DISCUSSION PAPER SERIES

DP11304

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INDUSTRIAL ORGANIZATION

ISSN 0265-8003

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Discussion Paper 11304
Published 31 May 2016
Submitted 31 May 2016

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Abstract

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JEL Classification: D43, L13, L41

Keywords: Bundling, Tying, leverage, Dominance, Entry Barrier

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Acknowledgements
We thank Mark Armstrong, Marc Bourreau, Jay Pil Choi, Jacques Crémer, Andrew Daughety, Federico Etro, Michele Gori, Martin Hellwig, Bruno Jullien, Preston McAfee, Andras Niedermayer, Martin Peitz, Jennifer Reinganum, Patrick Rey, Bill Rogerson, Yossi Spiegel, Konrad Stahl and Michael Whinston for helpful comments and the participants of the presentations at CEPR Conference on Applied IO 2013 (Bologna), Conference "The Economics of Intellectual Property, Software and the Internet" 2013 (Toulouse), EARIE 2012 (Rome), EEA 2012 (Malaga), Georgia Institute of Technology, ICT Workshop 2013 (Evora), MaCCI Annual Conference 2013 (Mannheim), Max-Planck Institute (Bonn), Michigan State University, Northwestern-Toulouse IO workshop 2012, University of Cergy-Pontoise, Vanderbilt University, University of Venice, Yonsei University. Hurkens acknowledges financial support from the Ministry of Economy and Competitiveness Grant number ECO2015-67171-P.

Leveraging Dominance with Credible Bundling*

Sjaak Hurkens† Doh-Shin Jeon‡ Domenico Menicucci§

May 26, 2016

Abstract

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1 Introduction

Does bundling (or tying)[1] build a barrier to entry? This is a classic question addressed by the literature on the leverage theory of tying. The leverage theory is about a monopolist leveraging its market power in the tying product to the market of the tied product in order to monopolize it.[2] In reality, however, there are many cases in which the tying firm is not a pure monopolist but rather a dominant firm in the tying market, which is consistent with the legal standard in both continents across the Atlantic. In Europe, for tying to be anticompetitive, the tying firm must have a dominant position in the tying market. A similar standard is adopted in the U.S., where the tying firm must have a market power "to force the buyer into the purchase of a tied product" according to a U.S. Supreme Court decision.[3] In this paper we study bundling by a dominant multi-product firm and offer novel insights with regard to the leverage theory of tying.

Even if the tying product is patented (or copyrighted), it often faces competition. The U.S. Supreme Court made it clear that a mere fact that a tying product is patented does not support the presumption of market power.[4] Consider two of the most publicized cases of bundling: bundling of aircraft engine and avionics in the GE/Honeywell merger case, and Microsoft's bundling of Windows and Internet Explorer. In the proposed merger of GE and Honeywell, which was opposed by the European Commission (E.C.), GE was supposed to have 52.5% market share in the engine market and Honeywell 50-60% in avionics (Nalebuff, 2002). One of E.C.'s concerns was that the merged entity would drive out rivals by practicing bundling. In the Microsoft case, Windows was supposed to have 90% market share in the OS market for Intel-compatible PCs (Evans *et al.*, 2000). Furthermore, the two U.S. Supreme Court cases which first adopted the doctrine of leverage theory to make block booking (i.e. bundling of movies) per se illegal are about licensing movies to movie theaters (U.S. v. Paramount Pictures, 1948) or to TV stations (U.S. v Loew's, 1962). In particular, in the second case, block booking was practiced by six major distributors of pre-1948 copyrighted motion picture feature films and it would be hard to argue that any of the distributors was

[1]In this paper, we use the terms bundling and tying interchangeably.

[2]For instance, according to Whinston (1990), leverage theory of tying is that "tying provides a mechanism whereby a firm with monopoly power in one market can use the leverage provided by this power to foreclose sales in, and thereby monopolize, a second market." (p.837)

[3]Jefferson Parish Hospital District No. 2 v. Hyde, 466 U.S. 2 (1984).

[4]Illinois Tool Works v. Independent Ink, 547 U.S. 28 (2006).

a monopoly.

By contrast, the existing formalized theory of leverage considers a pure monopoly in the tying market. In the baseline model of the seminal paper, Whinston (1990) finds that pre-commitment to tying induces the incumbent to be aggressive and thereby reduces the rival's profit in the tied market. Hence, tying may build an entry barrier if there is a fixed cost of entry. Notice, however, that tying also reduces the profit of the incumbent when entry occurs. In this sense, (contractual) tying is not credible because the incumbent would undo the tying if the entry occurs.[5] Nalebuff (2004) finds that bundling two products reduces the profit of an entrant into a single market in a credible way. But this result crucially relies on the assumption that the incumbent is a Stackelberg leader in setting prices, since bundling increases the entrant's profit in a simultaneous pricing game.[6] The credibility of bundling is an important issue as, for instance, when the Court of First Instance rejected the European Commission's analysis of pure bundling for the GE/Honeywell case (even if it upheld the E.C.'s decision), it questioned the credibility of the merged entity in practicing bundling.

In this paper, we study bundling of a dominant multi-product firm (called firm A).[7] We mainly compare two *simultaneous* pricing games (the game of independent pricing and that of pure bundling) and show that bundling is credible as long as firm A's dominance is strong enough.[8] To isolate the main effects, we start by analyzing the baseline model in which firm A faces competition from another multi-product firm (called firm B) and has the same level of dominance in each market. After studying this model, we extend our analysis to the case in which firm A faces competition from specialists, as in the GE/Honeywell case. Finally, we provide conditions under which firm A can credibly leverage its dominance in tying market(s) to a tied market in which A faces a dominant rival.

In the baseline model, each of the two firms produces $n \geq 2$ products. We generalize the

[5]Whinston distinguishes technical tying from contractual one and the difference lies in that the cost of undoing tying is high for technical tying but low for contractual tying.

[6]Choi and Stefanadis (2001) and Carlton and Waldman (2002) consider a system monopoly and allow for entry in each complementary component, but their mechanism of leverage requires pre-commitment to bundling as in Whinston (1990). A notable exception is Peitz (2008), who builds an *example* in which the incumbent's tying is credible and reduces the rival's profit. Peitz still considers the tying of a pure monopolist.

[7]In a similar spirit, Calzolari and Denicolò (2015) study exclusive dealing contracts of a *dominant* firm.

[8]In Appendix, we also allow each firm to use mixed bundling (in the baseline model) and establish that if A is dominant enough then the unique equilibrium outcome is the outcome obtained under pure bundling.

two-dimensional Hotelling model of Matutes and Régibeau (1988) in three respects. First, in order to formalize A's dominance, we allow that each of A's products gives a higher value than B's products by $\alpha \geq 0$. Second, we suppose that consumers' locations on the Hotelling line are i.i.d., each with a log-concave symmetric density function; this family of densities includes the uniform density considered by Matutes and Régibeau (1988). Last, we allow for more than two products. Our main simplifying assumption is full coverage, which means that all consumers are served under either regime. Hence, the difference between complements and independent products does not appear in our model.

When we compare independent pricing with pure bundling in the baseline model, we obtain the following novel results: (i) for low levels of dominance, bundling reduces each firm's profit; (ii) for intermediate levels of dominance, bundling increases the dominant firm's profit but reduces the rival's profit; (iii) for high levels of dominance, bundling increases each firm's profit. These results imply that for an intermediate level of dominance, pure bundling of firm A is credible and builds a barrier to entry against B, which in turn provides a rationale for the use of contractual bundling in order to erect an entry barrier. In contrast, for very high levels of dominance, bundling does not build a barrier to entry against B (but is still profitable for A). In this case, if A had the power to dictate the terms of competition, the most effective way to deter entry would be to enforce competition in independent pricing, which is completely opposite to Whinston's insight.

The intuition for the above results can be provided in terms of two effects: *demand size effect* and *demand elasticity effect*. To explain the demand size effect, suppose that for each product, firm A is located at 0 and B at 1 on the Hotelling line. When firm A is dominant, $\alpha > 0$, and the location of the indifferent consumer is $x(\alpha) > 1/2$ under independent pricing. What matters under bundling is the distribution of the average location, which is *more-peaked* around the mean than the log-concave distribution of each individual location (Proschan, 1965). This means that, for each given $k \in (0, \frac{1}{2})$, the probability of a deviation from the mean larger than k is smaller for the average location than for any individual location. Consider the following thought experiment. Initially both firms set prices for each of their products. Now firm A bundles all its products, so that consumers must buy all or none of its products, without changing prices. That is, A sells the bundle at a price equal to the sum of the prices of the individual products. Then, the average location of the consumer indifferent between buying all products (the bundle) from A and from B is still equal to

$x(\alpha)$, but more-peakedness of the average distribution implies that bundling increases the demand of A (and hence increases A's profit) and reduces the demand (and profit) of B.[9] This demand size effect is positive for dominant firm A, but is weak if α is low or is high (it is zero if A has 100% market share under independent pricing), and is highest for intermediate values of α.

We now turn to the demand elasticity effect. The fact that the distribution of the average location is more-peaked than that of the individual location implies also that the demand under bundling is more elastic (resp. less elastic) if the average location of the indifferent consumer is close to the mean (resp. close to 1). The average location of the indifferent consumer gets closer to 1 as firm A's dominance increases. Therefore, bundling makes each firm more aggressive (less aggressive) if A's dominance is small (large). Therefore, for low levels of dominance the competition-intensifying demand elasticity effect dominates the demand size effect and hence bundling reduces every firm's profit; this generalizes the finding of Matutes and Régibeau (1988) for $\alpha = 0$. Likewise, for high levels of dominance the competition-softening demand elasticity effect dominates the demand size effect and bundling increases every firm's profit. For intermediate levels of dominance, bundling is profitable for A but reduces B's profit as A benefits from the demand size effect.

We find that the welfare effects of bundling is non-monotonic in the dominance level. On the one hand, pure bundling reduces welfare by increasing mismatch between consumer preferences and products. On the other hand, bundling increases welfare if it increases the market share of firm A, because firm A is not aggressive enough from a welfare point of view, both under independent pricing and bundling. Bundling reduces welfare for low and high levels of dominance because in these cases the increase in market share of firm A is either positive but small or negative. However, for intermediate values of α, the demand size effect is strong and A's increase in market share may (but need not) dominate the negative mismatch effect, in which case bundling improves welfare.

In Section 4, we consider competition between firm A (producing n different products) and specialists B1, .., Bn, where Bj ($j = 1, 2, ..., n$) denotes the firm specialized in product j. When firm B is separated into n specialists, bundling creates a Cournot complement problem: for any given price of A's bundle, the specialists choose prices higher then the

[9]Indeed, Fang and Norman (2006) find this demand size effect of bundling in a monopoly setting in which valuations are i.i.d., each with a symmetric and log-concave density.

price that would be chosen if they were integrated. This is because a specialist does not internalize the negative externalities on the demand for the other specialists when it raises its price. This implies that A's profit under bundling is strictly higher when the rival firms are separate than when they are integrated. In other words, the threshold level of dominance for bundling to be credible is lower when the rival firms are separate than when they are integrated. This result suggests that bundling would have been more likely credible for GE and Honeywell if the merger had been approved.

In Section 5, we depart from the baseline model by considering asymmetric dominance: firm A is dominant in some markets but not dominant in the tied market. We identify three different conditions under which tying is credible and hurts the rival in the tied market: (i) the tied market has less horizontal differentiation than the tying market,[10] (ii) the tying firm leverages dominance from multiple markets, (iii) the tying firm faces competition from specialists. These results are relevant to the antitrust policy of tying as antitrust authorities are mainly concerned about situations in which the tied product is inferior to or as good as the competing product. For instance, in cable TV market, U.S. senator McCain introduced a bill in 2013 to encourage the wholesale and retail unbundling of programming.[11] Our paper is relevant to the wholesale market in which channel conglomerates may use bundling to foreclose competing channels. In fact, Cablevision filed a lawsuit against Viacom as it considers that Viacom's obligation to acquire the bundle of core and suite networks forecloses Cablevision from distributing competing networks that consumers would likely prefer. Our paper is also relevant to the telecommunications in which bundled offers such as triple or quadruple play are important (Pereira and Vareda, 2013).

Finally, at the end of Section 5, we consider bundling of a pure monopolist (i.e., A does not face competition in the tying market) and rediscover the findings of Whinston (1990) that bundling is never credible regardless of whether the products are complements or can be independently consumed.

The bundling (or tying) literature can be divided into three categories.[12] The first

[10] We find that in the case of uniform distribution, tying is not credible in the setting of asymmetric dominance if the degree of horizontal differentation is the same in both markets. This is because firm A's total profit under independent pricing increases with the asymmetry in dominance $\alpha_1 - \alpha_2$ when its aggregate dominance $\alpha_1 + \alpha_2$ is kept constant.

[11] See Crawford and Yurukoglu (2012) for an empirical analysis of retail bundling in cable TV.

[12] See Choi (2011) for a recent survey of bundling literature.

one includes papers that study bundling as a price discrimination device for a monopolist (Schmalensee, 1984, McAfee et al. 1989, Salinger 1995, Armstrong 1996, Bakos and Brynjolfsson, 1999, Fang and Norman, 2006, Chen and Riordan, 2012 and Menicucci, Hurkens and Jeon, 2015). The second is about competitive bundling where entry and exit is not an issue (Matutes and Régibeau 1988, Economides, 1989, Carbajo, De Meza and Seidmann, 1990, Chen 1997, Denicolo 2000, Nalebuff, 2000, Armstrong and Vickers, 2010, Carlton, Gans, and Waldman, 2010, Thanassoulis 2011, Hahn and Kim 2012, Kim and Choi, 2015 and Zhou, 2015). The last is about the leverage theory of bundling in which the main motive of bundling is to deter entry or induce the exit of rival firms in the competitive segment of the market (Whinston, 1990, Choi and Stefanadis 2001, Carlton and Waldman, 2002, Nalebuff, 2004, Peitz 2008, Jeon and Menicucci, 2006, 2012).[13]

We contribute to the last two categories. Studying bundling by a dominant firm and its implications on entry barriers, we bridge the gap between the real world and the leverage theory which studies a monopolist's bundling. Most importantly, we identify circumstances under which dominance in tying market(s) can be credibly leveraged into a tied market dominated by a rival. This result provides a justification for the use of contractual bundling to deter entry. We also contribute to the theory of competitive bundling by building a general framework that includes as a special case the model of Matutes and Régibeau (1988) and showing that the level of dominance of a firm is a crucial parameter such that their finding is completely reversed for strong dominance. Hahn and Kim (2012) also extend Matutes and Régibeau (1988) by introducing cost asymmetry, which generates results similar to those in our Proposition 3. However, there are a number of important differences. First (and the most important), we are mainly motivated by solving a puzzle in the literature on leverage theory of bundling, namely, to identify conditions under which bundling is credible and builds an entry barrier. In contrast, they focus only on the compatibility literature. Second, we consider the class of log-concave and symmetric densities, which includes the uniform density they assume. Hence, our proofs are based on general properties of log-concave and

[13] Jeon and Menicucci (2006, 2012) study competitive bundling and derive implications on entry barriers. They consider a common agency setting under complete information in which firms sell portfolios of digital products and find that in the absence of the buyer's budget constraint, all equilibria under bundling are efficient and each firm obtains a profit equal to the social marginal contribution of its portfolio (and hence bundling does not build any entry barrier). However, in the presence of the budget constraint, bundling builds an entry barrier since it allows firms with large portfolios to capture all the budget.

symmetric densities while their proofs are based on direct computations. Third, we give a unified intuition based on the demand size effect and the demand elasticity effect that explains the finding of Matutes and Régibeau (1988) as a special case, while their intuition is mainly based on the effects identified by Matutes and Régibeau; in particular, they do not mention the demand size effect (or its equivalent). Last, we provide various extensions that identify conditions under which credible leverage of dominance into a dominated market occurs.[14]

More recent research has addressed some of our questions in settings with symmetric firms. Precisely, Kim and Choi (2015) extend Matutes and Regibeau (1988) by allowing for $n \geq 3$ symmetric firms, and assuming that competition occurs on a torus (i.e., the Cartesian product of two circumferences) rather than on Hotelling segments. They prove that if $n \geq 4$, then there exists a way to symmetrically locate the firms on the torus which results in bundling generating higher profits than independent pricing. Zhou (2015) studies a model with $m \geq 2$ symmetric products and $n \geq 2$ symmetric firms, but instead of adopting a spatial competition model he uses the random utility framework of Perloff and Salop (1985). He proves that if the number of firms is sufficiently large, then bundling increases the firms' profits with respect to independent pricing. For both papers, the results are a consequence of a demand elasticity effect, which makes the demand under bundling less elastic and thus softens competition, as it occurs in our model for large dominance.

The paper is organized as follows. We present our model in Section 2, analyze how bundling affects competition between two generalist firms in Section 3. Section 4 extends it to competition between a generalist and n specialists. Section 5 identifies conditions for credible leverage of dominance into a dominated market. We conclude in Section 6. The proofs of the results in Sections 3 and 4 are gathered in Appendix A and the rest in on-line Appendix B.

2 Model

We consider competition between two multi-product (generalist) firms A and B, each producing $n > 1$ different products, to address the question of whether bundling helps a firm to

[14] Further differences are that we analyze mixed bundling and consider $n > 2$ products. Our work was done independently from Hahn and Kim (2012).

increase its static profit and to foreclose the other firm. Let ij denote product j produced by firm i, for i =A,B and $j = 1, \ldots, n$. Each consumer has unit demand for each product j.

We consider a model of both vertical and horizontal differentiation. Regarding the latter, let $s_j \in [0, 1]$ denote a generic consumer's location in terms of product j. For each product, firm A is located at $y_A = 0$ and firm B is located at $y_B = 1$ on a Hotelling segment. The gross utility that a consumer with location s_j obtains from consuming product ij is given by $v_{ij} - t_j|s_j - y_i|$, where $v_{ij} > 0$ is the same for all consumers, $t_j > 0$ is the usual (product specific) transportation cost parameter and $|s_j - y_i|$ denotes the distance between the consumer's and firm i's location. Utility is assumed to be additive over different products. We assume that v_{ij} is sufficiently high so that every consumer consumes one of the two competing products in market j for any $j = 1, \ldots, n$. Hence, the products can be interpreted not only as products that can be independently consumed, but also as complements.[15]

For each firm i, let c_{ij} denote its marginal cost of production for product ij. We assume that c_{Aj} and c_{Bj} are large enough that each consumer buys at most one unit of product j, for each j.A crucial role is played by the difference in social surplus $\alpha_j = (v_{Aj} - c_{Aj}) - (v_{Bj} - c_{Bj})$, representing a combination of vertical differentiation and cost asymmetry. We say that firm A is dominant in market j when $\alpha_j > 0$, and α_j is A's advantage or dominance level. Without loss of generality, we simplify notation by setting $c_{Aj} = c_{Bj} = 0$ for all j and interpret equilibrium prices as profit margins.

We assume that $s_1, ..., s_n$ are identically and independently distributed, each with support $[0, 1]$, c.d.f. F and p.d.f. f such that $f(s) > 0$ for all $s \in (0, 1)$. Moreover, we assume that f is differentiable, symmetric around 1/2, and log-concave, i.e., $\log(f)$ is a concave function. This implies that f is weakly increasing on $[0, 1/2]$ and weakly decreasing on $[1/2, 1]$. It also implies that $\log(F)$ and $\log(1 - F)$ are both concave, so that both $-F/f$ and $(1 - F)/f$ are decreasing.[16] For technical and expositional reasons we assume further that f is analytic on [0,1].

Let p_{ij} be the price charged by firm i for product ij under independent pricing. Under bundling, P_i denotes the price charged by firm i for the bundle of its products. We study the two following games of simultaneous pricing played by firms A and B:

[15] In the case of complements, one firm's decision to bundle its products has the same effect as the firm's choice to make its products incompatible with the products of the other firm. This interpretation is followed in Matutes and Régibeau (1988).

[16] See for example Bagnoli and Bergstrom (2005).

- Game of independent pricing [**IP**]: firm A chooses p_{Aj} and firm B chooses p_{Bj} for all $j = 1, \ldots, n$.
- Game of pure bundling [**PB**]: firm A chooses P_A for its bundle of n products and firm B chooses P_B for its bundle of n products.

In section A.7 of the appendix, we consider a third game in which $n = 2$, $\alpha_1 = \alpha_2$ and firms can use mixed bundling. We show that when firm A has a sufficiently large advantage, the equilibrium outcome of mixed bundling is the same as that of pure bundling.[17]

In Section 3, we study our baseline model of $\alpha_1 = ... = \alpha_n > 0$ and compare the two games. In addition, the analysis of Section 3 allows us to study the following three-stage game of entry:

- Stage one: firm B chooses between entering or not.
- Stage two: if B has entered, each firm chooses between IP and PB.
- Stage three: firms compete in the game determined by their choices at stage two.

Notice that if in stage two at least one firm has chosen PB, then competition in stage three occurs between the two pure bundles.[18] Therefore competition in independent prices occurs if and only if both firms have chosen IP. Also, there always exists an equilibrium in which both firms choose PB at stage two, but it may involve playing a weakly dominated strategy. We impose that firms do not play weakly dominated strategies, therefore (IP, IP) is the outcome only if this is the preferred outcome for both firms.

3 Competition between generalists: symmetric markets

In this section, we consider the baseline model of competition between firm A and firm B where all n markets are symmetric: firm A has a symmetric dominance (i.e., $\alpha = \alpha_1 = ... = \alpha_n > 0$) and the transportation cost is symmetric (i.e., $t = t_1 = ... = t_n > 0$). After

[17]For a similar result under monopolistic bundling, see Menicucci *et al.* (2015).

[18]Indeed, suppose that firm A, for instance, has chosen PB and firm B has chosen IP. Then each consumer either buys the pure bundle of A or the n products of firm B, which are therefore viewed as a bundle.

studying the game of independent pricing and that of pure bundling, we compare the two. We consider asymmetric dominance levels and transportation costs in Section 5.

3.1 Preliminaries: log-concave density

If $s_1, \ldots, s_n$ are $n > 1$ random variables that are i.i.d., each with support $[0,1]$ and with log-concave density function f, then their average, $\tilde{s} = (s_1 + \ldots + s_n)/n$, is distributed with density function f_n which is also log-concave (see An, 1998, Cor. 1). If, moreover, f is symmetric around 1/2 (that is, $f(s) = f(1-s)$ for each $s \in [0,1]$), then the distribution of the average is symmetric and strictly more-peaked around 1/2 (the mean) than that of each original variable (Proschan, 1965).[19] [20] That is, for any $z \in (0, 1/2)$,

$$\int_z^{1-z} f(s)ds < \int_z^{1-z} f_n(s)ds.$$

The c.d.f. of the average is obtained recursively, starting from $F_1(x) = F(x)$ and using $F_k(x) = \int_{-\infty}^{\infty} f(s) F_{k-1}\left(\frac{kx-s}{k-1}\right) ds$ for $k = 2, ..., n$, until

$$F_n(x) = \int_{-\infty}^{\infty} f(s) F_{n-1}\left(\frac{nx-s}{n-1}\right) ds.$$

The p.d.f. of the average is $f_n(x) = F_n'(x)$, hence

$$f_n(x) = \int_{-\infty}^{\infty} \frac{n}{n-1} f(s) f_{n-1}\left(\frac{nx-s}{n-1}\right) ds.$$

Note that the more-peakedness of the distribution of the average can be equivalently expressed as $F_n(x) > F(x)$ for all $x \in (1/2, 1)$. In fact, we have $F_k(x) > F_{k-1}(x)$ for all $x \in (1/2, 1)$, for $k = 2, ..., n$.

For later reference we point out two properties of the density functions of the individual and the average locations. We will use them later when comparing the outcomes of independent pricing with pure bundling.

Lemma 1. *Let f be a log-concave density function which is symmetric around 1/2 with support $[0,1]$. Let f_n be the density function of the average of $n(> 1)$ independently and*

[19] In fact, for all $t_1, \ldots, t_n > 0$, the weighted average $(\sum_{j=1}^n t_j s_j)/\sum_{j=1}^n t_j$ is distributed with a log-concave density function that is more-peaked around the mean than that of the original variable.

[20] Observe that for density functions that are not log-concave, the average is not necessarily more-peaked than the original distribution (see for instance the Cauchy distribution). This explains our restriction to log-concave densities.

identically distributed random variables that are distributed according to the density function f. Then we have:

(i) $f_n(1/2) > f(1/2)$;

(ii) $\lim_{s \uparrow 1} \frac{f_n(s)}{f(s)} = 0$.

3.2 Independent Pricing

When firms compete in independent prices, we can consider each market in isolation. Moreover, since markets are symmetric, we suppress the index to product j and restrict attention to price competition on the Hotelling segment where consumers are distributed with density f and bear transportation cost t per unit of distance. Recall that firm A (B) is located at 0 (1) and firm A offers a product that is valued α higher than that of firm B. Hence, given prices p_A and p_B, the indifferent consumer is located at[21]

$$x(\alpha, p_A, p_B) = \frac{1}{2} + \sigma(\alpha - p_A + p_B), \tag{1}$$

where $\sigma = 1/(2t)$ (unless, of course, p_A and p_B are such that every type of consumer prefers A, or every type prefers B). For simplicity, we will often suppress the arguments and simply write x for the location of the indifferent consumer.

We suppose for now that the distribution and the parameters are such that independent pricing leads to an interior equilibrium, i.e., both firms obtain positive market share.[22] Then the first-order conditions must be satisfied at the equilibrium prices. Since marginal costs are assumed to be zero, the profit functions are

$$\pi_A = p_A F(x), \qquad \pi_B = p_B(1 - F(x)),$$

and the first-order conditions are[23]

$$0 = F(x) - \sigma p_A f(x), \qquad 0 = 1 - F(x) - \sigma p_B f(x).$$

[21] As the distribution of locations is atomless, the way this indifference is broken does not affect the results.

[22] Proposition 1 characterizes when an interior equilibrium exists.

[23] Notice that $\frac{d\pi_A}{dp_A} = 0$ suffices to maximize π_A because $\frac{d\pi_A}{dp_A} = F(x)[1 - \sigma p_A \frac{f(x)}{F(x)}]$, and because log-concavity of F implies that $\frac{f(x)}{F(x)}$ is decreasing in x, and thus increasing in p_A. Hence, if p_A^* solves the first-order condition, then $\frac{d\pi_A}{dp_A} < 0$ for $p_A > p_A^*$ and $\frac{d\pi_A}{dp_A} > 0$ for $p_A < p_A^*$. A similar argument reveals that $\frac{d\pi_B}{dp_B} = 0$ suffices to maximize π_B with respect to p_B.

If p_A^*, p_B^* are the equilibrium prices and x^* denotes the equilibrium location of the indifferent consumer, then we have

$$x^* = \frac{1}{2} + \sigma\alpha - \sigma(p_A^* - p_B^*) = \frac{1}{2} + \sigma\alpha + \frac{1 - 2F(x^*)}{f(x^*)}.$$

Hence, the equilibrium location of the indifferent consumer is a fixed point of the mapping:

$$X^\alpha : x \mapsto \frac{1}{2} + \sigma\alpha + \frac{1 - 2F(x)}{f(x)}. \tag{2}$$

Notice that $\frac{1-2F(x)}{f(x)} = -\frac{F(x)}{f(x)} + \frac{1-F(x)}{f(x)}$. As we mentioned in Section 2, log-concavity of f implies that both $-\frac{F(x)}{f(x)}$ and $\frac{1-F(x)}{f(x)}$ are decreasing. Hence, X^α is weakly decreasing, $X^\alpha(1/2) > 1/2$, and a unique fixed point $x^* < 1$ exists, provided that $\lim_{x \to 1} X^\alpha(x) < 1$. The equilibrium prices are then also unique. Clearly, at $\alpha = 0$ we have $x^* = 1/2$ and Proposition 1(i) establishes that x^* is increasing in α, hence $x^* > 1/2$ for $\alpha > 0$. If α is sufficiently large and $f(1) > 0$, then $x^* = 1$.

Proposition 1 (Independent Pricing)**.** *(i) Suppose that $(\sigma\alpha - 1/2)f(1) < 1$. Then the independent pricing game has a unique and interior equilibrium, characterized by the unique fixed point $x^*(\alpha)$ of X^α, and $x^*(\alpha) \in [\frac{1}{2}, 1)$. The function $x^*(\alpha)$ is increasing and concave for $\alpha \geq 0$. The equilibrium prices (in each market) are*

$$p_A^*(\alpha) = \frac{F(x^*(\alpha))}{\sigma f(x^*(\alpha))}, \qquad p_B^*(\alpha) = \frac{1 - F(x^*(\alpha))}{\sigma f(x^*(\alpha))}.$$

The equilibrium profits (in each market) are

$$\pi_A^*(\alpha) = \frac{F(x^*(\alpha))^2}{\sigma f(x^*(\alpha))}, \qquad \pi_B^*(\alpha) = \frac{(1 - F(x^*(\alpha)))^2}{\sigma f(x^*(\alpha))}.$$

p_A^* and π_A^* are increasing in α, while p_B^* and π_B^* are decreasing in α.

(ii) Suppose that $(\sigma\alpha - 1/2)f(1) \geq 1$. Then the independent pricing game has a unique equilibrium, and it is such that firm A's market share is 1. The equilibrium prices and profits (in each market) are

$$p_A^*(\alpha) = \pi_A^*(\alpha) = \alpha - 1/(2\sigma), \qquad p_B^*(\alpha) = \pi_B^*(\alpha) = 0.$$

3.3 Pure Bundling

The analysis performed for independent pricing straightforwardly extends to the case in which firms A and B compete under bundling. Given P_A, P_B chosen by the firms, let $p_A =$

P_A/n, $p_B = P_B/n$ and notice that our analysis does not require $\alpha_1 = \ldots = \alpha_n$, as it applies even though the alphas are different but α below is equal to $(\alpha_1 + \ldots + \alpha_n)/n$. Let x_n denote the average location of the indifferent consumer. Hence, consumers with average location $\tilde{s} < x_n$ (respectively, $\tilde{s} > x_n$) will buy the bundle from firm A (respectively, firm B). Precisely, x_n is given by:

$$x_n = \frac{1}{2} + \sigma\left(\alpha - p_A + p_B\right).$$

The equilibrium bundle prices are found in a way very similar to the analysis of independent pricing, since the game under bundling can be considered as a competition between two firms each offering one product – in fact, a bundle. The only difference is that the underlying density function is the density f_n of the average location, and not the density f of the individual location. Let us thus define

$$X_n^\alpha : x \mapsto \frac{1}{2} + \sigma\alpha + \frac{1 - 2F_n(x)}{f_n(x)}. \tag{3}$$

Since f_n is log-concave, we obtain (as above) that X_n^α is decreasing in x. Since $\lim_{x \to 1} f_n(x) = 0$, X_n^α always admits a unique fixed point $x_n^*(\alpha) < 1$, and equilibrium prices and profits can be expressed in terms of this fixed point. Hence, under pure bundling we always obtain a unique and interior equilibrium in which both firms have positive market share.[24]

Proposition 2 (Pure Bundling)**.** *The pure bundling pricing game has a unique equilibrium, characterized by the unique fixed point $x_n^*(\alpha)$ of X_n^α, and $x_n^*(\alpha) \in [\frac{1}{2}, 1)$. The function $x_n^*(\alpha)$ is increasing and concave for $\alpha \geq 0$. The equilibrium bundle prices are*

$$P_{n,A}^*(\alpha) = \frac{nF_n(x_n^*(\alpha))}{\sigma f_n(x_n^*(\alpha))}, \qquad P_{n,B}^*(\alpha) = \frac{n(1 - F_n(x_n^*(\alpha)))}{\sigma f_n(x_n^*(\alpha))}.$$

The total equilibrium profits are

$$\Pi_{n,A}^*(\alpha) = \frac{nF_n(x_n^*(\alpha))^2}{\sigma f_n(x_n^*(\alpha))}, \qquad \Pi_{n,B}^*(\alpha) = \frac{n(1 - F_n(x_n^*(\alpha)))^2}{\sigma f_n(x_n^*(\alpha))}.$$

$P_{n,A}^*$ and $\Pi_{n,A}^*$ are increasing in α while $P_{n,B}^*$ and $\Pi_{n,B}^*$ are decreasing in α.

[24] We find at work here the same principle which makes it optimal to exclude some consumers for a multi-product monopolist: see Armstrong (1996).

3.4 Independent pricing vs. pure bundling

We now study how bundling affects each firm's profit in comparison to independent pricing. We find that the impact of bundling on profits depends crucially on the level of dominance that firm A has over firm B. We will show that for low levels of dominance both firms are hurt by bundling, for high levels of dominance both firms gain from bundling, and only the dominant firm gains from bundling for intermediate levels of dominance. The impact of bundling can be decomposed into two effects, the demand size effect and the demand elasticity effect. We first provide an intuition, based on these two effects, of our main result and then formally establish the result.

3.4.1 Heuristic Intuition

Demand size effect. Suppose that firms A and B sell two products (i.e., $n = 2$) independently and set prices (for each of them) p_A and p_B. Suppose furthermore that at these prices firm A has a market share larger than one half, that is the indifferent consumer in each market is located at $x > 1/2$. The demand for each product of firm A equals $F(x)$. Now assume that both firms bundle their two products at a price equal to the sum of the previous prices, that is $P_A = 2p_A$ and $P_B = 2p_B$. Then the indifferent consumer is the consumer whose average location is equal to x. The demand for A's bundle is $F_2(x)$. Since the distribution of the average location is more-peaked around 1/2 than the distribution of the individual location for any symmetric log-concave density f, we have $F_2(x) > F(x)$ for each $x \in (1/2, 1)$. Hence, the demand after bundling, for given prices, increases for the dominant firm A and decreases for the dominated firm B, unless firm A covers the whole market to start with (*i.e.*, when $x = 1$). (See right panel of Figure 2.) In particular, these arguments apply when p_A and p_B are equal, respectively, to p_A^* and p_B^*, their equilibrium values under independent pricing, since then $x^* > 1/2$ for $\alpha > 0$.

Demand elasticity effect. After bundling, firms will have incentives to change their prices away from $P_A = 2p_A^*$ and $P_B = 2p_B^*$. Whether they want to charge higher or lower prices depends on how bundling affects demand elasticity, which in turn depends on the location of the marginal consumer and hence on the level of A's dominance. From Figure 2, for low levels of dominance (that is, when x^* is not much larger than 1/2), bundling makes the demand more elastic: a given decrease in the average price of a bundle generates a higher boost in demand than the same decrease in the prices of single products because the distribution of the average location is more-peaked around 1/2 than the distribution of individual locations. On the other hand, for high levels of dominance (that is, when x^* is close to 1), bundling makes the demand less elastic: because $f_2(x)/f(x)$ converges to zero as x goes to one, for x close enough to one, a given decrease in the average price of a bundle generates a smaller boost in demand than the same decrease in the prices of single products. In summary, bundling changes the elasticity of demand such that firms compete more aggressively for low levels of dominance, but less aggressively for high levels of dominance. Figure 1 illustrates this for the case of the uniform distribution.

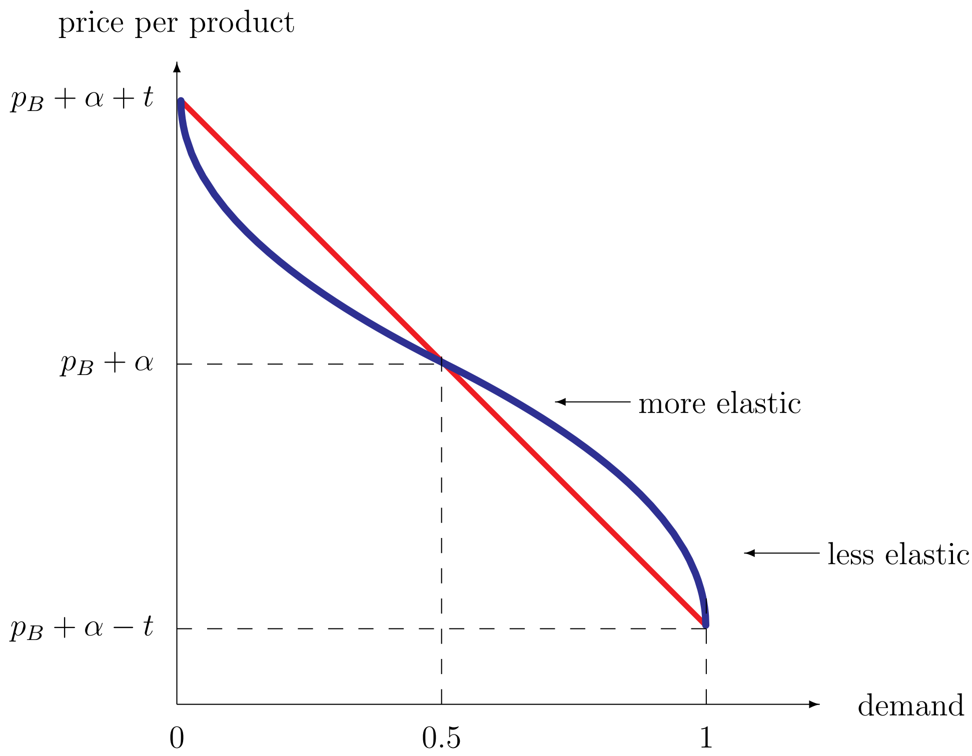

Figure 1: Demand for each of A's products when sold independently (thinner red line) and sold as bundle (thicker blue line), for given price per product p_B of firm B.

More precisely, at equilibrium prices for each firm the elasticity of demand of the own product with respect to the own price must be equal to one. For instance, at the independent pricing equilibrium, the elasticity of demand for firm A equals

$$\varepsilon_A = -\frac{D'(p)p}{D(p)} = \frac{\sigma f(x^*)p_A^*}{F(x^*)} = 1.$$

When firms A and B bundle their products without changing prices, elasticity equals

$$\tilde{\varepsilon}_A = \frac{\sigma f_2(x^*)p_A^*}{F_2(x^*)} = \frac{f_2(x^*)F(x^*)}{f(x^*)F_2(x^*)}.$$

For relatively low levels of dominance (that is, x^* slightly larger than 1/2), this elasticity is strictly larger than 1 because $F(1/2) = F_2(1/2)$ and $f_2(1/2) > f(1/2)$ from Lemma 1(i). This implies that firm A has strong incentives to lower its price (per product). On the other hand, if firm A has large dominance, then $F(x^*)$ and $F_2(x^*)$ are both close to 1 but $f_2(x^*)/f(x^*)$ is close to zero (by Lemma 1(ii)), implying that $\tilde{\varepsilon}_A$ is (much) smaller than 1. Hence firm A will want to increase its price. Similar arguments show that firm B has an incentive to lower (increase) its price for low (high) levels of dominance.

Combining the two effects. For very low levels of dominance, the demand elasticity effect dominates the demand size effect because the latter is negligible, implying that bundling reduces each firm's profit. As the level of A's dominance increases, the competition-strengthening demand elasticity effect becomes weaker such that there is a first cut-off level of dominance above which bundling increases A's profit due to the demand size effect. At this cut-off level, bundling still reduces B's profit because B suffers from both the demand size effect and the demand elasticity effect. As the level of A's dominance further increases, the demand elasticity effects starts to soften competition and the demand size effect is weak for large α. Hence it is quite intuitive that there is a second cut-off level of dominance (which is larger than the first cut-off) above which bundling increases each firm's profit.

3.4.2 Analysis

Let us first consider the extreme case where firm A has no advantage over firm B, that is, $\alpha = 0$. In this case, there is no demand size effect and each obtains half of the market both with independent pricing and with bundling. However, because of the competition-strengthening demand elasticity effect, the equilibrium bundle prices are strictly lower than the sum of the equilibrium prices under independent pricing. Precisely, because $f(1/2) < f_n(1/2)$ and $F(1/2) = F_n(1/2)$ hold, we have

$$np_A^* = \frac{nF(1/2)}{\sigma f(1/2)} > \frac{nF_n(1/2)}{\sigma f_n(1/2)} = P_{n,A}^*.$$

Likewise $np_B^* > P_{n,B}^*$. Bundling thus hurts both firms when firms are symmetric.

By continuity, bundling still hurts both firms for low but positive values of α, despite the fact that firm A benefits from the demand size effect. In order to see that firm A indeed obtains strictly higher market share under bundling, observe that

$$\frac{d}{d\alpha}F(x^*(\alpha)) = f(x^*(\alpha))\frac{dx^*}{d\alpha},$$

while

$$\frac{d}{d\alpha}F_n(x_n^*(\alpha)) = f_n(x_n^*(\alpha))\frac{dx_n^*}{d\alpha}.$$

Evaluating these expressions at $\alpha = 0$ yields $f(1/2)\sigma/3$ and $f_n(1/2)\sigma/3$, respectively (see the proof of Proposition 1 in the appendix). The latter expression is strictly higher than the former, so that we indeed conclude that $F_n(x_n^*(\alpha)) > F(x^*(\alpha))$ for small $\alpha > 0$.

Now consider the opposite extreme case where firm A has such a huge advantage that firm B obtains higher market share under bundling than under independent pricing. It is immediate to see that such levels of dominance exist when $f(1) > 0$[25] because then firm B has zero market share under independent pricing for high levels of dominance (Proposition 1(ii)), while it obtains always a positive market share under bundling (Proposition 2). We show in Lemma 2 below that when firm B obtains higher market share under bundling, both firms benefit from bundling as it increases the profits of both of them.

To analyze the case of intermediate levels of α, we introduce some notation for the sets of dominance levels where firm A's and B's market share and total profit are higher under bundling than under independent pricing. After defining these sets, we establish some inclusion relations between them.

Definition 1. **(i)** $\mathcal{A}_{MS}^+ = \{\alpha \geq 0 \ : \ F_n(x_n^*(\alpha)) \geq F(x^*(\alpha))\}$

(ii) $\mathcal{A}_{\pi A}^+ = \{\alpha \geq 0 \ : \ \Pi_{n,A}^*(\alpha) \geq n\pi_A^*(\alpha)\}$

(iii) $\mathcal{A}_{\pi B}^+ = \{\alpha \geq 0 \ : \ \Pi_{n,B}^*(\alpha) \geq n\pi_B^*(\alpha)\}$

Let furthermore $\mathcal{A}_K^- = [0,\infty) \setminus \mathcal{A}_K^+$ *for* $K \in \{MS, \pi A, \pi B\}$

Lemma 2. *We have the following strict superset relations between the various dominance level sets:*

$$\mathcal{A}_{\pi A}^+ \supset \mathcal{A}_{\pi B}^+ \supset \mathcal{A}_{MS}^-.$$

[25] But in fact, we prove their existence also if $f(1) = 0$.

In words, the lemma says that if firm B obtains a strictly higher market share under bundling, then it benefits from bundling and that whenever firm B benefits from bundling, so does firm A. Moreover, there are levels of dominance for which firm B benefits from bundling despite obtaining lower market share and there are levels of dominance for which only firm A benefits from bundling. This complements our earlier observation that bundling hurts both firms for low levels of dominance.

An immediate consequence of this lemma is that there are three regions of dominance levels with distinct effects of bundling on the firms' profits. Namely, for $\alpha \in \mathcal{A}^-_{\pi A}$, both firms are hurt by bundling, and $\mathcal{A}^-_{\pi A}$ includes values of α close to zero. For $\alpha \in \mathcal{A}^+_{\pi B}$, both firms benefit from bundling and $\mathcal{A}^+_{\pi B}$ includes large values of α. Finally, for α in the intermediate region $\mathcal{A}^+_{\pi A} \cap \mathcal{A}^-_{\pi B}$, only firm A benefits from bundling ($\mathcal{A}^+_{\pi A} \cap \mathcal{A}^-_{\pi B} \neq \varnothing$ by Lemma 2). We conjecture that these sets are convex but have been unable to prove it for general log-concave probability density functions f and number of products n.[26] We therefore summarize our result as follows.

Proposition 3 (Independent pricing vs. bundling)**.**

(i) There exist threshold levels $0 < \underline{\alpha} \leq \alpha_{\pi A} < \alpha_{\pi B} \leq \bar{\alpha}$ *such that bundling strictly benefits firm A and hurts firm B when* $\alpha \in (\alpha_{\pi A}, \alpha_{\pi B})$*, strictly hurts both firms when* $\alpha \in [0, \underline{\alpha})$ *and strictly benefits both firms when* $\alpha > \bar{\alpha}$*.*

(ii) If dominance level sets $\mathcal{A}^-_{\pi A}$ *and* $\mathcal{A}^-_{\pi B}$ *are convex, then* $\underline{\alpha} = \alpha_{\pi A}$ *and* $\alpha_{\pi B} = \bar{\alpha}$ *so that (a) profits of firm A are strictly higher under bundling if and only if* $\alpha > \alpha_{\pi A}$ *and (b) profits of firm B are strictly higher under bundling if and only if* $\alpha > \alpha_{\pi B}$*.*

Illustration through uniformly distributed locations In the special case of the uniform distribution, we have $f(x) = 1$, $F(x) = x$ and, for $x \geq 1/2$, $f_2(x) = 4(1 - x)$, $F_2(x) = 1 - 2(1 - x)^2$. These functions are depicted in Figure 2, clearly illustrating the more-peakedness of F_2. Fixing $t = 1$, we can get explicit expressions for equilibrium prices and profits under independent and bundling pricing when $n = 2$ by substituting $x^* = (3 + \alpha)/6$ and $x_2^* = \left(7 + \alpha - \sqrt{9 - 2\alpha + \alpha^2}\right)/8$ into the expressions of Propositions 1 and 2. It is easily verified (numerically) that the inequality $\Pi^*_{2,A}(\alpha) > 2\pi^*_A(\alpha)$ holds if and only if $\alpha > 1.415$. The inequality $\Pi^*_{2,B}(\alpha) > 2\pi^*_B(\alpha)$ holds if and only if $\alpha > 2.376$.

[26] We can numerically show this to be true for the uniform distribution and $2 \leq n \leq 32$ and for symmetric beta-distributions $f(s) = (s(1 - s))^\beta$ for integers $1 \leq \beta \leq 10$ and $n = 2$.

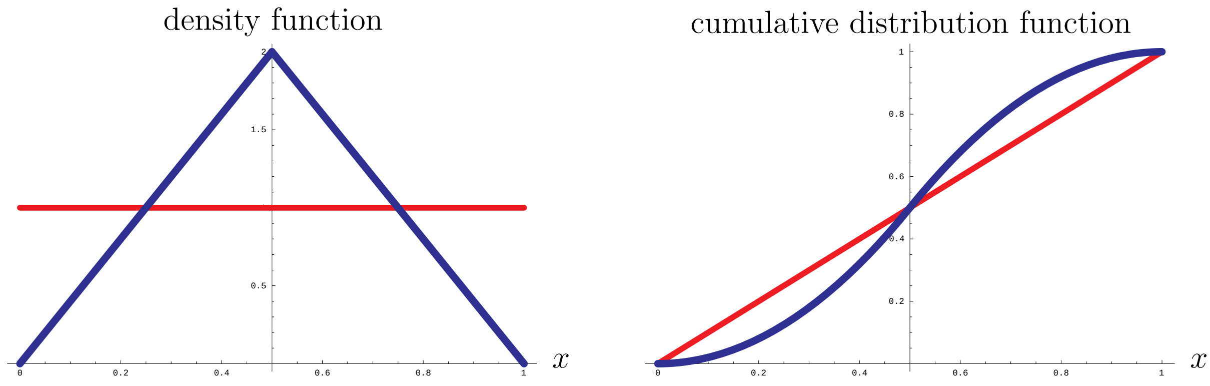

Figure 2: Density and distribution functions for the uniform case. The thicker (blue) graphs correspond to average valuations (bundling).

Let $\underline{\alpha}_n$ (respectively, $\bar{\alpha}_n$) denote the cutoff value for which firm A (respectively B) is indifferent between independent pricing and pure bundling when there are n products. Then $\underline{\alpha}_2 = 1.41$, $\underline{\alpha}_3 = 1.42$, $\underline{\alpha}_4 = 1.39$, $\underline{\alpha}_8 = 1.29$, $\underline{\alpha}_{16} = 1.19$ and $\bar{\alpha}_2 = 2.38$, $\bar{\alpha}_3 = 2.54$, $\bar{\alpha}_4 = 2.64$, $\bar{\alpha}_8 = 2.77$, $\bar{\alpha}_{16} = 2.85$. Bundling more and more products makes the region of credible bundling wider, which plays a role for the result described in Proposition 7.[27]

3.5 Entry deterrence

We are now in a position to solve the three-stage game outlined in section 2 by backward induction. In the last stage firms choose the equilibrium prices corresponding to the bundling game whenever at least one firm has chosen PB. This is so because if, for example, firm A chooses PB and firm B chooses IP, then effectively competition will be in bundles, and equilibrium prices will be $P^*_{n,A}$ for A's bundle, and $P^*_{n,B}/n$ for each of B's individual products. Substituting equilibrium payoffs from Stage 3 yields the following game to be played in Stage 2.

	PB	IP
PB	$\Pi^*_{n,A}, \Pi^*_{n,B}$	$\Pi^*_{n,A}, \Pi^*_{n,B}$
IP	$\Pi^*_{n,A}, \Pi^*_{n,B}$	$n\pi^*_A, n\pi^*_B$

Of course, in this reduced form game, both firms have a weakly dominant strategy. It is a weakly dominant strategy for firm A to choose PB when $\alpha \in \mathcal{A}^+_{\pi A}$ and to choose IP

[27]However, note that $\underline{\alpha}_n$ is not monotonic at $n = 3$ as we have $\underline{\alpha}_3 > \underline{\alpha}_2 > \underline{\alpha}_4$.

otherwise.Similarly, it is a weakly dominant strategy for firm B to choose PB when $\alpha \in \mathcal{A}^+_{\pi B}$ and to choose IP otherwise.

The bundling outcome will prevail when $\alpha \in \mathcal{A}^+_{\pi A}$ and it will then be enforced by firm A. For very low levels of dominance ($\alpha \in \mathcal{A}^-_{\pi A}$), firm A may be tempted to threaten to use bundling in order to deter entry, as it would lower firm B's profit. However, such a threat is not credible because firm A would choose independent pricing once B has entered. For (very high) levels of dominance for which bundling is profitable for firm B, firm A cannot use bundling to deter entry. In fact, in order to deter entry, firm A would need to threaten to use IP. Not only is this not credible, but firm B can in fact force bundling by choosing PB unilaterally. Only for intermediate levels of dominance, bundling is profitable for A (and thus credible) and hurts firm B. Hence, bundling can then be used as a foreclosure strategy if it reduces B's profit below B's entry cost.

Assuming that each firm will use its weakly dominant strategy in Stage 2, firm B will enter (by incurring the fixed cost of entry $K > 0$) when $n\pi^*_B - K > 0$ for $\alpha \in \mathcal{A}^-_{\pi A}$ and also when $\Pi^*_{n,B} - K > 0$ for $\alpha \in \mathcal{A}^+_{\pi A}$. Otherwise firm B will stay out.

3.6 Social welfare

In this subsection we study and compare static social welfare (defined as the sum of producers' and consumers' surplus) under independent pricing and bundling, under the assumption that $f(1) > 0$. We find that the effects of bundling are non-monotonic in the dominance level. Recall from Propositions 1 and 2 that the market share is always interior under bundling while, under independent pricing, firm A's share becomes one when $(\sigma\alpha - 1/2)f(1) \geq 1$. We have:

Proposition 4 (Welfare)**.** *(i) Both under independent pricing and under bundling, the market share of firm A is too low from the point of view of social welfare for any $\alpha > 0$ as long as it is interior.*

(ii) Suppose that $f(1) > 0$. Then bundling reduces social welfare both for $\alpha (\geq 0)$ small enough and for α (≥ 0) large enough (in particular, when $(\sigma\alpha - 1/2)f(1) \geq 1$). For intermediate values of α, bundling may increase or reduce social welfare.

In order to understand how Proposition 4(i) is obtained, notice that in the market for a single product j, and for a given location x of the indifferent consumer, social welfare under independent pricing is given by $W(x) = \alpha F(x) - T(x)$ (omitting v_{Bj}) where $T(x) = t\int_0^x zf(z)dz + t\int_x^1 (1-z)f(z)dz$ is the total transportation cost incurred by all consumers in that market, which is increasing for $x > 1/2$. Likewise, given the average location x of the indifferent consumer, social welfare under bundling is given by $W_n(x) = \alpha F_n(x) - T_n(x)$ where $T_n(x) = t\int_0^x zf_n(z)dz + t\int_x^1 (1-z)f_n(z)dz$. Both W and W_n are maximized by the same location $x_w = \min\{\frac{1}{2} + \sigma\alpha, 1\}$. However, in equilibrium the dominant firm is not aggressive enough, because $x^*(\alpha) < x_w$ and $x_n^*(\alpha) < x_w$ for any $\alpha > 0$,[28] except when $(\sigma\alpha - 1/2)f(1) \geq 1$ (in which case $x^*(\alpha) = x_w = 1$ under independent pricing).

From the formulations of W and W_n, we infer that bundling increases the first term in welfare if and only if it increases A's market share, *i.e.* if and only if $F_n(x_n^*(\alpha)) > F(x^*(\alpha))$. This inequality holds for α not too large (see the proof of Lemma 2). For the second term in W and W_n, note that $T(x) < T_n(x)$ holds for any $x \in [\frac{1}{2}, 1)$ since consumers cannot freely consume their preferred combinations of products under bundling. For small values of α, the difference $\alpha F_n(x_n^*(\alpha)) - \alpha F(x^*(\alpha))$ is positive but small. Hence, the negative effect of bundling on transportation costs dominates and we have $W(x^*(\alpha)) > W_n(x_n^*(\alpha))$. For large values of α, we have already noticed that $x^*(\alpha) = x_w = 1$ and hence again $W(x^*(\alpha)) > W_n(x_n^*(\alpha))$. However, for some intermediate values of α bundling may significantly increase the market share of firm A, such that $W_n(x_n^*(\alpha)) > W(x^*(\alpha))$ even though $T_n(x_n^*(\alpha)) > T(x^*(\alpha))$. For instance, for the uniform distribution, $W_2(x_2^*(\alpha)) > W(x^*(\alpha))$ if and only if $\alpha \in (1.071t, 2.306t)$. Our comparison above has been static in the sense that we have considered a given duopolistic market structure. However, we know from Subsection 3.5 that bundling may help firm A to erect an entry barrier against firm B. For instance, for the uniform distribution and $n = 2$, bundling is credible and reduces B's profit for $\alpha \in (1.415t, 2.376t)$, which largely overlaps with the interval for which bundling increases static welfare. Therefore, one should be very cautious in generating policy implications on bundling from static welfare analysis.

[28]This occurs because the function X^α introduced in (2) is such that $X^\alpha(\frac{1}{2} + \sigma\alpha) < \frac{1}{2} + \sigma\alpha$. A similar remark applies to X_n^α introduced in (3).

3.7 Extension to correlated valuations

We briefly discuss below how our results extend to the case of correlation in tastes. To introduce positive correlation in tastes,[29] suppose that a fraction $\rho \in (0, 1]$ of consumers have perfectly correlated locations, while the rest have locations independently and identically distributed as above. Given (ρ, ρ') satisfying $1 \geq \rho > \rho' > 0$, the distribution of the average location for ρ is less peaked than that of the average location for ρ'. Therefore, a greater positive correlation weakens both the demand size effect and the demand elasticity effect but the two effects still exist except for the limit case of perfect correlation. In the latter case, the distribution of the average location is identical to that of each individual location, which implies that bundling has no effect. In the case of the uniform distribution with $n = 2$, we find that the two threshold values of α decrease with $\rho \in (0, 1)$.

4 Competing against specialists

Up to now, we have considered competition between two generalists. In this section, we consider competition between a generalist firm A and specialists B1, .., Bn where Bj ($j = 1, 2, ..., n$) denotes the firm specialized in product j. For instance, in the case of GE-Honeywell merger, the merged firm would compete against engine specialists and avionics specialists. In the absence of bundling, competition in market j between A and Bj occurs as we have described in Section 3.2.

Under pure bundling of A's products, it is as if the specialists offer a bundle of their products at the price $\sum_{j=1}^{n} p_{Bj}$ and consumers choose between A's bundle and the bundle of the specialists. As each specialist chooses its price non-cooperatively, bundling creates a Cournot complement problem: for any given price of A's bundle, the specialists choose prices too high relative to the price that would be chosen by a generalist firm B, because each specialist does not internalize the negative externality on the demand for the other specialists when it raises its price. This implies that A's profit under bundling is strictly higher when the rival firms are separate than when they are integrated. However, it is not clear whether the rivals' profits under bundling will be lower than when they are integrated. On the one hand, they set too high prices due to the Cournot complement problem, but on

[29] For instance, Gandal *et al.* (2013) find strong positive correlation of consumer values for spreadsheets and word-processors when they empirically study Microsoft's bundling of the two products.

the other hand, this in turn induces A to charge a higher price because prices are strategic complements.

Firm A produces all products and sells them in a bundle at total price np_A. Firms B1,...,Bn produce one good each, sold at prices $p_{B1}, ..., p_{Bn}$. The indifferent consumer has average location y_n where

$$y_n = \frac{1}{2} + \sigma \left(\alpha - p_A + \frac{1}{n} \sum_{j=1}^{n} p_{Bj} \right), \tag{4}$$

with $\sigma = 1/(2t)$. We denote with y_n^{**} the equilibrium average location of the indifferent consumer, and we can argue as in subsection 3.2 to conclude that y_n^{**} is the unique fixed point of the mapping

$$Y_n^\alpha : y \mapsto \frac{1}{2} + \sigma\alpha + \frac{n - (n+1)F_n(y)}{f_n(y)} \tag{5}$$

and y_n^{**} determines the equilibrium prices and profits as described by next proposition.

Proposition 5 (Specialists). *The pure bundling pricing game against specialists has a unique equilibrium, characterized by the unique fixed point $y_n^{**}(\alpha)$ of Y_n^α, and $y_n^{**}(\alpha) \in (\frac{1}{2}, 1)$. The function $y_n^{**}(\alpha)$ is increasing and concave for $\alpha \geq 0$. The equilibrium prices per product are*

$$p_{n,A}^{**}(\alpha) = \frac{F_n(y_n^{**}(\alpha))}{\sigma f_n(y_n^{**}(\alpha))}, \qquad p_{n,Bj}^{**}(\alpha) = \frac{n(1 - F_n(y^{**}(\alpha)))}{\sigma f_n(x^{**}(\alpha))} \quad \text{for } j = 1, ..., n.$$

The equilibrium profits are

$$\Pi_{n,A}^{**}(\alpha) = \frac{nF_n(y_n^{**}(\alpha))^2}{\sigma f_n(y_n^{**}(\alpha))}, \qquad \Pi_{n,Bj}^{**}(\alpha) = \frac{n(1 - F_n(y_n^{**}(\alpha)))^2}{\sigma f_n(y_n^{**}(\alpha))} \quad \text{for } j = 1, ..., n.$$

$p_{n,A}^{**}$ and $\Pi_{n,A}^{**}$ are increasing in α while $p_{n,B}^{**}$ and $\Pi_{n,Bi}^{**}$ are decreasing in α.

Note that $Y_n^\alpha(y) - X_n^\alpha(y) = (n-1)(1 - F_n(y))/f_n(y) > 0$ for all $y \in (0, 1)$. Hence, $y_n^{**} > x_n^*$, that is the indifferent consumer is further away from firm A when A competes against specialists than when A faces a generalist opponent. This immediately implies the following results.

Corollary 1 (Cournot complement). *In comparison with the competition in bundles against a generalist, bundling by a generalist who competes against specialists,*

(i) leads to higher prices by both the generalist and the specialists and

(ii) yields the generalist higher demand and profit.

Consider the case where consumers are uniformly distributed and there are two products. Recall that with generalists, bundling is profitable for firm A when $\alpha > 1.145t$ while bundling is profitable for firm B when $\alpha > 2.376t$. In the case of specialists, bundling is profitable for firm A when $\alpha > 0.307t$ while it is profitable for each firm Bj when $\alpha > 2.092t$. However, when there are $n = 3$ products, with generalists bundling is profitable for firm A when $\alpha > 1.425$ while bundling is profitable for firm B when $\alpha > 2.541$. In the case of specialists, bundling is profitable for firm A when $\alpha > 0.093$ while it is profitable for each firm Bj when $\alpha > 2.595$. Therefore, when B is separated into specialists, it greatly expands the range of dominance under which bundling is credible but reduces the rivals' profits. Moreover, under bundling, the joint profits of the specialists may be higher than when they are integrated. For instance, when $n = 2$, for $2.092t < \alpha < 2.376t$, the joint profit of the specialists under bundling is higher than their profit under independent pricing, which in turn is higher than the generalist B's profit under bundling.

5 Credible leverage of dominance to a dominated market

In this section, we consider situations in which A is dominant in some markets but is (weakly) dominated in the tied market and identify three different circumstances in which A can credibly leverage dominance from the former to the latter: asymmetric transportation costs, multiple dominant products, competition against specialists. These results are relevant to antitrust policy of tying as antitrust authorities are mainly concerned about situations in which the tied product is inferior to or as good as the competing product. After identifying the three conditions, we consider the bundling of a pure monopolist in the tying market and show that tying is not credible.

With symmetric markets the effect on profits of bundling could be decomposed into a demand size and demand elasticity effect. However, the demand size effect does not straightforwardly extend to asymmetric markets because bundling at equal total price reduces demand for the tying product and increases demand for the tied product. When total demand (in terms of total units sold) stays the same, bundling reduces profits when the profit margin in the tying market is higher than in the tied market, which is typically the case. In particular, for the profit of firm A to increase it is necessary (but not sufficient) that firm A enjoys a positive "total" demand size effect.

In this section, we assume that each consumer's location for product j is independently and uniformly distributed for $j = 1, ..., n$. When we say that A is dominant in market j, we mean $0 < \alpha_j \leq 3t_j$. Then, under independent pricing, A's market share belongs to $(1/2, 1)$ for $0 < \alpha_j < 3t_j$ and is equal to 1 at $\alpha_j = 3t_j$. In other words, we limit the value of α_j such that B's market share is positive except for the limit of $\alpha_j = 3t_j$. This approach should be contrasted with the case in which A is pure monopolist in market j, which is similar to assuming $\alpha_j = v$.

As a benchmark, consider the case of symmetric transportation cost $t = t_1 = t_2$ with $n = 2$. In this case, we can show that bundling is not credible as it reduces A's profit for any $\alpha_1 \in [0, 3t]$ and $\alpha_2 = 0$. It has to do with the fact that each firm's profit from product j is convex with respect to α_j under independent pricing. Therefore, when the asymmetry $\alpha_1 - \alpha_2$ increases while keeping the aggregate dominance $\alpha_1 + \alpha_2$ constant, each firm's total profit increases under independent pricing. This makes bundling less profitable (or more unprofitable) for each firm with respect to independent pricing.

5.1 Asymmetric transportation cost

Suppose that transportation costs are asymmetric such that $t_1 > t_2$. Then, we have:

Proposition 6. *Suppose that (x_1, x_2) is uniformly distributed over $[0, 1]^2$, $t_1 > t_2$, and $\alpha_1 = 3t_1$, $\alpha_2 = 0$. If $t_1/t_2 > 2.257$, then bundling increases A's profit but decreases B's profit.*

By continuity, the result holds even when α_2 is slightly negative and α_1 slightly below $3t_1$. The proposition reveals that when firm A has a large advantage in one market and has no advantage in the other market, then tying allows it to credibly leverage its dominance from the former to the latter if the tying products are sufficiently differentiated relative to the tied products. The intuition for this result can again be decomposed into two effects. Given $\alpha_1 = 3t_1$ and $\alpha_2 = 0$, the equilibrium prices under IP are $p^*_{A1}(3t_1) = 2t_1$, $p^*_{A2}(0) = t_2$, $p^*_{B1}(3t_1) = 0$, $p^*_{B2}(0) = t_2$. Now consider bundling without changing the price levels: $P_A = 2t_1 + t_2$, $P_B = t_2$. At these prices, the profit of A is $2t_1 + (\frac{3}{4} - \frac{1}{8t_1}t_2)t_2$, which is greater than $\pi^*_{A1} + \pi^*_{A2} = 2t_1 + \frac{1}{2}t_2$ for any $t_1 > t_2$, and increases with t_1/t_2.This does not imply that tying is profitable because of the demand elasticity effect. Precisely, we have $\partial\Pi_B/\partial P_B = -t_2/8t_1 < 0$, which means that firm B has an incentive to reduce its price and this reduces the profit of firm A since $\partial\Pi_A/\partial P_B = 1/4 + t_2/8t_1 > 0$. But the values of $\frac{\partial\Pi_B}{\partial P_B}$ and of $\frac{\partial\Pi_A}{\partial P_B}$ suggest that the greater is t_1/t_2, the lower are both firm B's incentive to reduce P_B and the reduction in Π_A due to the reduction in P_B. This explains why bundling increases A's profit if t_1/t_2 is large.

5.2 Leverage of multiple products

Suppose now that A and B compete in three different markets and A is dominant in two markets: $\alpha_1 = \alpha_2 \equiv \alpha > \alpha_3 = 0$. Consider symmetric transportation cost $t_1 = t_2 = t_3 = 1$. Then, we have:

Proposition 7. *Consider competition between two generalist firms when each of them produces three products. Suppose that x_j is independently and uniformly distributed over $[0,1]$ for $j = 1, 2, 3$. Assume $t_1 = t_2 = t_3 = 1$ and $\alpha_1 = \alpha_2 \equiv \alpha \in (0,3]$ and $\alpha_3 = 0$. Then bundling always reduces firm B's profit but increases firm A's profit if and only if $\alpha > 2.496$.*

Consider $\alpha = 3$. Then, if we examine the demand size effect, bundling increases the demand for product A3 from a half to 47/48, which in turn increases A's total profit from 4.5 to 4.896. Since bundling allows firm A to increase substantially his share in the tied market at the cost of losing a small share in the tying markets, bundling remains credible even if firm B reacts by reducing its price. But bundling is detrimental to B as bundling reduces its profit from 0.5 to 0.053. This extension to more than two products with asymmetric dominance is particularly relevant to upstream channel bundling in cable TV industry and triple (or quadruple) play offers in telecommunication sector.

5.3 Competition against specialists

Suppose now that firm A competes against two specialists as in Section 5. Assume $t = t_1 = t_2 = 1$ and $\alpha_1 > \alpha_2 = 0$. We have:

Proposition 8. *Consider competition between a generalist firm A and two specialists B1 and B2. Suppose that (x_1, x_2) is uniformly distributed over $[0,1]^2$ and assume $t_1 = t_2 = 1$ and $\alpha_1 \in (0,3)$ and $\alpha_2 = 0$. Then, pure bundling always decreases B2's profit and the joint* *profit of B1 and B2. In addition, for* $\alpha_1 > 0.701$*, pure bundling increases A's profit; for* $\alpha_1 > 1.159$*, pure bundling increases B1's profit.*

The proposition should be contrasted with the fact that if B1 and B2 are integrated, tying is never profitable for A for $\alpha_1 \in (0, 3)$ and $\alpha_2 = 0$. Therefore, the proposition essentially captures the Cournot complement effect which arises when the competing firms are separated. Then, for $\alpha_1 > 0.701$, tying is profitable and reduces both B2's profit and the joint profit of B1 and B2.

5.4 Bundling of a pure monopolist

Consider now bundling of a pure monopolist: A faces no competition in market 1 but faces competition from B in market 2. In order to make the result comparable to Proposition 6, we consider $t_1 \geq t_2$ and $\alpha_2 = 0$. Using v rather than v_{A1}, full coverage under independent pricing occurs if $v \geq 2t_1$, but we make a slightly stronger assumption $v \geq 3t_1 - t_2$. In this setting, we need to distinguish the case of independent products from that of perfect complements. We find that in both cases, tying is never credible.

Proposition 9. *Suppose that firm A faces no competition in market 1 but faces competition from B in market 2. Suppose that* (x_1, x_2) *is uniformly distributed over* $[0, 1]^2$ *and assume* $v \geq 3t_1 - t_2$*,* $t_1 \geq t_2$ *and* $\alpha_2 = 0$*. Regardless of whether the products can be independently consumed or perfect complements, pure bundling reduces each firm's profit.*

The proposition captures Whinston (1990)'s findings in our model. Under independent pricing, pure monopoly in market 1 is equivalent to competing against firm B when $\alpha_1 = v$ as long as $v > 3t_1$.[30] Hence, A has a much higher profit margin in market 1 when it is pure monopolist than when it is dominant. When the products can be independently consumed, this implies that both the demand size effect and the demand elasticity effects are negative for both firms. Hence, bundling makes each firm more aggressive and thereby reduces each firm's profit. For instance, firm A becomes aggressive as it wants to expand the sale of the high margin monopoly product. It is surprising that this result is similar to the result we obtain when $\alpha_1 = \alpha_2 = 0$ and $t_1 = t_2$. In both cases, the demand elasticity effect is negative while the demand size effect is zero for the case of no dominance but negative for the case of pure monopolist.

[30]Then, in both cases, the profit in market 1 is equal to $v - t_1$ under independent pricing.

When the products are perfect complements, pure bundling means that only A's system is available and hence B's profit is zero. However, A does not gain anything by excluding B as was discovered by Whinston (1990).[31]

6 Conclusion

We contribute to the leverage theory of tying by studying the leverage of a dominant firm instead of a pure monopolist in the tying market. We find that the dominant firm benefits from a positive demand size effect of bundling, which makes its bundling credible as long as the demand elasticity effect is not too negative. By contrast, in the case of pure monopolist, we find that both the demand size effect and the demand elasticity effect are negative, which makes its bundling not credible. We identify three different circumstances in which a dominant firm in tying market(s) can credibly leverage dominance to foreclose a dominant firm in a tied market: asymmetric transportation costs, multiple dominant products, competition against specialists. In reality, some of these forces can coexist. For instance, when a channel conglomerate bundles several strong channels with some weak ones against multiple small rivals, the last two forces are combined to make bundling even more credible and powerful. Our findings provide a justification for the use of contractual bundling for foreclosure purposes.

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[31] See Proposition 3.

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A Appendix

A.1 Proof of Lemma 1

(i) We first show the result for $n = 2$. Note that

$$f_2(1/2) = 2\int_0^1 f(s)^2 ds = 4\int_0^{1/2} f(s)^2 ds > 4\int_{1/4}^{1/2} f(s)^2 ds. \tag{6}$$

Next, observe that for a log-concave function f we have $\log[f(a)f(b)] = \log[f(a)]+\log[f(b)] \leq 2\log[f(\frac{a+b}{2})] = \log[f(\frac{a+b}{2})^2]$ for any a and b in $(0,1)$, hence

$$f(a)f(b) \leq f(\frac{a+b}{2})^2. \tag{7}$$

In particular, taking $b = 1/2$ and $a = 2s - 1/2$ in (7) yields $f(s)^2 \geq f(2s-1/2)f(1/2)$ for $s > 1/4$. And thus

$$\int_{1/4}^{1/2} f(s)^2 ds \geq f(1/2)\int_{1/4}^{1/2} f(2s-1/2)ds = f(1/2)\int_0^{1/2}\frac{1}{2}f(y)dy = f(1/2)/4.$$

Combining this with (6) we obtain $f_2(1/2) > f(1/2)$.

In order to prove the result for $n > 2$, recall that F_j is more peaked than F_{j-1} for each $j \geq 2$, hence $F_j(x) > F_{j-1}(x)$ for each $x \in (\frac{1}{2}, 1)$. Since $F_j(1/2) = F_{j-1}(1/2)$, it follows that $f_j(\frac{1}{2}) \geq f_{j-1}(\frac{1}{2})$. Hence, we conclude that $f_j(1/2) \geq f_2(1/2) > f(1/2)$ for each $j \geq 2$.

(ii) We denote the i-th derivative of a function g by $g^{(i)}$, such that $g^{(0)} = g$. For $j \geq 2$, let $k_j \geq 1$ be such that $f_j^{(i)}(1) = 0$ for $i = 0, 1, \ldots, k_j - 1$ and $f_j^{(k_j)}(1) \neq 0$. Regarding $j = 1$, we define k_1 as k_j above if $f(1) = 0$; we define $k_1 = 0$ if $f(1) > 0$. We will prove that $k_{j+1} = k_j + k_1 + 1$ for all $j \geq 1$. By L'Hôpital's rule this implies $\lim_{x\uparrow 1} f_{j+1}(x)/f_j(x) = 0$ for each $j \geq 1$, hence $\lim_{x\uparrow 1}\frac{f_n(x)}{f(x)} = 0$.

We have that

$$f_{j+1}(x) = \int_{(j+1)x-j}^{1} \frac{j+1}{j} f(s) f_j\left(\frac{(j+1)x - s}{j}\right) ds$$

hence

$$f_{j+1}^{(i)}(x) = \int_{(j+1)x-j}^{1} \left(\frac{j+1}{j}\right)^{i+1} f(s) f_j^{(i)}\left(\frac{(j+1)x - s}{j}\right) ds$$

and $f_{j+1}^{(i)}(1) = 0$ for $i = 1, ..., k_j$. However, for $m \geq 1$ we find

$$\begin{aligned} f_{j+1}^{(k_j+m)}(x) &= -\sum_{h=1}^{m}(j+1)^{m-h+1}\left(\frac{j+1}{j}\right)^{k_j+h} f^{(m-h)}((j+1)x-j) f_j^{(k_j+h-1)}(1) + \\ &\int_{(j+1)x-j}^{1}\left(\frac{j+1}{j}\right)^{k_j+m+1} f(s) f_j^{(k_j+m)}\left(\frac{(j+1)x-s}{j}\right) ds \end{aligned}$$

Hence,

$$f_{j+1}^{(k_j+m)}(1) = -\sum_{h=1}^{m}(j+1)^{m-h+1}\left(\frac{j+1}{j}\right)^{k_j+h} f^{(m-h)}(1) f_j^{(k_j+h-1)}(1)$$

and $f_{j+1}^{(k_j+m)}(1) = 0$ if $m \leq k_1$, but $f_{j+1}^{(k_j+k_1+1)}(1) = -(j+1)^{k_1+1}\left(\frac{j+1}{j}\right)^{k_j+1} f^{(k_1)}(1) f_j^{(k_j)}(1) \neq 0$. Therefore $k_{j+1} = k_j + k_1 + 1$. ■

A.2 Proof of Proposition 1

(i) The condition of the proposition implies that $\lim_{x\to 1} X^{\alpha}(x) < 1$, so that the fixed point $x^*(\alpha)$ is smaller than 1.

Now we show that $x^*(\alpha)$ is increasing and concave for $\alpha \geq 0$. By taking the derivative w.r.t. α on both sides of the equation $X^{\alpha}(x^*(\alpha)) = x^*(\alpha)$, one obtains immediately

$$\frac{dx^*(\alpha)}{d\alpha} = \frac{\sigma}{3 + \frac{1-2F(x^*(\alpha))}{f(x^*(\alpha))}\frac{f'(x^*(\alpha))}{f(x^*(\alpha))}}.$$

Moreover, it follows that $\frac{dx^*(\alpha)}{d\alpha}$ is a strictly positive and weakly decreasing function of α: First note that both $(1-2F(x))/f(x)$ and $f'(x)/f(x)$ are non-positive for $x \geq 1/2$. Next observe that both functions are decreasing because of log-concavity of f. The product of these two functions is thus positive and increasing. Since $x^*(\alpha)$ is increasing in α, it follows that $\frac{dx^*(\alpha)}{d\alpha}$ is decreasing.

Next, the equilibrium price of firm A is increasing in α because (i) $x^*(\alpha)$ is increasing in α and (ii) $F(x)/f(x)$ is increasing in x by log-concavity. The equilibrium profit of firm A is then also increasing because both equilibrium price (as seen above) and market share $(F(x^*(\alpha)))$ are increasing. Similarly, the equilibrium price and profit for firm B are decreasing.

(ii) In this case, no interior equilibrium exists. Necessarily, $p_B^* = 0$ and firm A corners the market. The highest price to corner the market, given $p_B^* = 0$, is $p_A^* = \alpha - 1/(2\sigma)$. Clearly firm A has no incentive to set a lower price (as demand cannot be increased). Firm A has also no incentive to increase its price because the marginal profit, evaluated at p_A^*, equals

$$\frac{d\pi_A}{dp_A} = F(1) - \sigma p_A^* f(1) \leq 0,$$

where the inequality follows directly from $(\sigma\alpha - 1/2)f(1) \geq 1$. By virtue of the remark in footnote 23, it follows that $\frac{d\pi_A}{dp_A} < 0$ for any $p_A > p_A^*$. ■

A.3 Proof of Proposition 2

The proof is similar to that of Proposition 1 and therefore omitted. ■

A.4 Proof of Lemma 2

The proof consists of several steps. In step 1 we prove that both firms gain from bundling for dominance levels for which firm A obtains lower market share under bundling. In step 2 we prove the weak set relations for dominance levels for which firm A obtains higher market share under bundling. In step 3 we prove the strictness of the set relations. This is easy in case $f(1) > 0$, but requires additional steps 3.1 and 3.2 in case $f(1) = 0$.

It will be convenient to define some auxiliary dominance level sets. Let

$$\mathcal{A}^+_{DENS} = \{\alpha \geq 0 : \; f_n(x^*_n(\alpha)) \geq f(x^*(\alpha))\}, \qquad \mathcal{A}^-_{DENS} = \{\alpha \geq 0 : \; f_n(x^*_n(\alpha)) < f(x^*(\alpha))\},$$

$$\mathcal{A}^+_{PA} = \{\alpha \geq 0 : \; P^*_{n,A}(\alpha) \geq np^*_A(\alpha)\}, \qquad \mathcal{A}^+_{PB} = \{\alpha \geq 0 : \; P^*_{n,B}(\alpha) \geq np^*_B(\alpha)\}.$$

Step 1. We first establish that if the dominance level belongs to $\mathcal{A}^-_{MS}$, then both firms will set higher total prices and obtain higher profits under bundling. Let $\bar{\alpha} \in \mathcal{A}^-_{MS}$ be such a dominance level, that is $F_n(x^*_n(\bar{\alpha})) < F(x^*(\bar{\alpha}))$. As the distribution of the average location is more peaked, this implies that $x^*(\bar{\alpha}) > x^*_n(\bar{\alpha})$. From (2) and (3) we know that for any α such that $(\sigma\alpha - 1/2)f(1) < 1$ we have[32]

$$x^*_n(\alpha) - x^*(\alpha) = \frac{1 - 2F_n(x^*_n(\alpha))}{f_n(x^*_n(\alpha))} - \frac{1 - 2F(x^*(\alpha))}{f(x^*(\alpha))}. \tag{8}$$

In particular, for $\bar{\alpha}$ the left-hand side of (8) is negative. Eq. (8) can then only hold if $f_n(x^*_n(\bar{\alpha})) < f(x^*(\bar{\alpha}))$. That is, $\bar{\alpha} \in \mathcal{A}^-_{DENS}$. Using the expressions for equilibrium prices of firm B from Propositions 1 and 2 we conclude that $P^*_{n,B}(\bar{\alpha}) > np^*_B(\bar{\alpha})$. As firm B also obtains higher market share under bundling, firm B's profit is higher under bundling as well. Now firm A could set bundle price $P_{n,A} = np^*_A$ and obtain higher market share, and thus higher profits than what he obtains in the independent pricing equilibrium. The optimal bundle price for firm A yields at least as much profit. As we know that in equilibrium firm A obtains less market share than under independent pricing, the optimal bundle price must be such that $P^*_{n,A}(\bar{\alpha}) > np^*_A(\bar{\alpha})$.

Step 2. Next we focus on dominance levels for which firm A obtains higher market share under bundling, that is $\alpha \in \mathcal{A}^+_{MS}$. We will show that

$$(\mathcal{A}^+_{\pi A} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^+_{PA} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^+_{PB} \cap \mathcal{A}^+_{MS}) \supseteq (\mathcal{A}^+_{\pi B} \cap \mathcal{A}^+_{MS}).$$

It is straightforward that $\mathcal{A}^+_{\pi A} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^+_{PA} \cap \mathcal{A}^+_{MS}$. Namely, for dominance levels for which firm A sets higher total price and obtains higher market share under bundling, profits are automatically higher under bundling.

[32]If $(\sigma\bar{\alpha} - 1/2)f(1) \geq 1$, then Proposition 1(ii) applies and thus $\Pi^*_{n,B}(\bar{\alpha}) > n\pi^*_B(\bar{\alpha}) = 0$, $P^*_{n,B}(\bar{\alpha}) > np^*_B(\bar{\alpha}) = 0$. Minor changes to the arguments below establish that $\Pi^*_{n,A}(\bar{\alpha}) > n\pi^*_A(\bar{\alpha})$, $P^*_{n,A}(\bar{\alpha}) > np^*_A(\bar{\alpha})$.

Note that for $\alpha \in \mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS}$, $P^*_{n,A} = nF_n(x^*_n(\alpha))/(\sigma f_n(x^*_n(\alpha))) > nF(x^*(\alpha))/(\sigma f(x^*(\alpha))) = np^*_A$, because the numerator is larger (and positive) and the denominator is strictly smaller (but positive) on the left-hand side. This shows that $\mathcal{A}^+_{PA} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS}$.

Note that for $\alpha \in \mathcal{A}^+_{DENS} \cap \mathcal{A}^+_{MS}$, $P^*_{n,B} = n(1 - F_n(x^*_n(\alpha)))/(\sigma f_n(x^*_n(\alpha))) \leq n(1 - F(x^*(\alpha)))/(\sigma f(x^*(\alpha))) = np^*_B$, because the numerator is smaller (and positive) and the denominator is larger (and positive) on the left-hand side. Moreover, the inequality must be strict. Namely, the inequality could only be binding when both $f(x^*(\alpha)) = f_n(x^*_n(\alpha))$ and $F(x^*(\alpha)) = F(x^*_n(\alpha))$. But this would imply that $x^*(\alpha) = x^*_n(\alpha)$ because of (8). However, this is incompatible with $F(x^*(\alpha)) = F_n(x^*_n(\alpha))$ because $F_n(x) > F(x)$ for any $x \in (1/2, 1)$. This proves that $\mathcal{A}^-_{DENS} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^+_{PB} \cap \mathcal{A}^+_{MS}$.

It is straightforward that $\mathcal{A}^+_{PB} \cap \mathcal{A}^+_{MS} \supseteq \mathcal{A}^+_{\pi B} \cap \mathcal{A}^+_{MS}$. Namely, for dominance levels for which firm B obtains higher total profit under bundling, despite having smaller market share under bundling, it must be that the total price is higher under bundling.

Step 3. We show that the set relations are strict. We know that $f_n(x^*_n(0)) > f(x^*(0))$, but if $f(1) > 0$ and $\alpha \geq 1/(\sigma f(1)) + 1/(2\sigma)$ then necessarily $f_n(x^*_n(\alpha)) < f(1) = f(x^*(\alpha))$.[33] There must then exist a level $\alpha_{DENS} > 0$ for which $f_n(x^*_n(\alpha_{DENS})) = f(x^*(\alpha_{DENS}))$. In the hypothetical case that there exist multiple such levels, we choose the maximal one. We claim that $\alpha_{DENS} \in \mathcal{A}^+_{MS} \cap \mathcal{A}^+_{\pi A} \cap \mathcal{A}^-_{\pi B}$.

It is clear that $\alpha_{DENS} \in \mathcal{A}^+_{MS}$. Namely, suppose it is not true. Then firm B has strictly higher market share under bundling, and thus both firms would obtain higher profits under bundling (from Step 1). However, $f(x^*(\alpha_{DENS})) = f_n(x^*_n(\alpha_{DENS}))$ and $F(x^*(\alpha_{DENS})) > F_n(x^*_n(\alpha_{DENS}))$ contradict $\Pi^*_{n,A}(\alpha_{DENS}) > n\pi^*_A(\alpha_{DENS})$ (from Propositions 1 and 2.).

Using again Propositions 1 and 2, it easily follows that $\alpha_{DENS} \in \mathcal{A}^+_{\pi A}$. It must also be true that firm B has strictly lower profits under bundling. Namely, profits for firm B could at best be equal under bundling, but this would require that firm B's market share is exactly the same under both pricing regimes. We have already seen before that this is impossible as it would imply that $x^*_n(\alpha_{DENS}) = x^*(\alpha_{DENS})$ by Eq. (8), and thus $F_n(x) = F(x)$ for $x = x^*_n(\alpha_{DENS})$. This thus proves the strictness of the first superset relation.

In order to prove the second, let $\alpha_{MS} > 0$ be such that market shares of the two firms are equal under both regimes, that is, $F(x^*(\alpha_{MS})) = F_n(x^*(\alpha_{MS}))$. Such a level exists if $f(1) > 0$ as firm B has a strictly lower market share under bundling for small positive dominance

[33]We show below in Step 3.2 that if $f(1) = 0$, then $f_n(x^*_n(\alpha)) < f(x^*(\alpha))$ still holds for a large α.

levels, while for $\alpha \geq 1/(\sigma f(1)) + 1/(2\sigma)$ his market share is zero under independent pricing (Prop. 1) but positive under bundling (Prop. 2).[34] In the hypothetical case that there exist multiple levels of dominance with this property, we choose the maximal one. We claim that $\alpha_{MS} > \alpha_{DENS}$, and therefore $f_n(x_n^*(\alpha_{MS})) < f(x^*(\alpha_{MS}))$, which implies $\alpha_{MS} \in \mathcal{A}_{\pi B}^+$ even though $\alpha \notin \mathcal{A}_{MS}^-$

This follows easily from, on the one hand, observing that $\alpha_{MS} = \alpha_{DENS}$ leads to a contradiction, as again by Eq. (8) we would deduce that $x_n^*(\alpha_{MS}) = x^*(\alpha_{MS})$, which is impossible. On the other hand, $\alpha_{MS} < \alpha_{DENS}$ is impossible because of the assumption that α_{MS} has been chosen as the maximal level of dominance for which market shares are equal under the two regimes. It implies that for higher levels, in particular for α_{DENS}, market share is strictly lower for firm A under bundling. But we have already established before that $\alpha_{DENS} \in \mathcal{A}_{MS}^+$. ■

Step 3.1 Suppose that $f(1) = 0$. We prove that for $j = 1, ..., n-1$, we have $F_j(x_j^*(\alpha)) > F_{j+1}(x_{j+1}^*(\alpha))$ for large α. It then follows that for large α, $F(x^*(\alpha)) > F_n(x_n^*(\alpha))$.
Define the strictly increasing function $W_j(x) = x + \frac{2F_j(x)-1}{f_j(x)}$. Given $\alpha > 0$, we know that $x_j^*(\alpha)$ is such that $W_j(x_j^*(\alpha)) = \frac{1}{2} + \sigma\alpha$. For a large α, both $x_j^*(\alpha)$ and $x_{j+1}^*(\alpha)$ are close to 1. Thus, given x close to 1, we select $y(x)$ as the unique y such that $F_{j+1}(y) = F_j(x)$. We prove that $W_{j+1}(y(x)) > W_j(x)$ for x close to 1, hence $W_{j+1}(y(x_j^*(\alpha))) > W_j(x_j^*(\alpha)) = \frac{1}{2} + \sigma\alpha$ for a large α, which implies $x_{j+1}^*(\alpha) < y(x_j^*(\alpha))$ and thus $F_{j+1}(x_{j+1}^*(\alpha)) < F_{j+1}(y(x_j^*(\alpha))) = F_j(x_j^*(\alpha))$ for large α.

In order to prove $W_{j+1}(y(x)) > W_j(x)$, we notice that $F_{j+1}(y(x)) = F_j(x)$ makes the inequality equivalent to $(y(x) - x)f_{j+1}(y(x)) + (2F_j(x) - 1)[1 - \frac{f_{j+1}(y(x))}{f_j(x)}] > 0$. We prove that $\lim_{x\uparrow 1} \frac{f_{j+1}(y(x))}{f_j(x)} = 0$, hence $\lim_{x\uparrow 1} \Big((y(x) - x)f_{j+1}(y(x)) + (2F_j(x) - 1)[1 - \frac{f_{j+1}(y(x))}{f_j(x)}]\Big) = 1$.

As in the proof of Lemma 1, for $\ell = 1, ..., n$ we let $k_\ell \geq 1$ be such that $f_\ell^{(i)}(1) = 0$ for $i = 0, 1, \ldots, k_\ell - 1$ and $f_\ell^{(k_\ell)}(1) \neq 0$. Furthermore, we set $a_\ell = (-1)^{k_\ell} f_\ell^{(k_\ell)}(1)/(k_\ell!) > 0$ and $b_\ell = a_\ell/(k_\ell + 1) > 0$, such that $b_\ell < a_\ell$. Then use Taylor's formula to obtain

$$f_\ell(x) = a_\ell(1-x)^{k_\ell} + \eta_{f_\ell}(x) \quad (9)$$

$$1 - F_\ell(x) = b_\ell(1-x)^{k_\ell+1} + \eta_{F_\ell}(x) \quad (10)$$

with

$$\lim_{x\uparrow 1} \frac{\eta_{f_\ell}(x)}{(1-x)^{k_\ell}} = \lim_{x\uparrow 1} \frac{\eta_{F_\ell}(x)}{(1-x)^{k_\ell+1}} = 0 \quad (11)$$

[34]We show below in Step 3.1 that if $f(1) = 0$, then $F(x^*(\alpha)) > F_n(x_n^*(\alpha))$ still holds for a large α.

Let $\varepsilon > 0$ be close enough to zero to satisfy $\varepsilon < b_\ell < a_\ell$ for $\ell = 1, ..., n$, and let $\delta > 0$ be such that for all $x \in (1-\delta, 1)$ we have

$$\begin{aligned} |\eta_{f_\ell}(x)| &< \varepsilon(1-x)^{k_\ell} && (12) \\ |\eta_{F_\ell}(x)| &< \varepsilon(1-x)^{k_\ell+1} && (13) \end{aligned}$$

Of course, if x is close to 1 then both x and $y(x)$ belong to $(1-\delta, 1)$. From (10) and (13) with $\ell = j+1$ we obtain

$$(b_{j+1} - \varepsilon)(1-y)^{k_{j+1}+1} < 1 - F_{j+1}(y) < (b_{j+1} + \varepsilon)(1-y)^{k_{j+1}+1}$$

Therefore $y(x)$, the solution to $F_{j+1}(y) = F_j(x)$, satisfies

$$\frac{1 - F_j(x)}{b_{j+1} + \varepsilon} < (1 - y(x))^{k_{j+1}+1} < \frac{1 - F_j(x)}{b_{j+1} - \varepsilon} \tag{14}$$

Similarly, from (10) and (13) (with $\ell = j$) we obtain

$$(b_j - \varepsilon)(1-x)^{k_j+1} < 1 - F_j(x) < (b_j + \varepsilon)(1-x)^{k_j+1} \tag{15}$$

Combining (14) and (15) we thus conclude that

$$\frac{b_j - \varepsilon}{b_{j+1} + \varepsilon}(1-x)^{k_j+1} < (1 - y(x))^{k_{j+1}+1} < \frac{b_j + \varepsilon}{b_{j+1} - \varepsilon}(1-x)^{k_j+1} \tag{16}$$

Similarly, using (9) and (12) we obtain

$$\frac{f_{j+1}(y(x))}{f_j(x)} < \frac{(a_{j+1} + \varepsilon)(1 - y(x))^{k_{j+1}}}{(a_j - \varepsilon)(1-x)^{k_j}} \tag{17}$$

We conclude that

$$\frac{f_{j+1}(y(x))}{f_j(x)} < \left(\frac{a_{j+1} + \varepsilon}{a_j - \varepsilon}\right) \left(\frac{b_j + \varepsilon}{b_{j+1} - \varepsilon}\right)^{\frac{k_{j+1}}{k_{j+1}+1}} (1-x)^{\frac{k_{j+1} - k_j}{k_{j+1}+1}}$$

which proves that

$$\lim_{x \uparrow 1} \frac{f_{j+1}(y(x))}{f_j(x)} = 0$$

because $k_{j+1} > k_j$. ■

Step 3.2 Suppose that $f(1) = 0$. We prove that for $j = 1, ..., n-1$, we have $f_{j+1}(x^*_{j+1}(\alpha)) < f_j(x^*_j(\alpha))$ if $f(1) = 0$ and α is large. It then follows that for large α, $f_n(x^*_n(\alpha)) < f(x^*(\alpha))$.

Given x close to 1, we select $z(x)$ as the unique $z \in (\frac{1}{2}, x)$ such that $f_{j+1}(z) = f_j(x)$. We prove that $W_{j+1}(z(x)) < W_j(x)$ for x close to 1, hence $W_{j+1}(z(x^*_j(\alpha))) < W_j(x^*_j(\alpha)) = \frac{1}{2} + \sigma\alpha$

for a large α, which implies $x^*_{j+1}(\alpha) > z(x^*_j(\alpha))$ and thus $f_{j+1}(x^*_{j+1}(\alpha)) < f_{j+1}(z(x^*_j(\alpha))) = f_j(x^*_j(\alpha))$. The inequality $W_{j+1}(z(x)) < W_j(x)$ reduces to $f_j(x)(z(x) - x) < 2[F_j(x) - F_{j+1}(z(x))]$, and since $z(x) < x$ it suffices to prove that $F_j(x) > F_{j+1}(z(x))$. We know from the proof of Step 3.1 that if the equality $F_{j+1}(z(x)) = F_j(x)$ holds for x close to 1, then $f_{j+1}(z(x)) < f_j(x)$. In order to obtain $f_{j+1}(z(x)) = f_j(x)$ it is necessary to decrease $z(x)$, which implies $F_{j+1}(z(x)) < F_j(x)$. ■

A.5 Proof of Proposition 3

(i) Define $\underline{\alpha} = \min \mathcal{A}^+_{\pi A}$, $\alpha_{\pi A} = \sup \mathcal{A}^-_{\pi A}$, $\alpha_{\pi B} = \min \mathcal{A}^+_{\pi B}$ and $\bar{\alpha} = \sup \mathcal{A}^-_{\pi B}$.

(ii) This follows straightforwardly from (i). ■

A.6 Proof of Proposition 5

Given y_n defined in (4), the profit functions are

$$\pi_A = p_A n F_n(y_n), \qquad \pi_{Bj} = p_{Bj}(1 - F_n(y_n)) \quad \text{for } j = 1, ..., n$$

and the first-order conditions are

$$0 = F_n(y_n) - \sigma p_A f_n(y_n), \qquad 0 = 1 - F_n(y_n) - \frac{\sigma}{n} p_{Bj} f_n(y_n) \quad \text{for } j = 1, ..., n$$

Given the equilibrium average location y^{**}_n, if $p^{**}_{n,A}, p^{**}_{n,B}$ are the (symmetric) equilibrium prices then $p^{**}_{n,A} = F_n(y^{**}_n)/(\sigma f_n(y^{**}_n))$ and $p^{**}_{n,B} = n(1 - F_n(y^{**}_n))/(\sigma f_n(y^{**}_n))$. Hence

$$y^{**}_n = \frac{1}{2} + \sigma\alpha - \sigma(p^{**}_{n,A} - p^{**}_{n,B}) = \frac{1}{2} + \sigma\alpha + \frac{n - (n+1)F_n(y^{**}_n)}{f_n(y^{**}_n)}.$$

that is y^{**}_n is the fixed point of (5).Notice that $\frac{n-(n+1)F_n(y)}{f_n(y)} = -\frac{F_n(y)}{f_n(y)} + n\frac{1-F_n(y)}{f_n(y)}$, hence Y^α_n is weakly decreasing. Moreover, $Y^\alpha_n(1/2) > 1/2$, $\lim_{y \uparrow 1} Y^\alpha_n(y) = -\infty$, therefore a unique fixed point exists for Y^α_n in the interval $(\frac{1}{2}, 1)$.

A.7 Mixed Bundling in the baseline model

Here, we consider the baseline model with $n = 2$ and study the case in which each firm is allowed to practice mixed bundling. This means that firm i $(= A, B)$ chooses a price P_i for the bundle of its own products and a price $p_i = p_{ij}$ for each single product $j = 1, 2$. Thus each consumer buys the bundle of a firm i and pays P_i, or buys one object from each firm and pays $p_A + p_B$. The main result is that when α is sufficiently large, we find the same equilibrium outcome described by Proposition 2 under pure bundling because for firm A a pure bundling strategy is superior to any alternative strategy when it has a large advantage over firm B. Moreover, we show that the same result holds when A competes with specialists B1 and B2.

Without loss of generality, we assume that $P_i \leq 2p_i$ holds for $i = A, B$ and that each consumer willing to buy both products of i buys i's bundle. As a consequence, each consumer chooses one alternative among AA, AB, BA, BB, where for instance AB means buying products $A1$ and $B2$. In order to describe the preferred alternative of each type of consumer, we introduce

$$s' \equiv \frac{1}{2} + \frac{\alpha + P_B - p_A - p_B}{2t} \text{ and } s'' \equiv \frac{1}{2} + \frac{\alpha + p_A + p_B - P_A}{2t}$$

where $s' \leq s''$ holds from $P_A \leq 2p_A$ and $P_B \leq 2p_B$.[35]

We find:

- Type (s_1, s_2) buys AA if and only if $s_1 \leq s''$, $s_2 \leq s''$, $s_1 + s_2 \leq s' + s''$.
- Type (s_1, s_2) buys AB if and only if $s_1 \leq s'$, $s_2 > s''$.
- Type (s_1, s_2) buys BA if and only if $s_1 > s''$, $s_2 \leq s'$.
- Type (s_1, s_2) buys BB if and only if $s_1 > s'$, $s_2 > s'$, $s_1 + s_2 > s' + s''$.

Let $S_{ii'}$ and $\mu_{ii'}$ denote, respectively, the set of types who choose ii' and the measure of $S_{ii'}$ for $ii' = AA$, AB, BA, BB. Note that $\mu_{AB} = \mu_{BA}$, and moreover $\mu_{AB} > 0$ if $0 < s'$ and $s'' < 1$;[36] $\mu_{AB} = 0$ (as in Section 3.3) if $s' \leq 0$ and/or $s'' \geq 1$.[37] In either case, the firms' profits are given by

$$\pi_A = P_A \mu_{AA} + 2p_A \mu_{AB}; \qquad \pi_B = P_B \mu_{BB} + 2p_B \mu_{AB}.$$

[35]Precisely, s' is such that a consumer located at $(s_1, s_2) = (s', 1)$ (at $(s_1, s_2) = (1, s')$) is indifferent between the alternatives BB and AB (between the alternatives BB and BA). Likewise, s'' is such that a consumer located at $(s_1, s_2) = (s'', 0)$ (at $(s_1, s_2) = (0, s'')$) is indifferent between the alternatives AA and BA (between the alternatives AA and AB).

[36]The expressions for $\mu_{AA}, \mu_{AB}, \mu_{BB}$ are found in the proof of Proposition 10.

[37]Precisely, if $s' < 0$ then each type of consumer prefers BB to AB (and to BA). If $s'' > 1$, then each type of consumer prefers AA to AB (and to BA).

Given a equilibrium $(p_A^*, P_A^*, p_B^*, P_B^*)$ with the corresponding measures, $\mu_{AA}^*, \mu_{AB}^*, \mu_{BB}^*$ for S_{AA}, S_{AB}, S_{BB}, we say that it is a *mixed bundling equilibrium* if $\mu_{AB}^* > 0$ and that it is a *pure bundling equilibrium* if $\mu_{AB}^* = 0$. It is almost immediate to see that a pure bundling equilibrium exists for any values of parameters as, for each firm, pure bundling is a best response to pure bundling.[38] Next proposition establishes that no mixed bundling equilibrium exists when the dominance of firm A is sufficiently strong. In fact, this result also holds if firm A faces two specialist opponents B1 and B2, that is in each equilibrium firm A plays a pure bundling strategy, such that each consumer either buys firm A's bundle or products B1 and B2, at least as long as we consider symmetric equilibria such that $p_{A1} = p_{A2}$ and $p_{B1} = p_{B2}$. The reason is that when A faces two specialists such that $p_{B1} = p_{B2}$, A's pricing problem coincides with A's problem when A faces a generalist and $P_B = 2p_B$. Hence he has the same incentive to avoid mixed bundling strategies, as we describe immediately after the proposition.

Proposition 10. *Consider the mixed bundling game with $n = 2$. Then both if firm A faces a generalist opponent or two specialists opponents, we have that*
(i) there exists no mixed bundling equilibrium if $f(1) > 0$ and $\alpha \geq t + \frac{t}{f(1)}$;
(ii) when f is the uniform density, there exists no mixed bundling equilibrium if $\alpha \geq \frac{9}{8}t$.

Proposition 10(i) relies on proving that if α is sufficiently large and (p_A, P_A, p_B, P_B) are such that $\mu_{AB} > 0$, then $s'' < 1$ and it is profitable for A to reduce P_A. A small reduction in P_A reduces A's revenue from inframarginal consumers but attracts some marginal consumers. When α is large, the inequality $s'' < 1$ implies that P_A is large. Hence, it follows that the revenue increase (which is proportional to the initial P_A) from the marginal consumers dominates the revenue decrease from inframarginal consumers (which is proportional to the reduction in P_A). This explains why it is profitable to reduce P_A until s'' reaches the value of 1 to make $\mu_{AB} = 0$.[39]

[38] Let $P_{2,A}^*, P_{2,B}^*$ be the equilibrium prices from Proposition 2. Under mixed bundling, $(p_A^*, P_{2,A}^*, p_B^*, P_{2,B}^*)$ is an equilibrium if p_A^* and p_B^* are large enough, as for firm $A\,(B)$ it is impossible to induce any type of consumer to choose AB or BA since $P_B = P_{2,B}^*$ and a large p_B imply $s' < 0$ for any $p_A \geq 0$, thus $S_{AB} = S_{BA} = \emptyset$ ($P_A = P_{2,A}^*$ and a large p_A imply $s'' > 1$ for any $p_B \geq 0$, thus $S_{AB} = S_{BA} = \emptyset$).

[39] Proposition 10(i) is linked to a result in Menicucci, Hurkens, Jeon (2015) (MHJ henceforth) about the optimality of pure bundling for a two-product monopolist. In our duopoly setting, given (p_B, P_B) chosen by firm B, the problem of maximizing A's profit with respect to (p_A, P_A) is equivalent to the problem of

In the case of the uniform distribution, the lower bound on α from Proposition 10(i) is $t + \frac{t}{f(1)} = 2t$, but Proposition 10(ii) relies on some particular features of the uniform distribution to establish that no mixed bundling equilibrium exists if $\alpha \geq \frac{9}{8}t$.[40] In order to see how this stronger result is obtained, fix p_B, P_B arbitrarily and let M_A denote the set of (p_A, P_A) such that $\mu_{AB} > 0$. Whereas Proposition 10(i) is proved by showing that $\frac{\partial \pi_A}{\partial P_A}$ is negative at each $(p_A, P_A) \in M_A$ if $\alpha \geq t + \frac{t}{f(1)} = 2t$, for the uniform distribution we can show that if $\alpha \in [\frac{9}{8}t, 2t)$, there exists no $(p_A, P_A) \in M_A$ such that $\frac{\partial \pi_A}{\partial P_A} = 0$ and $\frac{\partial \pi_A}{\partial p_A} = 0$ are both satisfied; therefore no mixed bundling strategy is optimal for firm A when $\alpha \in [\frac{9}{8}t, 2t)$.

It is interesting to notice that a well-established result in the literature is that mixed bundling reduces profits with respect to independent pricing, at least for symmetric firms: see Armstrong and Vickers (2010) and references therein.[41] Propositions 3(i) and 10(i), conversely, prove that if one firm's dominance over the other is strong enough, that is if $\alpha \geq t + \frac{t}{f(1)}$ and $\alpha > \bar{\alpha}$, then mixed bundling boils down to pure bundling, and each firm's profit is larger under mixed bundling than under independent pricing.

We below provide the proof of Proposition 10(i) and leave the proof of part (ii) to the on-line appendix B.

Proof of Proposition 10 (i)

In the case that $0 < s'$ and $s'' < 1$, each of the sets S_{AA}, S_{AB}, S_{BB} has a positive measure as follows:

$$\mu_{AA} = F(s')F(s'') + \int_{s'}^{s''} F(s' + s'' - s_1) f(s_1) ds_1; \qquad \mu_{AB} = F(s')[1 - F(s'')]; \tag{18a}$$

$$\mu_{BB} = [1 - F(s')][1 - F(s'')] + \int_{s'}^{s''} [1 - F(s' + s'' - s_1)] f(s_1) ds_1. \tag{18b}$$

maximizing the profit of a two-product monopolist facing a consumer with suitably distributed valuations and such that the consumer enjoys a synergy of $2p_B - P_B \geq 0$ if she consumes both objects. Since MHJ do not allow for synergies, strictly speaking Proposition 10(i) is not a corollary of the results in MHJ.

[40] Numeric analysis suggests that (i) no mixed bundling NE exists as long as $\alpha \geq 0.72t$; (ii) when a mixed bundling NE exists, the firms' equilibrium profits are lower than under independent pricing.

[41] Armstrong and Vickers (2010) explain this result by referring to firms' incentives to compete fiercely for the consumers which choose to buy both products from the same firm. This is closely related to the strong demand elasticity effect we find when $\alpha = 0$, that is when the firms are symmetric.

Therefore, given $\pi_A = P_A\mu_{AA} + 2p_A\mu_{AB}$, we find

$$\begin{aligned}\frac{\partial\pi}{\partial P_A} &= \mu_{AA} + P_A[2F(s')f(s'') + \int_{s'}^{s''} F(s'+s''-s_1)f(s_1)ds_1](-\frac{1}{2t}) - 2p_A F(s')f(s'')(-\frac{1}{2t}) \\ &= F(s')f(s'')\left[\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t}\right] + \int_{s'}^{s''} f(s_1)f(s'+s''-s_1)\left[\frac{F(s'+s''-s_1)}{f(s'+s''-s_1)} - \frac{P_A}{2t}\right]ds_1\end{aligned}$$

and we prove that $\frac{\partial\pi}{\partial P_A} < 0$, given $s'' < 1$

- First, we prove that $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t} < 0$. Since f is log-concave, it follows that $\frac{F}{f}$ is increasing and $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t}$ is decreasing in P_A. Since the inequality $s'' < 1$ is equivalent to $p_A + p_B - t + \alpha < P_A$, it follows that $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t} < \frac{1}{f(1)} + \frac{t-p_B-\alpha}{t}$, and the latter expression is negative given $\alpha \geq t + \frac{t}{f(1)}$.
- Now we prove that $\frac{F(s'+s''-s_1)}{f(s'+s''-s_1)} - \frac{P_A}{2t} < 0$ for each $s_1 \in [s', s'']$. Since f is log-concave, it follows that $\frac{F(s'+s''-s_1)}{f(s'+s''-s_1)}$ is decreasing in s_1, and at $s_1 = s'$ we obtain the value $\frac{F(s'')}{f(s'')} - \frac{P_A}{2t}$, which is negative since it is smaller than $\frac{F(s'')}{f(s'')} - \frac{P_A}{t} + \frac{p_A}{t} < 0$, given $2p_A \geq P_A$.

B On-line appendix [NOT FOR PUBLICATION]

B.1 Proof of Proposition 6

From the example immediately after Proposition 3, we see that under independent pricing the total profit of A is $2t_1 + \frac{1}{2}t_2$, and the total profit of B is $\frac{1}{2}t_2$.

Given P_A, P_B, let $p_A = \frac{1}{2}P_A$ and $p_B = \frac{1}{2}P_B$. Then a consumer located in (s_1, s_2) buys the bundle of A if and only if $3t_1 - t_1s_1 - t_2s_2 - 2p_A \geq -t_1(1-s_1) - t_2(1-s_2) - 2p_B$, which is equivalent to $2t_1 + \frac{1}{2}t_2 + p_B - p_A \geq t_1s_1 + t_2s_2$. Therefore we need the c.d.f. for $t_1s_1 + t_2s_2$, which we denote G:

$$\Pr\{t_1s_1 + t_2s_2 \leq k\} = G(k) = \begin{cases} \frac{1}{2t_1t_2}k^2 & \text{if } 0 \leq k \leq t_2 \\ \frac{1}{t_1}(k - \frac{1}{2}t_2) & \text{if } t_2 < k < t_1 \\ 1 - \frac{1}{2t_1t_2}(t_1+t_2-k)^2 & \text{if } t_1 \leq k \leq t_1 + t_2 \end{cases}$$

and its density is

$$g(k) = \begin{cases} \frac{1}{t_1t_2}k & \text{if } 0 \leq k \leq t_2 \\ \frac{1}{t_1} & \text{if } t_2 < k < t_1 \\ \frac{1}{t_1t_2}(t_1+t_2-k) & \text{if } t_1 \leq k \leq t_1 + t_2 \end{cases}$$

Then $\Pi_A = 2p_A G(2t_1 + \frac{1}{2}t_2 + p_B - p_A)$, $\Pi_B = 2p_B(1 - G(2t_1 + \frac{1}{2}t_2 + p_B - p_A))$ and we obtain

$$\begin{aligned}
\frac{\partial \Pi_A}{\partial p_A} &= 2G(2t_1 + \frac{1}{2}t_2 + p_B - p_A) - 2p_A g(2t_1 + \frac{1}{2}t_2 + p_B - p_A) \\
\frac{\partial \Pi_B}{\partial p_B} &= 2(1 - G(2t_1 + \frac{1}{2}t_2 + p_B - p_A)) - 2p_B g(2t_1 + \frac{1}{2}t_2 + p_B - p_A)
\end{aligned}$$

and in three steps we establish that the equilibrium prices are

$$p^*_{2,A} = \frac{5}{8}t_1 - \frac{5}{16}t_2 + \frac{3}{16}Q, \qquad p^*_{2,B} = -\frac{1}{8}t_1 + \frac{1}{16}t_2 + \frac{1}{16}Q \tag{19}$$

with $Q = \sqrt{4t_1^2 + t_2^2 + 28t_1t_2}$.

Step 1 No equilibrium is such that $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ belongs to $[0, t_2]$.

If $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ belongs to $[0, t_2]$, then $p^*_{2,A}, p^*_{2,B}$ solve

$$\begin{aligned}
2 \cdot \frac{1}{2t_1t_2}(2t_1 + \frac{1}{2}t_2 + p_B - p_A)^2 - 2p_A \frac{2t_1 + \frac{1}{2}t_2 + p_B - p_A}{t_1t_2} &= 0 \\
2(1 - \frac{1}{2t_1t_2}(2t_1 + \frac{1}{2}t_2 + p_B - p_A)^2) - 2p_B \frac{(2t_1 + \frac{1}{2}t_2 + p_B - p_A)}{t_1t_2} &= 0
\end{aligned}$$

hence $p_A = \frac{1}{4}t_1 + \frac{1}{16}t_2 + \frac{1}{16}\sqrt{16t_1^2 + t_2^2 + 40t_1t_2}$, $p_B = -\frac{5}{4}t_1 - \frac{5}{16}t_2 + \frac{3}{16}\sqrt{16t_1^2 + t_2^2 + 40t_1t_2}$. But $2t_1 + \frac{1}{2}t_2 + p_B - p_A = \frac{1}{2}t_1 + \frac{1}{8}t_2 + \frac{1}{8}\sqrt{16t_1^2 + 40t_1t_2 + t_2^2}$ is greater than t_2. Therefore no equilibrium exists such that $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ is in $[0, t_2]$.

Step 2 No equilibrium is such that $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ belongs to (t_2, t_1).

If $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ belongs to (t_2, t_1), then $p^*_{2,A}, p^*_{2,B}$ solve

$$\begin{aligned}
2t_1 + p_B - 2p_A &= 0 \\
-t_1 - 2p_B + p_A &= 0
\end{aligned}$$

hence $p_A = t_1$, $p_B = 0$. But $2t_1 + \frac{1}{2}t_2 + p_B - p_A = t_1 + \frac{1}{2}t_2$, which is greater than t_1.

Step 3 There exists an equilibrium such that $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ belongs to $[t_1, t_1 + t_2]$, and $\frac{1}{2}t_2 > \Pi^*_{2,B}$, $2t_1 + \frac{1}{2}t_2 < \Pi^*_{2,A}$ if $t_1/t_2 > 2.57$.

If $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A}$ belongs to $[t_1, t_1 + t_2]$, then $p^*_{2,A}, p^*_{2,B}$ solve

$$\begin{aligned}
2\left(1 - \frac{1}{2t_1t_2}(\frac{1}{2}t_2 - t_1 - p_B + p_A)^2\right) - 2p_A \frac{1}{t_1t_2}(\frac{1}{2}t_2 - t_1 - p_B + p_A) &= 0 \\
2\left(1 - (1 - \frac{1}{2t_1t_2}(\frac{1}{2}t_2 - t_1 - p_B + p_A)^2)\right) - 2p_B \frac{1}{t_1t_2}(\frac{1}{2}t_2 - t_1 - p_B + p_A) &= 0
\end{aligned}$$

and then we obtain the prices in (19). As a consequence, $2t_1 + \frac{1}{2}t_2 + p^*_{2,B} - p^*_{2,A} = \frac{5}{4}t_1 - \frac{1}{8}Q + \frac{7}{8}t_2$, and it is simple to see that it belongs to $[t_1, t_1 + t_2]$. Moreover,

$$\Pi^*_{2,A} = \frac{1}{8}(10t_1 - 5t_2 + 3Q)\left(1 - \frac{1}{128t_1t_2}(t_2 - 2t_1 + Q)^2\right), \qquad \Pi^*_{2,B} = \frac{1}{1024t_1t_2}(t_2 - 2t_1 + Q)^3$$

It is useful to define $r = t_1/t_2$, and then we find $\frac{1}{2}t_2 - \Pi^*_{2,B} = \left(\frac{1}{2} - \frac{(\sqrt{4r^2+28r+1}+1-2r)^3}{1024r}\right) t_2$, and $\frac{1}{2} - \frac{(\sqrt{4r^2+28r+1}+1-2r)^3}{1024r} > 0$ for each $r > 1$, thus $\frac{1}{2}t_2 - \Pi^*_{2,B} > 0$. Likewise, we have $2t_1 + \frac{1}{2}t_2 - \Pi^*_{2,A} = \left(2r + \frac{1}{2} - \frac{1}{8}(10r - 5 + 3\sqrt{4r^2+28r+1}\right)\left(1 - \frac{1}{128r}(1 - 2r + \sqrt{4r^2+28r+1})^2)\right) t_2$, which is positive for $r \in (1, 2.257)$, is negative for $r > 2.257$.

B.2 Proof of Proposition 7

From the example immediately after Proposition 3, we see that under independent pricing the profit is $\pi^*_{A1}(\alpha) + \pi^*_{A2}(\alpha) + \pi^*_{A3}(0) = 4(\frac{3+\alpha}{6})^2 + \frac{1}{2}$ for firm A, $\pi^*_{B1}(\alpha) + \pi^*_{B2}(\alpha) + \pi^*_{B3}(0) = 4(\frac{3-\alpha}{6})^2 + \frac{1}{2}$ for firm B. Regarding bundling, we can apply Proposition 2 with

$$F_3(x) = \begin{cases} \frac{9}{2}x^3 & \text{if } 0 \le x < \frac{1}{3} \\ -9x^3 + \frac{27}{2}x^2 - \frac{9}{2}x + \frac{1}{2} & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ \frac{9}{2}x^3 - \frac{27}{2}x^2 + \frac{27}{2}x - \frac{7}{2} & \text{if } \frac{2}{3} \le x < 1 \end{cases}, \qquad f_3(x) = \begin{cases} \frac{27}{2}x^2 & \text{if } 0 \le x < \frac{1}{3} \\ -27x^2 + 27x - \frac{9}{2} & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ \frac{27}{2}x^2 - 27x + \frac{27}{2} & \text{if } \frac{2}{3} \le x < 1 \end{cases}$$

after replacing α with $\frac{2}{3}\alpha$. In particular, the unique fixed point for $X_3^{2\alpha/3}$ is[42]

$$x_3^*(\frac{2}{3}\alpha) = \begin{cases} \begin{array}{c} \frac{1}{10}\alpha + \frac{1}{2} - \frac{1}{2}r - \frac{1}{2r}(\frac{1}{100}\alpha^2 + \frac{1}{20}) + \frac{1}{2}i\sqrt{3}\left(r - \frac{1}{r}(\frac{1}{100}\alpha^2 + \frac{1}{20})\right) \\ \text{with } r = \frac{1}{10}\sqrt[3]{\alpha^3 - 5\alpha + 5\sqrt{-\alpha^4 - 2\alpha^2 - 5}} \end{array} & \text{if } 0 \le \alpha < \frac{11}{9} \\ \begin{array}{c} \frac{1}{10}\alpha + r + \frac{1}{r}(\frac{1}{100}\alpha^2 - \frac{1}{50}\alpha + \frac{1}{100}) + \frac{9}{10} \\ \text{with } r = \frac{1}{10}\sqrt[3]{3\alpha - 3\alpha^2 + \alpha^3 - \frac{209}{9} + \frac{20}{9}\sqrt{-27\alpha + 27\alpha^2 - 9\alpha^3 + 109}} \end{array} & \text{if } \frac{11}{9} \le \alpha < 3 \end{cases}$$

This allows to compute $\Pi^*_{3,A}(\frac{2}{3}\alpha)$ and $\Pi^*_{3,B}(\frac{2}{3}\alpha)$. Numerical computations show the result. ■

B.3 Proof of Proposition 8

From the example immediately after Proposition 3, we see that under independent pricing the profit is $\pi^*_{A1}(\alpha_1) + \pi^*_{A2}(0) = 2(\frac{3+\alpha_1}{6})^2 + \frac{1}{2}$ for firm A, and $\pi^*_{B1}(\alpha_1) = 2(\frac{3-\alpha_1}{6})^2$ for firm $B1$, $\pi^*_{B2}(0) = \frac{1}{2}$ for firm $B2$.

In order to find the profit under bundling, we can rely on Proposition 5 and find that $\Pi^{**}_{2,A}(\alpha_1/2) = \frac{4F_2(y_2^{**}(\alpha_1/2))^2}{f_2(y_2^{**}(\alpha_1/2))}$, $\Pi^{**}_{2,B1}(\alpha_1/2) = \Pi^{**}_{2,B2}(\alpha_1/2) = \frac{4(1-F_2(y_2^{**}(\alpha_1/2)))^2}{f_2(y_2^{**}(\alpha_1/2))}$, and $y_2^{**}(\alpha_1/2)$ is

[42] We find that $x_3^*(\frac{2}{3}\alpha)$ is a piecewise defined function because the fixed point of $X_3^{2\alpha/3}$ given $F_3(x) = -9x^3 + \frac{27}{2}x^2 - \frac{9}{2}x + \frac{1}{2}$ and $f_3(x) = -27x^2 + 27x - \frac{9}{2}$ is smaller than $\frac{2}{3}$ if and only if $\alpha \in [0, \frac{11}{9})$. For $\alpha \in [\frac{11}{9}, 3)$, the fixed point of $X_3^{2\alpha/3}$ given $F_3(x) = \frac{9}{2}x^3 - \frac{27}{2}x^2 + \frac{27}{2}x - \frac{7}{2}$ and $f_3(x) = \frac{27}{2}x^2 - 27x + \frac{27}{2}$ is greater or equal to $\frac{2}{3}$.

the fixed point of the function $Y_2^{\alpha_1/2}(y) = \frac{1}{2} + \frac{1}{2}\frac{\alpha_1}{2} + \frac{2-3\left(1-2(1-y)^2\right)}{4(1-y)}$ in the interval $(\frac{1}{2}, 1)$, that is $y_2^{**}(\alpha_1/2) = \frac{1}{20}\alpha_1 + \frac{9}{10} - \frac{1}{20}\sqrt{\alpha_1^2 - 4\alpha_1 + 44}$. Numerical computations show the result. ■

B.4 Proof of Proposition 9

B.4.1 Independent objects

For the case of independent objects, under independent pricing firm A charges $v - t_1$ for product 1 and has profit $v - t_1$; A and B each charge t_2 for product 2, with profit $\frac{1}{2}t_2$ each.

Under bundling, type (s_1, s_2) chooses between $2v - t_1 s_1 - t_2 s_2 - P$ (here we write P instead of P_A) and $v - t_2 + t_2 s_2 - p$ (here we write p instead of p_B) and chooses to buy the bundle of A if and only if $s_1 \leq \frac{1}{2} + \frac{\theta}{2t_2} - \frac{t_1}{2t_2}s_1$, with $\theta = v - P + p$.

The case of $t_1 \in (t_2, 2t_2]$ In this case we find that the demand functions for the bundle of A is

$$D_A(P,p) = \begin{cases} 0 & \text{if } \theta \leq -t_2 \\ \frac{(\theta+t_2)^2}{4t_1t_2} & \text{if } -t_2 < \theta \leq t_1 - t_2 \\ \frac{2t_2-t_1+2\theta}{4t_2} & \text{if } t_1 - t_2 < \theta \leq t_2 \\ 1 - \frac{(t_1+t_2-\theta)^2}{4t_1t_2} & \text{if } t_2 < \theta \leq t_1 + t_2 \\ 1 & \text{if } t_1 + t_2 < \theta \end{cases} \tag{20}$$

and $D_B(P,p) = 1 - D_A(P,p)$. The profit functions are $\Pi_A = PD_A(P,p)$ and $\pi_B = pD_B(P,p)$.

Step 1 No equilibrium exists such that $-t_2 < \theta \leq t_1 - t_2$.

When $-t_2 < \theta \leq t_1 - t_2$, the first order conditions are

$$\frac{(v-P+p+t_2)^2}{4t_1t_2} - \frac{(v-P+p+t_2)}{2t_1t_2}P = 0, \qquad 1 - \frac{(v-P+p+t_2)^2}{4t_1t_2} - \frac{(v-P+p+t_2)}{2t_1t_2}p = 0$$

and yield $P = \frac{1}{8}v + \frac{1}{8}t_2 + \frac{1}{8}\sqrt{2vt_2 + t_2^2 + 16t_1t_2 + v^2}$, $p = -\frac{5}{8}v - \frac{5}{8}t_2 + \frac{3}{8}\sqrt{2vt_2 + t_2^2 + 16t_1t_2 + v^2}$. Hence $\theta = \frac{1}{4}v - \frac{3}{4}t_2 + \frac{1}{4}\sqrt{v^2 + 2vt_2 + t_2^2 + 16t_1t_2}$, but $\theta \leq t_1 - t_2$ reduces to $\sqrt{v^2 + 2vt_2 + t_2^2 + 16t_1t_2} < 4t_1 - v - t_2$, which is violated since $v \geq 3t_1 - t_2$. Hence, no equilibrium is such that $-t_2 < \theta \leq t_1 - t_2$.

Step 2 No equilibrium exists such that $t_1 - t_2 < \theta \leq t_2$.

When $t_1 - t_2 < \theta \leq t_2$, the first order conditions are

$$\frac{2t_2 - t_1 + 2(v-P+p)}{4t_2} - \frac{1}{2t_2}P = 0, \qquad \frac{2t_2 + t_1 - 2(v-P+p)}{4t_2} - \frac{1}{2t_2}p = 0$$

and yield $P = \frac{1}{3}v - \frac{1}{6}t_1 + t_2, p = \frac{1}{6}t_1 - \frac{1}{3}v + t_2$. Hence $\theta = \frac{1}{3}v + \frac{1}{3}t_1$, but $t_1 - t_2 < \theta \leq t_2$ reduces to $v \leq 3t_2 - t_1$, which fails to hold since $v \geq 3t_1 - t_2$. Hence, no equilibrium is such that $t_1 - t_2 < \theta \leq t_2$.

Step 3 There exists an equilibrium such that $t_2 < \theta \leq t_1 + t_2$, and for each firm the profit is smaller than under independent pricing.

When $t_2 < \theta \leq t_1 + t_2$, the first order conditions are

$$\begin{aligned} 1 - \frac{(t_1 + t_2 - (v - P + p))^2}{4t_1 t_2} - \frac{t_1 + t_2 - (v - P + p)}{2t_1 t_2} P &= 0, \\ \frac{(t_1 + t_2 - (v - P + p))^2}{4t_1 t_2} - \frac{t_1 + t_2 - (v - P + p)}{2t_1 t_2} p &= 0 \end{aligned}$$

and yield $P = \frac{5}{8}v - \frac{5}{8}t_1 - \frac{5}{8}t_2 + \frac{3}{8}\sqrt{(v - t_1 - t_2)^2 + 16t_1 t_2}$, $p = -\frac{1}{8}v + \frac{1}{8}t_1 + \frac{1}{8}t_2 + \frac{1}{8}\sqrt{(v - t_1 - t_2)^2 + 16t_1 t_2}$. Hence $\theta = \frac{1}{4}v + \frac{3}{4}t_1 + \frac{3}{4}t_2 - \frac{1}{4}\sqrt{(v - t_1 - t_2)^2 + 16t_1 t_2}$ and $t_2 < \theta \leq t_1 + t_2$ is satisfied. Then we find

$$\begin{aligned} \Pi_A^{**} &= -(v - t_1 - t_2) \frac{-2vt_1 - 2vt_2 + t_1^2 + t_2^2 - 82t_1 t_2 + v^2}{128t_1 t_2} \\ &\quad + \frac{(v - t_1 - t_2)^2 + 36t_1 t_2}{128t_1 t_2} \sqrt{-2vt_1 - 2vt_2 + t_1^2 + t_2^2 + 18t_1 t_2 + v^2} \end{aligned}$$

and simple manipulations reveal that $v - t_1 + \frac{1}{2}t_2 > \Pi_A^{**}$ is equivalent to $w_A(v) > 0$, with $w_A(v) = 3v^3 + (-13t_1 - 9t_2) v^2 + (17t_1^2 + 158t_1 t_2 + 9t_2^2) v + 143t_1 t_2^2 - 311t_1^2 t_2 - 7t_1^3 - 3t_2^3$. Since $w_A'(v) = 9v^2 + 2(-13t_1 - 9t_2) v + 17t_1^2 + 158t_1 t_2 + 9t_2^2 > 0$ for $v > 3t_1 - t_2$, it follows that $w_A(3t_1 - t_2) = (2t_1^2 + 16t_1 t_2 + 24t_2^2)(4t_1 - t_2) > 0$ implies that $w_A(v) > 0$ for each $v \geq 3t_1 - t_2$.

Regarding firm B, we find

$$\begin{aligned} \Pi_B^{**} &= -(v - t_1 - t_2) \frac{-2vt_1 - 2vt_2 + t_1^2 + t_2^2 + 14t_1 t_2 + v^2}{128t_1 t_2} \\ &\quad + \frac{(v - t_1 - t_2)^2 + 4t_1 t_2}{128t_1 t_2} \sqrt{-2vt_1 - 2vt_2 + t_1^2 + t_2^2 + 18t_1 t_2 + v^2} \end{aligned}$$

and simple manipulations reveal that $\frac{1}{2}t_2 > \Pi_B^{**}$ is equivalent to $w_B(v) > 0$, with $w_B(v) = v^3 - 3(t_2 + t_1) v^2 + (18t_1 t_2 + 3t_1^2 + 3t_2^2) v + 17t_1 t_2^2 - t_1^3 - 17t_1^2 t_2 - t_2^3$. Since $w_B'(v) = 3v^2 - 6(t_1 + t_2) v + 18t_1 t_2 + 3t_1^2 + 3t_2^2 > 0$ for each $v > 3t_1 - t_2$, it follows that $w_B(3t_1 - t_2) = (2t_1^2 + 8t_2^2)(4t_1 - t_2) > 0$ implies that $w_B(v) > 0$ for each $v > 3t_1 - t_2$.

The case of $t_1 > 2t_2$ In this case we find that the demand function for the bundle of A is slightly different with respect to (20):

$$D_A(P,p) = \begin{cases} 0 & \text{if } \theta \le -t_2 \\ \frac{(\theta+t_2)^2}{4t_1t_2} & \text{if } -t_2 < \theta \le t_2 \\ \frac{\theta}{t_1} & \text{if } t_2 < \theta \le t_1 - t_2 \\ 1 - \frac{(t_1+t_2-\theta)^2}{4t_1t_2} & \text{if } t_1 - t_2 < \theta \le t_1 + t_2 \\ 1 & \text{if } t_1 + t_2 < \theta \end{cases}$$

and $D_B(P,p) = 1 - D_A(P,p)$. The profit functions are $\Pi_A = PD_A(P,p)$ and $\pi_B = pD_B(P,p)$ as before.

Step 1 No equilibrium exists such that $-t_2 < \theta \le t_2$.

We can argue as in Step 1 above to rule out the existence of equilibria such that $-t_2 < \theta \le t_2$, since $\theta = \frac{1}{4}v - \frac{3}{4}t_2 + \frac{1}{4}\sqrt{v^2 + 2vt_2 + t_2^2 + 16t_1t_2}$ violates $\theta \le t_2$.

Step 2 No equilibrium exists such that $t_2 < \theta \le t_1 - t_2$.

When $t_2 < \theta \le t_1 - t_2$, the first order conditions are

$$\frac{v - P + p}{t_1} - \frac{1}{t_1}P = 0, \qquad 1 - \frac{v - P + p}{t_1} - \frac{1}{t_1}p = 0$$

and yield $P = \frac{1}{3}v + \frac{1}{3}t_1$, $p = \frac{2}{3}t_1 - \frac{1}{3}v$. Hence $\theta = \frac{1}{3}v + \frac{1}{3}t_1$, but $\theta \le t_1 - t_2$ fails to hold since $v > 3t_1 - t_2$.

Step 3 There exists an equilibrium such that $t_1 - t_2 < \theta \le t_1 + t_2$, but $\Pi_A^{**} < v - t_1 + \frac{1}{2}t_2$ and $\Pi_B^{**} < \frac{1}{2}t_2$.

We know from step 3 above that the first order conditions yield P, p such that $\theta = \frac{1}{4}v + \frac{3}{4}t_1 + \frac{3}{4}t_2 - \frac{1}{4}\sqrt{(v - t_1 - t_2)^2 + 16t_1t_2}$, and this θ satisfies $t_1 - t_2 < \theta \le t_1 + t_2$. Moreover, the arguments provided above establish that $v - t_1 + \frac{1}{2}t_2 > \Pi_A^{**}$ and $\frac{1}{2}t_2 > \Pi_B^{**}$ also when $t_1 > 2t_2$.

B.4.2 Perfect complements

In this case, product 1 of firm A is essential and therefore bundling of firm A leaves B with zero sales. Then the optimal bundle price of A is P_A^{**} which maximizes $P_A \Pr\{v - t_1s_1 - t_2s_2 - P_A \ge 0\}$. It is immediate to see that under independent pricing the profit of firm A is not smaller than the profit under bundling. For each p_{B2}, let A choose $p_{A1} = P_A^{**}$ and $p_{A2} = 0$. Then, a consumer such that $v - t_1s_1 - t_2s_2 - P_A^{**} \ge 0$ buys product A1, either with

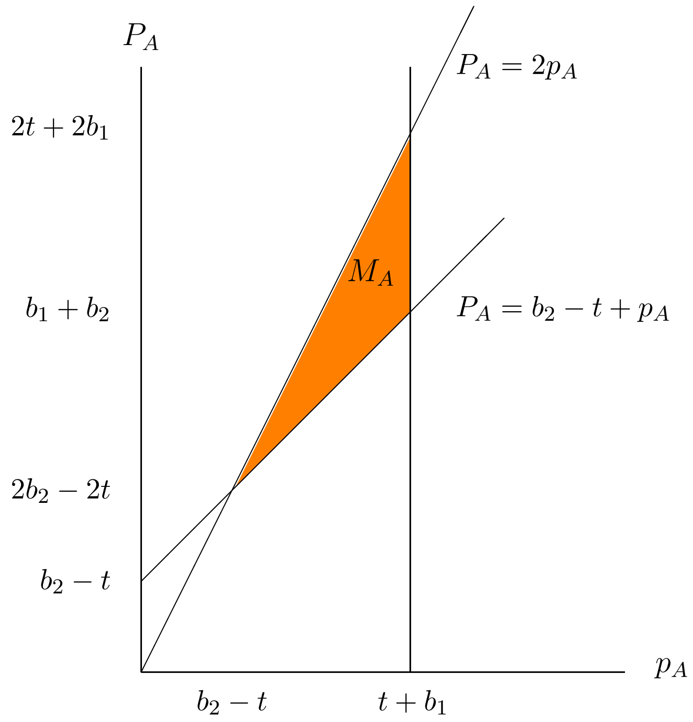

Figure 3: Mixed bundling strategies for firm A.

A2 or with B2, hence the demand (and the profit) for firm A is at least as large as under bundling. Such demand is in fact larger (and so A's profit) if there are consumers such that $v - t_1 s_1 - t_2 s_2 - P_A^{**} < 0$ and $v - t_1 s_1 - t_2(1 - s_2) - P_A^{**} - p_{B2} \geq 0$.

B.5 Proof of Proposition 10(ii)

Given $b_1 \equiv P_B - p_B + \alpha$, $b_2 \equiv p_B + \alpha \geq b_1$, we say that firm A plays a *pure bundling strategy* if and only if $p_A \geq b_1 + t$ and/or $P_A \leq b_2 - t + p_A$ because $\mu_{AB} = 0$ in either of these cases.[43] Given b_1, b_2, we define M_A as the set of (p_A, P_A) such that $\mu_{AB} > 0$, that is

$$M_A = \{(p_A, P_A) : p_A < b_1 + t, \quad b_2 - t + p_A < P_A \leq 2p_A\}.$$

We say that A plays a *mixed bundling strategy* if $(p_A, P_A) \in M_A$. Notice that M_A is non-empty if and only if $b_1 > -t$ and $b_2 < 2t + b_1$: see Figure 3.

[43]Precisely, $x' \leq 0$ if and only if $p_A \geq b_1 + t$; $x'' \geq 1$ if and only if $P_A \leq b_2 - t + p_A$.

Using (18), for each $(p_A, P_A) \in M_A$ we have

$$\pi_A = \frac{1}{8t^2}\begin{pmatrix} P_A^3 + 4p_A^3 - 2\left(b_1 + b_2 + 2t\right) P_A^2 - 6p_A^2 P_A - 4\left(b_1 - b_2 + 2t\right) p_A^2 + 8\left(b_1 + t\right) P_A p_A \\ +(2t^2 + 4tb_2 + b_2^2 + 2b_1 b_2 - b_1^2)P_A - 4(b_2 - t)(t + b_1)p_A \end{pmatrix}$$

and

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \frac{1}{8t^2}\left(12p_A^2 - 4\left(3P_A + 4t - 2b_2 + 2b_1\right) p_A + 8\left(b_1 + t\right) P_A - 4(b_2 - t)(t + b_1)\right) \\ \frac{\partial \pi_A}{\partial P_A} &= \frac{1}{8t^2}\left(3P_A^2 - 4\left(2t + b_1 + b_2\right) P_A - 6p_A^2 + 8\left(b_1 + t\right) p_A + 2t^2 + 4tb_2 + b_2^2 + 2b_2 b_1 - b_1^2\right). \end{aligned}$$

Since $\alpha \geq \frac{9}{8}t$ implies $b_1 > \frac{9}{8}t$, we consider the following set $\mathcal{B}$ of possible values for (b_1, b_2): $\mathcal{B} = \{(b_1, b_2) : \frac{9}{8}t < b_1 \leq b_2 < 2t + b_1\}$. We prove that for each $(b_1, b_2) \in \mathcal{B}$ it is never a best reply for firm A to play (p_A, P_A) in M_A, that is the best reply of firm A is a pure bundling strategy. The proof is organized in three steps. In Step 1 we prove that for firm A playing independent pricing (that is, $P_A = 2p_A$) in M_A is suboptimal. A mixed bundling strategy for firm A can thus be optimal only if it lies in the interior of M_A, which implies that the first (and second) order conditions must be satisfied. However, in Step 2 we show that if $(p_A, P_A) \in M_A$ is such that $\frac{\partial \pi_A}{\partial p_A} = 0$, then P_A must be larger than a suitable $\bar{P}_A$, while in Step 3 we show that $\frac{\partial \pi_A}{\partial P_A} = 0$ implies that P_A must be smaller than $\bar{P}_A$. Hence, it must be optimal for firm A to play a pure bundling strategy whenever $b_2 \geq \frac{9}{8}t$.

Step 1 Suppose that $(b_1, b_2) \in \mathcal{B}$. Playing $(p_A, P_A) \in M_A$ such that $P_A = 2p_A$ is not a best reply for firm A because either $\frac{\partial \pi_A}{\partial p_A} > 0$ and/or $\frac{\partial \pi_A}{\partial P_A} < 0$.
We start by evaluating $\frac{\partial \pi_A}{\partial p_A}$ and $\frac{\partial \pi_A}{\partial P_A}$ at $P_A = 2p_A$ and we find

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A} &= \frac{1}{t^2}\left(-\frac{3}{2}p_A^2 + (b_2 + b_1)p_A - \frac{1}{2}\left(b_2 - t\right)\left(t + b_1\right)\right) \equiv z(p_A), \\ \frac{\partial \pi_A}{\partial P_A} &= \frac{1}{t^2}\left(\frac{3}{4}p_A^2 - (t + b_2)p_A + \frac{1}{8}\left(2b_2 b_1 + b_2^2 + 4tb_2 + 2t^2 - b_1^2\right)\right) \equiv Z(p_A). \end{aligned}$$

Notice that if $(p_A, P_A) \in M_A$, then $p_A \in (b_2 - t, b_1 + t)$. Let p_A^* denote the larger solution to $z(p_A) = 0$, that is $p_A^* = \frac{1}{3}(b_1 + b_2 + \sqrt{(b_2 - t)^2 + (b_1 + t)\left(2t + b_1 - b_2\right)})$, and $b_2 - t < p_A^* < b_1 + t$ since $z(b_2 - t) = \frac{1}{2t^2}\left(b_2 - t\right)\left(b_1 - b_2 + 2t\right) > 0$ and $z(b_1 + t) = -\frac{1}{2t^2}\left(b_1 + t\right)\left(b_1 - b_2 + 2t\right) < 0$ in $\mathcal{B}$. In fact, from $z(b_2 - t) > 0 = z(p_A^*)$ we infer that $z(p_A) > 0$ for $p_A \in (b_2 - t, p_A^*)$. This implies that (p_A, P_A) such that $P_A = 2p_A$ and $p_A \in (b_2 - t, p_A^*)$ is not a best reply for A since it is profitable to increase p_A.

For $p_A \in [p_A^*, b_1+t)$ we prove that $Z(p_A) < 0$. This implies that (p_A, P_A) such that $P_A = 2p_A$ and $p_A \in [p_A^*, b_1+t)$ is not a best reply for A since it is profitable to reduce P_A. We find $Z(b_1+t) = -\frac{1}{8t^2}(b_2-b_1)(2t+b_1-b_2+2t+4b_1) \le 0$ in $\mathcal{B}$ and

$$Z(p_A^*) = -\frac{(2t+b_2-b_1)\left(b_2+b_1+4\sqrt{(b_2-t)^2+(b_1+t)(2t+b_1-b_2)}\right)-12t^2}{24t^2}$$

which now we prove to be negative in $\mathcal{B}$. Precisely, we define $\xi_1(b_1,b_2) \equiv (2t+b_2-b_1)(b_2+b_1+4\sqrt{(b_2-t)^2+(b_1+t)(2t+b_1-b_2)})$ and show that

$$\xi_1(b_1,b_2) > 12t^2 \qquad \text{for} \quad \text{any} \quad (b_1,b_2) \in \mathcal{B}. \tag{21}$$

To this purpose we prove below that $\frac{\partial \xi_1}{\partial b_2} > 0$ in $\mathcal{B}$, and $\xi_1(b_1,b_1) = 4t(b_1+2\sqrt{b_1^2+3t^2}) > 12t^2$ for any $b_1 > t$ implies (21). Precisely, $\frac{\partial \xi_1}{\partial b_2} = 2b_2+2t+\frac{6b_1^2+8b_2^2-10b_2b_1+14b_1t-10tb_2}{\sqrt{(b_2-t)^2+(b_1+t)(2t+b_1-b_2)}}$ and $\frac{\partial \xi_1}{\partial b_2} > 0$ in $\mathcal{B}$ since $\xi_2(b_1,b_2) \equiv 6b_1^2+8b_2^2-10b_2b_1+14b_1t-10tb_2 > 0$ in $\mathcal{B}$.[44] ■

Step 2 Suppose that $(b_1,b_2) \in \mathcal{B}$. If $(p_A,P_A) \in M_A$ is such that $\frac{\partial \pi_A}{\partial p_A} = 0$, then $P_A \ge \bar{P}_A$, for a suitable $\bar{P}_A$.

For the equation $\frac{\partial \pi_A}{\partial p_A} = 0$ in the unknown p_A, there exists at least a real solution if and only if $P_A \le \frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$ or $P_A \ge \frac{2}{3}(b_1+b_2+\sqrt{(b_1+t)(b_2-t)}) \equiv \bar{P}_A$. We now prove that if (p_A,P_A) is such that $\frac{\partial \pi_A}{\partial p_A} = 0$ and $P_A \le \frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$, then $(p_A,P_A) \notin M_A$; therefore $\frac{\partial \pi_A}{\partial p_A} = 0$ implies $P_A \ge \bar{P}_A$.

First notice that $\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$ is smaller than b_1+b_2 and in fact it is sometimes smaller than $2b_2-2t$ for some $(b_1,b_2) \in \mathcal{B}$. If $\frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)}) > 2b_2-2t$, then the line $P_A = \frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$ has a non-empty intersection with M_A, and we find that (i) at $p_A = P_A-b_2+t$ (i.e., along the south-east boundary of M_A) $\frac{\partial \pi_A}{\partial p_A} = \frac{1}{2}(b_2-t)(b_1+b_2-P_A)$, which is positive given $P_A \le \frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$; (ii) $\frac{\partial \pi_A}{\partial p_A}$ is decreasing with respect to p_A for $p_A \le \frac{1}{2}P_A+\frac{1}{3}(b_1-b_2)+\frac{2}{3}t$, and $P_A-b_2+t < \frac{1}{2}P_A+\frac{1}{3}(b_1-b_2)+\frac{2}{3}t$ given $P_A \le \frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$. Therefore $\frac{\partial \pi_A}{\partial p_A} > 0$ for each $(p_A,P_A) \in M_A$ such that $P_A \le \frac{2}{3}(b_1+b_2-\sqrt{(b_1+t)(b_2-t)})$, and in fact for each $(p_A,P_A) \in M_A$ such that $P_A < \bar{P}_A$. ■

Step 3 Suppose that $(b_1,b_2) \in \mathcal{B}$ and that $b_2 \ge \frac{9}{8}t$. If $(p_A,P_A) \in M_A$ is a best reply for firm A, then $P_A < \bar{P}_A$.

The equation $\frac{\partial \pi_A}{\partial P_A} = 0$ is quadratic and convex in P_A. In order to satisfy the second order condition, the best reply for firm A must be such that P_A is equal to the smaller solution of

[44] Minimizing ξ_2 over the closure of $\mathcal{B}$ yields the minimum point $b_1 = t$, $b_2 = \frac{5}{4}t$, with $\xi_2(t, \frac{5}{4}t) = \frac{15}{2}t^2 > 0$.

$\frac{\partial \pi_A}{\partial P_A} = 0$. We now show that $\frac{\partial \pi_A}{\partial P_A} < 0$ at $P_A = \bar{P}_A$, which implies that the smaller solution to $\frac{\partial \pi_A}{\partial P_A} = 0$ is smaller than $\bar{P}_A$. We find

$$\frac{\partial \pi_A}{\partial P_A} = -\frac{3}{4t^2}p_A^2 + \frac{b_1+t}{t^2}p_A + \frac{2b_2b_1 - 7b_1^2 - b_2^2 - 20tb_1 + 2t^2 - 16t\sqrt{(b_2-t)(b_1+t)}}{24t^2} \equiv W(p_A)$$

and notice that $\bar{P}_A < b_1 + b_2$; therefore W is defined for $p_A \in (\frac{1}{2}\bar{P}_A, \bar{P}_A - b_2 + t)$. We prove that $W(p_A) < 0$ for each $p_A \in (\frac{1}{2}\bar{P}_A, \bar{P}_A - b_2 + t)$, and to this purpose we notice that W is maximized with respect to p_A at $p_A = \begin{cases} \frac{2}{3}t + \frac{2}{3}b_1 & \text{if } b_2 \leq \frac{3-\sqrt{5}}{2}b_1 + \frac{5-\sqrt{5}}{2}t \\ \frac{1}{2}\bar{P}_A & \text{if } b_2 > \frac{3-\sqrt{5}}{2}b_1 + \frac{5-\sqrt{5}}{2}t \end{cases}$.

- If $b_2 \leq \frac{3-\sqrt{5}}{2}b_1 + \frac{5-\sqrt{5}}{2}t$, then $b_1 \leq \sqrt{5}t$ in order to satisfy $b_1 \leq b_2$, and $W(\frac{2}{3}t + \frac{2}{3}b_1) = \frac{1}{12t^2}(5t^2 - 2b_1t - \frac{1}{2}b_2^2 + b_2b_1 + \frac{1}{2}b_1^2 - 8t\sqrt{(b_1+t)(b_2-t)}) \equiv \xi_3(b_1, b_2)$, which is decreasing in b_2 and $\xi_3(b_1, b_1) = \frac{1}{12t^2}(5t^2 - 2tb_1 + b_1^2 - 8t\sqrt{b_1^2 - t^2})$ is negative for $b_1 \in [\frac{9}{8}t, \sqrt{5}t]$.

- If $b_2 > \frac{3-\sqrt{5}}{2}b_1 + \frac{5-\sqrt{5}}{2}t$, then we evaluate $W(\frac{1}{2}\bar{P}_A) = \frac{1}{24t^2}(4t^2 - 10tb_1 + 6tb_2 - b_1^2 - 3b_2^2 + 4b_1b_2 - 4(2t - b_1 + b_2)\sqrt{(b_1+t)(b_2-t)})$, and we prove it is negative. Precisely, we show that

$$\xi_4(b_1, b_2) \equiv 4(2t - b_1 + b_2)\sqrt{(b_2-t)(b_1+t)} - 4t^2 + 10tb_1 - 6tb_2 + b_1^2 + 3b_2^2 - 4b_1b_2$$

is positive, and from $b_1 + t > b_2 - t$ we obtain $\xi_4(b_1, b_2) > 4(2t - b_1 + b_2)(b_2 - t) - 4t^2 + 10tb_1 - 6tb_2 + b_1^2 + 3b_2^2 - 4b_1b_2 = b_1^2 + 7b_2^2 - 8b_1b_2 - 12t^2 + 14tb_1 - 2tb_2 \equiv \xi_5(b_1, b_2)$. It is immediate that ξ_5 is increasing with respect to b_2, and $\xi_5(b_1, \frac{3-\sqrt{5}}{2}b_1 + \frac{5-\sqrt{5}}{2}t) = -\frac{1}{2}(13\sqrt{5} - 27)b_1^2 + (61 - 23\sqrt{5})tb_1 - \frac{1}{2}(33\sqrt{5} - 71)t^2 > 0$ for $b_1 \in (\frac{9}{8}t, \sqrt{5}t)$; $\xi_5(b_1, b_1) = 12t(b_1 - t) > 0$. ■