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ABSTRACT

This paper considers the estimation of a number of commonly used single-equation linear models, all of which have rationally expected future explanatory variables. Fully efficient and less efficient instrumental variable estimators are proposed in each case. The choice of estimation method is usually represented as a trade-off between efficiency on the one hand and robustness and computational convenience on the other. It is shown in this paper that there is a more fundamental issue which must influence the choice of estimator, namely the type of solution that the model possesses. The construction of an efficient estimation method depends on whether or not the model has a unique solution and often this will not be known *a priori*. Preliminary estimation by instrumental variable methods can be used to resolve this question. Various tests are proposed in the paper.

Whiteman's solution method is used to determine the types of solution that are possible for each model. It is shown how these solutions can be written as both backwards and forwards solutions and the parameter restrictions which are required to obtain unique solutions.

JEL classification: 211

Keywords: rational expectations models, estimation, instrumental variables, Whiteman's solution method

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NON-TECHNICAL SUMMARY

There are now a large number of alternative methods of estimating linear econometric models which contain rational expectations variables. Researchers have sought estimation techniques which are statistically efficient, easy to compute and of general applicability. Despite many claims to the contrary, it appears that it is not possible to satisfy all three criteria. In particular, increased efficiency is likely to be achieved only at the expense of ease of computation and generality. In a 1982 paper the author showed how to use standard estimation procedures to obtain efficient estimates with the least computational cost. Unfortunately, these procedures lack generality: they are only available for models in which the only variables about which rational expectations are formed are dated in the current period. If rational expectations are formed concerning the value of a variable during some future period difficulties arise. The purpose of this paper is to extend the analysis by carrying out a systematic analysis of a number of models which have future dated rational expectations.

It is shown that before one can devise an efficient method of estimation it is necessary to know what type of solution the model has. Three types of solution are possible in rational expectations models: a globally stable solution, which in general is not unique; a saddlepoint solution and a globally unstable solution. The last two may or may not be unique. They may not even exist. Each of these solutions (where they exist) can be given both a backward and forward representation. These solutions can be expressed in a fairly general form, which may give the impression that a general way of obtaining an efficient estimator is also available. But each type of solution imposes a different set of restrictions on coefficients in the general representation, and efficient estimation requires that these restrictions are taken into account. The exception is the case of globally stable solutions which impose no restrictions and as a result are not unique. Thus a generally efficient method of estimation is not available for all solutions.

Unless there is some reason to impose a particular type of solution on the model, efficient estimation is not possible. One rationale for selecting a particular solution *a priori* is to assume that the solution is unique. Efficient estimation will then be possible but will require knowledge of the restrictions needed to be imposed. In the absence of such a rationale it will be necessary to discover what type of solution the model possesses before obtaining an efficient estimator. This will require preliminary estimation and hypothesis testing.

In this paper we analyse a number of commonly used types of model. The solution of each model is obtained using Whiteman's modification of Muth's method of undetermined coefficients. Different ways of representing these solutions are discussed and these representations are related to previous solutions that have appeared in the literature. Each solution is given one or more, backward and forward representations, and the restrictions associated with each representation and each type of solution are given. For each model and each type of solution both fully efficient and less efficient estimators are proposed. Inefficient estimators are usually obtained by replacing expectations with realisations. This in effect creates a model in which there is measurement error, the error in this case being the difference between the expected and realised values of the variable. Instrumental variable estimators are therefore appropriate in this case. Although these instrumental variable estimators are not in general fully efficient, the two-step, two-stage least squares estimator of Cumby, Huizinga and Obstfeld will be efficient within the class of instrumental variable estimators.

Due to its ease of computation and robustness, instrumental variable estimation is the obvious way to obtain preliminary estimates of the model prior to deriving fully efficient estimates. An alternative approach is to assume at the outset that a unique model solution exists and obtain fully efficient estimates of the model immediately. The coefficient restrictions imposed in order to obtain the unique solution can then be tested by re-estimating the model without the restrictions and carrying out either a Likelihood Ratio or a Lagrange Multiplier test.

1. Introduction

The choice of estimating linear models with rational expectations of endogenous variables by fully asymptotically efficient or by less efficient but consistent methods involves a trade-off. When the model is correctly specified efficient methods have the more desirable statistical properties and, moreover, will actually compute rational expectations.[1] A drawback is that efficient methods are usually not robust to misspecification. Imposing incorrect *a priori* restrictions either on the structural equations or on the information set will often impair the optimal properties of efficient estimators and even make them inconsistent. Another disadvantage is the greater computational complexity and cost of efficient methods. However, with the widespread availability of massive computing power computational cost is less of a problem than it used to be. Though the increasing preference for micro computers may entail computational constraints.

The most commonly used of the generally less efficient estimation methods is the errors in variables or instrumental variables method, McCallum (1976) and Wickens (1982).[2] Here the rational expectations are replaced in the structural equations by their realised values thereby creating measurement errors which are

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[1]Fully efficient estimators have been discussed for particular models by several authors, but general results have been obtained most notably by Hansen and Sargent (1980) and (1981), and by Wallis (1980). More recently Watson (1985) has proposed the use of recursive solution methods. Fair and Taylor (198) have considered the estimation of non-linear rational expectations models by iterative methods.

[2]See also Chow (1983) and Cumby, Huizinga and Obstfeld (1983).

the corresponding innovations. The equation is then estimated by the instrumental variable estimator with the information set (or a sub-set) providing the instrumental variables. In general the estimator will be consistent but not fully efficient and will involve the use of unrestricted predictions rather than rational expectations or model restricted predictions. In an important class of cases, however, this estimator while not fully rational will be asymptotically efficient, see Wickens (1982). This is when the model is globally stable and there are no rationally expected future endogenous variables in the structural system. Furthermore, consistency, but not efficiency, will be maintained even if an incomplete information set is used. The instrumental variable method can also be shown to be consistent when there are future rational expectations of endogenous variables in the model. Instrumental variable estimation is in general more robust to misspecification than fully efficient estimators and is very much easier to compute. For these reasons it is often preferred in practice. There is, however, an even more important reason why we might wish to use this estimator.

It is shown in this paper that until we know what type of rational expectations solution our model has we are not able to select the appropriate fully efficient estimator. In general linear models with future rational expectations can have three types of solution: globally stable, unstable and saddlepoint solutions. Globally stable solutions will not usually be unique while the other two may or may not be unique depending on the particular model. In order to construct an efficient estimator we need to know whether or not we have a unique solution. (In some cases we need to know more than this.) Unless there is good reason to assume a particular type of solution for the model, we will not know what the characteristics of the solution are and, in particular, if it is unique. Where a model is derived from a formal optimisation programme there is often a natural unique solution, or we impose conditions to produce one. In contrast it is possible to construct less efficient estimators which do not depend on the uniqueness of the solution. We can then use these estimates to determine what sort of solution we have and then, if desired, use this information to obtain fully efficient estimates. When no solution exists, even this will not be appropriate.

A number of different models are considered in this paper. One type is 'static' in that it contains no lagged endogenous variables. Another is 'dynamic' in that it has lagged endogenous variables. A third type permits expectations to be taken with respect to differently dated information sets. For every model investigated the following procedure is followed. First, the full solution is obtained for each possible outcome using Whiteman's (1983) method. For every feasible solution, two approaches to estimation are examined: a direct approach for which a fully efficient estimator is derived and the errors in variables approach for which both fully efficient and less efficient instrumental variable estimators are proposed.

The paper is set out as follows. In Section 2 we state the main assumptions of the paper and consider the basic static model. The corresponding 'dynamic' model is examined in Section 3. These models are then generalised in Section 4 to include higher order future expectations and in Section 5 to allow for information sets which have different dates. These results and some possible generalisations are discussed in Section 6.

2. The basic 'static' model

Consider the model

$$y_t = \alpha E_t y_{t+1} + \sum_{i=1}^{k} \beta_i x_{it} + e_t \qquad t = 1,\ldots,T \qquad (2.1)$$

where the x_{it} are stationary exogenous variables possibly including current and lagged values of an exogenous variable, e_t is i.i.d. $(0,\sigma_e^2)$ and is uncorrelated with the x_{it}. $E_t y_{t+1} = E(y_{t+1}/\Omega_t)$ where Ω_t is an information set which includes the x_{it}, possibly together with other contemporaneous exogenous variables and variables dated $t-i$ or before. It is also assumed to contain e_t which is known to agents but not to the econometrician.[1] The assumption of stationary x's can be weakened to allow certain forms of non-stationarity. The implications of this change will be discussed below. For notational convenience it will be assumed that there is only one exogenous variable (i.e. $k=1$). Extending the results to $k > 1$ is trivial. Thus (2.1) is specialised to

$$y_t = \alpha E_t y_{t+1} + \beta x_t + e_t \qquad (2.2)$$

It is assumed that x_t is generated by

$$x_t = \sum_{s=0}^{\infty} \theta_s \varepsilon_{t-s} \qquad (2.3)$$

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[1] This assumption can easily be modified and e_t can be excluded from the information set. The results of Section 5 can be specialised and then re-interpreted as excluding e_t from the information set. Including e_t is tantamount to assuming that the structural error contains variables whose individual importance is too small to be represented separately by the econometrician.

where ε_t is i.i.d. $(0, \sigma_\varepsilon^2)$, $E(\varepsilon_t e_t) = 0$ for all t, $\theta = 1$,

$\sum_{s=0}^{\infty} \theta_s^2 < \infty$ and $\sum_{s=0}^{\infty} \theta_s z^s$ is an analytical function on the open disk.

Although the analysis below is carried out with x_t represented by a moving average process, it is straightforward to re-interpret (2.3) as arising from an ARMA or an autoregressive process. Virtually the same general solution for y_t occurs, see McCallum (1985).[1] For estimation purposes, it will often be preferable to seek an ARMA or AR representation for x_t. Accordingly, where reference is made subsequently during estimation to (2.3), it is usually assumed that this has been derived from a general ARMA representation.[2]

We shall obtain the solution of (2.2) by applying the method of Whiteman (1983).[3] We can then consider alternative methods of estimation. The general solution of y_t which is appropriate for all

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[1]McCallum argues that one advantage of a minimal ARMA representation for x_t is that it can help eliminate bubbles in the solution for y_t.

[2]The asumption that $E\,\varepsilon_t e_t = 0$ for all t together with the assumptions that ε and e are i.i.d. processes impose strong exogeneity on x. It is possible to reduce this assumption to one of weakly exogenous x without altering the main estimation results. For example, lagged values of y (or e) could be allowed to determine x_t as well as lagged values of x and ε. Dropping $E\,\varepsilon_t e_t = 0$ would make x fully endogenous. As we wish to deal with single equation models in this paper we shall maintain this assumption. Moreover, it avoids certain identification problems, see Pudney (1981).

[3]Whiteman compares his solution method with most of the others in the literature. More recently Broze, Gourieroux and Szafarz (1984) have proposed a new procedure.

of the models we shall examine can be written

$$y_t = \sum_{s=0}^{\infty} a_s e_{t-s} + \sum_{s=0}^{\infty} b_s \varepsilon_{t-s} \tag{2.4}$$

where $\sum_{s=0}^{\infty} a_s^2 < \infty$, $\sum_{s=0}^{\infty} b_s^2 < \infty$ and $A(z) = \sum_{s=0}^{\infty} a_s z^s$

and $B(z) = \sum_{s=0}^{\infty} b_s z^s$ are analytic on an open disk. If L is the lag

operator $L^s x_t = x_{t-s}$ for any x_t, $s \geq 0$, then (2.3) and (2.4) can be

rewritten

$$x_t = \theta(L)\varepsilon_t \tag{2.5}$$

$$y_t = A(L)e_t + B(L)\varepsilon_t \tag{2.6}$$

where $\theta(L) = \sum_{s=0}^{\infty} \theta_s L^s$. Using the Weiner-Kolmogorov formulae

$$\begin{aligned} E_t\, x_{t+s} &= \sum_{i=0}^{\infty} \theta_{s+i}\, \varepsilon_{t-i} \\ &= \left[\frac{\theta(L)}{L^s}\right]_+ \varepsilon_t \\ &= L^{-s}\left[\theta(L) - \sum_{i=0}^{s-1} \theta_i L^i\right] \varepsilon_t \end{aligned} \tag{2.7}$$

it follows that

$$E_t y_{t+1} = L^{-1}(A(L) - a_0)e_t + L^{-1}(B(L)-b_0)\varepsilon_t \tag{2.8}$$

Substituting (2.5), (2.6) and (2.8) into (2.2) gives

$$\begin{aligned} A(L)e_t + B(L)\varepsilon_t = {} & \alpha L^{-1}((A(L) - a_0)e_t + (B(L) - b_0)\varepsilon_t) \\ & + \beta\theta(L)\varepsilon_t + e_t \end{aligned} \tag{2.9}$$

Equating terms in e_t and ε_t and introducing the z-transform, (2.9) implies that

$$z(1 - A(z)) + \alpha(A(z) - a_0) = 0 \tag{2.10}$$

$$z(\beta\theta(z) - B(z)) + \alpha(B(z) - b_0) = 0 \tag{2.11}$$

Hence,

$$A(z) = \frac{\alpha a_0 - z}{\alpha - z} \tag{2.12}$$

$$B(z) = \frac{\alpha b_0 - \beta z\theta(z)}{\alpha - z} \tag{2.13}$$

implying that the coefficients a_0 and b_0 are free parameters yet to be determined. Equations (2.12) and (2.13) are analytic for $|z| < 1$ if and only if $|\alpha| \geq 1$, otherwise $A(z)$ and $B(z)$ have a removable singularity at $\alpha \in \{z : |z| < 1\}$; see Whiteman p.7. We consider these cases in turn.

<u>Case 1: $|\alpha| \geq 1$</u>

In this case the model is globally stable, implying that the path back to equilibrium following a disturbance from it is not unique. In Wickens (1982) only globally stable solutions were considered. The coefficients a_0 and b_0 defined in (2.12) and (2.13) are not uniquely determined in this case and, moreover, there is no obvious way to choose them; see Whiteman Ch.3 for further discussion of the point. From (2.12) and (2.13) the solution for y_t is

$$y_t = \frac{[\alpha_0 - L]}{\alpha - L} \cdot e_t + \frac{[\alpha b_0 - \beta\theta(L)L]}{\alpha - L} \varepsilon_t$$

$$= \frac{1}{\alpha} y_{t-1} - \frac{\beta}{\alpha} x_{t-1} + a_0 e_t - \frac{1}{\alpha} e_{t-1} + b_0 \varepsilon_t \tag{2.14}$$

a) <u>Efficient Estimation</u>

Estimation of (2.14) can be carried out basically in two different ways depending on whether or not ε_t is treated as part of a composite disturbance.

(i) Composite disturbance including ε_t

If ε_t is assumed to be unknown to the econometrician (even though we have assumed that x_t is included in the information set) then the error term of equation (2.14) can be represented by the MA(1) process $v_t - \lambda v_{t-1}$ where λ lies within the unit circle and is a root of[1]

$$\lambda^2 - \frac{\left[(a_0^2 + \alpha^{-2})\sigma_e^2 + b_0^2\sigma_\varepsilon^2\right]\lambda}{a_0\sigma_e^2 / \alpha} + 1 = 0$$

and $\sigma_v^2 = \left[(a_0^2 + \alpha^{-2})\sigma_e^2 + b_0^2\sigma_\varepsilon^2\right] \Big/ (1+\lambda^2)$. Equation (2.14) could therefore be estimated as a dynamic model with y_{t-1} and x_{t-1} as the explanatory variables and the remaining right hand side variables forming an unrestricted MA(1) error. In the covariance matrix of the resulting disturbance the variance term is $\left[a_0^2 + \alpha^{-2}\right]\sigma_e^2 + b_0^2\sigma_\varepsilon^2$, the first order autocovariance is $a_0\sigma_e^2/\alpha$ and the other autocovariances are zero. The non-uniqueness of a_0 and b_0 is not therefore a problem if we assume that somehow agents have determined a_0 and b_0 but in a manner unknown to the econometrician who therefore regards a_0 and b_0 as unknown.

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[1]See also Ansley, Spivey and Wrobleski (1977).

There are two important disadvantages to the approach of including ε_t as part of a composite disturbance. Estimates of a_0 and b_0 are not obtainable and the estimates of the other coefficients are not fully efficient.

(ii) Separating ε_t from the disturbance

Fully efficient estimates, together with estimates of a_0 and b_0, can be obtained by estimating equations equations (2.3) and (2.14) simultaneously and taking account of the implied cross equation restrictions. The efficiency gain arises from the reduction in the variance of the disturbance, see also Pagan (1984). A computationally more attractive procedure which would also be asymptotically efficient is to estimate ε_t from (2.3), to substitute this estimate into (2.14) and then to estimate the resulting equation by non-linear methods. Note that ε_t is just the innovation in x_t. In effect this estimation procedure would be including ε_t in (2.14) as an observed (albeit constructed) variable. The error in estimating ε_t would be incorporated in the new disturbance term but as it is uncorrelated with all of the explanatory variables no problems are created thereby. The coefficient of ε_t in (2.14) provides an efficient estimator of b_0 and re-writing the disturbance of (2.14) as $a_0e_t - \frac{1}{\alpha a_0} a_0e_{t-1}$ we can obtain an estimate of a_0 from the coefficients of a_0e_{t-1} and y_{t-1}.

b) Instrumental Variable Estimation

An alternative to the direct estimation of (2.14) is to use the errors in variables approach.[1] This involves replacing $E_t y_{t+1}$ in equation (2.2) by y_{t+1} and estimating the resulting equation by instrumental variables. Since rational expectations implies that

$$y_{t+1} = E_t y_{t+1} + u_{t+1} \tag{2.15}$$

where u_{t+1} is the innovation in y_{t+1}, substituting $E_t y_{t+1}$ for y_{t+1} in (2.2) gives

$$y_t = \alpha y_{t+1} + \beta x_t + e_t - \alpha u_{t+1} \tag{2.16}$$

From (2.14) we can show that

$$E_t y_{t+1} = \frac{1}{\alpha} y_t - \frac{\beta}{\alpha} x_t - \frac{1}{\alpha} e_t$$

and hence

$$u_{t+1} = a_0 e_{t+1} + b_0 \varepsilon_{t+1}$$

implying that equation (2.5) can be re-written

$$y_t = \alpha y_{t+1} + \beta x_t - \alpha b_0 \varepsilon_{t+1} + e_t - \alpha a_0 e_{t+1} \tag{2.17}$$

Comparing equation (2.17) with the solution (2.14) it can be seen that (2.17) is just another way of writing (2.14). Replacing t in (2.17) by $t-1$ and renormalising on the new y_t (the previous y_{t+1}) gives (2.14). Thus (2.17) is also a solution to (2.2) for the case

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[1] McCallum (1976) and Wickens (1982).

$|\alpha| \geq 1$. The rational expectations solution methods of Gourieroux, Laffont and Monfort (1982) and Broze, Gourieroux, Szafarz (1984) are in fact based on such an errors in variables approach. In obtaining (2.17) we have used our knowledge of the solution (2.14) in deriving the relationship between u_{t+1} and e_{t+1} and ε_{t+1}. In particular, we have taken account of a_0 and b_0 being undetermined when $|\alpha| \geq 1$. Without knowledge of the solution it would not have been possible to evaluate u_{t+1} in this way solely on the basis of (2.16). We return to this point for the case $|\alpha| < 1$.

The estimation of equations like (2.17) by instrumental variable methods has been considered by Cumby, Huizinga and Obstfeld (1983) and Hayashi and Sims (1983). The simplest but least efficient method of IV estimation is to incorporate ε_{t+1} into a composite MA(1) disturbance as above and to estimate the resulting equation by the ordinary IV estimator. The explanatory variables will be y_{t+1} and x_t, and y_{t+1} will need instrumenting. Although the MA(1) structure of the disturbance term has been ignored during estimation, it must be taken into account when deriving the limiting distribution of the IV estimator. If the model is written

$$y = X\beta + u \tag{2.18}$$

where $E(u) = 0$, $E(uu') = \Sigma$, X is a $T \times k$ and Z is a $T \times \ell$ $(\ell > k)$ matrix of instruments, the IV estimator for β is

$$\hat{\beta}_{IV} = (H'X)^{-1}H'y \tag{2.19}$$

where $H = Z(Z'Z)^{-1}Z'X$ with asymptotic covariance matrix

$$V_{IV} = \text{plim}\ T^{-1}(H'X)^{-1}\ H'\Sigma H(X'H)^{-1} \tag{2.20}$$

Valid instrumental variables include lagged values of y_t and current and lagged values of x_t. Using (2.14) and the fact that e_t and ε_t are white noise processes it can be shown, for example, that for the asymptotic covariance between y_{t-1} and the disturbance of (2.17) is

$$
\begin{aligned}
& - \text{plim}\ T^{-1}\ \Sigma y_{t-1}\alpha \left[a_0 e_{t+1} - \frac{1}{\alpha} e_t + b_0 \varepsilon_{t+1}\right] \\
& = - \text{plim}\ T^{-1}\Sigma \left[\frac{1}{\alpha} y_{t-2} - \frac{\beta}{\alpha} x_{t-1} + a_0 e_{t-1} - \frac{1}{\alpha} e_{t-2} + b_0 \varepsilon_{t-1}\right] \\
& \qquad \alpha \left[a_0 e_{t+1} - \frac{1}{\alpha} e_t + b_0 \varepsilon_{t+1}\right] = 0
\end{aligned}
$$

implying that y_{t-1} is a valid instrumental variable.

A more efficient IV estimator of (2.17) than the ordinary IV estimator is the two-step two-stage least squares (2S2SLS) estimator of Cumby, Huizinga and Obstfeld (1983). This estimator takes account of the MA(1) structure of the disturbance of (2.17) and is given by

$$\hat{\beta} = (X'Z\hat{\Omega}^{-1}Z'X)^{-1}X'\hat{\Omega}^{-1}Z'y \tag{2.21}$$

where $\hat{\Omega}$ is a consistent estimate of $\Omega = \text{plim}\ T^{-1}Z'uu'Z$. The limiting distribution of $T(\hat{\beta}-\beta)$ is $N(0,V)$ where $V = \text{plim}\ T(X'Z\Omega^{-1}Z'X)^{-1}$. Alternative choices of $\hat{\Omega}$ have been proposed by Hansen (1982) and Cumby, Huizinga and Obstfeld. Recalling that we can write $u_t = v_t - \lambda v_{t+1}$ and that $E(uu') = \Sigma$, we can also use $\hat{\Omega} = T^{-1}Z'\hat{\Sigma}Z$ where we replace λ and σ_v^2 in Σ by consistent estimators. One way to implement this is to note that

$$\Sigma = (1+\lambda^2)\sigma_v^2 \begin{bmatrix} 1 & -\gamma & 0 \ldots 0 \\ \gamma & 1 & -\gamma \ldots 0 \\ 0 \ldots\ldots & -\gamma & 1 \end{bmatrix} = (1+\lambda^2)\,\sigma_v^2\Sigma^*$$

where $\gamma = -\lambda/(1+\lambda^2)$ is the first order autocorrelation coefficient of u_t. Using a consistent estimator $\hat{\gamma}$ of γ and noting that $\hat{\beta}$ is homogeneous of degree zero in $T^{-1}(1+\lambda^2)\sigma_v^2$, instead of $\hat{\Omega}$, we can use $\hat{\Omega}^* = Z'\hat{\Sigma}^*Z$ where $\hat{\Sigma}^*$ is Σ^* with γ replaced by $\hat{\gamma}$. The resulting estimator of β will have the same asymptotic distribution as $\hat{\beta}$. Cumby, Huizinga and Obstfeld show that (2.21) is asymptotically efficient in the class of instrumental variable estimators.

It may be noted that the most efficient estimator for models with autoregressive errors is Theil's (1958) generalised instrumental variable estimator which is given by[1]

$$\hat{\beta}_{GIV} = (\tilde{H}'X)^{-1}\tilde{H}'y$$

where $\tilde{H} = \Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-1}Z'\Sigma^{-1}X$. For (2.17), however, this would not be consistent for the same choice of instruments because $\text{plim}\, T^{-1}Z'\Sigma^{-1}u \neq 0$. To see this we note that for $v_{T+1} = 0$

$$\begin{bmatrix} u_1 \\ \vdots \\ u_T \end{bmatrix} = \begin{bmatrix} 1 & -\lambda & 0 \ldots 0 \\ 0 & 1 & -\lambda \ldots 0 \\ & & \ddots \quad -\lambda \\ 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ \\ v_T \end{bmatrix}$$

or $u = AV$ and that $E(uu') = E(Avv'A') = \sigma_v^2 AA' = \Sigma$.

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[1]See also, Wickens (1969).

Moreover, $\Sigma_v^{-1} = \sigma^{-2} A'^{-1} A^{-1}$ and

$$A^{-1} = \begin{bmatrix} 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & \lambda & \dots & 0 \\ & & & \ddots & \lambda \\ 0 & & & & 1 \end{bmatrix}$$

Hence for Z_i, the ith column of Z, we have

$$\text{plim } T^{-1} Z_i' \Sigma^{-1} u_i = \text{plim } T^{-1} \sigma^{2} (A^{-1} Z_i)' A_i^{-1}) u$$

$$= \sigma_v^{-2} \text{ plim } T^{-1} \sum_{t=1}^{T-2} (\lambda Z_{i,t} + (1+\lambda^2) z_{i,t+1} + \lambda z_{i,t+2}) u_t$$

$$\neq 0$$

This follows because $u_t = e_t - \alpha a_0 e_{t+1} - \alpha b_0 \varepsilon_{t+1}$ and hence for $Z_{it} = x_t$ we have $E(Z_{i,t+1}\varepsilon_{t+1}) \neq 0$, while for $Z_{it} = y_{t-1}$ we have $E(Z_{it+1}e_t)$, $E(Z_{i,t+2}e_{t+1}) \neq 0$. Only if we exclude y_{t-1}, y_{t-2} and x_t from the instrumental variables will the GIV estimator be consistent. Comparisons of asymptotic efficiency with 2S2SLS are then complicated by the fact that the set of instrumental variables is different for each estimator. The inconsistency of $\hat{\beta}_{GIV}$ is not altered if we drop the restriction $v_{T+1} = 0$ and calculate a more accurate expression for $\text{plim } T^{-1} Z_i \Sigma^{-1} u$ that does not exclude terms for T-1 and T.

In our discussion of IV estimation we have considered the case where ε_{t+1} has been incorporated into a composite disturbance u_t. It would, however, be possible to separate ε_{t+1} from the disturbance and to use in our estimating procedure information about ε_{t+1} obtained from (2.3), the process generating x_t. An obvious way to do this is to replace ε_{t+1} in (2.17) by an estimate $\hat{\varepsilon}_{t+1}$ obtained from prior estimation of (2.3). The estimating equation would then become

$$y_t = \alpha y_{t+1} + \beta x_t - \alpha b_0(\varepsilon_{t+1} - \hat{\varepsilon}_{t+1}) + w_t \qquad (2.22)$$

where $w_t = e_t - \alpha a_0 e_{t+1} - \alpha b_0(\varepsilon_{t+1} - \hat{\varepsilon}_{t+1})$. We may note that $\hat{\varepsilon}_{t+1}$ and the instruments used above will be asymptotically uncorrelated with w_t. Hence, we can estimate (2.22) by either the ordinary IV estimator or by 2S2SLS. The resulting estimators will be at least as efficient as the corresponding estimators which incorporate ε_{t+1} into a composite disturbance, u_t.

Case 2: $|\alpha| < 1$

In this case the model is unstable. To achieve equilibrium following a disturbance, y_t must jump straight to its new equilibrium. In terms of (2.12) and (2.13) the coefficients a_0 and b_0 will not now be free parameters. They can be selected to make $A(\alpha)$ and $B(\alpha)$ analytic by imposing the condition that their residues are zero. Thus, from (2.12), a_0 is chosen to satisfy

$$\lim_{z\to\alpha} (\alpha-z)\, A(z) = \alpha(a_0-1) = 0$$

implying that $a_0 = 1$. From (2.13), b_0 is chosen to satisfy

$$\lim_{z\to\alpha} (\alpha-z)\, B(z) = \alpha b_0 - \alpha\beta\theta(\alpha) = 0$$

implying that $b_0 = \beta\theta(\alpha)$. Hence from (2.3) and (2.6), the solution of y_t is

$$y_t = \frac{(\alpha.1 - L)}{\alpha - L}\, e_t + \frac{(\alpha\beta\theta(\alpha) - \beta\theta(L)L)}{\alpha - L}\, \varepsilon_t$$

$$= e_t + \beta\, \frac{\left[1 - \alpha L^{-1}\theta(\alpha)\theta(L)^{-1}\right]}{1 - \alpha L^{-1}}\, x_t$$

Defining $L^{-s}x_t = E_t x_{t+s}$ it can be shown that (see Hansen and Sargent (1980) or Whiteman p.57)

$$\frac{1 - \alpha L^{-1}\theta(\alpha)\theta(L)^{-1}}{1 - \alpha L^{-1}} \cdot x_t = \sum_{s=0}^{\infty} \alpha^s E_t X_{t+s} \qquad (2.23)$$

Hence

$$y_t = \beta \sum_{s=0}^{\infty} \alpha^s E_t x_{t+s} + e_t \qquad (2.24)$$

It is clear form (2.24) that y_t is always in equilibrium conditional on the given information.

a) Efficient Estimation

The estimation of α and β could now be based on (2.24). This can be carried out using (2.7) to substitute for terms in $E_t x_{t+s}$ to give

$$y_t = \beta \sum_{i=0}^{\infty} \left[\sum_{s=0}^{\infty} \alpha^s \theta_{s+i} \right] \varepsilon_{t-i} + e_t \tag{2.25}$$

$$= \beta \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-i} + e_t$$

where $\gamma_i = \sum_{s=0}^{\infty} \alpha^s \theta_{s+i}$. Truncating the summation of (2.3), and hence (2.25), estimating (2.3) and (2.25) jointly by non-linear maximum likelihood, and imposing the across equation non-linear restrictions associated with γ_i would produce an asymptotically efficient estimator of α and β.

In practice truncating the summation in (2.25) will not be necessary if x_t can be expressed as a finite ARMA (p, q) process. The solution for y_t can then be expressed in the form

$$y_t = \sum_{i=0}^{p-1} h_i x_{t-i} + \sum_{i=0}^{q-1} k_i \varepsilon_{t-i} + e_t \tag{2.26}$$

where the coefficients h_i, k_i are functions of the model parameters.[1] Estimating (2.26) jointly with (2.3) and imposing the non-linear restrictions associated with h_i, k_i will enable asymptotically efficient estimates to be obtained.

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[1] See the Appendix for further details of the derivation of equation (2.26)

b) Instrumental Variable Estimation

An alternative is to use the errors in variable approach and to estimate (2.16) by an instrumental variable estimator. The innovation u_{t+1} will be different from before. From the solution, equation (2.24), we find that

$$E_t\, y_{t+1} = \beta \sum_{s=0}^{\infty} \alpha^s E_t x_{t+s+1}$$

and hence

$$u_{t+1} = \beta \sum_{s=0}^{\infty} \alpha^s (E_{t+1} - E_t) x_{t+s+1} + e_{t+1}$$

$$= \beta \left[\sum_{s=0}^{\infty} \alpha^s \theta_s \right] \varepsilon_{t+1} + e_{t+1}$$

$$= \beta\gamma_0 \varepsilon_{t+1} + e_{t+1}$$

Equation (2.16) can therefore be written

$$y_t = \alpha y_{t+1} + \beta x_t - \alpha\beta\gamma_0 \varepsilon_{t+1} + e_t - \alpha e_{t+1} \tag{2.27}$$

This has the same structure as (2.17), the corresponding equation in case 1. The difference is that a_0 and b_0 do not appear in (2.27) because they are now uniquely determined as functions of the basic parameters of the model. Like (2.17) equation (2.27) is an alternative way of writing the solution for y_t. Setting t equal to $t-1$ and renormalising on the new y_t enables (2.27) to be re-written as

$$y_t = \frac{1}{\alpha} y_{t-1} - \frac{\beta}{\alpha} x_{t-1} + \beta\gamma_0 \varepsilon_t + e_t - \frac{1}{\alpha} e_{t-1}. \tag{2.28}$$

This is a backward version of the solution analagous to (2.14) but without the indeterminacy created by the presence of a_0 and b_0. Equation

(2.28) can also be derived from (2.14) by imposing the restrictions $a_0 = 1$ and $b_0 = \beta\theta(\alpha)$.

Alternative instrumental variable estimates of (2.27) can be obtained in the same was as for (2.17). Another more efficient class of IV estimators is available if we replace ε_{t+1} in (2.27) by an estimate $\hat{\varepsilon}_{t+1}$ obtained as before and treat $\hat{\varepsilon}_{t+1}$ as an observable variable. A further gain in efficiency is possible if account is taken of the restriction between the coefficients of y_{t+1} and e_{t+1}.

Fully efficient estimates can also be obtained by estimating (2.28) jointly with (2.3) using the maximum likelihood estimator and taking into account the cross-equation restrictions. It should be noted that the coefficient of y_{t-1} in (2.28) will be greater than unity. Normally when estimating a difference equation we make the assumption that the equation is stable. But because $|\alpha| < 1$, (2.28) is not stable and hence is not a conventional difference equation. Depending on the algorithm this may affect the convergence properties of the program used to compute the maximum likelihood estimates. It does not, however, influence the existence of maximum likelihood estimates. Conventional theory also makes the assumption that the model is stable when deriving the distribution of the maximum likelihood estimator. Clearly the standard proofs will not therefore apply here. In particular, y_t will need to be solved forwards not backwards. Once this is done there is no problem about deriving the distribution of the maximum likelihood estimator.

Drawing together our results so far, we have found that whether or not the model (2.2) is stable or unstable will influence both the solution and the method of obtaining efficient estimates. An efficient estimator requires that all of the available information is taken into account and without prior knowledge of whether the model is stable we are unable to do this. In contrast the instrumental variable estimators proposed do not depend on prior knowledge of the stability of the model. This suggests that we should first estimate the model

$$y_t = \alpha y_{t+1} + \beta x_t + v_t$$

by instrumental variables possibly allowing for the fact that v_t is a leading MA(1) variable and then use the IV estimate of α to determine whether (2.2) has a stable or unstable solution. Depending on the outcome we can then choose the appropriate fully efficient estimator. As we shall see, this approach is appropriate for all of the models considered in this paper.[1]

In deriving the solution for y_t above using Whiteman's method we have assumed that x_t is stationary. This assumption can be relaxed somewhat. If $|\alpha| < 1$ we can obtain the solution for y_t given by (2.24) directly from (2.2) using successive substitution methods to eliminate $E_t y_{t+1}$. Thus for $n > 0$

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[1]Chow (1983) has also proposed a general method of estimating RE models based on replacing expectations with realisations and including the innovations as explanatory variables. In effect his procedure amounts to the unrestricted estimation of (2.14) excluding any consideration of possible restrictions which can arise in the case of unique (and other) solutions. In general Chow's approach will not therefore lead to efficient estimation. See also Chow and Reny (1985).

$$y_t = \alpha^n E_t y_{t+n} + \beta \sum_{s=0}^{n-1} \alpha^s E_t x_{t+s} + \sum_{s=0}^{n-1} \alpha^s E_t e_{t+s} \tag{2.29}$$

and as $n \to \infty$ (2.29) converges to (2.24). This solution procedure has not made any explicit assumptions about the stationarity of x_t. In fact, all we require is that the expression $\sum_{s=0}^{\infty} \alpha^s E_t x_{t+s}$ exists. If, for example, $x_t = \mu x_{t-1} + \epsilon_t$ then $\sum_{s=0}^{\infty} \alpha^s E_t x_{t+s} = x_t/(1-\alpha\mu)$. The required condition, therefore, is that $|\alpha\mu| < 1$ and not that $|\mu| < 1$. In particular, for $1 > \alpha > 0$ we can have $\frac{1}{\alpha} > \mu > 1$.

3. Models with lagged dynamics

In this section we extend the basic model by including lagged values of the dependent variable as explanatory variables. We begin by examining the special case where

$$y_t = \alpha E_t y_{t+1} + \delta y_{t-1} + \beta x_t + e_t \tag{3.1}$$

The results can easily be generalised to allow for the inclusion of more than one exogenous variable and, with a little more effort, to permit higher order lags in y. We make the same assumptions as before and where possible we will use the previous notation.

The solution of (3.1) is obtained using the same procedure as before. It can be written in the general form (2.6). From (2.5) - (2.8) we can express (3.1) as

$$\begin{aligned} A(L)e_t + B(L)\varepsilon_t = {} & \alpha L^{-1}((A(L) - a_0)\, e_t + (B(L) - b_0)\varepsilon_t) \\ & + \delta L(A(L)e_t + B(L)\varepsilon_t) + \beta\theta(L)\varepsilon_t + e_t \end{aligned} \tag{3.2}$$

This implies that

$$A(z) = \frac{\alpha a_0 - z}{\alpha - z + \delta z^2} \tag{3.3}$$

$$B(z) = \frac{\alpha b_0 - z\beta\theta(z)}{\alpha - z + \delta z^2} \tag{3.4}$$

Let η_1 and η_2 denote the roots of the quadratic equation

$$\alpha - z + \delta z^2 = 0 \tag{3.5}$$

then $\eta_1\eta_2 = \alpha/\delta$, $\eta_1 + \eta_2 = 1/\delta$. Assuming these roots are real, three possibilities exist: (i) $|\eta_1|, |\eta_2| \geq 1$

(ii) $|\eta_1| \geq 1$, $|\eta_2| < 1$

and (iii) $|\eta_1|, |\eta_2| < 1$

Case 1: $|\eta_i| > 1$, $i = 1,2$

In this case the model is globally stable and will converge non-uniquely on a new equilibrium following any disturbance. a_0 and b_0 are again free parameters and from (2.6) the solution for y_t can be written

$$(\alpha - L + \delta L^2)\, y_t = (\alpha a_0 - L)\, e_t + (\alpha b_0 - \beta\theta(L)L)\, \varepsilon_t$$

or

$$y_t = \frac{1}{\alpha} y_{t-1} - \frac{\delta}{\alpha} y_{t-2} - \frac{\beta}{\alpha} x_{t-1} + a_0 e_t - \frac{1}{\alpha} e_{t-1} + b_0 \varepsilon_t \qquad (3.6)$$

The disturbance term of (3.6) is the same as that of (2.14).

a) Efficient Estimation

Estimation of (3.6) may be carried out using similar methods to those proposed for (2.14) because the additional presence of y_{t-2} creates no new issues of note.

b) Instrumental Variable Estimation

Using the errors in variables approach we replace $E_t y_{t+1}$ in (3.1) by y_{t+1} to give

$$y_t = \alpha y_{t+1} + \delta y_{t-1} + \beta x_t + e_t - \alpha u_{t+1} \qquad (3.7)$$

From (3.6) the innovation $u_{t+1} = a_0 e_{t+1} + b_0 \varepsilon_{t+1}$ and hence

$$y_t = \alpha y_{t+1} + \delta y_{t-1} + \beta x_t - \alpha b_0 \varepsilon_{t+1} + e_t - \alpha a_0 e_{t+1} \qquad (3.8)$$

which apart from the presence of y_{t-1} is identical to (2.17). Again

(3.8) is an alternative way of writing the solution for y_t. Equation (3.6) can be derived from (3.8) by setting t equal to $t-1$ and renormalising on y_t.

Equation (3.8) can be estimated by instrumental variable methods in the same way as (2.17). We may note that lagged values of y_t remain valid instruments. To see this, consider

$$\operatorname{plim} \frac{1}{T} \sum_t y_{t-1} (e_t - \alpha u_{t+1})$$

$$= \operatorname{plim} \frac{1}{T} \Sigma \left(\frac{1}{\alpha} y_{t-2} - \frac{\theta}{\alpha} y_{t-3} - \frac{\beta}{\alpha} x_{t-1} + a_0 e_{t-1} - \frac{1}{\alpha} e_{t-2} + b_0 \varepsilon_{t-1}\right)$$

$$(e_t - \alpha a_0 e_{t+1} - \alpha b_0 \varepsilon_{t+1})$$

$$= 0$$

Case 2: $|\eta_1| \geq 1$, $|\eta_2| < 1$

In this case we have a saddlepoint solution. There is a unique stable manifold, or saddlepoint path, along which convergence to equilibrium takes place and, following a disturbance, y_t jumps straight on to that path, see Wickens (1985). The coefficients a_0 and b_0 can be selected uniquely to achieve this by making $A(\eta_2)$ and $B(\eta_2)$ analytic through imposing the condition that the residues of A(z) and B(z) at $z = \eta_2$ satisfy

$$\lim_{z \to \eta_2} (\eta_2 - z) A(z) = \frac{\alpha a_0 - \eta_2}{\theta(\eta_1 - \eta_2)} = 0$$

$$\lim_{z \to \eta_2} (\eta_2 - z) B(z) = \frac{\alpha b_0 - \eta_2 \beta \theta(\eta_2)}{\theta(\eta_1 - \eta_2)} = 0$$

It follows that we select $a_0 = \eta_2/\alpha$ and $b_0 = \eta_2\beta\theta(\eta_2)/\alpha$. We note that $A(\eta_1)$ and $B(\eta_1)$ are analytic. From (2.6), (3.3) and (3.4) we can now express y_t as

$$y_t = \frac{(\eta_2-L)}{\delta(\eta_1-L)(\eta_2-L)}\cdot e_t + \frac{(\eta_2\beta\theta(\eta_2) - \beta\theta(L)L)}{\delta(\eta_1-L)(\eta_2-L)}\cdot \epsilon_t$$

Thus

$$\delta(\eta_1-L)y_t = e_t + \beta\,\frac{\left[1-\eta_2L^{-1}\theta(\eta_2)\theta(L)^{-1}\right]}{1-\eta_2L^{-1}}\;x_t$$

and the solution for y_t is

$$y_t = \frac{1}{\eta_1}\,y_{t-1} + \frac{\beta}{\delta\eta_1}\sum_{s=0}^{\infty}\eta_2^s\,E_tX_{t+s} + \frac{1}{\delta\eta_1}\,e_t \qquad (3.9)$$

In (3.9) the unstable root η_2 has been solved forward as in (2.24) and the stable root η_1 has been solved backwards as in (2.14).

It is instructive to re-write (3.9) as

$$\Delta y_t = \left[1 - \frac{1}{\eta_1}\right]\left[\bar{y}_t - y_{t-1}\right] \qquad (3.10)$$

where

$$\bar{y}_t = \frac{\beta}{\delta(\eta_1-1)}\sum_{s=0}^{\infty}\eta_2^s\,E_tx_{t+s} + \frac{1}{\delta(\eta_1-1)}\,e_t$$

Equation (3.10) shows that the solution (3.8) can be interpreted as a partial adjustment model with $\bar{y}_t$, the target value of y_t, based on information available in period t. In other words, movement along the saddlepoint path follows a partial adjustment mechanism and the jump onto the path is captured by a change in $\bar{y}_t$

It should be noted that $\tilde{y}_t$ will vary through time and, if the exogenous variables include current and lagged values, then a change in an exogenous variable will result in more than one change in $\tilde{y}_t$. For an example of this involving Dornbush's overshooting model of exchange rates, see Wickens (1984).

The partial adjustment model obtained here can be compared with the derivation of partial adjustment models of Brechling (1975), Eisner and Strotz (1963), Mortensen (1973) and Treadway (1971). They showed that if expectations are static, and the optimal solution of the decision variable is obtained as a result of maximising a multi-period objective function involving quadratic costs, then the dynamic behaviour of the decision variable could be represented by a partial adjustment mechanism. As here, the fundamental dynamic equation of the decision variable in this earlier work is a second order autoregressive process with one stable and one unstable root. The unstable root is eliminated by imposing the transversality condition and the stable root provides the adjustment parameter.

a) <u>Efficient Estimation</u>

As for equation (2.24), estimation of (3.8) can be carried out by expressing the distributed lead term as a distributed lag in ε_t. Thus (3.8) can be re-written

$$y_t = \frac{1}{\eta_1} y_{t-1} + \frac{\beta}{\delta\eta_1} \sum_{i=0}^{\infty} \phi_i \varepsilon_{t-i} + \frac{1}{\delta\eta_1} e_t \tag{3.11}$$

where $\phi_i = \sum_{s=0}^{\infty} \eta_2^s \theta_{s+i}$. Truncating the summation, estimating jointly with equation (2.3) and imposing the cross-equation restrictions implied by ϕ_i would provide asymptotically efficient estimates of η_1, η_2 and $\frac{\beta}{\delta\eta_1}$. From (3.5) we have $\delta\eta_1\eta_2 = \alpha$ and $\delta(\eta_1 + \eta_2) = 1$. Using these we can easily obtain estimates of the basic parameters of (3.1), namely α, β and δ. In an analogous argument to that used in Section 2 for equations (2.25) and (2.26), if x_t can be expressed as an ARMA (p,q) process, then truncation of (3.11) can be avoided by writing the solution of y_t as

$$y_t = \frac{1}{\eta_1} y_{t-1} + \sum_{i=0}^{p-1} h_i x_{t-i} + \sum_{i=0}^{q-1} k_i \varepsilon_{t-i} + \frac{1}{\delta\eta} e_t \qquad (3.12)$$

where again h_i; k_i are functions - different ones - of the basic parameters.[1] Asymptotically efficient estimates of the parameters of (3.1) can be obtained once more by joint non-linear estimation of (3.12) and (2.3) having re-written (2.3) as an ARMA process.

b) Instrumental Variable Estimation

This is based once more on equation (3.7). From (3.9) or (3.11) the innovation u_{t+1} can be expressed as

$$u_{t+1} = \frac{1}{\eta_1}(y_t - E_t y_t) + \frac{\beta}{\delta\eta_1} \sum_{s=0}^{\infty} \eta_2^s (E_{t+1} - E_t) x_{t+s+1} + \frac{1}{\delta\eta_1} e_{t+1}$$

$$= \frac{1}{\delta\eta_1}(\beta\phi_0\varepsilon_{t+1} + e_{t+1}) \qquad (3.13)$$

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[1] See the Appendix for further details of the derivation of (3.12)

Hence equation (3.7) becomes

$$y_t = \alpha y_{t+1} + \delta y_{t-1} + \beta x_t - \frac{\alpha\beta\phi_0}{\delta\eta_1}\varepsilon_{t+1} + e_t - \frac{\alpha}{\delta\eta_1}e_{t+1} \qquad (3.14)$$

which has the same structure as (3.8) and can be estimated by instrumental variables in the same way. Again lagged values of y_t are valid instruments since, for example,

$$\text{plim}\,\frac{1}{T}\;\Sigma\, y_{t-1}\,(e_t - \alpha u_{t+1})$$

$$= \text{plim}\,\frac{1}{T}\,\Sigma\left[\frac{1}{\eta_1}y_{t-2} + \frac{\beta}{\delta\eta_1}\sum_{s=0}^{\infty}\eta_2^s\,E_{t-1}x_{t+s-1} + \frac{1}{\delta\eta_1}e_{t-1}\right]$$

$$\left[e_t - \frac{\alpha}{\delta\eta_1}e_{t+1} - \frac{\alpha\beta\phi_0}{\delta\eta_1}\varepsilon_{t+1}\right]$$

$$= 0$$

Since (3.14) is an alternative way to (3.9) of expressing the solution for y_t and can be re-written as

$$y_t = \frac{1}{\alpha}y_{t-1} - \frac{\delta}{\alpha}y_{t-2} - \frac{\beta}{\alpha}x_{t-1} + \frac{\beta\phi_0}{\delta\eta_1}\varepsilon_t + \frac{1}{\delta\eta_1}e_t - \frac{1}{\alpha}e_{t-1} \qquad (3.15)$$

we can also obtain fully efficient estimates by estimating (3.15) and (2.3) jointly by maximum likelihood and methods taking into account the cross-equation parameter restrictions.

Case 3: $|\eta_i| < 1, \quad i = 1,2$

In this case it can be shown that no solution for y_t exists. Because both roots lie within the unit circle, $A(z)$ and $B(z)$, given by equations (3.3) and (3.4), will not be analytic but will each contain two singularities at η_1 and η_2. For $A(z)$ and $B(z)$ to be analytic for $|z| < 1$, we require $\lim_{z \to \eta_i} (\eta_i - z) A(z) = 0$ and $\lim_{z \to \eta_i} (\eta_i - z) B(z) = 0$ for $i = 1,2$. Thus, four conditions must be satisfied and there are only two free parameters a_0 and b_0 available to achieve this. A solution is not therefore possible for x_t defined by equation (2.3). Even if $\eta_1 = \eta_2$ the problem remains. See Whiteman p.15 for further discussion.

The results of this section can be generalised to models with higher order lags in y such as

$$y_t = \alpha E_t y_{t+1} + \sum_{s=1}^{m} \delta_s y_{t-s} + \beta x_t + e_t . \tag{3.16}$$

Provided (3.16) has not more than one unstable root only obvious amendments are required to the above results. In solution (3.6) the highest order lag becomes y_{t-m-1} and in equations (3.8), (3.9) and (3.14) it is y_{t-m}. If there is more than one unstable root then no solution to (3.16) will exist.

4. Models with more than one future expected variable

Another important extension of the basic model of Section 2 is where we have more than one rationally expected future variable. A general representation is

$$y_t = \sum_{s=1}^{n} \alpha_s E_t y_{t+s} + \beta x_t + e_t \tag{4.1}$$

We shall consider first the case where $n = 2$:

$$y_t = \alpha_1 E_t y_{t+1} + \alpha_2 E_t y_{t+2} + \beta x_t + e_t \tag{4.2}$$

The extension of these result to the general case of (4.1) is easily made. The previous assumptions about x_t and e_t will be maintained.

The solution of (4.2) is obtained as before. From (2.5) -(2.8) we can rewrite (4.2) as

$$\begin{aligned} A(L)e_t + B(L)\varepsilon_t = \alpha_1^{-1} & \{(A(L) - a_0)e_t + (B(L) - b_0)\varepsilon_t\} \\ & + \alpha_2 L^{-2}\{(A(L) - a_0 - a_1 L)e_t + (B(L) - b_0 - b_1 L)\varepsilon_t\} \\ & + \beta\theta)L\ \varepsilon_t + e_t \end{aligned} \tag{4.3}$$

Hence

$$A(z) = \frac{\alpha_2 a_0 + (\alpha_1 a_0 + \alpha_2 a_1)z - z^2}{\alpha_2 + \alpha_1 z - z^2} \tag{4.4}$$

$$B(z) = \frac{\alpha_2 b_0 + (\alpha_1 b_0 + \alpha_2 b_1)z - \beta z^2 \theta(z)}{\alpha_2 + \alpha_1 z - z^2} \tag{4.5}$$

Equations (4.4) and (4.5) have four free parameters a_0, a_1, b_0 and b_1.

Let η_1 and η_2 denote the roots of $z^2 - \alpha_1 z - \alpha_2 = 0$ and assume they are real. Three possibilities exist: (i) $|\eta_1|, |\eta_2| \geq 1$, (ii) $|\eta_1| \geq 1$, $|\eta_2| < 1$ and (iii) $|\eta_1|, |\eta_2| < 1$. We consider each in turn.

<u>Case 1: $|\eta_i| \geq 1$, $i = 2$</u>

The model is globally stable in this case and the four free parameters cannot be chosen uniquely. From (2.6) we can write y_t as

$$(\alpha_2+\alpha_1 L-L^2)y_t = (\alpha_2 a_0 + (\alpha_1 a_0 + \alpha_2 a_1)L-L^2)e_t$$

$$+ (\alpha_2 b_0 + (\alpha_1 b_0+\alpha_2 b_1)L-\beta\theta(L)L^2)\varepsilon_t$$

or

$$y_t = -\frac{\alpha_1}{\alpha_2} y_{t-1} - \frac{1}{\alpha_2} y_{t-2} - \frac{\beta}{\alpha_2} x_{t-2} + a_0 e_t + \left[\frac{\alpha_1 a_0}{\alpha_2} + a_1\right] e_{t-1} - \frac{1}{\alpha_2} e_{t-2}$$

$$+ b_0 \varepsilon_t + \left[\frac{\alpha_1 b_0}{\alpha_2} + b_1\right]\varepsilon_{t-1} \qquad (4.6)$$

a) <u>Efficient Estimation</u>

Estimation of (4.6) can be achieved using suitably amended methods to those proposed for (2.14). The terms in e_t, e_{t-1}, e_{t-2}, ε_t and ε_{t-1} could be treated as a composite MA(2) disturbance and an unrestricted MA(2) estimator applied with y_{t-1}, y_{t-2} and x_{t-2} as explanatory variables. Fully efficient estimates are derived by the joint estimation of (2.3) and (4.6) in which only the terms in e_t, e_{t-1} and e_{t-2} are incorporated into the disturbance. Alternatively, ε_t and ε_{t-1} can be replaced in (4.6) with estimates obtained from prior estimation of (2.3).

b) Instrumental Variable Estimation

The errors in variables approach can be implemented by re-writing (4.2) as

$$y_t = \alpha_1 y_{t+1} + \alpha_2 y_{t+2} + \beta x_t + (e_t - \alpha_1 u_{t+1} - \alpha_2 u_{t+2}) \qquad (4.7)$$

where $u_{t+i} = y_{t+i} - E_t y_{t+i} (i = 1,2)$. From (4.6)

$$u_{t+1} = a_0 e_{t+1} + b_0 \varepsilon_{t+1}$$

$$u_{t+2} = \frac{-\alpha_1}{\alpha_2} u_{t+1} + a_0 e_{t+2} + \left[\frac{\alpha_1 a_0}{\alpha_2} + a_1\right] e_{t+1} + b_0 \varepsilon_{t+2} + \left[\frac{\alpha_1 b_0}{\alpha_2} + b_1\right] \varepsilon_{t+1}$$

$$= a_0 e_{t+2} + a_1 e_{t+1} + b_0 \varepsilon_{t+2} + b_1 \varepsilon_{t+1}$$

Hence equation (4.7) can be written

$$y_t = \alpha_1 y_{t+1} + \alpha_2 y_{t+2} + \beta x_t + e_t - (\alpha_1 a_0 + \alpha_2 a_1) e_{t+1}$$
$$- \alpha_2 a_0 e_{t+2} - (\alpha_1 b_0 + \alpha_2 b_1) \varepsilon_{t+1} - \alpha_2 b_0 \varepsilon_{t+2} \qquad (4.8)$$

Equation (4.8) is an alternative way of expressing the solution of (4.6). By setting t equal to $t-1$ in (4.8) and re-normalising on the new y_t we obtain (4.6) once more. Equation (4.8) can be estimated consistently by the ordinary IV estimator with y_{t+1}, y_{t+2} and x_t as the explanatory variables and x_t and lagged values of y and x as instrumental variables. More efficient estimates are provided by the 2S2SLS estimator taking account of the composite MA(2) disturbance involving e_t, e_{t+1}, e_{t+2}, ε_{t+1}, ε_{t+2}. If the terms ε_{t+1}, ε_{t+2} are replaced by etimates and treated as explanatory variables, then these two IV estimators can be applied to the resulting equation with an efficiency gain relative to the corresponding previous IV estimator.

Case 2: $|\eta_1| > 1$, $|\eta_2| < 1$

This is a saddlepoint solution, though unlike that of the model of section 3, it is not unique because two parameters are still free. To see this, we note that if we impose the condition that the residues in A(z) and B(z) be analytic at $z = \eta_2$ we find that

$$\lim_{z\to\eta_2} A(z) = \frac{\alpha_2 a_0 + (\alpha_1 a_0 + \alpha_2 a_1)\eta_2 - \eta_2^2}{\alpha_2 + \alpha_1\eta_2 - \eta_2^2} = 0 \tag{4.9}$$

$$\lim_{z\to\eta_2} B(z) = \frac{\alpha_2 b_0 + (\alpha_1 b_0 + \alpha_2 b_1)\eta_2 - \beta\theta(\eta_2)\eta_2^2}{\alpha_2 + \alpha_1\eta_2 - \eta_2^2} = 0 \tag{4.10}$$

These provide two linear restrictions on the choice of the parameters:

$$(\alpha_2 + \alpha_1\eta_2)\, a_0 + (\alpha_2\eta_2)a_1 = \eta_2^2 \tag{4.11}$$

and

$$(\alpha_2 + \alpha_1\eta_2)\, b_0 + (\alpha_2\eta_2)b_1 \;=\; \beta\theta(\eta_2)\eta_2^2 \qquad (4.12)$$

Hence without two further restrictions we have in effect two free parameters.

The solution for y_t can therefore be written

$$y_t = \frac{\alpha_2 a_0 + (\alpha_1 a_0 + \alpha_2 a_1)L - L^2}{-(\eta_1 - L)(\eta_2 - L)} \cdot e_t$$

$$+ \frac{\alpha_2 b_0 + (\alpha_1 b_0 + \alpha_2 b_1)L - \beta\theta(L)L^2}{-(\eta_1 - L)(\eta_2 - L)} \cdot \varepsilon_t$$

Noting that $\eta_1 + \eta_2 = \alpha_1$ and $\eta_1\,\eta_2 = -\alpha_2$ we can show that

$$(\eta_1 - L)y_t = \left[\eta_1 a_0 + \frac{\eta_2(a_0 - \eta_1 a_1) - L}{1 - \eta_2 L^{-1}}\right] e_t$$

$$+ \left[\eta_1 b_0 + \frac{\eta_2(b_0 - \eta_1 b_1)}{1 - \eta_2 L^{-1}}\right] \varepsilon_t - \frac{\beta L^{-1}}{1 - \eta_2 L^{-1}}\; x_t$$

Since

$$Le_t/(1 - \eta_2 L^{-1}) = \sum_{s=0}^{\infty} \eta_2 E_{t-1} e_{t+s-1}$$

$$L^{-1}x_t/(1-\eta_2 L^{-1}) = \sum_{s=0}^{\infty} \eta_2^{-s} E_t x_{t+s+1} + \theta(\eta_2) \sum_{s=0}^{\infty} \eta_2^{-s-1} E_t \epsilon_{t+i+2}$$

and $E_t e_{t+s} = E_t \epsilon_{t+s} = 0$ for $s > 0$, we can write the solution for y_t as

$$y_t = \frac{1}{\eta_1} y_{t-1} + \frac{\alpha_1 a_0 + \alpha_2 a_1}{\eta_1} e_t - \frac{1}{\eta_1} e_{t-1}$$

$$+ \frac{\alpha_1 b_0 + \alpha_2 b_1}{\eta_1} \epsilon_t - \frac{\beta}{\eta_1} \sum_{s=0}^{\infty} \eta_2^{-s} E_t x_{t+s+1}$$

Eliminating a_1 and b_1 using (4.11) and (4.12) gives

$$y_t = \frac{1}{\eta_1} y_{t-1} + \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_t - \frac{1}{\eta_1} e_{t-1}$$

$$+ \left[b_0 + \frac{\beta\theta(\eta_2)\eta_2}{\eta_1}\right] \epsilon_t - \frac{\beta}{\eta_1} \sum_{s=0}^{\infty} \eta_2^{-s} E_t x_{t+s+1} \qquad (4.13)$$

It can also be written as

$$y_t = \frac{1}{\eta_1} y_{t-1} + \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_t - \frac{1}{\eta_1} e_{t-1} + \sum_{i=0}^{\infty} \lambda_i \epsilon_{t-1} \qquad (4.14)$$

where $\lambda_0 = -\beta(\theta_0 + (\eta_2^2 - 1)\theta(\eta_2))/\alpha_2$ and $\lambda_i = -\beta \sum_{s=0}^{\infty} \theta_{s+i+1} \eta_2^{s}/\eta_1$ $(i>0)$

a) Efficient Estimation

The joint estimation of (4.14) and (2.3) taking account of the cross-equation restrictions and imposing an MA(1) error on (4.14) will therefore provide fully asymptotically efficient estimates. If x_t is an ARMA (p,q) process an alternative way of writing (4.13) is

$$y_t = \frac{1}{\eta} y_{t-1} + \sum_{i=0}^{p-1} h_i x_{t-i-1} + \sum_{i=0}^{q-1} k_i \varepsilon_{t-i} + \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_t - \frac{1}{\eta_1} e_{t-1} \qquad (4.15)$$

where as before in (2.26) and (3.12) h_i, k_i are functions of the basic parameters.[1] Fully efficient estimation now involves jointly estimating (4.15) and (2.3) again taking account of all cross-equation restrictions. It should be noted that in both of these estimators we are unable to exploit the fact that y_{t-1} and e_{t-1} have the same coefficient because a_0 is an unknown parameter. The disturbance should therefore be specified as an unrestricted MA(1) process.

b) Instrumental Variable Estimation

The errors in variables approach involves estimating equation (4.7) by instrumental variables. The innovations u_{t+1} and u_{t+2} can be evaluated using (4.13) or (4.14). Thus

$$u_{t+1} = \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_{t+1} + \lambda_0 \varepsilon_{t+1}$$

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[1]See the Appendix for further details of the derivation of (4.15). Note that the term ε_t in (4.19) arises from the last term in (4.13) and not the distributed lead.

and

$$u_{t+2} - \frac{1}{\eta_1} u_{t+1} + \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_{t+2} - \frac{1}{\eta_1} e_{t+1} + \lambda_0 \varepsilon_{t+2} + \lambda_1 \varepsilon_{t+1}$$

Hence equation (4.7) can be written

$$y_t = \alpha_1 y_{t+1} + \alpha_2 y_{t+2} + \beta x_t + e_t - \alpha_1 a_0 e_{t+1}$$

$$- \alpha_2 \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_{t+2} - (\eta_1 \lambda_0 + \alpha_2 \lambda_1) \varepsilon_{t+1} - \alpha_2 \lambda_0 \varepsilon_{t+2} \qquad (4.16)$$

This has the same structure as the corresponding equation for case 1, namely equation (4.8), and can be estimated in an identical manner. A gain in efficiency over these estimators is also possible. The main difference with case 1 is that here the unstable root has provided two extra restrictions, equations (4.11) and (4.12), which enable us to reduce the number of free parameters from four in (4.8) to two in (4.16), namely a_0 and b_0. Recalling that η_1 and η_2 satisfy $\eta_1 + \eta_2 = \alpha_1$, and $\eta_1 \eta_2 = -\alpha_2$, we can see that the terms in e in the disturbance of (4.16) provide an additional non-linear restriction.

Equation (4.16) can be re-written as the backward difference equation

$$y_t = - \frac{\alpha_1}{\alpha_2} y_{t-1} + \frac{1}{\alpha_2} y_{t-2} - \frac{\beta}{\alpha_2} x_{t-2} + \left[a_0 + \frac{\eta_2}{\eta_1}\right] e_t + \frac{\alpha_1 a_0}{\alpha_2} e_{t-1} - \frac{1}{\alpha_2} e_{t-2}$$

$$+ \lambda_0 \varepsilon_t + \left[\frac{\eta_1 \lambda_0 + \alpha_2 \lambda_1}{\alpha_2}\right] \varepsilon_{t-1}$$

which is equivalent to (4.6) but with two restrictions which has been used to eliminate a_1 and b_1. An alternative way of obtaining a fully efficient estimator is, therefore, to estimate this equation jointly with (2.3) taking into account the cross-equation parameter restrictions.

Case 3: $|\eta_i| < 1, \quad i = 1,2$

The model now has both roots unstable. As in case 2 of section 2 and case 2 of section 3, this enables the four parameters a_0, a_1, b_0 and b_1 to be determined uniquely. It implies that there is a unique solution for y_t, and hence path to equilibrium. In addition to imposing the condition applied in case 2 that the residues in $A(z)$ and $B(z)$ be analytic at $z = \eta_2$, we also require them to be analytic at $z = \eta_1$. This produces four restrictions: equations (4.9) and (4.10) which involve η_2 and two corresponding expressions in η_1. From these we can derive two further equations to (4.11) and (4.12) in η_1. These four equations determine the four parameters uniquely. We can show that

$a_0 = 1, \quad a_1 = 0, \quad b_0 = \beta(\eta_2\theta(\eta_2) - \eta_1\theta(\eta_1)) / (\eta_2 - \eta_1)$ and

$b_1 = \beta(\theta(\eta_2) - \theta(\eta_1)) / (\eta_2 - \eta_1)$. The solution for y_t is therefore

$$y_t = \frac{\alpha_2 + \alpha_1 L - L^2}{\alpha_2 + \alpha_1 L - L^2} \cdot e_t$$

$$+ \frac{\beta(\alpha_2(\eta_2\theta(\eta_2)-\eta_1\theta(\eta_1))+[\alpha_1(\eta_2)-\eta_2\theta(\eta_2))+\alpha_2(\theta(\eta_2)-\theta(\eta_1))]L-(\eta_2-\eta_1)\theta(L)L^2)}{(\eta_2-\eta_1)(\alpha_2+\alpha_1 L-L^2)} \cdot \varepsilon_t$$

$$= e_t - \frac{\beta \sum_{i=1}^{2} (-1)^i (\alpha_2\eta_i\theta(\eta_i) + [\alpha_1\eta_i\theta(\eta_i) + \alpha_2\theta(\eta_i)]L - \eta_i\theta(L))}{(\eta_2-\eta_1)(L-\eta_1)(L-\eta_2)} \cdot \varepsilon_t$$

$$= e_t - \frac{\beta \sum_{i=1}^{2} (-1)^i \; \{[\alpha_2 + L(\alpha_1 + \alpha_2/\eta_i)] \; \eta_i L^{-1} \theta(\eta_i)\theta(L)^{-1} - \eta_i L\}}{(\eta_2 - \eta_1)(L-\eta_1)(L-\eta_2)} \, . \, Lx_t$$

$$= e_t + \frac{\beta}{\eta_2 - \eta_1} \left[\frac{\eta_2(1 - \eta_2 L^{-1}\theta(\eta_2)\theta(L)^{-1}}{1-\eta_2 L^{-1}} - \frac{\eta_1(1-\eta_1 L^{-1}\theta(\eta_1)\theta(L)^{-1}}{1-\eta_1 L^{-1}} \right] x_t$$

$$= e_t + \frac{\beta}{\eta_2 - \eta_1} \left\{ \eta_2 \sum_{s=0}^{\infty} \eta_2^s \, E_t x_{t+s} - \eta_1 \sum_{s=0}^{\infty} \eta_1 E_t x_{t+s} \right\} \qquad (4.17)$$

Comparing (4.17) with (2.24), where we had only a single unstable root which we solved forward, here we solve both unstable roots forward to give a unique solution for y_t.

a) Efficient Estimation

The estimation of (4.17) is a straightforward generalisation of earlier results. One approach is to combine equation (4.17) with (2.3) to give

$$y_t = \beta \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-i} + e_t \qquad (4.18)$$

which looks like (2.25) except that here

$$\gamma_i = \sum_{s=0}^{\infty} (\eta_2^{s+1} - \eta_1^{s+1}) \; \theta_{s+i} \; / \; (\eta_2 - \eta_1).$$

Truncating the summation and estimating (4.18) jointly with (2.3) imposing the cross equation restrictions would give an asymptotically efficient estimator. Alternatively, if x_t can be expressed as an

ARMA (p,q) then (4.18) can be written in the form of (2.26) with the coefficients h_i, k_i suitably redefined. Thus

$$y_t = \sum_{i=0}^{p-1} h_i x_{t-i} + \sum_{i=0}^{q-1} k_i \varepsilon_{t-i} + e_t \tag{4.19}$$

which can be efficiently estimated jointly with (2.3).

b) Instrumental Variable Estimation

The errors in variables approach once more involves (4.7). the innovations u_{t+1} and u_{t+2} are now

$$u_{t+1} = e_{t+1} + \beta\gamma_0\varepsilon_{t+1}$$

and

$$u_{t+2} = e_{t+2} + e_{t+1} + \beta\gamma_0\varepsilon_{t+2} + \beta\gamma_1\varepsilon_{t+1}.$$

Hence (4.7) can be written

$$\begin{aligned} y_t = {} & \alpha_1 y_{t+1} + \alpha_2 y_{t+2} + \beta x_t - \beta(\alpha_1\gamma_0 + \alpha_2\gamma_1)\,\varepsilon_{t+1} \\ & - \beta\alpha_2\gamma_0\varepsilon_{t+2} + e_t - (\alpha_1 + \alpha_2)\,e_{t+1} - \alpha_2 e_{t+2} \end{aligned} \tag{4.20}$$

As in the cases of (4.8) and (4.16), equation (4.20) can be estimated by the IV estimator with y_{t+1}, y_{t+2} and x_t treated as explanatory variables and the remaining variables comprising a composite MA(2) disturbance. Equation (4.20) can be estimated more efficiently if the 2S2SLS estimator is used, or yet more efficiently if ε_{t+1} and ε_{t+2}

are treated as explanatory variables and estimated from (2.3). In the latter case a further gain in efficiency is possible if the restrictions connecting the coefficients of the explanatory variables and the MA(2) error process in e_t are taken into account.

As before, equation (4.20) can be re-written in backward form as

$$y_t = -\frac{\alpha_1}{\alpha_2} y_{t-1} + \frac{1}{\alpha_2} y_{t-2} - \frac{\beta}{\alpha_2} x_{t-2} + \beta\gamma_0\varepsilon_t + \left[\frac{\beta(\alpha_1\gamma_0 + \alpha_2\gamma_0)}{\alpha_2}\right] \varepsilon_{t-1}$$

$$+ \; e_t + \left[\frac{\alpha_1 + \alpha_2}{\alpha_2}\right] e_{t-1} - \frac{1}{\alpha_2} e_{t-2} \qquad (4.21)$$

An alternative fully efficient estimator is therefore the joint estimation of (4.21) and (2.3) using the maximum likelihood estimator and imposing the cross-equation restrictions.

The results of this section have concerned the estimation of (4.2) which, like (2.2), is a special case of (4.1). Generalising these results to (4.1) is fairly straightforward in principle. If all of the roots are stable then the solution for y_t will be an nth order difference equation in y_t with a single nth order lag on x_t, a backward looking MA(n) in e_t and MA(n-1) in ε_t. The solution for y_t in the case of m stable roots and (n-m) unstable roots (n > m > 0) will involve an mth order difference equation in y with a backward MA(m) process in e_t and MA(n-m-1) in ε_t together with (n-m) distributed lead terms in expectations of x. The solution of y_t

with all n roots unstable will just have n distributed lead terms in expectations of x. The direct estimation procedures will depend on the number of stable and unstable roots in the solution. In contrast the errors in variables approach will be based on the same estimating equation no matter how many stable and unstable roots there are. This equation will have n leading terms in y, e and ε.

If equation (4.1) also contains lagged values of y_t then we must add the same number of lags in y_t to these solutions.

5. Withholding equations - models with information dated t-1 or before

All of the models studied so far have had expectations taken conditional on information dated time t. Another important generalisation would be to introduce variables with expectations conditioned on information dated t-1 or before. Whiteman refers to models with such variables as withholding equations because they seem to imply that relevant information is concealed from agents.

By way of illustration we shall consider in detail the estimation of

$$y_t = \alpha E_t y_{t+1} + \gamma E_{t-1} y_t + \delta y_{t-1} + \beta x_t + e_t \qquad (5.1)$$

Afterwards the extension of these results to more general models will be discussed. We shall maintain the previous assumptions.

Equation (5.1) can be solved as before. It can be re-written

$$A(L)\, e_t + B(L)\varepsilon_t = (\alpha L^{-1} + \gamma)\{(A(L) - a_0)e_t + (B(L) - b_0)\varepsilon_t\}$$

$$+ \delta L(A(L)e_t + B(L)\varepsilon_t) + \beta\theta(L)\varepsilon_t + e_t$$

which implies that

$$A(z) = \frac{\alpha_0 - (1-\gamma a_0)z}{\alpha - (1-\gamma)z + \delta z^2} \qquad (5.2)$$

$$b(z) = \frac{\alpha b_0 + \gamma b_0 z - z\beta\theta(z)}{\alpha - (1-\gamma)z + \delta z^2} \qquad (5.3)$$

Let η_1 and η_2 denote the roots of the quadratic equation

$$\alpha - (1-\gamma)z + \delta z^2 = 0 \qquad (5.4)$$

Assuming these roots are real we have three possible outcomes, but only two, the globally stable and the saddlepoint, admit solutions and so we shall concentrate on these. The third, where both roots are unstable, does not provide a solution. The analysis for this case corresponds closely to Case 3 in Section 3 and therefore need not be repeated.

Case 1: $|\eta_i| > 1, \quad i = 1,2$

For this globally stable case a_0 and b_0 are again free parameters and so y_t can be written for any choice of a_0 and b_0 as

$$y_t = \frac{1-\gamma}{\alpha} y_{t-1} - \frac{\delta}{\alpha} y_{t-2} - \frac{\beta}{\alpha} x_{t-1} + b_0\varepsilon_t + \gamma \frac{b_0}{\alpha} \varepsilon_{t-1} + a_0 e_t - \frac{(1-\gamma a_0)}{\alpha} e_{t-1} \quad (5.5)$$

The principle difference between equations (5.5) and (3.6) is that (5.5) contains an extra term involving ε_{t-1} This is the first occasion on which lagged values of ε have appeared in the solution of y_t and is a feature of witholding equations.

a) Efficient Estimation

Equation (5.1) can be estimated by the joint non-linear estimation of equations (5.5) and (2.3). Rewriting (5.5) as

$$y_t = \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 x_{t-1} + \pi_4 \varepsilon_t + \pi_5 \varepsilon_{t-1} + v_t \quad (5.6)$$

where $v_t = e_t^* - \frac{(1-\gamma a_0)}{\alpha a_0} e_{t-1}^*$ and $e_t^* = a_0 e_t$, we note that, because a_0 is free, the disturbance of (5.5) is an unrestricted MA(1) process, and estimates of the basic parameters are obtained from $\alpha = (\pi_1 + \pi_5/\pi_4)^{-1}$ $\beta = -\alpha\pi_3$, $\gamma = \alpha\pi_5/\pi_4$ and $\delta = -\alpha\pi_2$.

An alternative estimation method is to replace ε_t and ε_{t-1} in (5.6) by estimates $\hat{\varepsilon}_t$ and $\hat{\varepsilon}_{t-1}$ obtained from prior estimation of (2.3) and then estimate the resulting equation

$$y_t = \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 x_{t-1} + \pi_4 \hat{\varepsilon} + \pi_5 \hat{\varepsilon}_{t-1} + v_t^* \qquad (5.7)$$

where

$$v_t^* = v_t + \pi_4(\varepsilon_t - \hat{\varepsilon}_t) + \pi_5 (\varepsilon_{t-1} - \hat{\varepsilon}_{t-1})$$

by computationally simpler single equation methods.

A difficulty which arises is that the disturbance v_t^* will be correlated with the explanatory variables y_{t-1} and x_{t-1} due to the presence of $\varepsilon_t - \hat{\varepsilon}_t$ and $\varepsilon_{t-1} - \hat{\varepsilon}_{t-1}$. To see this consider estimating the standard linear, equation (2.8), with X exogenous. The expectation of the cross-product of the dependent variable and the error in estimating the disturbance will be non zero:

$$\begin{aligned} Ey'(u-\hat{u}) &= E\,(X\beta+u)'(u - (I-X(X'X)^{-1} X')u) \\ &= E\, u'X(X'X)^{-1} X'u \\ &= k\sigma_u^2 \end{aligned}$$

where $\hat{u}$ is the OLS residual. This suggests that equation (5.7) be estimated by instrumental variables with both y_{t-1} and x_{t-1} instrumented. Either the ordinary IV or the 2S2SLS estimator can be used.

Valid instruments are y_{t-2} , y_{t-3} , ... and x_{t-2} , x_{t-3} , ... Consistent estimates of the parameters of (5.1) can be obtained from the coefficients of (5.7) as before. The 2S2SLS estimator will be less efficient than the fully efficient estimator due to the presence of $\varepsilon_t - \hat{\varepsilon}_t$ and $\varepsilon_{t-1} - \hat{\varepsilon}_{t-1}$ in the disturbance v_t^* and to using instrumental variables.

b) Instrumental Variables Estimation

The errors in variables approach to estimating equation (5.1) again involves replacing expectations by outcomes to give

$$y_t = \frac{\alpha}{1-\gamma} y_{t+1} + \frac{\delta}{1-\gamma} y_{t-1} + \frac{\beta}{1-\gamma} x_t + \frac{1}{1-\gamma} (e_t - \alpha(y_{t+1} - E_t y_{t+1}))$$

$$- \frac{\gamma}{1-\gamma} (y_t - E_{t-1} y_t)) \qquad (5.8)$$

From (5.5), the solution of y_t , we have

$$y_{t+1} - E_t y_{t+1} = a_0 e_{t+1} + b_0 \varepsilon_{t+1}$$

and

$$y_t - E_{t-1} y_t = a_0 e_t + b_0 \varepsilon_t .$$

Hence

$$y_t = \frac{\alpha}{1-\gamma} y_{t+1} + \frac{\delta}{1-\gamma} y_{t-1} + \frac{\beta}{1-\gamma} x_t - \frac{b_0}{1-\gamma} (\gamma\varepsilon_t + \alpha\varepsilon_{t+1})$$

$$+ \frac{1}{1-\gamma} ((1-\gamma a_0) e_t - \alpha a_0 e_{t+1}) \qquad (5.9)$$

which can be re-written

$$y_t = \lambda_1 y_{t+1} + \lambda_2 y_{t-1} + \lambda_3 x_t + \lambda_4 \varepsilon_t + \lambda_5 \varepsilon_{t+1} + \lambda_6 e_t + \lambda_4 e_{t+1} \qquad (5.10)$$

The solution (5.5) can once more be obtained from (5.10) by writing (5.10) in backward form.

In order to obtain unique point estimates of the parameters of (5.1) it will be necessary to treat ε_t and ε_{t+1} (or estimates $\hat{\varepsilon}_t$ and $\hat{\varepsilon}_{t+1}$) as separate explanatory variables and not include them in a composite disturbance term. We then have $\alpha = \lambda_1\lambda_5/(\lambda_5+\lambda_1\lambda_4)$, $\beta = \lambda_3(1-\gamma)$, $\gamma = \alpha\lambda_4/\lambda_5$ and $\delta = \lambda_2(1-\gamma)$.

Equation (5.10) can be estimated by the ordinary IV estimator if e_t and e_{t+1} are treated as a composite disturbance, or by the 2S2SLS estimator if account is taken during estimation of the equations MA(1) error. If $\hat{\varepsilon}_t$ and $\hat{\varepsilon}_{t+1}$ are used as explanatory variables then the disturbance term will also include $\varepsilon_t - \hat{\varepsilon}_t$ and $\varepsilon_{t+1} - \hat{\varepsilon}_{t+1}$. Moreover, the former will be correlated with x_t. For the first time in the errors in variable approach, therefore, x_t will need to be instrumented in addition to y_{t+1} Lagged values of y_t and x_t, as well as $\hat{\varepsilon}_t$ and $\hat{\varepsilon}_{t+1}$, can be used as instrumental variables.

<u>Case 2: $|\eta_1| > 1$, $|\eta_2| < 1$</u>

This produces a saddlepoint solution for y_t. The analysis is similar to case 2 of the models of Sections 3 and 4. The singularities at $A(\eta_2)$ and $B(\eta_2)$ can be removed by choosing a_0 and b_0 to satisfy

$$\lim_{z\to\eta_2} (\eta_2-z)\,A(z) = \frac{\alpha a_0 - (1-\gamma a_0)\eta_2}{\delta(\eta_1-\eta_2)} = 0$$

$$\lim_{z\to\eta_2} (\eta_2-z)\, B(z) = \frac{\alpha b_0 + \gamma b_0\eta_2 - \eta_2\beta\theta(\eta_2)}{\delta(\eta_1-\eta_2)} = 0$$

from which we find $a_0 = \eta_2/(\alpha+\gamma\eta_2)$ and $b_0 = \eta_2\beta\theta(\eta_2)/(\alpha+\gamma\eta_2)$.

From (2.6), (5.2) and (5.3) the solution for y_t is

$$y_t = \frac{\alpha(\eta_2-L)}{(\alpha+\gamma\eta_2)\delta(\eta_1-L)(\eta_2-L)} \cdot e_t + \frac{(\alpha+\gamma L)\eta_2\beta\theta(\eta_2) - (\alpha+\gamma\eta_2)L\beta\theta(L)}{(\alpha+\gamma\eta_2)\delta(\eta_1-L)(\eta_2-L)} \cdot \varepsilon_t$$

Hence

$$\delta(\alpha+\gamma\eta_2)(\eta_1-L)\, y_t = \alpha e_t + \frac{\beta(\alpha+\gamma L)(1-\eta_2L^{-1}\theta(\eta_2)\theta(L)^{-1})}{1-\eta_2L^{-1}}\, x_t - \beta\gamma Lx_t \quad (5.11)$$

Using (2.23), and noting again that replacing x_t in (2.23) by Lx_t implies that the terms E_tx_{t+s} becomes $E_{t-1}x_{t+s-1}$, we have

$$\eta_1\delta(\alpha+\gamma\eta_2)\left[y_t - \frac{1}{\eta_1}\, y_{t-1}\right] = \alpha e_t + \beta\left[\alpha \sum_{s=0}^{\infty} \eta_2^s\, E_tx_{t+s} + \gamma \sum_{s=0}^{\infty} \eta_2^s\, E_{t-1}x_{t+s-1} - \gamma x_{t-1}\right]$$

$$= \alpha e_t + \beta\left[\alpha \sum_{s=0}^{\infty} \eta_2^s\, E_tx_{t+s} + \gamma\eta_2 \sum_{s=0}^{\infty} \eta_2^s\, E_{t-1}x_{t+s}\right]$$

$$= \alpha e_t + \beta\left[(\alpha+\gamma\eta_2) \sum_{s=0}^{\infty} \eta_2^s E_tx_{t+s} - \gamma\eta_2 \sum_{s=0}^{\infty} \eta_2^s(E_t-E_{t-1})x_{t+s}\right].$$

From (5.4) $\delta\eta_1\eta_2 = \alpha$ and $\delta(\eta_1+\eta_2) = 1-\gamma$ and hence the solution for y_t is

$$y_t = \frac{1}{\eta_1}\, y_{t-1} + \frac{\beta}{\delta\eta_1} \sum_{s=0}^{\infty} \eta_2^s\, E_tx_{t+s} + \frac{1}{\gamma+\delta\eta_1}\, e_t + \frac{\beta\gamma\phi_0}{\delta(\alpha-\eta_1)}\, \varepsilon_t \quad (5.12)$$

where $\phi_i = \sum_{s=0}^{\infty} \eta_2^s \theta_{s+i}$. The corresponding solution for (3.1), which has no term in $E_{t-1}y_t$, is equation (3.9). Unlike (5.12), equation (3.9) has no term in ε_t. If $\gamma=0$ the two equations are, of course, the same.

a) Efficient Estimation

Estimation of (5.12) can be carried out after it is re-written as

$$y_t = \frac{1}{\eta_1} y_{t-1} + \frac{\beta\phi_0}{\gamma+\delta\eta_1} \varepsilon_t + \frac{\beta}{\delta\eta_1} \sum_{s=1}^{\infty} \phi_i \varepsilon_{t-i} + \frac{1}{\gamma+\delta\eta_1} e_t \qquad (5.13)$$

Equations (5.10) and (2.3) can now be estimated jointly. From estimates of η_1, η_2, $\beta/\delta\eta_1$, $\beta\phi_0/(\gamma+\delta\eta_1)$ and the θ_i it is possible to derive estimates of the parameters of (5.1) as follows. Let $\pi_1 = \beta/\delta\eta_1$, $\pi_2 = \beta\phi/(\gamma+\delta\eta_1)$ then $\alpha = \pi_1\eta_1\eta_2/(\pi_1\phi_0\eta_1+\pi_2\eta_2)$, $\beta = \pi_1\delta\eta_1$, $\delta = \alpha/\eta_1\eta_2$ and $\gamma = 1 - \delta(\eta_1+\eta_2)$. These estimates will be asymptotically efficient. Again there may be an ARMA equivalent of (5.13) which it would be preferable to estimate.

b) Instrumental Variable Estimation

The errors in variables approach is once more based on (5.8). From (5.13) we find that

$$y_{t+1} - E_t y_{t+1} = \frac{1}{\gamma+\delta\eta_1} e_{t+1} + \frac{\beta\phi_0}{\gamma+\delta\eta_1} \varepsilon_{t+1}$$

and

$$y_t - E_{t-1} y_t = \frac{1}{\gamma+\delta\eta_1} e_t + \frac{\beta\phi_0}{\gamma+\delta\eta_1} \varepsilon_t$$

and therefore (5.8) becomes

$$y_t = \frac{\alpha}{1-\gamma} \; y_{t+1} + \frac{\delta}{1-\gamma} y_{t-1} + \frac{\beta}{1-\gamma} x_t - \frac{\beta\phi_0}{(1-\gamma)(\gamma+\delta\eta_1)} (\gamma\varepsilon_t + \alpha\varepsilon_{t+1})$$

$$+ \frac{1}{(1-\gamma)(\gamma+\delta\eta_1)} (\delta\eta_1 e_t - \alpha e_{t+1}) \qquad (5.14)$$

Equation (5.14) has the same general form as (5.10) and can be estimated by instrumental variables in an identical manner. It would also be possible to exploit the parameter restrictions in (5.10) that arise in the ε and e terms by using a non-linear IV estimator.

Equation (5.14) can be re-written in backward form as

$$y_t = \frac{(1-\gamma)}{\alpha} y_{t-1} - \frac{\delta}{\alpha} y_{t-2} - \frac{\beta}{\alpha} x_{t-1} + \frac{\beta\phi_0}{\gamma+\delta\eta_1} \left[\frac{\gamma}{\alpha} \varepsilon_{t-1} + \varepsilon_t\right]$$

$$- \frac{1}{\gamma+\delta\eta_1} \left[\frac{\delta\eta_1}{\alpha} e_{t-1} - e_t\right]$$

which is identical to (5.5) if a_0 and b_0 are replaced by their constrained values. Estimating this backward solution jointly with (2.3) by maximum likelihood methods and imposing the cross-equation restrictions will, once more, provide an alternative way of obtaining fully efficient estimates.

The generalisation of these results to the case of higher order lags in y in equation (5.1) is fairly obvious from the results of Section 3. Two other generalisations are of interest. One is having further future expectations based on information dated t and $t-1$ such as $E_t y_{t+1}$, $E_t y_{t+2}$ .... and $E_{t-1} y_{t+1}$; $E_{t-1} y_{t+2}$.... The other is having variables with expectations based on information sets dated prior to t-1. Combining the two produces a large number of models. Whiteman and Broze, Gourieroux and Szafarz (1984) have discussed particular examples of such models.

The most general model considered by Whiteman (1983, p.35) is

$$E_{t-1}\sum_{s=-n}^{m} \alpha_s y_{t+s} + \sum_{j=0}^{k} \theta_j y_{t-j} = \beta x_t + e_t \tag{5.15}$$

The type of solution is determined as follows. The characteristic equation of (5.15) is

$$\sum_{j=0}^{J} \lambda_j z^j = \prod_{j=1}^{J} (z-\eta_j) = 0 \tag{5.16}$$

where $J = m + \max(n,k))$. Let there be r roots $\eta_1,\dots,\eta_r$ which lie inside the unit circle (i.e. there are r unstable roots), then, since there are $n+i$ coefficients to be determined, if

(a) $r = n+i$, there is a unique solution

(b) $r < n+i$, there is an infinity of solutions

(c) $r > n+i$, there is no solution.

Broze, Gourieroux and Szafarz consider the model

$$y_t + \sum_{k=0}^{K} \sum_{h=0}^{H} \alpha_{kh} E_{t-k} y_{t+h-k} = \beta x_t + e_t \tag{5.17}$$

where $\alpha_{00} = 0$.

If the characteristic equation of (5.17) is written as (5.16) where J = max expectational lead ($\leq K+H$), and r is the number of unstable roots then, if I = max lead when expectations are replaced in (5.17) by realisations, when

(a) $r = I$, there is a unique solution

(b) $r < I$, there is an infinity of solutions

(c) $r > I$, there is no solution.

Combining the two models (5.15) and (5.17) would imply adding $\sum_{j=0}^{g} \delta_j y_{t-j}$ to (5.17). This may alter the value of J by adding to the lags in y but otherwise the conditions (a) - (c) of Broze et al. would remain unchanged. The estimation of models such as (5.15) and (5.17) will not be considered here because there are a number of complications which require extensive discussion.

6. Conclusions

We have shown how, using Whiteman's solution method, fully efficient estimators can be derived for a number of different linear models with future rationally expected variables. The characteristics of each model and its solution - in particular whether it has a stable, unstable or saddlepoint solution - have been found to be an important factor in devising an efficient method of estimation. This is both a strength - because it entails using all of the available information during estimation - and a weakness -because it requires prior knowledge of the type of solution before estimation, and the estimator is usually made much more complex as a consequence of incorporating this information. We have also shown that it will usually be helpful to the estimation if the exogenous variables can be modelled as an ARMA process.

It is well known that the errors in variables approach can also be used for these models and that estimation can be carried out by instrumental variables. We have shown that if we incorporate the innovations into a composite disturbance term, then the ordinary IV estimator will always provide a consistent estimator of the parameters of the model and that lagged values of the endogenous variables are valid instrumental variables. More efficient IV estimators can be obtained if we include estimates of the innovations in the exogenous variables as explanatory variables instead of incorporating these innovations into a composite disturbance. Perhaps the main attraction of the instrumental variables estimators is that prior knowledge of the type of solution of the model is not required. Consistent estimates of the parameters of the model are provided no matter the type of solution. This suggests that the model should first be estimated by instrumental variables in order to determine the type of solution. This information can then be used to construct the appropriate efficient estimator. Tests for misspecification of the model can also be carried out most conveniently at the instrumental variable estimation stage.

Another approach which might be adopted is to assume at the outset that the model has a unique solution and to estimate the model using the appropriate fully efficient estimator. This assumption will often be in keeping with the derivation of the model. We have seen that unique solutions impose restrictions on the coefficients of the polynomial functions A(L) and B(L). To see whether these restrictions are valid we can re-optimise having removed the restrictions, and then carry out suitable tests for a difference in the value of the function optimised. If we are maximising a likelihood function then this would be a likelihood ratio test. Since we first compute the likelihood on the null hypothesis that the restrictions are valid, the step of steepest ascent from this constrained optimum will provide a Lagrange Multiplier type test.

The results derived above are for a wide class of single equation models. It should be straightforward to extend them to other types of linear models using this general approach. An important extension yet to be made is to the case of a system of linear equations. The above results indicate, however, that the instrumental variable estimation of equation systems would be fairly straightforward.

Appendix

Forming future expectations from an ARMA process

At several points in the main text - for example, equations (2.26), (3.12), (4.15) and (4.19) - solutions which involve distributed lead functions in expectations of x have been re-written as rational distributed lag functions of x and ε on the assumption that x can be represented by an ARMA (p,q) process. In this appendix we derive the basic result from which these formulations are obtained.

The following result is proved: if x_t is generated by the ARMA (p,q) process

$$\sum_{i=0}^{p} a_i x_{t-i} = \sum_{i=0}^{q} b_i \varepsilon_{t-i} \tag{A1}$$

where $a_0 = b_0 = 1$, the roots of $\sum_{i=0}^{p} a_i z^i$ all lie outside the unit circle $|z| = 1$ and ε_t is an i.i.d. $(0, \sigma_\varepsilon^2)$ process, then

$$\sum_{s=0}^{\infty} \alpha^s E_t x_{t+s} = \sum_{i=0}^{p-1} \lambda_i x_{t-i} + \sum_{i=0}^{q-1} \mu_i \varepsilon_{t-i} \tag{A2}$$

Equation (A1) can be written in companion from as the vector AR(1) process

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-p+1} \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_p \\ 1 & 0 & \cdots & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-p} \end{bmatrix} + \begin{bmatrix} 1 & b_1 & \cdots & b_q \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix} \qquad (A3)$$

or, more compactly, as

$$\underset{\sim}{x}_t = A\underset{\sim}{x}_{t-1} + B\underset{\sim}{\varepsilon}_t \qquad (A4)$$

where $\underset{\sim}{x}'_t = (x_t, x_{t-1}, \ldots, x_{t-p+1})$ and $\underset{\sim}{\varepsilon}'_t = (\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q})$.

Using the standard projection formula for an AR(1) process we have for $s > 0$

$$\underset{\sim}{x}_{t+s} = A^s\underset{\sim}{x}_t + \sum_{i=0}^{s-1} A^i B\underset{\sim}{\varepsilon}_{t+s-i} \qquad (A5)$$

from which we can select the element

$$x_{t+s} = CA^s\underset{\sim}{x}_t + \sum_{i=0}^{s-1} CA^i B\underset{\sim}{\varepsilon}_{t+s-i} \qquad (A6)$$

where $C = (1, 0, \ldots, 0)$ performs the selection. We can define D_{q-s} as a <u>row</u> vector which contains the last $q-s$ elements of the row vector $CA^i B$ and $F_{q-s} = (D_{q-s}, 0, \ldots, 0)$ as a size (q+1) row vector.

It follows that

$$E_t x_{t+s} = CA^s \underset{\sim}{x}_t + F_{q-s} \underset{\sim}{\varepsilon}_t \qquad (A7)$$

Hence we can prove the required result that

$$\sum_{s=0}^{\infty} \alpha^s E_t x_{t+s} = \left[\sum_{s=0}^{\infty} \alpha^s CA^s\right] \underset{\sim}{x}_t + \left[\sum_{s=0}^{\infty} \alpha^s F_{q-s}\right] \underset{\sim}{\varepsilon}_t \qquad (A8)$$

$$= \sum_{i=0}^{p-1} \lambda_i x_{t-i} + \sum_{i=0}^{q-1} \mu_i \varepsilon_{t-i} \qquad (A2)$$

where λ_i is the ith element of $\sum_{s=0}^{\infty} \alpha^s CA^s$ and μ_i is the ith element of $\sum_{s=0}^{q-1} \alpha^s F_{q-s}$. (NB: for $s \geq q$, F_{q-s} is a null vector).

This result can be applied to the equations in the main text as follows. Equation (2.26) is obtained by substituting (A2) into (2.24) to give $h_i = \beta\lambda_i$ and $k_i = \beta\mu_i$. Equations (3.12) is obtained similarly with η_2 replacing α in A(2). Equation (4.19) is derived as the difference between two expresses of the form of (A2) with η_2 and η_1 replacing α. The derivation of equation

(4.15) involves evaluating an expression of the form

$\sum_{s=0}^{\infty} \alpha^s E_t \ x_{t+s+1}$. From (A7) it can be shown that for $s > 0$

$$E_t \ x_{t+s+1} = CA^{s+1} \underset{\sim}{x}_t + F_{q-s-1} \underset{\sim}{\varepsilon}_t \tag{A9}$$

and hence

$$\sum_{s=0}^{\infty} \alpha^s E_t \ x_{t+s+1} = \sum_{i=0}^{\infty} \alpha^s CA^{s+1} \underset{\sim}{x}_t + \sum_{i=0}^{\infty} \alpha^s F_{q-s-1} \underset{\sim}{\varepsilon}_t \tag{A10}$$

$$= \sum_{i=0}^{p-1} \lambda_i x_{t-i} + \sum_{i=0}^{q-1} \mu \, \varepsilon_{i \ t-i} \tag{A11}$$

The derivation of (4.15) is now obvious.

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