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JEL Classification: C43, D91, O41, O47

Keywords: Growth measurement, Quantity indexes, NIPA, Fisher-Shell index, Embodied technical change

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Is the output growth rate in NIPA a welfare measure?*

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October 2016

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1 Introduction

Until the 90's the Bureau of Economic Analysis (BEA) featured in its National Income and Product Accounts (NIPA) a Laspeyres fixed-base quantity index to measure real output growth. The traditional fixed-base quantity index yields a reasonable good measurement of real growth provided that relative prices remain stable. Since the mid-80's and following the seminal contribution of Gordon (1990), who after controlling for quality improvements found that the prices of durable goods was permanently declining relative to the price of non-durable consumption goods and services, the BEA provides with a constant quality price index for equipment investment.[1] When the relative price of equipment declines, the weight of investment with respect to consumption in the Laspeyres index becomes obsolete quickly enough to have a relevant impact on growth measurement. The observed fast decline of the relative price of equipment, notably computers and peripheral equipment, has lead then the BEA to consider alternative measures of output growth. As a reaction, since the early 1990's, NIPA moved to a chained-type index built on the Fisher ideal index.[2] However, the theoretical legitimation of these measures has not yet been explored. Indeed, the modern economic theory of index numbers is largely built on the idea that index numbers have to reflect the underlying preferences of households in a well-defined technological environment. In this sense, the rationale of a method to measure real growth stems from the ability of the index to reflect changes in welfare in an appropriate and well-defined theoretical framework.

The present paper bridges modern macroeconomics and the economic theory of index numbers to show that the class of chain indexes used by NIPA properly reflect changes in welfare when applied to a dynamic general equilibrium economy with recursive preferences. In doing so, it evaluates the suitability of NIPA's methodology for measuring output growth in a general model economy with explicit preferences and technology. In

[1]Cummins and Violante (2002) contains a thorough review of the evolution of constant quality prices for equipment from 1947 to 2000 in the US.

[2]See Triplett (1992). National Accounts in other countries already calculated alternative measures of real growth, like a chained-type index based on the Laspeyres index in the Netherlands and Norway. European Union member states have also followed BEA: Commission Decision 98/715/EC established 2005 as the beginning of a period in which member states would progressively adapt their National Accounts. Among these changes stands out the publication of a chain index based on the Fisher ideal index.

this framework, preferences are defined over consumption streams, present and future, but NIPA is contrainted to use observable information and aggregates the main components of current final demand: consumption and investment. To examine the validity of this procedure, this paper notes that the Bellman equation provides with a representation of preferences over current consumption and investment. Then index number theory is applied to this representation of preferences to show that a Fisher-Shell true quantity index is equal to the Divisia index, which in turn is well approximated by the Fisher ideal chain index used by NIPA.[3] This means that the output growth rate in National Accounts is a welfare based measurement in the very precise sense of compensating variation.

The interest of the exercise also stems from understanding better the notion of real growth and its connection with welfare in models with more than one sector.[4] Growth theory has been reformulated in the late nineties in order to replicate the observed trend in the relative price of durable to nondurable goods. Based on Solow (1960), Greenwood et al (1997) propose a simple two-sector optimal growth model with investment specific technical change where productivity grows faster in the investment than in the consumption sector.[5] In this family of models, as in the data, investment grows faster than consumption, which raises the fundamental problem of measuring output growth. The general methodology suggested in this paper is then applied to the two-sector AK model proposed by Rebelo (1991), which replicates the empirical regularities referred to as above –see Felbermayr and Licandro (2005). Index number theory identifies then the growth rate of output with the Divisia index, meaning that the changes in NIPA's methodology mentioned above have led to the adoption of a real growth rate that is a welfare based measurement.

This theoretical framework sheds light on an old debate in the growth and growth accounting literature. The so-called Solow-Jorgenson controversy was revived by the

[3]See Fisher and Shell (1971) for a definition of a Fisher-Shell index and for a discussion about the conditions of its applicability.

[4]If all components of final demand grow at the same rate, aggregation is not an issue: the growth rate of the economy is the common rate of consumption and investment.

[5]Many other papers have followed. See, for example, Krusell (1998), Gort et al (1999), Greenwood et al (2000), Cummins and Violante (2002), Whelan (2003), Boucekkine et al (2003,2005), and Fisher (2006).

differing interpretations found in Hulten (1992) and Greenwood et al (1997). The controversy can be shown to boil down to the issue of the aggregation of consumption and investment when these are measured in different units and, more importantly, when its relative price has a trend. In our conceptual framework, it becomes clear that Greenwood et al (1997) take a path that is more consistent with the theory. However, implicitly, these authors –and others following like Oulton (2007) –develop a modern version of the paradigm that consumption, and consequently its growth rate, is the relevant measure of real growth.[6] In this paper, we claim that investment growth, as reflected in the Divisia index, also matters for output growth. Notice that NIPA's methodology stresses the fact that the growth rate of investment does contain information relevant to the welfare of the representative household since it reflects utility gains associated with postponed consumption. This is particularly relevant in a world where technical change is embodied in equipment goods, and hence where technical progress only materialize through the incorporation of new equipment.

This paper is organized as follows. Section 2 describes the economy with general recursive preferences and general technology. It applies index number theory to it and proves that the Fisher-Shell true quantity index is equal to the Divisia index. Section 3 illustrates it in the interesting case of the two-sector AK model economy. Finally, Section 4 discusses the main implications of our results and Section 5 concludes and suggests some possible extensions.

2 Measuring output growth

Consider a two-sector non-stochastic perfectly competitive dynamic general equilibrium economy with two goods, consumption and investment, and a general technology transforming capital and labor into these two goods. Firms hire capital and labor to produce them, and under the usual intertemporal budget constraint, a representative household chooses continuously consumption and investment in order to maximize intertemporal

[6] Greenwood et al (1997), in fact, is not a normative paper. It does perform the positive exercise of measuring the contribution of embodied technical change to US growth. However, in doing so, they measure output and its growth rate in units of consumption, de facto identifying real output growth with consumption growth. Cummins and Violante (2002) generalize the exercise and use standard NIPA methodology to the same objective, finding similar quantitative results. See also Greenwood and Jovanovic (2001). Section 4 discusses further these issues.

utility. All along this paper, we assume that preferences and technology are such that an equilibrium path exists and is unique.

In this paper, we try to understand the problem faced by a National Statistical Office (NSO) operating in this economy, that only observes current nominal consumption and investment, and the corresponding prices, but has no information about individual preferences, technology and future consumption. Let us finally assume that the NSO uses this information to measure the growth rate of both real consumption and real investment and, then, computes a chained Fisher ideal index of real output growth. As it is well know, in continuous time, this is equivalent to compute a Divisia index.

The general problem in national accounts is to find an index built out of observables at t, current consumption and investment, and the corresponding prices, that measures changes in real output. For our fictitious economy, we aggregate equilibrium consumption and investment by the mean of a Fisher-Shell true quantity index –controlling for changes in equilibrium prices. Section 2.2 shows that in this context the Fisher-Shell index is equal to the Divisia index, which in continuous time is equal to the Fisher ideal chain index –the one used in NIPA to measure GDP growth.[7] The resulting rate of output growth is then welfare based. Section 2.3 generalizes the result to heterogeneous households and Section 3.3 discuss the meaning of the statement that the growth rate of output is welfare based.

2.1 Bellman equation under recursive preferences

The economy evolves in continuous time. For any date $t \geq 0$ and any consumption path $C : [0, \infty) \to \mathbb{R}_+$ let ${}_tC$ denote the restriction of C to the interval $[t, \infty)$, preferences of the representative household are represented by some recursive utility function U generated by the differential equation

$$\frac{d}{dt}U({}_tC) = -f(c_t, U({}_tC)). \tag{1}$$

The generating function f is assumed to be differentiable with $f_1 > 0$ and $f_2 < 0$. Note that f_1 is marginal utility from current consumption, lost when we move an infinitesimal period of time ahead, and so the negative sign in (1). In turn, $f_2 < 0$ is related to the

[7]The property that in continuous time the Fisher ideal chain index is equal to the Divisia index is shown in Appendix A3.

implicit subjective discount rate.[8] For instance, the classical additively separable utility function is an important particular case of the general specification above in which

$$U({}_tC) = \int_t^{\infty} e^{-\rho(s-t)} u(c_s) ds$$

with $u'(c) > 0$, $u''(c) < 0$ and $\rho > 0$. Differentiate with respect to time t to write

$$\frac{d}{dt} U({}_tC) = -u(c_t) + \rho U({}_tC).$$

Hence, in this case, $f(c, u) = u(c) - \rho u$ and indeed $f_1(c, u) = u'(c) > 0$ while $f_2(c, u) = -\rho < 0$. Indeed, this clarify the interpretation given above that f_1 is the marginal utility from current consumption, lost when we move an infinitesimal period of time ahead, and f_2 is the return to household assets, which value is represented by $U({}_tC)$ and the discount rate is ρ.

Each instant t, a social planer chooses individual consumption c_t and per capita net investment $\dot{k}_t$ such that $(c_t, \dot{k}_t) \in \Gamma(k_t, \Theta_t)$ is feasible, where k_t is capital and Θ_t represents a vector of exogenous non-stochastic states. In the following, we assume that, for a given $k_t > 0$, there exists a unique consumption and investment path equilibrium $(c_s, \dot{k}_s)_{s \geq t}$ that maximizes $U({}_tC)$ subject to the technological constraint. Then, total utility is $U({}_tC)$ and the current change in welfare as measured by $U({}_tC)$ is simply given by (1).

In addition to the well known problem that preferences are not univocally represented by a utility function, we face here the additional problem, from an accounting point of view, that neither preferences nor foreseen consumption are observable by National Statistical Offices. In this context, we wish to build a quantity index that reflects changes in welfare using only current consumption c_t and current net investment $x_t = \dot{k}_t$, both observables at instant t; and all that matters of the level of k_t is summarized in the price of investment p_t as we will argue below. To this end, however, we shall need to express preferences as a function of variables observed at t. Since preferences are recursive, this amounts to express changes in welfare as a function of current consumption c_t and net investment x_t.

In other words, we need a representation of preferences over current consumption and current investment, and this is what the Bellman equation gives us. The original

[8] Epstein (1987) explores conditions under which a generating function f represents a recursive utility function U. Becker and Boyd (1997, chapter 1) motivates the study of general recursive preferences.

problem is to maximize $U({}_tC)$ subject to $(c_s, \dot{k}_s) \in \Gamma(k_s, \Theta_s)$ for all $s \geq t$, $k_t > 0$ given, where Θ_s is a vector of exogenous states that directly affect technology. The associated Bellman equation is

$$0 = \max_{(c,x)\in\Gamma(k_t,\Theta_t)} f(c, v(k_t, \Theta_t)) + v_1(k_t, \Theta_t)x + v_2(k_t, \Theta_t)\dot{\Theta}_t. \tag{2}$$

The intuition behind this equation becomes clear if one notes that along an optimal path $v(k_t, \Theta_t) = U({}_tC)$ so $\mathrm{d}v(k_t, \Theta_t)/\mathrm{d}t = v_1(k_t, \Theta_t)\dot{k}_t + v_2(k_t, \Theta_t)\dot{\Theta}_t = -f(c_t, U({}_tC)) = -f\big(c_t, v(k_t, \Theta_t)\big)$. Note as well that, in a sense, with all past actions summarized in k_t, the objective function in (2) is giving us the preference relation over consumption and investment at instant t.[9]

2.2 Fisher-Shell true quantity index

In this section, we show that in the dynamic general equilibrium framework developed in the previous section, the Divisia index is a true quantity index. In regard of the Bellman equation (2), preferences of the representative consumer over consumption and investment at instant t can be seen as represented by the function

$$w_t(c, x) \doteq f(c, v(k_t, \Theta_t)) + v_1(k_t, \Theta_t)x + v_2(k_t, \Theta_t)\dot{\Theta}_t.$$

To save notation, we write $w_t(c, x)$, but time enters this function only through the stock of capital k_t and the exogenous states Θ_t, both given at time t.

For a given state of the system, as represented by k_t and Θ_t, the function $w_t(c, x)$ can then be seen as a representation of individual preferences over consumption and investment, the last summarizing postponed consumption. To the extent that the exogenous states and the stock of capital will change along an equilibrium path, these preferences are time-dependent. This is precisely the building block of the true quantity index introduced by Fisher and Shell (1971). Since welfare comparisons must be done within the same preference map, the Fisher-Shell true quantity index proposes to fix not only prices but also preferences. In particular, it compares income today with the hypothetical level of income that would be necessary to attain the level of utility associated with tomorrow's income and prices with today's prices and today's preferences –as evaluated by $w_t(c, x)$. The remain of this section elaborates this idea.

[9]The planner solves a standard recursive program in which the state variable summarizes at each instant t all past information that could be relevant for today's decisions. For a brief exposition of recursive techniques in continuous time see Obstfeld (1992).

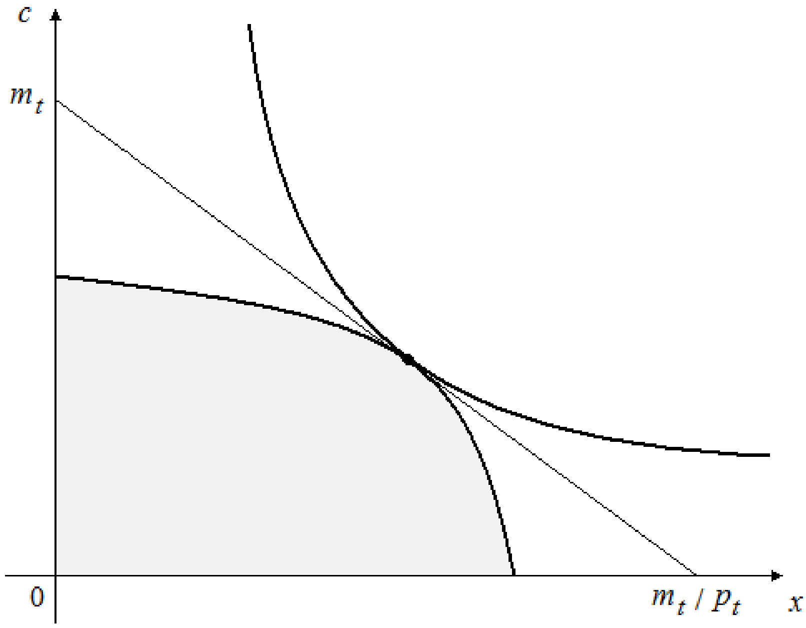

Figure 1: The production possibilities frontier and competitive prices

In the following, we adopt the consumption good as numeraire. Of course, the choice of the numeraire is inconsequential. Let us define equilibrium nominal net income at time t, along an equilibrium path for (c_t, x_t, p_t), as $m_t \doteq c_t + p_t x_t$. Under standard assumptions, optimal choices will lie in the boundary of $\Gamma(k_t, \Theta_t)$ so that there is a well-defined equilibrium price of net investment $p_t > 0$ expressed in units of consumption (see Figure 1). The constraint $(c, x) \in \Gamma(k_t, \Theta_t)$ can be replaced by the linear constraint $c + p_t x \leq m_t$ in the problem of the Bellman equation (see Figure 1). Hence, the associated indirect utility function of the representative household problem is defined as

$$u_t(m_t, p_t) \doteq \max_{c+p_t x \leq m_t} w_t(c, x)$$

while the expenditure function is

$$e_t(u_t, p_t) \doteq \min_{w_t(c,x) \geq u_t} c + p_t x.$$

Since comparisons must be done within the same preference map, the Fisher-Shell true quantity index fixes both prices and preferences. In particular, it compares income today m_t with the hypothetical level of income $\hat{m}_{t+h}$ that would be necessary to attain the level of utility $u_t(m_{t+h}, p_{t+h})$ associated with tomorrow's income and prices m_{t+h}, p_{t+h} with today's prices p_t and today's preferences as represented by e_t, u_t. This artificial level

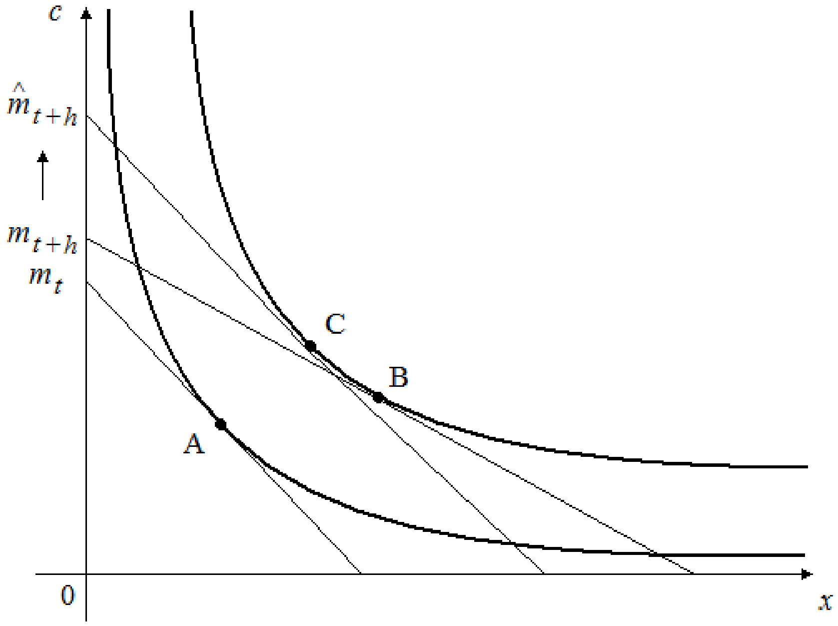

Figure 2: The Fisher-Shell true quantity index

of income is given by

$$\hat{m}_{t+h} = e_t\big(u_t(m_{t+h}, p_{t+h}), p_t\big).$$

The idea is illustrated in Figure 2. The preference map corresponds to instant t preferences as represented by w_t. Point A is the current situation at instant t. Point B is the choice using instant t preferences when we face instant $t+h$ prices p_{t+h} and income m_{t+h}. Point C represents the choice that maintains such level of utility but with instant t prices p_t. In the end, we compare two levels of income that correspond to the same price vector so it is clear that we are extracting price changes. In this particular case, the true quantity index is just reflecting the fact that the true output deflator is dropping with the price of investment, that is to say that income in real terms is growing more than nominal income m_{t+h}/m_t. The difference between $\hat{m}_{t+h}$ and m_{t+h} is a compensating variation measure stating by how much income would have to increase to compensate for not having the price of investment dropping.[10]

In continuous time, the reasoning is the same and the time gap h tends to zero. The

[10] If alternatively, the investment good were the numeraire, nominal income will be growing faster than the hypothetical income $\hat{m}$ and consumption price changes should be subtracted from nominal income growth to get real income growth. Indeed, the real growth rate will remain unchanged, since it does not depend on the choice of the numeraire.

instantaneous Fisher-Shell index is defined as

$$g_t^{\mathrm{FS}} \doteq \frac{d}{dh}\left.\frac{\hat{m}_{t+h}}{m_t}\right|_{h=0} = \frac{1}{m_t}\left.\frac{d\hat{m}_{t+h}}{dh}\right|_{h=0},$$

that is, the instantaneous growth rate of the factor defined above as h gets small.[11] To compute this index note that

$$\left.\frac{d\hat{m}_{t+h}}{dh}\right|_{h=0} = e_{1,t}\big(u_t(m_t,p_t),p_t\big)\Big(u_{1,t}(m_t,p_t)\dot{m}_t + u_{2,t}(m_t,p_t)\dot{p}_t\Big)$$

where subscripts denote the partial derivative with respect to the corresponding argument. To obtain an expression for all these derivatives let us go back to the dual and primal problems discussed above. Let μ be the Lagrange multiplier of the maximization problem in the definition of the indirect utility function, measuring the marginal contribution of income m to welfare w. We have, from the the primal problem

$$\begin{aligned} u_{1,t}(m_t,p_t) &= \mu \\ u_{2,t}(m_t,p_t) &= -\mu x_t, \end{aligned}$$

and, since the expenditure function is the inverse of the indirect utility function,

$$e_{1,t}(u_t,p_t) = \frac{1}{\mu}.$$

As expected, the marginal contribution of income to welfare, $\partial u/\partial m = \mu$, is equal to the inverse of the marginal contribution of utility u to total expenditure, $\partial e/\partial u = 1/\mu$. Moreover, the negative marginal contribution of prices to welfare is $\partial u/\partial p = -x\mu$, since an increase in prices reduces income by x units. These properties are critical for the result below and they are directly related to the *money metric utility* nature of the Fisher-Shell index, which defines the hypothetical income $\hat{m}$ using the expenditure function to valuate changes in utility after controlling for changes in prices.

Using the three conditions above in the definition of the Fisher-Shell index, we conclude that

$$g_t^{\mathrm{FS}} = \frac{\dot{m}_t - x_t\dot{p}_t}{m_t} = \frac{\dot{m}_t}{m_t} - \frac{p_t x_t}{m_t}\frac{\dot{p}_t}{p_t}.$$

Notice that the marginal terms e_1, u_1 and u_2 in the definition of the Fisher-Shell index simplify as a direct consequence of the properties discussed in the paragraph above; all

[11] Along an equilibrium path, in continuous time, it does not make a difference whether we define the true quantity index like we do or in terms of $m_t/\hat{m}_{t-h}$. See Appendix 5 for a rationale of this definition.

three are related to the marginal value of income μ. It is in this sense that money metric utility operates in the Fisher-Shell index. Since gains in welfare are measured as a compensating variation by comparing the artificial level of income $\hat{m}_{t+h}$ with the nominal income m_t, and prices enter linearly in the budget constraint, gains in welfare are equal to the change in nominal income (arbitrarily measured here in units of the consumption good) minus the contribution of prices to it (which comes only from the change of investment prices, weighted by the equilibrium (net) investment share).

Finally, differentiate the definition of nominal income $m_t = c_t + p_t x_t$ with respect to time and define the equilibrium share of net investment to net income as $s_t \doteq p_t x_t / m_t$ to write

$$\frac{\dot{m}_t}{m_t} = (1 - s_t)\,\frac{\dot{c}_t}{c_t} + s_t \frac{\dot{x}_t}{x_t} + s_t \frac{\dot{p}_t}{p_t},$$

which implies that

$$g_t^{\mathrm{FS}} = (1 - s_t)\,\frac{\dot{c}_t}{c_t} + s_t \frac{\dot{x}_t}{x_t} \doteq g_t^{\mathrm{D}}$$

where g_t^{D} denotes the Divisia index. We have then shown that, for all t, the Fisher-Shell index g_t^{FS} is equal to the Divisia index g_t^{D}. In this framework, by definition, the Divisia index is the average of the growth rates of consumption and net investment, weighted by their corresponding equilibrium shares in total net income.

We have then shown that in this framework the Divisia index is a true quantity index, and as such it is a welfare measure. The interpretation is straightforward. It is clear that g_t^{FS} is a measure of real growth since it is constructed as the growth rate of nominal income subtracting pure price changes, in this case the change of the relative price of investment p_t. The index only keeps changes in quantities. It is also clear that it is a true index because it is constructed from the representative household's preferences using standard theory.[12]

2.3 On household heterogeneity

The argument above was built under the assumption of a representative household. In this section, we show that the same reasoning applies to an economy where households

[12]This equivalence would come as no surprise to index number theorists. The Fisher ideal index is known to approximate in general some sort of true quantity index because both are bounded from above and below by the Laspeyres and Paasche indexes respectively. In continuous time, these indexes tend to each other as the time interval h tends to zero. Further, in general, the Divisia index coincides with the Fisher ideal index if the growth rates of consumption and investment are constant.

have both heterogeneous preferences and heterogeneous income. Critical in the result is the fact that the utility representation of preferences derived from the Bellman equation is quasilinear, belonging to the Gorman family.[13]

Let us assume that there is a continuum of heterogeneous households of unit mass with recursive preferences represented by the utility U_i generated by the differential equation

$$\frac{1}{dt}U_i({}_tC_i) = -f_i(c_{i,t}, U_i({}_tC_i)),$$

where ${}_tC_i$ represents the consumption path of household i. Let function f_i have the same properties as above. Let us also assume that, for this economy, an equilibrium exists and is unique. Notice that equilibrium will likely be different from the equilibrium with a representative household. In other words, the distribution of preferences and capital across individuals matters.

A distribution of capital maps any individual i at any instant t into a quantity of capital. We will denote by φ_t such a distribution. In the recursive competitive equilibrium representation of this economy, with exogenous state Θ_t and a distribution of capital φ_t, the problem of a household i with capital $k_{i,t}$ can be written as

$$\begin{aligned} 0 = \quad & \max \quad f_i(c_i, v_i(k_{i,t}, \Theta_t, \varphi_t)) + v_{i,1}(k_{i,t}, \Theta_t, \varphi_t)x_i + \pi_{i,t} \\ & \text{s.t.} \quad c_i + p_t x_i = m_{i,t} \end{aligned}$$

where p_t is the equilibrium price, common to all households, and $m_{i,t}$ is the equilibrium net income of individual i. $\pi_{i,t}$ represents the differential terms of $v_i(k_{i,t}, \Theta_t, \varphi_t)$ with respect to time that are exogenous to the problem of the consumer, i.e., those corresponding to Θ_t and φ_t.

As in section 2.2, the optimization problem of household i is associated to the instantaneous utility function over consumption and net investment

$$w_{i,t}(c_i, x_i) \doteq f_{i,t}(c_i) + x_i,$$

where $f_{i,t}(c_i) \doteq f_i(c_i, v_i(k_{i,t}, \Theta_t, \varphi_t))/v_{i,1}(k_{i,t}, \Theta_t, \varphi_t)$. Notice that we have subtracted $\pi_{i,t}$ from the right hand side of the Bellman equation and then divided it by $v_{i,1}(k_{i,t}, \Theta_t, \varphi_t)$. Since non of these two terms depend on c or x, such a transformation has no effect on the households program. Function $w_{i,t}(c_i, x_i)$ is maximized under the budget constraint $c_i +$

[13] See Gorman (1953, 1961).

$p_t x_i = m_{i,t}$, where, as said above, p_t is the equilibrium price and $m_{i,t}$, for all i, represents equilibrium household net income. Since this utility representation is quasilinear, it belongs to the Gorman family. It is easy to show that the indirect utility and expenditure functions become

$$u_{i,t}(m_{i,t}, p_t) = A_{i,t}(p_t) + m_{i,t}/p_t$$

$$e_{i,t}(u_{i,t}, p_t) = p_t\big(u_{i,t} - A_{i,t}(p_t)\big),$$

where $A_{i,t}(p_t)$ is defined below. In fact, from the household problem, optimal consumption c_i solves

$$f'_{i,t}(c_i) = 1/p_t.$$

Let us denote the implicit solution for c_i as $c_{i,t}(p_t)$. It is then easy to show that

$$A_{i,t}(p_t) = f_{i,t}\big(c_{i,t}(p_t)\big) - c_{i,t}(p_t)/p_t.$$

Let us define the artificial level of household i income as in section 2.2, i.e.,

$$\hat{m}_{i,t+h} = e_{i,t}\big(u_{i,t}(m_{i,t+h}, p_{t+h}), p_t\big) = p_t\Big(A_{i,t}(p_{t+h}) - A_{i,t}(p_t)\Big) + p_t/p_{t+h}\, m_{i,t+h},$$

which is linear on income due to the fact that preferences are quasilinear. Consistently with national accounts, let us define aggregate income as $m_t = \int_i m_{i,t}\mathrm{d}i$, which also measures per capita income since population has been normalized to unity. Let us now define aggregate hypothetical income consistently with the definition of per capita income as $\tilde{m}_t = \int_i \hat{m}_{i,t}\mathrm{d}i$. Using the results just above,

$$\tilde{m}_{t+h} = p_t\Big(\bar{A}_t(p_{t+h}) - \bar{A}_t(p_t)\Big) + p_t/p_{t+h}\, m_{t+h},$$

where

$$\bar{A}_t(p_t) = \int_i A_{i,t}(p_t)\mathrm{d}i.$$

Note that, in general, average hypothetical income $\tilde{m}_{t+h}$ at the equilibrium of the heterogeneous household economy, will be different from the hypothetical income $\hat{m}_{t+h}$ of the representative household economy at equilibrium, since these two economies will likely have a different equilibrium paths.

As in section 2.2, let us define the Fisher-Shell index for the economy with heterogeneous households as

$$\tilde{g}_t^{\mathrm{FS}} \doteq \frac{1}{m_t}\,\frac{d\tilde{m}_{t+h}}{dh}\bigg|_{h=0}.$$

Operating on the definition of $\tilde{m}_{it+h}$ above

$$\left.\frac{d\tilde{m}_{t+h}}{dh}\right|_{h=0} = \dot{m}_t + \Big(p_t \bar{A}_t'(p_t) - m_t/p_t\Big)\dot{p}_t.$$

Notice that

$$\bar{A}_t'(p_t) = \int_i A_{i,t}(p_t)di = \int_i \big(\underbrace{f_{i,t}'c_{i,t}' - 1/p_t c_{i,t}'}_{=0,\text{ since } f_{it}'=1/p_t} + c_{i,t}/p_t^2\big)di = c_t/p_t^2,$$

where $c_t = \int_i c_{it}\mathrm{d}i$ is per capita consumption. Then

$$\tilde{g}_t^{\mathrm{FS}} = \frac{\dot{m}_t}{m_t} - s_t\frac{\dot{p}_t}{p_t} = (1-s_t)\frac{\dot{c}_t}{c_t} + s_t\frac{\dot{x}_t}{x_t},$$

where $s_t \doteq p_t x_t/m_t$ as before. The Fisher-Shell index is, indeed, equal to the Divisia index, meaning that the growth rate in NIPA is a welfare measure irrespective of households being either homogeneous or heterogeneous. Of course, at equilibrium, consumption and investment may be growing at different rates than in the representative household model, and the saving rate may also be different. Consequently, even when the growth rate, as measured by the Divisia index is a welfare measure in both economies, these two economies may be growing at different rates.

Two assumptions are critical for the main result in this section, i.e., that the Fisher-Shell index is a Divisia index under heterogeneous households. First, as in the case of homogeneous households, under money metric utility, nominal income is the metric used to measure households' utility, implying that gains in welfare are measured as gains in nominal income minus inflation; the main principle used by national accounts. The second critical assumption is the use of the quasilinear representation of preferences that emerges from the Bellman equation representation of household preferences in the space of current consumption and current investment. This assumption is not critical at all in the case of a representative household; in facts, in Section 2.2, we show that the Fisher-Shell index is equal to the Divisia index for a general function $w(c,x)$. Indeed, it is critical in this section, since we profit from the quasi linearity representation of preferences to show that aggregate utility gains, as measured by the Fisher-Shell index, are equal to gains in nominal per capita income minus inflation.

This result comes at non surprise. By adopting aggregate nominal income as a norm for measuring aggregate output, the Fisher-Shell index implicitly assumes that the aggregate welfare function is utilitarian, giving to each household a weight proportional to its income. This clearly reflects in the definition of the artificial income measure $\tilde{m}_t$.

3 Embodied technical progress

As referred in the Introduction, following Gordon (1990)'s observation that quality adjusted equipment investment prices were permanently declining relative to the price of non-durable consumption goods and services, the Bureau of Economic Analysis (BEA) moved first to control for quality improvements in the measurement of investment prices, and second to a Fisher ideal chain index to measure output growth. The first change made investment to grow faster than non-durable consumption. As an undesirable consequence of trends in relative prices, the fixed-base quantity index used to measure GDP growth became obsolete fast enough to provided appropriate growth figures. In facts, in this case, fixed-base quantity indexes suffer from the well known substitution bias problem that tends to overestimate the weight of the fast growing items. The second change addresses this last problem by making the NIPA measure of output growth to be approximately equal to the Divisia index.

Almost contemporaneously, a new literature developed in macroeconomics aimed to accommodate growth theory to this new evidence. Greenwood et al (1997), in their seminal paper, extend the Ramsey model to a two sector (consumption and investment) growth model with two sources of technical progress, consumption and investment specific technical change (disembodied and embodied in capital goods, respectively). This model is able to replicate the permanent decline of the relative price of equipment investment, as well as the fact that investment grows faster than consumption (implying that the investment to output ratio is permanently growing). In this context, it is particularly clear that the aggregation issue is far from trivial since consumption and investment grow at different rates.

In this section, we describe a simple version of the two-sector AK model proposed by Rebelo (1991) and apply to it the Fisher-Shell index proposed in Section 2.2 to show that the BEA had good fundamental reasons to move to a chained-type quantity index of output growth. As shown in Felbermayr and Licandro (2005), the two-sector AK model is the simplest endogenous growth model that replicates the observed permanent decline in the relative price of equipment and the permanent increase in the investment to output ratio. We have preferred to use it instead of the original Greenwood et al (1997) model, since the AK model has the advantage of jumping to the balanced growth path from the initial time, which allows for an explicit solution of the value function. This is very useful to understand the role of money metric utility in the main statement of this paper that the growth rate in NIPA is a welfare measure.

3.1 The two-sector AK model

The model in this section is based on Rebelo (1991), follows Felbermayr and Licandro (2005) closely, and entails all the characteristics that are relevant to the present discussion in the simplest possible framework. The stock of machines at each instant t is k_t, from which a quantity $h_t \leq k_t$ is devoted to the production of the consumption good. Consumption goods technology is

$$c_t = h_t^{\alpha},$$

where $\alpha \in (0,1)$. The remaining stock $k_t - h_t \geq 0$ is employed in the production of new capital with a linear technology

$$\dot{k}_t = A(k_t - h_t)$$

where $A > 0$. The depreciation rate is assumed to be zero, which is unconsequential and simplifies notation. There is a given initial stock of capital $k_0 > 0$. Again, we will write $x_t = \dot{k}_t$ for net investment.

The representative household has preferences over consumption paths represented by[14]

$$\int_0^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma}\, \mathrm{e}^{-\rho t}\, \mathrm{d}t, \tag{3}$$

that is, the additive case mentioned above, where $\rho > 0$ is the subjective discount rate and $\sigma \geq 0$ the inverse of the intertemporal elasticity of substitution.

In the absence of market failures, equilibrium allocations are solutions to the problem of a planner aiming at maximizing household's utility subject to the technological constraints. The Bellman equation associated to the planner's problem is

$$\rho v(k_t) = \max_{x = A(k_t - c^{1/\alpha})} \frac{c^{1-\sigma}}{1-\sigma} + v'(k_t)x. \tag{4}$$

As shown in Felbermayr and Licandro (2005), the equilibrium growth rate of capital is

$$\gamma = \frac{A-\rho}{1-\alpha(1-\sigma)}.$$

[14]This is a particular case of the general preferences in Section 2.1. Here the correspondence Γ is defined for every $k \geq 0$ as the set $\Gamma(k)$ of pairs $(c, \dot{k})$ such that there exists h with $0 \leq h \leq k$, $c \leq h^{\alpha}$, and $\dot{k} \leq A(k-h)$.

From the feasibility constraints, it is clear that the growth rate of investment is also γ, and that $\alpha\gamma$ is the growth rate of consumption. Competitive equilibrium allocations are balanced growth paths from $t \geq 0$.

Returns to scale differ between sectors. Since $\alpha < 1$, as the stock of capital grows the investment sector becomes more productive with respect to the consumption goods sector. This difference in productivity causes the decline of investment prices relative to consumption goods prices. This difference in returns to scale can be interpreted in terms of the investment sector being more capital intensive than the consumption sector or, as put forth by Boucekkine et al (2003), as a consequence of strong spillovers in the production of investment goods.[15] From the feasibility constraints, we can obtain the competitive equilibrium price of investment in terms of consumption units as the marginal rate of transformation:

$$p_t = -\frac{dc_t}{dx_t} = -\frac{dc_t}{dh_t}\frac{dh_t}{dx_t} = \frac{\alpha}{A}h_t^{\alpha-1}.$$

Since the stock of machines used in the consumption goods sector grows at the constant rate γ, the price of investment relative to consumption decreases at rate $(\alpha-1)\gamma < 0$.

The competitive equilibrium allocation displays the regularities observed in actual data. Investment grows faster than consumption because $\gamma > \alpha\gamma$. The relative price of investment decreases at rate $(\alpha - 1)\gamma < 0$. Indeed, the nominal share of investment in income remains constant. To see this, let us take the consumption good as numeraire and define nominal income as in the general case as $m_t = c_t + p_t x_t$. From the equilibrium equations, one can show after some simple algebra that

$$s_t = \frac{p_t x_t}{m_t} = \frac{p_t x_t}{c_t + p_t x_t} = \frac{\alpha(A-\rho)}{\rho(1-\alpha) + \alpha\sigma A} \doteq s$$

for all $t \geq 0$. To be precise, s is the equilibrium share of investment in total income.

At this point it may be worth stressing that the choice of the consumption good as numeraire is inconsequential. The argument above follows equally if we choose to measure income in units of investment, $p_t^{-1}c_t + x_t$, or, for that matter, in any other arbitrary monetary unit provided that relative prices are respected –that is, that the

[15]Cummins and Violante (2002) observe that their measure of investment-specific technical change occurs first in information technology and then accelerates in other industries. They conclude that information technology is a "general purpose" technology, an interpretation that matches well with the spillovers' interpretation. See also Boucekkine et al (2005).

price of investment relative to consumption is p_t. This is important because identifying real growth with growth of nominal income is as arbitrary as the choice of the numeraire in which nominal income is expressed.

3.2 Measuring output real growth

In this section, we apply the general theory proposed in Section 2 to the two-sector AK model. As in the general case, in regard of the Bellman equation (4), the function

$$w_t(c, x) = \frac{c^{1-\sigma}}{1-\sigma} + v'(k_t)x$$

can be seen as representing preferences over contemporaneous consumption and investment. Again, the constraint in the Bellman equation (4) can be replaced by the budget constraint $c + p_t x \leq m_t$ because the budget line is tangent to the production possibilities frontier locally at the optimum. Notice that in this example the utility representation $w_t(c, x)$ changes over time only because the marginal value of capital does.

Let us define the indirect utility $u_t(m_t, p_t)$ and the expenditure function $e_t(u_t, p_t)$ as in Section 2. Recall that the Fisher-Shell true quantity index compares income today m_t with the hypothetical level of income $\hat{m}_{t+h}$ that would be necessary to attain the level of utility associated with tomorrow's income and prices m_{t+h}, p_{t+h} with today's prices p_t and today's preferences as evaluated by e_t, u_t. Denote again this artificial level of income as

$$\hat{m}_{t+h} = e_t\big(u_t(m_{t+h}, p_{t+h}), p_t\big).$$

From the definition of g_t^{FS} in Section 2, we conclude that, for all $t \geq 0$,

$$g_t^{\mathrm{FS}} = (1-s)\alpha\gamma + s\gamma = \frac{\alpha A(A-\rho)}{\rho(1-\alpha) + \alpha\sigma A}$$

As already said, the Fisher-Shell quantity index is equal to the Divisia index. As in the general case, the interpretation is straightforward: g^{FS} is a measure of real growth because it is constructed as the growth rate of nominal income substracting pure price changes, in this case the change of the relative price of investment p_t. The index only keeps changes in quantities. It is also clear that it is a true index because it is constructed from the representative household's preferences.

3.3 On money metric utility

What are the implications of money metric utility for the measurement of output growth? We argue in this section that money metric utility, implicit in the Fisher-Shell index, selects a particular representation of preferences that makes welfare grow at the rate g^{FS}. The particular representation depends crucially on preferences and technology. Let us develop this argument for the two-sector AK model, which has the virtue that equilibrium is always stationary allowing for an explicit solution for the value function.

Notice that the two-sector AK model jumps to the balanced growth path at the initial time. Consequently, a constant fraction of total capital will be permanently allocated to the production of consumption goods and capital will be permanently growing at the rate γ. After substituting the optima consumption path in (3), the value function can be written as

$$v(k_t) = Bk_t^{\alpha(1-\sigma)},$$

where $B > 0$ depends on the parameters of the model.

When the Fisher-Shell index is applied to the equilibrium path of this economy, the growth rate of output is measure by g^{FS}, as shown above. We argue in this paper that the growth rate of output, when measured by the mean of the Fisher-Shell index, is a welfare measure. This section shows that in the case of the two-sector AK model, in facts, g^{FS} measures the growth rate of welfare, in the sense that it is the growth rate of a particular representation of household preferences. In order words, money metric utility picks the particular representation of preferences that makes welfare, as measure by this particular representation, grow at the same rate as real output. The argument is the following. The utility function in the right hand side of (3) is one among many representations of the same preference order (constant intertemporal elasticity of substitution preferences). The Fisher-Shell index arbitrarily choses another, the one that grows at rate g^{FS} and adopts nominal income at some base year as its benchmark. We build the argument in two steps.

First, let us denote by $\hat{v}_t$ the equilibrium welfare of the representative agent at time t. Let us then make two assumptions concerning $\hat{v}_t$, consistently with the main implicit assumptions of the Fisher-Shell index. We assume first that at the initial time, $t = 0$, $\hat{v}_0 = (c_0 + p_0 x_0)/\rho$. This is the money metric utility assumption that the return to assets, the left hand side of the Bellman equation (4), is equal to nominal income at the base year (here, $t = 0$). The second assumption is that $\hat{v}_t$ grows at the rate g^{FS}, meaning that

$\dot{\hat{v}}_t/\hat{v}_t = g^{\mathrm{FS}}$. Then, for all $t \geq 0$,

$$\hat{v}_t = \hat{v}_0\, \mathrm{e}^{gt},$$

where $g = g^{\mathrm{FS}}$. Then, if a value function $\hat{v}(k_t)$ exists, such that, it is associated to an alternative representation of preferences in (3) consistently with the Fisher-Shell index, it has to be a potential function of k_t, with exponent g/γ. In the following step we show that such a representation exists.

Second, we adopt the following alternative representation of the constant intertemporal elasticity of substitution preferences in (3)

$$\hat{v}(k_t) = \max \left(\int_t^{\infty} \frac{c_s^{1-\sigma}}{1-\sigma}\ \mathrm{e}^{-\rho(s-t)}\ \mathrm{d}t \right)^{\frac{g}{\alpha\gamma(1-\sigma)}},$$

where $\hat{v}(k_t)$ is the associated value function. Since this new utility function represents the same preferences as those of the original two-sector AK model, the equilibrium path will be the same. Consequently, we can easily show that

$$\hat{v}(k_t) = C\, v(k_t)^{\frac{g}{\alpha\gamma(1-\sigma)}},$$

where $C \geq 0$ depends on the parameters of the model and income at $t = 0$.[16]

We have then show that the growth rate as measured by the Fisher-Shell index is a welfare measure in the sense that it is equal to the growth rate of a particular representation of household preferences. The choice of this representation directly results from the implicit assumptions in money metric utility that welfare is measured in units of nominal income at some base-time.

4 Discussion

In the framework of dynamic general equilibrium models, Section 2 shows that the Divisia index is, in fact, a true quantity index. This is of substantive interest since the Fisher ideal chain index used in actual National Accounts approximates well the Divisia index, implying that the growth rate of output in NIPA is welfare based. In this section, to make our main point clear, we refer to the two-sector AK model studied in Section 3 to

[16]Notice that $\frac{g}{\alpha\gamma(1-\sigma)}$ may be positive or negative depending on σ being smaller or larger than one, respectively. It is important to notice that B contains $1-\sigma$, meaning that its sign is positive or negative, depending also on σ being smaller or larger than one. This property of B extends C.

explain what we mean by that, but most of the arguments directly apply to the general model in Section 2.

On money metric utility. Notice that at equilibrium the welfare of the representative household, $v(k)$ in the Bellman equation (4), measures the value of capital. Then $\rho v(k)$ is the return to capital. From (4), the return to capital is equal to the utility of current consumption plus the value of current investment, priced at the marginal value of capital $v'(k)$. Of course, welfare as measured by $v(k)$ is defined in an arbitrary unit: monotonic transformations of $v(k)$ will change the level of utility leaving the preference map intact; consequently, the growth rate of different representations will not be necessarily the same. To overcome this problem, index number theory adopts a sensible norm to measure changes in welfare. In our context, it advocates for using observed income to measure the right hand side of the Bellman equation. Note that income as measured by National Accounts represents then the return to the stock of assets. Consequently, the Fisher-Shell quantity index and then the Divisia index are income compensating measures quantifying changes in the return to capital. Since the discount factor in (4) is time independent, the Divisia index also measures changes in welfare. Indeed, in the more general framework of recursive preferences, the rate of return is not necessarily constant, implying that changes in real income may be also due to changes in the subjective rate of return. In Section 3.3, we formally analyze this issue for the two-sector AK model and show that the growth rate of output as measured by the Fisher-Shell index is equal to the growth rate of welfare, explicitly writting the particular representation of preferences that grow at this rate.

Net National Product. In connection with these considerations, the use of the Bellman equation makes it clear why production in National Accounts is measured as final demand. Since present and future consumption is all that matter for welfare, and investment measures the value of future consumption, a welfare measure of output growth has to weight the growth rate of both final demand components. This interpretation is consistent with Weitzman (1976)'s claim that "net national product is a proxy for the present discounted value of future consumption."[17] In fact, his equation (10) is in spirit

[17]Weitzman's argument is developed in a simple optimal growth model with linear utility and the proof is based on the assumption that current income remains constant over time. In its own words, he gets "the right answer, although for the wrong reason." To be precise, using the main argument of the paragraph above, Weitzman's claim should be restated as "net national product is a proxy for the

equivalent to the Bellman equations (2) and (4), which rationalize our choice of taking current income as the proper norm in the Fisher-Shell true quantity index. It is important to point out that Weitzman (1976) is not about output growth and its relation to welfare gains in the growth process, but about the level of output and its relation to the level of welfare. In this sense, the non trivial question of the appropriate measurement of output growth has remained open until our days. The best result in this direction is in a subsequent paper by Asheim and Weitzman (2001). That paper builds a measure of the level of output and shows that output growth is a necessary and sufficient condition for welfare growth, but without providing any specific insight on how output growth should be measured. This papers gives a fundamental step ahead in this direction: by applying standard index number theory, we show that the precise way NIPA measures growth is welfare based. We can now clearly understand the meaning of a 2% increase in output, for example.[18]

Investment matters. The following example makes it more clear why investment matters in the definition of output growth. Consider a world with embodied technical progress –as the one in Greenwood and Yorukoglu (1997), for example. Let the consumption path in this economy be depicted as in Figure 3. In period T there is an unexpected permanent technological shock to the investment sector: embodied technical progress accelerates. New machines, if produced and added to the capital stock, can make the productivity in the consumption goods sector grow faster indefinitely. In our example, hence, after observing the unexpected acceleration of investment specific technical change in T, the consumer finds optimal to initially reduce consumption in order to increase investment and, then, profit from technical progress. In this world, at time T households welfare increases: the drop in consumption reflects the interest of the consumer in benefiting from faster growth thereon; if this move would have not increased her welfare, she would have chosen not to increase investment and remain in

return to capital, which value is equal to the present discounted value of future consumption."

[18]At this point it may be worth clarifying that, as pointed out by Weitzman (1976), it is not GDP but NNP what matters for welfare. Depreciated capital is a lost resource that does not contribute to welfare. It is in this sense that some authors claim that NNP is relevant for welfare and GDP for productivity –see the discussion in Oulton (2004). If the depreciation rate is constant, however, net and gross investment grow at the same rate. Indeed, when investment growth faster than consumption, NNP grows slower than GDP since the share of net investment on net income is smaller than the corresponding share of gross investment.

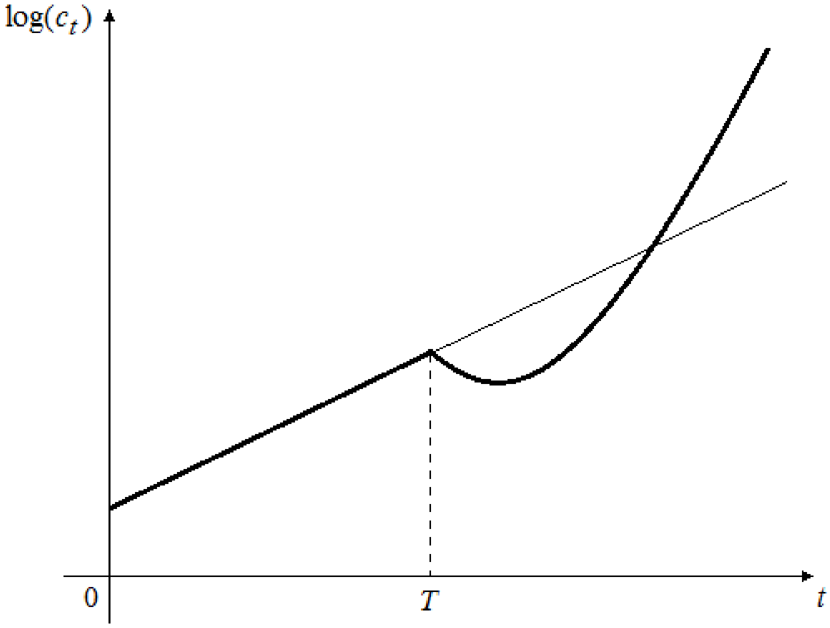

Figure 3: The manna economy versus embodied technical change

the original path with lower growth. Then, the consumption growth rate at time T does not measure welfare correctly. In fact, it has the opposite sign! However, the growth rate of output as measured by the Divisia index does, since it captures well the gains in welfare coming from the acceleration of technical progress and the associated optimal increase in investment. Remind that technical progress is assumed to be investment specific. Then, gains in productivity require new investments. The discussion above helps to illustrate why the growth rate of investment matters for output growth measurement. Faster growing investment today represents our best proxy for the preference for faster consumption growth tomorrow.

Paradox of endowment vs production economies. Moreover, it is very important to understand that a true quantity index of output growth is a welfare measure conditional on both preferences and technology, simultaneously. In other words, it does not reflect changes in welfare independently of the possibilities allowed by technology. We present below two examples that show the interplay between technology and preferences in the definition of output growth emerging from index number theory applied to this family of problems.

Consider the following example that clarifies further the meaning of measuring welfare changes. For the two-sector AK model in Section 3, take any configuration of parameters such that, for example, the growth rate of investment at equilibrium is 6% and the investment share is 20%. Let α be equal to 1/3. The Divisia index tells us that this economy will be growing at 2.8%, since consumption represents 80% of output and will grow at 2%. Alternatively, consider an *endowment* economy with exactly the same preferences and the same equilibrium consumption flow. In this economy, consumption is mana from haven. Indeed, a household would be indifferent between living in the AK or in the endowment economy, since she will get the same consumption path, that she will evaluate using the same preference map. In the endowment economy, indeed, index number theory will associate income to current consumption; the Divisia index will then measure output grow as consumption growth; 2% in our example. Why is it the case that two economies where people have identical preferences and face exactly the same consumption path do not grow at the same rate? Because a true quantity index takes current income as a norm and current income is defined differently; at any time, both economies share the same consumption utility, but investment goods are produced only in the production economy. These seemingly paradoxical example illustrate well the intimate relation between preferences (what we want to do) and technology (what we can do) when measuring output growth. Indeed, in this particular example, both measures of output growth are welfare based and consistent with NIPA's methodology. The example makes also clear the implications of measuring production as final demand: since there is no investment in the endowment economy, output growth becomes identical to consumption growth.

Subjective discount factor. In a first example, let us consider two different two-sector AK economies that only differ on their discount factors and their depreciation rates. For that, let extend the two-sector AK model to allow for capital depreciation, where δ is the depreciation rate. In this case, for all results in section 3.1 substsitute A for $A - \delta$. A patient and an impatient economy. To be more precise, let us assume $\rho_1 > \rho_2$ and $\delta_1 < \delta_2$, such that $\delta_1 + \rho_1 = \delta_2 + \rho_2$; the rest of parameters are common to both economies. Economy 2 is more patient than economy 1, but it is also less efficient in its depreciation technology. It is easy to check that investment will be growing at the same rate in both economies; as well as consumption. However, they will not have the same share of net investment[19]

$$
\begin{aligned}
s_1 = \frac{\alpha(A-\delta_1-\rho_1)}{\rho_1(1-\alpha)+\alpha\sigma(A-\delta_1)} &= \frac{\alpha(A-\delta_2-\rho_2)}{\rho_1(1-\alpha)+\alpha\sigma(A-\delta_1)} \\
&< \frac{\alpha(A-\delta_2-\rho_2)}{\rho_2(1-\alpha)+\alpha\sigma(A-\delta_2)} = s_2,
\end{aligned}
$$

where the equality follows from the imposition that $\delta_1+\rho_1=\delta_2+\rho_2$ and the inequality from $\rho_1>\rho_2$ (or equivalently from $\delta_1<\delta_2$). As expected, the patient economy saves more, even if it grows at the same rate of the impatient economy because its depreciation rate is larger. The Divisia index tell us then that the patient economy will grow faster than impatient economy, since it has a larger investment rate.

This example illustrates well that the patient economy weights future consumption more, so that values more than the other the same consumption growth rate. The Fisher-Shell and the Divisia indexes do reflect the differential in welfare gains, but the growth rate of consumption does not. In short, again, the growth rate of consumption is not a good measure of real growth because it is unable to reflect the welfare gains differences between these two economies, welfare gains derived from a different valuation of future consumption.

Growth accounting. To end this discussion, let us review the implications for growth accounting. In terms of model representations of actual economies, the introduction of more than one sector with different growth rates raises the practical and conceptual issue of how output growth has to be measured. The choice of the appropriate output growth rate affects every quantitative exercise based on the measurement of growth. This is the case in the literature on growth accounting under embodied technical change, the so-called Solow-Jorgenson controversy. To measure the contribution of investment specific technical change to growth, Hulten (1992) measures growth (his equation (7)) following Jorgenson (1966). He suggests a raw addition of consumption and investment units, calling the outcome quality-adjusted output. Using our notation, this strategy amounts to c_t+x_t. Greenwood et al (1997) note that, in their setting, adding consumption and effective investment turns the economy into a standard Solow (1960) growth model with no embodied technical change.[20] Greenwood et al (1997) correctly state that any aggregation requires the different quantities to be expressed in a common unit and they

[19]Remind from Section 3.3 that s is the share of net investment on net income.

[20]See Hercowitz (1998) for a review of the Solow-Jorgenson controversy.

adopt the consumption good as their standard. For this purpose, investment has to be multiplied by its relative price, in our notation their choice of output level would be $y_t = c_t + p_t x_t$.[21] Oulton (2004) generalizes the argument and suggests that output components have to be deflated by the consumption price index in order to measure growth. But this is indeed what Greenwood et al (1997) suggest when they identify non-durable production with real output and the real growth rate with the growth rate of consumption. What the present paper clarifies is that the issue is not the units used to measure real output *levels* but the choice of the right index of real output *growth*. In this sense, we follow Licandro et al (2002) and conclude that the "true" contribution of ETC to output growth, reflecting welfare changes, has to be measured using NIPA's methodology as in Cummins and Violante (2002).

A word of caution. We have to be careful in the way we interpret changes in the output growth rate. It is well-known in endogenous growth theory that raising the growth performance of an economy is costly, and that consequently there exists something as an optimal growth rate. Let us assume, for example, that an endogenous growth economy is at an optimal allocation growing at its optimal growth rate. Let us then assume that an uninformed government decides to introduce at time t_0 some incentives to promote growth, for example by subsidizing R&D. The economy will be then growing faster at the cost of a substantial welfare reduction at the initial time t_0. As shown in this paper, from t_0 the growth rate of output will be measuring welfare gains. However, it may be that the initial welfare lost is not necessarily capture by National Accounts, since changes in the value of assets are in general not registered.

5 Conclusions and extensions

This paper shows that a Fisher-Shell true quantity index when applied to a two-sector dynamic general equilibrium economy with general recursive preferences is equal to the Divisia index. Indeed, it turns out that the chained-type index used by National Accounts to compute real output growth is well approximated by the Divisia index. Consequently, real output growth in NIPA is a welfare measure. This result is illustrated in the frame-

[21]In their setting, this choice looks somewhat natural because the investment sector uses as input the consumption good. In their notation $y_t = c_t + p_t x_t$ is total output in the non-durable sector, even if only c_t is consumed and the remaining production $p_t x_t$ is allocated to the investment sector.

work of the two-sector AK model. This model replicates the well-know stylized facts that investment grows faster than consumption and that the relative price of investment permanently declines. Hence, it is the appropriate context to evaluate the shift to chain indexes by National Account. More important, changes in the growth rate of investment induced by changes in embodied technical progress turn out to be a relevant part of welfare increases along an equilibrium path. Investment then matters in the measurement of output growth. In general, this paper can be seen as a recall that index number theory has an important role to play clarifying the criteria with which we construct our indexes. In particular, this approach may be of great relevance for the recent debate on the use on index number theory to rationalize the use of the Penn World Tables (see Neary (2004) and van Veelen and van der Weide (2008)).

Let us finally comment on those dimensions in which this approach could be extended and those in which it will be hard to do. Broaden it to many durable and non-durable goods seems straightforward. The approach could also be applied to many forms of non-optimal equilibria. Notice that, in this case, the production possibility frontier will not be tangent to an indifference curve at equilibrium, and hence the generalization will not be straightforward. However, if the representative household is price taker in all markets, irrespective of the fact that prices are distorted, at equilibrium the budget constraint will be tangent to an indifference curve. Under theses circumstances, index number theory could be applied to compare different points in the equilibrium path in a similar way we did in Section 2. In particular, for a stationary economy moving from a distorted to a non distorted equilibrium, the Divisia index could be measuring the welfare gains period by period.

Note that this paper understands welfare changes as income compensating variations of a representative household. Yet, one could interpret the Divisia index to be measuring welfare changes of the average household in an economy with many different consumers. Actually, in the Bellman equation representation (4) utility is quasilinear on investment. Quasilinear preferences belong to the more general family of Gorman preferences, which can be aggregated and represented by those of a representative household.[22] In this sense, the growth rate in NIPA may be understood as a welfare based measurement even in worlds with heterogenous households. Indeed, things will be more complicated in overlapping generations economies.

[22]See Gorman (1953, 1961).

Appendix: Quantity indexes in continuous time

A1. Fixed-base quantity indexes

In this appendix we use the notation of our simple framework to review the methodological changes introduced by the BEA[23] whose extension to continuout time is no always straighforward. Traditional measures of real growth stem from fixed-base quantity indexes. The most common among these is the Laspeyres index. Let us choose consumption as the numeraire so that its price is normalized to one while the price of investment in consumption units is p_t. Let us fix prices at some base-time t and then compute the factor of change between t and $t+h$ as

$$\Pi_{t+h}^{t} = \frac{c_{t+h} + p_t x_{t+h}}{c_t + p_t x_t},$$

for all $h \geq 0$, where the superindex t in Π designate the base-time t and the subindex the current time $t+h$. In the jargon of National Accounts, Π_{t+h}^{t} is a volume index. The Laspeyres index g_{t+h}^{t} is the instantaneous growth rate of factor Π_{t+h}^{t} as a function of h (see Appendix A3). That is,

$$g_{t+h}^{t} = \frac{d\Pi_{t+h}^{t}}{dh} \frac{1}{\Pi_{t+h}^{t}} = \frac{\dot{c}_{t+h} + p_t \dot{x}_{t+h}}{c_{t+h} + p_t x_{t+h}},$$

which measures the real growth rate at $t+h$, for given base-time t. The Laspeyres index is popular because it is conceptually simple.

However, if the relative price of investment permanently declines and real investment permanently grows faster than real consumption, as observed in the data, the Laspeyres index tends to give too much weight to investment as we depart from the base-time t. In particular, since investment is growing faster than consumption, the Laspeyres growth rate tends to that of investment, therefore overstating real growth. Note that

$$g_{t+h}^{t} = \frac{c_{t+h}}{c_{t+h} + p_t x_{t+h}} \frac{\dot{c}_{t+h}}{c_{t+h}} + \frac{p_t x_{t+h}}{c_{t+h} + p_t x_{t+h}} \frac{\dot{x}_{t+h}}{x_{t+h}}. \tag{5}$$

It is easy to see that along an equilibrium path with constant income shares, the weight of consumption decreases and the weight of investment increases with h. This effect is

[23] Young (1992) is a non-technical presentation of the methodological changes introduced in NIPA. Whelan (2002, 2003) provides a more detailed guide into the new methods in use at BEA to measure real growth. For economic index number theory see Diewert (1993), Fisher and Shell (1998) and IMF (2004, chapter 17).

known in the index numbers literature as the *substitution bias*: the demand for goods whose price permanently decline displays higher growth in real terms. Quantity indexes based on past (relatively high) prices overweight these items, overstating the real growth rate. The effect is larger the farther we are from the base year.

The Paasche index uses current prices as a base, and hence tends to understate real growth as we go back in time. The factor is

$$\Pi_{t-h}^{t} = \frac{c_t + p_t x_t}{c_{t-h} + p_t x_{t-h}}$$

for all $h \geq 0$ and the growth rate

$$g_{t-h}^{t} = \frac{d\Pi_{t-h}^{t}}{dh}\frac{1}{\Pi_{t-h}^{t}} = \frac{c_{t-h}}{c_{t-h} + p_t x_{t-h}}\frac{\dot{c}_{t-h}}{c_{t-h}} + \frac{p_t x_{t-h}}{c_{t-h} + p_t x_{t-h}}\frac{\dot{x}_{t-h}}{x_{t-h}}. \tag{6}$$

As h grows, so $t-h$ decreases, the weight of consumption increases because x_{t-h}/c_{t-h} decreases.

In both cases these indexes yield poor measures of real growth when output components grow at different rates because of changing relative prices.[24]

A2. Chained-type quantity indexes

The introduction by the BEA of quality corrections in equipment prices in the mid-eighties revealed a persistent declining pattern in the price of equipment relative to the price of non-durable consumption goods. Since then, real investment appears to be growing much faster than real non-durable consumption. In this new scenario, fixed-base quantity indexes face a severe substitution bias problem. For this reason, the BEA moved to a chained-type index based on a Fisher ideal index computed for contiguous periods.[25]

In continuous time, let the factor of change in the interval $(t, t+h)$ be the geometric mean of the factors associated to the Laspeyres index with base t and the Paasche index with base $t+h$, that is

$$F_{t,t+h} = \left(\Pi_{t+h}^{t}\Pi_{t}^{t+h}\right)^{\frac{1}{2}}.$$

[24]Updating regularly the base is not a solution because it would imply a permanent revision of past growth performance. It posses the additional problem of multiple real growth measures for each period, each of them affected differently for the substitution bias depending on the associated base period.

[25]Diewert (1993) provides a clear explanation of the index suggested by Fisher (1922).

The Fisher ideal index is the growth rate of factor $F_{t,t+h}$ as a function of h. Computing the average compensates the overstatement of the Laspeyres index with the understatement of the Paasche index, thus reducing the impact of the selection bias.

Since the bases are updated every period, the growth factor over many periods is defined simply as the product of the intermediate one-period factors of growth. For example, over a time interval $[0, T]$ divided in N periods, the factor of growth between 0 and T is defined as

$$\Phi_{0,T} = \prod_{n=1}^{N} F_{n\frac{T}{N},(n+1)\frac{T}{N}}.$$

That is, the series $F_{n\frac{T}{N},(n+1)\frac{T}{N}}$ is "chained" to obtain the multiperiod factor. Chain indexes loose the multiplicative property that makes so easy working with Laspeyres indexes.[26] In exchange they reduce the substitution bias because it regularly updates the base. In continuous time, a chained-type index perfectly counterbalance the substitution bias. In the case of the Fisher ideal index, when $h \to 0$, as shown in Appendix A3, tends to the Divisia Index

$$g_t^{\mathrm{D}} = \frac{c_t}{c_t + p_t x_t}\frac{\dot{c}_t}{c_t} + \frac{p_t x_t}{c_t + p_t x_t}\frac{\dot{x}_t}{x_t},$$

which weights consumption and investment growth by their respective shares in income.[27] At any time, the growth rates of consumption and investment are weighted by their current shares, which are independent of any base-time. Even if there is a trend in relative prices, inducing the substitution of one good for another, the chained-type index allows weights to change continuously to avoid the emergence of any substitution bias.

A3. Quantity indexes in continuous time

In continuous time, let us define a growth factor Γ_t^{t+h}, interpreted as the gross rate of growth of an arbitrary variable between t and $t+h$. Let us then define the instantaneous

[26]Unlike fixed-base indexes, chain indexes do not have the multiplicative property. In general, the factor $F_{t,t+2h}$ does not coincide with the chained factor $F_{t,t+h} \times F_{t+h,t+2h}$. Just observe that $t+h$ prices play no role in the calculation of $F_{t,t+2h}$. Moreover, both the additive property, that income ratios add up to unity, and the property that income rations are bounded by unity do not hold. These issues are very well illustrated in Whelan (2002).

[27]As noted above, this is not surprising. In continuous time, as h goes to zero, the Laspeyres and Paasche indexes tend to each other, implying that chain indexes based on the Laspeyres, Paasche and Fisher ideal indexes coincide, and all are therefore equal to the Divisia index. In discrete time, however, only the Fisher ideal index approximates the Divisia index.

growth rate of the underline variable at time t as

$$g_t = \left.\frac{\mathrm{d}\Gamma_t^{t+h}}{\mathrm{d}h}\right|_{h=0}. \tag{7}$$

The definition in (7) is correct if one observes that the derivate of a factor of change of a continuous-time variable is equal to the growth rate of the variable itself. Let z_t be a continuous-time variable and fix some reference point at time t. The growth factor in this case is $\Gamma_t^{t+h} = z_{z+h}/z_t$. Let take the first derivate of it

$$\frac{\mathrm{d}\Gamma_t^{t+h}}{\mathrm{d}h} = \frac{\dot{z}_{t+h}}{z_t}.$$

When evaluated at $h = 0$

$$g_t = \left.\frac{\mathrm{d}\Gamma_t^{t+h}}{\mathrm{d}h}\right|_{h=0} = \left.\frac{\dot{z}_{t+h}}{z_t}\right|_{h=0} = \frac{\dot{z}_t}{z_t}.$$

This way of defining the instantaneous growth rate may look odd but it may be useful in those cases in which we have an index like Γ_t^{t+h} but no explicit variable giving rise to this index like z_t in this example. The Fisher ideal chain index is one of these cases.

Using the notation introduced in Section 2, the starting point is some nominal aggregate $c_t + p_t x_t$. Laspeyres quantity indexes use time t (the base-time) prices as weights based on the following growth factor

$$\mathcal{L}_t^{t+h} = \frac{c_{t+h} + p_t x_{t+h}}{c_t + p_t x_t}.$$

It does allow to compute the growth rate of output by putting all nominal values at base-time prices. Paasche indexes take current prices as weights by defining the growth factor as

$$\mathcal{P}_t^{t+h} = \frac{c_{t+h} + p_{t+h} x_{t+h}}{c_t + p_{t+h} x_t}.$$

Real output growth is measured at current $t + h$ prices.

Let us now use (7) to define the Laspeyres and Paasche indexes for the corresponding definitions of the growth factors. The definition in (7) implicitly assume that for any time t, t is the base-time; growth rates are defined independently of any arbitrary base-time, calculated for any time t as if t were the base-time, and then chained. This is the fundamental assumption of chain quantity indexes.

It is easy to see that in continuous time both Laspeyres and Paasche quantity indexes are equal to the Divisia index:

$$\left.\frac{\mathrm{d}\mathcal{L}_t^{t+h}}{\mathrm{d}h}\right|_{h=0} = \left.\frac{\mathrm{d}\mathcal{P}_t^{t+h}}{\mathrm{d}h}\right|_{h=0} = (1 - s_t)\frac{\dot{c}_t}{c_t} + s_t\frac{\dot{x}_t}{x_t},$$

where $s_t = \frac{p_t x_t}{c_t + p_t x_t}$.[28]

The Fisher ideal growth factor between t and $t+h$ is defined as

$$\mathcal{F}_t^{t+h} = \left(\mathcal{L}_t^{t+h}\mathcal{P}_t^{t+h}\right)^{\frac{1}{2}}. \tag{8}$$

Given that in continuous time, both Laspeyres and Paasche chain quantity indexes are equal to the Divisia index, it is easy to show that the Fisher ideal chain index is equal too.

The definition in equation (7) is also useful applied to the Fisher-Shell quantity index since we have a well-defined index $\hat{m}_{t+h}/m_t$.

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