DISCUSSION PAPER SERIES

DP12300
(v. 4)

MEASURING AGGREGATE ECONOMIC ACTIVITY

Akos Valentinyi, Georg Duernecker and Berthold Herrendorf

MACROECONOMICS AND GROWTH

ISSN 0265-8003

MEASURING AGGREGATE ECONOMIC ACTIVITY

Akos Valentinyi, Georg Duernecker and Berthold Herrendorf

Discussion Paper DP12300
First Published 13 September 2017
This Revision 06 May 2020

Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

MEASURING AGGREGATE ECONOMIC ACTIVITY

Abstract

Popular ways of measuring aggregate economic activity include the Fisher index of GDP, Weitzman's NDP in consumption units, and GDP in consumption units. We show that it matters which measure is used, because their growth rates differed considerably in the postwar U.S. We compare the properties of the measures in the two-sector growth model and show that the Fisher index of GDP is the preferable measure to use. It has a welfare interpretation under mild restrictions that allow the growth model to match the observed secular decline in the relative price of investment and the observed productivity growth slowdown.

JEL Classification: O41, O47, O51

Keywords: Fisher Index, Productivity Growth Slowdown, Weitzman's NDP

Akos Valentinyi - valentinyi.a@gmail.com
University of Manchester, CERS-HAS and CEPR

Georg Duernecker - duernecker@em.uni-frankfurt.de
University of Frankfurt, IZA and CEPR

Berthold Herrendorf - berthold.herrendorf@asu.edu
Arizona State University

Acknowledgements
Previous versions of this paper were entitled ``Quantity Measurement, Balanced Growth, and Welfare in Multi-Sector Growth Models''. For helpful comments and suggestions, we thank Timo Boppart, Ben Bridgman, Domenico Ferraro, Paul Gomme, Omar Licandro, Richard Rogerson, Gustavo Ventura, and the audiences at the ASU Conference on ``Productivity: Past, Present, and Future'', CEPR's European Summer Symposium in International Macroeconomics at Tarragona, European University Institute, the International Comparison Conference at Groningen, McGill University, the SED in Mexico City, the Swiss Macro Workshop, the Universities of Manchester, Munich, and Nottingham, and the Workshop on Structural Transformation and Macroeconomic Dynamics in Cagliari. Valentinyi thanks the Hungarian National Research, Development and Innovation Office (Project KJS K 124808). All errors are our own.

Measuring Aggregate Economic Activity*

Georg Duernecker (University of Frankfurt, CEPR, and IZA)

Berthold Herrendorf (Arizona State University)

Ákos Valentinyi (University of Manchester, CERS–HAS, and CEPR)

May 4, 2020

Abstract

Popular ways of measuring aggregate economic activity include the Fisher index of GDP, Weitzman's NDP in consumption units, and GDP in consumption units. We show that it matters which measure is used, because their growth rates differed considerably in the postwar U.S. We compare the properties of the measures in the two-sector growth model and show that the Fisher index of GDP is the preferable measure to use. It has a welfare interpretation under mild restrictions that allow the growth model to match the observed secular decline in the relative price of investment and the observed productivity growth slowdown.

Keywords: Fisher Index; Productivity Growth Slowdown; Weitzman's NDP.
JEL classification: O41; O47; O51.

*Previous versions of this paper were entitled "Quantity Measurement, Balanced Growth, and Welfare in Multi-Sector Growth Models". For helpful comments and suggestions, we thank Timo Boppart, Ben Bridgman, Domenico Ferraro, Paul Gomme, Omar Licandro, Richard Rogerson, Gustavo Ventura, and the audiences at the ASU Conference on "Productivity: Past, Present, and Future", CEPR's European Summer Symposium in International Macroeconomics at Tarragona, European University Institute, the International Comparison Conference at Groningen, McGill University, the SED in Mexico City, the Swiss Macro Workshop, the Universities of Manchester, Munich, and Nottingham, and the Workshop on Structural Transformation and Macroeconomic Dynamics in Cagliari. Valentinyi thanks the Hungarian National Research, Development and Innovation Office (Project KJS K 124808). All errors are our own.

1 Introduction

Assessing the performance of an economy requires a measure of aggregate economic activity. The U.S. Bureau of Economic Analysis ("BEA") uses real GDP calculated with the Fisher index as its headline aggregate statistic. In contrast, Weitzman (1976) suggested to use the net domestic product ("NDP") in consumption units, that is, the difference between current GDP and total depreciation, each expressed in units of current consumption. While NDP in consumption units is most popular in environmental economics, it is also widely used in macroeconomics.[1] Lastly, GDP in units of consumption is yet another measure of aggregate economic activity that is used in growth and business cycle theory; see for example Greenwood et al. (1997).

The widespread use of different measures of aggregate economic activity raises the question whether they give different assessments of economic performance. Table 1 documents for the postwar U.S. that the average annual growth rates differed widely. In particular, the average annual growth rate of GDP per worker measured with the Fisher index has been 0.41 and 0.35 percentage points higher than those of NDP and GDP in consumption units per worker, respectively. Since over long horizons, the resulting cumulative level differences are sizeable, it matters quantitatively which measure one uses. Moreover, as we will see below, while the growth rates of all three measures of aggregate economic activity started to slow down in the 1970s, the growth slowdown was much milder for the Fisher index of GDP than it was for the other two measures.

Table 1: Aggregate Economic Activity 1947–2018

Measure per worker	Average annual growth rate	Level after 70 years
Fischer index of GDP	1.68%	1.33
GDP in consumption units	1.33%	1.05
NDP in consumption units	1.27%	1.00

The sizeable differences among the annual growth rates of the three measures of aggregate economic activity raise the question which one growth theorists should use when they bring their models to the data. A common view in the literature is that GDP constructed with the Fisher index is appropriate for productivity analysis whereas NDP in units of consumption is

[1] See Heal and Kristrom (2005) and Fenichel et al. (2018) for reviews of the related environmental literature and Oulton (2004) and Bridgman (2018) for examples of the macro literature.

appropriate for welfare analysis; see for example Hulten (1992) and Oulton (2004). In contrast, the "hybrid" GDP in units of consumption plays a role mostly when constructing a balanced growth path and analyzing its properties because GDP constructed with the Fisher index is too cumbersome to work with theoretically. We assess the common view in the context of the standard two-sector growth model, in which one sector produces consumption and the other investment. The two-sector model is a natural starting point in our context because it captures secular changes in the relative price of investment to consumption, and so the question in which units to measure aggregate economic activity takes center stage. We use the discrete-time version of the model because NIPA data comes in discrete time and we assume that the model economy is closed because we focus on GDP and NDP.[2]

Our main contribution is to show that real GDP constructed with the Fisher index is the preferable measure of aggregate economic activity when one seeks to bring the two-sector growth model to the data. To arrive at that conclusion, we restrict the parameters so that the two-sector growth model captures the behavior of the major long-run moments of the postwar U.S. economy, including the marked productivity growth slowdown and the marked decline in the relative price of investment that occurred since the 1970s. We show that under the required parameter restrictions, real GDP measured with the Fisher index has a welfare interpretation but Weitzman's NDP in consumption units does not. We add the observation that GDP in consumption units does not have a known welfare interpretation even though it is often a useful concept when one seeks to theoretically analyze the two-sector model.

Turning now to the details of our analysis, we start with NDP in consumption units and derive a discrete-time version of Weitzman's (1976) result that the NDP in consumption units is proportional to permanent income (i.e., the return on the present discounted sum of current and future aggregate consumption). Proving the result in a growing economy requires constant consumption growth and a constant real interest rate [Oulton (2004)]. We therefore first construct a balanced growth path ("BGP") of the two-sector model along which all growth rates including those of NDP and GDP in consumption units are constant and the real interest rate

[2]To avoid confusion, note that Weitzman (1976) used the net *national* product ("NNP"), which acknowledges that the U.S. economy is open. As subsequent work by Oulton (2004) or Bridgman (2018), we focus on NDP because GDP is more commonly used than NDP and the numbers for NDP and GDP are very similar for the postwar U.S. We also note that we focus on the question how to measure aggregate *market* activity, which implies that we do not speak to the related question of how to measure *non-market* activity that takes place in the form of home production or public-goods provision.

is constant too. Afterwards, we derive the discrete-time version of Weitzman's result along the BGP, which establishes that NDP, not GDP, in consumption units has a welfare interpretation.

Despite its simplicity and intuitive appeal, there are important limitations of using NDP in consumption units as a measure of aggregate economic activity. To begin with, as with all money metrics, it is a cardinal welfare measure and it does not necessarily rank alternative equilibrium paths in the same way as the present discounted value of period utility would [Dasgupta (2009)]. Moreover, and perhaps more importantly in our context, deriving Weitzman's result requires that the growth rates of NDP and GDP in consumption units and of the relative price are all constant. We establish that these requirements are counterfactual for the postwar U.S.: starting in the 1970s, the growth rate of NDP and GDP in consumption units showed a marked slowdown; starting in the 1980s, the growth rate of the relative price of investment to consumption fell more and more strongly.

Next, we show that the counterfactual requirements can be avoided when we measure aggregate economic activity with real GDP constructed with the Fisher index.[3] We show that if the economy is on an equilibrium path, then the Fisher index approximates an ordinal welfare measure in discrete time, irrespective of whether or not the equilibrium path is a BGP. The welfare result is an adoption of the result of Diewert (1976) on superlative indexes to the two-sector growth model. Durán and Licandro (2017) were the first to do that for the continuous-time version. We provide a new proof for discrete time, which provides separate first-order approximations for the Laspeyres and the Paasche indexes.

To connect the two-sector model to the postwar U.S. economy, we construct an equilibrium path that does not restrict the equilibrium growth rates of GDP and NDP in consumption units and of the relative price of investment to be constant. Since changing growth rates violate the equilibrium concept of BGP, we need a weaker equilibrium concept. We follow Kongsamut et al. (2001) and use generalized balanced growth ("GBGP), which just requires that the real interest rate be constant while other variables may or may not grow at constant rates. We restrict the model parameters to match the observed constant trend TFP growth in the investment sector, to match the constant average ratio of consumption to investment expenditure, and to match the varying relative price of investment. We show that real GDP growth calculated with the Fisher index then exhibits a growth slowdown along the GBGP just like it did in the data.

[3] Note that the Fisher index is approximately equal to other superlative indexes such as the Törnqvist index. Therefore, the results we derive hold approximately also for Törnqvist index.

Our results leave the practical question how to bring the growth model to the data. After all, notwithstanding the attractive features that GDP constructed with the Fisher index has, there is no debate about the fact that it is too cumbersome to work with theoretically. We advocate to proceed in three steps: (i) construct a GBGP in the model that matches the changing growth rate of the relative price of consumption; (ii) construct GDP with the Fisher index using the model quantities and prices from the GBGP; (iii) compare the resulting measure of model GDP to data GDP. This procedure has several advantages. To begin with, it implies that the measure of aggregate economic activity – the Fisher index of GDP – is constructed in the same way in the model and in the data. Moreover, the model can generate the productivity growth slowdown of GDP per worker that is observed in the data. And lastly, the Fisher index of GDP is a measure of aggregate economic activity that has a welfare interpretation also when there is no BGP.

The organization of the rest of the paper is as follows. We first lay out the two-sector growth model and then study Weitzman's proposal to measure aggregate economy activity by using the NDP in consumption units. Afterwards, we study the BEA's practice of measuring aggregate economic activity by using the Fisher index of GDP. Lastly, we conclude. An Appendix contains all proofs and some background material.

2 Two-sector growth model

The two-sector version of the growth model goes back to Uzawa (1963). More recently, it has been developed in several directions by Greenwood et al. (1997), Gort et al. (1999), and Oulton (2007), among others. For ease of exposition, we employ the most basic version of the two-sector model with consumption and investment, noting that either one could be disaggregated further.

2.1 Environment

The representative household is endowed with initial capital $K_0 > 0$ and one unit of time in each period. Capital K_t accumulates according to

$$K_{t+1} = (1 - \delta)K_t + X_t,$$

where $\delta \in [0, 1]$ is the depreciation rate and X_t is investment.

The utility function is

$$\sum_{t=0}^{\infty} \beta^t \log(C_t),$$

where $\beta \in (0, 1)$ is the discount factor and C_t is consumption.

The sectoral production functions for consumption and investment are:

$$C_t = K_{ct}^{\theta}\left(A_{ct}L_{ct}\right)^{1-\theta}, \tag{1a}$$

$$X_t = K_{xt}^{\theta}\left(A_{xt}L_{xt}\right)^{1-\theta}, \tag{1b}$$

where $\theta \in (0, 1)$ is the capital-share parameter; K_{it} and L_{it} are sectoral capital and labor; changes in A_{it} reflect the exogenous, sector-specific, labor-augmenting technological progress. Having the same θ across sectors has the advantage that the production side aggregates. Herrendorf et al. (2015) established that having Cobb-Douglas production functions with equal capita-share parameters nonetheless captures the key features of labor reallocation in the postwar U.S.

Capital and labor are freely mobile between the sectors. Feasibility requires:

$$K_{ct} + K_{xt} \leq K_t,$$

$$L_{ct} + L_{xt} \leq L_t = 1.$$

2.2 Competitive equilibrium

A competitive equilibrium is a sequence of prices and an allocation such that: given prices, the allocation solves the household's problem and the firms' problems in each sector; markets clear. Since the two-sector model is well known, we state the standard most equilibrium properties without deriving them in detail. Herrendorf et al. (2014) provided the detailed steps of the derivation.

The household maximizes its utility subject to the budget constraint and the feasibility constraints:[4]

$$\max_{\{C_t,K_{t+1},B_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t \log(C_t) \tag{2}$$

$$\text{s.t.}\quad p_{ct}C_t + p_{kt}K_{t+1} + B_{t+1} = [(1-\delta_k)p_{xt} + r_t]K_t + (1+i_t)B_t + w_t,$$

$$C_t, K_{t+1} \geq 0,\quad B_t > -\underline{B} > -\infty,\quad K_0 > 0,\quad B_0 = 0.$$

The budget constraint is written in current dollars. B_t is the current-dollar value of the stock of a one-period government bond in period t and i_t is the corresponding interest rate. r_t and w_t denote the rental prices of capital and labor in current dollars. Lastly, p_{xt} and p_{ct} denote the prices of the sectoral outputs in current dollars.[5]

The first-order conditions imply the usual Euler equation and transversality condition:

$$\frac{p_{ct+1}C_{t+1}}{p_{ct}C_t} = \beta\frac{p_{xt+1}}{p_{xt}}\left(1-\delta+\frac{r_{t+1}}{p_{xt+1}}\right), \tag{3a}$$

$$\frac{p_{ct+1}C_{t+1}}{p_{ct}C_t} = \beta\left(1+i_t\right), \tag{3b}$$

$$0 = \lim_{t\to\infty}\beta^t\frac{p_{xt}K_{t+1}}{p_{ct}C_t}. \tag{3c}$$

Profit maximization in each sector implies that the rental prices for capital and labor, r_t and w_t, equal the marginal revenue products. For $i \in \{x, c\}$, this gives:

$$r_t = p_{it}\theta\left(\frac{K_{it}}{L_{it}}\right)^{\theta-1}A_{it}^{1-\theta}, \tag{4a}$$

$$w_t = p_{it}(1-\theta)\left(\frac{K_{it}}{L_{it}}\right)^{\theta}A_{it}^{1-\theta}. \tag{4b}$$

Combining the first-order conditions gives the usual result that the capital–labor ratios are

[4] For the household problem to be well defined, the objective function must be finite. With log-period utility, a sufficient condition is that consumption is bounded from above by a sequence that grows at a constant rate. That will be the case along the equilibrium paths we analyze below; along the BGP of Subsection 3.1, consumption itself grows at a constant rate; along the GBGP of Subsection 4.3, consumption will be bounded from above by a sequence that grows at a constant rate.

[5] As usual in growth models, only relative prices matter and the units are arbitrary. While we think of p_{it} as being denominated in current dollars for now, below we will chose consumption as the numeraire in the analysis of Weitzman's NDP and investment as the numeraire in the analysis of the Fisher index because doing so is more convenient.

equalized:

$$\frac{K_{xt}}{L_{xt}} = \frac{K_{ct}}{L_{ct}} = K_t, \tag{5}$$

where the last equality follow from the fact that $L_t = 1$. The relative price of consumption to investment is inversely related to relative sector TFPs:

$$\frac{p_{ct}}{p_{xt}} = \left(\frac{A_{xt}}{A_{ct}}\right)^{1-\theta}. \tag{6}$$

Panel a) of Figure 1 shows that the relative price of consumption to investment increased in the postwar U.S. We will therefore focus on the case $\widehat{A}_{xt} > \widehat{A}_{ct}$, where a "hat" denotes a growth factor. For example,

$$\widehat{A}_{xt+1} \equiv \frac{A_{xt+1}}{A_{xt}}.$$

Panel b) of the figure shows that the growth rate of the relative price started to accelerate in the 1980s. This feature of the relative price will turn out to be a crucial force behind the productivity growth slowdown.

In Figure 1, and all data work that follows, X_t is all investment including the purchases of consumer durables. In contrast, the BEA treats purchases of consumer durables as personal consumption expenditures, instead of investment expenditure. The BEA's practice is problematic because conceptually purchases of consumer durables are investment expenditures that ought to be capitalized. Capitalizing consumer durables is particularly important in our context because they importantly contribute to the decline of the relative price of investment, which takes center stage in our analysis. We therefore follow the methodology of Cooley and Prescott (1995) and modify the BEA's published NIPA data in two ways: (i) we reassign the flow of purchases of consumer durables from consumption to investment; (ii) we add an imputed service flow from the stock of consumer durables to consumption expenditure. Appendix A derives the imputation formula for the services flow from consumer durables in a more disaggregate version of the growth model that explicitly has consumer durables.

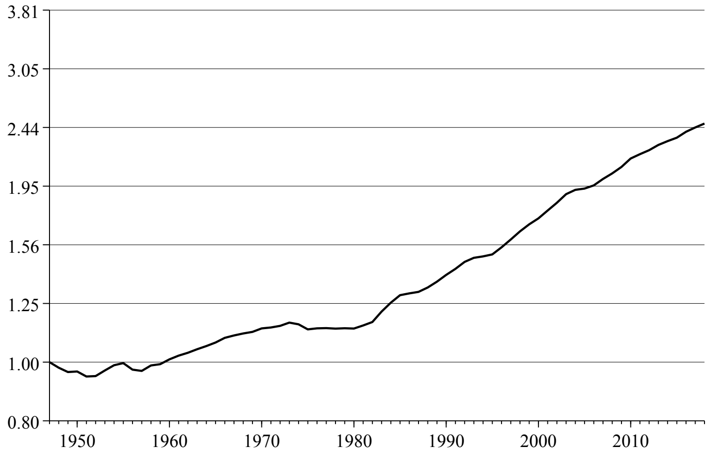
Combining (5)–(6), equations (1) become:

$$C_t = K_t^{\theta} A_{ct}^{1-\theta} L_{ct}, \tag{7a}$$

$$X_t = K_t^{\theta} A_{xt}^{1-\theta} L_{xt}. \tag{7b}$$

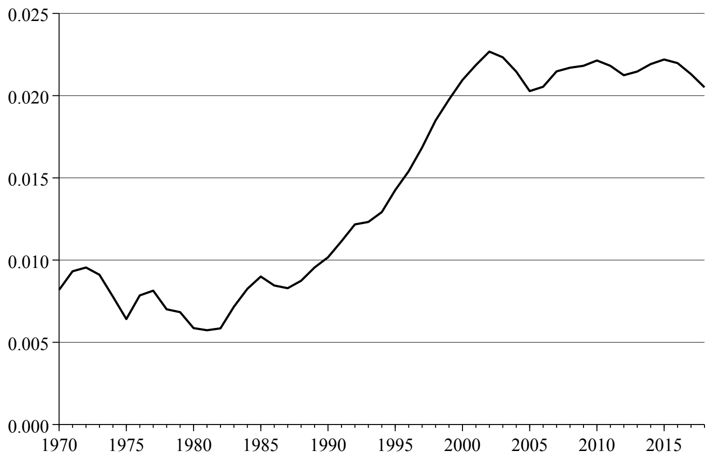
Figure 1: The Price of Consumption relative to Investment in the Postwar U.S. (log scale, 1947=1)

a) Levels in log scale (1947=1)



Source: NIPA, Bureau of Economic Analysis, own calculations.

b) Average annual growth rates over the preceding 20 years



Source: NIPA, Bureau of Economic Analysis, own calculations.

As usual with Cobb-Douglas production functions with equal share parameters, (6) and (7) imply the usual result that the expenditure ratio equals the labor ratio:

$$\frac{p_{ct}C_t}{p_{xt}X_t} = \frac{L_{ct}}{L_{xt}}.$$

Hence, we can restrict our attention to analyzing the properties of the expenditure ratio.

We are now ready to discuss how to measure aggregate economic activity in the model and connect the result to NIPA data. The goal is to find a measure that does have a welfare interpretation. We start with the suggestion of Weitzman (1976) to use the net domestic product in consumption units. NDP is applied both in the environmental literature and the macro literature; see Heal and Kristrom (2005) and Fenichel et al. (2018) reviews of the former and Oulton (2004) and Bridgman (2018) for examples of the latter. The idea of measuring aggregate economic activity as GDP in units of consumption has also gained traction more broadly in the macro literature; see for example Greenwood et al. (1997). While doing that may be helpful to analyze the two-sector growth model theoretically, there is no known proof that GDP in consumption units has a welfare interpretation. Therefore, we do not pursue this possibility further in the main part of the paper.

3 Weitzman's Net Domestic Product

Since Weitzman's NDP is in consumption units, it is convenient to choose consumption as the numeraire and set $p_{ct} = 1$ in this section. GDP in units of consumption is defined as:

$$Y_t^C \equiv C_t + p_{xt}X_t.$$

Using (6)–(7), we have that:

$$Y_t^C = K_t^{\theta} A_{ct}^{1-\theta}. \tag{8}$$

The net domestic product ("NDP") in consumption units results after subtracting depreciation in consumption units:[6]

$$Z_t^C \equiv Y_t^C - \delta p_{xt} K_t = K_t^\theta A_{ct}^{1-\theta} - \delta p_{xt} K_t.$$

Weitzman argued that the NDP in consumption units is a measure of aggregate welfare. The basic intuition behind his claim is that aggregate welfare surely depends on consumption, so current consumption must obviously be part of any aggregate welfare measure. Net investment must also be part of it because it increases future consumption. Weitzman (1976) also went beyond the basic intuition and gave a formal justification for using NDP in consumption. In particular, he showed in a continuous-time version of the growth model without TFP growth that NDP in consumption units is proportional to the maximized value of current and future consumption subject to feasibility. He also showed that one can reinterpret the result as implying that NDP in consumption units is proportional to permanent income.

Subsequent work by Weitzman (1997) and Oulton (2004) established that Weitzman's result holds as well if consumption grows at a constant rate as long as the real interest rate is constant. Moreover, Oulton (2004) showed that Weitzman's result holds as well if the real interest rate fluctuates but TPF growth is zero. Going down that path, however, is not useful in our context because, in at least one sector, TFP growth must be positive in order to match postwar U.S. data. Lastly, Barro (2019) has recently offered a new take on how to measure welfare, in which he also argues that current and future consumption is relevant for economic welfare.

Our goal is to establish Weitzman's result in discrete time. We first characterize a balanced growth path ("BGP") of the two-sector model along which all variables grow at constant rates (including zero). In particular, along the BGP, consumption grows at a constant rate and the real interest rate is constant along the BGP. We then derive the relationship between NDP in consumption units and permanent income along the BGP.

[6] NDP in consumption units is related to net national product ("NNP"). The difference is that NNP adjusts for net income from abroad whereas NDP does not. We use NDP here because our model economy is closed.

3.1 BGP with constant consumption growth

As above, we denote growth factors by hats. Moreover, given that consumption is the numeraire, we define the real interest rate factor as:

$$R_t \equiv (1 + i_t)\frac{P_{ct}}{P_{ct+1}}.$$

We will use the real interest rate factor below to calculate the present discounted value of current and future consumption, which is at the core of Weitzman's idea.

Proposition 1 *Suppose that $\widehat{A}_x$ and $\widehat{A}_c$ are constant. There is a BGP equilibrium along which $\{K_t, X_t, C_t, Y_t^C, Z_t^C\}_{t=0}^{\infty}$ grow at the following constant rates:*

$$\widehat{K}_t = \widehat{X}_t = \widehat{A}_x, \tag{9a}$$

$$\widehat{C}_t = \widehat{Y}_t^C = \widehat{Z}_t^C = \widehat{A}_x^{\theta}\widehat{A}_c^{1-\theta}. \tag{9b}$$

In addition, if $p_{ct} = 1$, then

$$r_t = \theta\left(\frac{K_t}{A_{ct}}\right)^{\theta-1}, \tag{9c}$$

$$w_t = (1-\theta)\left(\frac{K_t}{A_{ct}}\right)^{\theta}, \tag{9d}$$

$$R_t = \left(\frac{\widehat{A}_c}{\widehat{A}_x}\right)^{1-\theta}\left[1 - \delta + \theta\left(\frac{K_0}{A_{x0}}\right)^{\theta-1}\right]. \tag{9e}$$

R_t is constant and $\{r_t, w_t, p_{xt}\}_{t=0}^{\infty}$ grow at the following constant rates:

$$\widehat{r}_t = \widehat{p_{xt}} = \left(\frac{\widehat{A}_c}{\widehat{A}_x}\right)^{1-\theta}, \tag{9f}$$

$$\widehat{w}_t = \widehat{A}_x^{\theta}\widehat{A}_c^{1-\theta}. \tag{9g}$$

Proof. See Appendix B.1.

We are now ready to derive a relationship between the present discounted sum of current and future consumption and the initial NDP in consumption units along the previous BGP.

3.2 Weitzman's result in discrete time

We start by discussing the relationship between the allocations of two different problems: the household problem of maximizing the present discounted value of the sum of current and future *utility* subject to feasibility, which is stated in (2) with $p_{ct} = 1$; Weitzman's problem of maximizing the present discounted value of the sum of current and future *consumption* subject to feasibility:

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{C_t}{\mathcal{R}_t} \;\; \text{s.t.} \;\; p_{ct}C_t + p_{kt}K_{t+1} = [(1-\delta_k)p_{xt} + r_t]K_t + w_t, \tag{10}$$

$$C_t, K_{t+1} \geq 0, \quad K_0 > 0, \tag{11}$$

where $\mathcal{R}_t \equiv \Pi_{i=1}^{t} R_i$ for ≥ 1 and $\mathcal{R}_0 \equiv 1$. The following Lemma links the two problems to each other:[7]

Lemma 1 *Suppose the* $\{\mathcal{R}_t\}_{t=0}^{\infty}$ *implied by the solution to Problem* (2) *are the discount factors in Problem* (10). $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ *is part of the solution to Problem* (2) *if and only if it solves Problem* (10).

Proof. See Appendix B.2.

In what follows we will use the sequence $\{\mathcal{R}_t\}_{t=0}^{\infty}$ implied by the solution to Problem (2) as the discount factors in Problem (10).[8] Since, along the BGP, $R_t = R$ constant and $\widehat{C} = \widehat{A}_x^{\theta}\widehat{A}_c^{1-\theta}$, the previous lemma implies that the value of Problem (10) in period t is given by:

$$\sum_{s=t}^{\infty} \frac{C_s}{R^{s-t}} = C_t \sum_{s=t}^{\infty} \left(\frac{\widehat{A}_x^{\theta}\widehat{A}_c^{1-\theta}}{R} \right)^{s-t}.$$

The Euler equation from the BGP implies that:

$$\widehat{C} = \widehat{A}_x^{\theta}\widehat{A}_c^{1-\theta} = \beta R < R. \tag{12}$$

[7] Hulten (1992) states a similar result and cites Fisher (1930) for the basic idea of the proof.

[8] Note that the choice of the real interest factor with which future consumption is discounted is somewhat arbitrary. As long as the real interest rate factor is constant, Weitzman's result that NDP in consumption units is proportional to permanent income holds for any real interest rate factor.

Applying the formula of the geometric row and taking the limit for $t \to \infty$ therefore gives:

$$\sum_{s=t}^{\infty} \frac{C_s}{R^{s-t}} = \frac{R}{R - \widehat{A}_x^{\theta} \widehat{A}_c^{1-\theta}} C_t.$$

The feasibility constraint implies that along the BGP:

$$C_t = Z_t^C - (p_{xt} X_t - \delta p_{xt} K_t) = Z_t^C - p_{xt}(K_{t+1} - K_t) = Z_t^C - (\widehat{A}_x - 1) p_{xt} K_t. \tag{13}$$

Thus, we have shown that along the BGP:

$$\frac{R-1}{R} \sum_{s=t}^{\infty} \frac{C_s}{R^{s-t}} = \frac{R-1}{R - \widehat{A}_x^{\theta} \widehat{A}_c^{1-\theta}} \left[Z_t^C - (\widehat{A}_x - 1) p_{xt} K_t \right]. \tag{14}$$

In words, along the BGP, the return on the present discounted sum of current and future consumption ("permanent income") is proportional to the difference between the initial NDP in consumption units, Z_t^C, and the additional term $(\widehat{A}_x - 1) p_{xt} K_t$ also in consumption units. This result is closely related to the original result of Weitzman (1976). In a steady state of the model that Weitzman (1976) considered, $\widehat{A}_x = 1$ and Weitzman's original result obtains that permanent income equals the initial NDP in consumption units:

$$\frac{R-1}{R} \sum_{s=t}^{\infty} \frac{C_s}{R^{s-t}} = Z_t^C.$$

While that is no longer the case when sectoral TFP grows, along the BGP of Proposition 1, $\frac{R-1}{R} \sum_{s=t}^{\infty} \frac{C_s}{R^{s-t}}$ and Z_t^C still grow at the same factor $\widehat{A}_x^{\theta} \widehat{A}_c^{1-\theta}$.

Subsequent work in the continuous-time growth model generalized Weitzman's initial result to the case of constant TFP growth and showed that permanent income is still proportional to the level of initial NDP in consumption units. Appendix C offers a proof of the generalization in a continuous-time version of our model. While this has been proven before, our proof is somewhat more direct than existing proofs. Unfortunately, it is not obvious how to establish the corresponding result in discrete time with $\widehat{A}_x > 0$. We therefore leave it at equation (14). While it does not establish that permanent income and initial NDP in consumption units are proportional to each other, it does connect the two in discrete time.

3.3 Discussion

Despite its simplicity and intuitive appeal, NDP has important limitations. The first one is that NDP is a cardinal welfare measure that just adds up period consumption without taking account of curvature. One implication is that only in two special cases does NDP rank all consumption paths in the same way as the present discounted sum of period utilities: the marginal period utility is constant for all relevant consumption paths; the model economy is in steady equilibrium in which consumption is constant.[9] Along a BGP of our model, either one of these conditions is violated and there is no guarantee that the NDP ranks consumption paths in the same way as the present discounted sum of period utilities; see Oulton (2004) and Dasgupta (2009) for further discussion.[10]

An second limitation of NDP is that to derive its relationship with permanent income we needed to assume that consumption growth and the real interest rate are both constant. While Proposition 1 shows it is possible to construct a BGP with these properties, along the BGP $\widehat{p}_{xt}$ and $\widehat{Z}_t^C$ are both constant. Figures 1 and 2 show that this is counterfactual. In particular, relative price growth accelerated in the 1980s and NDP growth slowed down in the 1970s. If one is interested in matching these prominent features of the data, then one must give up the assumption that $\widehat{A_x}$ and $\widehat{A_c}$ are both constant. Unfortunately, that means that it is no longer clear that Z_t^C has a welfare interpretation, because known proofs all rest on the assumption of constant level or a constant growth rate of consumption.

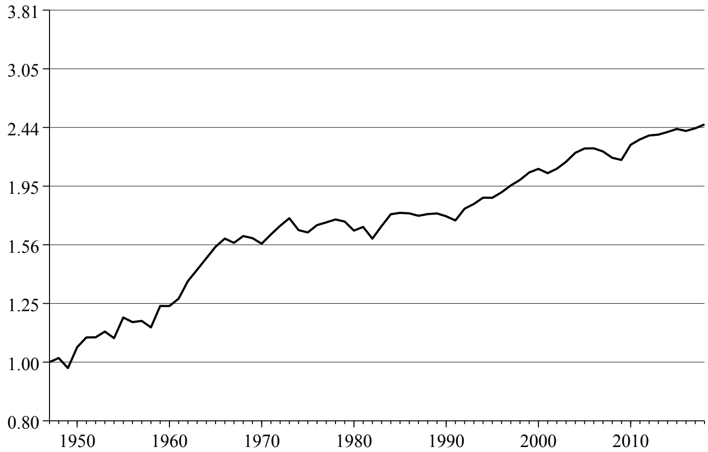
Given the limitations of NDP in consumption units, it is natural to consider an alternative measure of aggregate economic activity. We are particularly interested in a measure that has a welfare interpretation under more general conditions that allow for the slowdown in the growth of aggregate economic activity and the acceleration of the decline of the relative price of investment. We now establish that we can satisfy these requirements for real GDP constructed with the Fisher index.

[9] See Dasgupta and Mäler (2000) for further discussion.

[10] To avoid confusion, we should mention that Asheim and Weitzman (2001) showed that if utility is concave and real NDP is constructed with a Divisia consumption price index, then welfare increases if and only if real NDP in consumption units increases. While that means that welfare and NDP move in the same direction, it does not mean that NDP ranks different consumption path in the same way as does the present discounted sum of period utilities.

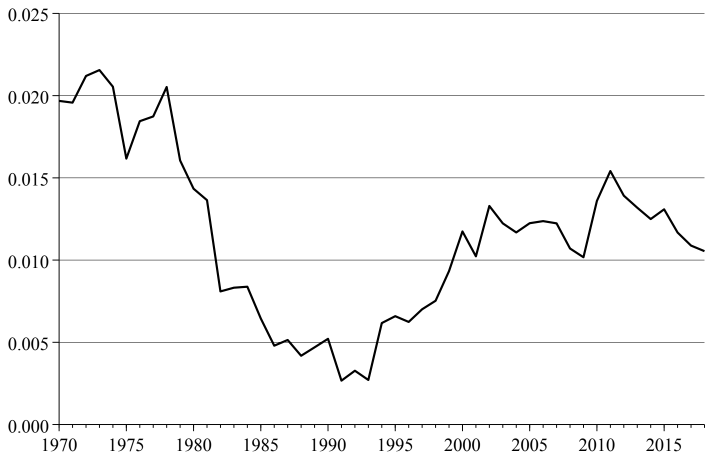
Figure 2: NDP per Worker in Consumption Units in the Postwar U.S.

a) Levels in log scale (1947=1)



Source: NIPA, Bureau of Economic Analysis, own calculations.

b) Average annual growth rates over the preceding 20 years



Source: NIPA, Bureau of Economic Analysis, own calculations.

4 Measuring GDP with the Fisher Index

4.1 Introducing the Fisher index

For any two adjacent periods, the Fisher quantity index is defined as the geometric average of the Laspeyres and Paasche (quantity) indexes:[11]

$$\widehat{Y}_t^F \equiv \sqrt{\widehat{Y}_t^L \cdot \widehat{Y}_t^P} \equiv \sqrt{\frac{p_{ct-1}C_t + p_{xt-1}X_t}{p_{ct-1}C_{t-1} + p_{xt-1}X_{t-1}} \cdot \frac{p_{ct}C_t + p_{xt}X_t}{p_{ct}C_{t-1} + p_{xt}X_{t-1}}}. \tag{15}$$

The definition immediately implies that GDP growth with the Fisher index is independent of the numeraire. To see this, just pull out p_{ct-1} and p_{ct} or p_{xt-1} and p_{xt} from the numerators and denominators of equation (15).

The GDP level in a particular year is obtained by choosing a reference year and then chaining the annual growth rates calculated with the Fisher index between the reference year and the year in question. For example, choosing year 0 as the reference year and denoting the nominal GDP of period 0 by Y_0 gives:

$$Y_t^F = Y_0 \cdot \widehat{Y}_1^F \cdot ... \cdot \widehat{Y}_t^F.$$

This is what the BEA calls real GDP in chained dollars.

4.2 The Fisher index as a measure of welfare changes

Our goal is to show that GDP constructed with the Fisher index is an ordinal welfare measure which is invariant to monotonic transformations of utility and which applies on and off the BGP equilibrium. The construction involves compensating expenditure and indifference between different allocations. Insisting on indifference is crucial because it is an ordinal, instead of a cardinal, concept that is invariant under monotonic transformations of utility. The basic underlying result is due to Diewert (1976), who showed that the Fisher index is a superlative index, that is, it is capable of providing an approximation to an arbitrary twice-continuously-differentiable utility function. The basic idea of how to apply this result in the dynamic context of the growth model is due to Licandro et al. (2002) and Durán and Licandro (2017). Here, we build on their results from continuous time and prove corresponding results in discrete-time ver-

[11]Fisher (1922) called it the ideal index. Triplett (1992) and Whelan (2002) offer detailed discussions of the properties of the Fisher index.

sion of the two-sector growth model. As in the previous case with Weitzman's result, achieving this presents some additional challenges.

We start by defining the expenditure functions and the indirect utility function needed to construct compensating expenditure. We choose X_t as the numeraire and set $p_{xt} = 1$. This is convenient because we will construct an equilibrium path under the assumption that $\widehat{A}_x$ is constant but $\widehat{A}_{ct}$ is not. Reflecting the new choice of numeraire, we denote GDP in units of investment as:

$$Y_t^X \equiv p_{ct} C_t + X_t = K_t^\theta A_{xt}^{1-\theta}. \tag{16}$$

The last equality follows again from (6)–(7).

We begin by stating the household's problem in recursive form. Denoting the value function by V, we have:

$$\begin{aligned} V(K_t, A_{xt}, A_{ct}) \equiv \max_{C_t, X_t} \Big\{ & \log(C_t) + \beta V(K_{t+1}, A_{xt+1}, A_{ct+1}) \\ \text{s.t} \quad & \frac{A_{xt}}{A_{ct}} C_t + X_t \leq K_t^\theta A_{xt}^{1-\theta}, \\ & (K_{t+1}, A_{xt+1}, A_{ct+1}) = \Big(X_t + (1-\delta) K_t, \widehat{A}_x A_{xt}, \widehat{A}_{ct} A_{ct} \Big) \Big\}, \end{aligned}$$

where we used that $A_{it+1} = \widehat{A}_{it} A_{it}$. We summarize the state variables by $S_t \equiv (K_t, A_{xt}, A_{ct})$ and write $V(S_t) = V(K_t, A_{xt}, A_{ct})$. We define an indirect utility function as:[12]

$$\begin{aligned} v(p_{ct}, Y_t^X; S_t) \equiv \max_{C_t, X_t} \Big\{ & \log(C_t) + \beta V(S_{t+1}) \\ \text{s.t} \quad & p_{ct} C_t + X_t \leq Y_t^X, \\ & S_{t+1} = (K_{t+1}, A_{xt+1}, A_{ct+1}) = \Big(X_t + (1-\delta) K_t, \widehat{A}_x A_{xt}, \widehat{A}_{ct} A_{ct} \Big) \Big\}. \end{aligned}$$

The definition of the indirect utility function drops the constraints $p_{ct} = A_{xt}/A_{ct}$ and $Y_t^X = K_t^\theta A_{xt}^{1-\theta}$ for period t, but leaves them in place for all subsequent periods. Hence, it implies the

[12]In dynamic contexts like ours there are two indirect utility functions: a period one and a present-value one. Our indirect utility function is a recursive formulation of the present-value indirect utility function, that is, the present value of the current and all future utilities that result under optimal behavior. In recursive formulation, that present value is a function of current income, current prices, and the current realizations of the state variables. To avoid confusion with the language used in Durán and Licandro (2017), we call their indirect value function an indirect utility function.

value of the program also for realizations of relative prices and income that are not consistent with equilibrium in period t. Similarly, the minimum-expenditure function for reaching the utility level v given prices p_{ct} is defined as:

$$e(p_{ct}, v; S_t) = \min_{C_t, X_t} \Big\{ p_{ct} C_t + X_t \\ \text{s.t. } \log(C_t) + \beta V(S_{t+1}) \geq v, \\ S_{t+1} = \Big(X_t + (1-\delta)K_t, \widehat{A}_x A_{xt}, \widehat{A}_{ct} A_{ct} \Big) \Big\}.$$

We now develop a measure of welfare changes that is based on compensating expenditure differences. The basic idea goes back to Fisher and Shell (1972), who generalized existing superlative indexes to situations in which preferences evolve over time. They emphasized that since utility is an ordinal concept, one must not compare the utility levels from periods $t-1$ and t. Instead, they calculated compensating expenditure levels by imposing indifference in terms of the *same* indirect utility function.[13] Durán and Licandro (2017) showed how to apply the true quantity index of Fisher and Shell (1972) to the two-sector growth model with general recursive preferences. The basic insight is that it does not matter whether the time dependence of $v(\cdot)$ and $e(\cdot)$ arises from evolving preferences, as in Fisher and Shell's model, or from evolving state variables, as in the growth model.[14] While Durán and Licandro (2017) used *continuous* time, we develop a true quantity index for *discrete* time. Using discrete time is both more natural for connecting model data to NIPA data and also is more cumbersome because it requires a careful distinction between different reference periods. A novelty of our work is that this leads to two perspectives: the backward-looking (forward-looking) perspective uses prices and realizations of the state variables from "today" ("yesterday").

The forward-looking perspective compares yesterday's observed expenditure, Y^X_{t-1}, with the compensating expenditure that the household needed yesterday to reach the same indirect utility as it gets from today's expenditure at today's prices.[15] Imposing indifference while keeping yesterday's state variables unchanged, this gives $e\Big(p_{ct-1}, v(p_{ct}, Y^X_t; S_{t-1}); S_{t-1}\Big)$. The following

[13] Although the original index of Fisher and Shell is a true cost-of-living index, it is straightforward to apply the underlying principles to the construction of the corresponding true quantity index.

[14] Fisher and Shell dismissed the forward-looking perspective because yesterday's tastes are no longer relevant today. In contrast, the forward-looking perspective is meaningful when yesterday's indirect utility function represents past realizations of the state variables.

[15] The latter is closely related to equivalent variation in static micro theory.

forward-looking true quantity index is:

$$\widehat{FS}_{t-1,t} \equiv \frac{e\Big(p_{ct-1}, v(p_{ct}, Y_t^X; S_{t-1}); S_{t-1}\Big)}{Y_{t-1}^X}.$$

The backward-looking perspective compares today's observed expenditure, Y_t^X, with the compensating expenditure that the household needs today to reach the same indirect utility as it gets from yesterday's expenditure at yesterday's prices.[16] Imposing indifference while keeping today's state variables unchanged, this gives $e\Big(p_{ct}, v(p_{ct-1}, Y_{t-1}^X; S_t); S_t\Big)$. The backward-looking true quantity index is:

$$\widehat{FS}_{t,t-1} \equiv \frac{Y_t^X}{e\Big(p_{ct}, v(p_{ct-1}, Y_{t-1}^X; S_t); S_t\Big)}.$$

The Fisher-Shell true quantity index is the geometric average of the forward- and backward-looking indexes:

$$\widehat{FS}_t \equiv \sqrt{\widehat{FS}_{t-1,t} \cdot \widehat{FS}_{t,t-1}}.$$

The next proposition states our main results that the Fisher quantity index first-order approximates the Fisher-Shell true quantity index. While this result is a discrete-time version of the one of Durán and Licandro (2017), we use a more direct method of proof that provides additional first-order approximations for the Laspeyres and the Paasche indexes.

Proposition 2 *Along an equilibrium path of the two-sector growth model,*

$$\begin{aligned}\widehat{FS}_{t-1,t} &\approx \widehat{Y}_t^L,\\ \widehat{FS}_{t,t-1} &\approx \widehat{Y}_t^P,\\ \widehat{FS}_t &\approx \widehat{Y}_t^F.\end{aligned}$$

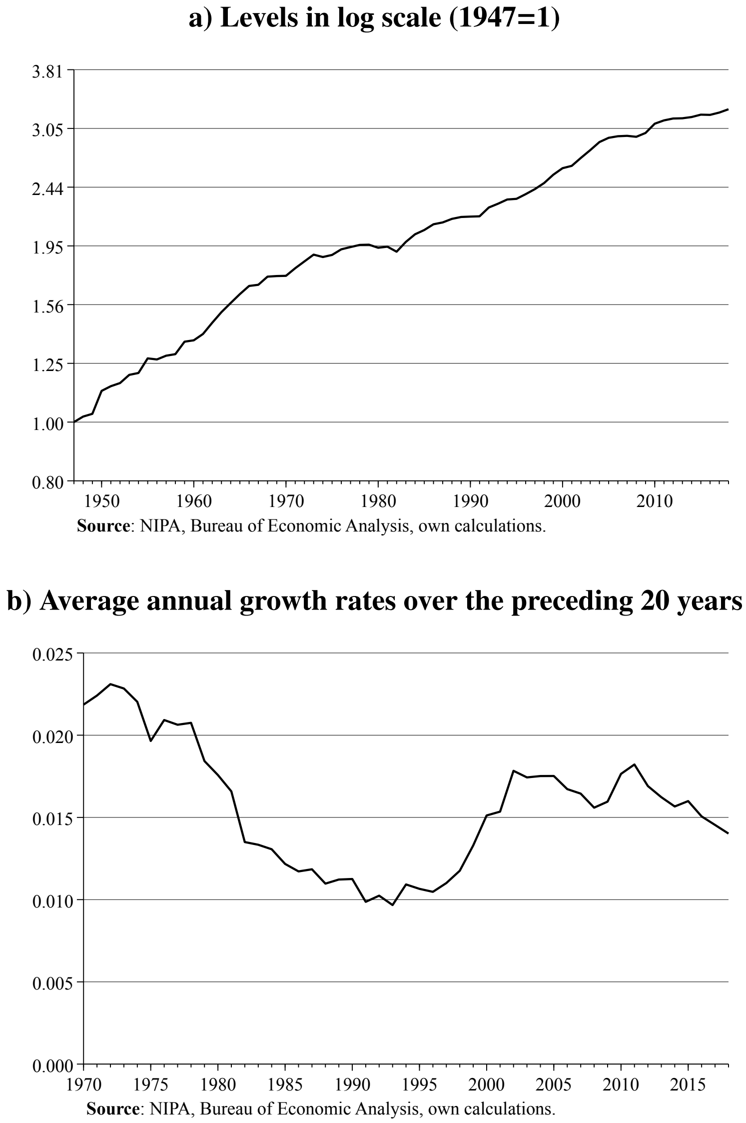
Proof. See Appendix B.3

In sum, we have established that the Fisher index first-order approximates the change in the Fisher-Shell index. We emphasize that this is an *ordinal* welfare measure that applies irrespective of whether the economy is on a BGP equilibrium. In particular, at no point in the above derivations did we assume that GDP or NDP per worker grow at a constant rate, which was essential for the proof of Weitzman's result. This is important because the growth of GDP per

[16]The latter is closely related to compensating variation in static micro theory.

worker as measured by the Fisher index slowed down since the 1970s; see Figure 3. In the next subsection, we will construct an equilibrium path that captures the growth slowdown.

Figure 3: GDP per Worker Measured with the Fisher Index in the Postwar U.S.



4.3 A GBGP that captures the growth slowdown

We now establish that one can avoid the counterfactual implications of the previous BGP by relaxing the assumption that $\widehat{A}_{ct}$ be constant. As long as $\widehat{A}_x$ remains constant as before, one can still construct an equilibrium path. The new equilibrium path does not have the counterfactual features of the previous one, because it permits the growth rates of the relative price and of

GDP measured in units of consumption (or in the Fisher index) to vary over time.

The new equilibrium path is not a BGP because only a subset of endogenous variables grow at constant rates. We therefore adopt the less stringent equilibrium concept generalized balanced growth path ("GBGP"), which goes back to Kongsamut et al. (2001):

Definition 1 A GBGP is an equilibrium path along which the real interest rate is constant.

Given that investment is the numeraire, the real interest rate factor now equals:

$$R_t \equiv (1 + i_t)\frac{P_{xt}}{P_{xt+1}}. \tag{17}$$

Proposition 3 *Suppose that $\widehat{A}_x$ is constant. There is a GBGP equilibrium path along which $\{K_t, X_t, C_t, Y_t^X, Y_t^C, Z_t^C\}_{t=0}^{\infty}$ grow at the following rates:*

$$\widehat{K}_t = \widehat{X}_t = \widehat{Y}_t^X = \widehat{A}_x, \tag{18a}$$

$$\widehat{C}_t = \widehat{Y}_t^C = \widehat{Z}_t^C = \widehat{A}_x^{\theta}\widehat{A}_{ct}^{1-\theta}. \tag{18b}$$

In addition, if $p_{xt} = 1$, then

$$r_t = \theta\left(\frac{K_t}{A_{xt}}\right)^{\theta-1}, \tag{18c}$$

$$w_t = (1-\theta)\left(\frac{K_t}{A_{xt}}\right)^{\theta}, \tag{18d}$$

$$R_t = 1 - \delta + \theta\left(\frac{K_0}{A_{x0}}\right)^{\theta-1}. \tag{18e}$$

Lastly, r_t and R_t are constant and $\{w_t, p_{ct}\}_{t=0}^{\infty}$ grow at the following rates:

$$\widehat{w}_t = \widehat{A}_x, \tag{18f}$$

$$\widehat{p}_{ct} = \left(\frac{\widehat{A}_x}{\widehat{A}_{ct}}\right)^{1-\theta}. \tag{18g}$$

Proof. See Appendix B.4.

An important implication of Proposition 3 is that $\widehat{Y}_t^C$ and $\widehat{Z}_t^C$ are not necessarily constant along the GBGP equilibrium, as they depend on $\widehat{A}_{ct}$ that no longer is restricted to be constant. Moreover, it turns out that $\widehat{Y}_t^C$ and $\widehat{Z}_t^C$ may now slow down.

Corollary 1 *Suppose that $\widehat{A}_{ct}$ declines, $\widehat{A}_x$ is constant, and $\widehat{A}_{ct} < \widehat{A}_x$. Along the GBGP: $\widehat{Y}_t^C < \widehat{Y}^X$; $\widehat{Y}_t^C = \widehat{Z}_t^C$ declines; $\widehat{Y}^X$ is constant.*

Proof. See Appendix B.5.

The previous result implies that it is possible to construct an equilibrium path which features a slowdown of both $\widehat{Y}_t^C$ and $\widehat{Z}_t^C$. For connecting the model with the data, this result is not very useful though because $\widehat{Z}_t^C$ has a known welfare interpretation only when it grows at a constant rate. GDP calculated with the Fisher index is more useful in this context, as it has a welfare interpretation also if the economy is not on a BGP. To characterize its growth rate, we rearrange the terms in (15) while using that $\widehat{Y}_t^C = \widehat{C}_t$ and $\widehat{Y}_t^X = \widehat{X}_t$ along the GBGP:

$$\widehat{Y}_t^F = \widehat{Y}_t^C \sqrt{\frac{\frac{p_{xt}X_t}{C_t}\frac{p_{xt-1}}{p_{xt}} + 1}{\frac{p_{xt-1}X_{t-1}}{C_{t-1}}\frac{p_{xt}}{p_{xt-1}} + 1}} = \widehat{Y}_t^X \sqrt{\frac{1 + \frac{p_{ct}C_t}{X_t}\frac{p_{ct-1}}{p_{ct}}}{1 + \frac{p_{ct-1}C_{t-1}}{X_{t-1}}\frac{p_{ct}}{p_{ct-1}}}}. \tag{19}$$

Using equation (6) together with the assumption that $\widehat{A}_x$ is constant and the fact that $p_{xt}X_t/C_t$ and $p_{ct}C_t/X_t$ are constant along the GBGP, we get that along the GBGP:

$$\widehat{Y}_t^F = \widehat{A}_x^{\theta}\widehat{A}_{ct}^{1-\theta} \sqrt{\frac{\frac{p_xX}{C}\frac{\widehat{A}_x}{\widehat{A}_{ct}} + 1}{\frac{p_xX}{C}\frac{\widehat{A}_{ct}}{\widehat{A}_x} + 1}} = \widehat{A}_x \sqrt{\frac{1 + \frac{p_cC}{X}\frac{\widehat{A}_{ct}}{\widehat{A}_x}}{1 + \frac{p_cC}{X}\frac{\widehat{A}_x}{\widehat{A}_{ct}}}}. \tag{20}$$

Using these equations, we can show that:

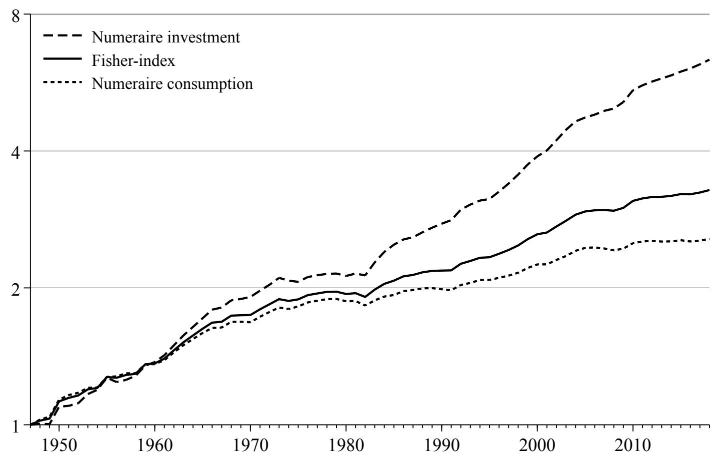
Proposition 4 *Suppose that $\widehat{A}_{ct}$ declines and $0 < \widehat{A}_{ct} < \widehat{A}_x$. Along the GBGP: $\widehat{Y}_t^C < \widehat{Y}_t^F < \widehat{Y}_t^X$; $\widehat{Y}_t^F$ declines.*

Proof. See Appendix B.6.

Proposition 4 raises the question of how large the differences between $\widehat{Y}_t^C$, $\widehat{Y}_t^X$, and $\widehat{Y}_t^F$ are in the data. Figure 4 shows that they are sizeable. This implies that it matters which measure we use. It also implies that it is crucial to use the same measure of GDP in both model and data. This point, of course, illustrates the importance of the more general principle that one must measure things in the same way in the model and in the data.[17] Figure 4 also shows that while

[17]Whelan (2003) emphasized this principle in the context of GDP measurement.

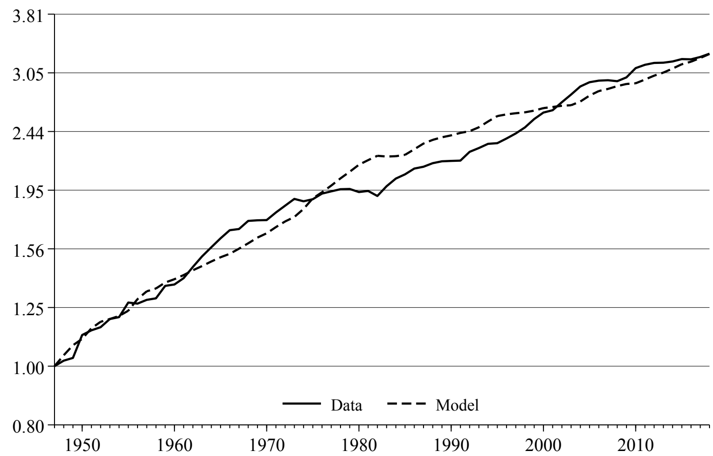
Figure 4: Different Measures of U.S. GDP per Worker



Source: NIPA, Bureau of Economic Analysis, own calculations.

GDP growth measured in the numeraire investment has a near constant long-run trend, GDP growth measured with the Fisher index indeed slows down, but not as much as GDP growth measured with the numeraire consumption.

Figure 5: Data versus model of GDP per worker measured with the Fisher index



Source: NIPA, Bureau of Economic Analysis, own calculations.

The question remains whether the GBGP can quantitatively generate the growth slowdown of the Fisher index of GDP per worker. Figure 5 plots the Fisher index of GDP per worker. The solid line represents the data whereas the dotted line represents the model along the GBGP, where GDP along the GBGP is calculated by fixing $\widehat{A}_{xt}$ and $p_{ct}C_t/X_t$ at their average values $\widehat{A}_x$ and p_cC/X and by choosing $\widehat{A}_{ct}$ so as to match the $\widehat{p}_{ct}$. Figure 5 shows that the GBGP of the model does generate the entire growth slowdown. We note though that in the model the growth slowdown starts about ten years later than in the data (that is, in the early 1980s versus the early

1970s). The discrepancy in the timing comes from the fact that along the GBGP of the model the sole driver of the growth slowdown in GDP per worker is the acceleration in the growth rate of the relative price of consumption; see the second equality of expression (20). Panel b) of Figure 1 shows that the acceleration starts in the early 1980s, which is about ten years after the start of the productivity growth slowdown. The initial slowdown in GDP per worker in the data is driven by additional low-frequency fluctuations in $\widehat{A}_{xt}$, which are noticeable in Figure 4 above. The GBGP misses them by construction because it requires $\widehat{A}_x$ to be constant.

4.4 Related literature on Baumol's cost disease

In the previous subsection, we have established that the two-sector growth model generates the growth slowdown of GDP per worker measured with the Fisher index, but that it does not get the timing exactly right because $\widehat{A}_t^X$, and therefore $\widehat{Y}_t^X$, show important low-frequency movements. The recent literature on structural change and Baumol's cost disease suggests that structural change may cause low-frequency movement in $\widehat{A}_{xt}$. The underlying reason is that at a more disaggregate level, investment and consumption are composites of goods and services. While the composition differs in that investment has a considerably higher goods share than does consumption, the usual patterns of structural change apply to the investment sector as they do to the consumption sector. In particular, as GDP increases, the share of value added from the goods sector in total investment expenditures goes down and from services sector goes up. As a result, the TFP of producing investment, A_{xt}, endogenously depends on the TFPs in the goods and services sectors. Since TFP growth is usually faster in the goods sector than in the services sector, structural change can cause a decline in $\widehat{A}_{xt}$. Herrendorf et al. (2020) established this claim analytically under the assumption that investment is a CES aggregator of goods and services.

Herrendorf et al. (2020) also explore the effects of structural change on the relative price of consumption and showed that a sizeable part of the accelerating increase in the relative price of consumption to investment is due to the fact that consumption expenditure contain a much larger share of services value added than investment expenditure. Faster TFP growth in the goods sector then endogenously increases the relative price of consumption to investment.

The slowdown in GDP per worker is closely related to the observation of Baumol (1967) that the costs of producing GDP increase and labor productivity decreases as the value-added share of services increases. Several recent papers discussed this observation. Ngai and Pissarides (2004) mentioned that Baumol's Cost Disease can lead to a GDP growth slowdown when GDP growth is calculated with constant relative prices. However, they did not pursue the growth slowdown further but framed their entire analysis in terms of a balanced growth path and constant GDP growth measured in a current numeraire. Moro (2015) provided an interesting model in which Baumol's Cost Disease reduces GDP measured with the Fisher index. His analysis differs from our analysis because he focused on the role of differences in the sectoral intermediate-input shares in a cross section of middle- and high-income countries. In independent work, Leon-Ledesma and Moro (2017) asked to what extent structural change may lead to violations of the Kaldor (1961) growth facts. In their simulation results, based on the model of Boppart (2014), structural change leads to a growth slowdown of GDP measured with the Fisher index. Lastly, Duernecker et al. (2017) study the natural follow up question whether GDP growth will slow further in the coming years. A particular worry is that the slowest-growing services industries could take over the economy. They find that substitutability within the service sector prevents that from happening.

5 Conclusion

We have studied two popular measures of aggregate economic activity: Weitzman's NDP in consumption units and the BEA's real GDP constructed with the Fisher index. We have established for the postwar US that it matters quantitatively which one is used because their growth rates differed substantially. We have compared their properties in a discrete-time version of the standard two-sector growth model with consumption and investment. Both measures can be given a welfare interpretation if the model is on an equilibrium path, but for the NDP the required parameter restrictions are so stringent that they rule out the observed productivity growth slowdown that has happened during the last decades. In contrast, for real GDP constructed with the Fisher index the required restrictions are less stringent and allow the model to generate the observed GDP growth slowdown.

We have concluded from our results that measuring real GDP with the Fisher index is preferable over using Weitzman's NDP. We have advocated to proceed in three steps when connecting aggregate economic activity in model to the data: (i) construct a GBGP in the model that matches the changing growth rate of the relative price of consumption; typically that will involve choosing a numeraire and expressing GDP in it; (ii) construct GDP with the Fisher index using the model quantities and prices from the GBGP; (iii) compare the resulting measures of model and data GDP. This procedure implies that the measure of aggregate economic activity – the Fisher index of GDP – is constructed in the same way in the model and in the data. Moreover, this procedure implies that the model can generate the productivity growth slowdown of GDP per worker that is observed in the data. And lastly, the Fisher index of GDP is a measure of aggregate economic activity that has a welfare interpretation.

Weitzman's NDP and real GDP constructed with the Fisher index abstracts from several relevant features of reality that affect welfare. Important examples are inequality, leisure, and life expectancy. Jones and Klenow (2016) proposed a broader welfare measure that takes these features into account and implemented it for a set of countries. Weitzman's NDP and real GDP constructed with the Fisher index also abstract from home production, which for many basic services is a close substitute to market production. Bridgman et al. (2018) proposed a way to impute the value of home production and they provided a broad measure of market and home production for around 30 countries. We leave integrating these additional relevant features of reality into the current framework is an important task for future research.

References

Asheim, Geir B. and Martin L. Weitzman, "Does NNP Growth Indicate Welfare Improvement?," *Economics Letters*, 2001, *73*, 233—239.

Barro, Robert J., "Double-Counting of Investment," Manuscript, Harvard University 2019.

Baumol, William J., "Macroeconomics of Unbalanced Growth: The Anatomy of the Urban Crisis," *American Economic Review*, 1967, *57*, 415–426.

Boppart, Timo, "Structural Change and the Kaldor Facts in a Growth Model with Relative Price Effects and Non-Gorman Preferences," *Econometrica*, 2014, *82*, 2167–2196.

Bridgman, Benjamin R., "Is Labor's Loss Capital's Gain? Gross versus Net Labor Share," *Macroeconomic Dynamics*, 2018, *22*, 2070–2087.

——, **Georg Duernecker, and Berthold Herrendorf**, “Structural Transformation, Marketization, and Household Production around the World,” *Journal of Development Economics*, 2018, *133*, 1–25.

Cooley, Thomas F. and Edward C. Prescott, “Economic Growth and Business Cycles,” in Thomas F. Cooley, ed., *Frontiers of Business Cycle Research*, Princeton, NJ: Princeton University Press, 1995.

Dasgupta, Parta and Karl-Göran Mäler, “Net National Product, Wealth, and Social Well-being,” *Environment and Development Economics*, 2000, *5*, 69–93.

Dasgupta, Partha, “The Welfare Economic Theory of Green National Accounts,” *Environmental and Resource Economics*, 2009, *42*, 3–38.

Diewert, W. Erwin, “Ecact and Superlative Index Numbers,” *Journal of Econometrics*, 1976, *4*, 115–145.

Duernecker, Georg, Berthold Herrendorf, and Ákos Valentinyi, “Structural Change within the Service Sector and the Future of Baumol’s Cost Disease,” Discussion Paper 12467, Centre for Economic Policy Research, London 2017.

Durán, Jorge and Omar Licandro, “Is Output Growth in NIPA a Welfare Measure?,” Manuscript 2017.

Fenichel, Eli P., Joshua K. Abbott, and Seong DoYun, “The Nature of Natural Capital and Ecosystem Income,” in Partha Dasgupta, Subhrendu K. Pattanayak, and V. Kerry Smith, eds., *Handbook of Environmental Economics*, Vol. 4, Elsevier, 2018, chapter 3, pp. 85–142.

Fisher, Franklin M. and Karl Shell, “Essay I: Taste and Quality Change in the Pure Theory of the True Cost-of-Living Index,” in Karl Shell, ed., *The Economic Theory of Price Indices – Two Essays on the Effects of Taste, Quality, and Technological Change*, Academic Press, 1972.

Fisher, Irving, *The Making of Index Numbers*, Boston: Houghton Mifflin, 1922.

——, *The Theory of Interest*, New York: Macmillan, 1930.

Gort, Michael, Jeremy Greenwood, and Peter Rupert, "Measuring the Rate of Technological Progress in Structures," *Review of Economic Dynamics*, 1999, *2*, 207–230.

Greenwood, Jeremy, Zvi Hercowitz, and Per Krusell, "Long-run Implication of Investment-Specific Technological Change," *American Economic Review*, 1997, *87*, 342–362.

Heal, Geoffrey and Bengt Kristrom, "National Income and the Environment," in Karl-Göran Mäler and Jeffrey R. Vincent, eds., *Handbook of Environmental Economics*, Vol. 3, Elsevier, 2005, chapter 22, pp. 1147–1217.

Herrendorf, Berthold, Christopher Herrington, and Ákos Valentinyi, "Sectoral Technology and Structural Transformation," *American Economic Journal: Macroeconomics*, 2015, *7*, 1–31.

—–, Richard Rogerson, and Ákos Valentinyi, "Growth and Structural Transformation," in Philippe Aghion and Steven N. Durlauf, eds., *Handbook of Economic Growth*, Vol. 2, Elsevier, 2014, pp. 855–941.

—–, —–, and —–, "Structural Change in Investment and Consumption: A Unified Approach," *forthcoming: Review of Economic Studies*, 2020.

Hulten, Charles R., "Accounting for the Wealth of Nations: The Net versus the Gross Output Controversy and its Ramifications," *Scandinavian Journal of Economics*, 1992, *94 (Supplement)*, S9–S24.

Jones, Charles I. and Peter J. Klenow, "Beyond GDP? Welfare across Countries and Time," *American Economic Review*, 2016, *106*, 2426–2457.

Kaldor, Nicholas, "Capital Accumulation and Economic Growth," in Friedrich A. Lutz and D. C. Hague, eds., *The Theory of Capital*, New York: St, Martin's Press, 1961, pp. 177–222.

Kongsamut, Piyabha, Sergio Rebelo, and Danyang Xie, "Beyond Balanced Growth," *Review of Economic Studies*, 2001, *68*, 869–882.

Leon-Ledesma, Miguel and Alessio Moro, "The Rise of Services and Balanced Growth in Theory and Data," Manuscript 2017.

Licandro, Omar, Javier Ruiz Castillo, and Jorge Durán, "The Measurement of Growth under Embodied Technical Change," *Recherches Économiques de Louvain*, 2002, *68*, 7–19.

Moro, Alessio, "Structural Change, Growth, and Volatility," *American Economic Journal: Macroeconomics*, 2015, *7*, 259–294.

Ngai, L. Rachel and Christopher A. Pissarides, "Structural Change in a Multisector Model of Growth," Discussion Paper 4763, CEPR 2004.

Oulton, Nicholas, "Productivity versus Welfare: or GDP versus Weitzman's NDP," *Review of Income and Wealth*, 2004, *50*, 329–355.

——, "Investment–specific Technological Change and Growth Accounting," *Journal of Monetary Economics*, 2007, *54*, 1290–1299.

Triplett, Jack E., "Economic Theory and BEA's Alternative Quantity and Price Indexes," *Survey of Current Business*, April 1992, pp. 49–52.

Uzawa, Hirofumi, "On a Two–sector Model of Economic Growth II," *Review of Economic Studies*, 1963, *30*, 105–118.

Weitzman, Martin L., "On the Welfare Significance of National Product in a Dynamic Economy," *Quarterly Journal of Economics*, 1976, *90*, 156–162.

——, "Sustainability and Technical Progress," *Scandinavian Journal of Economics*, 1997, *99*, 1–14.

Whelan, Karl, "A Guide to U.S. Chain–Aggregated NIPA Data," *Review of Income and Wealth*, 2002, *48*, 217–233.

——, "A Two–Sector Approach to Modeling U.S. NIPA Data," *Journal of Money, Credit, and Banking*, 2003, *35*, 627–656.

Appendix A: Imputing the Services Flow from the Stock of Consumer Durables

Consider a more disaggregated multi-sector model with consumption goods C_{gt}, consumption services C_{st}, capital K_t, and consumer durables D_t. Consumer durables are part of the capital stock and generate a services flow $u_t D_t$ in units of consumption services. There is also a one-period nominal bond. B_t denotes the current-dollar stock of the bonds and i_t denotes the interest rate in period t.

The household in the modified model solves:

$$\max_{\{C_{gt},C_{st},D_{t+1},K_{t+1},B_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_{gt}, C_{st} + u_t D_t)$$

$$\begin{aligned} \text{s.t.}\ \ & p_{gt}C_{gt} + p_{st}C_{st} + p_{dt}D_{t+1} + p_{xt}K_{t+1} + B_{t+1} \\ & = (1-\delta_d)p_{dt}D_t + [(1-\delta_k)p_{xt} + r_t]K_t + (1+i_t)B_t + w_t, \\ & C_{gt}, C_{st}, D_{t+1}, K_{t+1} \geq 0, \quad D_0, K_0, \quad B_t > -\underline{B} > -\infty, \quad B_0 = 0. \end{aligned}$$

Note that, as in the body of the paper, the budget constraint is written in current dollars.

The first-order conditions are:

$$\begin{aligned} C_{gt}: &\quad 0 = \beta^t \frac{\partial u(C_{gt}, C_{st} + u_t D_t)}{\partial C_{gt}} - \lambda_t p_{gt}, \\ C_{st}: &\quad 0 = \beta^t \frac{\partial u(C_{gt}, C_{st} + u_t D_t)}{\partial C_{st}} - \lambda_t p_{st}, \\ D_{t+1}: &\quad 0 = -\lambda_t p_{dt} + \lambda_{t+1}(1-\delta_d)p_{dt+1} + \beta^{t+1}\frac{u_{t+1}}{C_{t+1}}, \\ K_{t+1}: &\quad 0 = -\lambda_t p_{xt} + \lambda_{t+1}[(1-\delta_k)p_{xt+1} + r_{t+1}], \\ B_{t+1}: &\quad 0 = -\lambda_t + \lambda_{t+1}[1 + i_{t+1}]. \end{aligned}$$

Using the abbreviation:

$$u_{2t} \equiv \frac{\partial u(C_{gt}, C_{st} + u_t D_t)}{\partial C_{st}}$$

and consolidating the first-order conditions gives:

$$\frac{p_{st+1}}{p_{st}}\frac{u_{2t}}{u_{2t+1}} = \beta\frac{p_{dt+1}}{p_{dt}}\left[(1-\delta_d)+\frac{u_{t+1}p_{st+1}}{p_{dt+1}}\right], \tag{21}$$

$$\frac{p_{st+1}}{p_{st}}\frac{u_{2t}}{u_{2t+1}} = \beta\frac{p_{xt+1}}{p_{xt}}\left[(1-\delta_k)+\frac{r_{t+1}}{p_{xt+1}}\right], \tag{22}$$

$$\frac{p_{st+1}}{p_{st}}\frac{u_{2t}}{u_{2t+1}} = \beta[1+i_{t+1}]. \tag{23}$$

Equating (21) and (23) gives:

$$p_{dt+1}(1-\delta_d)+u_{t+1}p_{st+1} = p_{dt}(1+i_{t+1}). \tag{24}$$

Note that this can be rewritten into the standard asset pricing formula in discrete time:

$$p_{dt} = \frac{u_{t+1}p_{st+1}+(1-\delta_d)p_{dt+1}}{1+i_{t+1}}.$$

The price of durables today equals the discounted sum of the payoff plus the price of the un-depreciated durables tomorrow. Since everything is in current dollars here, we use the nominal interest rate to discount.

To obtain an equation for the imputation of the service flow from consumer durables in period t, we solve the pricing equation (24) for the service flow and multiply the result with the stock of durables, D_t:

$$p_{st}u_tD_t = \frac{p_{dt-1}}{p_{dt}}(1+i_t)p_{dt}D_t-(1-\delta_d)p_{dt}D_t. \tag{25}$$

The terms of the right-hand side of (25) are measurable. p_{dt}/p_{dt-1} is the inflation factor of durable goods, $p_{dt}D_t$ is the current-cost net stock of durable goods, and $\delta_d p_{dt}D_t$ is the current-cost depreciation of consumer durables. We calculate the interest rate using that equations (22) and (23) imply:

$$1+i_t = \frac{p_{xt}}{p_{xt-1}}\left[1+\frac{r_tK_{t+1}-\delta_kK_t}{p_{xt}K_t}\right].$$

p_{xt}/p_{xt-1} is the inflation factor of investment other than durables. We use the corporate accounts of NIPA to calculate the remaining terms. r_tK_{t+1} and δ_kK_t are current-price capital income and current-price depreciation. $p_{xt}K_t$ is the current-cost net stock of fixed assets in the corporate sector. We use 3-year moving averages of p_{dt}/p_{dt-1} and p_{xt}/p_{xt-1} to capture the expectations of future values.

Appendix B: Proofs of the Propositions

Appendix B.1: Proof of Proposition 1

We begin by eliminating prices and consolidating the equilibrium conditions so that the only unknowns are equilibrium quantities:

$$1 = \frac{C_t}{K_t^\theta A_{ct}^{1-\theta}} + \frac{X_t}{K_t^\theta A_{xt}^{1-\theta}}, \tag{26a}$$

$$\widehat{K}_{t+1} = \frac{X_t}{K_t} + 1 - \delta, \tag{26b}$$

$$\left(\frac{\widehat{A}_{xt+1}}{\widehat{A}_{ct+1}}\right)^{1-\theta} \widehat{C}_{t+1} = \beta\left[1 - \delta + \theta\left(\frac{K_{t+1}}{A_{xt+1}}\right)^{\theta-1}\right], \tag{26c}$$

$$0 = \lim_{t\to\infty} \beta^t \left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta} \frac{K_{t+1}}{C_t}. \tag{26d}$$

We need to show that if $\widehat{A}_x$ and $\widehat{A}_c$ are constant, then an equilibrium path exists along which all endogenous variables grow at constant rates. We do so by constructing a path for $\{K_t, X_t, C_t\}_{t=0}^{\infty}$ that satisfies (26a)–(26d) and has these properties. In particular, the path satisfied (9a) and the first part of (9b). We then show the additional claimed properties for $\{Y_t^C, Z_t^C, R_t, w_t, r_t, p_{xt}\}_{t=0}^{\infty}$, that is, R_t is constant and the second part of (9b) as well as (9c)–(9g) hold.

To construct $\{K_t, X_t, C_t\}_{t=1}^{\infty}$, we set $\widehat{K}_t = \widehat{A}_x$. We define $\{\widehat{X}_t\}_{t=1}^{\infty}$ such that equation (26b) is satisfied for all $t > 0$ if it is satisfied at $t = 0$. Since

$$\frac{X_t}{K_t} = \widehat{K}_{t+1} - (1 - \delta) = \widehat{A}_x - (1 - \delta),$$

this implies X_t/K_t must be constant. Thus, we set $\widehat{X}_t = \widehat{A}_x$. We define $\{\widehat{C}_t\}_{t=1}^{\infty}$ such that (26a) is satisfied for all $t > 0$ if it is satisfied at $t = 0$. Since $\widehat{K}_t = \widehat{X}_t = \widehat{A}_x$, (26a) implies that $C_t/(K_t^\theta A_{ct}^{1-\theta})$ must be constant. Hence, we set $\widehat{C} = \widehat{A}_x^\theta \widehat{A}_c^{1-\theta}$.

Next, we set (K_0, X_0, C_0) such that (26a)–(26b) hold at $t = 0$ and the Euler equation (26c) holds for all $t \geq 0$. Together with the previous growth factors, this uniquely determines $\{K_t, X_t, C_t\}_{t=0}^{\infty}$. Using consumption growth and that $K_{t+1}/A_{xt+1} = K_0/A_{x0}$, (26c) becomes:

$$\widehat{A_x} = \beta\left(1 - \delta + \theta\left(\frac{K_0}{A_{x0}}\right)^{\theta-1}\right).$$

We choose the unique solution $K_0 > 0$ given $A_{x0} > 0$. Given K_0, we set $X_0 \equiv [\widehat{A_x} - (1-\delta)]K_0$ to satisfy (26b) at $t = 0$. Given X_0 and K_0, we choose C_0 to satisfy (26a) at $t = 0$:

$$C_0 = \left(1 - \frac{X_0}{K_0^{\theta} A_{x0}^{1-\theta}}\right) K_0^{\theta} A_{c0}^{1-\theta}.$$

To show that the transversality condition (26d) holds, we substitute the growth factors for K_{t+1} and C_t into the right-hand side:

$$\beta^t \left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta} \frac{K_{t+1}}{C_t} = \beta^t \frac{K_0}{C_0}.$$

Since this converges to zero as $t \to \infty$, we have constructed an equilibrium path.

This leaves to show the statements about $\{r_t, w_t, R_t, p_{xt}, Y_t^C, Z_t^C\}_{t=0}^{\infty}$. The choice of numeraire, $p_{ct} = 1$, and equations (4) and (5) imply that:

$$\frac{r_t}{p_{xt}} = \theta K_t^{\theta-1} A_{xt}^{1-\theta},$$
$$w_t = (1-\theta) K_t^{\theta} A_{ct}^{1-\theta}.$$

The two Euler equations (3a) and (3b) imply that the real interest rate factor in units of consumption is given by:

$$R_t = \left(\frac{\widehat{A_c}}{\widehat{A_x}}\right)^{1-\theta} \left[1 - \delta + \theta\left(\frac{K_t}{A_{xt}}\right)^{\theta-1}\right],$$

which is constant. To obtain equations (9f)–(9g), note that since $\widehat{K} = \widehat{A_x}$ along the equilibrium path, the previous equations imply that $\widehat{w_t} = \widehat{A_x}^{\theta} \widehat{A_c}^{1-\theta}$ and $\widehat{r_t} = \widehat{p_{xt}}$. Equation (6) together with the assumption that the TFP growth rates are constant gives (9f).

Equation (8) and that $\widehat{K} = \widehat{A_x}$ imply that:

$$\widehat{Y_t^C} = \widehat{A_x}^{\theta} \widehat{A_c}^{1-\theta}.$$

Equation (6) together with together with the previous results for $\widehat{X}_t$ and $\widehat{Y}_t^C$ give the remaining equalities of (9b):

$$\widehat{Z}_t^C = \widehat{Y_t^C - \delta p_{xt} X_t} = \widehat{A}_x^{\theta} \widehat{A}_c^{1-\theta}.$$

QED

Appendix B.2: Proof of Lemma 1

The current-value Lagrangian for Problem (10) is:

$$\mathscr{L} = \sum_{t=0}^{\infty} \frac{1}{\mathcal{R}_t} \Big\{ C_t + \lambda_t \Big[A_{ct}^{1-\theta} K_t^{\theta} + (1-\delta) P_{xt} K_t - C_t - P_{xt} K_{t+1} \Big] \Big\}.$$

The first-order conditions are:

$$\frac{\partial \mathscr{L}}{\partial C_t} = \frac{1}{\mathcal{R}_t}(1 - \lambda_t) = 0, \tag{27a}$$

$$\frac{\partial \mathscr{L}}{\partial K_{t+1}} = -\frac{1}{\mathcal{R}_t} \lambda_t P_{xt} + \frac{1}{\mathcal{R}_{t+1}} \lambda_{t+1} \theta A_{ct+1}^{1-\theta} K_{t+1}^{\theta-1} + \frac{1}{\mathcal{R}_{t+1}} \lambda_{t+1} P_{xt+1}(1-\delta) = 0, \tag{27b}$$

$$\lim_{t\to\infty} \frac{1}{\mathcal{R}_t} \lambda_t P_{xt} K_{t+1} = 0. \tag{27c}$$

Consolidating implies:

$$R_{t+1} = \left(\frac{A_{xt}}{A_{ct}}\right)^{1-\theta} \left((1-\delta) \left(\frac{A_{ct+1}}{A_{xt+1}}\right)^{1-\theta} + \theta A_{ct+1}^{1-\theta} K_{t+1}^{\theta-1} \right), \tag{28a}$$

$$A_{ct}^{1-\theta} K_t^{\theta} - \delta \left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta} K_t = C_t + \left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta} (K_{t+1} - K_t). \tag{28b}$$

Since both problems have the same real interest rate, (28a) and (28b) are first-order for each of the two problems. Hence, $\{C_t, K_{t+1}\}_{t=0}^{\infty}$ is part of the solution to Problem (2) if and only if it solves Problem (10). **QED**

Appendix B.3: Proof of Proposition 2

We prove the claims by establishing that the Laspeyres and Paasche quantity indexes are first-order approximations to the forward-looking and backward-looking Fisher-Shell true quantity indexes:

$$\widehat{FS}_{t-1,t} \approx \frac{X_t + p_{ct-1}C_t}{Y^X_{t-1}}, \qquad \widehat{FS}_{t,t-1} \approx \frac{Y^X_t}{X_{t-1} + p_{ct}C_{t-1}}.$$

Two identities are helpful:

$$\frac{\partial e\Big(p_{ct}, v(p_{ct}, Y^X_t; S_t); S_t\Big)}{\partial v}\frac{\partial v(p_{ct}, Y^X_t; S_t)}{\partial p_{ct}} = -C_t, \tag{29}$$

$$\frac{\partial e\Big(p_{ct}, v(p_{ct}, Y^X_t; S_t); S_t\Big)}{\partial v}\frac{\partial v(p_{ct}, Y^X_t; S_t)}{\partial Y^X_t} = 1. \tag{30}$$

(29) follows from Roy's identity,

$$\left[\frac{\partial v(p_{ct}, Y^X_t; S_t)}{\partial Y^X_t}\right]^{-1}\frac{\partial v(p_{ct}, Y^X_t; S_t)}{\partial p_{ct}} = -C_t,$$

and (30). (30) follows by taking the derivative of $e_t(\cdot)$ with respect to Y^X_t and rearranging.

We establish that $\widehat{FS}_{t-1,t} \approx \widehat{Y}^L_t$ by showing that $e\Big(p_{ct-1}, v(p_{ct}, Y^X_t; S_{t-1}); S_{t-1}\Big) \approx X_t + p_{ct-1}C_t$. Interpreting $e\Big(p_{ct-1}, v(p_{ct}, Y^X_t; S_{t-1}); S_{t-1}\Big)$ as a function of (p_{ct}, Y^X_t) and linearizing it around (p_{ct-1}, Y^X_{t-1}) gives:

$$\begin{aligned} e\Big(p_{ct-1}, v(p_{ct}, Y^X_t; S_{t-1}); S_{t-1}\Big) \approx\ & e\Big(p_{ct-1}, v(p_{ct-1}, Y^X_{t-1}; S_{t-1}); S_{t-1}\Big) \\ & + \frac{\partial e\Big(p_{ct-1}, v(p_{ct-1}, Y^X_{t-1}; S_{t-1}); S_{t-1}\Big)}{\partial v}\frac{\partial v(p_{ct-1}, Y^X_{t-1}; S_{t-1})}{\partial p_{ct-1}}(p_{ct} - p_{ct-1}) \\ & + \frac{\partial e\Big(p_{ct-1}, v(p_{ct-1}, Y^X_{t-1}; S_{t-1}); S_{t-1}\Big)}{\partial v}\frac{\partial v(p_{ct-1}, Y^X_{t-1}; S_{t-1})}{\partial Y^X_{t-1}}(Y^X_t - Y^X_{t-1}). \end{aligned}$$

Using (30)–(29) and that $e\Big(p_{ct-1}, v(p_{ct-1}, Y^X_{t-1}; S_{t-1}); S_{t-1}\Big) = Y^X_{t-1}$ gives:

$$\begin{aligned} e\Big(p_{ct-1}, v(p_{ct}, Y^X_t; S_{t-1}); S_{t-1}\Big) &\approx e\Big(p_{ct-1}, v(p_{ct-1}, Y^X_{t-1}; S_{t-1}); S_{t-1}\Big) - C_{t-1}(p_{ct} - p_{ct-1}) + (Y^X_t - Y^X_{t-1}) \\ &= Y^X_t - C_{t-1}(p_{ct} - p_{ct-1}) \\ &= X_t + p_{ct-1}C_t + (C_t - C_{t-1})(p_{ct} - p_{ct-1}) \\ &\approx X_t + p_{ct-1}C_t, \end{aligned}$$

where the last step leaves out the second-order terms.

We establish that $\widehat{FS}_{t,t-1} \approx \widehat{Y}_t^p$ by showing that $e\Big(p_{ct}, v(p_{ct-1}, Y_{t-1}^X; S_t); S_t\Big) \approx X_{t-1} + p_{ct} C_{t-1}$. The proof follows by interpreting $e\Big(p_{ct}, v(p_{ct-1}, Y_{t-1}^X; S_t); S_t\Big)$ as a function of (p_{ct-1}, Y_{t-1}^X), linearizing it around (p_{ct}, Y_t^X), and following the same steps as before:

$$
\begin{aligned}
e\Big(p_{ct}, v(p_{ct-1}, Y_{t-1}^X; S_t); S_t\Big) \approx\; & e\Big(p_{ct}, v(p_{ct}, Y_t^X; S_t); S_t\Big) \\
& + \frac{\partial e\Big(p_{ct}, v(p_{ct}, Y_t^X; S_t); S_t\Big)}{\partial v} \frac{\partial v(p_{ct}, Y_t^X; S_t)}{\partial p_{ct}} (p_{ct-1} - p_{ct}) \\
& + \frac{\partial e\Big(p_{ct}, v(p_{ct}, Y_t^X; S_t); S_t\Big)}{\partial v} \frac{\partial v(p_{ct}, Y_t^X; S_t)}{\partial Y_t^X} (Y_{t-1}^X - Y_t^X) \\
=\; & Y_{t-1}^X - C_t (p_{ct-1} - p_{ct}) \\
\approx\; & X_{t-1} + p_{ct} C_{t-1}.
\end{aligned}
$$

QED

Appendix B.4: Proof of Proposition 3

As in Proposition 1, we consolidate the equilibrium conditions so that the only unknowns are equilibrium quantities:

$$1 = \frac{C_t}{K_t^\theta A_{ct}^{1-\theta}} + \frac{X_t}{K_t^\theta A_{xt}^{1-\theta}}, \tag{31a}$$

$$\widehat{K}_{t+1} = \frac{X_t}{K_t} + 1 - \delta, \tag{31b}$$

$$\left(\frac{\widehat{A}_{xt+1}}{\widehat{A}_{ct+1}}\right)^{1-\theta} \widehat{C}_{t+1} = \beta \left[1 - \delta + \theta \left(\frac{K_{t+1}}{A_{xt+1}}\right)^{\theta-1}\right], \tag{31c}$$

$$0 = \lim_{t\to\infty} \beta^t \left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta} \frac{K_{t+1}}{C_t}. \tag{31d}$$

We show that an equilibrium path exists by constructing a path $\{K_t, X_t, C_t\}_{t=0}^\infty$ such that X_t and K_t grow at constant factors, $L_t = 1$ is constant, and (31a)–(31d) are satisfied.

We first construct $\{K_t, X_t, C_t\}_{t=1}^\infty$. Set $\widehat{K}_t = \widehat{A}_x$, which is constant. We set $\{\widehat{X}_t\}_{t=1}^\infty$ such that equation (31b) is satisfied for all $t > 0$ if it is satisfied at $t = 0$. Since

$$\frac{X_t}{K_t} = \widehat{K}_{t+1} - (1 - \delta) = \widehat{A}_x - (1 - \delta),$$

this implies X_t/K_t must be constant. Thus, we set $\widehat{X_t} = \widehat{A_x}$. We set $\{\widehat{C_t}\}_{t=1}^{\infty}$ such that (31a) is satisfied for all $t > 0$ if it is satisfied at $t = 0$. Since $\widehat{K_t} = \widehat{X_t} = \widehat{A_x}$, (31a) implies that $C_t/(K_t^{\theta}A_{ct}^{1-\theta})$ must be constant. Hence, we set $\widehat{C_t} = \widehat{A_x^{\theta}}\widehat{A_{ct}^{1-\theta}}$.

Next, we set (K_0, X_0, C_0) such that (31b) holds at $t = 0$ and the Euler equation (31c) holds for all $t \geq 0$. Together with the previous growth factors, this uniquely determines $\{K_t, X_t, C_t\}_{t=0}^{\infty}$. Using consumption growth and that $K_{t+1}/A_{xt+1} = K_0/A_{x0}$, (31c) becomes:

$$\widehat{A_x} = \beta\left(1 - \delta + \theta\left(\frac{K_0}{A_{x0}}\right)^{\theta-1}\right).$$

We choose the unique solution $K_0 > 0$ given $A_{x0} > 0$. Given K_0, we then set $Y_0^X \equiv K_0^{\theta}A_{x0}^{1-\theta}$ and $X_0 \equiv [\widehat{A_x} - (1-\delta)]K_0$ to satisfy (31a) at $t = 0$. Given X_0 and K_0, we choose C_0 to satisfy (31a) at $t = 0$:

$$C_0 = \left(1 - \frac{X_0}{K_0^{\theta}A_{x0}^{1-\theta}}\right)K_0^{\theta}A_{c0}^{1-\theta}.$$

To show that the transversality condition (31d) holds, we substitute the growth factors for K_{t+1} and C_t into the right-hand side:

$$\beta^t\left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta}\frac{K_{t+1}}{C_t} = \beta^t\frac{K_0}{C_0}.$$

Since this converges to zero as $t \to \infty$, the transversality condition holds.

This leaves to show the statements about $\left\{Y_t^C, Y_t^X, r_t, w_t, p_{ct},\right\}_{t=0}^{\infty}$. Equations (8) and (16) and the previous results imply that:

$$\widehat{Y_t^C} = \widehat{A_x^{\theta}}\widehat{A_{ct}^{1-\theta}},$$
$$\widehat{Y_t^X} = \widehat{A_x}.$$

In addition, $Y_t^C = p_{ct}^{-1}Y_t^X$ and $Z_t^C = p_{ct}^{-1}(Y_t^X - \delta K_t)$. Since, $\widehat{Y^X}_t = \widehat{K_t}$, we also have $\widehat{Y_t^C} = \widehat{Z_t^C}$.

If $p_{ct} = 1$, then equations (4) and (5) imply that:

$$r_t = \theta\left(\frac{A_{xt}}{K_t}\right)^{1-\theta},$$
$$w_t = (1-\theta)K_t^{\theta}A_{xt}^{1-\theta}.$$

Since $\widehat{K}_t = \widehat{A}_x$ along the equilibrium path, $\widehat{w}_t = \widehat{A}_x$ and $\widehat{r}_t = 0$. Hence, $r_t - \delta$ constant. Lastly, the statement about p_{ct} follows directly from $p_{xt} = 1$ and equation (6). **QED**

Appendix B.5: Proof of Corollary 1

The first claim follows from the fact that along the GBGP:

$$\widehat{Y}_t^C = \widehat{K}_t^\theta \widehat{A}_{ct}^{1-\theta} = \widehat{A}_x^\theta \widehat{A}_{ct}^{1-\theta} < \widehat{A}_x = \widehat{Y}_t^X.$$

The second claim follows from the fact that along the GBGP $\widehat{Y}_t^C = \widehat{A}_x^\theta \widehat{A}_{ct}^{1-\theta}$ and the assumption that $\widehat{A}_x$ is constant and $\widehat{A}_{ct}$ declines. Thus, $\widehat{Y}_t^C$ slows down. Since $\widehat{Y}_t^C = \widehat{Z}_t^C$, this means that $\widehat{Z}_t^C$ slows down too. The third claim follows from the fact that along the GBGP $\widehat{Y}_t^X = \widehat{A}_x$ and $\widehat{A}_x$ is assumed constant. **QED**

Appendix B.6: Proof of Proposition 4

Given the assumptions, the first claim immediately follows from (20). Turning to the second claim, $\widehat{Y}_t^F$ slows down if and only if:

$$\widehat{Y}_t^F > \widehat{Y}_{t+1}^F.$$

Using that $\widehat{Y^X} = \widehat{A^X}$, the second equation of (20) implies that this is equivalent to:

$$\widehat{A^X}\sqrt{\frac{1 + \frac{p_c C}{X}\frac{\widehat{A}_{ct}}{\widehat{A}_x}}{1 + \frac{p_c C}{X}\frac{\widehat{A}_x}{\widehat{A}_{ct}}}} > \widehat{A^X}\sqrt{\frac{1 + \frac{p_c C}{X}\frac{\widehat{A}_{ct+1}}{\widehat{A}_x}}{1 + \frac{p_c C}{X}\frac{\widehat{A}_x}{\widehat{A}_{ct+1}}}}.$$

Multiplying out gives:

$$\left(X\widehat{A}_x + p_c C\boldsymbol{\widehat{A}_{ct}}\right)\left(X\widehat{A}_{ct}\widehat{A}_{ct+1} + p_c C\widehat{A}_x\boldsymbol{\widehat{A}_{ct}}\right) > \left(X\widehat{A}_x + p_c C\boldsymbol{\widehat{A}_{ct+1}}\right)\left(X\widehat{A}_{ct}\widehat{A}_{ct+1} + p_c C\widehat{A}_x\boldsymbol{\widehat{A}_{ct+1}}\right).$$

The only difference between the left-hand and the right-hand side are in the second terms of the respective sums, which are highlighted in boldface for convenience. On the left-hand side, they contain $\widehat{A}_{ct}$ whereas on the right-hand side they contain $\widehat{A}_{ct+1}$. Since all terms are positive on both sides, this implies that the inequality holds if and only if $\widehat{A}_{ct} > \widehat{A}_{ct+1}$, which happens when

$\widehat{A}_{ct}$ declines. **QED**

Appendix C: Derivation of Weitzman's Result in Continuous Time

In this appendix, we derive Weitzman's result in continuous time. To this end, we briefly state the key conditions of our two-sector in continuous time. We then derive the continuous-time version of (14). Finally, we show that it is equivalent to how Weitzman stated his result in continuous time.

Appendix C.1: Solving the household problem in continuous time

As in discrete time, the following equilibrium conditions hold in continuous time:

$$p_{xt} = \left(\frac{A_{ct}}{A_{xt}}\right)^{1-\theta}, \tag{32}$$

$$Y_t = A_{ct}^{1-\theta} K_t^{\theta}, \tag{33}$$

$$Z_t^C = A_{ct}^{1-\theta} K_t^{\theta} - \delta p_{xt} K_t. \tag{34}$$

Weitzman had the household maximize the present discounted sum of consumption subject to feasibility. In continuous time, this amounts to:[18]

$$\max \int_0^{\infty} C_t e^{-Rt} dt \quad \text{s.t.} \quad \dot{K}_t p_{xt} = A_{ct}^{1-\theta} K_t^{\theta} - p_{xt} \delta K_t - C_t. \tag{35}$$

Note that the equilibrium path differs from the ones studied in the main part of the paper because, in addition to using continuous time, the objective function in (35) now is the present discounted value of all period *consumption* instead of *utility*.

The current-value Hamiltonian associated with the household problem is:

$$\mathscr{H}_t \equiv C_t + \lambda_t \dot{K}_t = C_t + \lambda_t \left(\frac{A_{ct}^{1-\theta} K_t^{\theta}}{p_{xt}} - \delta K_t - \frac{C_t}{p_{xt}} \right).$$

[18]In the discrete-time model in the body of the paper, we used r_t and R_t to denote the rental price of capital and the real interest *rate factor*, respectively. For lack of obvious intuitive alternatives, we use R to denote the real interest *rate* in the continuous-time model. We trust that this does not lead to confusion.

The first-order conditions are:

$$\frac{\partial \mathscr{H}_t}{\partial C_t} = 1 - \frac{\lambda_t}{p_{xt}} = 0, \tag{36a}$$

$$\frac{\partial \mathscr{H}_t}{\partial K_t} = \lambda_t \left(\frac{\theta A_{ct}^{1-\theta} K_t^{\theta-1}}{p_{xt}} - \delta \right) = -\dot{\lambda}_t + \lambda_t R, \tag{36b}$$

$$\lim_{t\to\infty} e^{-Rt} \lambda_t K_t = 0. \tag{36c}$$

(36a) implies that the shadow price equals the relative price of investment in terms of consumption: $\lambda_t = p_{xt}$. Using this, we can write the equilibrium conditions as:

$$Z_t^C = A_{ct}^{1-\theta} K_t^{\theta} - \delta p_{xt} K_t = C_t + p_{xt} \dot{K}_t, \tag{37a}$$

$$\theta A_{ct}^{1-\theta} K_t^{\theta-1} - \delta p_{xt} = R p_{xt} - \dot{p}_{xt}, \tag{37b}$$

$$\lim_{t\to\infty} e^{-Rt} p_{xt} K_t = 0, \tag{37c}$$

$$p_{xt} = \left(\frac{A_{ct}}{A_{xt}} \right)^{1-\theta}. \tag{37d}$$

Appendix C.2: Constructing an equilibrium path in continuous time

From now on we focus on an equilibrium path along which the interest rate is constant. Equations (37b) and (37d) imply that:

$$R = \theta \left(\frac{A_{xt}}{K_t} \right)^{1-\theta} - \delta + \widetilde{p}_{xt} = \theta \left(\frac{A_{xt}}{K_t} \right)^{1-\theta} - \delta + (1-\theta) \left(\widetilde{A}_c - \widetilde{A}_x \right), \tag{38}$$

where "tildes" denote growth rates in continuous time. For example,

$$\widetilde{A}_i \equiv \frac{\dot{A}_{it}}{A_{it}}, \quad i \in \{c, x\}.$$

As in the analysis in discrete time, we assume that both growth rates are constant. If the interest rate is constant, then A_{xt}/K_t is also constant and

$$\widetilde{K} = \widetilde{A}_x. \tag{39}$$

Note that together with (38), this means that given A_0, we can choose K_0 so that the real interest rate equals any value, including the real interest rate that obtains along the BGP of the growth model of the main part of the paper, in which the household maximizes the present discounted value of *utility* instead of the present discounted value of *consumption*.

Equations (37a) and (37d) imply

$$\left(\frac{A_{xt}}{K_t}\right)^{1-\theta} - \delta = \left(\frac{A_{xt}}{A_{ct}}\right)^{1-\theta} \frac{C_t}{K_t} + \widetilde{K}. \tag{40}$$

The left-hand side and $\widetilde{K}$ are constant. Hence the first term on the right-hand side must also be constant, which is the case if and only if

$$\widetilde{C} = \theta\widetilde{A}_x + (1-\theta)\widetilde{A}_c. \tag{41}$$

Finally, the law of motion for capital accumulation,

$$K_{t+1} = X_t + (1-\delta)K_t,$$

implies that X_t/K_t is constant. Thus:

$$\widetilde{X} = \widetilde{A}_x. \tag{42}$$

We are now ready to derive Weitzman's result in continuous time. We start with a simple direct derivation that follows the exact same steps of the discrete-time derivation in the body of the paper. Abbreviating the constant growth rate of consumption in equation (41) by g, the present discounted value of consumption along BGP can be written as:

$$\int_t^{\infty} C_s e^{-R(s-t)} ds = C_t \int_t^{\infty} e^{-(R-g)(s-t)} ds = \frac{1}{R-g} C_t, \tag{43}$$

where we assumed that $\lim_{s\to\infty} e^{-(R-g)s} = 0$, which is the case if $R > g$. Using (38)–(41), this is equivalent to assuming that:

$$\theta\left(\frac{A_{x0}}{K_0}\right)^{1-\theta} - \delta - \widetilde{A}_x > 0. \tag{44}$$

Multiplying both sides of (43) with r and recalling that

$$C_t = A_{ct}^{1-\theta} K_t^{\theta} - (\delta + \widetilde{A}_x) p_{xt} K_t = Z_t^C - \widetilde{A}_x p_{xt} K_t,$$

we end up with:

$$R \int_t^{\infty} C_t e^{-R(s-t)} ds = \frac{R}{R-g} \left(Z_t^C - \widetilde{A}_x p_{xt} K_t \right). \tag{45}$$

(45) is the continuous-time equivalent of the discrete time result (14) from the main part of the paper. In particular, on the left-hand side, it has permanent income, i.e., the return on the present discounted value of consumption. Note that in continuous time, the interest payments occur in the current period and therefore do not need to be discounted. On the right-hand side, it has the NDP along with the additional term, that is, $-\widetilde{A}_x p_{xt} K_t$, which is the continuous-time equivalent to $-(\widehat{A}_x - 1) p_{xt} K_t$.

Appendix C.3: Weitzman's result in continuous time

We will now show that in continuous time (45) is equivalent to Weitzman's result that permanent income is proportional to the NDP in consumption units:

$$R \int_t^{\infty} C_s e^{-R(s-t)} ds \ \propto \ Z_t^C.$$

The first step is to derive the growth rate of the NDP along the BGP. The previous results imply that:

$$Z_t^C = A_{ct}^{1-\theta} K_t^{\theta} - \delta p_{xt} K_t = C_t + p_{xt} \dot{K}_t = C_t + p_{xt} \widetilde{K}_t K_t. \tag{46}$$

Taking the time derivative yields:

$$\dot{Z}_t^C = \dot{C}_t + \dot{p}_{xt} \dot{K}_t + p_{xt} \dot{\widetilde{K}}_t K_t + p_{xt} \widetilde{K} \dot{K}_t = C_t \widetilde{C}_t + p_{xt} \dot{K}_t \left(\widetilde{p}_{xt} + \widetilde{K}_t \right), \tag{47}$$

where we used that along the BGP $\widetilde{K}_t = \widetilde{A}_x$ is constant and so $\dot{\widetilde{K}} = 0$. Using that $p_{xt} \dot{K}_t = Z_t^C - C_t$, it follows that:

$$\widetilde{Z_t^C} = \frac{C_t}{Z_t^C}\widetilde{C} + \left(1 - \frac{C_t}{Z_t^C}\right)\left(\widetilde{p}_x + \widetilde{K}\right) = \left(\widetilde{p}_x + \widetilde{K}\right) + \frac{C_t}{Z_t^C}\left(\widetilde{C} - \widetilde{p}_x - \widetilde{K}\right).$$

(37d), (39), and (41) imply that

$$\widetilde{p}_x + \widetilde{K} = \widetilde{C} = (1-\theta)\widetilde{A}_c + \theta\widetilde{A}_x. \tag{48}$$

Hence:

$$\widetilde{Z_t^C} = g = (1-\theta)\widetilde{A}_c + \theta\widetilde{A}_x. \tag{49}$$

The next step is to totally differentiate Z_tC:

$$\begin{aligned}\frac{dZ_t^C}{dt} &= \frac{d\left(A_{ct}^{1-\theta}K_t^{\theta} - \delta p_{xt}K_t\right)}{dt}\\ &= (1-\theta)A_{ct}^{-\theta}\dot{A}_{ct}K_t^{\theta} + A_{ct}^{1-\theta}\theta K_t^{\theta-1}\dot{K}_t - \delta\dot{P}_{xt}K_t - \delta p_{xt}\dot{K}_t\\ &= (1-\theta)A_{ct}^{1-\theta}K_t^{\theta}\widetilde{A}_{ct} - \delta\dot{P}_{xt}K_t + \left(\theta A_{Xt}^{1-\theta}K_t^{\theta-1} - \delta\right)p_{xt}\dot{K}_t.\end{aligned}$$

Using (37d), (38), and (46), we get:

$$\begin{aligned}\frac{dZ_t^C}{dt} &= (1-\theta)\left(A_{ct}^{1-\theta}K_t^{\theta} - \delta p_{xt}K_t\right)\widetilde{A}_{ct} + (1-\theta)\delta p_{xt}K_t\widetilde{A}_{ct} - \delta p_{xt}K_t\widetilde{p}_{xt} + \left(R - \widetilde{p}_{xt}\right)p_{xt}\dot{K}_t \qquad (50)\\ &= (1-\theta)Z_t^C\widetilde{A}_{ct} + (1-\theta)\delta p_{xt}K_t\widetilde{A}_{xt} - p_{xt}\dot{K}_t\widetilde{p}_{xt} + R\left(Z_t^C - C_t\right)\\ &= \left((1-\theta)\widetilde{A}_{ct} + (1-\theta)\frac{\delta p_{xt}K_t}{Z_t^C}\widetilde{A}_{xt} - \frac{p_{xt}\dot{K}_t}{Z_t^C}\widetilde{p}_{xt}\right)Z_t^C + R\left(Z_t^C - C_t\right).\end{aligned}$$

It is helpful to define

$$\kappa_t \equiv \frac{\delta p_{xt}K_t}{Z_t}, \tag{51}$$

$$\gamma_t \equiv (1-\theta)\widetilde{A}_c + \kappa_t(1-\theta)\widetilde{A}_x - \frac{p_{xt}K_t}{Z_t}\widetilde{K}_t\widetilde{p}_{xt}. \tag{52}$$

Along an BGP, $\kappa_t = \kappa$ and $\gamma_t = \gamma$ are both constant. The reason ist that (48) and (49) imply that along an BGP $p_{xt}K_t$ and Z_t^C grow at the same rate g.

Collecting the results from the previous paragraphs, we have:

$$\frac{dZ_t^C}{dt} = \gamma Z_t^C + R\left(Z_t^C - C_t\right).$$

Multiplying both sides by e^{-Rt} and rearranging then gives:

$$\frac{d\left(Z_t^C e^{-Rt}\right)}{dt} = \frac{dZ_t^C}{dt} e^{-Rt} - RZ_t^C e^{-Rt} = \gamma Z_t^C e^{-Rt} - RC_t e^{-Rt}. \tag{53}$$

Integrating both sides, we obtain:

$$\lim_{s\to\infty} Z_s^C e^{-Rs} - Z_t^C e^{-Rt} = \gamma \int_t^\infty Z_s^C e^{-Rs} ds - R \int_t^\infty C_s e^{-Rs} ds.$$

Using that along an BGP $\lim_{s\to\infty} Z_s^C e^{-Rs} = Z_t^C \lim_{s\to\infty} e^{-(R-g)s} = 0$, this can be rewritten as:

$$Z_t^C = -\gamma Z_t^C \int_t^\infty e^{-(R-g)(s-t)} ds + R \int_t^\infty C_s e^{-R(s-t)} ds = -\frac{\gamma}{R-g} Z_t^C + R \int_t^\infty C_s e^{-R(s-t)} ds.$$

Hence, we have proven Weitzman's result:

$$R \int_t^\infty C_s e^{-R(s-t)} ds = \frac{R-g+\gamma}{R-g} Z_t^C. \tag{54}$$

On the face of it, this version of Weitzman's result differs from the previous version, (45). To begin with, (45) has the additional term $-\widetilde{A}_x p_{xt} K_t$ in parenthesis on the right-hand side, whereas (54) has only the NDP as the second term on the right-hand side. Moreover, the proportionality factors differ.

It turns out that (45) is nonetheless equivalent to (54) in continuous time:

$$\frac{R}{R-g}\left(Z_t^C - \widetilde{A}_x p_{xt} K_t\right) = \frac{R-g+\gamma}{R-g} Z_t^C.$$

To establish this, we substitute the definitions of g and γ into the previous equation:

$$\begin{aligned} R\left(A_{ct}^{1-\theta}K_t^{\theta} - (\delta + \widetilde{A}_x)p_{xt}K_t\right) &= \left(R - \theta\widetilde{A}_x + \kappa(1-\theta)\widetilde{A}_x - \frac{p_{xt}\dot{K}_t}{Z_t}\widetilde{p}_{xt}\right)\left(A_{ct}^{1-\theta}K_t^{\theta} - \delta p_{xt}K_t\right) \\ \iff \quad -R\widetilde{A}_x p_{xt}K_t &= -\theta\widetilde{A}_x\left(A_{ct}^{1-\theta}K_t^{\theta} - \delta p_{xt}K_t\right) + \delta p_{xt}K_t(1-\theta)\widetilde{A}_x - p_{xt}\dot{K}_t\widetilde{p}_{xt} \\ \iff \quad R\widetilde{A}_x p_{xt}K_t &= \theta A_{xt}^{1-\theta}K_t^{\theta-1}\widetilde{A}_x p_{xt}K_t - \delta\widetilde{A}_x p_{xt}K_t + \widetilde{p}_{xt}\widetilde{A}_x p_{xt}K_t \\ \iff \quad R &= \theta A_{xt}^{1-\theta}K_t^{\theta-1} - \delta + (1-\theta)\left(\widetilde{A}_c - \widetilde{A}_x\right), \end{aligned}$$

where we used (37d) and that $\widetilde{K} = \widetilde{A}_x$ along BGP. The last equation is just a restatement of the expression for the real interest rate, (38). Hence, in continuous time, (45) and (54) are the same. **QED**