_2$

In this case, x maximizes $v - e(y^*) - c$ in E_1 while x' maximizes the same object in $E_1 \cup E_2$. Thus, $\pi \leq \pi'$. By Proposition 3 (1), it must be $\pi = \pi'$. Since the monopolist can earn the same profit by selling x under w' , it must be $x = x'$ (so $x, x' \in E_1 \cap E_2$) by the assumption.

Case 3: $x \in E_1, x' \in E_3$

Since $x \in E_1$, $\pi = v(x) - e(z^*) - c(x) \leq u(x) - e(y^*) + e(z^*) - w - c(x)$. If x' was sold under

w , it would generate $u(x') - e(y^*) + e(z^*) - w - c(x')$, which must be smaller than π . Thus $u(x) - c(x) > u(x') - c(x')$.

Case 4: $x \in E_2 \cup E_3, x' \in E_1$

In this case, $\pi = u(x) - e(y^*) - e(z^*) - w - c(x) \leq v(x) - e(y^*) - c(x)$ and $\pi' = v(x') - e(y^*) - c(x')$. However, the monopolist could have got the profit π' by selling x' even under w so $\pi \geq \pi'$. By proposition 3 1, $\pi = \pi'$. By the assumption, $x = x'$.

Case 5: $x, x' \in E_2 \cup E_3$

In this case, $\pi = u(x) - e(y^*) + e(z^*) - w - c(x)$. By selling x' under w , the monopolist's profit would be $u(x') - e(y^*) + e(z^*) - c(x') - w$, which cannot be greater than π so $u(x) - c(x) \geq u(x') - c(x')$.

Finally, to see part (iii), note that consumer's welfare is total surplus minus monopolist's profit. Since total surplus is weakly increasing and profit is weakly decreasing, consumer's welfare is weakly increasing in w .

□