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JEL Classification: E13, J13, J14

Keywords: long term care, birth timing, childbearing age, family pol- icy, OLG models.

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Long-Term Care and Births Timing*†

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March 21, 2016

Abstract

Due to the ageing process, the provision of long-term care (LTC) to the dependent elderly has become a major challenge of our epoch. But our societies are also characterized, since the 1970s, by a postponement of births, which, by raising the intergenerational age gap, can affect the provision of LTC by children. In order to examine the impact of those demographic trends on the optimal policy, we develop a four-period OLG model where individuals, who receive children's informal LTC at the old age, must choose, when being young, how to allocate births along their life cycle. It is shown that, in line with empirical evidence, early children provide more LTC to their elderly parents than late children, because of the lower opportunity cost of providing LTC when being retired. When comparing the laissez-faire with the long-run social optimum, it appears that individuals have, at the laissez-faire, too few early births, and too many late births. We then study, in first-best and second-best settings, how the social optimum can be decentralized by encourageing early births, in such a way as to reduce the social burden of LTC provision.

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1 Introduction

The beginning of the 21st century is characterized by two fundamental demographic trends, which constitute the most recent corollaries of the demographic transition started two centuries ago (Lee 2003).

On the one hand, the ageing process raises the proportion of elderly persons in need of long-term care (LTC), i.e. persons who can no longer carry out daily activities such as eating, washing, etc. The number of dependent elderly in Europe (EU-27) is expected to grow from 38 million people in 2010 to 57 million in 2060 (EU 2012). The rise of LTC constitutes a major challenge for families, since two thirds of LTC is provided informally (Norton 2000). Forecasts from the EU suggest that a significant part of LTC provision will remain informal in the future (see Figure 1). Given the limited development of private LTC insurance markets, this constitutes also a major challenge for policy-makers.[1]

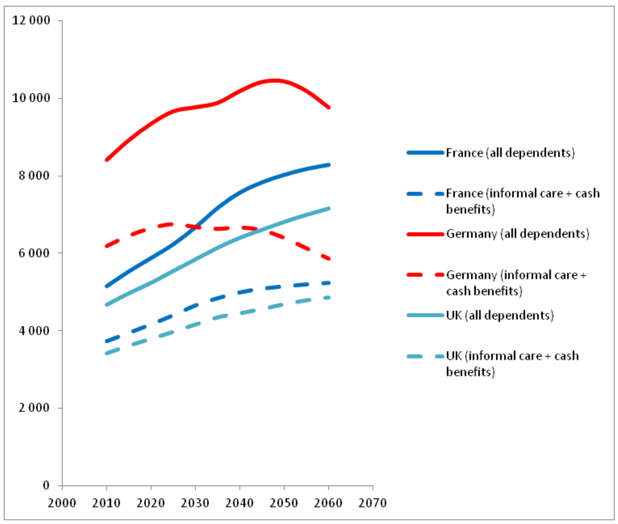

Figure 1: Number of dependent persons (total / relying on informal care and cash benefits), in thousands (source EU 2012)

On the other hand, our societies witness also, since the 1970s, a significant postponement of births (see Gustafsson 2001). Because of various reasons, such as the rise in education, medical advances or cultural norms, individuals tend to have their children later on in their life.[2] To illustrate that trend, Figure 2 shows the rise in the mean age at birth in different European countries. That rise is substantial: whereas the mean age at birth was below 27 years in France in the late 1970s, it is about 30 years today. The postponement of births influences the dynamics of the age-structure of the economy, and, hence, may affect the financial sustainability of PAYG pensions schemes.

[1] On this, see Cremer et al (2012).

[2] On the causes of births delay, see Cigno and Ermisch (1991) and Happel et al (1984).

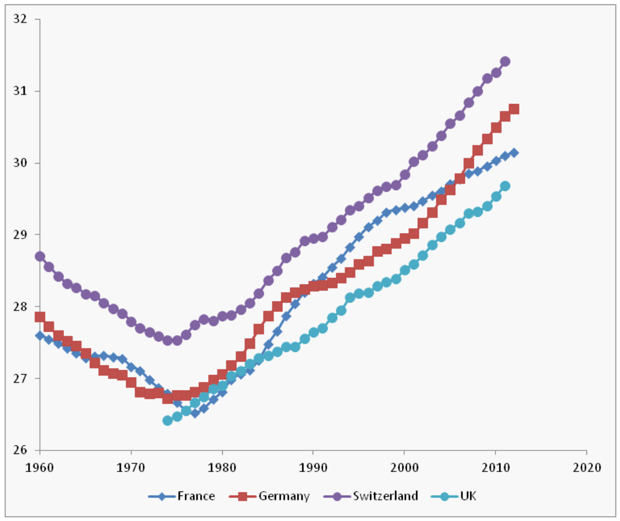

Figure 2: Mean age at birth (source: Human Fertility Database).

Although those two demographic trends may seem, at first glance, to be unrelated, those two phenomena are linked through various channels. First of all, a large volume of LTC services are informal, and provided by the family. Hence the rise of the demand for LTC imposes some pressure on the time constraints faced by informal care givers, such as spouses and children (see Korn and Wrede 2013). Secondly, the timing of births chosen by parents determines the age gap between parents and children, which constitutes a major determinant of the amount of LTC that children will provide to their parents once those will reach the old age. When the caregivers are children of the dependent elderly, the age of caregivers has a major impact on the opportunity cost of helping their elderly parents. Clearly, the oldest child of the family is, when his parent becomes dependent, quite old already, and close to retirement. Hence, for the first children born in a family, the opportunity cost of providing LTC is lower. On the contrary, younger children are, when their parent becomes dependent, younger, and are not close to retire, implying a higher opportunity cost of providing LTC.

Whereas those links may appear purely theoretical, empirical studies show that informal LTC is far from a marginal phenomenon. According to the SCAN Foundation (2012), informal caregivers spend, on average, 20.4 hours per week providing care. Moreover, the period during which informal care is given is, on average, equal to 4.6 years. Thus informal LTC really affects time constraints in real life. Regarding the impact of the age of children, recent studies, such as Fontaine et al (2007) showed, on the basis of SHARE data, that the behavior of the children of a dependent parent without spouse is far from symmetric, but, rather, varies with the age of the child and his/her involvement in the labor market. In particular, if younger children are working full time, older children tend to be more involved in the provision of LTC to their elderly parent, even though younger children also participate. Thus the age (and, hence, working status) of children providing LTC is an important variable, which is directly related to the fertility choices made by their parents when they were young.

The goal of this paper is to examine the relations between, on the one hand, the provision of LTC by children, and, on the other hand, the timing of births, by paying a particular attention to the unequal opportunity costs of providing LTC among children of unequal ages. We propose to study the conditions under which the timing of birth of children can be used by parents as an insurance device for LTC provision at the old age in an economy where there exist no social insurance, and where LTC is only provided informally through children.

For that purpose, we develop a 4-period OLG model where individuals become dependent at the old age (period 4), and where the dependent's health depends on the amount of (informal) LTC provided by their children. In order to study the fertility timing, we consider a model of lifecycle fertility, where parents can choose to have children in two fertility periods (i.e. periods 2 and 3). We consider the joint of decisions of the timing of births and of the provision of LTC to the elderly parents, within a dynamic model of capital accumulation.

Our analysis proceeds in two stages. Our model is first used to examine, at the laissez-faire, the relation between birth timing and LTC provision. We explore the conditions under which we can rationalize the empirical fact that older children provide more LTC than younger children. We first examine that issue at the temporary equilibrium, and, then, we characterize the stationary equilibrium, to examine to what extent our results are robust to the long-run variations of wages and interest rates, which affect the opportunity cost of providing LTC for children. In a second stage, we characterize the long-run social optimum, and we examine the decentralization of that social optimum by means of appropriate policy instruments. Our question are: what does the optimal family policy look like? Is the birth timing chosen by parents socially optimal in the long-run? Is the postponement of births desirable in times of a rise in LTC?

Anticipating on our results, we first show that, in general, early children provide, at the laissez-faire, more LTC to their elderly parents in comparison to late children, because of a lower opportunity cost of providing LTC. That tendency is shown to be persistent in the long-run. Then, we characterize the long-run social optimum. We show that individuals have in general, at the laissez-faire, too few children early in their life, and too many children later on in their life in comparison with the social optimum. The reason is that parents do not perfectly anticipate, when they plan the births of their children, that the chosen birth timing will affect the opportunity costs faced by their children when they will provide LTC 50 or 60 years later, once they will be dependent. As a consequence, the decentralization of the social optimum requires subsidies on early births aimed at encouraging parents to have children early, so as to minimize the opportunity cost of providing LTC in the future. When considering the second-best problem (and focusing, for simplicity, on an open economy where factor prices are taken as given), we show that the optimal (uniform) subsidy on early births depends not only on standard equity and efficiency concerns, but aims also at internalizing population composition effects, as well as at encouraging the provision of LTC to the dependent parents.

This paper pertains to several branches of the literature. First of all, it complements the literature on LTC and family games. That literature emphasized the crucial role played by the age of children in LTC provision, in particular when the child decides his geographical location (see Konrad et al 2002; Wakabayashi and Horioka 2009). The present paper is not about location choices, but focuses on the differences in the time cost of providing LTC due to the proximity with the retirement period.[3] Secondly, our paper is also in line with the literature focusing on the links between fertility choices and LTC informal provision. Cremer et al (2013) study the role of fertility as an insurance device in case of LTC risk at the old ages. Korn and Wrede (2013) examine the impact of LTC provision on time constraints and fertility choices. Our contribution with respect to that literature is to consider the timing of births. Thirdly, we also contribute to the literature on optimal public policy under LTC (see Jousten et al 2005; Pestieau and Sato 2006, 2008; Cremer and Pestieau 2010; Cremer and Roeder 2013). Those papers study the design of optimal LTC public insurance, but do not consider fertility choices. Finally, we also complement the literature on the timing of births in OLG models, such as d'Albis et al (2010) and Pestieau and Ponthiere (2014, 2015). Our contribution with respect to those papers is to highlight the effects of birth timing decisions on the informal provision of LTC.

The rest of the paper is organized as follows. The model is presented in Section 2. The laissez-faire (temporary equilibrium and intertemporal equilibrium) is studied in Section 3. Section 4 characterizes the long-run social optimum and studies its decentralization. The second-best optimal policy is studied in Section 5. Section 6 concludes.

2 The model

We consider a four-period OLG model. The duration of each period is normalized to 1. Each cohort is a continuum of individuals of size 1. Fertility is at the replacement level, that is, one child per agent.[4]

Period 1 is childhood. Period 2 is young adulthood, during which individuals work, consume, save resources and have $n_t \in]0,1[$ children.[5] During period 3, individuals consume and have $1 - n_t \in]0,1[$ children. In period 3, individuals work during a fraction of time $z \in]0,1[$, and enjoy a period of retirement of duration $1 - z$.[6] Period 4 is a period of old-age dependence, during which

[3] We also complement other papers, such as Pezzin et al (2007, 2009), which focus more on strategic motives for LTC provision, without considering endogenous fertility choices.

[4] This assumption, which amounts to suppose that, in the real world, the number of children per woman is equal to 2, is made for analytical simplicity. Fixing the quantity (*quantum*) of births allows us to better focus on the timing (*tempo*) of births.

[5] As usual in the economics of fertility, we treat the number of children as a real number. This is an obvious simplification of reality, where the number of births is a natural number.

[6] For the sake of analytical tractability, we treat z as an exogenous parameter, which is not chosen by agents. Allowing agents to choose z would, by raising the number of choice variables, complicate the analysis without bringing new insight regarding the timing of births. We will, throughout the paper, assess the robustness of our results to the level of z.

individuals need LTC. The health of the dependent elderly depends on the amount of informal LTC received from his children.[7]

There exist two types of agents, depending on the age of their parent:

- Type-E agents: children born from young parents (i.e. "early" children)
- Type-L agents: children born from older parents (i.e."late" children).

Within the population of young adults at time t, the proportion of type E agents is denoted by q_t ($0 \leq q_t \leq 1$). The proportion of agents of type L is $1 - q_t$.

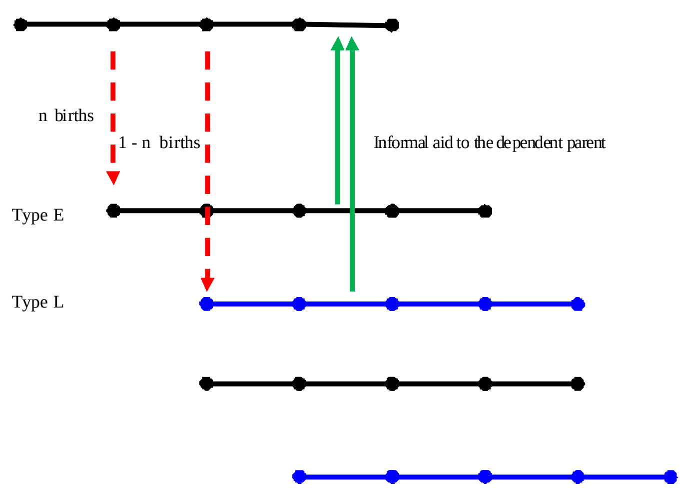

Figure 3: Type-E and Type-L agents in the OLG economy

Figure 3 illustrate the coexistence of agents of types E and L; those agents differ regarding the age of their parents: while type-E individuals have younger parents (i.e. the age-gap between them and their parents is low), type-L individuals have older parents (i.e. the age-gap between them and their parents is higher). As a consequence, those agents will also differ in terms of age at the time of providing LTC to their elderly parents. Type-E agents will be older when their parents will fall into dependence. On the contrary, type-L agents will be younger and fully active when their parents will become dependent.

[7] It should be stressed here that our 4-period OLG model is only a reduced-form model of the human life cycle. Dividing a life in 4 ages is an obvious simplification. Ideally, if one wanted to fit all ages of life, we would need a N-period OLG model.

2.1 Health production

At the old age (i.e. period 4), all individuals become dependent (whatever they are of type E or L). We assume here that there exists a unique level of dependence.[8] However, a dependent elderly can be more or less healthy, depending on the amount of LTC he receives. Throughout this paper, the health of the dependent elderly is assumed to be an increasing function of the amount of informal LTC that the dependent receives from his children.

The health of the dependent elderly of type i at time t, which is denoted by H_t^i, is modeled as a function:

$$H_t^i \equiv H\left(b_t^i\right) \tag{1}$$

where $H'(\cdot) > 0$ and $H''(\cdot) < 0$, and where b_t^i denotes the temporal aid received from children at the old age.[9]

The total LTC received at the old age b_t^i is assumed to be the sum of the temporal aid provided by each child:[10]

$$b_t^i = n_{t-2}^i a_t^E + (1 - n_{t-2}^i) a_t^L \tag{2}$$

where a_t^E denotes the amount of LTC provided by each of the n_{t-2}^i early children, whereas a_t^L is the amount of LTC provided by each of the $1 - n_{t-2}^i$ late children.

We assume a perfect substitutability between the informal LTC provided by early children and late children. This assumption seems plausible, since informal LTC requires only basic skills, and is thus a kind of homogeneous input in the production of the dependent's health: whether it is one child or another who gives some of his time to help his dependent parent does not seem to make a substantial difference as far as the health of the dependent is concerned.[11]

2.2 Preferences

When being adults, individuals care about their consumption in periods 2 and 3, denoted respectively by c_t^i and d_{t+1}^i, and about the number of early and late children, denoted respectively by n_t^i and $1 - n_t^i$. Individuals are also assumed to derive some utility from contributing to the health of their elderly parent, even though helping their parents implies also a cost in terms of time (see below).[12] At the old age (period 4), a person is dependent, and cares only about his health.

[8] This constitutes a simplification, since in real life there exist various degrees of dependence, which all require different amounts of LTC, depending on the needs of the patients. By assuming a unique level of dependence, we suppose that all patients have the same needs.

[9] The monotonicity of $H(\cdot)$ amounts to presuppose that caregivers always provide an amount of care that is inferior or equal to the needs of the patient.

[10] For the sake of analytical convenience, we focus here only on LTC provision by children, and exclude grandchildren from the provision of LTC.

[11] Note, however, that if we were considering also formal care, then issues of complementarity between different types of LTC could arise, but in our model there is only one type of LTC (informal), and perfect substitutability seems to be a plausible assumption in that context.

[12] We are aware that providing LTC generates substantial disutility for caregivers, and can also deteriorate their health (see Do et al 2015). However, for the sake of simplicity, we abstract here from those costs, and we only focus on the time cost of providing LTC.

The preferences of a young adult of type $i \in \{E, L\}$ are represented by the following utility function:[13]

$$u(c_t^i) + v\left(n_t^i\right) + u(d_{t+1}^i) + v(1 - n_t^i) + \varphi(a^i) + \gamma H(b_{t+2}^{ie}) \tag{3}$$

where we assume, as usual, that $u'(\cdot) > 0$ and $u''(\cdot) < 0$. We also suppose that children are desirable goods, i.e. $v'(\cdot) > 0$, and that births are also characterized by declining marginal utility, that is, that $v''(\cdot) < 0$.[14]

The function $\varphi(a^i)$, which reflects the individual's utility obtained from contributing to the health of his parent, is supposed to be increasing and concave (i.e. $\varphi'(\cdot) > 0$ and $\varphi''(\cdot) < 0$). The amount of LTC given to the dependent parent is denoted by a^i, which is equal to a_{t+1}^E for type E agents and to a_t^L for type L agents. We suppose here that the caregiver only values his *own* contribution to the health of his dependent parent through the function $\varphi(a^i)$, independently from the LTC provided by other children.[15]

The parameter $\gamma \in [0, 1]$ captures the extent to which young individuals take into account the impact of their fertility choices on their future health at the old age $H(\cdot)$. Full myopia occurs when $\gamma = 0$, whereas $\gamma = 1$ implies no myopia.

When making their choices, individuals do not know the amount of LTC that they will receive, at the old age, from their children, but form anticipations regarding those amounts. The amount of LTC is thus expressed in expected terms, i.e. b_{t+2}^{ie}, and is the sum of the expected amount of LTC from all children: $b_{t+2}^{ie} = n_t^i a_{t+2}^{Ee} + (1 - n_t^i) a_{t+2}^{Le}$. Hence, when characterizing the temporal equilibrium, we will condition this on given expectations a_{t+2}^{Ee} and a_{t+2}^{Le}.

2.3 Budget constraints

When an agent is of type E, his parent becomes dependent when he is in the third period of his life. Hence, he provides LTC to his dependent parent when being an old adult, that is, we have $a_t^E = 0$ and $a_{t+1}^E > 0$. Hence, assuming that a child costs a fraction σ in terms of time, the resource constraints are:

$$w_t\left(1 - \sigma n_t^E\right) = c_t^E + s_t^E \tag{4}$$

$$w_{t+1}^e z(1 - \sigma(1 - n_t^E) - a_{t+1}^E) + R_{t+1}^e s_t^E = d_{t+1}^E \tag{5}$$

[13] We abstract here from pure time preferences, which may potentially affect the timing of births. We make that simplification to better focus on how purely economic factors, i.e. budget constraints and time constraints, affect the timing of births.

[14] Assuming that $v(\cdot)$ is concave amounts to suppose that parents like, to some extent, to smooth the number of births along their fertile period. Without the concavity of $v(\cdot)$, fertility choices would quite often be corner solutions, where all births take place at the same moment in life. Given that we are interested in differences in LTC provision according to the ranking of births, it makes sense to consider interior solutions for birth timing choices.

[15] That assumption differs from the one in van den Berg *et al* (2005), where the caregiver values the health of the dependent. Under that alternative assumption, the health of the dependent parent would be a public good, which would generate strategic interdependencies among children. Our assumption allows us to abstract from those interdependencies, since our main focus is on birth timing choices rather than on coordination failures among children (which is a general problem not specially related to birth timing).

where s_t^E is savings, w_t is the hourly wage earned at time t, w_{t+1}^e is the expected wage rate at time $t+1$, while R_{t+1}^e is equal to one plus the expected interest rate prevailing at time $t+1$. The associated intertemporal budget constraint is:

$$w_t\left(1-\sigma n_t^E\right)+\frac{w_{t+1}^e z(1-\sigma(1-n_t^E)-a_{t+1}^E)}{R_{t+1}^e}=c_t^E+\frac{d_{t+1}^E}{R_{t+1}^e} \tag{6}$$

When an agent is of type L, his parent becomes dependent when he is in the second period of his life. Hence, he provides LTC to his parent when being a young adult, that is, we have $a_t^L>0$ and $a_{t+1}^L=0$. Hence, we have:

$$w_t(1-\sigma n_t^L-a_t^L) = c_t^L+s_t^L \tag{7}$$

$$w_{t+1}^e z\left(1-\sigma(1-n_t^L)\right)+R_{t+1}^e s_t^L = d_{t+1}^L \tag{8}$$

The unique difference between the two sets of budget constraints lies in the moment of life at which children provide LTC to their elderly parents.[16] The associated intertemporal budget constraint is:

$$w_t(1-\sigma n_t^L-a_t^L)+\frac{w_{t+1}^e z\left(1-\sigma(1-n_t^L)\right)}{R_{t+1}^e}=c_t^L+\frac{d_{t+1}^L}{R_{t+1}^e} \tag{9}$$

From those constraints, it can be seen that the two types of children do not face the same opportunity cost of providing LTC, since the age at which they have to help their parents is not the same: early children provide LTC later on in their life, unlike late children, who provide LTC earlier in their life. Hence, although all agents have the same preferences, they will not necessarily behave in the same way, since agents face different time and budget constraints depending on whether they are early children or late children.

2.4 Production

The production process involves capital K_t and labour L_t, and exhibits constant returns to scale:

$$Y_t=F(K_t,L_t) \tag{10}$$

The total labor force is:

$$\begin{aligned} L_t &= q_t\left(1-\sigma n_t^E\right)+(1-q_t)\left(1-\sigma n_t^L-a_t^L\right)\\ &\quad +q_{t-1}z(1-a_t^E-\sigma(1-n_{t-1}^E))+(1-q_{t-1})z\left(1-\sigma(1-n_{t-1}^L)\right) \end{aligned} \tag{11}$$

For simplicity, it is assumed that there is full depreciation of capital after one period of use. Hence capital accumulates according to the law:

$$K_{t+1}=q_t s_t^E+(1-q_t)s_t^L \tag{12}$$

As usual, factors are paid at their marginal productivity:

$$w_t = F_L\left(K_t,L_t\right) \tag{13}$$

$$R_t = F_K\left(K_t,L_t\right) \tag{14}$$

[16] In the hypothetical case where there was no LTC provision (i.e. $a_{t+1}^E=a_t^L=0$), the two sets of budget constraints would be the same across types E and L.

3 The laissez-faire

Let us now consider how individuals of the two types will make their decisions in terms of consumption, fertility and LTC provision. For that purpose, we will first consider the temporary equilibrium, which is a situation where each agent maximizes his utility, conditionally on his beliefs regarding future production factor prices (i.e. w^e_{t+1} and R^e_{t+1}) and regarding the future LTC received from the children born early and late (i.e. a^{Ee}_{t+2} and a^{Le}_{t+2}). For the sake of simplification, we will assume, in the rest of this section, that all agents, whatever these are of type E or L, formulate exactly the same beliefs regarding future factor prices and future LTC received at the old age.

3.1 Temporary equilibrium

Conditionally on their anticipations, individuals of type E solve the problem:

$$\begin{array}{ll} \max_{c^E_t, d^E_{t+1} n^E_t, a^E_{t+1}} & u(c^E_t) + v\left(n^E_t\right) + u(d^E_{t+1}) + v\left(1 - n^E_t\right) + \varphi(a^E_{t+1}) + \gamma H(b^{Ee}_{t+2}) \\ & \text{s.t. } b^{Ee}_{t+2} = n^E_t a^{Ee}_{t+2} + (1 - n^E_t) a^{Le}_{t+2} \\ & \text{s.t. } w_t\left(1 - \sigma n^E_t\right) + \dfrac{w^e_{t+1} z(1 - \sigma(1 - n^E_t) - a^E_{t+1})}{R^e_{t+1}} = c^E_t + \dfrac{d^E_{t+1}}{R^e_{t+1}} \end{array}$$

First-order conditions (FOCs) yield:

$$u'(c^E_t) = R^e_{t+1} u'(d^E_{t+1}) \tag{15}$$

$$\left[\begin{array}{c} v'\left(n^E_t\right) - v'\left(1 - n^E_t\right) \\ +\gamma H'(b^{Ee}_{t+2})\left(a^{Ee}_{t+2} - a^{Le}_{t+2}\right) \end{array}\right] = u'(c^E_t)\sigma\left(w_t - \frac{w^e_{t+1} z}{R^e_{t+1}}\right) \tag{16}$$

$$\varphi'(a^E_{t+1}) = u'(c^E_t)\frac{w^e_{t+1} z}{R^e_{t+1}} \tag{17}$$

The first condition is the standard Euler equation, relating present and future consumption. The second condition characterizes the optimal birth timing. It says that the birth timing is optimally chosen when the marginal welfare gain from increasing the number of early births (LHS of the condition) is equal to the marginal welfare loss from increasing early births (RHS of the condition). Note that the LHS of that condition depends on the difference in terms of the expected LTC received at the old age, depending on the type of child (third term of the LHS). When parents anticipate equal help from early and late children, that term vanishes. If, on the contrary, parents expect to receive more help from their early children, this increases the marginal welfare gain from having early children *ceteris paribus*. The last condition characterizes the optimal level of LTC given to the parent. The marginal welfare loss of helping their elderly parent (RHS) is increasing with the retirement age z. Later retirement raises the opportunity cost of providing LTC.

Individuals of type L solve the following problem:

$$\max_{c_t^L, d_{t+1}^L n_t^L, a_t^L} \quad u(c_t^L) + v\left(n_t^L\right) + u(d_{t+1}^L) + v\left(1 - n_t^L\right) + \varphi(a_t^L) + \gamma H(b_{t+2}^{Le})$$
$$\text{s.t. } b_{t+2}^{Le} = n_t^L a_{t+2}^{Ee} + (1 - n_t^L) a_{t+2}^{Le}$$
$$\text{s.t. } w_t(1 - \sigma n_t^L - a_t^L) + \frac{w_{t+1}^e z\left(1 - \sigma(1 - n_t^L)\right)}{R_{t+1}^e} = c_t^L + \frac{d_{t+1}^L}{R_{t+1}^e}$$

FOCs yield:

$$u'(c_t^L) = R_{t+1}^e u'(d_{t+1}^L) \tag{18}$$

$$\left[\begin{array}{l} v'\left(n_t^L\right) - v'\left(1 - n_t^L\right) \\ +\gamma H'(b_{t+2}^{Le})\left(a_{t+2}^{Ee} - a_{t+2}^{Le}\right) \end{array}\right] = u'(c_t^L)\sigma\left(w_t - \frac{w_{t+1}^e z}{R_{t+1}^e}\right) \tag{19}$$

$$\varphi'(a_t^L) = u'(c_t^L) w_t \tag{20}$$

Comparing those conditions for the two types of agents reveals several things. Firstly, the condition for optimal fertility timing is the same across agents. Hence, given that those agents face the same market prices, the unique difference in birth timing comes from differences in the levels of consumption profiles. Individuals with larger consumption at the young age exhibit a lower marginal utility of consumption, which reduces the marginal welfare loss associated with early births (RHS), and encourages advancing births. Secondly, the RHS of the condition for optimal LTC is also different from its counterpart for type-E agents. If $\frac{w_{t+1}^e z}{R_{t+1}^e} < w_t$, late children face, *ceteris paribus*, a higher marginal welfare loss from providing LTC. As shown below, the lower opportunity cost of providing LTC explains why early children provide, at the laissez-faire, more LTC than late children.

Proposition 1 *Given the anticipated future prices w_{t+1}^e and R_{t+1}^e, the anticipated future levels of LTC received a_{t+2}^{Ee} and a_{t+2}^{Le}, the capital stock K_t and the partitions q_{t-1} and q_t, the temporary equilibrium is a vector $\left\{c_t^E, d_{t+1}^E, n_t^E, a_{t+1}^E, c_t^L, d_{t+1}^L, n_t^L, a_t^L, w_t, L_t\right\}$ satisfying the conditions:*

$$\begin{aligned}
u'(c_t^E) &= R_{t+1}^e u'(d_{t+1}^E) \text{ and } u'(c_t^L) = R_{t+1}^e u'(d_{t+1}^L) \\
u'(c_t^E)\sigma\left(w_t - \frac{w_{t+1}^e z}{R_{t+1}^e}\right) &= v'\left(n_t^E\right) - v'\left(1 - n_t^E\right) + \gamma H'(b_{t+2}^{Ee})\left(a_{t+2}^{Ee} - a_{t+2}^{Le}\right) \\
u'(c_t^L)\sigma\left(w_t - \frac{w_{t+1}^e z}{R_{t+1}^e}\right) &= v'\left(n_t^L\right) - v'\left(1 - n_t^L\right) + \gamma H'(b_{t+2}^{Le})\left(a_{t+2}^{Ee} - a_{t+2}^{Le}\right) \\
\varphi'(a_{t+1}^E) &= u'(c_t^E)\frac{w_{t+1}^e z}{R_{t+1}^e} \text{ and } \varphi'(a_t^L) = u'(c_t^L) w_t \\
w_t &= F_L\left(K_t, L_t\right) \\
L_t &= \left[\begin{array}{l} q_t\left(1 - \sigma n_t^E\right) + (1 - q_t)\left(1 - \sigma n_t^L - a_t^L\right) \\ +q_{t-1} z(1 - a_t^E - \sigma(1 - n_{t-1}^E)) \\ +(1 - q_{t-1}) z\left(1 - \sigma(1 - n_{t-1}^L)\right) \end{array}\right]
\end{aligned}$$

Under the assumption $\frac{w^e_{t+1}z}{R^e_{t+1}} < w_t$, individuals of type E provide, in comparison with type-L individuals, a larger amount of LTC to their elderly parents, they have more early children and consume also more than type-L individuals:

$$\begin{aligned} a^E_{t+1} &> a^L_t \text{ and } n^E_t > n^L_t \\ c^E_t &> c^L_t \text{ and } d^E_{t+1} > d^L_{t+1} \end{aligned}$$

Proof. See the Appendix. ■

Proposition 1 states that, provided $\frac{w^e_{t+1}z}{R^e_{t+1}} < w_t$, individuals born from young parents provide, at the laissez-faire, more LTC to the dependent parents, in comparison with individuals born from older parents. That condition is obtained by merely comparing the opportunity costs of providing LTC to an elderly parent for type-E and type-L agents. Providing LTC has an opportunity cost, since the time dedicated to helping the dependent parent cannot be worked. This is true for both types of agents. However, when $\frac{w^e_{t+1}z}{R^e_{t+1}} < w_t$, the opportunity cost of providing LTC is larger for type-L agents than for type-E agents. This implies that, although all individuals have the same preferences, it is the case that late children provide less LTC in comparison to early children.

The condition $\frac{w^e_{t+1}z}{R^e_{t+1}} < w_t$ is compatible with basic assumptions on the formation of expectations. If, for instance, one assumes that individuals form myopic anticipations on future factor prices, we have $w^e_{t+1} = w_t$ and $R^e_{t+1} = R_t$, so that the condition becomes $R_t > z$. Given that $z < 1$, it is sufficient that $R_t > 1$ to insure that the condition is satisfied. This amounts to assume that the capital stock lies on the left side of the Golden Rule capital level, that is, that the economy is in underaccumulation of capital, which is a standard assumption in the literature. Thus, under myopic anticipations, the condition $\frac{w^e_{t+1}z}{R^e_{t+1}} < w_t$ is satisfied, which implies that early children provide more LTC than late children.

That theoretical result is consistent with the empirical literature on informal LTC provision, which shows that the amount of LTC provided by older children exceeds the amount of LTC provided by younger children when younger children are still working full time whereas older children are not (Fontaine et al 2007).

The size of the inequality in LTC provision depends on the retirement age z. A lower z raises inequalities in informal LTC provided between early and late children. Indeed, in that case, the opportunity cost of providing LTC to the dependent parent is lower for early children in comparison with late children. However, note that, even if $z = 1$, we still have, under $\frac{w^e_{t+1}}{R^e_{t+1}} < w_t$, that early children provide more LTC than late children. Thus the level of z only affects the size of the gap in terms of LTC, without affecting the main prediction of the model, which is that early children provide more LTC than late children.

Finally, it should be stressed that the differences in the behavior of the two types of agents concern not only the LTC provision, but, also the fertility timing. As stated in Proposition 1, early children tend to have a larger proportion of children born early in life, whereas late children tend to have a larger proportion of children born later on in life. Those results come merely from the different time constraints faced by the two types of agents. Type-E children must provide LTC to their elderly parents when they are mature, and this leaves them plenty of time to have more children when they are younger. On the contrary, type-L children must provide LTC to their parents when they are young adults, and this encourages them to postpone births later on in their life.[17]

3.2 Intertemporal equilibrium

Up to now, we focused on the temporary equilibrium of the economy. At the temporary equilibrium, whether early children provide more LTC than late children depends on the levels of wages and interest rates. Given that those factor prices are likely to evolve over time, it makes sense to extend our study, to examine the intertemporal equilibrium of the economy. This section characterizes the intertemporal equilibrium with perfect foresight.

The intertemporal equilibrium with perfect foresight is a sequence of temporary equilibria for a given initial capital stock K_0 and a given partition q_0, sequence satisfying the following capital accumulation equation:

$$K_{t+1} = q_t s_t^E + (1 - q_t) s_t^L \tag{21}$$

The proportion of type E agents in each cohort follows the law:

$$q_t = \frac{q_{t-1} n_{t-1}^E + (1 - q_{t-1}) n_{t-1}^L}{\left(q_{t-1} n_{t-1}^E + (1 - q_{t-1}) n_{t-1}^L\right) + q_{t-2}(1 - n_{t-2}^E) + (1 - q_{t-2})(1 - n_{t-2}^L)} \tag{22}$$

Under perfect foresight, we have:

$$\begin{aligned} w_{t+1}^e &= w_{t+1} \text{ and } R_{t+1}^e = R_{t+1} \\ a_{t+2}^{Ee} &= a_{t+2}^E \text{ and } a_{t+2}^{Le} = a_{t+2}^L \end{aligned}$$

At intertemporal equilibrium, the labor market clears:

$$L_t = \left[\begin{array}{l} q_t \left(1 - \sigma n_t^E\right) + (1 - q_t)\left(1 - \sigma n_t^L - a_t^L\right) \\ + q_{t-1} z (1 - a_t^E - \sigma(1 - n_{t-1}^E)) + (1 - q_{t-1}) z \left(1 - \sigma(1 - n_{t-1}^L)\right) \end{array} \right]$$

as well as the market for goods:

$$F\left(K_t, L_t\right) = q_t c_t^E + (1 - q_t) c_t^L + q_{t-1} d_t^E + (1 - q_{t-1}) d_t^L + K_{t+1}$$

At the intertemporal equilibrium, the FOCs relative to individuals'utility

[17] The social reproduction process at work in our model (i.e. early children tend to have more children early, whereas late children tend to have more children later on in their life) is dictated by rationality. Parents, by choosing the age at which they have their children, determine the age gap between them and their descendants, which will constraint these in all their choices (through the time constraints), including fertility choices. Thus our model of rational choices can mimic social reproduction behaviors.

maximization problems are also satisfied:

$$
\begin{aligned}
u'(c_t^E) &= R_{t+1}u'(d_{t+1}^E) \text{ and } u'(c_t^L) = R_{t+1}u'(d_{t+1}^L) \\
u'(c_t^E)\sigma\left(w_t - \frac{w_{t+1}z}{R_{t+1}}\right) &= v'\left(n_t^E\right) - v'\left(1-n_t^E\right) + \gamma H'(b_{t+2}^E)\left(a_{t+2}^E - a_{t+2}^L\right) \\
u'(c_t^L)\sigma\left(w_t - \frac{w_{t+1}z}{R_{t+1}}\right) &= v'\left(n_t^L\right) - v'\left(1-n_t^L\right) + \gamma H'(b_{t+2}^L)\left(a_{t+2}^E - a_{t+2}^L\right) \\
\varphi'(a_{t+1}^E) &= u'(c_t^E)\frac{w_{t+1}z}{R_{t+1}} \text{ and } \varphi'(a_t^L) = u'(c_t^L)w_t
\end{aligned}
$$

Similarly, firms maximize their profits, so that:

$$
w_t = F_L\left(K_t, \cdot\right) \text{ and } R_t = F_K\left(K_t, \cdot\right)
$$

3.3 Stationary equilibrium

Let us now consider the stationary equilibrium. At the stationary equilibrium, the economy perfectly reproduces itself over time. The capital stock is constant, the composition of the population in the two types is constant, consumptions are constant over time, and informal care to the dependent parents is also constant over time, meaning that the amount of LTC that a child of a given type gives to his dependent parent is exactly equal to the LTC that this person will receive from a child of the same type as himself at the old age.

At steady-state, the following conditions are satisfied (time indexes are dropped):

$$
\begin{aligned}
K &= qs^E + (1-q)s^L \\
L &= \left[\begin{array}{l} q\left(n^L\left(\sigma - z\sigma\right) - n^E\left(\sigma - z\sigma\right) + a^L - za^E\right) \\ +1 - \sigma n^L - a^L + z - \sigma z + \sigma z n^L \end{array}\right]
\end{aligned}
$$

The resource constraint of the economy is thus:

$$
F\left(K, L\right) = q\left(c^E + d^E\right) + (1-q)\left(c^L + d^L\right) + K
$$

Factors are paid at their marginal productivity:

$$
w = F_L\left(K, L\right) \text{ and } R = F_K\left(K, L\right)
$$

All agents maximize their utility subject to their budget constraints:

$$
\begin{aligned}
u'(c^E) &= Ru'(d^E) \text{ and } u'(c^L) = Ru'(d^L) \\
u'(c^E)\sigma w\left(1 - \frac{z}{R}\right) &= v'\left(n^E\right) - v'\left(1-n^E\right) + \gamma H'(b^E)\left(a^E - a^L\right) \\
u'(c^L)\sigma w\left(1 - \frac{z}{R}\right) &= v'\left(n^L\right) - v'\left(1-n^L\right) + \gamma H'(b^L)\left(a^E - a^L\right) \\
\varphi'(a^E) &= u'(c^E)\frac{wz}{R} \text{ and } \varphi'(a^L) = u'(c^L)w
\end{aligned}
$$

Regarding the long-run composition of the population, imposing $q_t = q_{t-1} = q_{t-2}$ in the condition: $q_t = \frac{q_{t-1}n^E+(1-q_{t-1})n^L}{q_{t-1}n^E+(1-q_{t-1})n^L+q_{t-2}(1-n^E)+(1-q_{t-2})(1-n^L)}$ yields:

$$q = \frac{n^L}{1-n^E+n^L}$$

Proposition 2 summarizes key features of the stationary equilibrium.

Proposition 2 *The stationary equilibrium is a vector* $\left\{c^E, d^E, n^E, a^E, b^E, c^L, d^L, n^L, a^L, b^L, K, L, w, R, q\right\}$ *satisfying the conditions:*

$$\begin{aligned}
u'(c^E) &= Ru'(d^E) \text{ and } u'(c^L) = Ru'(d^L) \\
u'(c^E)w\sigma\left[1-\frac{z}{R}\right] &= v'\left(n^E\right) - v'\left(1-n^E\right) + \gamma H'(n^E a^E + (1-n^E)a^L)\left(a^E - a^L\right) \\
u'(c^L)w\sigma\left[1-\frac{z}{R}\right] &= v'\left(n^L\right) - v'\left(1-n^L\right) + \gamma H'(n^L a^E + (1-n^L)a^L)\left(a^E - a^L\right) \\
\varphi'(a^E) &= u'(c^E)\frac{wz}{R} \text{ and } \varphi'(a^L) = u'(c^L)w \\
K &= \left[\begin{array}{l} q\left(w\left(1-\sigma n^E\right) - c^E\right) \\ +(1-q)\left(w\left(1-\sigma n^L - a^L\right) - c^L\right) \end{array}\right] \\
L &= \left[\begin{array}{l} q\left(n^L\sigma\left(1-z\right) - n^E\sigma\left(1-z\right) + a^L - za^E\right) \\ +1 - \sigma n^L - a^L + z - \sigma z + \sigma z n^L \end{array}\right] \\
w &= F_L\left(K, L\right) \text{ and } R = F_K\left(K, L\right) \\
q &= \frac{n^L}{1-n^E+n^L}
\end{aligned}$$

At the stationary equilibrium, and assuming $R > z$, type-E agents provide more LTC to their parents, in comparison with type-L agents. They also have more early children, consume more and benefit from more LTC at the old age:

$$\begin{aligned}
a^E &> a^L \text{ and } n^E > n^L \\
c^E &> c^L \text{ and } d^E > d^L \\
b^E &> b^L
\end{aligned}$$

Proof. See the Appendix. ■

Proposition 2 states that the tendency of early children to provide more LTC to their elderly parent in comparison to late children is not a short-run phenomenon. Actually, in the long-run, we find, under the condition $R > z$, the same inequality $a^E > a^L$. Given $z < 1$, the condition $R > z$ is quite weak, and holds when the economy is in underaccumulation of capital (i.e. $R > 1$).

An interesting interpretation of the results prevailing at the stationary equilibrium concerns the comparison of the LTC provided by two children of the *same* parent, i.e. a_t^E and a_t^L. Since consumptions per period for each type E and L are constant over time at the stationary equilibrium, it is possible to show, from the FOCs for LTC provision, that, under $R > z$, early children (i.e. who are older) provide more LTC than late children (i.e. who are younger), despite the fact that all those children have exactly the same preferences. The differential in LTC provision comes only from differences in the opportunity cost of providing LTC, which is larger for late children.

Note also that, even if we considered the case where individuals work fully in their third period (i.e. $z = 1$), the condition stated in Proposition 2 would still hold, provided $R > 1$, which is a weak assumption. Hence, the fact that early children provide more LTC than late children is robust to the postulated retirement age. However, it remains true that the level of z affects the size of the gap between the amounts of LTC provided by the different types of children.

Finally, it should be stressed that, at the stationary equilibrium, it is not only true that early children provide more LTC to their dependent parents than late children; it is also true that type-E agents, once old and dependent, *receive* more LTC from their own children, in comparison with type-L agents (i.e. $b^E > b^L$). The intuition behind that result is that type-E agents have more early births, which, given that early children provide more LTC, leads them to receive themselves a higher amount of LTC at the old age.

3.4 Existence of a stationary equilibrium

Proposition 2 characterizes a stationary equilibrium in our economy, without exploring the issue of the existence of such an equilibrium. Given that we will, throughout the rest of this paper, assume that a stationary equilibrium exists, it is important to show that assuming the existence of a stationary equilibrium is not a too strong assumption. This is the task of the present subsection.

Given the presence of heterogeneity due to birth timing choices, the study of the existence of a stationary equilibrium is not trivial.[18] An additional difficulty lies in the fact that agents'decisions depend on expectations regarding not only future prices, but, also, regarding the amount of LTC provided by children in the future. All this makes the study of the existence of stationary equilibria quite complex. Hence, in order to discuss the existence of a stationary equilibrium, we will have to impose some particular functional forms on the health production process, on preferences and on the production of goods.

For that purpose, we suppose that the health production function $H(\cdot)$ is:

$$H\left(b_t^i\right) = \log\left(b_t^i\right) \tag{23}$$

We assume that preferences are log-linear. The utility of a type-i agent is:

$$(1-\delta)\log(c_t^i) + \delta\log\left(n_t^i\right) + (1-\delta)\log(d_{t+1}^i) + \delta\log\left(1 - n_t^i\right) + \eta\log(a^i) + \gamma\log(b_{t+2}^{ie}) \tag{24}$$

with $\delta \in [0, 1]$ and $\eta > 0$.

We also suppose a Cobb Douglas production function, where $\alpha \in]0, 1[$:

$$Y_t = AK_t^{\alpha}L_t^{1-\alpha} \tag{25}$$

Proposition 3 examines the existence of a stationary equilibrium under those functional forms.

[18] On this, see Pestieau and Ponthiere (2014, 2015).

Proposition 3 *Consider our economy with a log-linear utility function and a Cobb-Douglas production function.*

Suppose $\gamma = 0$ *(myopia of young agents regarding their future health) and* $\sigma = 0$ *(no time cost of children).*

Suppose $2(1-\delta) > \eta$ *and* $2(1-\delta)(1-\alpha)z + z\eta > \alpha\eta$.

Denote $\Theta \equiv 2\eta[1-\delta][[4(1-\delta)+\eta](1+z)\alpha + z(1-\alpha)[2(1-\delta)+\eta]]$ *and* $\Gamma \equiv z[(1-\alpha)[2(1-\delta)+\eta]+\alpha\eta]$.

If:

$$\Gamma\eta^2 + 4[1-\delta]^2\alpha\eta < \Theta < \Gamma 4[1-\delta]^2 + \alpha\eta^3$$

then there exists at least one stationary equilibrium with perfect foresight such that $0 < 1 - a^i < 1 \forall i$.

Proof. See the Appendix. ■

Proposition 3 states a sufficient condition for the existence of a stationary equilibrium with perfect foresight regarding future production factor prices (wages and interest rates), and for which the amount of LTC provided by the two types of children is an interior solution. Note that this condition does not guarantee the uniqueness of that equilibrium, but only its existence.

Another limitation of Proposition 3 is that it focuses on economies where children are not costly and where young parents are fully myopic regarding the impact of their fertility choices on the amount of LTC received at the old age. Note, however, that examining the conditions for the existence of a stationary equilibrium while relaxing those assumptions would require an entire paper on its own. We prefer thus to end the positive study on the existence of a stationary equilibrium here, and to shift to our analysis of optimal policy.

4 Long-run social optimum

In order to characterize the long-run social optimum, we consider the standard utilitarian social welfare function. The goal of the social planner is here to select consumption, fertility and informal LTC variables, as well as some stock of capital, to maximize the lifetime welfare prevailing at the stationary equilibrium.[19]

The social planning problem of the utilitarian planner differs from the problems faced by agents in several important dimensions. First, the social planner considers the maximization of social welfare in the long run (i.e. at the stationary equilibrium), and considers thus a time horizon quite different from the ones faced by individuals. Second, the social planner does not suffer from myopia,

[19] Throughout this section, we assume, for the sake of simplicity, that the retirement age z is a exogenous parameter that the social planner takes as given. The reason why we keep z as exogenous here is twofold. First, allowing the government to choose z would bring a strong asymmetry between the choice variables of individuals and of the social planner. This would make the policy discussion more oriented towards the choice of the optimal retirement age, whereas the main topic of this paper is not the retirement decision, but the timing of births. Second, assuming a fixed retirement age simplifies the picture, but that assumption is not so strong, since, in real economies, the retirement age is quite rigid, and can hardly be adjusted by governments.

so that γ is set to 1 in the social planning problem. Besides those classical differences, another, more original difference concerns the fact that the social planner takes into account the impact of fertility choices on the composition of the population in the long-run. In other words, the social planner internalizes compositional externalities related to individual fertility choices.

4.1 The social planner's problem

The social planner's problem can be written by means of the Lagrangian:

$$\max_{\substack{c^E,d^E,a^E,n^E\\ c^L,d^L,a^L,n^L,K}} \begin{array}{l} \frac{n^L}{1-n^E+n^L}\left[\begin{array}{c} u(c^E)+v(n^E)+u(d^E)+v(1-n^E) \\ +H\left(n^E a^E+(1-n^E)a^L\right)+\varphi(a^E) \end{array}\right] \\ +\frac{1-n^E}{1-n^E+n^L}\left[\begin{array}{c} u(c^L)+v(n^L)+u(d^L)+v(1-n^L) \\ +H\left(n^L a^E+(1-n^L)a^L\right)+\varphi(a^L) \end{array}\right] \\ +\lambda\left[\begin{array}{c} F\left(K,\left[\begin{array}{c} \frac{n^L\left[n^L(\sigma-z\sigma)-n^E(\sigma-z\sigma)+a^L-za^E\right]}{1-n^E+n^L} \\ +1-\sigma n^L-a^L+z-\sigma z+\sigma z n^L \end{array}\right]\right) \\ -\frac{n^L}{1-n^E+n^L}\left(c^E+d^E\right)-\frac{1-n^E}{1-n^E+n^L}\left(c^L+d^L\right)-K \end{array}\right] \end{array}$$

where λ is the Lagrange multiplier associated to the resource constraint.

FOCs for consumptions are: :

$$u'(c^E)=u'(c^L)=u'(d^E)=u'(d^L)=\lambda \tag{26}$$

The social optimum involves thus an equalization of consumption levels across time periods and across types of individuals.

The FOC for K is:

$$F_K\left(K,\left[\begin{array}{c} \frac{n^L\left[n^L(\sigma-z\sigma)-n^E(\sigma-z\sigma)+a^L-za^E\right]}{1-n^E+n^L} \\ +1-\sigma n^L-a^L+z-\sigma z+\sigma z n^L \end{array}\right]\right)=1 \tag{27}$$

This condition is the standard Golden Rule of capital accumulation (Phelps 1961), in the case where there is no cohort growth and where there is full depreciation of capital.

The FOCs for a^E and a^L are:

$$\begin{aligned} & \frac{n^L}{1-n^E+n^L}\left[H'(\cdot)n^E+\varphi'(a^E)\right]+\frac{1-n^E}{1-n^E+n^L}H'(\cdot)\,n^L \\ = \; & \lambda F_L(K,\cdot)\left[\frac{zn^L}{1-n^E+n^L}\right] \end{aligned} \tag{28}$$

$$\begin{aligned} & \frac{1-n^E}{1-n^E+n^L}\left[H'(\cdot)(1-n^L)+\varphi'(a^L)\right]+\frac{1-n^E}{1-n^E+n^L}H'(\cdot)\,n^L \\ = \; & \lambda F_L(K,)\left[\frac{1-n^E}{1-n^E+n^L}\right] \end{aligned} \tag{29}$$

Further simplifications yield:

$$H'(\cdot)n^E+\varphi'(a^E)+H'(\cdot)\left(1-n^E\right) = u'(c)F_L(K,\cdot)\,z \tag{30}$$

$$H'(\cdot)\,n^L+\varphi'(a^L)+H'(\cdot)(1-n^L) = u'(c)F_L(K,\cdot) \tag{31}$$

The LHS is the marginal social welfare gain from providing LTC to the elderly parents, whereas the RHS is the marginal social welfare loss from providing LTC. Obviously, that loss depends on how large the marginal productivity of labour is. Moreover, the marginal social welfare loss from providing LTC is larger for individuals of type L than for individuals of type E.

Denoting

$$\Delta \equiv \left[\begin{array}{c} u(c^E) + v(n^E) + u(d^E) + v(1-n^E) + H\left(n^E a^E + (1-n^E)a^L\right) + \varphi(a^E) \\ -\left[u(c^L) + v(n^L) + u(d^L) + v(1-n^L) + H\left(n^L a^E + (1-n^L)a^L\right) + \varphi(a^L)\right] \end{array} \right]$$

the FOCs for n^E can be written as:

$$\begin{aligned} & v'(n^E) - v'(1-n^E) + H'(\cdot)\left(a^E - a^L\right) \\ = \quad & -u'(c)F_L(K,\cdot)\left[\frac{\left(a^L - za^E\right) - (\sigma - z\sigma)}{(1-n^E+n^L)}\right] - \frac{\Delta}{(1-n^E+n^L)} \end{aligned} \tag{32}$$

while the FOC for n^L is:

$$\begin{aligned} & v'(n^L) - v'(1-n^L) + H'(\cdot)\left(a^E - a^L\right) \\ = \quad & -u'(c)F_L(K,\cdot)\left[\frac{\left(a^L - za^E\right) - (\sigma - \sigma z)}{(1-n^E+n^L)}\right] - \frac{\Delta}{(1-n^E+n^L)} \end{aligned} \tag{33}$$

Note that the two conditions have the same RHS. Hence we have:

$$\begin{aligned} & v'(n^E) - v'(1-n^E) + H'\left(n^E a^E + (1-n^E)a^L\right)\left(a^E - a^L\right) \\ = \quad & v'(n^L) - v'(1-n^L) + H'\left(n^L a^E + (1-n^L)a^L\right)\left(a^E - a^L\right) \end{aligned}$$

Denote the function present in the LHS and RHS by $\psi(x) \equiv v'(x) - v'(1-x) + H'\left(xa^E + (1-x)a^L\right)\left(a^E - a^L\right)$. We have $\psi'(x) < 0$ on the interval $[0,1]$ for $a^E \lesseqgtr a^L$. Given that the function $\psi(x)$ is strictly monotone, and that it appears on both sides of the above condition, it must be the case that its argument must take the same value on both sides. From which it follows that, at the social optimum, we have equal fertility across types: $n^E = n^L = n$. Hence, the optimal long-run proportion of early children is:

$$q = \frac{n}{1-n+n} = n \tag{34}$$

In the light of this, the FOC for n becomes:

$$\begin{aligned} & v'(n) - v'(1-n) + H'\left(na^E + (1-n)a^L\right)\left(a^E - a^L\right) \\ = \quad & u'(c)F_L(K,\cdot)\left[\sigma(1-z) - \left(a^L - za^E\right)\right] - \left[\varphi(a^E) - \varphi(a^L)\right] \end{aligned} \tag{35}$$

The LHS of that condition is similar to the LHS of the condition for fertility at the laissez-faire. However, the RHS differs first by the fact that $R = 1$ at the social optimum, and by the addition of two terms: $-u'(c)F_L(K,\cdot)\left[a^L - za^E\right]$ and $-\left[\varphi(a^E) - \varphi(a^L)\right]$.

Therefore, the FOCs for informal LTC become:

$$\varphi'(a^E) = u'(c)F_L(K,\cdot)\,z - H'\left(na^E + (1-n)a^L\right) \tag{36}$$
$$\varphi'(a^L) = u'(c)F_L(K,\cdot) - H'\left(na^E + (1-n)a^L\right) \tag{37}$$

The last term of RHS is the same. First term identical up to z. Under $z < 1$, we have $\varphi'(a^E) < \varphi'(a^L)$, implying $a^E > a^L$. Thus, it is not only the case that early children provide, at the laissez-faire, more LTC to their parents in comparison to late children; it is also the case, under a utilitarian social welfare function, that early children *should*, at the social optimum, provide more LTC than late children.[20] The following proposition summarizes our results.

Proposition 4 *The long-run social optimum is a vector* $\left\{c^{E*}, c^{L*}, d^{E*}, d^{L*}, a^{E*}, a^{L*}, b^{E*}, b^{L*}, n^{E*}, n^{L*}, K^*, L^*, q^*\right\}$ *such that:*

$$\begin{aligned}
c^{E*} &= c^{L*} = d^{E*} = d^{L*} = c^* \\
n^{E*} &= n^{L*} = n^* \\
b^{E*} &= b^{L*} = b^* = n^*a^{E*} + (1-n^*)a^{L*} \\
v'(n^*) - v'(1-n^*) + H'(b^*)\left(a^{E*} - a^{L*}\right) &= \left[\begin{array}{l} u'(c^*)F_L(K^*,\cdot)\left[\sigma(1-z) - \left(a^{L*} - za^{E*}\right)\right] \\ -\left[\varphi(a^{E*}) - \varphi(a^{L*})\right] \end{array}\right] \\
F_K(K^*,\cdot) &= 1 \\
\varphi'(a^{E*}) &= u'(c^*)F_L(K^*,\cdot)\,z - H'(b^*) \\
\varphi'(a^{L*}) &= u'(c^*)F_L(K^*,\cdot) - H'(b^*) \\
\varphi'(a^{L*}) - \varphi'(a^{E*}) &= u'(c^*)F_L(K^*,\cdot)(1-z) > 0 \implies a^{E*} > a^{L*} \textit{ if } z < 1 \\
L^* &= q^*\left(a^{L*} - za^{E*}\right) + 1 - \sigma n^* - a^{L*} + z - \sigma z + \sigma z n^* \\
q^* &= n^*
\end{aligned}$$

Proof. See above. ■

At the utilitarian social optimum, early and late children have the same consumption patterns and fertility patterns. All types of individuals receive also, at the old age, the same amount of LTC, unlike what prevails at the laissez-faire. The unique difference between type-E and type-L agents is that early children should, at the social optimum, provide a larger LTC than late children. The intuition goes as follows: from the perspective of social welfare, it is better that early children provide more LTC, since the social opportunity cost of helping their elderly parent is lower for them, as long as $z < 1$.[21]

Let us now compare the long-run social optimum with the stationary equilibrium achieved at the laissez-faire. Proposition 5 summarizes our findings.

Proposition 5 *Comparing the laissez-faire (i) under $R > z$ with the social optimum (i$*$), we obtain:*

[20] Note, however, that, in the extreme case where individuals work fully in their third period (i.e. $z = 1$), then it would be socially optimal that both types of children provide the same amount of LTC to their parents.

[21] The gap between a^{E*} and a^{L*} depends on how large z is. When z is low, the LTC gap is large, whereas, when z is close to 1, the LTC gap becomes tiny.

- *$K^i \lesseqgtr K^{i*}$ when $R \gtreqless 1$ prevails at the laissez-faire.*
- *$c^{i*} = d^{i*}$, whereas $c^i \lesseqgtr d^i$ when $R \gtreqless 1$ prevails at the laissez-faire.*
- *If*

$$\begin{array}{rcl} u'(c^E)F_L(K,\cdot)\dfrac{z}{R} & > & u'(c^*)F_L(K^*,\cdot)\,z - H'(b^*) \\ u'(c^L)F_L(K,\cdot) & > & u'(c^*)F_L(K^*,\cdot) - H'(b^*) \end{array}$$

we have:

$$a^{E*} > a^E \text{ and } a^{L*} > a^L$$

- *And if*

$$\left[\begin{array}{l} u'(c^E)F_L(K,\cdot)\,\sigma \\ -\gamma H'\left(b^E\right)\left(a^E - a^L\right) \end{array}\right] > \left[\begin{array}{l} u'(c^*)F_L(K^*,\cdot)\left(\sigma - a^{L*} + za^{E*}\right) \\ -\left[\varphi(a^{E*}) - \varphi(a^{L*})\right] - H'(b^*)\left(a^{E*} - a^{L*} + \sigma\right) \end{array}\right]$$

we have:

$$n^* > n^E > n^L$$

- *We also have, under those conditions,*

$$b^* > b^E > b^L$$

Proof. See the Appendix. ■

The social optimum exhibits several important differences in comparison with the laissez-faire. First, it exhibits a different level of capital, as well as flat consumption profiles, unlike at the laissez-faire. Secondly, an important difference between the laissez-faire and the social optimum concerns the amount of LTC provision. The first two conditions in Proposition 5 state that, for respectively individuals of types E and L, the marginal welfare loss from providing LTC at the laissez-faire (LHS) exceeds the marginal social welfare loss from providing LTC (RHS). Those conditions are likely to be satisfied, since individuals, when deciding LTC provision at the laissez-faire, do not take into account the impact of LTC provision on the health of the dependent parent (unlike the social planner, who takes that impact into account). This is likely to lead to an under-provision of LTC at the laissez-faire for both types of agents. Thus, under weak conditions, individuals of both types tend to provide too little LTC to their elderly parents, in comparison with the social optimum.

Another difference between the laissez-faire and the social optimum concerns the timing of births. The second part of Proposition 5 states a sufficient condition under which individuals of both types have, at the laissez-faire, too few early births and too many late births in comparison with the social optimum.[22] That condition is likely to be satisfied, especially when parents exhibit strong

[22] That sufficient condition compares, for an individual of type E, the marginal welfare loss from having children earlier at the laissez-faire with the marginal social welfare loss from having births earlier.

myopia at the laissez-faire (i.e. a low γ). Indeed, in that case, the LHS of that condition is likely to be positive, whereas all the terms of the RHS - except the first one - are unambiguously negative. Thus, under that condition, the long-run social optimum exhibits a larger proportion of early births in comparison with the laissez-faire. The underlying intuition goes as follows. The marginal social welfare loss associated to the provision of LTC is larger among late children than among early children. But at the laissez-faire, parents do not take this into account when deciding the timing of births. This explains why they have too many late births in comparison with what is socially optimal.

As a consequence of $a^{E*} > a^E$, $a^{L*} > a^L$ and $n^* > n^E > n^L$, all dependent elderly individuals receive, at the long-run social optimum, an amount of LTC that is *equal* for all individuals (unlike at the laissez-faire), and higher than the laissez-faire level. This rise in the amount of LTC received has two origins: first, all children provide more LTC at the social optimum than at the laissez-faire; second, the social optimum involves also, in comparison with the laissez-faire, a higher proportion of early children, who, once adult, provide more LTC.

4.2 Decentralization and optimal policy

Let us now examine how the long-run social optimum can be decentralized. For that purpose, we design policy instruments in such a way that individual decisions at the laissez-faire would, under those policy instruments, lead to the social optimum. The following proposition summarizes our results.

Proposition 6 *The long-run social optimum can be decentralized by means of:*

- *Intergenerational lump-sum transfers allowing the capital stock to reach its optimal level K^*.*
- *Intra-generational lump-sum transfers allowing the equalization of consumption levels across types E and L.*
- *Subsidies on early births θ^E and θ^L equal to:*

$$\begin{aligned}\theta^{i*} &= F_L\left(K^*,\cdot\right)\left[\left(a^{L*}-za^{E*}\right)\right]+\frac{\varphi(a^{E*})-\varphi(a^{L*})}{u'(c^*)}\\ &\quad+\frac{\left(a^{E*}-a^{L*}\right)\left[H'\left(n^*a^{E*}+(1-n^*)a^{L*}\right)-\gamma H'\left(n^ia^{E*}+(1-n^i)a^{L*}\right)\right]}{u'(c^*)}\end{aligned}$$

- *Subsidies on LTC to the elderly parents equal to:*

$$\mu^{E*}=\mu^{L*}=\frac{H'\left(n^*a^{E*}+(1-n^*)a^{L*}\right)}{u'(c^*)}$$

Proof. See the Appendix. ■

The decentralization of the long-run social optimum requires to subsidize early births, in order to induce parents to advance births in a way compatible with long-run social welfare maximization. The decentralization of the long-run social optimum also requires to subsidize LTC. The intuition goes as follows. At the laissez-faire, individuals' provision of LTC is purely driven by their own interests. However, the dependent parents benefit a lot from the LTC provided by their children, thanks to improvements in their health. The social planner takes into account those welfare gains, and subsidize children's LTC to an extent that depends on (non-internalized) welfare gains. Thus the subsidy fully internalizes welfare externalities induced by children's provision of LTC.

5 The second-best problem

In this section, we explore the decentralization of the second-best social optimum, that is, under a limited set of available policy instruments. For that purpose, we will consider an economy at the stationary equilibrium, at which $q = \frac{n^L}{1-n^E+n^L}$ and $1-q = \frac{1-n^E}{1-n^E+n^L}$. Moreover, given that our interest lies on the design of family policy, we will consider here, for the sake of simplicity, that the economy is open, so that wages and interest rates are fixed.[23]

We will consider three policy instruments: a tax on labour earnings τ, a demogrant T and a subsidy on early children θ. The last instrument can be used by the government to reduce the cost of early children. For the sake of analytical convenience, the cost of children is here defined in terms of goods (and not in terms of time, as in the previous sections).

5.1 Laissez faire

Under those policy instruments, parents of types E and L choose savings s^i, informal LTC a^i and fertility n^i and $1-n^i$ conditionally on policy instruments τ, T and θ. In what follows, we assume complete myopia of agents, namely $\gamma = 0$. This makes the analysis simpler, but it does not change the qualitative nature of results compared to a case of partial myopia ($0 < \gamma < 1$).

Type E's problem is:

$$\begin{aligned} \max_{s^E, n^E, a^E} \quad & u\left(w(1-\tau) - s^E - \sigma n^E(1-\theta) + T\right) \\ & + u\left(zw(1-\tau)(1-a^E) + s^E - \sigma(1-n^E)\right) \\ & + v(n^E) + v(1-n^E) + \varphi(a^E) \end{aligned}$$

FOCs are:

$$u'(c^E) = u'(d^E) \tag{38}$$

$$u'(c^E)\sigma(1-\theta) = v'(n^E) - v'(1-n^E) \tag{39}$$

$$u'(d^E)zw(1-\tau) = \varphi'(a^E) \tag{40}$$

The subsidy θ tends to encourage earlier births.

[23] We also suppose, for the sake of simplicity, that $R = 1$.

Type L's problem is:

$$\begin{aligned}\max_{s^L,n^L,a^L} \quad & u\left(w(1-\tau)(1-a^L)-s^L-\sigma n^L(1-\theta)+T\right) \\ & +u\left(zw(1-\tau)+s^L-\sigma(1-n^L)\right) \\ & +v(n^L)+v(1-n^L)+\varphi(a^L)\end{aligned}$$

FOCs are:

$$u'(c^L) = u'(d^L) \tag{41}$$
$$u'(c^L)\sigma(1-\theta) = v'(n^L)-v'(1-n^L) \tag{42}$$
$$u'(c^L)w(1-\tau) = \varphi'(a^L) \tag{43}$$

Those conditions are similar to the ones of agents of type E, except regarding the marginal cost of providing LTC, which is here larger (given $z<1$).

5.2 Second best solution

Under the presence of instruments τ, θ and T, the social planner chooses the optimal levels of savings, fertility and long term care for both types of agents. The demand functions obtained from the laissez-faire are, for $i \in \{E, L\}$:

$$\begin{aligned} s^i &= s^i(\tau,\theta,T) \\ n^i &= n^i(\tau,\theta,T) \\ a^i &= a^i(\tau,\theta,T)\end{aligned}$$

The second-best planning problem can be written as the following Lagrangian:

$$\begin{aligned}\pounds \;=\; & q[u\left(w(1-\tau)-s^E-\sigma n^E(1-\theta)+T\right)+u\left(wz(1-\tau)(1-a^E)+s^E-\sigma(1-n^E)\right) \\ & +v(n^E)+v(1-n^E)+\varphi(a^E)+H(\hat{b}^E)] \\ & +(1-q)[u\left(w(1-\tau)(1-a^L)-s^L-\sigma n^L(1-\theta)+T\right)+u\left(wz(1-\tau)+s^L-\sigma(1-n^L)\right) \\ & +v(n^L)+v(1-n^L)+\varphi(a^L)+H(\hat{b}^L)] \\ & +\mu\left[\tau\left(q\left(w+(1-a^E)zw\right)+(1-q)(w(1-a^L)+zw\right)-\theta\sigma\left(qn^E+(1-q)(n^L)\right)-T\right]\end{aligned}$$

where $\hat{b}^i = a^E n^i + a^L(1-n^i)$ and $q = \frac{n^L}{1+n^L-n^E}$, while μ is the Lagrange multiplier of the government's budget constraint.

Using the envelope theorem and the laissez-faire FOCs, we obtain:

$$\frac{\partial \pounds}{\partial s^i} = 0 \tag{44}$$

$$\frac{\partial \pounds}{\partial a^E} = qH'(\hat{b}^E)n^E + (1-q)\,H'(\hat{b}^L)n^L - \mu\tau qzw \tag{45}$$

$$\frac{\partial \pounds}{\partial a^L} = (1-q)\,H'(\hat{b}^L)\left(1-n^L\right) + qH'(\hat{b}^E)(1-n^E) - \mu\tau(1-q)w \tag{46}$$

$$\frac{\partial \pounds}{\partial n^E} = \left[U^E - U^L + \mu\tau w\left(a^L - za^E\right)\right]\frac{\partial q}{\partial n^E} + qH'(\hat{b}^E)(a^E - a^L) \\ -\mu\theta\sigma\left[q + (n^E - n^L)\frac{\partial q}{\partial n^E}\right] \tag{47}$$

$$\frac{\partial \pounds}{\partial n^L} = \left[U^E - U^L + \mu\tau w\left(a^L + za^E\right)\right]\frac{\partial q}{\partial n^L} + (1-q)\,H'(\hat{b}^L)(a^E - a^L) \\ -\mu\theta\sigma\left[1 - q + (n^E - n^L)\frac{\partial q}{\partial n^L}\right] \tag{48}$$

5.3 Earning tax

Let us introduce the following notations (using the expectation operator E):

$$\begin{aligned} \bar{y} &\equiv w\left(q + (1-q)\left(1 - a^L\right)\right) + wz\left(q(1-a^E) + (1-q)\right) \\ Eu' &\equiv q\left[u'(c^E) + u'(d^E)\right] + (1-q)\left[u'(c^L) + u'(d^L)\right] \\ Eu'y &\equiv q\left[u'(c^E)w + u'(d^E)zw(1-a^E)\right] + (1-q)\left[u'(c^L)w(1-a^L) + u'(d^L)zw\right] \end{aligned}$$

In order to derive the formula for the optimal tax rate τ, we consider the derivatives of the compensated Lagrangian with respect to that fiscal instrument. That derivative amounts to the effect of a marginal change of τ on the Lagrangian when the change in τ is compensated by a variation in T, in such a way as to maintain the government's budget balanced.

Assuming that cross derivatives in compensated terms are negligible (i.e. $\frac{\partial \tilde{n}^E}{\partial \tau} = \frac{\partial \tilde{n}^L}{\partial \tau} = \frac{\partial \tilde{a}^E}{\partial \theta} = \frac{\partial \tilde{a}^L}{\partial \theta} = 0$), that derivative can be written as:

$$\frac{\partial \tilde{\pounds}}{\partial \tau} = \frac{\partial \pounds}{\partial \tau} + \frac{\partial \pounds}{\partial T}\bar{y} = -cov(u', y) + A - \mu\tau w\left[qz\frac{\partial \tilde{a}^E}{\partial \tau} + (1-q)\frac{\partial \tilde{a}^L}{\partial \tau}\right] \tag{49}$$

where

$$A \equiv \frac{\partial \tilde{a}^E}{\partial \tau}\left[qH'(\hat{b}^E)n^E + (1-q)\,H'(\hat{b}^L)n^L\right] \\ +\frac{\partial \tilde{a}^L}{\partial \tau}\left[qH'(\hat{b}^E)(1-n^E) + (1-q)\,H'(\hat{b}^L)(1-n^L)\right]$$

and where

$$\frac{\partial \tilde{a}^i}{\partial \tau} \equiv \frac{\partial a^i}{\partial \tau} + \frac{\partial a^i}{\partial T}\frac{\partial T}{\partial \tau} = \frac{\partial a^i}{\partial \tau} + \frac{\partial a^i}{\partial T}\bar{y}$$

is the effect of a change of τ on the amount of LTC provided by type-i agents to their elderly parents when that change in τ is compensated by a variation in the demogrant T in such a way as to maintain the government's budget balanced.

Equalizing $\frac{\partial \tilde{\pounds}}{\partial \tau}$ to 0 and isolating τ yields the following tax formula:

$$\tau = \frac{-cov(u', y) + A}{\mu w\left[qz\frac{\partial \tilde{a}^E}{\partial \tau} + (1-q)\frac{\partial \tilde{a}^L}{\partial \tau}\right]} \tag{50}$$

To interpret that earning tax formula, we can distinguish between three terms. The first term of the numerator is an equity term. Given that individuals with higher incomes benefit from a lower marginal utility of consumption, we have $cov(u', y) < 0$, so that this first term is positive, and pushes towards a higher earning tax. The second term of the numerator, A, captures the incidence of the earning tax on the provision of LTC by children. By taxing labor earnings, the opportunity cost of providing LTC to elderly parents is reduced, which is likely to raise the amount of LTC provided. That term pushes towards more taxation of labor. Finally, the denominator of the tax formula is a standard efficiency term, which captures the incidence of the tax on the tax base.

5.4 Family allowances

Let us now derive the formula for the subsidy on early children θ. For that purpose, we will use the following notations:

$$\begin{aligned} \bar{n}_E &\equiv \sigma\left(qn^E + (1-q)n^L\right) \\ Eu'_E &\equiv qu'(c^E) + (1-q)u'(c^L) \\ Eu'_E n_E &\equiv \sigma\left(qn^E u'(c^E) + (1-q)n^L u'(c^L)\right) \end{aligned}$$

In order to derive the formula for the optimal child allowance θ, we consider, as above, the derivatives of the compensated Lagrangian with respect to that subsidy. That derivative amounts to the effect of a marginal change of θ on the Lagrangian when the change in θ is compensated by a variation in T, in such a way as to maintain the government's budget balanced.

Assuming that cross derivatives in compensated terms are negligible (i.e. $\frac{\partial \tilde{n}^E}{\partial \tau} = \frac{\partial \tilde{n}^L}{\partial \tau} = \frac{\partial \tilde{a}^E}{\partial \theta} = \frac{\partial \tilde{a}^L}{\partial \theta} = 0$), and using the FOCs of the laissez-faire, that derivative can be written as:

$$\frac{\partial \tilde{\pounds}}{\partial \theta} = \frac{\partial \pounds}{\partial \theta} + \frac{\partial \pounds}{\partial T}\bar{n}_E = Eu'_E n_E - \bar{n}_E Eu'_E + B + C - \theta D \tag{51}$$

where

$$\begin{aligned} B &\equiv \left[U^E - U^L + \mu\tau(wa^L - zwa^E)\right]\left[\frac{\partial q}{\partial n^E}\frac{\partial \tilde{n}^E}{\partial \theta} + \frac{\partial q}{\partial n^L}\frac{\partial \tilde{n}^L}{\partial \theta}\right] \\ C &\equiv \left(a^E - a^L\right)\left[qH'(\hat{b}^E)\frac{\partial \tilde{n}^E}{\partial \theta} + (1-q)\,H'(\hat{b}^L)\frac{\partial \tilde{n}^L}{\partial \theta}\right] \\ D &\equiv \mu\sigma\left[\frac{\partial \tilde{n}^E}{\partial \theta}\left(q + \left(n^E - n^L\frac{\partial q}{\partial n^E}\right)\right) + \frac{\partial \tilde{n}^L}{\partial \theta}\left(1 - q + \left(n^E - n^L\frac{\partial q}{\partial n^L}\right)\right)\right] \end{aligned}$$

and where: $\frac{\partial \tilde{n}^E}{\partial \theta} \equiv \frac{\partial n^E}{\partial \theta} + \frac{\partial n^E}{\partial T}\frac{\partial T}{\partial \theta} = \frac{\partial n^E}{\partial \theta} + \frac{\partial n^E}{\partial T}\bar{n}_E$ and $\frac{\partial \tilde{n}^L}{\partial \theta} \equiv \frac{\partial n^L}{\partial \theta} + \frac{\partial n^L}{\partial T}\frac{\partial T}{\partial \theta} = \frac{\partial n^L}{\partial \theta} + \frac{\partial n^L}{\partial T}\bar{n}_E$.

Hence the tax formula for the early births subsidy is:

$$\theta = \frac{cov(u'_E, n_E) + B + C}{D} \tag{52}$$

That formula includes four terms. The first term of the numerator is the equity term. The intuition behind that term goes as follows. A family allowance on early births is socially desirable if first-period consumption and first-period births are negatively correlated. Actually, if individuals with lower first-period consumption also have more early children, it makes sense, on equity grounds, to subsidize early births, as a way to redistribute resources towards the disadvantaged individuals.[24] The term B at the numerator gives the effect of composition (q) on overall utility and on earning tax revenue. That term reflects that the optimal subsidy on early births depends on its incidence on the long-run composition of the population. If the decentralized population involves a too large proportion of late children from a social perspective, this justifies to subsidize early births, to make the population closer to its optimal structure in the long-run. The term C reflects the incidence of family allowances on the amount of LTC provided by children. In a hypothetical case where the amounts of LTC provided by early and late children would be equal, that term would vanish, since affecting the timing of births would then be neutral for the provision of LTC. But as early children face a lower opportunity cost of providing LTC, we have $a^E - a^L > 0$, implying that this term is likely to be positive. Finally, the term D is the standard efficiency term. It is also expected to be positive. Our results are summarized in Proposition 7.

Proposition 7 *Assuming a small open economy with $R = 1$, and assuming that the only available instruments are a tax on earnings τ, a subsidy on early births θ and a demogrant T, we obtain that, under negligible cross derivatives, the optimal τ and the optimal θ are given by:*

$$\tau = \frac{-cov(u', y) + A}{\mu w \left[qz \frac{\partial \tilde{a}^E}{\partial \tau} + (1-q) \frac{\partial \tilde{a}^L}{\partial \tau} \right]} \quad \text{and } \theta = \frac{cov(u'_E, n_E) + B + C}{D}$$

where

$$\begin{aligned}
A &\equiv \frac{\partial \tilde{a}^E}{\partial \tau} \left[qH'(\hat{b}^E) n^E + (1-q) H'(\hat{b}^L) n^L \right] \\
&\quad + \frac{\partial \tilde{a}^L}{\partial \tau} \left[qH'(\hat{b}^E)(1-n^E) + (1-q) H'(\hat{b}^L)(1-n^L) \right] \\
B &\equiv \left[U^E - U^L + \mu\tau(wa^L - zwa^E) \right] \left[\frac{\partial q}{\partial n^E} \frac{\partial \tilde{n}^E}{\partial \theta} + \frac{\partial q}{\partial n^L} \frac{\partial \tilde{n}^L}{\partial \theta} \right] \\
C &\equiv (a^E - a^L) \left[qH'(\hat{b}^E) \frac{\partial \tilde{n}^E}{\partial \theta} + (1-q) H'(\hat{b}^L) \frac{\partial \tilde{n}^L}{\partial \theta} \right] \\
D &\equiv \mu\sigma \left[\frac{\partial \tilde{n}^E}{\partial \theta} \left(q + \left(n^E - n^L \frac{\partial q}{\partial n^E} \right) \right) + \frac{\partial \tilde{n}^L}{\partial \theta} \left(1 - q + \left(n^E - n^L \frac{\partial q}{\partial n^L} \right) \right) \right]
\end{aligned}$$

[24] On that equity effect, see Pestieau and Ponthiere (2013), who study, in a static model with endogenous birth timing, the design of optimal family allowances when parents differ in wage levels.

Proof. See above. ■

In sum, our analysis of the second-best problem identifies the various determinants of the optimal subsidy on early births. Besides standard equity and efficiency terms, the level of the optimal subsidy on early births depends also on the incidence of that allowance on the composition of the population, as well as on its incidence on the amount of LTC provided to the dependent parents.

6 Conclusions

The rise of LTC and the postponement of births are two key demographic trends of the last 40 years. This paper proposed to analyze, within a dynamic OLG model with endogenous birth timing and old-age dependence, the formal relation between those two phenomena.

Our results show that the timing of births matters for informal LTC provision. Because of their proximity with retirement, early children face, in comparison with late children, a lower opportunity cost of providing LTC to their parents. Thus, even if all children have the same preferences, they provide unequal amounts of LTC, because they are not at the same moment in their life, and, hence, do not face the same time constraints. Our model can thus rationalize the stylized fact that early children provide more LTC than late children.

On the normative side, we showed that, at the utilitarian optimum, early children provide more LTC than late children. Moreover, the social optimum involves a different birth timing than the laissez-faire: it is, in general, socially desirable, for the sake of long-run social welfare maximization, to increase the proportion of early births, and to reduce the proportion of late births, in comparison with the laissez-faire. Thus, in the context of LTC provision, decentralized fertility choices do not coincide with what maximizes social welfare in the long-run. As a consequence, early births should, to some extent, be encouraged, since these allow the society to benefit from cheaper informal LTC provision from children of dependents who are already retired or close to retirement.

This paper, by focusing on the link between fertility choices and informal LTC provision, left aside some important determinants of the timing of births. As this is well-known, fertility decisions and education decisions are strongly related, in the sense that more educated persons have, on average, children later on in their life. That dimension was not taken into account in our model, whose goal was to explore the link between fertility timing and informal LTC provision, and to show that having children earlier can be regarded as some form of insurance for receiving more LTC at the old age. Our framework also involved other limitations. Firstly, we focused only on the provision of informal LTC, without considering the potential interactions with formal LTC provision. We also left aside any kind of monetary transfers. Secondly, an important limitation of this work is that it takes into account the impact of informal LTC provision on the health of the dependent elderly, but it ignores the impact of informal LTC provision on the health of caregivers. That impact was shown to be negative in the recent study by Do et al (2015). Introducing this additional effect could limit the extent to which governments should encourage early births. Thirdly, our analysis kept the retirement age as fixed, whereas it could be argued that rational individuals would also choose when they retire, depending on the LTC needs of their parents.[25] Here we deliberately left the retirement decision aside, since our main focus was on the choice of the timing of births. Fourthly, our analysis supposed that total fertility is constant, and equal to the replacement fertility level. As a consequence, we neglected "size effects", i.e. the potential impact of the number of births on LTC provision. It is not clear to see how large those size effects are, but allowing a varying number of births would constitute an interesting - but also more complex - extension of our framework.

In sum, the recent demographic trends observed - the rise of LTC and the postponement of births - raise deep challenges to both economists and policy-makers. Given that those two trends are strongly linked, it follows that these invite a consistent family policy aimed at dealing with these both issues in an adequate way. The strong link between those two demographic trends of the 21st century invites a joint design of fertility policies and LTC policies.

7 References

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8 Appendix

8.1 Proof of Proposition 1

The first part of the proposition is obtained from the FOCs of the problems faced by individuals of types E and L. Regarding the second part of the proposition, this can be obtained by considering the following three cases:

- Case 1: $c_t^E = c_t^L = c_t$. Then $d_{t+1}^E = d_{t+1}^L = d_{t+1}$ from the FOCs for optimal savings and $n_t^E = n_t^L$ from the FOCs for optimal fertility. Hence, from the FOC for LTC, we have:

$$\varphi'(a^E_{t+1}) = u'(c_t)\frac{w^e_{t+1}z}{R^e_{t+1}} \text{ and } \varphi'(a^L_t) = u'(c_t)w_t$$

Substituting for $u'(c_t)$, we have:

$$\varphi'(a^E_{t+1}) = \varphi'(a^L_t)\frac{w^e_{t+1}z}{w_t R^e_{t+1}} \implies a^E_{t+1} = \varphi'^{-1}\left(\varphi'(a^L_t)\frac{w^e_{t+1}z}{w_t R^e_{t+1}}\right)$$

Under $c^E_t = c^L_t = c_t$, $d^E_{t+1} = d^L_{t+1} = d_{t+1}$ and $n^E_t = n^L_t$, we obtain, from the comparison of the budget constraints of the two types, that:

$$a^E_{t+1} = \frac{R^e_{t+1}w_t}{w^e_{t+1}z}a^L_t$$

Collecting those things, we obtain, by transitivity of equality:

$$\varphi'^{-1}\left(\varphi'(a^L_t)\frac{w^e_{t+1}z}{w_t R^e_{t+1}}\right) = a^L_t\frac{R^e_{t+1}w_t}{w^e_{t+1}z}$$

which is a contradiction. Indeed, taking the transform $\varphi'(\cdot)$ on both sides, we obtain:

$$\varphi'(a^L_t)\frac{w^e_{t+1}z}{w_t R^e_{t+1}} = \varphi'\left(a^L_t\frac{R^e_{t+1}w_t}{w^e_{t+1}z}\right)$$

which is false except when $\frac{R^e_{t+1}w_t}{w^e_{t+1}z} = 1$. Hence this case cannot arise.

- Case 2: $c^E_t < c^L_t$. Then $d^E_{t+1} < d^L_{t+1}$ and $n^E_t < n^L_t$. From the FOC for LTC, it seems that two subcases can arise: either $a^E_{t+1} > a^L_t$ or $a^E_{t+1} \leq a^L_t$. But further analysis shows that only the first subcase is possible. To show this, rewrite the two budget constraints as follows:

$$\begin{aligned} w_t + \frac{w^e_{t+1}z}{R^e_{t+1}} - \left(w_t\sigma n^E_t + \frac{w^e_{t+1}z(\sigma(1-n^E_t))}{R^e_{t+1}}\right) - w^e_{t+1}z\frac{a^E_{t+1}}{R^e_{t+1}} &= c^E_t + \frac{d^E_{t+1}}{R^e_{t+1}} \\ w_t + \frac{w^e_{t+1}z}{R^e_{t+1}} - \left(w_t\sigma n^L_t + \frac{w^e_{t+1}z\left(\sigma(1-n^L_t)\right)}{R^e_{t+1}}\right) - w_t a^L_t &= c^L_t + \frac{d^L_{t+1}}{R^e_{t+1}} \end{aligned}$$

Suppose that $c^E_t < c^L_t$ and $a^E_{t+1} \leq a^L_t$. Then the RHS of the second condition is higher than the RHS of the first condition. On the LHS, the first two terms are identical across conditions. Given $\frac{w^e_{t+1}z}{R^e_{t+1}} < w_t$ and $n^E_t < n^L_t$, the term in brackets term is higher, in absolute value, in the second condition, implying that more is subtracted there. But if $a^E_{t+1} \leq a^L_t$, the last term of the LHS is also larger in absolute value, implying also that more is subtracted there. Hence, the LHS of the second condition is unambiguously lower than the LHS of the first condition. But since the RHS must be, under our assumptions, unambiguously larger in the second condition than in the first one, a contradiction is reached. Thus the case $a^E_{t+1} \leq a^L_t$ cannot arise.

- Case 3: $c_t^E > c_t^L$. Then $d_{t+1}^E > d_{t+1}^L$ and $n_t^E > n_t^L$. Hence, from the FOC for LTC, i.e.

$$\varphi'(a_{t+1}^E) = u'(c_t^E)\frac{w_{t+1}^e z}{R_{t+1}^e} \text{ and } \varphi'(a_t^L) = u'(c_t^L)w_t$$

we have that $u'(c_t^E) < u'(c_t^L)$. Hence, under $\frac{w_{t+1}^e z}{R_{t+1}^e} < w_t$, the RHS of the first condition is unambiguously lower than the RHS of the second condition, implying that $a_{t+1}^E > a_t^L$. Only that case can arise.

8.2 Proof of Proposition 2

The first part of Proposition 2 is obtained from the FOCs of the problems faced by agents of types E and L. Regarding the second part of Proposition 2, this is obtained by the same kind of argument as at the temporary equilibrium (Proposition 1). We consider three cases, as for the temporary equilibrium:

- Case 1: $c^E = c^L$. Then $d^E = d^L$ and $n^E = n^L$.
- Case 2: $c^E < c^L$. Then $d^E < d^L$ and $n^E < n^L$.
- Case 3: $c^E > c^L$. Then $d^E > d^L$ and $n^E > n^L$.

The proof is the same as for Proposition 1, except that we replace w_t and w_{t+1}^e by w and R_{t+1}^e by R. But the result is the same: cases 1 and 2 lead to contradictions, and only case 3 is possible. In that case, we obtain, from the FOC for LTC, that $a^E > a^L$. Given $n^E > n^L$ and $a^E > a^L$, it follows that the LTC received by a type E, i.e. $b^E = n^E a^E + (1-n^E)a^L$ exceeds the LTC received by a type L, i.e. $b^L = n^L a^E + (1-n^L)a^L$.

8.3 Proof of Proposition 3

Conditionally on their anticipations, and assuming that $\sigma = 0$, individuals of type E solve the following problem:

$$\begin{aligned}\max_{s_t^E, n_t^E, a_{t+1}^E} \quad & (1-\delta)\log(w_t - s_t^E) + \delta\log\left(n_t^E\right) + (1-\delta)\log(R_{t+1}^e s_t^E + w_{t+1}^e z(1-a_{t+1}^E)) \\ & +\delta\log\left(1-n_t^E\right) + \eta\log(a_{t+1}^E) + \gamma\log(n_t^E a_{t+2}^{Ee} + (1-n_t^E)a_{t+2}^{Le})\end{aligned}$$

FOCs are:

$$\begin{aligned}\frac{1}{(w_t - s_t^E)} &= \frac{R_{t+1}^e}{(R_{t+1}^e s_t^E + w_{t+1}^e z(1-a_{t+1}^E))} \\ \frac{\delta}{n_t^E} - \frac{\delta}{\left(1-n_t^E\right)} &= \frac{\gamma\left(a_{t+2}^{Le} - a_{t+2}^{Ee}\right)}{(n_t^E a_{t+2}^{Ee} + (1-n_t^E)a_{t+2}^{Le})} \\ \frac{(1-\delta)w_{t+1}^e z}{(R_{t+1}^e s_t^E + w_{t+1}^e z(1-a_{t+1}^E))} &= \frac{\eta}{a_{t+1}^E}\end{aligned}$$

The FOC for birth timing is quite complex, since it depends on expected future amounts of LTC to be provided by children in the future. However, if one assumes myopic parents, who ignore the impact of fertility on future health (i.e. $\gamma = 0$), the FOC for birth timing yields: $n_t^E = 1/2$. Hence

$$a_{t+1}^E = \frac{\eta(R_{t+1}^e s_t^E + w_{t+1}^e z)}{[(1-\delta)w_{t+1}^e z + \eta w_{t+1}^e z]} \text{ and } s_t^E = \frac{w_t\left[(1-\delta)+\eta\right] - \frac{z w_{t+1}^e}{R_{t+1}^e}(1-\delta)}{2(1-\delta)+\eta}$$

Hence

$$a_{t+1}^E = \frac{\eta\left[R_{t+1}^e\left(\frac{w_t[(1-\delta)+\eta] - \frac{z w_{t+1}^e}{R_{t+1}^e}(1-\delta)}{2(1-\delta)+\eta}\right) + w_{t+1}^e z\right]}{[(1-\delta)w_{t+1}^e z + \eta w_{t+1}^e z]}$$

Consider now Type L agents. Their problem is:

$$\begin{aligned}\max_{s_t^L, n_t^L, a_t^L} \quad & (1-\delta)\log(w_t(1-a_t^L) - s_t^L) + \delta\log\left(n_t^L\right) + (1-\delta)\log(R_{t+1}^e s_t^L + w_{t+1}^e z) \\ & +\delta\log\left(1-n_t^L\right) + \eta\log(a_t^L) + \gamma\log(n_t^L a_{t+2}^{Ee} + (1-n_t^L)a_{t+2}^{Le})\end{aligned}$$

FOCs are:

$$\begin{aligned}\frac{1}{R_{t+1}^e(w_t(1-a_t^L) - s_t^L)} &= \frac{1}{(R_{t+1}^e s_t^L + w_{t+1}^e z)} \\ \frac{\delta}{n_t^L} - \frac{\delta}{\left(1-n_t^L\right)} &= \frac{\gamma\left(a_{t+2}^{Le} - a_{t+2}^{Ee}\right)}{(n_t^L a_{t+2}^{Ee} + (1-n_t^L)a_{t+2}^{Le})} \\ \frac{(1-\delta)w_t}{(w_t(1-a_t^L) - s_t^L)} &= \frac{\eta}{a_t^L}\end{aligned}$$

Here again, when $\gamma = 0$, we have: $n_t^L = 1/2$. Hence

$$a_t^L = \frac{\eta(w_t - s_t^L)}{[(1-\delta)w_t + \eta w_t]} \text{ and } s_t^L = \frac{w_t(1-\delta) - \frac{z w_{t+1}^e}{R_{t+1}^e}\left[(1-\delta)+\eta\right]}{2(1-\delta)+\eta}$$

Hence

$$a_t^L = \frac{\eta\left[w_t - \left(\frac{w_t(1-\delta) - \frac{z w_{t+1}^e}{R_{t+1}^e}[(1-\delta)+\eta]}{2(1-\delta)+\eta}\right)\right]}{[(1-\delta)w_t + \eta w_t]}$$

When $n_t^E = n_t^L = 1/2$, the dynamics of heterogeneity, given by:

$$q_t = \frac{q_{t-1}n_{t-1}^E + (1-q_{t-1})n_{t-1}^L}{\left(q_{t-1}n_{t-1}^E + (1-q_{t-1})n_{t-1}^L\right) + q_{t-2}(1-n_{t-2}^E) + (1-q_{t-2})(1-n_{t-2}^L)}$$

collapses to:

$$q_t = \frac{q_{t-1}\frac{1}{2} + (1-q_{t-1})\frac{1}{2}}{\left(q_{t-1}\frac{1}{2} + (1-q_{t-1})\frac{1}{2}\right) + q_{t-2}(1-\frac{1}{2}) + (1-q_{t-2})(1-\frac{1}{2})} = \frac{1}{2}$$

Thus the partition of the population adjusts instantaneously to 1/2, independently from initial conditions.

Hence the capital accumulation equation is:

$$\begin{aligned}
K_{t+1} &= q_t s_t^E + (1-q_t) s_t^L \\
&= \frac{1}{2}\frac{w_t\left[(1-\delta)+\eta\right] - \frac{z w_{t+1}^e}{R_{t+1}^e}(1-\delta)}{2(1-\delta)+\eta} + \frac{1}{2}\frac{w_t(1-\delta) - \frac{z w_{t+1}^e}{R_{t+1}^e}\left[(1-\delta)+\eta\right]}{2(1-\delta)+\eta} \\
&= \frac{w_t(1-\delta) - \frac{z w_{t+1}^e}{R_{t+1}^e}(1-\delta)}{2(1-\delta)+\eta} + \frac{1}{2}\frac{\eta}{2(1-\delta)+\eta}\left(w_t - \frac{z w_{t+1}^e}{R_{t+1}^e}\right)
\end{aligned}$$

Remind that factor prices are $w_t = F_L(\cdot) = AK_t^{\alpha}(1-\alpha)L_t^{-\alpha}$ and $R_t = F_K(\cdot) = A\alpha K_t^{\alpha-1}L_t^{1-\alpha}$, where total labour is:

$$L_t = q_t + (1-q_t)\left(1-a_t^L\right) + q_{t-1}z(1-a_t^E) + (1-q_{t-1})z$$

Let us define, in intensive terms, k_t as capital stock per labor unit: $k_t \equiv \frac{K_t}{L_t}$. Hence factor prices can be rewritten as:

$$w_t = F_L(\cdot) = Ak_t^{\alpha}(1-\alpha) \text{ and } R_t = F_K(\cdot) = A\alpha k_t^{\alpha-1}$$

From the formula for a_t^L, we have: $1-a_t^L = \frac{w_t 2[1-\delta] - \frac{z w_{t+1}^e}{R_{t+1}^e}\eta}{[2(1-\delta)+\eta]w_t}$. Hence, at the stationary equilibrium, and substituting for factor prices, we have:

$$1-a^L = \frac{2\left[1-\delta\right] - \frac{zk^{1-\alpha}}{A\alpha}\eta}{\left[2(1-\delta)+\eta\right]}$$

We have that $1-a^L$ is decreasing in k. When $k \to 0$, we have: $1-a^L = \frac{2[1-\delta]}{[2(1-\delta)+\eta]} = \bar{a} < 1$. Thus we always have $1-a^L < 1$. We have $1-a^L = 0$ when $k = \left[\frac{2A\alpha[1-\delta]}{z\eta}\right]^{\frac{1}{1-\alpha}}$. Let us denote this level of k by $\bar{k}$. This is the maximum k, beyond which $a^L = 1$ and $1-a^L = 0$. Thus we have:

$$0 < 1-a^L < 1 \iff k < \left[\frac{2A\alpha\left[1-\delta\right]}{z\eta}\right]^{\frac{1}{1-\alpha}}$$

Take now a_{t+1}^E. From the above formula, we have: $1-a_{t+1}^E = \frac{2w_{t+1}^e z[1-\delta] - R_{t+1}^e w_t \eta}{w_{t+1}^e z[2(1-\delta)+\eta]}$. Hence, at the stationary equilibrium, we have:

$$1-a^E = \frac{2z\left[1-\delta\right] - A\alpha k^{\alpha-1}\eta}{z\left[2(1-\delta)+\eta\right]}$$

Hence, when k increases, $1-a^E$ increases. We have $1-a^E = 0$ when $k = \left[\frac{A\alpha\eta}{2z[1-\delta]}\right]^{\frac{1}{1-\alpha}}$. Let us denote this level of k by $\tilde{k}$. This is the minimum level of

k, below which $a^E = 1$ and $1 - a^E = 0$. When $k \to +\infty$, we have: $1 - a^E = \frac{2[1-\delta]}{[2(1-\delta)+\eta]} = \bar{a} < 1$. Thus we always have $1 - a^E < 1$. Thus we have:

$$0 < 1 - a^E < 1 \iff k > \left[\frac{A\alpha\eta}{2z\left[1-\delta\right]}\right]^{\frac{1}{1-\alpha}}$$

Hence total labour is thus:

$$L_{t+1} = q_{t+1} + (1-q_{t+1})\left(\frac{w^e_{t+1}2\left[1-\delta\right] - \frac{zw^e_{t+2}}{R^e_{t+2}}\eta}{\left[2(1-\delta)+\eta\right]w^e_{t+1}}\right) + q_t z\left(\frac{2w^e_{t+1}z\left[1-\delta\right] - R^e_{t+1}w_t\eta}{w^e_{t+1}z\left[2(1-\delta)+\eta\right]}\right) + (1-q_t)z$$

Hence, given that, at a stationary equilibrium, $q_t = q_{t-1} = 1/2$, this can be rewritten as:

$$L_t = \frac{1}{2} + \frac{1}{2}\left[\frac{w^e_{t+1}2\left[1-\delta\right] - \frac{zw^e_{t+2}}{R^e_{t+2}}\eta}{\left[2(1-\delta)+\eta\right]w^e_{t+1}}\right] + \frac{1}{2}z\left[\frac{2w^e_{t+1}z\left[1-\delta\right] - R^e_{t+1}w_t\eta}{w^e_{t+1}z\left[2(1-\delta)+\eta\right]}\right] + \frac{1}{2}z$$

Hence, given that $w_t = w^e_{t+2} = Ak^\alpha(1-\alpha)$ and $R^e_{t+1} = A\alpha k^{\alpha-1}$, we have:

$$L_{t+1} = \frac{\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right) - \frac{zk^{1-\alpha}}{A\alpha}\eta - A\alpha k^{\alpha-1}\eta}{2\left[2(1-\delta)+\eta\right]}$$

Given that $K_{t+1} = k_{t+1}L_{t+1} = q_t s^E_t + (1-q_t)s^L_t$, one can rewrite the capital accumulation in intensive terms as:

$$k_{t+1} = \frac{q_t s^E_t + (1-q_t)s^L_t}{\left[\frac{[4(1-\delta)+\eta](1+z) - \frac{zk^{1-\alpha}}{A\alpha}\eta - A\alpha k^{\alpha-1}\eta}{2[2(1-\delta)+\eta]}\right]}$$

Substituting for savings, we obtain that a sufficient condition for the existence of a stationary equilibrium with perfect foresight on factor prices is that there exists a $k > 0$ such that:

$$k = \frac{\frac{Ak^\alpha(1-\alpha)(1-\delta) - \frac{zk(1-\alpha)}{\alpha}(1-\delta)}{2(1-\delta)+\eta} + \frac{1}{2}\frac{\eta}{2(1-\delta)+\eta}\left(Ak^\alpha(1-\alpha) - \frac{zk(1-\alpha)}{\alpha}\right)}{\frac{[4(1-\delta)+\eta](1+z) - \frac{zk^{1-\alpha}}{A\alpha}\eta - A\alpha k^{\alpha-1}\eta}{2[2(1-\delta)+\eta]}}$$

That condition can be rewritten as:

$$\underbrace{k\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right) + \frac{z(1-\alpha)\left[2(1-\delta)+\eta\right]}{\alpha}\right]}_{\Phi(k)}$$
$$= \underbrace{Ak^\alpha\left[\left(1-\alpha\right)\left[2(1-\delta)+\eta\right] + \alpha\eta\right] + \frac{zk^{2-\alpha}}{A\alpha}\eta}_{\Psi(k)}$$

Remind that $0 < 1 - a^L < 1 \iff k < \bar{k} = \left[\frac{A\alpha}{z}\frac{2[1-\delta]}{\eta}\right]^{\frac{1}{1-\alpha}}$ and $0 < 1 - a^E < 1 \iff k > \tilde{k} = \left[\frac{A\alpha}{z}\frac{\eta}{2[1-\delta]}\right]^{\frac{1}{1-\alpha}}$. Suppose $\frac{2[1-\delta]}{\eta} > 1$. Hence $\left[\frac{A\alpha}{z}\frac{2[1-\delta]}{\eta}\right]^{\frac{1}{1-\alpha}} > \left[\frac{A\alpha}{z}\frac{\eta}{2[1-\delta]}\right]^{\frac{1}{1-\alpha}}$ Hence $\bar{k} > \tilde{k}$. Hence the discussion on the existence of a stationary equilibrium must focus on the subspace where $k \in \left[\tilde{k}, \bar{k}\right]$.

Regarding the LHS, we have: $\Phi'(k) = \left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right) + \frac{z(1-\alpha)[2(1-\delta)+\eta]}{\alpha}\right] > 0$. We have thus $\Phi(\tilde{k}) < \Phi(\bar{k})$. Regarding the RHS, we have: $\Psi'(k) = A\alpha k^{\alpha-1}\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right] + \frac{(2-\alpha)zk^{1-\alpha}}{A\alpha}\eta > 0$. We have thus $\Psi(\tilde{k}) < \Psi(\bar{k})$.

A stationary equilibrium exists when there is an intersection of $\Phi(k)$ and $\Psi(k)$. A sufficient condition for such an intersection, and, hence, for the existence of a stationary equilibrium, is an inversion of ranking, that is, either $\Phi(\tilde{k}) < \Psi(\tilde{k})$ and $\Phi(\bar{k}) > \Psi(\bar{k})$ or $\Phi(\tilde{k}) > \Psi(\tilde{k})$ and $\Phi(\bar{k}) < \Psi(\bar{k})$. By continuity, one of those conditions would guarantee the existence of at least one intersection between $\Phi(k)$ and $\Psi(k)$.

We have:

$$\begin{aligned}
\Phi(\tilde{k}) \quad &\lesseqgtr \quad \Psi(\tilde{k}) \\
\iff \quad &\left[\frac{A\alpha}{z}\frac{\eta}{2\left[1-\delta\right]}\right]^{\frac{1}{1-\alpha}}\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)+\frac{z(1-\alpha)\left[2(1-\delta)+\eta\right]}{\alpha}\right] \\
\lesseqgtr \quad & A\left[\frac{A\alpha}{z}\frac{\eta}{2\left[1-\delta\right]}\right]^{\frac{\alpha}{1-\alpha}}\left[\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]+\frac{z\left[\frac{A\alpha}{z}\frac{\eta}{2[1-\delta]}\right]^{\frac{2-2\alpha}{1-\alpha}}}{A^2\alpha}\eta\right]
\end{aligned}$$

Hence

$$\begin{aligned}
&2\eta\left[1-\delta\right]\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)\alpha+z(1-\alpha)\left[2(1-\delta)+\eta\right]\right] \\
\lesseqgtr \quad & 4\left[1-\delta\right]^2 z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]+\alpha\eta^3 \qquad \text{(Condition C1)}
\end{aligned}$$

We have also:

$$\begin{aligned}
\Phi(\bar{k}) \quad &\lesseqgtr \quad \Psi(\bar{k}) \\
\iff \quad &\left[\frac{A\alpha}{z}\frac{2\left[1-\delta\right]}{\eta}\right]^{\frac{1}{1-\alpha}}\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)+\frac{z(1-\alpha)\left[2(1-\delta)+\eta\right]}{\alpha}\right] \\
\lesseqgtr \quad & A\left[\frac{A\alpha}{z}\frac{2\left[1-\delta\right]}{\eta}\right]^{\frac{\alpha}{1-\alpha}}\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]+\frac{z\left[\frac{A\alpha}{z}\frac{2[1-\delta]}{\eta}\right]^{\frac{2-\alpha}{1-\alpha}}}{A\alpha}\eta
\end{aligned}$$

Hence

$$\begin{aligned}
&2\eta\left[1-\delta\right]\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)\alpha+z(1-\alpha)\left[2(1-\delta)+\eta\right]\right] \\
\lesseqgtr \quad & z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]\eta^2+4\left[1-\delta\right]^2\alpha\eta \qquad \text{(Condition C2)}
\end{aligned}$$

- Case 1: $\Phi(\tilde{k}) < \Psi(\tilde{k})$ and $\Phi(\bar{k}) > \Psi(\bar{k})$

  We have:

$$\begin{aligned} & 2\eta\left[1-\delta\right]\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)\alpha+z(1-\alpha)\left[2(1-\delta)+\eta\right]\right] \\ < \quad & 4\left[1-\delta\right]^{2} z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]+\alpha\eta^{3} \end{aligned}$$

  and

$$\begin{aligned} & 2\eta\left[1-\delta\right]\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)\alpha+z(1-\alpha)\left[2(1-\delta)+\eta\right]\right] \\ > \quad & z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]\eta^{2}+4\left[1-\delta\right]^{2}\alpha\eta \end{aligned}$$

- Case 2: $\Phi(\tilde{k}) > \Psi(\tilde{k})$ and $\Phi(\bar{k}) < \Psi(\bar{k})$

We have:

$$\begin{aligned} & 2\eta\left[1-\delta\right]\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)\alpha+z(1-\alpha)\left[2(1-\delta)+\eta\right]\right] \\ > \quad & 4\left[1-\delta\right]^{2} z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]+\alpha\eta^{3} \end{aligned}$$

and

$$\begin{aligned} & 2\eta\left[1-\delta\right]\left[\left[4\left(1-\delta\right)+\eta\right]\left(1+z\right)\alpha+z(1-\alpha)\left[2(1-\delta)+\eta\right]\right] \\ < \quad & z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right]\eta^{2}+4\left[1-\delta\right]^{2}\alpha\eta \end{aligned}$$

Let us compare the two conditions: C1 and C2. These have the same LHS. The RHS would be the same if $\eta^2 = 4\left[1-\delta\right]^2$ or $\eta = 2\left[1-\delta\right]$

We suppose here $\frac{2[1-\delta]}{\eta} > 1$ or $\eta < 2\left[1-\delta\right]$. Hence if $z\left[(1-\alpha)\left[2(1-\delta)+\eta\right]+\alpha\eta\right] > \alpha\eta$, the RHS of condition C1 is larger than the RHS of condition C2. Hence if the LHS is larger than the RHS of C1, it is also larger than the RHS of C2. Thus it is impossible that $\Phi(\tilde{k}) > \Psi(\tilde{k})$ and $\Phi(\bar{k}) < \Psi(\bar{k})$. Thus, for an intersection, we need that the LHS is lower than the RHS of condition C1, but larger than the RHS of condition C2. This coincides with Case 1: $\Phi(\tilde{k}) < \Psi(\tilde{k})$ and $\Phi(\bar{k}) > \Psi(\bar{k})$.

8.4 Proof of Proposition 5

The result regarding the capital stock is derived from the Golden Rule condition, in comparison with $R \gtreqless 1$ at the laissez-faire. The difference concerning the shape of consumption profiles follows from comparing the FOCs at the social optimum and at the laissez-faire, under $R \gtreqless 1$ at the laissez-faire.

Regarding the amount of informal LTC provided to the dependent parent, the FOCs are, at the laissez faire:

$$\varphi'(a^E) = u'(c^E)F_L\left(K,\cdot\right)\frac{z}{F_K\left(K,\cdot\right)} \text{ and } \varphi'(a^L) = u'(c^L)F_L\left(K,\cdot\right)$$

whereas these are, at the optimum:

$$\begin{aligned} \varphi'(a^{E*}) &= u'(c^*)F_L\left(K^*,\cdot\right)z - H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right) \\ \varphi'(a^{L*}) &= u'(c^*)F_L\left(K^*,\cdot\right) - H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right) \end{aligned}$$

Provided the conditions:

$$\begin{aligned} u'(c^E)F_L(K,\cdot)\frac{z}{F_K(K,\cdot)} &> u'(c^*)F_L(K^*,\cdot)\,z - H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right) \\ u'(c^L)F_L(K,\cdot) &> u'(c^*)F_L(K^*,\cdot) - H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right) \end{aligned}$$

hold, we obtain $a^{E*} > a^E$ and $a^{L*} > a^L$.

Let us now consider fertility. At the laissez-faire, the FOC for n^i was:

$$v'\left(n^i\right) - v'\left(1-n^i\right) = u'(c^i)F_L(K,\cdot)\,\sigma\left[1-\frac{z}{R}\right] - \gamma H'\left(n^i a^E + (1-n^i)a^L\right)\left(a^E - a^L\right)$$

whereas the FOC for the socially optimal n^i is:

$$v'(n^*) - v'(1-n^*) = \begin{array}{l} u'(c^*)F_L(K^*,\cdot)\left[\sigma(1-z) - \left(a^{L*} - z a^{E*}\right)\right] - \left[\varphi(a^{E*}) - \varphi(a^{L*})\right] \\ -H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right)\left(a^{E*} - a^{L*}\right) \end{array}$$

The LHS is the same. It is decreasing in n. But the RHS differs. Hence, a sufficient condition for $n^* > n^i$ is:

$$\begin{array}{l} u'(c^i)F_L(K,\cdot)\,\sigma\left[1-\frac{z}{R}\right] \\ -\gamma H'\left(n^i a^E + (1-n^i)a^L\right)\left(a^E - a^L\right) \end{array} > \begin{array}{l} u'(c^*)F_L(K^*,\cdot)\left[\sigma(1-z) - \left(a^{L*} - z a^{E*}\right)\right] \\ -\left[\varphi(a^{E*}) - \varphi(a^{L*})\right] \\ -H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right)\left(a^{E*} - a^{L*}\right) \end{array}$$

For type E, that sufficient condition can be rewritten as:

$$\begin{array}{l} u'(c^E)F_L(K,\cdot)\,\sigma + u'(c^*)F_L(K^*,\cdot)\,\sigma z \\ -\gamma H'\left(n^E a^E + (1-n^E)a^L\right)\left(a^E - a^L\right) \end{array} > \begin{array}{l} u'(c^*)F_L(K^*,\cdot)\,\sigma + u'(c^E)F_L(K,\cdot)\,\sigma\frac{z}{R} \\ -u'(c^*)F_L(K^*,\cdot)\left(a^{L*} - z a^{E*}\right) - \left[\varphi(a^{E*}) - \varphi(a^{L*})\right] \\ -H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right)\left(a^{E*} - a^{L*}\right) \end{array}$$

Given that we assumed above that $u'(c^E)F_L(K,\cdot)\frac{z}{R} > u'(c^*)F_L(K^*,\cdot)\,z - H'\left(n^* a^E + (1-n^*)a^L\right)$, that inequality still holds if we replace $u'(c^E)F_L(K,\cdot)\,\sigma\frac{z}{R}$ by $u'(c^*)F_L(K^*,\cdot)\,\sigma z - \sigma H'\left(n^* a^E + (1-n^*)a^L\right)$.

Hence, after simplifications, the condition becomes:

$$\begin{array}{l} u'(c^E)F_L(K,\cdot)\,\sigma \\ -\gamma H'\left(n^E a^E + (1-n^E)a^L\right)\left(a^E - a^L\right) \end{array} > \begin{array}{l} u'(c^*)F_L(K^*,\cdot)\left(\sigma - a^{L*} + z a^{E*}\right) - \left[\varphi(a^{E*}) - \varphi(a^{L*})\right] \\ -H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right)\left(a^{E*} - a^{L*} + \sigma\right) \end{array}$$

Under that condition, we have $n^* > n^E$. Hence, given that, at the laissez-faire under $R > z$, we have $n^E > n^L$, that condition also guarantees that $n^* > n^E > n^L$.

Finally, since $n^* > n^E > n^L$, $a^{E*} > a^E$ and $a^{L*} > a^L$, we also have:

$$b^* = n^* a^{E*} + (1-n^*)a^{L*} > n^E a^E + (1-n^E)a^L > n^L a^E + (1-n^L)a^L$$

implying $b^* > b^E > b^L$.

8.5 Proof of Proposition 6

Once the Golden Rule capital level is reached, we have $F_K(K,\cdot) = R = 1$, which automatically makes individual consumption profiles flat: $c^i = d^i$. The lump-sum transfers also guarantee that all consumption levels are equal across types, i.e. $c^E = c^L$ and $d^E = d^L$.

Regarding the subsidies on early births, remind that the FOC for fertility choices becomes, under such a subsidy:

$$\begin{aligned} v'\left(n^i\right) - v'\left(1-n^i\right) &= u'(c^i)F_L(K,\cdot)\,\sigma\left[1 - \frac{z}{F_K(K,\cdot)}\right] - u'(c^i)\theta^i \\ &\quad -\gamma H'\left(n^i a^E + (1-n^i)a^L\right)\left(a^E - a^L\right) \end{aligned}$$

Given that the goal is to have n^i satisfying:

$$v'(n^i) - v'(1-n^i) = \begin{array}{l} u'(c^*)F_L(K^*,\cdot)\,\sigma(1-z) - u'(c^*)F_L(K^*,\cdot)\left[\left(a^{L*} - za^{E*}\right)\right] \\ -\left[\varphi(a^{E*}) - \varphi(a^{L*})\right] - H'(\cdot)\left(a^{E*} - a^{L*}\right) \end{array}$$

it is easy to see that the decentralization requires an equalization of the two RHS. When the capital stock equals its Golden Rule level ($R = 1$), this implies:

$$\theta^i = \frac{\left[\begin{array}{l} u'(c^i)F_L(K,\cdot)\,\sigma(1-z) - u'(c^*)F_L(K^*,\cdot)\,\sigma(1-z) \\ +u'(c^*)F_L(K^*,\cdot)\left[\left(a^{L*} - za^{E*}\right)\right] + \left[\varphi(a^{E*}) - \varphi(a^{L*})\right] \\ +H'(\cdot)\left(a^{E*} - a^{L*}\right) - \gamma H'\left(n^i a^E + (1-n^i)a^L\right)\left(a^E - a^L\right) \end{array}\right]}{u'(c^i)}$$

It is easy to see that, under the condition stated in Proposition 5, i.e. $\left[\begin{array}{l} u'(c^E)F_L(K,\cdot)\,\sigma \\ -\gamma H'\left(b^E\right)\left(a^E - a^L\right) \end{array}\right] > \left[\begin{array}{l} u'(c^*)F_L(K^*,\cdot)\left(\sigma - a^{L*} + za^{E*}\right) \\ -\left[\varphi(a^{E*}) - \varphi(a^{L*})\right] - H'(b^*)\left(a^{E*} - a^{L*} + \sigma\right) \end{array}\right]$, we have $\theta^i > 0$.

If some subsidies on LTC bring $a^E = a^{E*}$ and $a^L = a^{L*}$ (see below), and if lump sum transfers equalize all consumptions to c^*, this formula vanishes to:

$$\theta^i = F_L(K^*,\cdot)\left(a^{L*} - za^{E*}\right) + \frac{\varphi(a^{E*}) - \varphi(a^{L*}) + \left(a^{E*} - a^{L*}\right)\left[\begin{array}{l} H'\left(n^* a^{E*} + (1-n^*)a^{L*}\right) \\ -\gamma H'\left(n^i a^{E*} + (1-n^i)a^{L*}\right) \end{array}\right]}{u'(c^*)}$$

Finally, regarding the LTC to the parents, the FOCs would become, under a subsidy μ^i:

$$\varphi'(a^E) = u'(c^E)\frac{F_L(K,\cdot)\,z}{F_K(K,\cdot)} - u'(c^E)\mu^E \text{ and } \varphi'(a^L) = u'(c^L)F_L(K,\cdot) - u'(c^L)\mu^L$$

Given that the optimal a^E and a^L satisfy:

$$\varphi'(a^{E*}) = u'(c^*)F_L(K^*,\cdot)\,z - H'(\cdot) \text{ and } \varphi'(a^{L*}) = u'(c^*)F_L(K^*,\cdot) - H'(\cdot)$$

It is straightforward to deduce, given $F_K(K,\cdot) = 1$, that $\mu^E = \mu^L = \frac{\varphi'(\cdot)}{u'(c^*)}$.