# DISCUSSION PAPER SERIES

DP11593

## FIRM DYNAMICS IN THE NEOCLASSICAL GROWTH MODEL

Omar Licandro

***MACROECONOMICS AND GROWTH***

ISSN 0265-8003

# FIRM DYNAMICS IN THE NEOCLASSICAL GROWTH MODEL

*Omar Licandro*

Discussion Paper DP11593
Published 01 November 2016
Submitted 01 November 2016

Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

# FIRM DYNAMICS IN THE NEOCLASSICAL GROWTH MODEL


## Abstract

This paper integrates firm dynamics theory into the Neoclassical growth framework. It embeds selection into an otherwise standard dynamic general equilibrium model of one good, two production factors (capital and labor) and competitive markets. Selection relies on firm specific investment: i) capital is a fixed production factor --playing the role of an entry cost, ii) the productivity of capital is firm specific, but observed after investment, iii) firm specific capital is partially reversible --its opportunity cost plays the same role as fixed production costs. At equilibrium, aggregate technology is Neoclassical, but the average quality of capital is endogenous and positively related to selection; due to capital irreversibility, the marginal product of capital is larger than the user cost and capital depreciation positively depends on selection. At steady state, output per capita and welfare both raise with selection; rendering capital more reversible or increasing the variance of the idiosyncratic shock both raise selection, productivity, output per capita and welfare.



JEL Classification: O3, O4

Keywords: Firm Dynamics, Entry and Exit, selection, Neoclassical growth model, Ramsey, Hopenhayn, Capital irreversibility



Omar Licandro - omar.licandro@nottingham.ac.uk
*Nottingham University and CEPR*



Acknowledgements
I would like to thank David de la Croix, Giammario Impullitti and Frances Obiols for their comments and suggestions. Alessandro Ruggiere suggested the proof of Proposition 3 and Emilio Espino suggested the endogenous growth extension. The author acknowledges financial support from the Spanish Ministry of Economy and Competitiveness (ECO2010-17943 and ECO2013-48884-C3-1-P).

# Firm Dynamics in the Neoclassical Growth Model*

**Omar Licandro**

University of Nottingham

(omar.licandro@nottingham.ac.uk)

**October 2016**

**Abstract**: This paper integrates firm dynamics theory into the Neoclassical growth framework. It embeds selection into an otherwise standard dynamic general equilibrium model of one good, two production factors (capital and labor) and competitive markets. Selection relies on firm specific investment: i) capital is a fixed production factor –playing the role of an entry cost, ii) the productivity of capital is firm specific, but observed after investment, iii) firm specific capital is partially reversible –its opportunity cost plays the same role as fixed production costs. At equilibrium, aggregate technology is Neoclassical, but the average quality of capital is endogenous and positively related to selection; due to capital irreversibility, the marginal product of capital is larger than the user cost and capital depreciation positively depends on selection. At steady state, output per capita and welfare both raise with selection; rendering capital more reversible or increasing the variance of the idiosyncratic shock both raise selection, productivity, output per capita and welfare.

**Keywords**: Firm dynamics, Entry and Exit, Selection, Neoclassical growth model, Ramsey, Hopenhayn, Capital irreversibility.

**JEL classification numbers**: O3, O4.

---

*I would like to thank David de la Croix, Giammario Impullitti and Frances Obiols for their comments and suggestions. Alessandro Ruggiere suggested the proof of Proposition 3 and Emilio Espino suggested the endogenous growth extension. The author acknowledges financial support from the Spanish Ministry of Economy and Competitiveness (ECO2010-17943 and ECO2013-48884-C3-1-P).

# 1 Introduction

The Neoclassical growth model is the cornerstone of modern macroeconomics. Its origins can be traced to the contributions of Robert Solow [23] and Frank Ramsey [21], among many others. Its power relies on its ability to replicate in a simple and stylized framework the salient evidence on economic growth, the well-known Kaldor [16] stylized facts.

The Neoclassical growth model builds on, among others, the assumption of a representative firm. During the last decades, indeed, following the seminal contributions of Hopenhayn [11] and Jovanovic [15], a new framework has emerged designed to understand the implications of a growing evidence on firm dynamics and market behavior.[1] In this literature, selection plays a fundamental role in explaining market performance. Firms are heterogenous, and competition creates and destroys firms and jobs, moving resources from low productive to high productive firms; the exit-entry process (the extensive margin) and the expansion-contraction of incumbent firms (the intensive margin) govern market behavior raising average productivity and wages.

In the spirit of both the Neoclassical growth theory and the theory of firm dynamics, this paper studies a competitive dynamic general equilibrium model with heterogeneous firms. We will refer to it as the Ramsey-Hopenhayn model (RH in the following). Firms' technology is assumed to use a fixed heterogeneous production factor (capital) and a flexible homogeneous factor (labor) to produce a sole good, which as usual may be consumed or invested. To produce, firms have to pay an entry cost: the price of the fixed factor. The productivity of firms, more precisely the productivity of firms' capital, is unknown to firms before entry. For the sake of simplicity, as in Melitz [20], we assume that idiosyncratic productivity is time invariant, i.e., firms face no other productivity shock that the initial one. In addition to that, capital is assumed to be partially irreversible. When a firm decides to close down, the productivity of its capital is so low that no other firm would like to buy it. Used capital has, indeed, a scrap value, which is smaller than the replacement value of capital. The fact that capital has an opportunity cost makes low productive firms exit. Consequently, fixed production costs are not required to generate endogenous exit at equilibrium.[2]

[1] See Bartelsman and Doms [1] for a revision of the empirical literature.

[2] We borrow this set of assumptions from Gabler and Licandro [7].

Using standard aggregation theory, this paper shows that aggregate technology at equilibrium is Neoclassical with the average value of capital being positively related to selection. In the limit, the RH model encompasses the Neoclassical model in the cases of homogeneous firms and/or fully reversible capital. This aggregation result is in line with aggregation in the vintage capital literature, as in Solow [24] and Solow et al [25], where different vintage technologies collapse on a Neoclassical production function.[3] Indeed, heterogeneity in the firm dynamics and vintage capital models are of a different nature.

In the RH model, selection is costly due to capital irreversibility, which makes the user cost of capital larger than its rental rate as derived by Jorgenson [14]. The difference between them is what Chirinko and Schaller [5] call the irreversibility premium, due here to the fact that some new units of capital are optimally scrapped straightaway since they are not productive enough to be profitably used in the production process. In this sense, selection is costly since it requires some capital units, the less productive, to be destroyed. The irreversibility premium relates here to the fact that this framework displays both physical and economic depreciation, which is in line with capital obsolescence in the vintage capital literature (See Malcomson [19] and Boucekkine et al [3]). At the stationary equilibrium, the RH model shares most properties of the Neoclassical theory, but endogenous selection positively affects the value of capital at the cost of increasing the depreciation rate. Nonetheless, consistently with the First Welfare Theorem, changes in the environment that promote selection are shown to increase steady state consumption and welfare.

Most of the literature on firm dynamics is based on a critical assumption: firms have to pay a fixed production cost to produce. In a general equilibrium model of this nature, a raise in the fixed production cost (all other parameters given) increases selection. But raising fixed production cots make the economy less efficient, reducing output and welfare. We formally develop this argument in Section 3. Compared to the RH model proposed in this paper, a more standard model of firm dynamics is clearly

[3]Aggregation of vintages technologies has been extensively used in the more recent literature on embodied technical progress or investment specific technical change. See the seminal paper by Greenwood et al [10], as well as the endogenous growth extensions by Krusell [17] and Boucekkine et al [2]. Similar aggregation results are pointed out by Hopenhayn [12] when relating the literature on firm dynamics with the recent literature on misallocation (Restuccia and Rogerson [22], for example).

less efficient, since capital is fully irreversible and in addition production entails fixed production costs. More important, we show that an increase in fixed production costs generates more selection at the cost of rendering the economy less efficient. The opposite arrives in the RH model, where increasing the degree of capital reversibility generates more selection, by rendering the economy more efficient. Even if for both economies the First Welfare Theorem applies, and both competitive equilibria are Pareto optimal, the needed change in the environment that raises selection have opposite effects on the fundamentals. In the RH model, a raise in capital reversibility makes the economy more efficient, but in the more standard firm dynamic model a raise in fixed production costs worsens the fundamentals.

The paper also relates to the literature on occupational choice that follows the seminal contribution of Lucas [18]. In line with the survey in Hopenhayn [12], our framework could be extended to study different types of distortions that affect the efficient allocation of productive resources across firms.

The paper is organized in the following way. Under the assumption that there is no technical change, Sections 2 describes the Ramsey-Hopenhayn economy and studies its steady state equilibrium. Section 3 studies the stationary equilibrium of a more standard firm dynamics economy with sunk entry costs and fixed production costs. Section 4 extends the RH model to the case of disembodied technical progress and endogenous growth, and Section 5 concludes.

# 2 The RH Model

## 2.1 Economy

The model in this paper is a competitive dynamic general equilibrium model with endogenous entry and exit. There is a unit measure of identical households. Each household member offers inelasticity one unit of labor at any time $t$. The representative household maximizes utility

$$U = \int_0^\infty u(c_t)\, e^{-\rho t}\, dt \tag{1}$$

subject to a standard budget constraint. Instantaneous utility $u(c_t)$ is Neoclassical, where $c_t$ represents per capita consumption. The subjective discount rate $\rho > 0$.

As in Hopenhayn [11], a continuum of heterogeneous firms produce a sole homogeneous good under perfect competition. Goods production is allocated to consumption and investment. Firms' technology employs a fixed production factor (capital) and a flexible production factor (labor). Capital plays the role of an entry cost; firms need to buy one unit of capital to enter the market and then produce. Capital is firm specific and partially irreversible, in the sense that when a firm close down it only recovers a fraction $\theta \in (0, 1)$ of its capital. We will refer to $\theta$ as the scrap value of capital. Technology has decreasing returns to the flexible factor, labor. Firm's productivity is heterogeneous; it is randomly drawn after entry and remains constant over time.

Let us be more precise on the specification of firms' technology. Each firm requires one unit of capital to produce and is characterized by a firm-specific productivity $z$. Output of a $z$-firm is given by

$$y(z) = F\big(z, \ell(z)\big). \tag{2}$$

Technology $F(z, \ell)$ is assumed to be $\mathcal{C}^1$, increasing in both arguments, concave and homogeneous of first degree. Variables $y(z)$ and $\ell(z)$ denote equilibrium output and employment, respectively, of a firm with productivity $z$.

To fix ideas, let us interpret the firm specific productivity $z$ as being embodied in firm's capital and reveled after investment, as if different capital units have different qualities represented by $z$. In facts, after entry, productivity $z$ is drawn from the continuous density $\varphi(z)$, for $z$ in the support $\mathcal{Z} \subset \Re^+$ –the associated cumulative distribution is denoted by $\Phi(z)$. In the following, we denote by $\zeta \geq 0$ and $\omega > \zeta$ to the lower and upper bounds of the support $\mathcal{Z}$. Expected productivity at entry is assumed to be one without any loss of generality. Firms may exit for two different reasons. First, endogenously, when its value is smaller than the scrap value of capital $\theta$. Second, firms also exit at the exogenous rate $\delta$, $\delta > 0$, in which case the scrap value of capital is zero. Under these conditions, as it will be shown below, there exists a strictly positive productivity cutoff $z^* \in \mathcal{Z}$, such that the distribution of firms at the stationary equilibrium, as in Melitz (2003), is the truncated density

$$\phi(z) = \frac{\varphi(z)}{1 - \Phi(z^*)} \tag{3}$$

for $z \in \mathcal{Z}$ and $z \geq z^*$.

## 2.2 Stationary Equilibrium

**Household behavior.** Even if firms face idiosyncratic risk, households fully diversify it by buying the market portfolio. Consequently, the optimal behavior of the representative household is given by the Euler equation

$$\frac{\dot{c}_t}{c_t} = \sigma_t(r_t - \rho), \tag{4}$$

where $\sigma_t$ is the intertemporal elasticity of substitution and $r_t$ the equilibrium interest rate. At steady state $r_t = \rho$.

**Firms' behavior and labor market clearing.** A firm with productivity $z$ solves

$$\max_{\ell(z)} F\big(z, \ell(z)\big) - w\ell(z),$$

for a given wage rate $w$. The production good is the numeraire. From the first-order-condition associated to this problem, the optimal labor demand reads

$$F_2\big(z, \ell(z)\big) = w \quad \Rightarrow \quad \ell(z) = G^{-1}(w)z,$$

where $G\big(\frac{\ell}{z}\big) \equiv F_2\big(1, \frac{\ell}{z}\big) = F_2(z, \ell)$. Remind that $F(.)$ is homogeneous of first degree, then its first derivatives are homogeneous of degree zero.

The labor market clearing condition is

$$n \int_{z \geq z^*} \ell(z)\phi(z)dz = 1,$$

where $n$ represents the mass of operative firms per capita, which is equal to the stock of capital measured in physical units (in units of the production good). After substitution of individual labor demands $\ell(z)$ into the labor market clearing condition, we get

$$nG^{-1}(w) \int_{z \geq z^*}^{\omega} z\phi(z)dz = 1.$$

The equilibrium wage rate can then be written as

$$w = F_2(k, 1), \tag{5}$$

where $k = \bar{z}n$ measures capital per capita in quality adjusted units.[4] Remind that a firm requires one unit of capital to produce and idiosyncratic productivity $z$ measures

[4]This way of measuring capital is consistent with national accounts, where after a long debate following Gordon [9]'s seminal work, investment is deflated using constant quality price indexes.

the quality of this capital unit, with

$$\bar{z} = \int_{z \geq z^*} z\phi(z)dz$$

representing the average quality of capital for the whole economy. Since $F_{22}(.) < 0$, the wage rate is positively related to capital per capita $k$ which crucially depends on the average quality of capital $\bar{z}$. Selection makes average capital more productive, raising wages. As usual in this literature, the raise in wages reallocates labor from less to more productive firms.

Substituting the equilibrium wage rate into the labor demand function, it becomes

$$\ell(z) = \frac{z}{k} = \left(\frac{1}{n}\right)\frac{z}{\bar{z}},$$

where $1/n$ represents average labor per firm; labor is distributed across firms according to the relative productivity $z/\bar{z}$. More productive firms hire more labor in order to equalize the marginal productivity of labor across firms. Within firm reallocation operates here through the intensive margin: high productive firms employ more workers than low productive firms. Selection, as shown below, operates through the extensive margin.

**Selection and free entry.** Since $F(z, \ell)$ is homogeneous of first degree, it is easy to show that stationary profits are linear on $z$ at equilibrium

$$\pi(z) = F_1(k, 1)z. \tag{6}$$

Since profits are the return to capital, they are equal to the marginal product of aggregate capital per worker, $F_1(k, 1)$, weighted by the productivity of the firm, $z$. We discuss this issue in more detail below.

The value of the $z$-firm at steady state is

$$v(z) = \begin{cases} \pi(z)/(\rho + \delta) & \text{if } z \geq z^* \\ \theta & \text{otherwise,} \end{cases}$$

which, for $z \geq z^*$, is monotonically increasing in $z$. Consequently, the exit threshold $z^*$ is determined by the exit condition

$$\pi(z^*) = \theta(\rho + \delta) \quad \Rightarrow \quad z^* = \frac{\theta(\rho + \delta)}{F_1(k, 1)}. \tag{EC}$$

A firm with productivity $z < z^*$ exits, since the value of producing is lower than the scrap value of capital. Exit is then the result of capital not being productive enough relative to its scrap value. It becomes clear now why under partial capital reversibility there is no need of fixed production costs to the cutoff productivity $z^*$ be interior.

Let us assume that there is a infinite mass of potential entrants. New firms have to buy one unit of capital as represented by the entry cost at the right-hand-side of the equation (FE) below. The free entry condition reads

$$\Phi(z^*)\ \theta + \big(1 - \Phi(z^*)\big)\ \pi(\bar{z})/(\rho + \delta) = 1. \tag{FE}$$

With probability $\Phi(z^*)$ a newly created firm draws a productivity smaller than $z^*$, in which case it closes down and recovers the scrap value of capital $\theta$. Otherwise, with probability $1 - \Phi(z^*)$ it becomes operative. In the latter case, its value is equal to the expected value of operative firms, which is equal to $\pi(\bar{z})/(\rho + \delta)$ since profits are linear on $z$.

Combining the (EC) and (FE) conditions, we get

$$\frac{\theta \bar{z}}{z^*} = \frac{1 - \theta \Phi(z^*)}{1 - \Phi(z^*)}, \tag{EC-FE}$$

which only depends on $z^*$, since $\bar{z}$ only depends on it too. It is easy to see that the (EC-FE) condition does not depend on the particular form of the production function $F(.)$. Proposition 1 below shows that the RH model admits one and only one stationary solution for $z^*$. Proposition 2 shows that $z^*$ is an increasing function of $\theta$; selection then raises with capital reversibility. More important, Proposition 2 shows that the ratio $z^*/\theta$ increases with $\theta$. Since, as we show below, $z^*/\theta$ affects positively steady state capital, output and consumption per capita, Proposition 2 also implies that more selective economies produce and consume more in per capita terms. Finally, Proposition 3 shows that a larger variance of the productivity distribution is associated with more selection, which increases both stationary output and consumption per capita. As a consequence, the RH model shows unambiguous welfare gains from selection.

The main properties of the equilibrium cutoff $z^*$ are studied in the propositions below.

**Proposition 1** *For $\theta \in (\zeta, 1)$, where $\zeta$ is the lower bound of the productivity support $\mathcal{Z}$, there exists one and only one $z^*$, $z^*$ in the interior of $\mathcal{Z}$, that solves (EC-FE).*

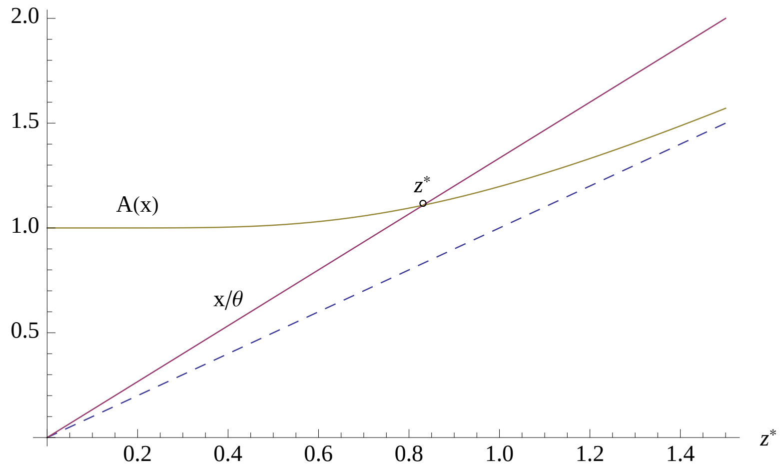


Figure 1: Determination of the cutoff productivity $z^*$

**Proof**: Inverting both sides of (EC-FE) and multiplying them by $\bar{z}$, $z^*$ is determined by

$$\frac{z^*}{\theta} = \frac{1 - \Phi(z^*)}{1 - \theta\Phi(z^*)}\, \bar{z}(z^*) < \bar{z}(z^*).$$

Multiply then both sides by $\theta\big(1-\theta\Phi(z^*)\big)$, move the term $\theta z^*\Phi(z^*)$ to the right-hand-side and use the definition of $\bar{z}$ to rewrite the equilibrium condition as

$$\frac{z^*}{\theta} = \int_{z\geq z^*} z\varphi(z)dz + z^*\Phi(z^*) \equiv \mathcal{A}(z^*). \tag{EC-FE'}$$

The left-hand-side of (EC-FE') is linear on $z^*$, crosses the origin and has a slope $1/\theta > 1$. Concerning the right-hand-side, $\mathcal{A}(\zeta) = 1$ (remind that the entry distribution has unit mean) and $\lim_{x->\omega} \mathcal{A}(x)/x = 1$, which is strictly smaller than $1/\theta$. Under the assumption that $\theta > \zeta$, it is easy to see that $\mathcal{A}(x)$ crosses $x/\theta$ at least once in the interior of $\mathcal{Z}$. The first derivative of $\mathcal{A}(x)$ is

$$\mathcal{A}'(x) = \Phi(x) \in (0, 1).$$

Since the slope of $\mathcal{A}(x)$ is smaller than the slope of the left-hand-side for all $x > \zeta$, $\mathcal{A}(x)$ can only crosses $x/\theta$ once, which completes the proof.□

Equation (EC-FE) defines $z^*$ as an implicit function of $\theta$, say $z^*(\theta)$, which functional form depends on the functional form of the entry distribution $\Phi(.)$. (EC-FE) can be seen as an arbitrage condition: A firm exits if and only if the expected return of reentering is larger than the current return. For this reason, as shown in the proof above, $z^* < \theta\bar{z}$

(remind that the firm will expect getting $\bar{z}$ with probability $1 - \Phi(z^*) < 1$). Figure 1 represents both sides of the equilibrium condition (EC-FC'), the diagonal (dashed line) and the equilibrium value of $z^*$ (the projection of point $z^*$ in the x-axis).

It is easy to see that there is no selection in an economy with full capital irreversibility. When the distribution admits a lower bound productivity $\zeta = 0$ and $\theta = 0$, the solution with full irreversibility is $z^*(0) = 0$. In the case of Figure 1, the $x/\theta$ line collapses to the vertical axis and point $z^*$ moves to $(0, 1)$. Otherwise, when the distribution admits $\zeta > 0$ and $\theta$ is at its minimum admisible value $\zeta$, $\mathcal{A}(x) = x/\theta$ only admits $z^* = \zeta$ as a solution. On the other extreme, when $\theta$ goes to one, $z^*$ goes to the upper bound of the support $\omega \in \mathcal{Z}$, $\omega < \infty$. The $z^*/\theta$ locus in Figure 1 would move towards the diagonal cutting the $\mathcal{A}(x)$ locus at $\omega$. In case $\omega$ is unbounded, $z^*$ tends to infinity. When capital is fully reversible, firms would always like to try again and again in order to reach the maximum posible productivity $\omega$.

The fact that the cutoff productivity is increasing in $\theta$ can be easily observed in Figure 1. Notice that an increase in $\theta$ moves the $x/\theta$ locus to the right, raising $z^*$. Indeed, as shown in the proposition below, $z^*$ raises more than proportional.

**Proposition 2** *For* $\theta \in (\zeta, 1)$, $\frac{dz^*}{d\theta}\frac{\theta}{z^*} > 1$.

**Proof**: Take logs and totally differentiate the equilibrium condition $x = \theta\mathcal{A}(x)$, use the result above that $\mathcal{A}'(x) = \Phi(x)$ and reorganize terms to get

$$\frac{dx/x}{d\theta/\theta} = \left(1 - \frac{\Phi(x)x}{\mathcal{A}(x)}\right)^{-1},$$

which is larger than one by definition of $\mathcal{A}(x)$. □

What is the role of idiosyncratic uncertainty in the selection process? The follow proposition shows that selection is increasing in the dispersion of productivity across firms.

**Proposition 3** *If the support of the initial distribution is bounded from above, $z^*$ is increasing in the variance of the entry distribution.*

**Proof**: Let the support $\mathcal{Z}$ of the entry distribution be the line segment $(\zeta, \omega)$, $\zeta \geq 0$ and $\omega < \infty$. Firstly, integrate by parts the integral at the right-hand-side of (EC-FE'), and multiply both sides by $\theta$, to get

$$z^* = \theta\left(\omega - \int_{z^*}^{\omega} \Phi(z)dz\right). \tag{7}$$

An equilibrium is a fixed point of this relation (see Proposition 1).

Let the initial distribution be a mean preserving spread (remind that the mean of the entry distribution is supposed to be one) and make it explicit by writing it as $\Phi(z;\sigma)$, where $\sigma$ is the standard deviation.

Let us now consider two distributions of the same family that have different standard deviations $\sigma_1 < \sigma_2$. In this case, for any $z^* \in (\zeta, \omega)$,[5]

$$\int_{\zeta}^{z^*} \big(\Phi(z;\sigma_2) - \Phi(z;\sigma_1)\big) dz \geq 0, \quad \Rightarrow \quad \int_{z^*}^{\omega} \big(\Phi(z;\sigma_2) - \Phi(z;\sigma_1)\big) dz \leq 0.$$

Consequently, the right-hand-side of (7) will move to the right when $\sigma$ moves from $\sigma_1$ to $\sigma_2$, increasing $z^*$.□

**Aggregate Technology.** Per capita output results from the aggregation of firms' production (remind that population has been normalized to unity):[6]

$$y = n \int_{z \geq z^*} y(z)\phi(z)dz = F(k,1) \equiv f(k). \tag{8}$$

Remind that at equilibrium $F\big(z, \ell(z)\big) = F(k,1)z/k$, since $\ell(z) = z/k$ and $F(.)$ is homogeneous of first degree. Notice that aggregate technology is Neoclassical and has the same functional form as the individual firm's technology. The only relevant difference with respect to the Neoclassical model is that the average quality of capital, as measured by $\bar{z}$, instead of being normalized to one is endogenous and, as shown above, increasing in selection. It is interesting to notice that, from equation (5), labor is paid at its marginal productivity, since $w = F_2(k,1) = f(k) - kf'(k)$.

**Capital per capita and the User Cost of Capital.** Since capital is heterogeneous, the return to capital depends on capital's productivity. Individual profits in (6) can be written as

$$\pi(z) = zf'(k),$$

[5]See Diamond and Stiglitz (1974).

[6]Due to the fact that the average quality of capital, and then investment, is endogenously given by $\bar{z}$, in strict sense, investment and consumption are different goods. At the stationary equilibrium, the price of investment relative to consumption is $1/\bar{z}$. In this sense, $y$ measures per capita output in consumption units and $\bar{z}y$ in capital units.

meaning that, conditional on the firm being operative, firm's capital is paid at its marginal productivity. Profits are the return to firms' capital, being equal to the marginal product of per worker aggregate capital $k$ multiplied by the firm specific capital productivity $z$. The return to capital is larger, the more productive capital is.

Indeed, from the free entry condition (FE), using equilibrium profits just above, we get

$$\bar{z} f'(k) = \underbrace{\frac{1-\theta \Phi(z^*)}{1-\Phi(z^*)}}_{\text{irreversibility premium}} (\rho+\delta) > \rho+\delta.$$

Adding one unit of output to the stock of capital has a marginal productivity $\bar{z} f'(k)$. As usual, marginal productivity has to be equal to the user cost of capital, represented here by the right-hand-side of the above equation. Jorgenson [14]'s type user cost of capital, $\rho+\delta$, is multiplied here by a factor measuring the irreversibility premium.[7] With probability $\Phi(z^*)$ the firm will get a bad productivity draw, in which case, it will optimally close down and send its capital to scrap getting in return less than the replacement cost of capital.

From the exit condition (EC), using again equilibrium profits (6), we get

$$z^* f'(k) = \theta(\rho+\delta) < \rho+\delta.$$

The cost of destroying the marginal unit of capital, the less productive, is $z^* f'(k)$. The opportunity cost of doing it, indeed, is the return on the scrap value, represented at the the right-hand-side of the equation above. The user cost of the marginal unit of capital is smaller than the Jorgensonian user cost of capital, smaller than the user cost of the average unit of capital.

Given the equilibrium cutoff productivity, we can use the (EC) condition to solve for the aggregate capital per capita, implicit in

$$f'(k) = \theta / z^* (\rho+\delta).$$

From Proposition 2, we know that the more reversible capital is, the lower $\theta/z^*$, making aggregate capital per capita to be larger. This is fundamentally due to the fact that the average productivity of capital $\bar{z}$ is larger. As a direct implication, output per capital as measured by $y$ is larger too.

[7] In line with Chirinko and Schaller [5].

To fix ideas, for a given equilibrium cutoff $z^*$, the equilibrium value of capital can be written as

$$k = g(z^*),$$

where $g(z^*)$ result from inverting the relation $f'(k) = \theta(\rho+\delta)/z^*$. Since technology $F(.)$ is assumed to be concave, $f''(k) < 0$, implying that the capital stock per capita is growing with selection. This is particularly true, from Proposition 2 above, if selection results from an increase in the degree of capital reversibility $\theta$. Moreover, from Proposition 3, idiosyncratic uncertainty opens new opportunities that require some degree of capital reversibility to be realized, raising both capital and output per capita.

**Scrapping.** Let $e$ be the mass of entrants per capita. At the stationary equilibrium net entry equals exit, i.e.,

$$e\left(1 - \Phi(z^*)\right) = \delta n.$$

Remind that entrants that draw a productivity smaller than $z^*$ automatically exit. At steady state, $e = \frac{\delta n}{1-\Phi(z^*)} > \delta n$, implying that, as usual in this literature, selection makes endogenous gross entry be larger than exogenous exit.

In terms of production, $y = c + \left(1 - \theta\Phi(z^*)\right) e$, since the scrapped capital $\theta$ of the $\Phi(z^*)\, e$ exiting firms is reemployed in production as an input. Substituting $e$ using the stationary equilibrium condition above, output can be written as

$$y = c + \underbrace{\frac{1 - \theta\Phi(z^*)}{1 - \Phi(z^*)}\,\delta}_{\text{effective depreciation}} \quad n.$$

Given that selection destroys part of the newly created capital, the less productive, and the destroyed capital is only partially reversible, the effective depreciation rate is larger than $\delta$. The difference between $\delta$ and the effective depreciation rate is due to partial reversibility. Selection adds economic depreciation to the physical depreciation of capital, in a similar way as obsolescence do in the vintage capital literature (see Malcomson [19]).

Since $y = f(k)$ and $k = \bar{z}n$, using the (EC-FE) condition, the feasibility condition at the steady state equilibrium reads

$$f(k) = c + \frac{\theta}{z^*}\,\delta k,$$

where output per capita is allocated to consumption and investment, both measures in consumption units. Notice that the relative price of investment is $1/\bar{z}$, implying that the second term on the right-hand-side is identical to $n$ multiplied by the effective depreciation rate as defined above (use the (EC-FE) condition to show it).

**Neoclassical Model.** The RH model encompasses the Neoclassical model in two particular limit cases. In the first case, the variance of the distribution $\Phi(z)$ is assumed to be zero. This is equivalent to assume that there is a representative firm with productivity equal to one. At the stationary equilibrium, $\bar{z} = 1$, implying that $k = n$; the real wage is $w = f(k) - kf'(k)$, equal to the marginal product of labor. From the (FE) condition $f'(k) = \rho + \delta$; the marginal product of capital is equal to the Jorgensonian user cost of capital. Notice that under firm homogeneity irreversibility plays no role: the firms' value is larger than the scrap value $\theta$, making all firms optimally stay in the market irrespective of how low $\theta$ is.

In the second case, the degree of capital reversibility $\theta$ is assumed to be one, implying that capital is fully reversible. To the stationary equilibrium be well defined, let us assume that the support $\mathcal{Z}$ is bounded above by $\omega < \infty$ and the exogenous exit rate $\delta = 0$. By combining (EC) with (FE), it is easy to see that under $\theta = 1$, $z^* = \bar{z}$, which by definition of $\bar{z}$ can only be true if both $z^*$ and $\bar{z}$ are equal to $\omega$. Since capital is fully reversible, whatever the initial distribution of capital is, all firms with productivity smaller than $\omega$ will eventually scrap their capital and the economy will converge to a stationary state with a degenerate productivity distribution, all firms' productivity being equal to $\omega$. Under these assumptions, $k = n\omega$, $w = f(k) - kf'(k)$ and $\rho + \delta = \omega f'(k)$, corresponding to the marginal product of labor and capital, respectively. Notice that, even if firms are initially heterogeneous, they become homogeneous at the stationary equilibrium with selection moving all firms to the frontier technology.

Since at the stationary equilibrium, as shown above, $k = g(z^*)$, $g'(z^*) > 0$, the steady state capital per capita of the RH model will be bounded by these two alternative Neoclassical stationary equilibria. Indeed, irrespective of the fact that capital per capita in the RH can be larger or smaller than in the Neoclassical model (depending on the definition), it is unambiguously increasing in selection.

**Welfare.** From the feasibility condition above, it is easy to see that selection positively affects consumption per capita, i.e.,

$$\frac{\partial c}{\partial(\theta/z^*)} = \left(f'(k) - \frac{\theta\delta}{z^*}\right)\frac{\partial k}{\partial(\theta/z^*)} - \delta k = \frac{\theta\rho}{z^*}\frac{\rho+\delta}{f''(k)} - \delta k < 0. \tag{W}$$

From Proposition 2, an increase in capital reversibility, $\theta$, reduces $\theta/z^*$. Stationary consumption per capita, and then welfare, are both increasing in selection, in particular when more selection is a consequence of larger capital reversibility. More efficient economies, those with larger $\theta$, are more selective, have a larger capital per capita, produce and consume more at the stationary equilibrium. For a similar argument, steady state welfare is also bigger in economies with larger dispersion of the idiosyncratic productivity shock.

# 3 Fixed production costs and sunk entry costs

This section studies the main properties of an alternative general equilibrium model. Let us refer to it as the standard Hopenhayn model (SH), standard in the particular sense that instead of assuming partial capital reversibility, we assume that there is a sunk entry cost (capital is fully irreversibly) and a fixed production cost $\lambda$, $\lambda > 0$, which for convenience we measured in units of the consumption good.[8] In this alternative framework, selection is positively related to fixed production costs. But, an environment with larger fixed production costs is less efficient. Nevertheless, in the RH model selection is positively related to capital reversibility, with the polar property that an environment with larger capital reversibility is more efficient.

Under the previous assumptions, net revenues $\pi(z)$ adopt the same functional form as profits in the previous section

$$\pi(z) = f'(k)z.$$

The new exit condition, indeed, becomes

$$\pi(z^*) = \lambda \quad \Rightarrow \quad f'(k)z^* = \lambda.$$

[8]Assuming that the production cost $\lambda$ is fixed in units of the consumption good instead of labor is not critical for the main arugment, but simplifies the analysis by allowing us to use the main results of the paper. The critical difference between the standard Hopenhayn type of model and the Ramsey-Hopenhayn model relies on assuming a fixed production cost instead of partial capital reversibility.

A firm with productivity $z \geq z^*$ will be active, since profits are positive at the stationary equilibrium.

The new free entry condition is

$$\big(1 - \Phi(z^*)\big)\big(f'(k)\bar{z} - \lambda\big)/(\rho + \delta) = 1.$$

Since capital is a sunk cost, with probability $\Phi(z^*)$ the firm close down and gets nothing in return. Otherwise, it expects to get the discounted flow of expected profits $\big(\pi(\bar{z}) - \lambda\big)/(\rho + \delta)$.

Combining the new exit and free entry conditions, we get

$$\frac{\rho + \delta + \lambda}{\lambda}\, z^* = \mathcal{A}(z^*),$$

with $\mathcal{A}(z^*)$ defined as in Proposition 1. Notice that this equation is identical to (EC-FE') in the proof of Proposition 1, with $\frac{\lambda}{\lambda+\rho+\delta} \in (0, 1)$ instead of $\theta$. Consequently, Propositions 1 to 3 all apply to the SH model too.

To fix ideas, let us assume that the productivity distribution at entry is Pareto,

$$\Phi(z) = 1 - \left(\frac{\zeta}{z}\right)^{\kappa},$$

with tail parameter $\kappa > 1$, and lower bound productivity $\zeta = (\kappa - 1)/\kappa$ in order to the expected productivity at entry be one (the same normalization as in the previous section). The cutoff productivity of the RH model, emerging from equation (EC-FE) is

$$z^* = \left(\frac{\theta}{(1-\theta)(\kappa - 1)}\right)^{\frac{1}{\kappa}} \zeta,$$

which is increasing in $\theta$. Take logs and differentiate the equation above (taking $\kappa$ and $\zeta$ as given) to show that

$$\frac{dz^*/z^*}{d\theta/\theta} = \frac{1}{\kappa(1-\theta)},$$

which is larger than one, since $\theta \in (\zeta, 1)$. This result is just an application of Proposition 2. The ratio $z^*/\theta$ is then increasing in $\theta$. We can finally use the (EC) condition to solve for $k$, which positively depends on $z^*/\theta$. Consequently, an increase in $\theta$ makes the RH economy more selective and more capital intensive. This will reflect on larger output and welfare at steady state.

Let see what the equivalent result is in the SH economy. Using the equations above,

$$z^* = \left(\frac{\lambda}{(\rho + \delta)(\kappa - 1)}\right)^{\frac{1}{\kappa}} \zeta.$$

Indeed, it is easy to see that

$$\frac{dz^*/z^*}{d\lambda/\lambda} = \frac{1}{\kappa} < 1,$$

since $\kappa > 1$; the ratio $z^*/\lambda$ is decreasing in $\lambda$. Given the exit condition above, $k$ is positively related to $z^*/\lambda$, which is decreasing in $\lambda$. Then, an increase in $\lambda$ generates more selection, at the cost of a reduction in capital per worker. The fundamental reason is that an increase in the degree of reversibility in the RH model raises the value of capital, but an increase in the fixed production cost in the SH model reduces it, since production will be more costly. In these two economies, the incentives for capital accumulation move in opposite directions when the economy faces more selection.

Let us now define output as value added

$$y = n \int_{z \geq z^*} y(z)\phi(z)dz - n\lambda = f(k) - n\lambda.$$

Fixed production costs $\lambda$ are (internal) inputs used in the production process that needs to be subtracted to define firm's output as $y(z) - \lambda$. Let us combine the definition of $k$, i.e. $k = n\bar{z}$, and the property of the Pareto distribution that $\bar{z} = \kappa/(\kappa - 1)\ z^*$, to write $n = \frac{\kappa-1}{\kappa}\frac{k}{z^*}$. Let us then use the exit condition $f'(k) = \lambda/z^*$ to get[9]

$$y = f(k) - \frac{\kappa - 1}{\kappa}\ f'(k)\ k \equiv g(k).$$

Notice that its first derivative is

$$g'(k) = \frac{f'(k)}{\kappa} - \frac{\kappa - 1}{\kappa}\ f''(k)\ k > 0.$$

Indeed, output is increasing in $k$, even if an increase in $k$ may be raising fixed production costs. Since $k$ is decreasing with selection, output is decreasing too, and consequently welfare.

Compared to the RH model in the previous section, this economy is clearly less efficient, since capital is fully irreversible and in addition production requires the fixed production cost $\lambda$. More important, an increase in $\lambda$ generates more selection at the cost of rendering the economy less efficient. The opposite arrives in the RH model, where increasing $\theta$ to generate more selection, renders the economy more efficient. Even if for

[9] Of course, conditions may be imposed on firm's technology $f(k)$ in order to the aggregate technology $g(k)$ be Neoclassical. Notice, however, that the aggregate technology depends on the productivity distribution, which is not the case in the model with capital reversibility.

both economies the First Welfare Theorem applies, and both competitive equilibria are Pareto optimal, the needed change in the environment that raises selection have opposite effects on the fundamentals. In the RH model, a raise in $\theta$ makes the economy more efficient, but in the SH model a raise in $\lambda$ worsens the fundamentals.

# 4 Extensions

**Disembodied Technical Progress** The model can be easily extended to the case of exogenous labor saving (disembodied) technical progress. For this purpose, let us assume that the technology of a firm $z$ is

$$y_t(z) = F\big(z, A_t \ell_t(z)\big),$$

where $A_t = A\ \mathrm{e}^{\gamma t}$, $A > 0$, and $\gamma > 0$ representing the rate of technical change. In this case, a balanced growth path exists where per capita output, capital and consumption, all grow at the rate $\gamma$. It is easy to see that the cutoff productivity $z^*$ is determined as before by the (EC-FE), which does not dependen on technology. Propositions 1 to 3, then, apply and selection plays the same role as in the previous sections. It is important to notice that under disembodied technical progress, the position of a firm in the productivity distribution remains unchanged at a balanced growth path. The equilibrium distribution is then given by the truncated distribution (3). As a consequence, the average quality of capital $\bar{z}$ remains constant at the balanced growth path, meaning that all raise in $k_t$ is due to an increase in capital when measured in physical units.[10]

**Endogenous Growth** The aggregation argument developed above applies to any firm's technology $x = F(z, \ell)$, increasing in and homogeneous of first degree on both arguments $z$ and $\ell$, and concave on $\ell$, but not necessarily concave on $z$. It then applies,

[10] The case of embodied technical progress is less straightforward, belonging to the vintage capital literature (see Boucekkine et al [4] and Gilchrist and Williams [8]) with within-vintage heterogeneity. The natural extension will be to assume that the average productivity at entry grows at the rate $\gamma$, meaning that on average newer vintages are more productive. As time goes, the equilibrium cutoff productivity will be moving to the right. Since it is common to all vintages, the equilibrium distribution will not simply be a truncation of the entry distribution.

in particular, to the firm's technology

$$x = Az + H(z, \ell),$$

where $H(z, \ell)$ is is assumed to be increasing in, concave and homogeneous of first degree on both arguments $z$ and $\ell$. Using a similar argument as above, it aggregates at equilibrium to the aggregate technology

$$y = Ak + h(k),$$

where $h(k) = H(k, 1)$. Selection, as in the RH model, only depends on the scrap value of capital $\theta$ and the parameters of the productivity distribution at entry, $\Phi(z)$. For a given average productivity $\bar{z}$, if $h(.)$ is a Neoclassical production function, the economy behaves at the stationary equilibrium as in Jones and Manuelli [13]. At the unique (asymptotic) balance growth path, under the assumption that household preferences are constant intertemporal elasticity of substitution, consumption grows at the endogenous stationary rate

$$g = \sigma(A\bar{z} - \rho - \delta).$$

In this framework, selection, by making the productivity of capital larger, positively affects the stationary growth rate of the economy.

# 5 Conclusions

This paper extends the Neoclassical growth model to accommodate for firm heterogeneity in the spirit of the literature on firm dynamics. In this framework, selection raises the value of capital and per capita output. More important, it increases steady state welfare.

The argument also applies to endogenous growth models of the AK type (see [13]). In this case, selection has a positive effect on the endogenous growth rate by increasing the equilibrium expected return to capital accumulation.

# References

[1] Bartelsman, Eric J. and Mark Doms (2000) "Understanding Productivity: Lessons from Longitudinal Microdata," Journal of Economic Literature, 38(3), 569–94.

[2] Boucekkine, Raouf, Fernando del Río and Omar Licandro (2003) "Embodied Technological Change, Learning-by-Doing and the Productivity Slowdown," Scandinavian Journal of Economics, 105(1), 87–98.

[3] Boucekkine, Raouf, Fernando del Río and Omar Licandro (2005) "Obsolescence and modernization in the growth process," Journal of Development Economics, 77(1), 153–171.

[4] Boucekkine, Raouf, Marc Germain and Omar Licandro (1997) "Replacement Echoes in the Vintage Capital Growth Model," Journal of Economic Theory, 74, 333–348.

[5] Chirinko, Robert and Huntley Schaller (2009) "The irreversibility premium," Journal of Monetary Economics, 56, 390–408.

[6] Diamond, Peter and Joseph Stiglitz (1974) "Increases in Risk and in Risk Aversion," Journal of Economic Theory, 8, 337–360.

[7] Gabler, Alain and Omar Licandro (2009) "Firm Dynamics and the Importance of the Embodied Question," Barcelona GSE WP 397.

[8] Gilchrist, Simon and John Williams (2000) "Putty-Clay and Investment: A Business Cycle Analysis", Journal of Political Economy, 108(5), 928–960.

[9] Gordon, Robert (1990) *The measurement of durable goods prices*, Chicago University Press.

[10] Greenwood, Jeremy, Zvi Hercowitz, and Per Krusell (1997) "Long-run implications of investment-specific technological change," American Economic Review, 87(3), 342-362.

[11] Hopenhayn, Hugo (1992) "Entry, Exit, and Firm Dynamics in Long Run Equilibrium," Econometrica, 60(5), 1127–1150.

[12] Hopenhayn, Hugo (2014) "Firms, Misallocation, and Aggregate Productivity: A Review," Annual Review of Economics, 6, 735–70.

[13] Jones, Larry and Rodolfo Manuelli (1997) "The sources of growth," Journal of Economic Dynamics and Control, 21(1), 75–114.

[14] Jorgenson, Dale (1963) "Capital Theory and Investment Behavior," American Economic Review, 53(2), .247–259.

[15] Jovanovic, Boyan (1996) "Selection and the Evolution of Industry," Econometrica, 50(3), 649–670.

[16] Kaldor, Nicholas (1961) "Capital Accumulation and Economic Growth," in F.A. Lutz and D.C. Hague eds. *The Theory of Capital*, St. Martins Press, 177–222.

[17] Krusell, Per (1998) "Investment-Specific R and D and the Decline in the Relative Price of Capital," Journal of Economic Growth, 3(2), 131–141.

[18] Lucas, Robert (1978) "On the Size Distribution of Business Firms," The Bell Journal of Economics, 9(2), 508–523

[19] Malcomson, James (1975) "Replacement and the rental value of capital equipment subject to obsolescence," Journal of Economic Theory, 10(1), 24–41

[20] Melitz, Marc J. (2003) "The Impact of Trade on Intra-Industry Reallocations and Aggregate Productivity," Econometrica, 71(5), 1695–1725.

[21] Ramsey, Frank (1928) "A Mathematical Theory of Saving," Economic Journal, 38(152), 543–559.

[22] Restuccia, Diego and Richard Rogerson (2008) "Policy distortions and aggregate productivity with heterogeneous establishments," Review of Economic dynamics, 11(4), 707–720.

[23] Solow, Robert (1956) "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, 70(1), 65–94.

[24] Solow, Robert (1956) "Substitution and Fixed Proportions in the Theory of Capital," Review of Economic Studies, 29, 207–218.

[25] Solow, Robert, James Tobin, C. Von Weizsacker and M. Yaari (1966) "Neoclassical Growth with Fixed Factor Proportions," Review of Economic Studies, 33, 79–15.