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**PATERNALISM, CULTURAL
TRANSMISSION AND DIFFUSION ON
COMPLEX NETWORKS**

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***INDUSTRIAL ORGANIZATION and
PUBLIC ECONOMICS***



Centre for Economic Policy Research

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Abstract

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JEL Classification: C73, L14 and O33

Keywords: cultural transmission, diffusion, mean-field and social networks

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1 Introduction

Economists increasingly investigate the relationships between culture and socio-economic activities. For instance, a number of recent empirical analyses recognize the impact of cultural characteristics on economic outcomes such as production, trade and growth (Bowles, 1998; Guiso et al., 2006; Tabellini, 2008; Benhabib et al., 2010). Similarly, several experimental studies highlight the importance of cultural differences to understand systematic variations in the way individuals play various types of games of cooperation and trust

(see, for example, Henrich et al., 2004; Gächter et al., 2010). Consistent with these views, studying the mechanisms leading to long run cultural persistence and differences becomes therefore a key question. In this respect, increasing evidence suggest that cultural differentiation and persistence are actually sustained through processes of cultural transmission that crucially depend on the social network structures in which individuals are embedded (Dohmen et al., 2012; Henrich and Broesch, 2011; Patacchini and Zenou, 2011; Schotter and Sopher, 2003).

Starting from these observations, the purpose of this paper is analyze cultural transmission in large scale social structures, and to merge the two main approaches economists and social scientists have developed to study the transmission and diffusion of cultural traits in complex social networks, namely the epidemiological and the cultural transmission frameworks.

The first approach considers cultural traits as *viruses* transmitted in a population and develops epidemiology models to discuss how the diffusion of cultural traits depends on the topological properties of the social networks in which agents are embedded (Pastor-Satorras and Vespignani, 2001, 2005). Generally focusing on complex network structures,¹ the main idea of these models is to consider the existence of two traits in the population, one benchmark trait and one *infected* trait. At each point of time, agents in the natural trait have a probability of being infected that depends on the share of infected neighbors, while infected agents recover to the natural trait at a given exogenous rate. Models such as the standard Susceptible-Infected-Susceptible (SIS) or Susceptible-Infected-Recovered (SIR) typically follow this logic. These models usually find threshold levels on the connectivity of the network that allow for the full diffusion or the persistence of cultural diversity, as, for example, López-Pintado (2008, 2012), Jackson and Lopez-Pintado (2013), and Galeotti and Rogers (2013). For some nice overviews of the literature on diffusion in complex network see also Vega-Redondo (2007) and Jackson (2010).

The second approach builds upon evolutionary models of cultural transmission as developed for instance by Cavalli-Sforza and Feldman (1981) and Boyd and Richerson (1985). These models emphasize the role of various channels of cultural transmission: vertical or parental socialization, horizontal or peer socialization, and oblique socialization. Cultural transmission models then study how the biased combination of these transmission processes generate population level dynamics of cultural change within and across societies. This line of research has been extended in economics with the introduction of purposeful and motivated vertical cultural socialization efforts interacting with the other oblique/horizontal transmission channels (see Bisin and Verdier, 2010, for a survey). According to this perspective, cultural transmission is not exogenously governed by mechanistic rules of learning and socialization, but depends explicitly on parents' optimal social actions, and their specific motivations for cultural transmission. In the typical setting, parents are assumed to care about their children's adopted cultural trait and

¹Actually there is an important literature also in non random network in which the network's adjacency matrix can be computed and the cultural dynamics depends on the spectral properties of the graph as, for example, Golub and Jackson (2010, 2012)

to have a desire to see this adopted trait close to their own trait, an assumption called *paternalism or imperfect empathy* (Bisin and Verdier, 2001). Long run cultural persistence (polymorphic cultural equilibria) is then shown to exist when a *cultural substitution* property is satisfied. This property essentially reflects the fact that parents have an incentive at the margin to devote more effort to transmit their own trait to their offspring when the frequency of that trait becomes smaller in the population. This basic setting has been applied to various socio-economic contexts². Some natural recent extensions have highlighted the importance of social networks in these types of cultural transmission dynamics (Brueckner and Smirnov, 2007; Buechel et al., 2014; Panebianco, 2014). These models however only consider finite size populations or groups, and therefore may not be fully adapted to understand long run cultural patterns for large scale societies connected in many and complex ways.³

Our paper merges the epidemiological and the cultural transmission frameworks to analyze cultural dynamics in complex networks, and provides four main methodological contributions.

First, compared to the epidemiological literature in which transition rates from one trait to the other exogenously depend on the network connectivity, in our framework the transition rates are derived from an explicit cultural transmission process as the one in Bisin and Verdier (2001), integrating the two crucial dimensions of vertical and oblique socialization into the epidemiological literature.

Second, instead of having an unidirectional process of infection with subsequent recovery to the original state, we analyze a two-way symmetric epidemic SIS model, where both the infection and recovery rates are related to the topology of the network. In our setting, the recovery rate is not independent of the social network. Indeed agents, in order to be recovered, need to be "infected back" to the original state through a topology dependent cultural transmission technology.

Third, by modeling transition probabilities as the result of parental choices, the model allows the rates of infection and recovery to be naturally endogenous and shaped by the paternalistic motivations for socialization of the parents as well as the opportunity costs structure of such socialization. This provides some interesting economic interpretations of comparative statics on the three pillars of cultural transmission in our model : motivation, social structure and economic technology.

Finally, as we discuss below, endogenous vertical socialization can be also interpreted as a mechanism that produce endogenous homophily rates.

Our setting is as follows. We consider a population of non overlapping generations of individuals distributed along a fixed complex network endowed with two cultural types. Cultural transmission proceeds through vertical lines from parents to offsprings and oblique lines depending on the type distribution of neighbors' parent generation. Parents care about children' socialization through two channels: They enjoy spending some time with children and they have some paternalistic motivations to transmit their own

²See in particular the references in Bisin and Verdier (2010).

³See Bisin and Verdier (2001) for a complete survey of the economic extensions.

trait to their offsprings. Parents, then, can divide their time between socialization and some other socio-economic activity (like work or production). An important element of our framework is to acknowledge the fact that the technology of this alternative activity depends both on the cultural type of the parent as well as on his degree of social connectivity. This implies immediately that the opportunity cost of cultural transmission also depends in subtle ways on the interaction between cultural dimensions, paternalistic motivations and the structure of the social network. In such a context, we analyze the long run patterns of cultural evolution and discuss the conditions for cultural homogenization or not on the network.

Our main results are the following:

First, we consider the case in which cultural transmission is not driven by paternalistic motivations, but by a "joy of spending time" with one's own children. The transmission of traits comes then as a simple by-product of the time spent together. In such a context, we identify the emergence of a "*social structure*" bias that determines the direction of cultural change. This bias is directly related to the way the opportunity cost of parental socialization (the economic structure) interacts with the social network (the social structure). Because of such interactions, vertical socialization rates naturally depend on the topology of social connections, and vary accordingly with the degree of nodes on the network. As a consequence, the trait that across neighbors is relatively more successfully transmitted tends to have a positive bias in the cultural diffusion process along the network structure. Crucially, this bias is much affected by the degree distribution of neighbors and the way vertical transmission rates vary with connectivity. For some social networks, the *social structure* bias need not *always* favor the diffusion of one trait at the expense of the other. In such cases, long run cultural heterogeneity prevails in the networked population.

Second, using a standard mean-field approximation argument, we show that long run cultural heterogeneity is preserved when social connections and vertical socialization rates interact in such a way that the network degree distribution and the patterns of vertical socialization rates satisfy two balancing conditions across the network nodes. It is important to point out that in standard SIR/SIS models there is a unique condition. This uniqueness is driven by the fact that there is a unique trait to be infected to. In our context, being a two-way infection model, the standard connectivity threshold property of standard endemic models is replaced by the analysis of two balancing conditions that need to be satisfied to get long run cultural heterogeneity. To provide additional insights on these balancing conditions and their implications, we analyze the cases of free scale degree distribution and of an alternative environment such that the interaction between the social structure and socialization costs of socialization implies simple dichotomous thresholds effects.

Third, we consider the case in which parents have paternalistic motivations for cultural transmission as in Bisin and Verdier (2001). In such situations, the "social structure" bias effect combines with the additional "*cultural substitution*" effect. Our set-up shows that cultural diversity is then always preserved: the cultural substitutability effect more than compensates any social bias directly induced by the social network structure. The in-

tuition behind this result is the following. Although the network structure is formally assumed to be exogenous, paternalistic motivations for transmission tend to bias the process of information diffusion as if there were some endogenous pattern of homophily along the effective nodes of cultural transmission on the network. Because of the cultural substitutability effect, this "induced" effect of cultural transmission homophily for a particular cultural trait is in turn inversely related to the average cultural population frequency of that trait on the network. It favours therefore the diffusion of the cultural trait of the smaller group. As a consequence, any exogenous unbalancedness of the social structure that favours directly the diffusion of a particular trait, is compensated by the endogenous effective network transmission homophily effect of the other trait. This in turns leads to the maintenance of long run cultural heterogeneity for all types of social network structures.

Fourth, the introduction of endogenous purposeful socialization rates allows us to investigate how paternalistic motivations for cultural transmission interact with the social structure for the long run evolution of cultural groups. Specifically, we show with a simple example that a small increase of paternalistic motivations in cultural transmission tends to increase the long run equilibrium frequency of the minority group, whenever that group is sufficiently negatively biased by the social structure of the network. This example illustrates vividly how subjective motivations for socialization may mitigate direct social structure bias effects in the diffusion of cultural values.

Beyond the aforementioned literature on epidemiology and cultural evolutionary models, this paper also connects to the agent-based models literature investigating the spread and emergent properties of norm behaviors on large social networks (Axelrod, 1997; Axelrod et al., 2000; Durrett and Levin, 2005; Nakamaru and Levin, 2004). Following the standard socio-biology evolutionary literature on cultural transmission, cultural transmission in these models occurs through specific social exposure to alternative role models in the network and exogenous revision rules of cultural types. These transmission rules are not necessarily derivable from rational behavior based on optimization and strategic individual actions in a particular context. The consequences on long run cultural patterns are then generally analyzed through computer simulations. Our approach differs and complements that literature along three dimensions. First, we emphasize the role of purposeful actions by agents on cultural diffusion, allowing us to investigate the impact of motivational forces (preferences) and socio-economic opportunity costs (technologies) on the structure of cultural steady states. Second, our perspective provides explicit analytical conditions that characterize, in simple but natural cases, the configurations of network structures and socio-economic technologies that are compatible (or not) with long run cultural heterogeneity. Finally our economic approach illustrates how specific motivations for cultural transmission (ie. paternalism) may mitigate or complement the impact of network structure on cultural transmission that these models emphasize.

The plan of the paper is the following. Section 2 presents the general model of cultural transmission on a complex network. Section 3 considers the case where parents do not have paternalistic motivations to transmit their traits and characterizes the patterns of long run

cultural steady states, depending on the nature of the *social structure* bias induced by the network. Section 4 provides a discussion of simple examples of large network structures and their interactions with the socio-economic environment of the individuals. Section 5 considers the case of cultural transmission by paternalistic parents with a discussion of some comparative statics highlighting the role of motivational paternalistic forces and social network structures on long run cultural steady states. Finally section 7 concludes with some areas for future research.

2 The Model

Consider a society of n agents (with n large enough) located in a random network g . Let $q \equiv q(k)_{k=0}^{\infty}$ be the degree distribution of the network, where k is the degree of nodes. The society is partitioned in two groups of agents characterized by a distinct cultural trait A or B . Denote $p_{k,t}^i \in (0, 1)$ the share of individuals of type $i \in \{A, B\}$ in the subset of agents with degree k at time t . We assume that links are formed using a standard degree-based sampling process such that the degree distribution among neighboring nodes is given by a function $\xi(k) \propto q(k)k^\alpha$, with $\alpha \in [0, 1]$. Through normalization, $\xi(k) = \frac{q(k)k^\alpha}{\sum_k q(k)k^\alpha}$.⁴ Denoting by \tilde{p}_t^A the expected fraction of neighbors' of cultural type A in the network at some time t , one gets:

$$\tilde{p}_t^A = \sum_{k=0}^{\infty} \xi(k) p_{k,t}^A \quad (1)$$

Let us now focus on how transition probabilities among types are built. Following Bisin and Verdier (2001), we consider a model of intergenerational cultural transmission on a network in which cultural transmission arises from the combination of parents' endogenous vertical socialization rates and oblique socialization through the network.

Specifically, at each point in time t , with exogenous instantaneous probability λdt each agent of type $i \in \{A, B\}$ with a position of degree k on the network has an offspring taking his place in the network. Conversely with probability $1 - \lambda dt$, the agent remains in place. When a child appears, the latter has to be socialized to a particular cultural trait $j \in \{A, B\}$ through vertical and oblique transmission. The probability of successful vertical socialization is directly related to the "parenting" time $\tau_{k,t}^i$ the parent (of type i and degree k) spends with the offspring rather than on own production. With the complementary probability, vertical transmission does not succeed and the child is socialized by oblique transmission, picking up a random cultural role model from the neighborhood of the parent. Denote by $P_{k,t}^{ij}$ the probability, at time t , that the child adopts the cultural trait j , when having a parent with cultural trait i and degree k . Using a mean-field approximation, we consider that the neighbors' cultural type distribution is homogenous across agents, and thus oblique socialization to a specific type will be successful with

⁴In some examples we will use $\alpha = 1$ so that $\xi(k) = \frac{q(k)k}{\bar{k}}$, where \bar{k} is the average degree

a probability reflecting the expected fraction of neighbors' of that cultural type in the network (in our case, \tilde{p}_t^A or $\tilde{p}_t^B = 1 - \tilde{p}_t^A$).⁵ The probabilities $P_{k,t}^{ij}$ can then be written as:

$$P_{k,t}^{ii} = \tau_{k,t}^i + (1 - \tau_{k,t}^i)\tilde{p}_t^i \quad (2)$$

$$P_{k,t}^{ij} = (1 - \tau_{k,t}^i)\tilde{p}_t^j \quad (3)$$

Altruism motivates parents to exert effort to socialize their offspring. This altruism however is assumed to be 'paternalistic' in the sense that parents wish to transmit their own trait, and do not just internalize their children's preferences or some measure of their success. Specifically, we assume that a parent with degree k and type i has the following linear quadratic altruistic utility function:

$$W_{k,t}^i = Q_k^i - \frac{(\tau_{k,t}^i - \bar{\tau}_{k,t}^i)^2}{2c_k^i} + \beta[P_{k,t}^{ii}V^{ii} + P_{k,t}^{ij}V^{ij}] \quad (4)$$

where, V^{ij} is the utility that each parent of type i derives from a child of type j , with $i, j \in \{A, B\}$, and Q_k^i , $\bar{\tau}_{k,t}^i$, c_k^i and β are positive constant parameters. Equation (4) can be rewritten extensively as:

$$W_{k,t}^i = Q_k^i - \frac{(\tau_{k,t}^i - \bar{\tau}_{k,t}^i)^2}{2c_k^i} + \beta[\tau_{k,t}^i + (1 - \tau_{k,t}^i)\tilde{p}_t^i]V^{ii} + [(1 - \tau_{k,t}^i)(1 - \tilde{p}_t^i)]V^{ij} \quad (5)$$

The first two terms in (5) capture in a simple way the utility of a parent devoting parenting time τ_k^i without a specific motivation for paternalistic socialization, as we discuss in details in Section 3.2. It reflects the outcome of a tradeoff of time allocation between alternative activities in the household (production, leisure, interactions with friends and family,...). As a matter of fact it can be seen as a simple quadratic approximation of a time allocation problem in which the productivity of alternative activities to parenting are affected both by the nature of the cultural trait (type i) and the social connectivity (the degree k) of the parent individual. The parameter $\bar{\tau}_k^i$ is then simply the optimal parenting time without a specific motivation for paternalistic cultural transmission. It follows immediately that the opportunity cost of parenting time depends, through $\bar{\tau}_k^i$ and c_k^i , on the degree k and the cultural type i of the parent. This dependence could be positive reflecting the idea that social connections reduce this opportunity cost of vertical transmission (a "social prestige-based" idea of cultural transmission). This dependence could be also negative if indeed social connections tend to crowd out the parenting time parents devote to their offsprings. In Section 3.2, we provide a microfounded context

⁵Notice that although we consider here a fixed social network structure, the mean field approximation mimics the case in which each new generation is able to redraw randomly its network connections maintaining the degree unchanged. Thus, under this restriction, the network structure may be interpreted as partially changing.

showing precisely how the opportunity cost of socialization depends on social connectivity according to the technology and the cultural traits that parents have.

The last two terms of (5) reflect the paternalistic motivation of parents to transmit their trait to their kids with β the altruistic weight, and $[\tau_k^i + (1 - \tau_k^i)\tilde{p}_t^i]V^{ii} + [(1 - \tau_k^i)(1 - \tilde{p}_t^i)]V^{ij}$, the expected gain to transmit one's trait. Paternalistic motivation translates into assuming that $V^{ii} \geq V^{ij}$, if $i \neq j$.⁶

In mean field approximation, the time-continuous approximation for the dynamics of the $p_{k,t}^A$ is therefore given by

$$\dot{p}_{k,t}^A = \lambda [p_{k,t}^A(\tau_k^A - 1)(1 - \tilde{p}_t^A) + (1 - p_{k,t}^A)(1 - \tau_k^B)\tilde{p}_t^A] \quad (6)$$

3 Cultural Dynamics without Paternalism

It is useful to consider first the case of cultural evolution when parents do not have paternalistic motivations for cultural transmission, (ie. $V^{ii} = V^{ij} = V^i$, if $i \neq j$). This will highlight most clearly the role of social connectivity and its implications for long run cultural heterogeneity in the population. In such a case, the maximization problem of a parent of type i and degree k collapses to

$$\max_{\tau_k^i} Q_k^i - \frac{(\tau_k^i - \bar{\tau}_k^i)^2}{2c_k^i} + \beta V^i \quad (7)$$

and the optimal parenting time is trivially given by $\tau_k^i = \bar{\tau}_k^i$.

We make the assumption that the vertical distribution rates $\{\bar{\tau}_k^A\}_{k \geq 0}$ and $\{\bar{\tau}_k^B\}_{k \geq 0}$ are uniformly bounded away from 0 and 1 across the network. Formally:

$$\begin{aligned} \textbf{Assumption 1} & : \quad \exists (\tilde{\tau}_{\min}^A, \tilde{\tau}_{\max}^A, \tilde{\tau}_{\min}^B, \tilde{\tau}_{\max}^B) \in]0, 1[^4 : \\ & \forall k \in [1, +\infty[, \bar{\tau}_k^A \in [\tilde{\tau}_{\min}^A, \tilde{\tau}_{\max}^A] \text{ and } \bar{\tau}_k^B \in [\tilde{\tau}_{\min}^B, \tilde{\tau}_{\max}^B] \end{aligned}$$

This property basically prevents the existence of trivially irrelevant or completely absorbing nodes for the process of vertical transmission.

3.1 Balancing Conditions and Cultural Steady States

At a steady state equilibrium of (6), one should have

⁶Some evidence in support of ‘paternalistic altruism’ can also be derived from socio-economic surveys. For instance, in response to NORC’s General Social Survey’s question, ‘Which three of the qualities listed would you say are the most desirable for a child to have?’, ‘obedience’ is cited on average across the sample more than, (in order) ‘self-control’, ‘success’, ‘studiousness’, ‘cleanliness’, and less often only than ‘honesty’.

For a natural selection explanation of paternalistic forms of altruism, see also Bisin and Verdier (1998).

$$p_k^A = \frac{\tilde{p}^A(1 - \bar{\tau}_k^B)}{\tilde{p}^A(1 - \bar{\tau}_k^B) + (1 - \tilde{p}^A)(1 - \bar{\tau}_k^A)} \quad (8)$$

Plugging (1) into (8) the steady state value \tilde{p}^A should satisfy:

$$\tilde{p}^A = \sum_{k=0}^{\infty} \xi(k) \frac{\tilde{p}^A(1 - \bar{\tau}_k^B)}{\tilde{p}^A(1 - \bar{\tau}_k^B) + (1 - \tilde{p}^A)(1 - \bar{\tau}_k^A)} \equiv \sum_{k=0}^{\infty} \xi(k) \varphi(k, \tilde{p}^A) \equiv \Phi(\tilde{p}^A) \quad (9)$$

out of which one may recover p_k^A from (8).

To discuss the nature of the cultural steady states on the network, it is useful to define the following **Balancing Conditions (BC)**:

$$\text{condition(BCa)} : \sum_{k=0}^{\infty} \xi(k) \frac{(1 - \bar{\tau}_k^B)}{(1 - \bar{\tau}_k^A)} > 1 \quad (10)$$

$$\text{condition(BCb)} : \sum_{k=0}^{\infty} \xi(k) \frac{(1 - \bar{\tau}_k^A)}{(1 - \bar{\tau}_k^B)} > 1 \quad (11)$$

Examination of (9) provides the following proposition that characterizes the full structure of average steady state points of our process of cultural evolution.

Proposition 1 : *Assume that assumption 1 holds. Then:*

- i) The dynamic system (6) always admits two homomorphic steady states: $p_k^A = \tilde{p}^A = 0$ for all k , and $p_k^A = \tilde{p}^A = 1$ for all k ;*
- ii) When both (10) and (11) are satisfied, there exists a unique polymorphic steady state $\tilde{p}^A \in (0, 1)$ satisfying equations (8) and (9). The polymorphic steady state is locally stable;*
- iii) When only (10) is satisfied, there is no polymorphic steady state and only the homomorphic steady state $\tilde{p}^A = 1$ is locally stable;*
- iv) When only (11) is satisfied, there is no polymorphic steady state and only the homomorphic steady state $\tilde{p}^A = 0$ is locally stable;*
- v) When both (10) and (11) are not satisfied, there exists a unique polymorphic steady state $\tilde{p}^A \in (0, 1)$. The polymorphic steady state is locally unstable.*

Proof. : See the Appendix. ■

Essentially, inequality (10) (respectively (11)) describes the condition for the existence of a *social structure* cultural bias for the initial diffusion of trait A (respectively trait B). Intuitively such conditions indicate that the number of agents better socializing to trait A and their socialization efforts need to be balanced with the number of agents better socializing to trait B and their socialization efforts. When both conditions are satisfied, the extreme points $\tilde{p} = 0$ or $\tilde{p} = 1$ are locally unstable and the system does not converge

towards cultural homogeneity. These conditions are also sufficient for the existence of a unique locally stable polymorphic equilibria $\tilde{p} \in (0, 1)$ such that $\dot{\tilde{p}} = 0$.⁷

Our approach clearly connects to the standard epidemiology literature in social networks (Pastor-Satorras and Vespignani, 2001, 2005; López-Pintado, 2008, 2012). In such models, the recovery rate from infection is exogenous and degree independent and a typical endemic long run steady state exists when the infection rate satisfies a threshold condition that depends on the connectivity of the network. In the current context, we have a two-way cultural transmission model where "infection" rates (going say from trait A to trait B) and "recovery" rates (going from trait B to trait A) are both degree dependent through vertical transmission. It turns out therefore that the condition for the sustainability of long run cultural heterogeneity, which would be the equivalent to a long term endemic outcome, depends on two-threshold conditions (10) and (11). This suggests that the connectivity of the social network and the shape of vertical transmission in the network should be somewhat balanced in order to satisfy these two conditions.

Importantly, the balancing conditions BCs highlight the fact that what matters for the existence of a "social structure" transmission bias is the way network connectivity affects the *relative* pattern of vertical transmission rates across the two cultural traits (ie. the distribution of $(1 - \bar{\tau}_k^A)/(1 - \bar{\tau}_k^B)$ or its inverse). To get long run cultural heterogeneity, such distribution should be relatively balanced across the network to ensure that the bias that the "social structure" creates in favor of one trait in some part of the network gets compensated somewhere else by a transmission bias favoring the other trait. It is also clear that if $\bar{\tau}_k^A > \bar{\tau}_k^B, \forall k$, or vice-versa, then no polymorphic equilibrium is either feasible. A special case of such situation would be if social connectivity is irrelevant to vertical socialization (ie. $\bar{\tau}_k^A = \bar{\tau}^A$ and $\bar{\tau}_k^B = \bar{\tau}^B$ for all k). In such a case obviously, only one trait is generically present in the long run⁸. Hence a necessary condition to observe a polymorphic equilibrium in this context is the fact social connectivity matters for parenting time and that the distribution of the vertical socialization rates across the two cultural traits crosses at some critical degree threshold \tilde{k} (See Figure 1).

The distribution $\{\bar{\tau}_k^i\}_{k \in N}$ of vertical transmission rates may be shaped by exogenous features of the social environment, such as specific positions on the social network that may confer advantages or disadvantages to transmit a particular cultural trait. The following section provides a simple economic example illustrating these features.

⁷Note however that $\dot{\tilde{p}} = 0$ is a necessary but not a sufficient condition for $\dot{p}_k = 0, \forall k$. Indeed when $\tilde{p} = 0$ is unstable, then the homogeneous steady state $p_k = 0, \forall k$ is also unstable. As well when $\tilde{p} = 1$ is unstable, then also $p_k = 1, \forall k$ is unstable. However it is possible to have $\dot{\tilde{p}} = 0$ but $\dot{p}_k \neq 0$ for some k with some cycles arising.

⁸In the limit case where $\bar{\tau}^A = \bar{\tau}^B$, the steady state cultural distribution of traits remains indeterminate.

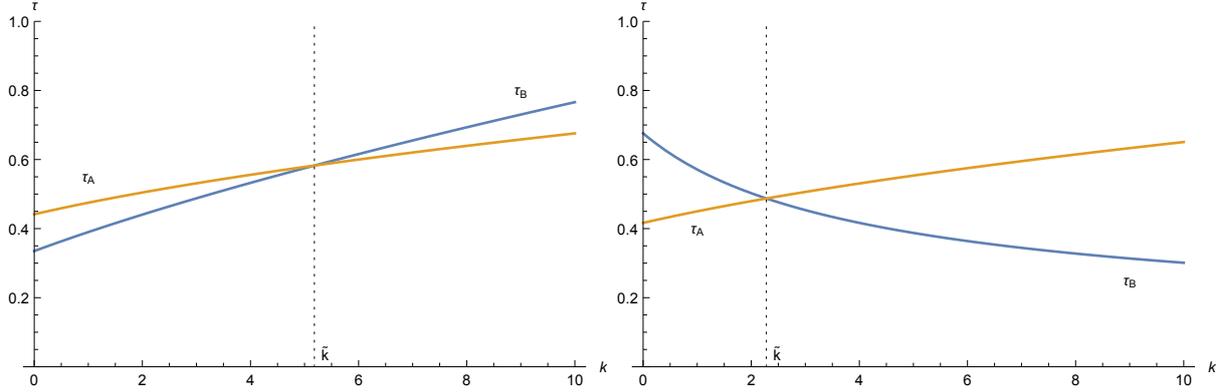


Figure 1: Two Examples of distributions of $\{\tau_k^A\}_{k=1}^\infty$ and $\{\tau_k^B\}_{k=1}^\infty$, and the critical degree threshold \tilde{k} .

3.2 A Simple Time Allocation Model of vertical transmission

In this section we present a simple microeconomic time allocation example that highlights how social connectivity can affect the shape of $\{\bar{\tau}_k^i\}_{k \in N}$. This model provides then a simple economic rationale for the shape of our first quadratic term $Q_k^i - (\tau_k^i - \bar{\tau}_k^i)^2 / 2c_k^i$ in (5).

Assume that individuals are endowed with one unit of time. At each point of time an agent with cultural trait $i \in \{A, B\}$ and degree k can allocate this unit of time between production and parenting (when he has a child). Production involves a labor technology with marginal productivity $\phi^i(k)$ that depends both on the cultural type of the individual and his social connectivity to the network. We assume that social interactions generate positive spillovers, and therefore that $\phi^i(\cdot)$ is an increasing function in the degree k of the individual. We also assume that the cultural trait B is more "congruent" to social connectivity than the cultural trait A , namely that individuals endowed with culture B are more likely to enjoy the spillovers of social interactions than individuals endowed with trait A . Formally we assume that $\phi^{A'}(k) < \phi^{B'}(k)$ for all $k \geq 0$. We also assume the following boundary conditions $\phi^A(0) > \phi^B(0)$ and $\lim_{\infty} \phi^A(k) < \lim_{\infty} \phi^B(k)$ that ensure that social connectivity can revert the ranking of productive efficiency across the two cultural traits.

An agent of type $i \in \{A, B\}$ cares about individual consumption and eventually about parenting when he has a child. Specifically he has the following utility function (with a "joy of parenting time" argument):

$$u^i(c^i, \tau^i, n^i) = c^i + n^i H^i(\tau^i) \quad (12)$$

where c^i is individual consumption, τ^i is parenting time, $n^i \in \{0, 1\}$ is the number of offsprings and $H^i(\tau^i)$ is an increasing strictly concave function reflecting the direct utility of parenting when one has an offspring (ie. when $n^i = 1$). Using the production technology

$\phi^i(\cdot)$, This can be rewritten as

$$U_k^i(\tau^i, n^i) = \phi^i(k)(1 - \tau^i) + n^i H^i(\tau^i) \quad (13)$$

Obviously parenting only occurs for individuals who have an offspring. It is then easy to see that the second order approximation the direct utility of agent of type i with degree k writes as

$$U_k^i(\tau_k^i) = Q_k^i - \frac{(\tau_k^i - \bar{\tau}_k^i)^2}{2c_k^i} \quad (14)$$

with $Q_k^i = U_k^i(\bar{\tau}_k^i, 1)$, $\bar{\tau}_k^i = H^{i-1}(\phi^i(k))$ and $c_k^i = -1/H^{i''}(\bar{\tau}_k^i)$.

The assumptions on the labor productivity function $\phi^i(\cdot)$ ensure that the vertical transmission rates sequences $\{\bar{\tau}_k^i\}_{k \in N}$ are decreasing in the degree k . Moreover, if the direct utility of parenting is independent of the cultural type and quadratic, ie. $H^i(\tau^i) = H(\tau^i) = \frac{1}{c}[\tau^i - \frac{(\tau^i)^2}{2}]$, then it is easy to see that there exists a unique threshold $\tilde{k} \in]0, +\infty[$ such that $\bar{\tau}_k^A \geq \bar{\tau}_k^B$ if and only if $k \geq \tilde{k}$.⁹ In this way the transmission rate functions $\bar{\tau}_k^A$ and $\bar{\tau}_k^B$ cross at some threshold \tilde{k} and may satisfy the necessary condition for observing long run cultural heterogeneity.

4 Examples

The two balancing conditions for the existence of a cultural polymorphic steady state in (10)-(11), combine in a complex way both the social structure of the network and the degree dependent structure of vertical socialization rates. To get more insights on such conditions, we consider now two examples of complex networks and vertical socialization rates, and discuss how these patterns of social connectivity and vertical socialization combine to ensure the existence of long cultural diversity in the population.

In the first example we consider a social and technological environment associated to fixed degree distribution, a free scale distributions, but allowing for generic vertical socialization rates. The second example takes the alternative route to discuss the case of a production technology that generates a specific two-valued distribution of vertical transmission rates across nodes but allowing, on the other hand, for any type of degree distribution structure for the network. These two examples provide parametrized sequences

⁹If the functions $H^i(\tau)$ are quadratic and cultural type dependent (ie. $H^i(\tau) = \frac{1}{c_i}(\tau - \frac{\tau^2}{2})$), then we need to impose the following conditions

$$\phi^B(0) < \frac{c^A}{c^B} \cdot \phi^A(0), \quad \lim_{\infty} \phi^B(k) < \frac{c^A}{c^B} \cdot \lim_{\infty} \phi^A(k)$$

and

$$\phi^{A'}(k) < \frac{c^B}{c^A} \cdot \phi^{B'}(k)$$

to ensure the existence of a unique threshold $\tilde{k} \in]0, +\infty[$ such that $\bar{\tau}_k^A \geq \bar{\tau}_k^B$ if and only if $k \geq \tilde{k}$.

$\{\bar{\tau}_k^i\}_{k \in N}$ of vertical transmission rates and precise analytical balanced conditions for the existence of long run cultural heterogeneity.

4.1 Scale-free Degree distributions

In this example we fix the degree distribution and we study which cultural transmission efforts shapes are compatible with long run cultural heterogeneity. We assume that the degree distribution of the network is scale-free, so that $q(k) = \sigma k^{-\gamma}$ where $\gamma \in (2, 3)$. Such network structure has been found to be empirically relevant in a number of social situations (see Albert and Barabási, 2002; Barabási and Bonabeau, 2003, for some references). We build upon on our simple microfounded model of parental time allocation of section 4. We assume that the degree dependent labor productivity $\phi^i(k)$ takes the following shape $\phi^i(k) = \theta k^{\rho^i}$ for the two cultural traits $i \in \{A, B\}$, with $\rho^B > \rho^A > 0$, capturing the fact that individuals endowed with culture B are enjoying more than individuals with culture A the positive spillovers of network connectivity. Finally, we assume that the direct utility of parental care in (12) is quadratic and given by $H^i(\tau^i) = \frac{1}{c^i} \left(\tau^i - \frac{(\tau^i)^2}{2} \right)$ with $c^A > c^B$ (parental care is more valued *per se* by individuals of type B compared to individuals of type A). With such specifications, one obtains immediately the vertical socialization rates as:

$$1 - \bar{\tau}_k^i = c^i \theta k^{\rho^i} \quad \text{for } i \in \{A, B\} \quad (15)$$

Looking at conditions (10) and (11), the crucial variable to be analyzed is $r(k) \equiv \frac{(1-\tau_k^B)}{(1-\tau_k^A)}$, which here writes as

$$r(k) \equiv \frac{c^B}{c^A} k^{(\rho^B - \rho^A)} \quad (16)$$

As we have seen above, the existence of a polymorphic equilibrium depends on how this ratio $r(k)$ changes with the degree. Depending on parameters c^A , c^B , ρ^A , ρ^B , the ratio $r(k)$ can take different shapes.

Looking at ξ_k , we assume that $\alpha = 1$, so that $\xi_k = \frac{q(k)k}{\bar{k}}$. Then we can write the balancing conditions for long run cultural heterogeneity as (see Appendix 2):

$$\frac{(\gamma - 2)}{c^A} < \frac{(\rho^B - \rho^A)}{(c^A - c^B)} < \frac{(\gamma - 2)}{c^B} \quad \text{with } \gamma > 2 + \rho_B - \rho_A \quad (17)$$

Since for free scale networks one has $\gamma \in (2, 3)$, it follows that we need to impose $\rho_B - \rho_A \in (0, 1)$, that is the difference of the impact of social connectivity on labor production should not be too large across the two cultural traits.

First note that, for every γ , the set of parameters that satisfies (17) is a nonempty set of strictly positive measure. Notice also that there is always a degree critical threshold \tilde{k} such that for $k \leq \tilde{k} = (c^A/c^B)^{\frac{1}{\rho_B - \rho_A}}$ we have $r(k) \geq 1$. Therefore, the distribution of

vertical transmission rates should be such that nodes with degree $k > \tilde{k}$ (resp. $k < \tilde{k}$) are more successful at transmitting trait A (resp. trait B). It follows also that various critical threshold configurations \tilde{k} of the relative vertical transmission rates of the two traits induce the existence of a stable polymorphic cultural steady state on the network. Note finally that condition (17) states that the relative cross cultural impact of social connectivity $\rho^B - \rho^A$ on the opportunity cost of socialization of type B should not be too highly unbalanced relative to the relative cultural benefit of parental care $c^A - c^B$ to allow for long run cultural heterogeneity on the network.

4.1.1 Balancing Conditions and Stochastic Dominance

The scale free case allows us to perform some comparative statics based on the statistical properties of the degree distribution. We look at changes in parameters that determine a First Order Stochastic Dominance. At the same time, we cannot proceed here with an analysis of Second Order Stochastic Dominance since, given $\gamma \in (2, 3)$, the variance is not finite (we will do it in the next example). Note that

$$\langle k \rangle = \int_1^\infty \sigma k^{1-\gamma} dk = \frac{\sigma}{2-\gamma} k^{2-\gamma} \Big|_1^\infty = \frac{\sigma}{\gamma-2} \quad (18)$$

It is then clear that, taking two degree distribution $q(k)$ and $q'(k)$ with parameters σ , σ' , γ and γ' , then $q(k)$ FOSD $q'(k)$ if $\sigma > \sigma'$ or if $\gamma < \gamma'$. The conditions (17) can be rewritten as

$$(\gamma-2) \frac{c^A - c^B}{c^A} < \rho^B - \rho^A < (\gamma-2) \frac{c^A - c^B}{c^B} \quad (19)$$

For convenience, denote, $\underline{\eta} = (\gamma-2) \frac{c^A - c^B}{c^A}$ and $\bar{\eta} = (\gamma-2) \frac{c^A - c^B}{c^B}$. It is then immediate to see that if $q(k)$ FOSD $q'(k)$, then $\underline{\eta} > \underline{\eta}'$ and $\bar{\eta} > \bar{\eta}'$. That is to say that the interval of values of $\rho^B - \rho^A$ such that a polymorphic equilibrium exists shifts down as the mean increases. In order to understand the mechanism behind this, remember that $\tau_k^A \leq \tau_k^B$ if and only if $k \leq \tilde{k} = (c^A/c^B)^{\frac{1}{\rho^B - \rho^A}}$ and that $\rho^B - \rho^A$ measures how social connectivity tends to increase the relative ability of a node to socialize to A type rather than to B type. Consider first the upper threshold $\bar{\eta}'$. Above this threshold there is a homomorphic equilibrium in which only A types exist. Now, if $q(k)$ FOSD $q'(k)$, then under $q(k)$ there is a larger number of nodes with high degree, those who are exactly the ones relatively better at socializing to the A type. Thus at $\bar{\eta}'$, instead of observing a balancing between socialization of A and B we observe only A types present in the long run. This implies that the upper bound of the balancing conditions has to be smaller, therefore $\bar{\eta} < \bar{\eta}'$ and the upper threshold shifts down. Consider now the lower threshold $\underline{\eta}'$. For values of $\rho^B - \rho^A$ below that level, B types individuals invade the whole society, while at $\underline{\eta}'$ there is balance with A types agents. However if the mean increases, A type agents start to have an advantage over B types so that the balancing threshold has to decrease. As a consequence of this, it follows that the whole balancing condition interval in (19) shifts

down.¹⁰

4.2 Two-valued distributions of vertical transmission rates

In this example we fix cultural transmission efforts and we analyze which degree distributions are compatible with long run cultural heterogeneity. Assume that the shape of the degree dependent labor productivity $\phi^i(k)$ takes the following form:

$$\begin{aligned}\phi^i(k) &= \underline{\phi}^i \text{ for } k \leq \tilde{k} \\ &= \bar{\phi}^i \text{ for } k > \tilde{k}\end{aligned}\tag{20}$$

with $\underline{\phi}^i < \bar{\phi}^i$ and $\tilde{k} > 1$. Social connectivity generates positive spillovers to an agent only when his degree is larger than a threshold value \tilde{k} . Assume as well that $\underline{\phi}^B < \underline{\phi}^A < \bar{\phi}^A < \bar{\phi}^B$. At the margin, the effect of social connectivity on productivity is stronger for B types than for A types, while A types start with a higher productivity than B types. Also assume again that the direct utility of parental care is quadratic $H^i(\tau^i) = \frac{1}{c^i} \left(\tau^i - \frac{(\tau^i)^2}{2} \right)$ and for simplicity $c^A = c^B = c$. Then given this specification, one gets a very simple two value distribution of vertical transmission rates for each type $i \in \{A, B\}$:

$$\begin{aligned}\bar{\tau}_k^i &= \tau_1^i = 1 - \underline{\phi}^i c \text{ for } k \leq \tilde{k} \\ &= \tau_2^i = 1 - \bar{\phi}^i c \text{ for } k > \tilde{k}\end{aligned}\tag{21}$$

implying that type B individuals have an advantage in terms of vertical transmission rates at low levels of social connectivity. (ie. for $k \leq \tilde{k}$, $\bar{\tau}_k^A \equiv \tau_1^A < \tau_1^B \equiv \bar{\tau}_k^B$, while for $k > \tilde{k}$ then $\bar{\tau}_k^A \equiv \tau_2^A > \tau_2^B \equiv \bar{\tau}_k^B$). Define, for any network degree distribution, $U_1 \equiv \sum_{k=0}^{\tilde{k}} \xi(k)$, $U_2 \equiv 1 - U_1 = \sum_{k=\tilde{k}}^{\infty} \xi(k)$, $x_1 \equiv \frac{(1-\tau_1^B)}{(1-\tau_1^A)} = \frac{\underline{\phi}^B}{\underline{\phi}^A}$, and $x_2 \equiv \frac{(1-\tau_2^B)}{(1-\tau_2^A)} = \frac{\bar{\phi}^B}{\bar{\phi}^A}$ so that $x_2 > 1 > x_1$. After some algebra, one can then rewrite conditions (10) and (11) as:

$$\frac{x_2 - 1}{x_2 - x_1} x_1 < U_1 < \frac{x_2 - 1}{x_2 - x_1}\tag{22}$$

which reads as

$$\frac{\left(\bar{\phi}^B - \bar{\phi}^A \right) \underline{\phi}^B}{\underline{\phi}^A \bar{\phi}^B - \underline{\phi}^B \bar{\phi}^A} < U_1 < \frac{\left(\bar{\phi}^B - \bar{\phi}^A \right) \underline{\phi}^A}{\underline{\phi}^A \bar{\phi}^B - \underline{\phi}^B \bar{\phi}^A}\tag{23}$$

Given the values of \tilde{k} , and $\underline{\phi}^B < \underline{\phi}^A < \bar{\phi}^A < \bar{\phi}^B$, condition (23) restricts the family of degree distributions such that a polymorphic equilibrium is shown. This restriction is

¹⁰However we must point out that the two thresholds do not shift down by the same amount because of the different magnitudes of the socialization efforts. In particular the upper threshold decreases faster than the lower threshold so that, as the mean of the degree distribution increases, the size of the interval actually shrinks.

based on the network statistics U_1 . Given (9), the value of the polymorphic steady state $\tilde{p}^A \in (0, 1)$ can then be easily obtained exactly in the interval identified by condition (23), as well as the stationary long run cultural frequency distribution of type A $(p_k^A)_{k \in N}$ across the network. It is also immediate to see that such distribution will also take dichotomous values (ie. $p_k^A = p_1^A$ for $k \leq \tilde{k}$ and $p_k^A = p_2^A$ for $k > \tilde{k}$). It is interesting to note that the larger the differential impact of social connectivity across cultural traits on productivity (ie the larger $\bar{\phi}^B - \bar{\phi}^A$ and $\underline{\phi}^A - \underline{\phi}^B$), the larger the range of values of the network statistic U_1 such that the balanced conditions will be satisfied and therefore the larger the likelihood of having a long run polymorphic cultural equilibrium.

4.2.1 Balancing Conditions and Stochastic Dominance

The two-valued vertical transmission rate distribution example can also provide some insights on how the long run polymorphic steady state changes when the network degree distribution changes. Consider two degree distributions $q(k)$ and $q'(k)$, such that $q'(k)$ has a First Order Stochastic Dominance over $q(k)$. Since $\tilde{k} = \tilde{k}'$, then

$$U_1 \equiv \sum_{k=0}^{\tilde{k}} \xi(k) > \sum_{k=0}^{\tilde{k}} \xi'(k) \equiv U_1' \quad (24)$$

Since under $q'(k)$ the number of agents better socializing to trait A is higher, the long run polymorphic steady state equilibria are such that $\tilde{p}'_A > \tilde{p}$. Intuitively, trait A is more (resp. less) successfully transmitted than trait B at high (resp. low) degree nodes. A first order stochastic dominance shift tends to reduce the mass of low degree nodes and increase the mass of high degree nodes. This in turn changes the "social structure" bias in favor of trait A . As a consequence the average steady state fraction of trait A in the network increases.

Consider now the case in which $q'(k)$ is a mean-preserving spread of $q(k)$. If $\tilde{k} < \bar{k}$. Then $U_1 \equiv \sum_{k=0}^{\tilde{k}} \xi(k) < \sum_{k=0}^{\tilde{k}} \xi'(k) \equiv U_1'$. Keeping the mean of the degree distributions unchanged, under $q'(k)$ the number of agents better socializing to trait B is larger than under the other distribution $q(k)$. Consequently, cultural transmission should promote more the diffusion of trait B in the network, and $\tilde{p}'_A < \tilde{p}_A$. If $\tilde{k} > \bar{k}$, by a similar argument one gets one gets the reverse result $\tilde{p}'_A > \tilde{p}_A$.

5 Cultural Dynamics with Paternalism

In this section we complete our analysis by considering the case in which parents also have paternalistic motivations to transmit their trait (ie. $\Delta V^i = V^{ii} - V^{ij} > 0$, for $i, j \in \{A, B\}$ and $i \neq j$). We also impose the following assumption

$$\textbf{Assumption 2: } \Delta V^i \in \left[\frac{\bar{\tau}_k^{-i} - \bar{\tau}_k^i}{c_k^i}, \frac{1 - \bar{\tau}_k^i}{c_k^i} \right] \forall i, k$$

where $\bar{\tau}_k^{-i}$ represents the effort of the type other than i . As it will be clear from the proof of proposition 2, the upper bound represents a condition for an interior equilibrium to exist and to exclude the case of extremely high motivations to vertical socialization that would induce a corner solution in terms of socialization rates. The lower bound is necessary to ensure the presence of a polymorphic equilibrium.¹¹

Recalling (5), the optimization problem of a parent now writes as

$$\max_{\tau_k^i} Q_k^i - \frac{(\tau_k^i - \bar{\tau}_k^i)^2}{2c_k^i} + \beta[\tau_k^i + (1 - \tau_k^i)\tilde{p}_t^i]V^{ii} + [(1 - \tau_k^i)(1 - \tilde{p}_t^i)] V^{ij} \quad (25)$$

The solution is straightforward:

$$\tau_k^i = \bar{\tau}_k^i + c_k(1 - \tilde{p}^i)\Delta V^i, \text{ with } i \in \{A, B\} \quad (26)$$

The paternalistic motivation induces the so-called *cultural substitutability* property (Bisin and Verdier 2001): τ_k^i is negatively related to the average fraction \tilde{p}_i of his own group on the network. Moreover the upper bound of Assumption 2 imposes that $\tau_k^i < 1$.

The cultural dynamics are again determined by (6). Specifically the characterization of the steady states is now obtained from the following equations:

$$\tilde{p}^A = \sum_{k=0}^{\infty} \xi(k)\varphi(k, \tilde{p}^A) \equiv \Phi(\tilde{p}^A) \quad (27)$$

where $\varphi(k, \tilde{p}^A)$ is given by

$$\varphi(k, \tilde{p}^A) = \frac{G(\Delta V^B c_k^B, \tilde{p}^A)}{G(\Delta V^B c_k^B, \tilde{p}^A) + G(\Delta V^A c_k^A, 1 - \tilde{p}^A)} \quad (28)$$

with

$$G(\Delta V^B c_k^B, \tilde{p}^A) = \tilde{p}^A(1 - \bar{\tau}_k^B - c_k^B \Delta V^B \tilde{p}^A) \quad (29)$$

$$G(\Delta V^A c_k^A, 1 - \tilde{p}^A) = (1 - \tilde{p}^A) [1 - \bar{\tau}_k^A - c_k^A \Delta V^A (1 - \tilde{p}^A)] \quad (30)$$

The steady state points $\tilde{p}^A \in [0, 1]$ are such that

$$\Psi(\tilde{p}^A) = \Phi(\tilde{p}^A) - \tilde{p}^A = 0 \quad (31)$$

One then gets the following result:

¹¹Technically, the upper bound should be written as

$$\Delta V^i < \frac{1 - \bar{\tau}_k^i}{c_k^i \tilde{p}_t^i} \forall i, k$$

However, being sufficient conditions, these are satisfied, for every \tilde{p}_t^i , if $\Delta V^A < \frac{1 - \bar{\tau}_k^A}{c_k^A}$ and $\Delta V^B < \frac{1 - \bar{\tau}_k^B}{c_k^B}$.

Proposition 2 : *Assume that parents have paternalistic motivations to transmit their trait vertically and that assumption 2 holds. The dynamic system (6) always admits two homomorphic steady states: $p_k^A = \tilde{p}^A = 0$ for all k , and $p_k^A = \tilde{p}^A = 1$ for all k . For all complex network structures, these two homomorphic steady states are locally unstable and there exists at least one locally stable polymorphic steady state $\tilde{p}^A \in (0, 1)$.*

Proof. See the appendix. ■

This proposition generalizes the result of polymorphic equilibria of Bisin and Verdier (2001) to complex networks and, at the same time, completely endogenizes the recovery rate of epidemic models. The intuition is the following. Close to full homogeneity to type $i \in \{A, B\}$, given that cultural transmission is costly, each individual in the population free rides on oblique cultural transmission and spends little vertical effort of socialization. Indeed, close to the homomorphic steady state $\tilde{p}^A = 0$ (resp. $\tilde{p}^A = 1$) each individual of type A (resp. B) is almost sure that oblique transmission through society will satisfy his paternalistic motivation to see his offspring sharing his own trait. Therefore he does not undertake any further vertical socialization rate other than the effort chosen in absence of paternalistic motivations. Conversely, the very few individuals of the other cultural trait have strong incentives to transmit vertically their own type, as they know that this is probably the only way for them to see their offspring get their trait in a society mainly composed of the other cultural type. This feature implies that across the network, the few individuals of the minority trait have a vertical transmission bias compared to the majority trait. This force is enough to diffuse in the network and to destabilize the long run cultural homomorphic steady states. Continuity of the function $\Phi(k, \tilde{p}^A)$ then ensures that there also exists a locally stable polymorphic steady state $\tilde{p}^A \in (0, 1)$ on the network.

Interestingly, the process of endogenous socialization is very much like inducing an endogenous homophily effect on the effective network of role models for cultural transmission. Although the network structure here is assumed to be exogenous for production and alternative activities, paternalistic motivations for cultural transmission induce parents to bias the cultural role models children have access to on the network, putting endogenously more weight on the family role model. Young individuals tend to be more influenced by models similar to the one of their previous generation. This force tends to maintain in the long run some cultural heterogeneity in the population.

Note that in proposition 2, the existence of a polymorphic cultural steady state is ensured under *any* complex network structure when assumption 2 is satisfied. An important element of assumption 2 is the fact that there is a lower bound condition for paternalistic motivations. Indeed such motivation for cultural transmission has to be strong enough to ensure that the cultural substitutability effect always overcomes *any* social structure bias effect that can be induced by any network structure. Obviously for network structures satisfying the balancing conditions (10) and (11), a standard continuity argument ensures that long run cultural heterogeneity is also preserved for low values of paternalistic motivations for cultural transmission.

5.1 The Effects of Socialization Motivation on long run cultural equilibria

The last section showed that independently from of social structure in which cultural transmission occurs, strong enough paternalistic motivations push in the direction of the existence of a polymorphic long run cultural outcome. In this section, we discuss further how the pattern of cultural heterogeneity depends on the interaction between the network structure (different degree distributions) and the intensity of parental motives to transmit their traits. While a general analysis at any possible equilibrium level of motivation and any network structure cannot be performed, we focus on our two-valued example for vertical socialization rates as defined in section 4.2 by equations(21)-(23).

To appreciate the effect of the network structure and of the cultural substitution effects, we impose a symmetry assumption on paternalistic motivations, so that the results are not driven by asymmetries favouring one cultural group or the other. Specifically we consider the case where $\Delta V = V^{ii} - V^{ij} = V^{jj} - V^{ji}$.

By the proof of proposition 2 we have that

$$\tilde{p}^A = U_1 \frac{G_1^B}{G_1^B + G_1^A} + (1 - U_1) \frac{G_2^B}{G_2^B + G_2^A} \equiv \Phi(\tilde{p}^A) \quad (32)$$

where $G_i^B = \tilde{p}^A(1 - \tau_i^B - \frac{\Delta V}{c}\tilde{p}^A)$ and $G_i^A = (1 - \tilde{p}^A)(1 - \tau_i^A - \frac{\Delta V}{c}(1 - \tilde{p}^A))$. This can be written as: .

$$\begin{aligned} \tilde{p}^A = & U_1 \frac{\tilde{p}^A(\underline{\phi}^B - \Delta V \tilde{p}^A)}{\tilde{p}^A(\underline{\phi}^B - \Delta V \tilde{p}^A) + (1 - \tilde{p}^A)(\underline{\phi}^A - \Delta V(1 - \tilde{p}^A))} \\ & + (1 - U_1) \frac{\tilde{p}^A(\bar{\phi}^B - \Delta V \tilde{p}^A)}{\tilde{p}^A(\bar{\phi}^B - \Delta V \tilde{p}^A) + (1 - \tilde{p}^A)(\bar{\phi}^A - \Delta V(1 - \tilde{p}^A))} \end{aligned} \quad (33)$$

Consider then the effect of rising the parental incentives to socialize ΔV , starting from a situation of long run cultural heterogeneity with no paternalism in cultural transmission¹². Then the following proposition holds:

Proposition 3 *The effect of a marginal increase of paternalism in cultural transmission is to favor the minority group whenever that group is sufficiently negatively biased by the social structure. Formally, there exists a vector $(\underline{U}_1, \bar{U}_1) \in [0, 1]^2$ satisfying $\underline{U}_1 < \bar{U}_1$ and the balance conditions (23) and such that :*

$$\left(\frac{\partial \tilde{p}^A}{\partial \Delta V} \right)_{\Delta V=0} < 0 \text{ when } U_1 < \underline{U}_1 \quad (34)$$

$$\left(\frac{\partial \tilde{p}^A}{\partial \Delta V} \right)_{\Delta V=0} > 0 \text{ when } U_1 > \bar{U}_1 \quad (35)$$

¹²In this example, we undertake a comparative static exercise for degree network distributions satisfying (23) (ie. such that there exists a long run polymorphic cultural equilibrium even without paternalism, ie. when $\Delta V = 0$). Hence we do not need to assume that ΔV has to satisfy the lower bound restriction of assumption H2.

Proof. See the appendix. ■

This result illustrates that, to some extent, paternalistic motivations tend to act as substitutes to the "social structure bias" associated to the process of cultural transmission. When the social network structure induces a pattern of opportunity costs of parenting that is biased strongly enough against one cultural trait, then paternalistic motivations for socialization creates a countervailing force that tends to mitigate this social structure bias in cultural transmission.

6 Concluding Remarks

In this paper, we presented a process of cultural transmission in a context where agents are connected on a given complex network and motivated by paternalistic concerns to transmit their cultural traits. This is an attempt to bridge two literatures using, respectively, an epidemic and a cultural transmission approach. We first identify a "social structure bias" in cultural transmission that the topology of society induces for the existence and stability of long run cultural patterns in that society. This bias is closely related on how the connectivity of the social network interacts with the opportunity costs of family socialization. and the resulting vertical transmission rates in society. We then pointed out that when vertical cultural transmission rates and social connections are structurally distributed across the network in a sufficiently balanced way, then long run cultural diversity can be observed even without family socialization motivated by paternalistic concerns. In turn, strong enough paternalistic motivations for cultural transmission are able to compensate any social structure unbalancedness that would otherwise lead to homogenous cultural societies.

These results provide some first steps to an interesting perspective on how social structures and motivations can deeply interact for the evolution of preferences in large scale complex societies. Several extensions can be thought for future research. First of all, our approach remains analytic, using standard mean field approximations. An interesting extension would be of course to incorporate these ideas of motivational cultural transmission into large scale and simulated agent-based models of cultural diffusion. These models so far typically assume exogenous rules of cultural transmission and evolution across the nodes of the social network. Introducing in these frameworks explicit purposeful socialization efforts by cultural parents and role models would allow more systematic insights on the respective roles of "agents' motivations" versus "social structure" on the cultural dynamics of large scale societies.

Second, our paper also typically emphasizes one side of the relation between social structure and preferences : from exogenous social structures to long run patterns of preferences. Clearly, one may also think about the other side of the interaction: social connections are often endogenous and emerge out of a process of explicit socio-economic choices. In such a case, one may expect the distribution of preferences and motivations to affect in return the topology of social structures. This could create interesting social

patterns of homophily as observed in real social networks. Analyzing these two-way interactions in a tractable framework is certainly an important direction for future research. We hope that our approach can be a stepping stone in that direction.

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Appendix 1: Proof of Proposition 1

Consider the following functions

$$\begin{aligned}\varphi(k, p) &= \frac{p(1 - \tau_k^B)}{p(1 - \tau_k^B) + (1 - p)(1 - \tau_k^A)} \\ \Phi(p) &\equiv \sum_{k=0}^{\infty} \xi(k) \varphi(k, p) \\ \Psi(p) &= \Phi(p) - p\end{aligned}$$

then simple computations provide that

$$\frac{\partial^3 \varphi(k, p)}{\partial p^3} = 6 \frac{(1 - \tau_k^B)(1 - \tau_k^A) (\tau_k^B - \tau_k^A)^2}{[p(1 - \tau_k^B) + (1 - p)(1 - \tau_k^A)]^2} > 0$$

Hence

$$\Phi'''(p) = \Psi'''(p) = \sum_{k=0}^{\infty} \xi(k) \frac{\partial^3 \varphi(k, p)}{\partial p^3} > 0$$

and $\Phi''(p) = \Psi''(p)$ are strictly increasing in p for all $p \in [0, 1]$.

Existence results:

i) A steady state of the system (6) is clearly a fixed point of $\Phi(p)$ or a point p such that $\Psi(p) = 0$. Clearly $\Psi(0) = \Psi(1) = 0$ Hence the system (6) always admits the two homomorphic steady states: $p_k^A = \tilde{p}^A = 0$ for all k , and $p_k^A = \tilde{p}^A = 1$ for all k .

ii) When both (10) and (11) are satisfied, we have that

$$\begin{aligned}\Psi'(0) &= \Phi'(0) - 1 = \sum_{k=0}^{\infty} \xi_k \frac{(1 - \tau_k^B)}{(1 - \tau_k^A)} - 1 > 0 \\ \text{and } \Psi'(1) &= \Phi'(1) - 1 = \sum_{k=0}^{\infty} \xi_k \frac{(1 - \tau_k^A)}{(1 - \tau_k^B)} - 1 > 0\end{aligned}$$

Given that $\Psi(0) = \Psi(1) = 0$, it follows that there is $p_1 > 0$ but close enough to 0 such that $\Psi(p_1) \simeq \Psi(0) + p_1\Psi'(0) > 0$. As well there is $p_2 < 1$ but close enough to 1 such that $\Psi(p_2) \simeq \Psi(1) - (1 - p_2)\Psi'(0) < 0$. Hence by continuous differentiability of the function $\Psi(p)$, there is an interior point $\tilde{p}^A \in]0, 1[$ such that $\Psi(\tilde{p}^A) = 0$ and $\Psi'(\tilde{p}^A) < 0$.

Note as well that that when both (10) and (11) are satisfied, there necessarily exists a unique $p_h \in]0, 1[$ such that $\Psi''(p_h) = 0$. Indeed, given that $\Psi''(p)$ is strictly increasing in p , p_h should be unique if it exists. Suppose then that p_h does not exist. Then it follows that $\Psi''(p)$ does not change sign on $[0, 1]$. If $\Psi''(p) > 0$, then $\Psi'(p)$ is strictly increasing in p . As $\Psi'(0) > 0$, then for all p $\Psi'(p) > 0$ which contradict the fact that there is a \tilde{p}^A such that $\Psi'(\tilde{p}^A) < 0$. If $\Psi''(p) < 0$, then $\Psi'(p)$ is strictly decreasing in p . As $\Psi'(\tilde{p}^A) < 0$, this implies that $\Psi'(p) < 0$ for all $p \in [\tilde{p}^A, 1]$. This contradicts the fact that $\Psi'(1) > 0$. Hence it follows that under (10) and (11) satisfied, there exists a unique $p_h \in]0, 1[$ such that $\Psi''(p_h) = 0$. From this it follows that $\Psi'(p)$ is decreasing on $[0, p_h]$ and then increasing on $[p_h, 1]$.

Suppose now that there is more than one interior point $\tilde{p}^A \in]0, 1[$ such that $\Psi(\tilde{p}^A) = 0$. Denote by \tilde{p}_0^A the smallest one and by \tilde{p}_1^A the next one. Then given $\Psi'(0) > 0$, that one should necessarily have $\Psi'(\tilde{p}_0^A) < 0$ and $\Psi'(\tilde{p}_1^A) > 0$. But it follows necessarily that $p_h < \tilde{p}_1^A$. Hence for all $p \in [\tilde{p}_1^A, 1]$, $\Psi'(p) > \Psi'(\tilde{p}_1^A) > 0$. Hence $\Psi(p)$ is increasing for all $p \in [\tilde{p}_1^A, 1]$ and therefore $\Psi(p) > \Psi(\tilde{p}_1^A) = 0$. In particular this implies that $\Psi(1) > 0$, a contradiction. Hence finally there is a unique interior point $\tilde{p}^A \in]0, 1[$ such that $\Psi(\tilde{p}^A) = 0$.

The preceding discussion provides the existence of a unique polymorphic steady state $\tilde{p}^A \in (0, 1)$.

iii) When (10) is satisfied and (11) is not, then $\Psi'(0) > 0 > \Psi'(1)$. Then obviously we cannot have $\Psi''(p) > 0$. Two possible cases remain. First suppose that for all $p \in [0, 1]$ $\Psi''(p) < 0$, then $\Psi'(p)$ is strictly decreasing in p . If there exists an interior solution to $\Psi(\tilde{p}^A) = 0$, it should be unique. But then this means that for $p \in]\tilde{p}^A, 1]$ $\Psi(p) < \Psi(\tilde{p}^A) = 0$. A contradiction with $\Psi(1) = 0$.

The second case is when there exists a unique $p_h \in]0, 1[$ such that $\Psi''(p_h) = 0$. In such a case $\Psi'(p)$ is decreasing on $[0, p_h]$ and then increasing on $[p_h, 1]$. This implies that $\Psi'(p_h) < 0$ (otherwise one would have $\Psi'(1) > \Psi'(p_h) \geq 0$, a contradiction). But then $\Psi'(p_h) < 0$ and $\Psi'(0) > 0 > \Psi'(1)$ implies that there exists a unique $p_h'' \in]0, p_h[$ such that $\Psi'(p_h'') = 0$. On $[p_h'', 1]$, the function $\Psi'(p)$ has a negative sign, implying that $\Psi(p) > 0$ for $p \in [p_h'', 1]$. Similarly for $p \in]0, p_h''[$ one has $\Psi'(p) > 0$. Hence for $p \in]0, p_h''[$, $\Psi(p) > \Psi(0) = 0$. It follows in the end that for all $p \in]0, 1[$ $\Psi(p) > 0$. Hence there cannot exist an interior solution to $\Psi(p) = 0$ and there is no polymorphic steady state.

iv) When (11) is satisfied and (10) is not, a symmetric argument as in iii) provides the result that again there is no polymorphic steady state and only the homomorphic steady state $\tilde{p}^A = 0$ is locally stable.

v) When both (10) and (11) are not satisfied, a symmetric argument as ii) shows that there exists a unique polymorphic steady state $\tilde{p}^A \in (0, 1)$.

Local stability of steady states: The full cultural dynamics are described by the following set of equations:

$$\dot{p}_{k,t}^A = \lambda [p_{k,t}^A(\tau_k^A - 1)(1 - \tilde{p}_t^A) + (1 - p_{k,t}^A)(1 - \tau_k^B)\tilde{p}_t^A]$$

with

$$\tilde{p}_t^A = \sum_{k=0}^{\infty} \xi(k) p_{k,t}^A$$

Thus

$$\begin{aligned} \dot{\tilde{p}}_t^A &= \lambda \sum_{k=0}^{\infty} \xi(k) [(1 - p_{k,t}^A)(1 - \tau_k^B)\tilde{p}_t^A - p_{k,t}^A(1 - \tau_k^A)(1 - \tilde{p}_t^A)] \\ &= \lambda \sum_{k=0}^{\infty} \xi(k) (\varphi(k, \tilde{p}_t^A) - \tilde{p}_{k,t}^A) \cdot [(1 - \tau_k^B)\tilde{p}_t^A + (1 - \tau_k^A)(1 - \tilde{p}_t^A)] \end{aligned}$$

Given assumption H1, we have the following derivations:

- Close to the steady state $[p_k^A]_k = 0$,

$$\varphi(k, \tilde{p}_t^A) \simeq \varphi'(k, 0)\tilde{p}_t^A = \frac{(1 - \tau_k^B)}{(1 - \tau_k^A)}\tilde{p}_t^A$$

thus

$$\dot{\tilde{p}}_t^A \simeq \lambda \sum_{k=0}^{\infty} \xi(k) (\varphi'(k, 0)\tilde{p}_t^A - \tilde{p}_{k,t}^A) (1 - \tau_k^A)$$

and

$$\lambda \tilde{p}_t^A (1 - \bar{\tau}_{\max}^A) \cdot \sum_{k=0}^{\infty} \xi(k) (\varphi'(k, 0) - 1) < \dot{\tilde{p}}_t^A < \lambda \tilde{p}_t^A (1 - \bar{\tau}_{\min}^A) \cdot \sum_{k=0}^{\infty} \xi(k) (\varphi'(k, 0) - 1)$$

or

$$\lambda \tilde{p}_t^A (1 - \bar{\tau}_{\max}^A) \cdot \Psi'(0) < \dot{\tilde{p}}_t^A < \lambda \tilde{p}_t^A (1 - \bar{\tau}_{\min}^A) \cdot \Psi'(0) \quad (36)$$

- Similarly close to the steady state $[p_k^A]_k = 1$, we have:

$$\varphi(k, \tilde{p}_t^A) \simeq 1 - \varphi'(k, 1)(1 - \tilde{p}_t^A) = 1 - \frac{(1 - \tau_k^A)}{(1 - \tau_k^B)}(1 - \tilde{p}_t^A)$$

thus

$$\dot{\tilde{p}}_t^A \simeq \lambda \sum_{k=0}^{\infty} \xi(k) (-\varphi'(k, 1)(1 - \tilde{p}_t^A) + (1 - \tilde{p}_{k,t}^A)) (1 - \tau_k^B)$$

and after substitution

$$-\lambda(1 - \tilde{p}_t^A)(1 - \bar{\tau}_{\max}^B) \cdot \Psi'(1) < \dot{\tilde{p}}_t^A < -\lambda(1 - \tilde{p}_t^A)(1 - \bar{\tau}_{\min}^B) \cdot \Psi'(1) \quad (37)$$

- Finally close to an interior steady state $[p_k^{A*}]_k$ one has

$$\varphi(k, \tilde{p}_t^A) \simeq \tilde{p}^{A*} + \varphi'(k, \tilde{p}^{A*})(\tilde{p}_t^A - \tilde{p}^{A*})$$

$$\varphi'(k, \tilde{p}^{A*}) = \frac{\tilde{p}^{A*}(1 - \tau_k^B)(1 - \tau_k^A)}{[\tilde{p}^{A*}(\tau_k^A - \tau_k^B) + (1 - \tau_k^A)]^2}$$

and

$$\dot{\tilde{p}}_t^A \simeq \lambda \sum_{k=0}^{\infty} \xi(k) [\varphi'(k, \tilde{p}^{A*})(\tilde{p}_t^A - \tilde{p}^{A*}) + \tilde{p}^{A*} - \tilde{p}_{k,t}^A] \cdot [(1 - \tau_k^B)\tilde{p}^{A*} + (1 - \tau_k^A)(1 - \tilde{p}^{A*})]$$

Thus posing

$$\begin{aligned} 1 - \bar{\tau}_{\max} &= \min \{ (1 - \bar{\tau}_{\max}^B), (1 - \bar{\tau}_{\max}^A) \} \\ 1 - \bar{\tau}_{\min} &= \max \{ (1 - \bar{\tau}_{\min}^B), (1 - \bar{\tau}_{\min}^A) \} \end{aligned}$$

one gets after substitution

$$-\lambda(\tilde{p}^{A*} - \tilde{p}_t^A)(1 - \bar{\tau}_{\max}) \cdot \Psi'(\tilde{p}^{A*}) < \dot{\tilde{p}}_t^A < -\lambda(\tilde{p}^{A*} - \tilde{p}_t^A)(1 - \bar{\tau}_{\min}) \cdot \Psi'(\tilde{p}^{A*}) \quad (38)$$

The local stability of the corner steady states 0 and 1 and the interior steady state \tilde{p}^{A*} (whenever it exists) follows directly from the inequalities (36), (37) and (38) and the fact that the balanced conditions BCa) and BCb) rewrite as $\Psi'(0) > 0$ and $\Psi'(1) > 0$.

- For case ii) the two homomorphic steady states are obviously locally unstable as $\Psi'(0) > 0$ and $\Psi'(1) > 0$. The polymorphic steady state is locally stable as $\Psi'(\tilde{p}^A) < 0$.

- for case iii) the homomorphic steady state $\tilde{p}^A = 1$ is locally stable (as $\Psi'(1) < 0$) while the homomorphic steady state $\tilde{p}^A = 0$ is instable (as $\Psi'(0) > 0$).

- for case iv) the homomorphic steady state $\tilde{p}^A = 0$ is locally stable (as $\Psi'(0) < 0$) while the homomorphic steady state $\tilde{p}^A = 1$ is instable (as $\Psi'(1) > 0$).

- For case v) The two homomorphic steady states 0 and 1 are locally stable (as $\Psi'(0) < 0$ and $\Psi'(1) < 0$) and the polymorphic steady state is locally unstable (as $\Psi'(\tilde{p}^A) > 0$).

QED.

Appendix 2: Free Scale Examples

Consider a Free Scale degree distribution. Call $r(k) = \beta k^{\Delta\rho}$ with $\beta \equiv c^B/c^A$ and $\Delta\rho \equiv \rho^B - \rho^A$. We can write the continuous approximation of condition (10) as follows:

$$\int_1^\infty kq(k)\rho(k)dk > \bar{k} \quad (39)$$

Using the scale free distributions, since $\gamma > 2$ we have

$$\bar{k} = \int_1^\infty \sigma k^{1-\gamma} dk = \frac{\sigma}{2-\gamma} \Big|_1^\infty = \frac{\sigma}{\gamma-2} \quad (40)$$

We can then write condition (10) as

$$\int_1^\infty \sigma \beta k^{1+\Delta\rho-\gamma} dk > \frac{\sigma}{\gamma-2} \quad (41)$$

Now, if $\gamma > 2 + \eta$

$$\int_1^\infty \sigma \beta k^{1+\Delta\rho-\gamma} dk = \frac{\sigma \beta}{2 + \Delta\rho - \gamma} k^{2+\Delta\rho-\gamma} \Big|_1^\infty = \frac{\sigma \beta}{2 + \Delta\rho - \gamma} > \frac{\sigma}{\gamma-2}$$

so that condition (10) is now rewritten as

$$\Delta\rho > (\gamma - 2)(1 - \beta) \quad (42)$$

Notice that, given $\gamma \in (2, 3)$, the condition $\gamma > 2 + \Delta\rho$ would imply that $\Delta\rho < 1$. Consider now (11). Its continuous approximation can be rewritten as

$$\int_1^\infty kq(k) \frac{1}{\rho(k)} dk > \bar{k} \quad (43)$$

We can then write condition (11) as

$$\int_1^\infty \frac{\sigma}{\beta} k^{1-\eta-\gamma} dk > \frac{\sigma}{\gamma-2} \quad (44)$$

if $\gamma > 2 - \Delta\rho$

$$\int_1^\infty \frac{\sigma}{\beta} k^{1-\Delta\rho-\gamma} dk > \frac{\sigma}{\gamma-2} = \frac{\sigma}{\beta} \frac{1}{2 - \Delta\rho - \gamma} k^{2-\Delta\rho-\gamma} \Big|_1^\infty = \frac{\sigma}{\beta} \frac{1}{2 + \Delta\rho - \gamma} > \frac{\sigma}{\gamma-2}$$

so that condition (11) is now rewritten as

$$\Delta\rho < \frac{(\gamma - 2)(1 - \beta)}{\beta} \quad (45)$$

Thus, if if $\gamma > 2 + \Delta\rho$ we can write balancing conditions as

$$(\gamma - 2)(1 - \beta) < \Delta\rho < \frac{(\gamma - 2)(1 - \beta)}{\beta} \text{ with } \beta \in (0, 1). \quad (46)$$

By substituting back $\beta \equiv c^B/c^A$ and $\Delta\rho \equiv \rho^B - \rho^A$ results immediately follow.

Appendix 3: Proof of proposition 2.

- The cultural dynamics are again determined by (6).

The characterization of the steady states is obtained from the following equations:

$$\tilde{p}^A = \sum_{k=0}^{\infty} \xi(k) \varphi(k, \tilde{p}^A) \equiv \Phi(\tilde{p}^A)$$

with $\varphi(k, \tilde{p}^A)$ given by

$$\varphi(k, \tilde{p}^A) = \frac{G(\Delta V^B c_k^B, \tilde{p}^A)}{G(\Delta V^B c_k^B, \tilde{p}^A) + G(\Delta V^A c_k^A, 1 - \tilde{p}^A)} \quad (47)$$

and

$$\begin{aligned} G(\Delta V^B c_k^B, \tilde{p}^A) &= \tilde{p}^A (1 - \bar{\tau}_k^B - c_k^B \Delta V^B \tilde{p}^A) \\ G(\Delta V^A c_k^A, 1 - \tilde{p}^A) &= (1 - \tilde{p}^A) [1 - \bar{\tau}_k^A - c_k^A \Delta V^A (1 - \tilde{p}^A)] \end{aligned}$$

The steady state points $\tilde{p}^A \in [0, 1]$ are such that

$$\Psi(\tilde{p}^A) = \Phi(\tilde{p}^A) - \tilde{p}^A = 0$$

Obviously for $\gamma < 1$, $\Psi(0) = 0$ and $\Psi(1) = 0$ and the dynamic system (??) always admits two homomorphic steady states: $p_k^A = \tilde{p}^A = 0$ for all k , and $p_k^A = \tilde{p}^A = 1$ for all k . It is just a matter of algebra to see also that

$$\frac{\partial \varphi(k, 0)}{\partial p} = \frac{1 - \bar{\tau}_k^B}{1 - \bar{\tau}_k^A - c_k^A \Delta V^A}$$

and

$$\frac{\partial \varphi(k, 1)}{\partial p} = \frac{1 - \bar{\tau}_k^A}{1 - \bar{\tau}_k^B - c_k^B \Delta V^B}$$

Hence, to have a stable polymorphic equilibrium we need

$$\Phi'(0) = \sum_{k=0}^{\infty} \xi(k) \frac{\partial \varphi(k, 0)}{\partial p} = \sum_{k=0}^{\infty} \xi(k) \frac{1 - \bar{\tau}_k^B}{1 - \bar{\tau}_k^A - c_k^A \Delta V^A} > 1$$

and

$$\Phi'(1) = \sum_{k=0}^{\infty} \xi(k) \frac{\partial \varphi(k, 1)}{\partial p} > \sum_{k=0}^{\infty} \xi(k) \frac{1 - \bar{\tau}_k^A}{1 - \bar{\tau}_k^B - c_k^B \Delta V^B} > 1$$

These inequalities are both verified if

$$\begin{aligned} \Delta V^A &> \frac{\bar{\tau}_k^B - \bar{\tau}_k^A}{c_k^A}, \forall k \\ \Delta V^B &> \frac{\bar{\tau}_k^A - \bar{\tau}_k^B}{c_k^B}, \forall k \end{aligned}$$

If these are merged together with the upper bound of H2 we get

$$\begin{aligned} \Delta V^A &\in \left[\frac{\bar{\tau}_k^B - \bar{\tau}_k^A}{c_k^A}, \frac{1 - \bar{\tau}_k^A}{c_k^A} \right], \forall k \\ \Delta V^B &\in \left[\frac{\bar{\tau}_k^A - \bar{\tau}_k^B}{c_k^B}, \frac{1 - \bar{\tau}_k^B}{c_k^B} \right], \forall k \end{aligned}$$

so that if these conditions are satisfied then $\Psi'(0) > 0$ and $\Psi'(1) > 0$. Given that $\Psi(0) = \Psi(1) = 0$, it again follows that there is $p_1 > 0$ but close enough to 0 such that $\Psi(p_1) \simeq \Psi(0) + p_1 \Psi'(0) > 0$. As well there is $p_2 < 1$ but close enough to 1 such that $\Psi(p_2) \simeq \Psi(1) - (1 - p_2) \Psi'(0) < 0$. Hence by continuous differentiability of the function $\Psi(p)$, there is an interior point $\tilde{p}^A \in]0, 1[$ such that $\Psi(\tilde{p}^A) = 0$ and $\Psi'(\tilde{p}^A) < 0$. By a reasoning similar to appendix 1, it is immediate to see that the two homomorphic steady states are locally unstable and there is at least one locally stable polymorphic steady state $\tilde{p}^A \in (0, 1)$. **QED.**

Appendix 4: Proof of proposition 3

Consider the polymorphic steady state equilibrium $\tilde{p}^A(0)$ with no paternalism incultural transmission (ie. when $\Delta V = 0$). This is given by

$$\tilde{p}^A = U_1 \frac{\tilde{p}^A(\underline{\phi}^B)}{\tilde{p}^A(\underline{\phi}^B) + (1 - \tilde{p}^A)(\underline{\phi}^A)} + (1 - U_1) \frac{\tilde{p}^A(\bar{\phi}^B)}{\tilde{p}^A(\bar{\phi}^B) + (1 - \tilde{p}^A)(\bar{\phi}^A)}$$

or

$$\tilde{p}^A(0) = \frac{\underline{\phi}^A}{\underline{\phi}^A - \underline{\phi}^B} - U_1 \frac{\bar{\phi}^B \underline{\phi}^A - \underline{\phi}^B \bar{\phi}^A}{(\bar{\phi}^B - \bar{\phi}^A)(\underline{\phi}^A - \underline{\phi}^B)} \quad (48)$$

Now call $H_1 = \frac{G_1^B}{G_1^B + G_1^A}$ and $H_2 = \frac{G_2^B}{G_2^B + G_2^A}$. Then $\frac{\partial \Phi(\tilde{p}^A)}{\partial \Delta V} = U_1 \frac{\partial H_1}{\partial \Delta V} + (1 - U_1) \frac{\partial H_2}{\partial \Delta V}$. Now it is just a matter of algebra to see that

$$\frac{\partial H_1}{\partial \Delta V} > 0 \text{ iff } \frac{\underline{\phi}^B}{\underline{\phi}^A} > \frac{\tilde{p}^A}{1 - \tilde{p}^A} \quad (49)$$

$$\frac{\partial H_2}{\partial \Delta V} > 0 \text{ iff } \frac{\overline{\phi}^B}{\overline{\phi}^A} > \frac{\tilde{p}^A}{1 - \tilde{p}^A} \quad (50)$$

or

$$\frac{\partial H_1}{\partial \Delta V} > 0 \text{ iff } \tilde{p}^A < \frac{\underline{\phi}^B}{\underline{\phi}^A + \underline{\phi}^B} = \underline{p} \quad (51)$$

$$\frac{\partial H_2}{\partial \Delta V} > 0 \text{ iff } \tilde{p}^A < \frac{\overline{\phi}^B}{\overline{\phi}^A + \overline{\phi}^B} = \overline{p} \quad (52)$$

Hence noting that given that $\underline{\phi}^B < \underline{\phi}^A < \overline{\phi}^A < \overline{\phi}^B$ we have

$$\underline{p} = \frac{\underline{\phi}^B}{\underline{\phi}^A + \underline{\phi}^B} < \frac{\overline{\phi}^B}{\overline{\phi}^A + \overline{\phi}^B} = \overline{p}$$

Thus $\frac{\partial \Phi(\tilde{p}^A)}{\partial \Delta V} > 0$ when $\tilde{p}^A < \underline{p}$ while $\frac{\partial \Phi(\tilde{p}^A)}{\partial \Delta V} < 0$ when $\tilde{p}^A > \overline{p}$. Hence $\frac{\partial \tilde{p}^A}{\partial \Delta V} > 0$ when $\tilde{p}^A < \underline{p}$ while $\frac{\partial \tilde{p}^A}{\partial \Delta V} < 0$ when $\tilde{p}^A > \overline{p}$. Looking at the specific case with no paternalism in cultural transmission, and given (48), it is then easy to see that there exists \underline{U}_1 and $\overline{U}_1 \in [0; 1]$ satisfying $\underline{U}_1 < \overline{U}_1$ and conditions (23) and such that :

$$\begin{aligned} \left(\frac{\partial \tilde{p}^A}{\partial \Delta V} \right)_{\Delta V=0} &< 0 \text{ when } U_1 < \underline{U}_1 \\ \left(\frac{\partial \tilde{p}^A}{\partial \Delta V} \right)_{\Delta V=0} &> 0 \text{ when } U_1 > \overline{U}_1 \end{aligned}$$

QED. ■