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SPATIAL COMPETITION IN QUALITY

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ABSTRACT

Spatial Competition in Quality*

We develop a model of vertical innovation in which firms incur a market entry cost and position themselves in the quality space. Once established, firms compete monopolistically, selling to consumers with heterogeneous tastes for quality. We establish existence and uniqueness of the pricing game in such vertically differentiated markets with a potentially large number of active firms. Turning to firms' entry decisions, exogenously growing productivities induce firms to enter the market sequentially at the top end of the quality spectrum. We spell out the conditions under which the entry problem is replicated over time so that each new entrant improves incumbent qualities in fixed proportions. Sequential market entry overcomes the asymmetry of the location problem, which unavoidably arises in the quality spectrum because of its top and bottom ends. Our main technical contribution lies in handling this asymmetry, a feature absent in Salop (1979) and other circular representations of Hotelling (1929) and Lancaster (1966).

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1 Introduction

Hotelling's classic "location" paradigm is widely used to reflect generic product characteristics. The well-studied formalism of the spacing model, however, does not apply to competition in quality. By its very definition, quality requires that individuals agree on the ranking of varieties and, in particular, that individually preferred *ideal varieties* coincide.[1]

Aware of the spacing model's fundamental misfit in addressing competition in quality, Mussa and Rosen (1978), Gabszewicz and Thisse (1979 and 1980), and Shaked and Sutton (1982 and 1983) pioneered research on vertically differentiated markets in which natural oligopolies prevail. Shaked and Sutton (1982 and 1983) call the characteristic leading to such market outcomes the *finiteness property*. Its key element is that marginal production costs increase only moderately in quality, which implies that the highest quality firm supplies the entire market.[2] When this condition is violated and consumers differ in their individual ranking of variety-price pairs, Shaked and Sutton (1983) observe that competition in quality is "reminiscent of the 'location' paradigm" by Hotelling.

In this paper, we analyze endogenous firm entry into vertically differentiated markets when the finiteness property is violated. Specifically, we model firms' costly market entry with endogenous quality choices. Exogenous productivity growth makes ever-higher qualities affordable so that firms enter the market sequentially at the top end of the quality spectrum. We specify conditions under which the entry problem of each new entrant is replicated and which imply that new qualities improve upon existing qualities by a fixed proportion. While our modeling choice with the sequential market entry of firms introduces new technical difficulties, it enables us to overcome the asymmetry of the location problem that unavoidably arises in the quality space: the quality spectrum has a top and a bottom end. Our main technical contribution lies in handling the border conditions that arise (which are absent in the circular world of the related Lancaster (1966) model).

As the building block of our modeling strategy we choose a framework based on Mussa

[1] Bils and Klenow (2001) and Broda and Romalis (2009) document that in most good categories, there are pronounced quality differences in the type of goods that households with different incomes consume, suggesting that quality differentiation is present in most industries. Additionally, the international trade literature has documented that most industries are characterized by a high degree of vertical specialization. Richer countries sell goods of higher quality (approximated with unit values in Schott (2004) and Hummels and Klenow (2005) and directly estimated by Hallak and Schott (2011) and Khandewal (2010)). Higher-quality goods are mostly imported by high income countries (Hallak (2006)).

[2] See also Shaked and Sutton (1984), Anderson et al. (1992), and Sutton (2007a) and (2007b).

and Rosen (1978), Gabszewicz and Thisse (1979), and Shaked and Sutton (1982). These models feature firms, each of which holds a blueprint of a unique quality of an otherwise homogeneous good. All consumers value quality but differ in their "valuation" and thus in their willingness to pay for quality.

Our analysis proceeds in two steps. In the first, we focus on monopolistic competition in the quality space for given entry of any number of firms and arbitrary spacings, assuming that the *finiteness property* is violated. Our basic assumption for production technology is that the marginal production cost is convex in quality, which guarantees the survival of many firms.[3] We document that the analogue of the "transportation cost" in Lancaster (1966) arises from the convexity of the cost schedule.[4] We characterize the pricing strategies and profits of firms that compete à la Bertrand, taking as given each firm's quality, i.e., its location in the quality space. Specifically, we prove the existence of a unique pure-strategy pricing equilibrium if valuations of quality are uniformly distributed.[5] We note that in contrast to horizontally differentiated markets, "undercutting" does not arise in vertically differentiated markets.[6]

The second part of our analysis endogenizes firms' location choice in the quality space. Firms can incur a fixed cost to improve upon existing qualities, and they are granted a perpetual patent to produce their particular quality. Exogenously growing productivities reduce entry and production costs at equal rates. We prove that, in this setup, there is a dynamic equilibrium in which each new entrant chooses a quality that is a constant

[3] We are not the first to note that a globally convex marginal cost schedule with respect to quality guarantees that a large number of firms can co-exist in equilibrium (see Gabszewicz and Thisse (1980), Shaked and Sutton (1982), and, in particular, the discussion in Anderson et al. (1992)). Close to the current paper is the work by Zweimüller and Brunner (2005), who study innovation incentives in a model that features many firms supplying different qualities.

[4] Consider, for example, three firms producing $q_1 < q_2 < q_3$ under a marginal cost schedule that is convex in quality (i.e., the cost increment per quality difference between the production of q_3 and q_2 is larger than the cost increment between q_1 and q_2). Next, consider the range of consumers whose willingness to pay for additional quality exceeds the first increment but falls short of the second. These consumers receive a surplus by buying q_2 at the marginal production cost instead of buying either q_1 or q_3 at any price exceeding the respective marginal production costs. Because the producers of q_1 and q_3 never sell below their marginal cost, the producer of good q_2 enjoys positive market power. In this way, the convexity of the marginal cost schedule generates market power for individual firms.

[5] Non-existence of equilibria in pure strategies in oligopolistic markets is pervasive. See Vives 2001, p. 126 for a simple example with a duopoly and, more generally, for an overview of firm pricing in oligopolistic markets as the number of firms grows large (Chapters 5 and 6).

[6] D'Aspremont et al. (1979) discuss the assumptions on the "transportation cost" that are necessary to guarantee the existence of an equilibrium in the classical Hotelling framework.

percentage higher than the incumbent technology leader. We analyze the resulting degree of product market competition as a function of market size and entry costs. Specifically, larger markets induce more frequent firm entry and a higher density of quality supply because higher sales and profits allow for the more rapid recovery of setup costs. Surprisingly, an equal percentage increase in the marginal cost of production for all firms is associated with a more densely supplied market. This result holds because markups are proportional to costs. Thus, when production costs rise for all firms, profits actually increase for any given quality spacing. Excess profits cannot exist in equilibrium, and consequently, firms must exhibit denser quality spacing and "tougher" competition.

Our analysis adds to the sizeable literature on quality competition derived from Gabszewicz and Thisse (1979), Mussa and Rosen (1978), and Shaked and Sutton (1982 and 1983) that focuses on vertically differentiated markets in which natural oligopolies prevail (for a survey of this literature, see Anderson et al. (1992)). Such markets are dominated by a limited number of "market leaders." Shaked and Sutton (1984) and Sutton (2007a and 2007b) analyze the case of one-firm environments, whereas Champsaur and Rochet (1989) address that of duopolies. This literature assumes that the marginal cost of production increases only moderately with quality, which enables high-quality firms to price low-quality competitors out of the market.[7]

The main technical contributions of this paper are the solution of the price game and the entry game for the case of monopolistic competition in vertically differentiated markets. The approach of the present paper differs from the existing literature, first in the assumption concerning the underlying production technology, and secondly in the fact that entry is costly. We analyze the case where the marginal cost of production increases sufficiently in quality so that multiple firms can coexist, and explicitly model the firms' quality choice under the assumption of costly market entry, which is standard in models of monopolistic competition.[8]

[7] Lahmandi-Ayed (2000 and 2004) extensively discusses the technological conditions that induce natural oligopolies.

[8] We believe that the many-firm case in vertically differentiated industries is an empirically important one. Multiple empirical studies in the field of international trade document that nearly all manufacturing industries are characterized by many coexisting firms with heterogeneous prices and profits and that this heterogeneity can, to a large extent, be explained by an underlying heterogeneity in product quality. See, in particular, Khandelwal (2010) and Kugler and Verhoogen (2010), but also Baldwin and Harrigan (2011), Johnson (2012), Verhoogen (2008), and Hallak and Schott (2009). Moreover, many industry-level studies document the frequent coexistence of a technological leader and multiple lagging firms. For example,

Our analysis aims at a quasi-stationary, periodic equilibrium in which each firm enters the industry as the technological leader and successively transits through the product cycle as it becomes superseded by further innovators. The advantage of such a dynamic entry game is that we only need to analyze the entry problem of one firm at a time. The quality choice and product cycle of all subsequent entrants is then isomorphic. In particular, we avoid the problems that arise in a simultaneous-entry game, as in Vogel (2008). In fact, the resulting complications would be substantial in our setup, because the clear ranking of the quality line prevents us from using the symmetry properties that arise in Salop (1979) or other circular representations of Hotelling (1929) and Lancaster (1966). In a quality setup any attempt to "close the circle" must fail, as it amounts to identifying the highest-quality good with the lowest-quality good.

Nevertheless, our analysis of the firm's location choice closely relates to Hotelling's classic "location" paradigm. This and the related ideal-variety approach of Lancaster (1966) is the cornerstone of the location choice. These frameworks have been extensively studied and widely used in various fields of the profession.[9] Only recently, however, Vogel (2008) has extended this setup to firms of heterogeneous productivities.[10] The present paper extends some of these insights to the quality space. At the same time, it overcomes the technical difficulties of the entry game that are shared by models of vertical and horizontal differentiation.

The remainder of this paper is structured as follows. In Section 2, we develop a theoretical model of competition in the quality space and examine its static predictions. We next analyze free entry decisions and stationary equilibria in 3 and 4 before concluding in Section 5.

Aizcorbe and Kortum (2005) document how, in the semiconductor industry, the innovation of increasingly powerful chips coincides with the continued production of less advanced chips.

[9] For a survey of this literature see Anderson et al. (1992).

[10] Vogel (2008) also presents an extension of his model allowing for quality differentiation of goods, thereby demonstrating that his results concerning the location of firms in the horizontal product space also hold when firms can additionally choose the quality of their output.

2 A Model of Monopolistic Competition in the Quality Space

In this section, we analyze how firms compete monopolistically in the quality space for given locations. We adopt a setup in the spirit of Mussa and Rosen (1978), Gabszewicz and Thisse (1979), and Shaked and Sutton (1982), wherein consumers value quality at a linear rate. We assume that the marginal cost of production increases sufficiently with quality, thus violating the finiteness property, so that lower-valuation consumers in equilibrium prefer to buy goods other than those of the current technological leader. Within this framework, we can analyze the static determinants of prices and profits for a given quality spacing.

2.1 Preferences

There are two goods: a homogeneous good D and a good Q that is differentiated in quality and supplied at different quality levels $\{q_n\}_{n \in S}$. Individuals derive utility from consumption of the two goods. Their utility is linear in the quantity of the homogeneous good. Each individual consumes either one or zero units of the differentiated good Q. Apart from this binary decision, if the consumer chooses to buy a Q good, she can also choose between different qualities q of this good. Specifically, the individual's utility function is defined as:[11]

$$u_v(q, d) = v \cdot q + d \tag{1}$$

where v is a parameter that determines the desire to consume quality (and the willingness to pay it). In the following, we will therefore call v the valuation of quality, or simply the *valuation*. Notice that no consumption of good Q is equivalent to consumption of quality zero at price zero.

The total number of individuals equals L. These individuals value quality differently, i.e., they have different values of v. We define the resulting cumulative density function of valuations as

$$G(v) : [0, \infty) \rightarrow [0, 1] \tag{2}$$

which is assumed to have bounded support, i.e. $supp(G) = [\underline{v}, \bar{v}]$, where $0 \leq \underline{v} < \bar{v} < \infty$.

[11] We take this preference structure from Mussa and Rosen (1978); this structure is also similar to the formulation of Shaked and Sutton (1982), who assume a multiplicative structure between the homogeneous and the differentiated good.

2.2 Production

Good D is produced competitively with constant returns to scale and labor as the only factor. A normalized wage thus implies that good D is the numéraire. The production technology of the Q-type goods depends on the quality level produced. A firm that enters the Q-market to produce the quality $q \in (0, \infty)$ must acquire a blueprint at the fixed cost of

$$F(q) = \phi q^{\theta} \tag{3}$$

labor units. We thus assume that blueprints for higher qualities are always more expensive.

Having acquired a blueprint for quality q, a firm can produce its quality of good Q at the constant marginal cost of

$$c(q) = \varphi q^{\theta} \tag{4}$$

labor units. The parameters $\phi, \varphi > 0$ govern production costs. We assume that fixed cost of entry and marginal cost are proportional to each other and are both increasing and convex in quality.

$$\theta > 1$$

2.3 Optimal Pricing

Our aim is to characterize an equilibrium in which firms enter the industry at the optimal quality level and subsequently engage in monopolistic pricing. The equilibrium is solved through backward induction: we first determine the prices at given quality levels and subsequently analyze entry decisions.

We begin by characterizing the general pricing solution for an arbitrary distribution of a countable set of qualities. The price of quality q_i is denoted by $p_i(q_i)$. For notational simplicity, we set $p_n = p(q_n)$ and $c_n = c(q_n)$, where q_n is the quality level produced by firm n. We index firms by $n \in \{0, -1, -2, ..., -N\}$ and order firms by their quality level so that firm 0 produces the highest quality level q_0, and all further quality levels satisfy $q_{n-1} < q_n$. We start with the assumption that prices are such that the consumers with a specific valuation are indifferent between buying from firm n and $n+1$, a conjecture we prove below.[12]

[12] Notice that we implicitly assume that the set of firms is countable. By making this assumption, we anticipate that, in the equilibrium of the subsequent entry game, firms must recoup their setup costs with

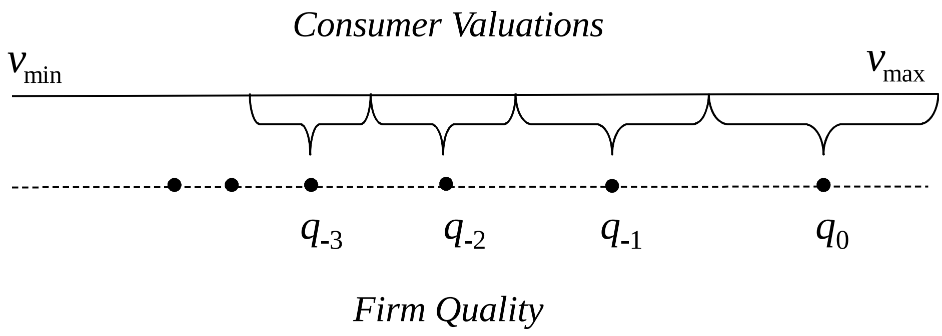

Figure 1: Segmentation of the consumer/valuation space by quality levels.

Firms compete in prices, i.e., each firm sets the price for its quality to maximize its operating profits, while taking other firms' prices as given. Under preferences (1), a consumer with valuation v is indifferent between two goods q_n and q_{n+1} if and only if their prices p_n and p_{n+1} are such that $vq_{n+1} - p_{n+1} = vq_n - p_n$. Thus, given $G(v)$ from (2) and given the prices $\{p_n\}_{n\leq 0}$, the n^{th} firm sells to all consumers with valuations v in the interval $[v_n,\ v_{n+1}]$, where

$$v_n = \begin{cases} \bar{\mathrm{v}} & \text{if } n = 1 \\ \dfrac{p_n - p_{n-1}}{q_n - q_{n-1}} & \text{if } n < 1 \\ \underline{\mathrm{v}} & \text{if } n = -N \end{cases} \tag{5}$$

The firms' market shares are thus $[v_n, v_{n+1}]$, and the market is partitioned as shown in Figure 1: higher-valuation consumers tend to buy from high-quality producers. Each firm (except the top and bottom quality producer) has two direct competitors and sells to a range of consumers who value quality sufficiently highly to buy from the firm in question rather than the direct lower-competitor but do not value quality highly enough to buy from the higher-quality competitor.

Because each consumer with valuation $v \in [v_n, v_{n+1}]$ demands one unit of the variety produced by firm n, firm n sells $G(v_{n+1}) - G(v_n)$ units of its good and solves the maximization problem:

$$\max_{p_n} (p_n - c_n) \left[G(v_{n+1}) - G(v_n)\right] L \qquad s.t.\ (5) \tag{6}$$

monopoly rents. Under Bertrand competition and positive setup cost, this assumption implies that firms must be located at positive distances from each other. Thus, the number of firms is necessarily countable.

The optimality condition of this problem is

$$G(v_{n+1}) - G(v_n) - (p_n - c_n)\left[\frac{G'(v_{n+1})}{q_{n+1} - q_n} + \frac{G'(v_n)}{q_n - q_{n-1}}\right] = 0 \tag{7}$$

where expressions (5) apply. At $\underline{\mathrm{v}}$ and $\bar{\mathrm{v}}$ (the constant limits of the distribution of valuations), the derivatives in (7) are set to zero ($G'(\underline{\mathrm{v}}) \equiv 0$; $G'(\bar{\mathrm{v}}) \equiv 0$).

Under the assumption that G is uniform, the system (7) characterizes a unique equilibrium in pure strategies of the price game for any distribution of qualities q_n. Specifically, we formulate the following lemma.

Proposition 1 *If G is the uniform distribution on $[\underline{v},\bar{v}]$, then the system (7) has a unique equilibrium in pure strategies.*

Proof. *See Appendix* ■

We point out that the results of Proposition 1 are strong when comparing them to corresponding results in setups with a horizontally differentiated market à la Hotelling. Indeed, the corrsponding pricing stage of Hotelling's location game is plagued by non-uniqueness and even non-existence in pure strategies. Specifically, D'Aspremont et al. (1979) disprove Hotelling's original existence claim in the classical location game of vertical differentiation with two firms, showing that "no [pure-strategy] equilibrium price solution will exist when both sellers are not far enough from each other."[13] With this pitfall of the horizontal location model in mind, and given the parallels between the setups of horizontal and vertical differentiation, one might indeed expect that non-uniqueness and non-existence affect the present paper's setup as well. In that sense, the result of Proposition 1 that, independently of firm locations, a pure strategy pricing equilibrium exists for uniform distributions of valuations is surprising.

The reason for the strong result regarding existence and uniqueness of equilibria in vertically differentiated markets lies in the model's specification through heterogeneity of valuations, i.e. of the consumer's willingness to pay for quality. As long as this willingness is continuously distributed, a marginal decrease in a firm's price increases its market share only marginally – and, indeed, at a constant rate. This feature leads to continuity and

[13] This specific result derives from a discontinuity in profits and hence in the best response function when undercutting the adjacent competitor and taking over its entire market. It is worth stressing that in this classical model, this pathological outcome affects the model also under uniform distribution of consumers.

concavity of profits in prices, which is key to derive existence and uniqueness through the relevant theorems.[14] The feature survives, in particular, when a firm charges a price that is low enough so that it marginally drives one of its direct competitors out of the market.[15] By contrast, in models of horizontal differentiation, the *undercutting* of prices from direct competitors induces discrete jumps of market shares and profits and thus complicates the analysis.[16]

A quite restrictive precondition of Proposition 1 is obviously the assumption of a uniform G. It is noteworthy, however, that the literature concerned with Hotelling's location model has struggled to abandon both uniformity and considering general distributions of consumers. Early attempts to relax this assumption with specific distributions include Neven (1986) and Osborne and Pitchik (1987). Anderson et al. (1997) observe that "one assumption, which is clearly unrealistic, has been left virtually untouched by the tools of theorists. This is the condition that the consumers are uniformly distributed [...]" The authors analyze the location game under the assumption that the "density is not 'too asymmetric' and not 'too concave'." Up to today, only special cases of non-uniform distributions have been treated in the analysis of location choice of the traditional setup of spatial competition (for a recent overview see Biscaia and Mota (2012)). To establish results of more generality, we prove existence and uniqueness under less restrictive regularity conditions on G.[17] In our later entry game, however, we will build upon the existence and uniqueness of the pricing game in the case of uniformly distributed tastes.

With the existence (and conditional uniqueness) of the pricing game according to (7),

[14]In accordance to this intuition, we show in the appendix that Proposition 1 generalizes to settings in which G is non-uniform but exhibits some smoothness properties.

[15]This statement can be checked directly with the help of the expression (5). Observe that firm n's direct competitor (say $n+1$) leaves the market when $v_{n+2} = v_{n+1}$ or, with (5) $p_n = [p_{n+1}(q_{n+2} - q_n) - p_{n+2}(q_{n+1} - q_n)] / (q_{n+2} - q_{n+1})$ holds. Up to this point, firm n's market segment $v_{n+1} - v_n$ is continuous in p_n and any further reduction of p_n increases its market share $v_{n+2} - v_n$ continuously. Moreover, the second derivative of profits in (6) is negative in both cases, proving concavity.

[16]D'Aspremont et al. (1979) document that an equilibrium exists in the Hotelling setup when assuming quadratic transport costs. See also Vogel (2008). Reny (1999) defines conditions under which a pure strategy Nash equilibrium exists in discontinuous games. The central condition of *better-reply security* is generally satisfied in Bertrand's classical price-competition. In our setup, the discontinuity of profits, which typically challenges games of Bertrand-competition, does not arise. Therefore, earlier theorems apply, establishing uniqueness in the case of uniform distributions G.

[17]See Claim A in the appendix.

we can write the operating profits as

$$\pi_n = (p_n - c_n)^2 \left[\frac{G'(v_{n+1})}{q_{n+1} - q_n} + \frac{G'(v_n)}{q_n - q_{n-1}} \right] L \tag{8}$$

With the characterization of prices and operating profits, in turn, an important regularity of equilibrium prices and profits emerges.

Lemma 1 *Let $\{q_n\}_{n\leq 0}$, (4), (5) and (7) define a system with equilibrium prices $\{p_n\}_{n\leq 0}$ and profits $\{\pi_n\}_{n\leq 0}$. For given $\chi > 0$, consider the transformed system defined by $q_n' = \chi q_n$, $\varphi_n' = \chi^{1-\theta}\varphi_n$, (4), (5) and (7). This transformed system has the solution $\{p_n'\}_{n\leq 0}$ and $\{\pi_n'\}_{n\leq 0}$ satisfying*

$$p_n' = \chi p_n \qquad \textit{and} \qquad \pi_n' = \chi \pi_n \quad \forall n.$$

Proof. *Prices p_n' solve the transformed optimality conditions (7). The relation for π_n' follows from (5) and (8).* ■

The Lemma states that, if quality levels and marginal productivities change in suitable proportions (and (4) follows suit), then equilibrium prices and profits constitute a constant proportion of the marginal production costs. These regularities lead to a particularly convenient location pattern when firms chose their locations endogenously in a dynamic setting – namely, proportional spacing.

For the remainder of the paper, we restrict the setup to the case of uniformly distributed valuations, which guarantees, according to Proposition 1, a unique price equilibrium. Formally, we write G from (2) as

$$G(v) = \begin{cases} 0 & \text{if } v < \underline{\mathrm{v}} \\ (v - \bar{\mathrm{v}})/(\underline{\mathrm{v}} - \bar{\mathrm{v}}) & \text{if } v \in [\underline{\mathrm{v}}, \bar{\mathrm{v}}] \\ 1 & \text{if } v > \bar{\mathrm{v}} \end{cases} \tag{9}$$

Before turning to the actual entry game, we shall take a moment to look at the equilibrium prices under uniformly distributed valuations and a regular spacing that will later prove important.

2.4 Pricing with Equal Relative Spacing

In this subsection, we solve the equilibrium prices under two restrictions. The first condition is that the minimum consumer valuation is zero ($\bar{\mathrm{v}}$= 0), which we assume to save space.

In the Appendix we compute equilibrium prices and provide comparative statics for the case of uniform distributions on [$\underline{v}$,$\bar{v}$] with $\underline{v}>0$. Our second, seemingly specific condition is that qualities exhibit a particularly regular pattern, which we will call the *equal relative spacing* property.

$$\gamma q_{n-1} = q_n \qquad \forall\ n. \tag{10}$$

If (10) holds, each quality is a constant fraction higher than the immediately preceding one. Our focus on this special case (10) will be justified further below.

In the case of uniformly distributed valuations and equal relative spacing, the equilibrium prices of firms can be solved explicitly and the system (7) becomes

$$p_n = \begin{cases} \frac{1}{2}\left[c_0 + \left(1-\frac{1}{\gamma}\right) q_0 \bar{v} + p_{-1}\right] & \text{if } n = 0 \\ \frac{1}{2}\left[c_n + \frac{1}{\gamma+1} p_{n+1} + \frac{\gamma}{\gamma+1} p_{n-1}\right] & \text{if } n < 0 \end{cases} \tag{11}$$

In this case we can formulate the following proposition:

Lemma 2 *Assume equal relative spacing in quality, i.e., (10) holds. Then prices are*

$$p_n = \lambda^n A + \alpha c_n \qquad \forall \quad -N \leq n \leq 0 \tag{12}$$

where

$$\alpha = \frac{\gamma+1}{2(\gamma+1)-\gamma^{\theta}-\gamma^{1-\theta}} \tag{13}$$

$$\lambda = \gamma + 1 + \sqrt{\gamma^2+\gamma+1} \tag{14}$$

$$A = \frac{\lambda}{2\lambda-1}\left(\left[1-\alpha\left(2-\gamma^{-\theta}\right)\right] c_0 + \frac{\gamma-1}{\gamma} q_0 \bar{v}\right). \tag{15}$$

Proof. Substitution $u_n = p_n - \alpha c_n$ into (11) yields

$$2\left[u_n + \alpha c_n\right] = c_n + \frac{1}{\gamma+1}\left[u_{n+1} + \alpha c_{n+1}\right] + \frac{\gamma}{\gamma+1}\left[u_{n-1} + \alpha c_{n-1}\right]$$

for $n > 1$. With $\alpha = (1+\gamma)/\left[2(1+\gamma)-\gamma^{\theta}-\gamma^{1-\theta}\right]$ this is $2(\gamma+1)u_n = u_{n+1} + \gamma u_{n-1}$. The equation

$$X^2 - 2(\gamma+1)X + \gamma = 0 \tag{16}$$

has two roots, $\lambda = \left[\gamma + 1 + \sqrt{\gamma^2+\gamma+1}\right]$ larger than unity and $\mu = \left[\gamma + 1 - \sqrt{\gamma^2+\gamma+1}\right]$, smaller than unity. The general solution to the second-order recursive series is thus

$$p_n = A\lambda^n + B\mu^n + \alpha c_n. \tag{17}$$

Since $\lim_{n\to-\infty} p_n$ must be finite and non-negative, $B = 0$. Equation (17) for $n = 0$ is $2p_0 = c_0 + (q_0 - q_{-1})\bar{v} + p_{-1}$ and, in combination with equation (17) for $n = -1$ implies

$$2\left(A + \alpha c_0\right) = c_0 + \left(q_0 - q_{-1}\right)\bar{v} + \left(A/\lambda + \alpha c_{-1}\right)$$

Solving for A proves the claim. ■

Lemma 2 shows that the equilibrium price of firm n under equal relative spacing (12) features a constant markup term $(\alpha - 1)$ and an auxiliary term $\lambda^n A$.

The auxiliary term $\lambda^n A$ may be of either sign. This term derives from the fact that the first order condition of firm 0 is different from the first order condition of all other firms since firm 0 faces only one competitor instead of two. It depends on the relative scarcity of quality compared to valuations. In particular, if $\bar{v}$ is relatively small and close to v_0, firm 0 sells its top quality only to a small set of consumers. This implies that its demand elasticity is relatively large, since small price decreases induce losses of large fractions of its market. Consequently, its markup is relatively small and A is small or negative. Conversely, if $\bar{v}$ is very large, then the top firm serves a large market segment, faces a low demand elasticity and charges high markups. The corresponding value A in (15) is thus positive.

Overall, the relative markups $(\lambda/\gamma^\theta)^n A + \alpha - 1$ of prices $p_n = ((\lambda/\gamma^\theta)^n A + \alpha)c_n$ must obviously be positive. To verify that they are, we observe that the equivalence

$$\alpha > 0 \Leftrightarrow \lambda > \gamma^\theta \tag{18}$$

holds, which can be checked by $\alpha > 0 \Leftrightarrow 2(\gamma + 1) > \gamma^\theta + \gamma^{1-\theta}$ and $2(\gamma + 1) = \lambda - \gamma/\lambda$ (by (16)). Thus the contribution of A to the markup of low quality firms ($n \to -\infty$) is negligible if and only if γ is small in the sense that $\lambda > \gamma^\theta$ holds. If $\lambda < \gamma^\theta$, instead, (18) implies $\alpha < 0$. In that case, all markups are positive, since, first, $A + \alpha > 1$ is required if the top quality firm is to produce and secondly, the relative markup $(\lambda/\gamma^\theta)^n A + \alpha$ decreases in n.

Lemma 2 has shown that the price schedule (12) decomposes into two components – a constant markup over production costs and an auxiliary term stemming from the border condition. In addition, the closed form solution (12) allows us to derive comparative statics with respect to the parameters, which we formulate in the following corollary.

Corollary 1 *Assume that (10) holds. Then,*
(i) prices p_n, markups p_n/c_n and the slope of the pricing schedule $(p_{n+1} - p_n)$ are increasing

in $\overline{v}$ for all n.
(ii) prices p_n and the slope of the pricing schedule $(p_{n+1} - p_n)$ are increasing in q_0 and φ for all n.
(iii) markups p_n/c_n are decreasing in φ for all n.

Proof. *see Appendix* ■

Two parts of Corollary 1 describe the dependence of prices on, first, the distribution of consumer valuations and, secondly, on the set of qualities supplied. Neither of these dependencies have a correspondence in the model of horizontal differentiation with a circular setup as in Salop (1979). Specifically, the corresponding equilibria under horizontal differentiation are invariant to uniform taste shifts, because a firm's losses at one end are exactly offset by gains at the other end of the market. In the current setup of vertical differentiation, however, this irrelevance does not hold. Thus, consider first an upward shift of the range of valuations (Corollary 1 (i)). Higher valuations shift demand towards the top quality. Consequently, the firm producing the top quality serves a larger market segment which translates into a less elastic demand and results in an increased markup. This price increase of the top quality firm increases all prices in the market, but the upward shift of valuations benefits higher quality firms relatively more, so that the pricing schedule steepens (all $p_{n+1} - p_n$ increase).[18]

Consider next a shift in the set of qualities offered – more precisely, an increase in q_0 at constant γ – which is equivalent to an equal percentage increase of all qualities supplied. Just as in the case of upwardly shifting demand, firms react to this shift in the set of qualities with price increases. However, the nature of the shift in supply on prices is different now: the increases in price simply reflect the increases in production costs that firms need to pass on to consumers. Notice that, due to the convexity of costs in quality, this effect is stronger for higher quality firms so that the prices for higher qualities increase by more (all $p_{n+1} - p_n$ increase).

Parts (ii) and (iii) of Corollary 1 show how technology impacts prices. Not surprisingly, a uniform increase in production costs of all firms increases the prices of all firms. This

[18] In the appendix, we document a similar comparative static for the lower bound of the distribution of the valuations: increasing the lowest valuation decreases all prices and decreases the slope of the pricing schedule. We empirically examine the importance of the support of income distributions on markups in a model of how quality is priced to market in Auer et al. (2014).

increase in prices induces consumers to substitute towards lower qualities. Thus the market share of the top-quality firm shrinks and its market power falls, inducing the top firm to charge lower markups. This increase in competition at the top end of qualities, in turn, propagates downwards and lowers the prices of all firms (but affects high-quality firms relatively more). Overall, the increase in production costs is thus imperfectly passed on to consumers.

For comparison, we note that in the classical Hotelling model (for given entry), a uniform change in the cost parameter leaves markups unaltered because it does not affect the relative trade-off between varieties. The key reason for the differences of the impact of the parameters on equilibrium prices is that in our setup with a vertically differentiated market, the border condition of the top firm and thus all relative prices are affected.[19]

Corollary 2 *Assume that (10) holds. Then, the equilibrium cutoff-valuations v_n are increasing in $\overline{v}$, φ and q_0.*

Proof. *Follows directly from (5) and Corollary 1 (i) and (ii).* ■

Corollary 2 shows that the arrival of consumers with a high valuation for quality "crowds down" other consumers: if there is an entry by high-valuation consumers ($\overline{v}$ increases), all prices increase but the prices of high quality firms increase the most. Consequently, cutoffs move up, i.e. the pre-existing set of consumers downgrades its quality. Similarly, a proportional decrease in the marginal cost of all firms leads consumers to choose higher qualities.

Notice also that, parallel to Hotelling's setup, the market size impacts neither firms' prices nor their market segments, since the recursive pricing formula (11) and, indeed, the generic optimality condition (7) are independent of L.

Finally, we can assess the impact of entry of an additional firm on the markups of existing ones and on consumers' equilibrium quality choice. The pricing rule allows us to make an intuitive and simple statement regarding the effect of entry on the markups of existing firms.

[19] By contrast, in Vogel (2008) transportation costs do impact markups for a given number of entrants due to their effect on the equilibrium spacing: specifically, higher transportation costs mitigate the importance of heterogeneity in productivity and thus the equilibrium degree of isolation between firms.

Lemma 3 *Assume that (10) holds. The entry of an additional firm at the top end of the quality spectrum (at $q_1 = \gamma q_0$)*
(i) weakly decreases the markup of all established firms,
(ii) weakly flattens the pricing schedule $(p_{n+1} - p_n)$, and
(iii) weakly decreases cutoff valuations v_n.

Proof. (i) Denote the parameter from (15) before (after) the entry of the additional firm with A ($\tilde{A}$). If $\tilde{A} + \alpha < 1$, the new entrant does not produce and there is no effect on prices. Thus, assume that $\tilde{A} + \alpha > 1$ holds. Denoting the price of firm k before and after entry of the additional firm with p_k and $\tilde{p}_k$, respectively, we have

$$\frac{\tilde{p}_k}{p_k} = \frac{\widetilde{A}\lambda^{k-1} + \alpha c_k}{A\lambda^k + \alpha c_k} = \frac{\lambda^{-1}\left(\left[1 - \alpha\left(2 - \gamma^{-\theta}\right)\right] + \frac{\gamma - 1}{\gamma}\gamma^{1-\theta}\frac{q_0}{c_0}\bar{\mathrm{v}}\right)\frac{c_0}{c_k} + \frac{2\lambda - 1}{\lambda^{k+1}}\alpha}{\left(\left[1 - \alpha\left(2 - \gamma^{-\theta}\right)\right]c_0 + \frac{\gamma - 1}{\gamma}\frac{q_0}{c_0}\bar{\mathrm{v}}\right)\frac{c_0}{c_k} + \frac{2\lambda - 1}{\lambda^{k+1}}\alpha}$$

By $\lambda, \gamma, \theta > 1$ the numerator is larger than the denominator. Since both are positive, this proves the statement. (ii) by (i), it also holds that

$$(\tilde{p}_{k+1} - \tilde{p}_k) - (p_{k+1} - p_k) = \left(\lambda^{k+1-1} - \lambda^{k-1}\right)\widetilde{A} - \left(\lambda^{k+1} - \lambda^k\right)A \leq 0.$$

Finally, note that (iii) holds by (ii) and (5). ■

The Lemma shows that entry of a new technological leader crowds the top end of the market, increasing competition and lowering prices. For a given $\bar{\mathrm{v}}$, the new technological leader faces a smaller market segment and thus a higher demand elasticity than the old technological leader did before the entry.[20]

Lemma 3 also establishes that entry weakly flattens the pricing schedule $(p_{n+1} - p_n)$ and decreases the cutoff valuations v_n. By lowering the cost to consumers of increasing the quality of the consumed good, the entry of a new technological leader increases the equilibrium cutoff valuations v_n not only for top-valuation consumers, but (weakly) for all consumers.

[20] Technically, entry at the top reduces the constant from A to $\widetilde{A}$. The incumbent technological leader becomes firm -1, and is thus "isolated" from the new constant. If the constant is negative, this isolation can be beneficial. However, Lemma 3 documents that the total effect is always such that prices (and thus markups) of pre-existing firms weakly decrease once entry happens. Also note that the Lemma distinguishes two cases. First, the additional firm engages in production and impacts the whole market by depressing markups. Secondly, it does not pay for the additional firm to produce and sell its goods, and consequently leaves the market unaffected. Since this second case may indeed occur, the entry of additional firms decreases the markup of any preexisting firm only weakly.

This section has established comparative statics of prices for given entry and analyzed the impact entry of an additional firm on prices. In all cases, we have considered the special case of equal relative spacing. We did not, of course, choose the special case of equal relative spacing by chance: endogenous firm entry generates exactly the pattern described in (10). We now turn to the entry game and the resulting endogenous quality spacing.

3 Endogenous Spacing with Exogenous Growth

This section analyzes the firms' endogenous choice of qualities. Upon market entry, firms form expectations about future profits, anticipating the price equilibria described in Proposition 1. We remind the reader that we assume that valuations are distributed uniformly on the interval $[\underline{v},\bar{v}]$ according to (9). By Proposition 1 this assumption grants uniqueness of the price equilibrium for all potential quality choices.[21]

We will show that in a dynamic version of the general setup as described above, free entry supports equilibria with equal relative spacing of the firms, endogenously generating quality levels that satisfy (10).

We introduce a dynamic dimension into our model by assuming that time is continuous and that productivity in the Q-sector grows at the constant rate a, which is exogenously given.[22] Indexing the cost parameters φ and ϕ with time subscripts, we can write

$$\varphi_t = e^{-at}\varphi \qquad \text{and} \qquad \phi_t = e^{-at}\phi. \tag{19}$$

Our analysis aims at a stationary equilibrium in which each firm enters the industry as the technological leader and successively transits through the product cycle as it becomes superseded by further innovators. The advantage of such a dynamic entry game is that we only need to analyze the entry problem of one firm at a time. In particular, we avoid the problems that arise in a simultaneous-entry game, as in Vogel (2008).[23]

[21] As a general principle, price equilibria do not need to be unique for firms to compute expected profits and entry decisions. In fact, the results of the current section apply conveniently when defining expectations about future price equilibria. However, we focus on uniform distributions of valuations for simplicity's sake and to avoid distracting discussions regarding these expectations.

[22] In the working paper version of this paper, we endogenize the rate of technical progress a by assuming that aggregate technological progress in the Q-sector is a byproduct of firms' costly R&D activity. In this framework, we document that introducing consumer heterogeneity has important consequences for the nature of "creative destruction" and the mechanisms through which innovating firms create aggregate innovation (see Klette and Kortum (2004)).

[23] In fact, the resulting complications would be tremendous in our setup, because the clear ranking of the

In the entry game, firms decide not only which quality to invent but also when to enter the market. We assume that, at each point in time, there is a pool of potential entrants who may receive a perpetual monopoly to produce the good of quality level q when incurring the fixed cost (3). Driven by competition among potential entrants, a firm innovates as soon as innovation generates a profit flow with a net present value at least as large as the innovation cost.

We begin the analysis with the situation described in the previous section: the set of active firms is $\{0, -1, -2, ...\}$ and the firms are ranked in ascending order by quality so that a higher firm index corresponds to a firm producing higher quality $q \in \{q_n\}_{n \leq 0}$. Moreover, all existing qualities q_n satisfy (10).

We assume that a plant established to produce quality q_m automatically holds the blueprints for all qualities between q_{m-1} and q_m, where q_{m-1} is the next lowest quality level. This assumption restricts the entry of additional firms to quality levels above the pre-existing ones ($q_{m+1} \geq q_m$).[24]

Firms that gradually establish themselves at the top end of the quality spectrum are indexed with $m \geq 1$. Let t_m be the entry date of the m^{th} *additional* firm (implying $0 \leq t_1 \leq t_2 \leq ..$) and q_m denote its quality level ($q_0 \leq q_1 \leq q_2 \leq ..$). It will prove convenient to express the quality choice of the m^{th} entrant relative to the highest quality of all incumbents (q_{m-1}) as

$$\gamma_m = q_m / q_{m-1} \qquad\qquad m \geq 1.$$

At time $\tau \in [t_{m+k}, t_{m+k+1})$, the set of quality levels supplied to the market is $\{q_n\}_{n \leq m+k}$. Current prices are determined by equation (7) and depend on all currently produced quality levels and on productivities by equation (19). Consequently, at time $\tau \in [t_{m+k}, t_{m+k+1})$,

quality line prevents us from using the symmetry properties that arise in models such as Salop (1979) that are based on Hotelling (1929) and Lancaster (1966) and assume that the characteristic space is formed like a circular street or the beach surrounding an island. In a quality setup, however, any attempt to "close the circle" must fail, as such an attempt would amount to identifying the highest-quality good with the lowest-quality good.

[24] We note that this is a strong assumption. In an unrestricted entry game we cannot exclude a priori that an entrant would choose a quality below that of the technological leader. However, we focus on the case of top quality entry only, since in international patent law entrants are granted a patent only if innovation contains an element of novelty that improves upon existing technology and that patents thereby also grant current technology leaders protection from future entry by lower quality competitors (see World Intellectual Property Organization (2004)). Chor and Lai (2013) label this the "inventive step requirement" and theoretically examine its welfare-maximizing level in a model of Schumpeterian growth following Aghion and Howitt (1992).

the operating profits of the m^{th} additional firm, given by equation (8), are a function of the qualities $\{q_n\}_{n\leq m+k}$ and time τ. Note that productivity depends on time t through the term $e^{a\tau}$ only and we can thus express time dependence through $e^{a\tau}$. Formally, operating profits of the firm m at time τ are thus[25]

$$\pi_m\left(e^{a\tau}, q_{m+k}, \gamma_{m+k}, \gamma_{m+k-1}, \gamma_{m+k-2}, ..., \gamma_1, \gamma\right) \qquad \tau \in [t_{m+k}, t_{m+k+1}).$$

Defining the product

$$\Gamma_{m,k} = \prod\nolimits_{j=1}^{k} \gamma_{m+j} \tag{20}$$

yields $q_{m+k} = \Gamma_{m,k} q_m$ so that at time t_m, the present value of the flow of operating profits for a potential entrant can be expressed as

$$\Pi(\gamma_m, t_m) = \sum_{k\geq 0} \int_{t_{m+k}}^{t_{m+k+1}} e^{-r(\tau - t_m)} \pi_m\left(e^{a\tau}, \Gamma_{m,k}\gamma_m\Gamma_{0,m-1}q_0, \gamma_{m+k}, \gamma_{m+k-1}, ..., \gamma_1, \gamma\right) d\tau, \tag{21}$$

where the parameter r is the constant rate at which firms discount future profits.

We are now in position to formulate firms' entry decisions. The m^{th} firm chooses its entry date (t_m) and its location on the quality line (γ_m). With the second choice, it maximizes the present value of profits at time t_m (21) net of costs (3). Given the spacing $\gamma_{m-1}, \gamma_{m-2}, ..., \gamma_1, \gamma$, and conditional on the entry date t_m, the m^{th} optimal quality choice is

$$\begin{aligned}\hat{\gamma}_m\left(\gamma_{m-1}, ..., \gamma_1, \gamma\right) = \arg\max_{\tilde{\gamma}\geq 1} \Bigg\{ \sum\nolimits_{k\geq 0} \int_{t_{m+k}}^{t_{m+k+1}} e^{-r(\tau - t_m)} \pi_m \Big(e^{a\tau}, \tilde{\Gamma}_{m,k}\tilde{\gamma}\Gamma_{0,m-1}q_0, \hat{\gamma}_{n+k}, \hat{\gamma}_{n+k-1} \cdots \\ \cdots, \hat{\gamma}_{n+1}, \tilde{\gamma}, \gamma_{n-1}, ..., \gamma_1, \gamma\Big) d\tau - F_{t_m}(\tilde{\gamma}\Gamma_{0,m-1}q_0) \Bigg\}\end{aligned} \tag{22}$$

Here, $\tilde{\Gamma}_{m,k}$ denotes, similarly to (20), the product of the k future optimal relative spacing parameters, given that the m^{th}-entrant plays $\tilde{\gamma}$:

$$\tilde{\Gamma}_{m,k} = \prod\nolimits_{j=1}^{k} \hat{\gamma}_{m+j}\left(\hat{\gamma}_{m+j-1}, \hat{\gamma}_{m+j-2}, \cdots \tilde{\gamma}_m, \gamma_{m-1}, .., \gamma_1, \gamma\right).$$

[25] Notice that prices and profits π_m are unique by Lemma 1 and assumption (9). In cases of general distribution functions and multiple pricing equilibria to the system (7), one may read π_m as the expected profit with some exogenous probability assigned to each of the equilibria. Under the additional assumption that the realization of the equilibria is not path-dependent, the expected profits depend on incumbent qualities only and the following analysis applies.

Notice that for $j \geq 1$, all of the location choices $\hat{\gamma}_{m+j}$ (and $\tilde{\Gamma}_{m,j}$) and the entry dates t_{m+j} are functions of the choice of firm m. For expositional purposes, however, the arguments $\hat{\gamma}_{m+j}(\tilde{\gamma})$, $\tilde{\Gamma}_{m,j}(\tilde{\gamma})$, $t_{m+j}(\tilde{\gamma})$ are suppressed in (22) and further down. The m^{th} firm's entry date is determined by the free entry condition, *i.e.*, the requirement $\Pi(\gamma_m, t_m) \geq F(\gamma_m, q_{m-1})$. Formally, we write

$$t_m = \inf\left\{ t \geq t_{m-1} \;\middle|\; \sup_{\tilde{\gamma}\geq 1}\left[\sum\nolimits_{k\geq 0}\int_{t_{m+k}}^{t_{m+k+1}} e^{-r(\tau - t_m)}\pi_m\left(e^{a\tau}, \tilde{\Gamma}_{m,k}\tilde{\gamma}\Gamma^*_{0,m-1}q_0, \hat{\gamma}_{m+k}, \hat{\gamma}_{m+k-1}, \ldots \right.\right.\right.$$
$$\left.\left.\left. \ldots, \hat{\gamma}_{m+1}, \tilde{\gamma}, \gamma^*_{m-1}, \gamma^*_{m-2} \ldots, \gamma^*_1, \gamma\right) d\tau - F_t(\tilde{\gamma}\Gamma^*_{0,m-1}q_0)\right] \geq 0\right\} \tag{23}$$

where the asterisk * denotes the equilibrium locations:

$$\gamma^*_1 = \hat{\gamma}_1(\gamma) \qquad \text{and} \qquad \gamma^*_k = \hat{\gamma}_k(\gamma^*_{k-1}, \gamma^*_{k-2}, \ldots, \gamma^*_1, \gamma) \tag{24}$$

and $\Gamma^*_{0,k}$ is defined parallel to (20) as the product of the equilibrium γ^*_j

$$\Gamma^*_{0,k} = \prod\nolimits_{j=1}^{k} \gamma^*_j.$$

Optimal quality choices (22) and the free entry conditions (23) of all entrants ($m \geq 1$) determine the equilibrium of the entry game. The first important result of this section concerns the solution of the system (22) - (23) and is formulated in the following Proposition.

Proposition 2 *Let $(\theta, \phi, \varphi, L, r, a)$ be any combination of positive parameters. Then,*

(i) for any given q_m and $\{\gamma_n\}_{n\leq m}$, the entry date t^*_{m+1} and the relative location

$$\gamma^*_{m+1}(\theta, \phi, \varphi, L, r, a, \{\gamma_n\}_{n\leq m}) \in (1, \infty)$$

are unique. The choice γ^*_{m+1} does not depend on q_m.

(ii) there is a $\bar{\gamma}(\theta, \phi, \varphi, L, r, a) > 1$ so that, if $\gamma_n = \bar{\gamma}$ for all $n \leq m$, then

$$\gamma_{m+1}(\theta, \phi, \varphi, L, r, a, \{\gamma_n\}_{n\leq m}) = \bar{\gamma}$$

holds. In this case, time intervals between consecutive entries are constant and equal to

$$t^*_{m+1} - t^*_m = \Delta = \ln(\bar{\gamma})\,(\theta - 1)\,/a. \tag{25}$$

Proof. See Appendix. ∎

Part (i) of Proposition 2 shows that there is a unique equilibrium with finite relative spacing ($\gamma_m = q_m/q_{m-1}$). Interestingly, the relative spacing of the $m+1^{th}$ firm does not depend on the level of previous qualities, $\{q_n\}_{n\leq m}$, but just on their relative spacing $\{\gamma_n\}_{n\leq m}$. This result is a reflection of Lemma 1, and ultimately a result of the scaling property of the model. A new entrant $m+1$ compensates a proportional increase in existing qualities q_n ($n \leq m$) by postponing the entry date, which lowers entry costs and marginal costs. The entry date is delayed, but location decisions are unaffected.

Part (ii) of Proposition 2 shows that equilibria can arise that exhibit a particularly convenient and regular spacing pattern: equal relative spacing $\gamma_n = \bar{\gamma}$ applies to qualities of all incumbents but also for those of all future entrants. Notice, however, that part (ii) of the proposition does not involve uniqueness. Specifically, the unique dynamic equilibrium from part (i) does not guarantee that the equilibrium with equal relative spacing is unique. Just as in growth models where unique investment strategies – given a set of relevant state variables – do not guarantee unique steady states, uniqueness in part (i) does not translate into uniqueness in part (ii).

The lack of uniqueness should not come as a major surprise as it is a typical property of models of strategic complementarity in entry. Many comparable models of free entry to markets are subject to the same indeterminacy (see Capozza and Van Order (1980) and, more recently, Vogel (2008)). In general, some sort of strategic complementarity is involved when multiple equilibria arise (see also Cooper and John (1988)). In our current modeling setup, this would mean that a dense spacing of the existing qualities would make it relatively more attractive for new entrants to choose a small γ on its own (thus replicating the dense spacing). At first sight, such an effect may seem counterintuitive. However, considering the simultaneous choice of location and entry date, one consideration of the current entrant is to delay the entry of the next entrant. One way to do that is to commit to lower future prices. Such a commitment can be reached if preexisting qualities are densely spaced since in this case, the impact of the additional entrant on the incumbent top-firm's pricing rule is lower. As long as the value of commitment rises with the density of preexisting qualities relative to the loss in instantaneous profits, strategic complementarities can potentially arise, thus giving rise to the possibility of multiple equilibria.

An equilibrium described in Proposition 2 (ii), which is characterized by $\gamma_n = \bar{\gamma}$ for all integers n, can be labeled an Equal Relative Spacing Equilibrium (ERSE). Since the proposition is silent about its uniqueness, in the following analysis we refer to *the* ERSE as one equilibrium (out of possibly several equilibria) with the minimal spacing $\bar{\gamma}$. Notice that (as argued in the proof of Proposition 2), under a preexisting spacing parameter equal to one ($\gamma = 1$), the optimal spacing of the first entrant $\gamma^*(\gamma)$ from (24) satisfies $\gamma^*(\gamma) > \gamma$ for all $\gamma \in (1, \bar{\gamma})$. Therefore, at the minimal symmetric $\bar{\gamma}$, characterized by $\gamma^*(\bar{\gamma}) = \bar{\gamma}$, the following inequality holds:

$$\left.\frac{d\gamma^*(\gamma)}{d\gamma}\right|_{\gamma=\bar{\gamma}} < 1. \tag{26}$$

We note that under some conditions uniqueness of the ERSE follows directly – for example if the distribution of valuations is sufficiently compressed (the ratio $\bar{\mathrm{v}}/\underline{\mathrm{v}}$ is small). In that case, there is a unique optimal γ for all initial conditions, which characterizes the ERSE. To verify this statement, consider entry of firm 1 under any set $\{q_0, q_{-1}, ...\}$ of qualities of the existing firms. By Proposition 2 (i), the relative location choice of firm 1, γ_1, is independent of q_0. Moreover, if the ratio $\bar{\mathrm{v}}/\underline{\mathrm{v}}$ is small enough, γ_1 is independent of $q_{-1}, q_{-2}, ...$ as well, because at most two firms are simultaneously active in that case.[26] Consequently, firm 1's relative location choice γ_1 is independent of all previous qualities q_n ($n \leq 0$) and is therefore unique. Hence, there is only one optimal γ for all entrants, which means that the relative spacing parameters of all entrants are identical.[27]

Turning to the general case, however, we consider ERSE for which (26) holds. Existence of such an equilibrium is established in Proposition 2 (ii) and we can derive the following Lemma, which exploits the scaling properties of the model to establish regularities of the solution to the optimal entry problem of the firms.

Lemma 4 *Let $\bar{\gamma}$ be the spacing parameter of the ERSE. Then,*

(i) for given $\phi/(\varphi L)$, $\bar{\gamma}$ is independent of ϕ, φ and L, so that $\bar{\gamma}(\phi/(\varphi L))$.

(ii) the transformation $(\phi, \varphi) \to \chi \cdot (\phi, \varphi)$ ($\chi > 0$) postpones entry dates by $\ln(\chi)/a$.

(iii) $\bar{\gamma}$ is constant under the transformation $(r', a', L') = \chi \cdot (r, a, L)$ with $\chi > 0$.

[26] Using (7) and (9), prices and profits are quickly computed for three firms $n = 0, -1, -2$. For $\underline{\mathrm{v}}$ close to $\bar{\mathrm{v}}$, the firms' qualities ($q_0 = \gamma^2 q_{-2}$, $q_{-1} = \gamma q_{-2}$ and q_{-2}) need to be sufficiently close to each other for all firms to be active. In particular, $\gamma \to 1$ as $\underline{\mathrm{v}} \to \bar{\mathrm{v}}$. This implies zero profits in the limit and contradicts free entry under any $\phi > 0$.

[27] Technically, the intertemporal links decouple and the dynamic system effectively becomes a static one: the entrant's problem of choosing the optimal γ is independent of the model's state variables ($q_0, q_{-1}, ...$) and is thus replicated for each new entrant.

(iv) the transformation $\tilde{G}(x) = G(\chi x)$ $(\chi > 0)$ leaves $\bar{\gamma}$ unchanged and postpones entry dates by $\ln(1/\chi)/a$.

Proof. *(i)* Operating profits π are linear in L and setup costs F are linear in ϕ. Thus, when replacing $\phi' = \phi/L$, population L factors out of the slanted brackets in (22) and the square brackets in (23). Consequently, the solution to problem (22) - (24) and thus $\bar{\gamma}$ depends on $\phi' = \phi/L$ only. Similarly, operating profits are, by Lemma 1, linear in φ under the transformation $q_n' = q_n' \varphi^{1/(\theta-1)}$ that leaves the relative spacing unchanged. Hence, replacing $\phi' = \phi/\varphi$ in (22) and (23) shows that $\bar{\gamma}$ depends on ϕ' only.

(ii) Consider the transformation $(\phi', \varphi') = \chi \cdot (\phi, \varphi)$. Adding the time transformation $t'' = t + \ln(\chi)/a$ implies $(\phi'', \varphi'') = \chi^{-1}(\phi', \varphi') = (\phi, \varphi)$ by (19), replicating the original problem and thus its solution.

(iii) Given entry dates and locations, the net present profits (21) are constant under rescaling of time $t \to \chi t$. Thus firm entry remains unchanged. Finally, the time transformation is equivalent to the transformation $(r', a', L') = \chi \cdot (r, a, L)$ (see (22) and (23)).

(iv) The transformation $\tilde{G}(x) = G(\chi x)$ and $(\phi', \varphi') = \chi(\phi, \varphi)$ leaves the optimality conditions (7) unchanged. Since the time transformation $t' = t + \ln(1/\chi)/a$ is equivalent to the transformation $(\phi', \varphi') = \chi(\phi, \varphi)$ (see (ii) above) this shows the claim. ■

Technically, part (i) of Lemma 4 shows that the density of spacing $\bar{\gamma}$ is only affected by the costs ϕ and φ and the market size L only through the the ratio $\phi/(\varphi L)$. Intuitively, since prices are independent of the market size L, the operating profits are proportional to L. Thus, an increase in the fixed costs of market entry, ϕ, can be entirely offset by a corresponding percentage increase in the market size without any impact on the equilibrium spacing. Similarly, a proportional increase in market entry cost ϕ and marginal production cost φ by the factor χ does not matter for optimal spacing. In such a case, however, entrants need to let time pass until costs have dropped according to (19). After a period of length $\ln(1/\chi)/a$, the original level of costs is reached and the entry problem is replicated. This is stated in part (ii) of Lemma 4.

Lemma 4 (iii) concerns the choice of units of time. Equal percentage changes in a and r are isomorphic to changes in time units. Moreover, when doubling the units of time, the flow of profits is "thinner". However, when this thinning out of profits is compensated by an increase in market size L, the total effect on relative spacing of qualities vanishes. This statement also shows that for a fixed ratio of r/a (e.g. if $r = a$), the spacing property of $\bar{\gamma}$ depends on the ratio a/L only. Together with part (i) of the lemma, we can write $\bar{\gamma}(a\phi/(\varphi L))$ in this case.

Finally, Lemma 4 (iv) shows that stretching (or compressing) the distribution of valuations v does not affect the equilibrium spacing. Stretching the distribution G (with $\chi \in (0,1)$) increases each consumer's willingness to pay for quality and thus simply advances the entry of firms. Together with (i) and (ii), this statement can be reformulated in an intuitive way as an equivalence between the valuations v and costs ϕ and φ. A proportional increase in consumers' willingness to pay for quality and the costs of quality leaves the equilibrium qualities unchanged and the entry date of each given quality occurs earlier by $\ln(1/\chi)/a$ units of time.

This section has derived a novel result about the regularity of firm spacing (Proposition 1) and the relative impact of the model's key parameters (Lemma 2). In the next section we will discuss the ERSE and provide comparative statics.

4 Properties of the ERSE

With Lemma 2 and Lemma 3, we can illustrate dynamics of the equilibrium described in Proposition 2 (ii). The prices and profits of a firm producing q_0 evolve as depicted by the blue and green lines in Figure 2. Each continuous section represents prices and profits when no innovation occurs. Innovations occur at regular intervals; the entry dates are marked by the vertical dotted lines. At these equidistant dates of entry, firm 0's markup over marginal cost and thus its operating profit both drop by a discrete amount because the new competitor reduces the incumbents' market power and its market share.

Two opposing forces shape the overall trend, which is decreasing in the long-run. First, for a given set of firms, the profit flow for the top-quality firm is increasing as productivities increase over time, making production of all goods cheaper, spurring demand for quality and thus driving up the market share of the top quality firm. Second, productivity growth also implies that each firm's market segment narrows. In particular, as time passes, the firm serves consumers with lower and lower valuations. It becomes squeezed to the bottom end of the quality distribution, serving an ever smaller segment. Therefore, the profit flow drops to zero in the limit.

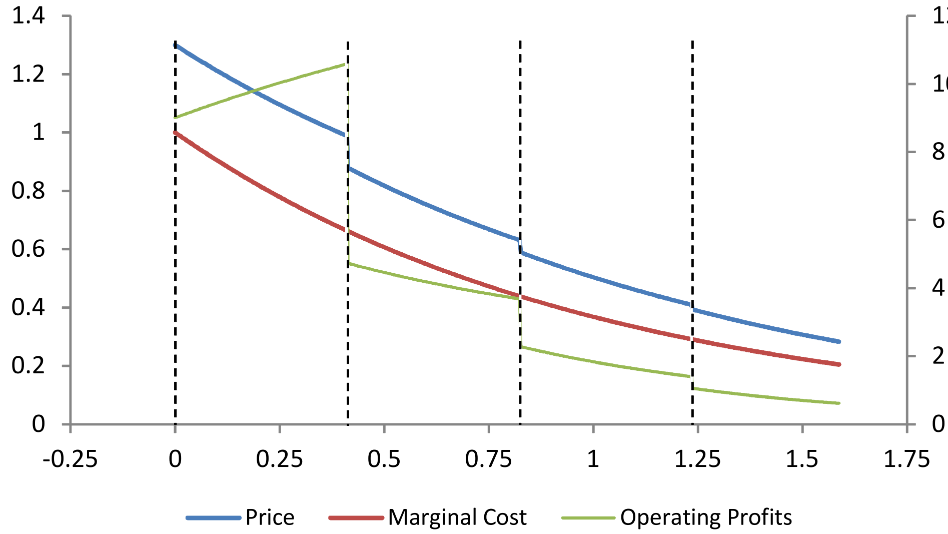

Figure 2: Profit Flow over the Life Cycle of the Firm Entering at $t^* = 0$.

In the equilibrium depicted in Figure 2, an innovating firm immediately starts producing once it enters the market. However, if entry is sufficiently cheap relative to the marginal productivity, the zero profit condition may also force firms to enter the industry preemptively: at the moment of entry, the production of the top quality is too costly so that there is no demand for it even when sold at marginal costs.

The resulting product life cycle with preemptive entry is depicted in Figure 3. In this equilibrium, further technological progress is needed to reduce the cost of the highest quality good before it is actually produced. In Figure 3 we depict such an equilibrium, where firms only sell a while after entry. In this equilibrium, at the moment when productivities are just high enough so that the consumer with the maximum valuation buys from the firm that entered at t_0^*, firm 0 sells at marginal costs (the firm's demand is infinitely elastic because it sells to a negligible set of consumers). Thereafter, the profit flow of firm 0 increases steeply: the set of its consumers grows with productivity growth. As it sells to a growing range of consumers, its demand becomes less elastic so its markup increases. At each date at which a new competitor starts producing (which occurs always some while after innovation), the increased competition reduces markups and profits of the incumbent firm. Again, the respective dates are marked by vertical dotted lines. In the limit, the firm approaches a constant markup. However, because the firm's market share converges

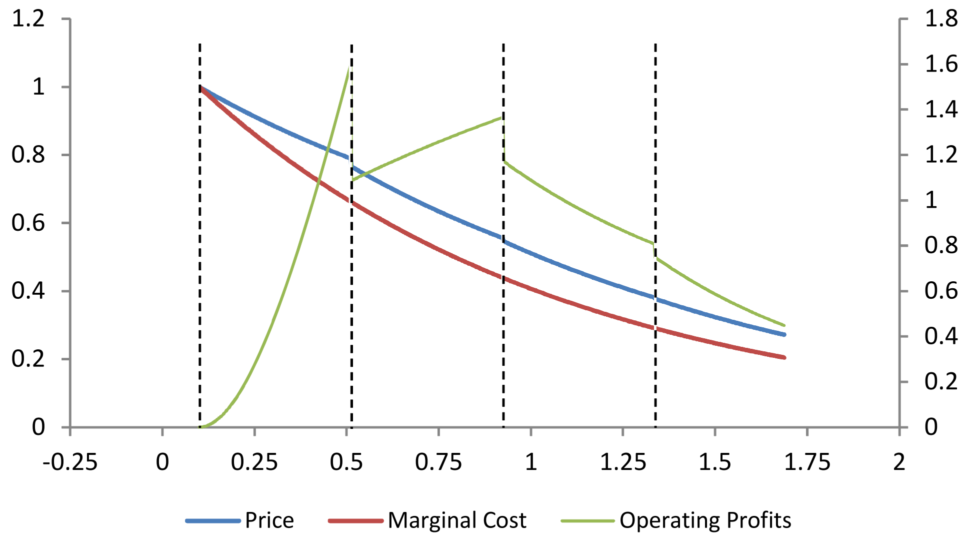

Figure 3: Profit Flow of the Firm Entering at t_0^* under Pre-Emptive Entry

to zero in the long run, its profit flow approaches 0.

Our next aim is to conduct comparative statics with regard to the model's parameters. As a preparatory step, we write for the relative markup of the highest quality firm

$$A+\alpha-1=\frac{\lambda}{2\lambda-1}\left\{\frac{\gamma-1}{\gamma}\frac{q_0\bar{v}}{c_0}+\left(\frac{\alpha}{\gamma^{\theta}}-1\right)-\frac{\alpha-1}{\lambda}\right\}.$$

With the explicit formula for the prices (12), the operating profits from (8) are

$$\pi_n=\begin{cases}\gamma\left(\gamma-1\right)\left(\dfrac{A+\alpha-1}{\gamma-1}\right)^2\dfrac{c_0^2}{q_0}\dfrac{L}{\bar{v}} & \text{if } n=0\\[2ex] \left(\gamma^2-1\right)\left(\dfrac{A\left(\lambda/\gamma^{\theta}\right)^n+\alpha-1}{\gamma-1}\right)^2\dfrac{c_n^2}{q_n}\dfrac{L}{\bar{v}} & \text{if } n<0\end{cases}\tag{27}$$

Observe with (13) and (15) that the limit

$$\lim_{\gamma\to1}\frac{A+\alpha-1}{\gamma-1}=\frac{1}{\sqrt{3}}\left\{\frac{q_0\bar{v}}{c_0}-\theta\right\}$$

is finite. Thus, in the case of equal relative spacing, the operating profits are, by (13) - (15) and the limit above, continuously differentiable for all $\gamma\geq1$ and satisfy, moreover

$$\pi_n\to0\qquad(\gamma\to1).$$

Next, and very importantly, we can sign the slope of the ERSE's location, i.e., the function $\bar{\gamma}(\phi)$ for some (technical) parameter restrictions.

Proposition 3 *Assume $\underline{v}$=0 and let* $0 < \underline{\phi} < \bar{\phi} < \infty$*. Then, there is a* $r_0 > 0$ *so that* $\bar{\gamma}(\phi)$ *is weakly increasing on* $[\underline{\phi}, \bar{\phi}]$ *for all* $r \in [0, r_0]$*.*

Proof. See Appendix. ■

In combination with the proofs of the existence and uniqueness of the pricing game and existence of the entry equilibrium (see Proposition 2), Proposition 3 represents the main result of our analysis. It establishes the comparative statics of the equilibrium degree of spacing with regard to the entry cost.[28]

The proposition shows that higher setup costs increase the relative spacing between quality levels. Intuitively, firms must be compensated for increases in setup costs by increased profits. The latter profit increases are brought about by larger market shares and by higher markups and, ultimately, by a wider spacing parameter $\bar{\gamma}$. Larger markets induce more frequent firm entry and a higher density of quality supply because higher sales and profits allow for a faster recovery of setup costs. Markups, in turn, are decreasing in the density of supply and are thus decreasing in market size.

Together with Lemma 2, Proposition 3 also determines the impact of market size (L) and marginal production costs (φ) on the spacing $\bar{\gamma}$ of the ERSE. In particular, increases in L and φ have similar effects on $\bar{\gamma}$ as do reductions in setup costs: all decrease the equilibrium spacing, $\bar{\gamma}$. Clearly, a larger market induces, *ceteris paribus*, higher profits and allows firms to generate more profit. At given setup costs, larger markets therefore experience more frequent entry of firms at closer distances; that is, the competitive pressure among firms rises.

Surprisingly, productivity growth at the margin (a decrease in marginal production costs φ) *increases* relative spacing. This adverse effect of marginal productivity growth on competitive pressure may appear somewhat puzzling. To understand the forces operating to this effect, observe that the preference specification developed in this paper generates, just as with preferences featuring a constant elasticity of substitution, relative firm markups $p_n/c_n - 1$ that are independent of costs (see prices (12)). Put differently, at a given relative

[28] The condition on the interest rate r establishes an upper bound for the instantaneous profits upon firm entry and is a technical requirement for the proof of Proposition 3.

spacing, operating profits constitute a constant share of revenues. Hence, when quality levels are constant, an increase in marginal productivity (or a drop in marginal costs) tends to curb revenues and thereby depresses operating profits.[29] As firms must cover their setup costs, however, the productivity gains that curb profits per consumer must come about with increases in market share, *i.e.*, with a wider equilibrium spacing. At the same time, this widening of relative spacing does increase relative markups. Hence, competitive pressure decreases as marginal productivity grows.[30]

It would be premature to infer welfare consequences based on the parameter $\bar{\gamma}$ alone (and its impact on markups), conjecturing, e.g., that an equal increase in setup costs ϕ and operating costs φ leaves the welfare levels of the economy unchanged. In fact it does not. Such a change in technology actually postpones innovation by Lemma 4 (ii) so that more time elapses until a product of a given quality is on the market. This delay means that individuals purchase lower-quality goods, which has a negative impact on consumer surplus.

5 Conclusion

In this study, we analyze the pricing game and endogenous firm entry into vertically differentiated markets, assuming that the relevant production technology is such that the *finiteness property* of Shaked and Sutton (1982 and 1983) is violated, i.e. that a potentially large number of firms can coexist alongside the technological leader.

We first examine monopolistic competition in the quality space for given entry of any number of firms and arbitrary spacings. Our basic assumption for production technology is that the marginal cost of production is convex with respect to quality, which guarantees the survival of many firms. We characterize the pricing strategies and profits of firms that compete à la Bertrand, taking as given each firm's quality, i.e., its location in the quality space. Specifically, we provide sufficient conditions for existence and uniqueness of a pure-strategy pricing equilibrium, proving that they are always met under uniform distribution of consumer valuations.

[29] This aggregate relationship does not, of course, mean that each single firm can raise its profits by artificially decreasing its productivity.

[30] Notice that this effect depends on the property of our model that demand does not react along an intensive margin. In particular, consumers do not react to price changes by consuming more or less, but instead by switching to other firms.

Secondly, we model the costly quality choices of firms entering the industry. Exogenous productivity growth makes ever higher qualities affordable for consumers, so firms sequentially enter the market at the top end of the quality spectrum. We specify the conditions under which the entry problem of each new entrant is replicated, which implies that each new quality level exceeds the current quality level by a fixed proportion. Although the firms' sequential market entry introduces technical difficulties, this model overcomes the asymmetry of the location problem that unavoidably arises in quality space: the quality spectrum has top and bottom ends. Our main technical contribution lies in handling the border conditions that therefore appear (and are absent from the circular world of Salop (1979) and other circular representations of Hotelling (1929) and Lancaster (1966)).

Because it provides a rich, yet reasonably tractable description of vertically differentiated industries featuring a large number of coexisting firms, we believe that our framework has many further applications in the fields of industrial organization, international trade, and economic growth.[31]

[31] For example, Auer et al. (2014) embed the preferences developed in this study in a multi-country world featuring costly trade and then both theoretically and empirically examine how firms price their goods differently depending on the density of quality competition in each market.

A Appendix – Proofs

Proof of Lemma 1. We first show that firm n's strategy space is convex, compact, and non-empty for all n. For prices exceeding $p_n^{\max} = \bar{v} q_n$, consumer demand is zero, independently of other prices. Hence, firms n's strategy lies within the convex and compact interval $[c_n, \bar{v} q_n]$. Firms' strategies are thus a subset of the convex and compact set $S = [c_0, \bar{v} q_0] \times [c_{-1}, \bar{v} q_{-1}] \times ... \times [c_N, \bar{v} q_N]$. Furthermore, firm n's profits are zero at $p_n = c_n$ or $v_{n+1} = v_n$. The latter equality is equivalent to

$$\bar{p}_n = \frac{(q_n - q_{n-1})\, p_{n+1} + (q_{n+1} - q_n)\, p_{n-1}}{q_{n+1} - q_{n-1}} \tag{28}$$

Convexity of $c(q)$ implies

$$\bar{p}_n = \frac{(q_n - q_{n-1})\, p_{n+1} + (q_{n+1} - q_n)\, p_{n-1}}{q_{n+1} - q_{n-1}} \geq \frac{(q_n - q_{n-1})\, c_{n+1} + (q_{n+1} - q_n)\, c_{n-1}}{q_{n+1} - q_{n-1}} > c_n$$

so that profits are non-negative on the interval $p_n \in [c_n, \bar{p}_n]$ and positive on $p_n \in (c_n, \bar{p})$.

The strategy space S_0 defined through $p_n \in [c_n, \bar{p}_n]\ \forall n \leq 0$ is a closed non-empty subset of S and is thus compact and non-empty. Finally, it is convex since for $\mathbf{p} = (p_n)_{n \leq 0}$ and $\mathbf{p}' = (p_n')_{n \leq 0}$ and $\lambda \in (0,1)$ we have for $\mathbf{p}'' = \lambda \mathbf{p} + (1-\lambda)\mathbf{p}'$

$$\lambda p_n + (1-\lambda) p_n' \leq \lambda \bar{p}_n + (1-\lambda) \bar{p}_n' = \bar{p}_n''$$

for all n, where the respective upper bound on strategies $\bar{p}_n^{('')}$ is defined parallel to (28).

The Gershgorin circle theorem states that under condition $a_{ii} + \sum_{j \neq i} |a_{ij}| < 0$ the matrix $A = (a_{ij})$ is negative definit. We aim to apply this theorem to the matrix A consisting of the elements $a_{nk} = d^2\pi_n/(dp_n dp_k) + d^2\pi_k/(dp_n dp_k)$, with descending indices $n, k = 0, -1, ..., N$. To save notation, we write $G_n = G(v_n)$, $G_n' = G'(v_n)$, and $\Delta q_n = q_n - q_{n-1}$. Using (7), we then compute (normalizing $L = 1$)

$$\frac{d^2\pi_n}{dp_n dp_k} = \begin{cases} G_{n+1}'/\Delta q_{n+1} & if \quad k = n+1 \\ -2\left[G_{n+1}'/\Delta q_{n+1} + G_n'/\Delta q_n\right] & if \quad k = n \\ G_n'/\Delta q_n & if \quad k = n-1 \end{cases}$$

and zero else. Notice that $d^2\pi_n/(dp_n dp_k) = d^2\pi_k/(dp_n dp_k)$ so that the matrix A is symmetric and its elements are

$$a_{nk} = 2\frac{d^2\pi_n}{dp_n dp_k}$$

This matrix satisfies the condition of the Gershgorin circle theorem.

Notice also that for given p_{n+1}, p_{n-1}, firm n's profit

$$\pi_n = (p_n - c_n)\left[G(v_{n+1}) - G(v_n)\right]$$

is continuous and convex in p_n, since

$$\frac{d^2\pi_n}{(dp_n)^2} = -2(p_n - c_n)\left[\frac{G'(v_{n+1})}{q_{n+1} - q_n} + \frac{G'(v_n)}{q_n - q_{n-1}}\right] < 0$$

As each firm's strategy space is convex, compact, and non-empty, each payoff function π_n is continuous and concave in p_n and $A = (a_{ij})$ is negative definit, we can now apply Theorem 2 in Rosen (1965) to conclude that the pricing equilibrium is unique. ■

Claim A. *Assume that there is a $\Gamma > 1$ so that that $q_{n+1}/q_n > \Gamma$ holds for all n. Assume further that G is twice continuously differentiable and $|G''/G'| < (\Gamma - 1)/(2\Gamma\bar{v})$ with $\bar{v}$ being the maximal valuation from (2). Then, the system (7) has a unique equilibrium in pure strategies.*

Proof. We use the same line of argument as in the preceeding proof above. As above, the strategy space S_0 defined through $p_n \in [c_n, \bar{p}_n]$ $\forall n \leq 0$ is a compact, non-empty and convex set. To apply the Gershgorin circle theorem, we compute (adopting the notation from above and normalizing $L = 1$)

$$a_{nk} = \frac{d^2\pi_n}{dp_n dp_k} = \begin{cases} \frac{G'_{n+1}}{\Delta q_{n+1}} - (p_n - c_n)\frac{G''_{n+1}}{(\Delta q_{n+1})^2} & if \;\; k = n+1 \\ -2\left[\frac{G'_{n+1}}{\Delta q_{n+1}} + \frac{G'_n}{\Delta q_n}\right] + (p_n - c_n)\left[\frac{G''_{n+1}}{(\Delta q_{n+1})^2} - \frac{G''_n}{(\Delta q_n)^2}\right] & if \;\; k = n \\ \frac{G'_n}{\Delta q_n} + (p_n - c_n)\frac{G''_n}{(\Delta q_n)^2} & if \;\; k = n-1 \end{cases}$$

Thus, defining $a_{nk} = d^2\pi_n/(dp_n dp_k) + d^2\pi_k/(dp_n dp_k)$, we calculate

$$\begin{aligned} a_{ii} + \sum_{j\neq i}|a_{ij}| \;\; = \;\; & -4\left[\frac{G'_{n+1}}{\Delta q_{n+1}} + \frac{G'_n}{\Delta q_n}\right] + 2(p_n - c_n)\left[\frac{G''_{n+1}}{(\Delta q_{n+1})^2} - \frac{G''_n}{(\Delta q_n)^2}\right] + \ldots \\ & \ldots + \left|\frac{G'_n}{\Delta q_n} + (p_n - c_n)\frac{G''_n}{(\Delta q_n)^2}\right| + \left|\frac{G'_n}{\Delta q_n} - (p_{n-1} - c_{n-1})\frac{G''_n}{(\Delta q_n)^2}\right| \\ & \ldots + \left|\frac{G'_{n+1}}{\Delta q_{n+1}} - (p_n - c_n)\frac{G''_{n+1}}{(\Delta q_{n+1})^2}\right| + \left|\frac{G'_{n+1}}{\Delta q_{n+1}} + (p_{n+1} - c_{n+1})\frac{G''_{n+1}}{(\Delta q_{n+1})^2}\right| \end{aligned}$$

or

$$\begin{aligned} a_{ii} + \sum_{j\neq i}|a_{ij}| \;\; \leq \;\; & -4\left[\frac{G'_{n+1}}{\Delta q_{n+1}} + \frac{G'_n}{\Delta q_n}\right] + 2(p_n - c_n)\left[\frac{G''_{n+1}}{(\Delta q_{n+1})^2} - \frac{G''_n}{(\Delta q_n)^2}\right] + \ldots \\ & \ldots + 2\left|\frac{G'_n}{\Delta q_n}\right| + \left|(p_n - c_n)\frac{G''_n}{(\Delta q_n)^2}\right| + \left|(p_{n-1} - c_{n-1})\frac{G''_n}{(\Delta q_n)^2}\right| \\ & \ldots + 2\left|\frac{G'_{n+1}}{\Delta q_{n+1}}\right| + \left|(p_n - c_n)\frac{G''_{n+1}}{(\Delta q_{n+1})^2}\right| + \left|(p_{n+1} - c_{n+1})\frac{G''_{n+1}}{(\Delta q_{n+1})^2}\right| \end{aligned}$$

or with $p_n < \bar{v} q_n$

$$a_{ii} + \sum_{j \neq i} |a_{ij}| < \frac{G'_{n+1}}{\Delta q_{n+1}} 2 \left[2\Gamma\bar{v} \frac{G''_{n+1}}{G'_{n+1}(\Gamma - 1)} - 1 \right] + \frac{G'_n}{\Delta q_n} 2 \left[2\Gamma\bar{v} \frac{G''_n}{G'_n(\Gamma - 1)} - 1 \right]$$

By assumtion $|G''/G'| < (\Gamma - 1)/(2\Gamma\bar{v})$ this expression satisfies

$$... < \frac{G'_{n+1}}{\Delta q_{n+1}} 2 [1 - 1] + \frac{G'_n}{\Delta q_n} 2 [1 - 1] = 0$$

Hence, $A = (a_{nk})_{nk}$ is negative definit.

The calculations also show that $d^2\pi_n/(dp_n)^2 = a_{ii}$ is negative so that profits π_n are continuous and concave in p_n. Theorem 2 in Rosen (1965) then proves the claim. ■

Proof of Corollary 1. 1 Write with (12)

$$\begin{aligned} p_n &= \lambda^n \frac{\lambda}{2\lambda - 1} \left(\frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} \alpha c_0 + \frac{\gamma - 1}{\gamma} q_0 \bar{v} \right) + \alpha c_n \\ p_{n+1} - p_n &= \lambda^n (\lambda - 1) \frac{\lambda}{2\lambda - 1} \left(-\frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} \alpha c_0 + \frac{\gamma - 1}{\gamma} q_0 \bar{v} \right) + \alpha(\gamma^\theta - 1) c_n \end{aligned}$$

(i) Follows by $\lambda, \gamma > 1$.

(ii) Compute

$$\frac{dp_n}{d\varphi} = \left[-\lambda^n \frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} \alpha c_0 + \alpha c_n \right] \frac{1}{\varphi} = \left[-(\lambda/\gamma^\theta)^n \frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} + 1 \right] \frac{c_n \alpha}{\varphi} \quad (29)$$

If $\lambda > \gamma^\theta$, then $\alpha > 0$ and the term in square brackets is

$$-(\lambda/\gamma^\theta)^n \frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} + 1 \geq -\frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} + 1 > -\frac{\lambda}{2\lambda - 1} \frac{\lambda - 1/\lambda}{\gamma + 1} + 1$$

Since λ solves (16), it satisfies $\lambda^2 = 2\lambda(\gamma + 1) - \gamma$ so that the expression on the right hand side is

$$-\frac{1}{2\lambda - 1} \frac{\lambda^2 - 1}{\gamma + 1} + 1 = -\frac{1}{2\lambda - 1} \frac{2\lambda(\gamma + 1) - \gamma - 1}{\gamma + 1} + 1 = -\frac{1}{2\lambda - 1} \frac{2\lambda - 1}{1} + 1 = 0$$

If, instead, $\lambda < \gamma^\theta$ then $\alpha < 0$ holds and

$$-(\lambda/\gamma^\theta)^n \frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} + 1 \leq -\frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} + 1 < -\frac{\lambda}{2\lambda - 1} \frac{\lambda - 1/\lambda}{\gamma + 1} + 1$$

The last expression is zero, as shown above. Hence, (29) is always positive since $\lambda > \gamma^\theta \Leftrightarrow \alpha > 0$.

Further,

$$\frac{d(p_{n+1} - p_n)}{d\varphi} = \left[-(\lambda/\gamma^\theta)^n (\lambda - 1) \frac{\lambda}{2\lambda - 1} \frac{\gamma^\theta - \gamma^{-\theta}}{\gamma + 1} + (\gamma^\theta - 1) \right] \frac{c_n \alpha}{\varphi}$$

The proof follows as above.

Observe with (12) that

$$\frac{dp_n}{dq_0} = \frac{dp_n}{d\varphi}\frac{\varphi\theta}{q_0} + \lambda^n \frac{\lambda}{2\lambda - 1}\frac{\gamma - 1}{\gamma}\bar{\mathrm{v}}$$

so that

$$\frac{d\left(p_{n+1} - p_n\right)}{dq_0} = \frac{d\left(p_{n+1} - p_n\right)}{d\varphi}\frac{\varphi\theta}{q_0} + \lambda^n(\lambda - 1)\frac{\lambda}{2\lambda - 1}\frac{\gamma - 1}{\gamma}\bar{\mathrm{v}}$$

Both expressions are positive by the statements $dp_n/d\varphi > 0$ and $d(p_{n+1} - p_n)/d\varphi > 0$ above.

(iii) and (iv) Follow directly from (12). ■

Proof of Proposition 2. Consider the location choice of the first entrant ($n = 1$) at given q_0 and $\{\gamma_n\}_{n\leq 0}$, where $\gamma_n = q_n/q_{n-1}$.

(i) Notice that t_m is bounded below by $t_{m-1} > -\infty$. Therefore, the infimum in (23) is well-defined and t_1 is unique. Consider firm 1's location choice $\gamma_1 = q_1/q_0$. For given t_1, the net present value of firm 1's future operating profits $\pi^1(\tau, \gamma_1, \{q_n\}_{n\leq 0})$ is

$$\int_{t_1}^{\infty} e^{-r(\tau - t_1)}\pi^1(\tau, \gamma_1, \{\gamma_n\}_{n\leq 0}, q_0)\ d\tau - \phi(\gamma_1 q_0)^{\theta}$$

By Lemma 1, this expression is homogeneous of degree θ in q_0. This proves that the maximazing γ_1 is independent of q_0. We can thus set $q_0 = 1$ wlog. Entry at $\gamma_1 = 1$ cannot be optimal since Bertrand competition would imply $\pi^1 = 0$. Similarly, entry at $\gamma_1 \to \infty$ is not optimal. To verify this statement, observe that firm 1's profits are bounded by profits in absence of competitors. A monopolist would sell to the market segment $[p'/q,\bar{\mathrm{v}}]$, charges its optimal price $p^{mon} = (\bar{\mathrm{v}}q + c)/2$, receiving profits that satisfy $\pi^{mon}(\tau) < \gamma_1(\bar{\mathrm{v}}/2)^2 L/(\bar{\mathrm{v}} - \underline{\mathrm{v}})$. Thus, firm 1's net profits are bounded by

$$\gamma_1(\bar{\mathrm{v}}/2)^2 L/(\bar{v} - \underline{\mathrm{v}}) \int_{t_1}^{\infty} e^{-r\tau} d\tau - \phi_{t_1}\gamma_1^{\theta}$$

which is negative as $\gamma_1 \to \infty$. Thus, net profits are maximal for a $\gamma_1 \in (1, \infty)$. Finally, for given t_1 the value γ_1 maximizing net profits is unique just as in generic maximization problems.

(ii) Assume that $\{q_n\}_{n\leq 0}$ satisfy (10) with prevailing γ and consider entry decision of firm $m = 1$. Observe that entry with $\gamma_1 = 1$ is not optimal since Bertrand competition would imply $\Pi_{t_1}(\gamma_1) = 0$ (regardless of t_1), which violates the free entry condition. Hence, $\gamma = 1$ implies $\gamma_1 > \gamma$.

We show next that $\gamma_1 < \gamma$ holds for γ large enough. If this were not the case, $\gamma_1 \geq \gamma$ for all $\gamma \geq 1$, which implies $\gamma_1 \to \infty$ as $\gamma \to \infty$. But $\gamma_1 \to \infty$ was shown above to contradict free entry. Consequently, $\gamma_1 < \gamma$ holds for γ large enough.

Writing the first entrant's relative location γ_1 as a function of the preexisting relative location parameter γ, i.e., writing $\gamma_1(\gamma)$ we thus have $\gamma_1(\gamma) > \gamma$ for $\gamma = 1$ and $\gamma_1(\gamma) < \gamma$ for γ large enough. Continuity then implies that there is a $\gamma > 1$ so that $\gamma_1 = \gamma$. We denote this by $\bar{\gamma}$. At this $\gamma = \bar{\gamma}$, the firm $n = 1$ locates in the quality space, extending equal relative spacing (10) to all $n \leq 1$.

Under $\gamma = \gamma_1 = \bar{\gamma}$, we call the spacing problem of the remaining additional firms ($n = 2, 3, ...$) the *residual spacing problem*. With the notation

$$\gamma'_n = \gamma_{n+1} \qquad (n \geq 1) \qquad q'_0 = \bar{\gamma} q_0 = q_1 \qquad \text{and} \qquad \tau' = \tau + a^{-1}(\theta - 1) \ln \bar{\gamma} \tag{30}$$

the residual spacing problem solves the corresponding system (22) - (24) above, where the relevant variables q', v' and γ' now bear a prime. Applying Lemma 1, (19) and (30) shows that

$$\pi_n \left(e^{a\tau}, \tilde{\Gamma}'_{n,k} \hat{\gamma}' \Gamma'^{(*)}_{0,n-1} q'_0, \gamma'_m, \gamma'_{m-1}, ..., \gamma'_1, \bar{\gamma} \right) = \bar{\gamma} \pi_n \left(e^{a\tau'}, \tilde{\Gamma}'_{n,k} \hat{\gamma}' \Gamma'^{(*)}_{0,n-1} q_0, \gamma'_m, \gamma'_{m-1}, ..., \gamma'_1, \bar{\gamma} \right)$$

Notice also that the setup cost (3) satisfies $F_{\tau'}(\cdot, q'_0) = \bar{\gamma}^{1-\theta} F_\tau(\cdot, \bar{\gamma} q_0) = \bar{\gamma} F_\tau(\cdot, q_0)$. Hence, $\bar{\gamma}$ factors out of the right hand side of (22) and of the square brackets in (23). Consequently, the solution of the residual spacing problem coincides with the original problem, implying $\gamma'_1 = \gamma_2 = \gamma_1 = \bar{\gamma}$. A simple induction argument completes the proof that $\gamma_n \equiv \bar{\gamma}$ for all $n \geq 1$.

Finally, (19) and the transformation (30) show that two consecutive entries occur at dates satisfying $\phi_{t_n} = \bar{\gamma}^{1-\theta} \phi_{t_n}$. With (19), this is $e^{-a(t_{n+1} - t_n)} = \bar{\gamma}^{1-\theta}$ and proves the second statement. ■

Proof of Proposition 3. We start with some preparatory steps. First, we define Ψ as net profits of the first entrant as a function of existing spacing γ, setup costs ϕ, entry date t and location choice $\hat{\gamma}$. Suppressing dependence of Ψ on parameters other than ϕ and normalizing $q_0 = 1$, we have

$$\Psi(\gamma, t, \hat{\gamma}, \phi) = \Pi(\hat{\gamma}, t) - \phi \left(\hat{\gamma} \right)^\theta$$

Free entry implies that equilibrium entry date and location $t^*(\gamma)$ and $\gamma^*(\gamma)$ satisfy

$$\Psi(\gamma, t^*(\gamma), \gamma^*(\gamma), \phi) = 0. \tag{31}$$

for all γ and ϕ. Optimality of the firm's location choice implies

$$\Psi_{\gamma^*}(\gamma, t^*(\gamma), \gamma^*(\gamma), \phi) = 0. \tag{32}$$

Taking derivatives of (31) w.r.t. γ and using (32) yields

$$\Psi_\gamma + \Psi_{t^*}\frac{dt^*}{d\gamma} = 0. \tag{33}$$

Next, we denote the relative spacing variable of the ERSE with $\bar{\gamma}(\phi)$. Along the path of all these ERSEs (31) holds, so that $\Psi(\bar{\gamma}(\phi), t^*(\bar{\gamma}(\phi), \phi), \bar{\gamma}(\phi), \phi) = 0$. Taking derivatives w.r.t. ϕ and using (33) yields

$$0 = \left[\Psi_\gamma + \Psi_{t^*}\frac{dt^*}{d\gamma}\right]\frac{d\bar{\gamma}}{d\phi} + \Psi_{t^*}\frac{\partial t^*}{\partial \phi} + \Psi_\phi = \Psi_{t^*}\frac{\partial t^*}{\partial \phi} + \Psi_\phi$$

which implies with $\Psi_\phi = -\bar{\gamma}^\theta$

$$\frac{\partial t^*}{\partial \phi} = \frac{\bar{\gamma}^\theta}{\Psi_{t^*}}. \tag{34}$$

Now, taking derivatives of (32) w.r.t. γ leads to

$$0 = \Psi_{\gamma^*\gamma} + \Psi_{\gamma^* t^*}\frac{dt^*}{d\gamma} + \Psi_{\gamma^*\gamma^*}\frac{d\gamma^*}{d\gamma}. \tag{35}$$

Along the path of all ERSE, (32) requires $\Psi_{\gamma^*}(\bar{\gamma}(\phi), t^*(\bar{\gamma}(\phi), \phi), \bar{\gamma}(\phi), \phi) = 0$. Taking derivatives w.r.t. ϕ yields

$$0 = \left[\Psi_{\gamma^*\gamma} + \Psi_{\gamma^* t^*}\frac{dt^*}{d\gamma} + \Psi_{\gamma^*\gamma^*}\right]\frac{d\bar{\gamma}}{d\phi} + \Psi_{\gamma^* t^*}\frac{\partial t^*}{\partial \phi} + \Psi_{\gamma^*\phi}$$

or, with (35),

$$\Psi_{\gamma^*\gamma^*}\left[1 - \frac{d\gamma^*}{d\gamma}\right]\frac{d\bar{\gamma}}{d\phi} = -\Psi_{\gamma^* t^*}\frac{\partial t^*}{\partial \phi} - \Psi_{\gamma^*\phi}$$

Equations (3) and (21) imply $\Psi_{t^*} = r\phi(\gamma^*)^\theta - \pi(t^*)$ and thus $\Psi_{t^*\gamma^*} = r\phi\theta(\gamma^*)^{\theta-1} - \pi_{\gamma^*}(t^*)$. Further, $\Psi_{\gamma^*\phi} = -\theta\left(\gamma^*\right)^{\theta-1}$ holds so that we have with (34) at the ERSE

$$\Psi_{\gamma^*\gamma^*}\left[1 - \frac{d\gamma^*}{d\gamma}\right]\frac{d\bar{\gamma}}{d\phi} = \left\{\pi_{\gamma^*}(t^*)\bar{\gamma}^\theta - \pi(t^*)\theta\bar{\gamma}^{\theta-1}\right\}\frac{1}{\Psi_{t^*}} \tag{36}$$

The second order condition of the firm's optimization requires $\Psi_{\gamma^*\gamma^*} < 0$, while (26) implies that the term in the square brackets is positive. Moreover, by definition of t^*, $\Psi_{t^*} > 0$ holds. Consequently, $\bar{\gamma}(\phi)$ is increasing (constant) in ϕ if and only if the expression in the slanted brackets on the right of (36) is negative (zero).

Whenever $\pi(t^*) = 0$ the expression on the right is zero and thus $\bar{\gamma}$ is constant in ϕ. We thus need to show that at the ERSE $d\ln(\pi(t^*)/(\gamma^*)^\theta)/d\gamma^* < 0$ holds for $\pi(t^*) > 0$. To this aim it is thus sufficient to show (remember $q_0 = 1$ so that $q_1 = \gamma^*$)

$$\frac{d}{dq_1}\ln(\pi(t^*)) < \theta/q_1$$

With profits $\pi_1 = L(p_1 - c_1)(\bar{\mathrm{v}} - v_1)/\bar{\mathrm{v}}$ and $v_1 = (p_1 - p_0)/(q_1 - q_0)$ (compare (5) and (6), shifting up indices) and the envelope theorem (thus neglecting the terms dp_1/dq_1) the condition above is equivalent to

$$\frac{-\dot{c}_1}{p_1 - c_1} + \frac{-1}{\bar{\mathrm{v}} - v_1}\left(\frac{-\dot{p}_0}{q_1 - q_0} - \frac{p_1 - p_0}{(q_1 - q_0)^2}\right) < \theta$$

where $\dot{x} \equiv dx/dq_1$. With (5) and firm 1's optimality condition $2p_1 = c_1 + (q_1 - q_0)\bar{\mathrm{v}} + p_0$ (see (7), given (9), shifting up indices) this condition is

$$\dot{p}_0 + v_1 - \dot{c}_1 < \theta\,(p_1 - c_1) \tag{37}$$

To compute $\dot{p}_0$ write the system (7) as

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -(q_0 - q_{-1}) & 2(q_1 - q_{-1}) & -(q_1 - q_0) & 0 \\ 0 & -(q_{-1} - q_{-2}) & 2(q_0 - q_{-2}) & \ldots \\ \ldots & 0 & \ldots & \ldots \end{pmatrix} p = \begin{pmatrix} c_1 + (q_1 - q_0)\bar{v} \\ (q_1 - q_{-1})c_0 \\ (q_0 - q_{-2})c_{-1} \\ \ldots \end{pmatrix} \tag{38}$$

where $p \equiv (p_1, p_0, \ldots)^t$. Taking derivatives w.r.t. q_1 yields

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -(q_0 - q_{-1}) & 2(q_1 - q_{-1}) & -(q_1 - q_0) & 0 \\ 0 & -(q_{-1} - q_{-2}) & 2(q_0 - q_{-2}) & \ldots \\ \ldots & 0 & \ldots & \ldots \end{pmatrix} \dot{p} + \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & -1 & 0 & \\ 0 & 0 & 0 & 0 & \\ \ldots & & & & \ldots \end{pmatrix} p = \begin{pmatrix} \dot{c}_1 + \bar{v} \\ c_0 \\ 0 \\ \ldots \end{pmatrix}$$

and evaluating at the ERSE leads to

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2(\gamma + 1) & -\gamma & 0 \\ 0 & -1 & 2(\gamma + 1) & -\gamma \\ \ldots & 0 & \ldots & \ldots \end{pmatrix} \dot{p} = \begin{pmatrix} \dot{c}_1 + \bar{\mathrm{v}} \\ \gamma\frac{-2p_0 + p_{-1} + c_0}{q_1 - q_0} \\ 0 \\ \ldots \end{pmatrix} \tag{39}$$

Replicating the proof of Proposition 2, we obtain that $\dot{p}_n$ satisfies $\dot{p}_n = \lambda^n \dot{p}_0$ with $\lambda = \gamma + 1 + \sqrt{\gamma^2 + \gamma + 1}$ for $n \leq 0$. The second row of (39) thus becomes

$$-\dot{p}_1 + [2(\gamma+1) - \gamma/\lambda]\, \dot{p}_0 = -\dot{p}_1 + \lambda \dot{p}_0 = -\gamma \frac{2p_0 - p_{-1} - c_0}{q_1 - q_0}$$

where we have used the fact that λ solves (16). Combining this equation with the first row of (39) ($2\dot{p}_1 - \dot{p}_0 = \dot{c}_1 + \bar{v}$) leads to

$$[2\lambda - 1]\, \dot{p}_0 = \dot{c}_1 + \bar{v} - 2\gamma \frac{2p_0 - p_{-1} - c_0}{q_1 - q_0} = \dot{c}_1 + \bar{v} - 2\frac{p_1 - 2p_0 + c_0}{q_1 - q_0}$$

where we used the second row of (38) in the last step. With $v_1 = (p_1 - p_0)/(q_1 - q_0)$ and $\bar{v} - v_1 = (p_1 - c_1)/(q_1 - q_0)$ (compare (5) and (11)), we have

$$\dot{p}_0 = \frac{1}{2\lambda - 1} \left\{ 3\bar{v} - 6v_1 + \dot{c}_1 + 2\frac{c_1 - c_0}{q_1 - q_0} \right\}$$

and hence

$$\dot{p}_0 + v_1 - \dot{c}_1 = \frac{1}{2\lambda - 1} \left\{ 3(\bar{v} - v_1) - 2\left(\dot{c}_1 - \frac{c_1 - c_0}{q_1 - q_0}\right) - [2\lambda - 4](\dot{c}_1 - v_1) \right\} \tag{40}$$

We show next that (37) holds for all $t \in [t^{**}, t^{**} + \delta]$ with $\delta > 0$ small enough and t^{**} defined as the date where $\bar{v} = v_1$ holds. At this date we have $p_1 = c_1$ by (11) so that

$$\dot{p}_0 + v_1 - \dot{c}_1 = \frac{-2}{2\lambda - 1} \left[\left(\dot{c}_1 - \frac{c_1 - c_0}{q_1 - q_0}\right) + [\lambda - 2](\dot{c}_1 - v_1) \right] < 0$$

where the last inequality holds by $\lambda > 2$ and

$$v_1 = \frac{c_1 - p_0}{q_1 - q_0} < \frac{c_1 - c_0}{q_1 - q_0} < \dot{c}_1$$

for all $\gamma > 1$, showing (37). By continuity, there is a $\delta > 0$ so that (37) holds for all $t \in [t^{**}, t^{**} + \delta]$.

Now, since π_1 from (27) is increasing in t, there is an $\varepsilon > 0$ so that $t \in (t^{**}, t^{**} + \delta)$ holds whenever $\pi_1 < \varepsilon$. This last condition holds for $r > 0$ small enough as $\Psi_{t^*} > 0$ implies $0 < -\pi_1 + r\Pi$ or

$$\pi_1 < r\phi\bar{\gamma}^{\theta} \tag{41}$$

Finally, we restrict the pair of parameters (r, ϕ) to the compact set $[0, r_1] \times [\underline{\phi}, \bar{\phi}]$. Hence, there are $\gamma_{\min}$ and $\gamma_{\max}$ with $1 < \gamma_{\min} < \gamma_{\max} < \infty$ so that $\bar{\gamma}$ is restricted to the compact set $[\gamma_{\min}, \gamma_{\max}]$. Consequently, there is a uniform $r_0 \leq r_1$ so that for all $(r, \phi) \in [0, r_1] \times [\underline{\phi}, \bar{\phi}]$ we have $\pi_1 < \varepsilon$ and (37) holds uniformly. This proves the statement. ■

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Appendix: Not For Publication

B Appendix – Lower Cutoff

We here generalize the solution of equilibrium prices (12). Specifically, we assume that the distribution of consumer valuations (2) is uniform on $[\underline{v},\bar{v}]$, i.e.

$$G(v) = \begin{cases} 0 & \text{if } v < \underline{v} \\ (v - \bar{v})/(\underline{v} - \bar{v}) & \text{if } v \in [\underline{v},\bar{v}] \\ 1 & \text{if } v > \bar{v} \end{cases} \tag{42}$$

holds. In addition, we assume that (10) still holds , so that each quality is a constant fraction higher than its immediately preceding one. In this case, the system (7) becomes

$$p_n = \begin{cases} \frac{1}{2}\left[c_0 + \left(1 - \frac{1}{\gamma}\right) q_0 \bar{v} + p_{-1}\right] & \text{if } n = 0 \\ \frac{1}{2}\left[c_n + \frac{1}{\gamma+1} p_{n+1} + \frac{\gamma}{\gamma+1} p_{n-1}\right] & \text{if } -N < n < 0 \\ \frac{1}{2}\left[c_{-N} + p_{-N+1} - (\gamma - 1)\, q_{-N} \underline{v}\right] & \text{if } n = -N \end{cases} \tag{43}$$

We can formulate the following proposition (noting that $(\lambda - 2\gamma) = 3\gamma\left(2 - \lambda^{-1}\right)^{-1}$).

Proposition 4 *Assume equal relative spacing in quality, i.e., (10) holds. Then prices are*

$$p_n = \lambda^n A_N + (\gamma/\lambda)^{N+n} B_N + \alpha c_n \qquad \forall \quad -N \leq n \leq 0 \tag{44}$$

where

$$\alpha = \frac{\gamma + 1}{2(\gamma + 1) - \gamma^{\theta} - \gamma^{1-\theta}} \tag{45}$$

$$\lambda = \gamma + 1 + \sqrt{\gamma^2 + \gamma + 1} \tag{46}$$

$$A_N = \frac{\lambda - 2\gamma}{3\gamma} \frac{\gamma a_0 + (\gamma/\lambda)^N b_N}{1 - \gamma^N \lambda^{-2N}} \tag{47}$$

$$B_N = \frac{\lambda - 2}{3\gamma} \frac{\lambda^{-N} \gamma a_0 + b_N}{1 - \gamma^N \lambda^{-2N}} \tag{48}$$

$$a_0 = \left(1 - \left(2 - \gamma^{-\theta}\right) \alpha\right) c_0 + (1 - 1/\gamma)\, q_0 \bar{v} \tag{49}$$

$$b_N = \left(1 - \left(2 - \gamma^{\theta}\right) \alpha\right) c_{-N} - (\gamma - 1)\, q_{-N} \underline{v} \tag{50}$$

and N is defined as

$$N = \max\left\{m \mid B_m + A_m \left(\lambda/\gamma^{\theta}\right)^{-m} + \alpha > 1\right\}.$$

Proof. Substitution $u_n = p_n - \alpha c_n$ into (43) yields

$$2\left[u_n + \alpha c_n\right] = c_n + \frac{1}{\gamma+1}\left[u_{n+1} + \alpha c_{n+1}\right] + \frac{\gamma}{\gamma+1}\left[u_{n-1} + \alpha c_{n-1}\right]$$

for $n > 1$. With $\alpha = (1+\gamma)/\left[2(1+\gamma) - \gamma^{\theta} - \gamma^{1-\theta}\right]$ this is $2(\gamma+1)u_n = u_{n+1} + \gamma u_{n-1}$. The equation

$$X^2 - 2(\gamma+1)X + \gamma = 0 \tag{51}$$

has two roots, $\lambda = \left[\gamma + 1 + \sqrt{\gamma^2+\gamma+1}\right]$ larger than unity and $\mu = \left[\gamma + 1 - \sqrt{\gamma^2+\gamma+1}\right]$, smaller than unity. The general solution to the second-order recursive series is thus

$$p_n = \tilde{A}\lambda^n + \tilde{B}\mu^n + \alpha c_n. \tag{52}$$

Equation (43) for $n = 0$ is $2p_0 = c_0 + (q_0 - q_{-1})\bar{v} + p_{-1}$ and, in combination with equation (52) for $n = -1$ implies

$$2\left(\tilde{A} + \tilde{B} + \alpha c_0\right) = c_0 + (q_0 - q_{-1})\,\bar{v} + \left(\tilde{A}\lambda^{-1} + \tilde{B}\mu^{-1} + \alpha c_{-1}\right),$$

while equation (43) for $n = -N$ is $2p_{-N} = c_{-N} + p_{-N+1} - (\gamma - 1)\,q_{-N}\underline{v}$, in combination with equation (52) for $n = -N+1$ hence implying that

$$2\left(\tilde{A}\lambda^{-N} + \tilde{B}\mu^{-N} + \alpha c_{-N}\right) = \left(c_{-N} + \tilde{A}\lambda^{-N+1} + \tilde{B}\mu^{-N+1} + \alpha c_{-N+1} - (\gamma-1)\,q_{-N}\underline{v}\right)$$

We can substitute μ using $\lambda\mu = \gamma$, which holds the solution by $\lambda^2 - 2(\gamma+1)\lambda + \gamma = 0$. Finally, solving for $\tilde{A}$ and $\tilde{B}$, replacing $A = \tilde{A}$ and $B = \tilde{B}\mu^{-N}$ proves (15). ■

Corollary 3 *Assume that (10) holds. Then,*
(i) prices p_n, markups p_n/c_n and the slope of the pricing schedule $(p_{n+1} - p_n)$ are increasing in $\bar{v}$ and decreasing in $\underline{v}$ for all n.
(ii) if $\gamma^{\theta} \leq \lambda$, the slope of the pricing schedule $(p_{n+1} - p_n)$ is increasing in φ for all n.
(iii) if $\gamma^{\theta} \leq \lambda$, the slope of the pricing schedule $(p_{n+1} - p_n)$ is increasing in q_0 for all n.
(iv) if $\bar{v} > \underline{v}\,(\lambda/\gamma)^N$ holds, markups p_n/c_n are decreasing in φ for all n.

Proof. (i) The statement for prices p_n follows since $\gamma > 1$ implies with (49) and (50) $da_N/d\bar{v} > 0$, $da_N/d\underline{v} = db_N/d\bar{v} = 0$ and $db_N/d\underline{v} < 0$ so that with (44) - (50) $dp_n/d\bar{v} > 0$

and $dp_n/d\underline{v} < 0$. The statement for markups p_n/c_n follows since $dc_n/d\overline{v} = dc_n/d\underline{v} = 0$. For the statement regarding $(p_{n+1} - p_n)$ compute with (44)

$$\frac{d(p_{n+1} - p_n)}{d\overline{v}} = (\lambda - 1)\lambda^n \frac{dA_N}{d\overline{v}} + (\gamma/\lambda - 1)(\gamma/\lambda)^{N+n} \frac{dB_N}{d\overline{v}}$$

Using (47) - (50), check that

$$\frac{dA_N}{d\overline{v}} = \frac{\lambda - 2\gamma}{3\gamma} \frac{\gamma}{1 - (\gamma/\lambda^2)^N} (1 - 1/\gamma) q_0 \quad \text{and} \quad \frac{dB_N}{d\overline{v}} = \frac{\lambda - 2}{3\gamma} \frac{\lambda^{-N}\gamma}{1 - (\gamma/\lambda^2)^N} (1 - 1/\gamma) q_0$$

By $\gamma/\lambda^2 < 1$, $d(p_{n+1} - p_n)/d\overline{v}$ is positive if and only if

$$[\lambda - 1][\lambda - 2\gamma]\lambda^n - [1 - \gamma/\lambda][\lambda - 2](\gamma/\lambda)^{N+n}\lambda^{-N}$$

is positive. Since $\lambda > 2\gamma$, all terms in squared brackets are positive so that the expression above is increasing in n and hence minimal for $n = -N$. It is thus sufficient to show that

$$[\lambda - 1][\lambda - 2\gamma] - [1 - \gamma/\lambda][\lambda - 2] > 0 \tag{53}$$

holds. But this inequality is satisfied since λ solves the identity $\lambda^2 - (2\gamma + 1)\lambda + 2\gamma = 0$ (see (51) in the proof of (44) in the appendix) so that the expression above equals

$$2\gamma + 2 - 2\gamma/\lambda > 0$$

This proves that $d(p_{n+1} - p_n)/d\overline{v} > 0$.

Next, compute

$$\frac{d(p_{n+1} - p_n)}{d\underline{v}} = (\lambda - 1)\lambda^n \frac{dA_N}{d\underline{v}} + (\gamma/\lambda - 1)(\gamma/\lambda)^{N+n} \frac{dB_N}{d\underline{v}}$$

Using (47) - (50), check that

$$\frac{dA_N}{d\underline{v}} = \frac{\lambda - 2\gamma}{3\gamma} \frac{-(\gamma/\lambda)^N}{1 - \gamma^N\lambda^{-2N}} (\gamma - 1) q_{-N} \text{ and } \frac{dB_N}{d\underline{v}} = \frac{\lambda - 2}{3\gamma} \frac{-1}{1 - \gamma^N\lambda^{-2N}} (\gamma - 1) q_{-N}$$

so that $d(p_{n+1} - p_n)/d\underline{v}$ is negative if and only if

$$-[\lambda - 1][\lambda - 2\gamma] + [1 - \gamma/\lambda][\lambda - 2]\left(\gamma/\lambda^2\right)^n < 0$$

holds. Since all terms in squared brackets are positive and $\gamma/\lambda^2 < 1$ this expression is maximal at $n = 0$ so that the statement holds by (50) above.

(ii) First notice with (45), (49) and (50) that

$$a_0 = -\beta c_0 + (1 - 1/\gamma) q_0 \overline{v} \quad \text{and} \quad b_N = \gamma\beta c_{-N} - (\gamma - 1) q_{-N}\underline{v}$$

where

$$\beta = \frac{\gamma^{\theta} - \gamma^{-\theta}}{2(\gamma+1) - \gamma^{\theta} - \gamma^{1-\theta}}$$

and compute with (44) - (50)

$$\begin{aligned}
\frac{d\left(p_{n+1}-p_n\right)}{d\varphi} &= \lambda^n\left(\lambda-1\right)\frac{\partial A_N}{\partial\varphi} - \left(\frac{\gamma}{\lambda}\right)^{N+n}\left(1-\frac{\gamma}{\lambda}\right)\frac{\partial B_N}{\partial\varphi} + \gamma^{\theta n}\alpha\left(\gamma^{\theta}-1\right)\frac{c_0}{\varphi} \\
&= \left[-\left(\lambda^n\left(\lambda-1\right)\left(\lambda-2\gamma\right)\frac{1-\left(\frac{\gamma^{1-\theta}}{\lambda}\right)^N}{1-\left(\frac{\gamma}{\lambda^2}\right)^N} + \ldots\right.\right. \\
&\quad \left.\left. \ldots + \left(\frac{\gamma}{\lambda}\right)^{N+n}\left(1-\gamma/\lambda\right)\left(\lambda-2\right)\lambda^{-N}\frac{\left(\frac{\gamma^{\theta}}{\lambda}\right)^{-N}-1}{1-\left(\frac{\gamma}{\lambda^2}\right)^N}\right)\frac{\beta}{3} + \gamma^{\theta n}\alpha\left(\gamma^{\theta}-1\right)\right]\frac{c_0}{\varphi}
\end{aligned}$$

Using $\lambda^2 = 2(\gamma+1)\lambda - \gamma$ from (51) we can substitute

$$\left(\lambda-1\right)\left(\lambda-2\gamma\right) = \lambda+\gamma \quad \text{and} \quad \left(1-\gamma/\lambda\right)\left(\lambda-2\right) = \gamma\frac{\lambda+1}{\lambda}$$

so that

$$\begin{aligned}
\frac{d\left(p_{n+1}-p_n\right)}{d\varphi} &= \left[-\left(\left(\lambda+\gamma\right)\frac{1-\left(\frac{\gamma^{1-\theta}}{\lambda}\right)^N}{1-\left(\frac{\gamma}{\lambda^2}\right)^N} + \ldots\right.\right. \\
&\quad \left.\left. \ldots + \left(\frac{\gamma}{\lambda^2}\right)^n\left(\frac{\gamma}{\lambda}\right)^N\gamma\frac{\lambda+1}{\lambda}\lambda^{-N}\frac{\left(\frac{\gamma^{\theta}}{\lambda}\right)^{-N}-1}{1-\left(\frac{\gamma}{\lambda^2}\right)^N}\right)\frac{\beta}{3} + \left(\frac{\gamma^{\theta}}{\lambda}\right)^n\alpha\left(\gamma^{\theta}-1\right)\right]\lambda^n\frac{c_0}{\varphi}
\end{aligned} \tag{54}$$

To prove the claim, we will show that the expression in square brackets is positive. As a first step, we look at the terms that depend on n:

$$\left(\frac{\gamma}{\lambda^2}\right)^n\left[-\gamma\frac{\lambda+1}{\lambda}\left(\frac{\gamma}{\lambda^2}\right)^N\frac{\left(\frac{\gamma^{\theta}}{\lambda}\right)^{-N}-1}{1-\left(\frac{\gamma}{\lambda^2}\right)^N}\left(\gamma^{\theta}-\gamma^{-\theta}\right)\right] + \left(\frac{\gamma^{\theta}}{\lambda}\right)^n\left[3(\gamma+1)\left(\gamma^{\theta}-1\right)\right] \tag{55}$$

at $n=-N$ this expression equals

$$\begin{aligned}
\ldots &= -\gamma\frac{\lambda+1}{\lambda}\frac{\left(\lambda/\gamma^{\theta}\right)^N-1}{1-\gamma^N\lambda^{-2N}}\left(\gamma^{\theta}-\gamma^{-\theta}\right) + 3\left(\frac{\gamma^{\theta}}{\lambda}\right)^{-N}\left(\gamma+1\right)\left(\gamma^{\theta}-1\right) \\
&= \left[-\gamma\frac{\lambda+1}{\lambda}\frac{1-\left(\lambda/\gamma^{\theta}\right)^{-N}}{1-\gamma^N\lambda^{-2N}}\frac{\gamma^{\theta}-\gamma^{-\theta}}{\gamma^{\theta}-1} + 3(\gamma+1)\right]\left(\frac{\gamma^{\theta}}{\lambda}\right)^{-N}\left(\gamma^{\theta}-1\right)
\end{aligned}$$

The expression in square brackets satisfies

$$
\begin{aligned}
-\gamma\frac{\lambda+1}{\lambda}\frac{1-(\gamma^{\theta}/\lambda)^{N}}{1-(\gamma/\lambda^{2})^{N}}\frac{\gamma^{\theta}-\gamma^{-\theta}}{\gamma^{\theta}-1}+3(\gamma+1) &> -\gamma\frac{\lambda+1}{\lambda}\frac{\gamma^{\theta}-\gamma^{-\theta}}{\gamma^{\theta}-1}+3(\gamma+1)\\
&> -\gamma\frac{\lambda+1}{\lambda}2+3(\gamma+1)>0
\end{aligned}
$$

Thus we have shown that, in absolute terms, the positive expression in the second squared brackets in (55) is larger than the negative term in the first. Thus, multiplying the negative term by $\left(\gamma/\lambda^{2}\right)$ and the positive term by $(\gamma^{\theta}/\lambda)$ increases the sum of both. Hence the expression in (55) is increasing in n and so is the expression in (54). Therefore it is sufficient to show that the expression in (54) is positive for $n=-N$. To this aim, we set $n=-N$ in (54), multiply by $\left(\gamma^{\theta}/\lambda\right)^{N}$ and show next that

$$
-\left((\lambda+\gamma)\frac{\left(\frac{\gamma^{\theta}}{\lambda}\right)^{N}-\left(\frac{\gamma}{\lambda^{2}}\right)^{N}}{1-\left(\frac{\gamma}{\lambda^{2}}\right)^{N}}+\gamma\frac{\lambda+1}{\lambda}\frac{1-\left(\frac{\gamma^{\theta}}{\lambda}\right)^{N}}{1-\left(\frac{\gamma}{\lambda^{2}}\right)^{N}}\right)(\gamma^{\theta}-\gamma^{-\theta})+3(\gamma+1)\left(\gamma^{\theta}-1\right)
$$

is positive. To do so, we use that $\dfrac{(\gamma^{\theta}/\lambda)^{N}-\gamma^{N}\lambda^{-2N}}{1-\gamma^{N}\lambda^{-2N}}+\dfrac{1-(\gamma^{\theta}/\lambda)^{N}}{1-\gamma^{N}\lambda^{-2N}}=1$ so that we can rewrite the expression above as

$$
\left(\frac{1-(\gamma^{\theta}/\lambda)^{N}}{1-\gamma^{N}\lambda^{-2N}}\left((\lambda+\gamma)-\gamma\frac{\lambda+1}{\lambda}\right)-(\lambda+\gamma)\right)\left(\gamma^{\theta}-\gamma^{-\theta}\right)+3\,(1+\gamma)\left(\gamma^{\theta}-1\right)
$$

Using now that $\dfrac{1-(\gamma^{\theta}/\lambda)^{N}}{1-\gamma^{N}\lambda^{-2N}}<1$ is increasing in N and $N\geq 1$, we have

$$
\begin{aligned}
\ldots &\geq \left(\frac{1-\gamma^{\theta}/\lambda}{1-\gamma/\lambda^{2}}\left((\lambda+\gamma)-\gamma\frac{\lambda+1}{\lambda}\right)-(\lambda+\gamma)\right)\left(\gamma^{\theta}-\gamma^{-\theta}\right)+3\,(1+\gamma)\left(\gamma^{\theta}-1\right)\\
&= -\left(\gamma^{\theta}+\gamma\right)\left(\gamma^{\theta}-\gamma^{-\theta}\right)+3\,(1+\gamma)\left(\gamma^{\theta}-1\right)
\end{aligned}
$$

To check that this last expression is positive, we divide by $(1+\gamma)\left(\gamma^{\theta}-1\right)$ and look at

$$
-\frac{\left(\gamma^{\theta}+\gamma\right)}{(1+\gamma)}\frac{\left(\gamma^{\theta}-\gamma^{-\theta}\right)}{(\gamma^{\theta}-1)}+3\geq-\frac{(\lambda+\gamma)}{(1+\gamma)}\frac{(\lambda-1/\lambda)}{(\lambda-1)}+3
$$

where we have used $\gamma^{\theta}\leq\lambda$. Simplifying further with $\lambda^{2}=2(\gamma+1)\lambda-\gamma$ from (51) leads to

$$
\begin{aligned}
-\frac{(\lambda+\gamma)}{(1+\gamma)}\frac{(\lambda-1/\lambda)}{(\lambda-1)}+3 &= -\frac{\lambda^{2}-1+\gamma\lambda-\gamma/\lambda}{(1+\gamma)(\lambda-1)}+3\\
&= -\frac{2(\gamma+1)\lambda-(\gamma+1)}{(1+\gamma)(\lambda-1)}-\frac{\lambda-1/\lambda}{(1+\gamma)(\lambda-1)}+3\\
&> -\frac{2\lambda-1}{\lambda-1}-\frac{\gamma\lambda}{\gamma\lambda+\lambda-(\gamma+1)}+3>-2-1+3=0
\end{aligned}
$$

The last inequality holds by $\lambda > (\gamma + 1)$ and proves the statement.

(iii) Notice with (44) - (50) that

$$\frac{d}{dq_0}\left(\frac{p_{n+1}-p_n}{q_0}\right) = \frac{d\left(p_{n+1}-p_n\right)}{d\varphi}\left(\frac{\varphi}{q_0^2}\frac{\theta-1}{\theta}\right)$$

Since furthermore,

$$\frac{d}{dq_0}\left(\frac{p_{n+1}-p_n}{q_0}\right) = \frac{1}{q_0}\frac{d\left(p_{n+1}-p_n\right)}{dq_0} - \frac{p_{n+1}-p_n}{q_0^2}$$

we have

$$\begin{aligned} \frac{d\left(p_{n+1}-p_n\right)}{dq_0} &= q_0\frac{d}{dq_0}\left(\frac{p_{n+1}-p_n}{q_0}\right) + \frac{p_{n+1}-p_n}{q_0} \\ &= q_0\frac{d\left(p_{n+1}-p_n\right)}{d\varphi}\left(\frac{\varphi}{q_0^2}\frac{\theta-1}{\theta}\right) + \frac{p_{n+1}-p_n}{q_0} \end{aligned}$$

which is positive by (ii).

(iv) Compute with (44) - (50)

$$\begin{aligned} \frac{d(p_n/c_n)}{d\varphi} &= \lambda^n\frac{d(A_N/c_n)}{d\varphi} + (\gamma/\lambda)^{N+n}\frac{d(B_N/c_n)}{d\varphi} \\ &= \frac{1}{3\gamma\left(1-\gamma^N\lambda^{-2N}\right)}\left\{\lambda^n(\lambda-2\gamma)\left[\gamma\frac{d(a_0/c_n)}{d\varphi} + (\gamma/\lambda)^N\frac{d(b_N/c_n)}{d\varphi}\right] + ...\right. \\ &\qquad \left. ... + (\gamma/\lambda)^{N+n}(\lambda-2)\left[\lambda^{-N}\gamma\frac{d(a_0/c_n)}{d\varphi} + \frac{d(b_N/c_n)}{d\varphi}\right]\right\} \\ &= \frac{(\gamma-1)/(\varphi c_n)}{3\gamma\left(1-\gamma^N\lambda^{-2N}\right)}\left\{\lambda^n(\lambda-2\gamma)[-q_0\bar{v} + (\gamma/\lambda)^N q_{-N}\underline{v}] + ...\right. \\ &\qquad \left. ... + (\gamma/\lambda)^{N+n}(\lambda-2)\left[-\lambda^{-N}q_0\bar{v} + q_{-N}\underline{v}\right]\right\} \end{aligned}$$

Straight forward simplification with $q_{-N} = \gamma^{-N}q_0$ leads to

$$\frac{d(p_n/c_n)}{d\varphi} = \frac{q_0\underline{v}\,(\gamma-1)/(\varphi c_n)}{3\gamma\left(1-\gamma^N\lambda^{-2N}\right)}\left\{\lambda^n(\lambda-2\gamma)\left[-\frac{\bar{v}}{\underline{v}} + \lambda^{-N}\right] + (\gamma/\lambda)^{N+n}(\lambda-2)\left[-\lambda^{-N}\frac{\bar{v}}{\underline{v}} + \gamma^{-N}\right]\right\}$$

Since $\bar{v}/\underline{v} > 1 > \lambda^{-N}$ holds always and $\bar{v}/\underline{v} > (\lambda/\gamma)^N$ by assumption, the term on the right is negative, which proves the statement. ■

C Appendix – Endogenous Growth

In the main body of this paper we have treated productivity growth (19) and interest rates as exogenous but have neglected economy-wide resource constraints. We address these shortcomings in this appendix. We show that a slight modification of the model we have developed here is compatible with individual optimization under balanced growth, which, in turn, is generated by spillover effects of innovation.

C.1 Extended Setup

We modify the preference setup introduced above in the following two ways. First, rather than focusing on only one Q-type sector, there is now a continuum of differentiated goods Q_j of total mass one ($j \in [0,1]$). We will refer to the different j as sectors. Each sector j comprises a set S_j of different quality levels $\{q_{nj}\}_{n \in S_j}$. The reason for introducing many such sectors is that in each sector, technology growth happens at discrete moments and the economy would thus display lumpy growth if it only consisted of a small number of sectors. With a continuum of sectors, aggregate growth is smooth, as is required for the existence of a balanced growth path.

Preferences. Individual utility is now derived from the quantity d of good D and the vector of consumed qualities $\mathbf{q} = \{q_j\}_{j \in [0,1]}$. Specifically, we modify the preference structure (1) to

$$u_v(\mathbf{q}, d) = v \int_0^1 q_j \, dj + d$$

where valuations v are distributed according to (2), just as in the previous sections. We can ensure that individuals with the highest valuation $\bar{\text{v}}$ consume a positive quantity of the homogeneous good D by choosing an adequately low value of $\bar{\text{v}}$. Consequently, all consumers consume good D, and the demand structure of Section 2 applies separately for each sector j. Thus, optimal consumption and pricing follows as developed above. Note also that resulting aggregate demand is identical across all sectors.

Intertemporal Optimization. Consumers trade off consumption between different periods. Instead of modeling this trade-off, we simply assume that under balanced growth, the economy's savings rate s is constant and positive.

Productivity. Production costs are equal across all Q-sectors. Within each sector j, the resulting equilibrium prices are determined by (7).

C.2 Technology Spillovers from Innovation

We assume that there is a generic productivity parameter B_t that depends on the qualities of all invented varieties in the Q-sectors:

$$B_t = b \int_0^1 \sum\nolimits_{n \in S_j} q_{nj} \; dj$$

Writing $q(t)$ for the variety invented at date t, we can rewrite the integral as

$$B_t = b \int_{-\infty}^{t} q(\tau) \; \Psi(\tau) \; d\tau$$

where $\Psi(.)$ is the rate at which the qualities are invented. On the balanced growth path, a constant number of qualities is invented in any two time intervals of equal length. Put differently, the rate at which qualities appear on the time-line is constant so that constant innovation activity generates constant productivity growth. We observe that within the time interval Δ from (25), each sector j experiences exactly one innovation. Hence, in each interval dt, there are $\Delta^{-1}dt$ qualities invented so that $\Psi = \Delta^{-1}$.

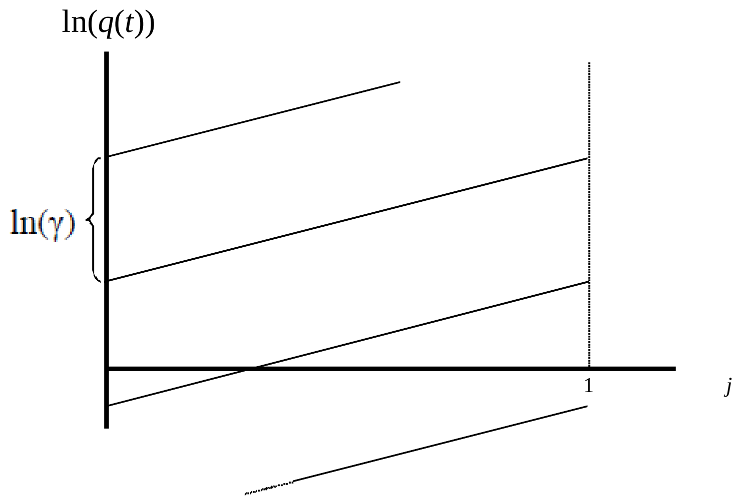

Figure 4

To determine the rate at which productivity grows, observe that within the period $[t, t + \Delta]$ exactly one invention is made in each sector so that $q(t + \Delta) = \bar{\gamma} q(t)$ holds true. Thus, with (25), the maximal quality $q(t)$ grows at the rate

$$g = a/(\theta - 1).$$

Figure 4 illustrates the (logged) quality levels of established firms on the industry-quality plane (j, q). Within each industry (for each j), firms establish in a regular equal log-spacing. If g is high ($q(t)$ grows fast), then the sloped line in Figure 4 extends rapidly to the upper left, replicating the regular pattern of the graph. The time required for each diagonal segment to establish is $\Delta = \ln(\gamma)/g$ by (25). Hence, a denser pattern ($\gamma > 0$ small) or a higher growth rate g correspondséto a faster pace at which the regular pattern extends to the top.

We chose the constant b to normalize $B_0 = 1$ and thus obtain

$$B_t = e^{gt} \tag{56}$$

Based on the definition of B_t we postulate the following type of productivity spillover. Production costs (3) and the marginal costs (4) of Q_j-production fall proportionately to $B_t^{\theta-1}$:

$$\varphi_t = \varphi B_t^{1-\theta} \qquad \text{and} \qquad \phi_t = \phi B_t^{1-\theta}$$

This assumption generates the dynamics specified in (19).

As the blueprint for a new quality requires $\phi_t q_t^\theta$ units of labor, and q_t grows at the rate g, the required labor units for a new quality grow at rate $g\theta + g(1-\theta) = g$. For the economy to be able to cover these resource requirements in the long run, we assume that labor endowments are multiplied by B_t, so that effective labor supply grows at the rate g. The population size is constant, so that nominal income grows at the rate g. On top of that effect, effective income growth materializes through improved quality.

C.3 Economic Aggregates

Income. Nominal labor income is pinned down by the constant productivity on the numéraire D and equals the size of the labor force, L. At each point in time total income Y_t consists of returns on past savings W_t plus labor income, i.e.,

$$Y_t = rW_t + B_t L. \tag{57}$$

On the balanced growth path, both components of income must grow at the same rate g.

Wealth. Aggregate wealth equals aggregate savings sY_t. By $Y_t = Y_o e^{gt}$ we can compute

$$W_t = \int_{-\infty}^{t} sY_\tau d\tau = sY_t/g \tag{58}$$

Investment. Within each time interval of length $\Delta = \ln(\gamma)/g$, the qualities of all industries are upgraded exactly once (see (25)). Thus, $\Delta^{-1} \cdot dt$ new blueprints appear in each infinitesimal time interval dt. The costs of these inventions are financed by the flow of investment

$$I_t = \Delta^{-1}\phi_t(q(t))^\theta = \Delta^{-1}\phi_o(q_o)^\theta \cdot e^{gt}. \tag{59}$$

where $q(t)$ is the quality invented at date t.

When writing the profits at time $t+\tau$ of the firm established at date t as $\pi(t, t+\tau)$, the free entry condition requires that

$$\phi_t(q(t))^\theta = \int_t^\infty e^{-r\tau}\pi(t, t+\tau)d\tau$$

Savings. By (58) the flow of aggregate savings is

$$S_t = gW_t. \tag{60}$$

C.4 Market Clearing

Aggregate savings equals aggregate investment. Since all investment is channeled to the invention of blueprints, this implies with (59) and (60)

$$gW_t = \Delta^{-1}\phi_t q_t^\theta = \Delta^{-1}\int_t^\infty e^{-r\tau}\pi(t, t+\tau)d\tau \tag{61}$$

Total firm profits at time t consist of the sum of current profits of all firms invented at t and before. These total profits cover the returns to investment, i.e.

$$rW_t = \int_0^\infty \pi(t-\tau, t)d\tau \tag{62}$$

To combine both expressions above, recall first that $q(t)$ represents the quality invented at date t. Moreover, two qualities established at t and $t-\tau$ satisfy the relation $q(t) = e^{g\tau}q(t-\tau)$. At time $t+\tau$, the producer of quality $q(t)$ faces the same market conditions as the producer of $q(t-\tau)$ at time t, except for the fact that its quality (and those of its competitors) is scaled by $e^{g\tau}$ and costs have dropped by $e^{-g(\theta-1)\tau}$ (see (19)). Thus, writing $\pi(t, t')$ for the profits at time t', which the firm established at time t makes ($t \leq t'$), we can apply Lemma 1 to establish $\pi(t, t+\tau) = e^{g\tau}\pi(\tau, t)$. Remember again that the flow of firms establishing at each point in time is Δ^{-1}. Multiplying the profits of the individual firm by this density, we can rewrite the integral in (61) as

$$\int_0^\infty \pi(t-\tau,t)\ d\tau = \Delta^{-1}\int_0^\infty e^{-g\tau}\pi(t,t+\tau)\ d\tau \tag{63}$$

Combining the identities (61) - (62) leads to

$$\int_0^\infty \left(ge^{-g\tau} - re^{-r\tau}\right)\pi(t,t+\tau)\ d\tau = 0,$$

which is solved by $r = g$.